

DERIVED INVARIANCE OF THE ALBANESE RELATIVE CANONICAL RING

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ABSTRACT. We show the derived invariance of various geometric invariants of smooth complex projective varieties governed by the Albanese map, including the relative canonical ring and the class of the relative canonical model in a suitable variant of the Grothendieck ring of varieties. Then we derive some applications to the derived invariance of Hodge numbers.

1. INTRODUCTION

This paper is concerned with the understanding of which aspects of the geometry of a smooth complex projective variety are preserved under exact equivalences of its derived category. One of the most fundamental results in this direction is the derived invariance of the canonical ring

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$$

(Orlov [O, Corollary 2.1.9]). In the first part of this paper we prove a similar result for the relative canonical ring under the Albanese morphism $a_X : X \rightarrow \text{Alb } X$

$$\mathcal{R}(a_X) = \bigoplus_{m \geq 0} a_{X*} \omega_X^{\otimes m}.$$

The second part of the paper is devoted to applications to the derived invariance of various geometric data. In particular we focus on motivic classes of irregular varieties in suitable variants of the Grothendieck ring of varieties introduced in the paper [IMOU], and on Hodge numbers.

Turning to a more detailed presentation, we recall that an exact equivalence $\Phi : D(X) \xrightarrow{\sim} D(Y)$ induces canonically an isomorphism of algebraic groups $\varphi : \text{Aut}^0 X \times \text{Pic}^0 X \xrightarrow{\sim} \text{Aut}^0 Y \times \text{Pic}^0 Y$, referred to as the Rouquier isomorphism [R]. By employing this isomorphism, in [PoS1, Theorem A] the authors show that the Picard varieties $\text{Pic}^0 X$ and $\text{Pic}^0 Y$ are (non-canonically) isogenous abelian varieties, but are in general not isomorphic. Hence the same happens for the Albanese varieties. In conclusion the Albanese relative canonical rings $\mathcal{R}(a_X)$ and $\mathcal{R}(a_Y)$ are sheaves over varieties which are in general non-isomorphic.

Our main result is that nevertheless $\mathcal{R}(a_X)$ and $\mathcal{R}(a_Y)$ are somewhat preserved by the (dual) Rouquier isomorphism. This is accomplished in a series of steps, of independent interest, leading to an auxiliary version of the Albanese relative canonical ring, better behaving under exact equivalence, as described below.

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We first consider the Iitaka fibration of X and more precisely one of its models

$$\tilde{X} \rightarrow Z_X$$

such that Z_X is smooth. The abelian subvariety $\text{Pic}^0 Z_X \xrightarrow{i} \text{Pic}^0 X$ does not depend on the model. Moreover the image of the Albanese map $a_{Z_X}: Z_X \rightarrow \text{Alb } Z_X$ coincides with the image of the natural morphism c_X (referred to as the *Albanese-Iitaka map*)

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \text{Alb } X \\ & \searrow c_X & \downarrow p_X \\ & & \text{Alb } Z_X \end{array} \quad (1.1)$$

where p_X is the projection dual of the inclusion i (see Subsection 2.5). Our first result is that the Albanese-Iitaka variety and the image of the Albanese-Iitaka map are preserved by the (dual) Rouquier isomorphism. In fact a more precise result holds true. This can be summarized as follows (see Theorem 3.0.1 for the precise statement).

Theorem A. *Let $\Phi: D(X) \rightarrow D(Y)$ be an exact equivalence. Then:*

- (a) *The subvariety $\text{Pic}^0 Z_X$ of $\text{Pic}^0 X$ is a derived invariant. More precisely, the Rouquier isomorphism sends $\text{Pic}^0 Z_X$ to $\text{Pic}^0 Z_Y$ inducing an isomorphism $\varphi: \text{Pic}^0 Z_X \xrightarrow{\sim} \text{Pic}^0 Z_Y$.*
- (b) *The dual isomorphism*

$$\hat{\varphi}: \text{Alb } Z_Y \xrightarrow{\sim} \text{Alb } Z_X \quad (1.2)$$

sends $c_Y(Y)$ to $c_X(X)$ (up to translation).

- (c) *More precisely, let*

$$\begin{array}{ccc} X & \xrightarrow{s_X} & X' \\ & \searrow c_X & \downarrow c'_X \\ & & \text{Alb } Z_X \end{array}$$

be the Stein factorization of the Albanese-Iitaka morphism c_X . By employing the same notation for the Stein factorization of the Albanese-Iitaka morphism of Y , there is an isomorphism $\psi: Y' \xrightarrow{\sim} X'$ such that the diagram

$$\begin{array}{ccc} X' & \xleftarrow{\sim} & Y' \\ \downarrow c'_X & \psi & \downarrow c'_Y \\ \text{Alb } Z_X & \xleftarrow{\hat{\varphi}} & \text{Alb } Z_Y \end{array}$$

is commutative.

Note: in the upcoming paper [CL, Theorem 3.0.1 (iii) and Remark 3.0.5], the authors prove further that the general fibers of s_X and s_Y are derived equivalent. Moreover, the equivalence Φ becomes a relative equivalence when restricted to the loci where both the morphisms $\psi^{-1} \circ s_X$ and s_Y are flat (however these further properties are not necessary in this paper).

The previous theorem enables us to prove the derived invariance of the relative canonical ring attached to the Albanese-Iitaka morphism:

$$\mathcal{R}(c_X) = \bigoplus_{m \geq 0} c_{X*} \omega_X^{\otimes m}.$$

More generally, we prove the invariance in the derived category of the complex

$$\mathcal{HA}(c_X) = \bigoplus_{m \in \mathbb{Z}, q \geq 0} \mathbf{R} c_{X*} (\Omega_X^q \otimes \omega_X^{\otimes m})[q], \quad (1.3)$$

whose cohomology can be thought as a relative version of the bigraded algebra

$$\bigoplus_{m \in \mathbb{Z}, k} \bigoplus_{p-q=k} H^p(X, \Omega_X^q \otimes \omega_X^{\otimes m})$$

in [O, Corollary 2.1.10]. We refer to §4.5 and §4.6 for the proof of the following theorem.

Theorem B. *Let $\Phi: D(X) \rightarrow D(Y)$ be an exact equivalence. Then the isomorphism (1.2) induces:*

(a) *an isomorphism of $\mathcal{O}_{\text{Alb } Z_Y}$ -algebras*

$$\widehat{\varphi}^* \mathcal{R}(c_X) \cong \mathcal{R}(c_Y);$$

(b) *isomorphisms of $\mathcal{O}_{\text{Alb } Z_Y}$ -modules*

$$\bigoplus_{p-q=k} \widehat{\varphi}^* R^p c_{X*} (\Omega_X^q \otimes \omega_X^{\otimes m}) \cong \bigoplus_{p-q=k} R^p c_{Y*} (\Omega_Y^q \otimes \omega_Y^{\otimes m}) \quad \forall m, k \in \mathbb{Z}.$$

Now it is time to state the announced result regarding the derived invariance of the relative canonical algebra with respect to the Albanese map. Roughly speaking, we prove that the Albanese relative canonical algebra is the pullback of a derived invariant algebra on the Albanese-Iitaka variety “weighted” by a derived invariant (finite) semigroup of topologically trivial line bundles.

Less informally, we prove two statements. The first one (which is independent on derived equivalences), is Theorem C(a) below. Roughly, its content is that the relative canonical ring $\mathcal{R}(a_X)$ is very close to being the pullback of a (finitely generated) graded $\mathcal{O}_{\text{Alb } Z_X}$ -algebra, denoted by \mathcal{U}_X , via the quotient map p_X of (1.1). More precisely, the graded components of the algebra \mathcal{U}_X are endowed with a canonical decomposition, and the relative canonical algebra $\mathcal{R}(a_X)$ is obtained from the pullback of \mathcal{U}_X by twisting the pullback of the summands appearing in such decompositions by torsion line bundles belonging to a certain finite subgroup S_X of $\text{Pic}^0 X$.

The second statement is Theorem C(b), telling that both the twisting semigroup S_X and the algebra \mathcal{U}_X are derived invariants. More precisely S_X and \mathcal{U}_X are sent to S_Y and \mathcal{U}_Y , respectively, by the Rouquier isomorphism and by the dual isomorphism (1.2). This second statement is in fact a variant of Theorem B(a). The precise statement (see Theorem 5.2.2) is as follows.

Theorem C. (a) *For any smooth projective variety X , and for all integers $m > 0$, there exists an essentially canonical decomposition*

$$a_{X*} \omega_X^{\otimes m} \cong \bigoplus_{i=1}^{N_X(m)} (p_X^* \mathcal{G}_{X,m,i}) \otimes P_{\alpha_{X,m,i}},$$

where $P_{\alpha_{X,m,i}}$ are line bundles parametrized by torsion points $\alpha_{X,m,i} \in \text{Pic}^0 X$. Moreover, the sheaves

$$\mathcal{U}_X^m := \bigoplus_{i=1}^{N_X(m)} \mathcal{G}_{X,m,i}$$

are the graded pieces of a finitely generated $\mathcal{O}_{\text{Alb } Z_X}$ -algebra \mathcal{U}_X .

- (b) Let $\Phi: D(X) \rightarrow D(Y)$ be an exact equivalence. For every $m > 0$ we have $N_X(m) = N_Y(m)$. Moreover, by keeping the setting and notation of Theorems A and B, we obtain

$$\varphi(\text{id}_X, \alpha_{X,m,i}) = (\text{id}_Y, \alpha_{Y,m,i})$$

(up to reordering). Finally, there is an isomorphism

$$\widehat{\varphi}^* \mathcal{U}_X \cong \mathcal{U}_Y$$

as $\mathcal{O}_{\text{Alb } Z_Y}$ -graded algebras.

Note that Orlov's theorem on the derived invariance of the canonical ring is recovered both from Theorem C and Theorem B(a) by taking H^0 .

Finally, we remark that in many circumstances the Rouquier isomorphism already sends $\text{Pic}^0 X$ to $\text{Pic}^0 Y$. For example this happens when $\text{Aut}^0 X$ is affine, or $\chi(X, \mathcal{O}_X) \neq 0$. In this case the statement of Theorem C is enormously simplified, because the Rouquier isomorphism induces a dual isomorphism $\widehat{\varphi}: \text{Alb } Y \rightarrow \text{Alb } X$ so that $\widehat{\varphi}^* \mathcal{R}(a_X) \cong \mathcal{R}(a_Y)$, see Corollary 8.1.4.

In the second part of the paper we draw the following applications of Theorem C.

Kodaira dimension. It follows from Theorem C that the Kodaira dimension of a general fiber of the Albanese map is a derived invariant. More precisely (see Corollary 5.4.1), we obtain an isomorphism between the canonical rings of the general fibers over points of $\text{Alb } X$ (resp. $\text{Alb } Y$) mapping to points of $\text{Alb } Z_X$ (resp. $\text{Alb } Z_Y$) corresponding via the isomorphism $\widehat{\varphi}$.

Motivic classes and Hodge numbers. More refined derived invariants are obtained in the case where the canonical sheaf is relatively big (respectively big and nef) with respect to the Albanese map. In the spirit of a recent paper of Ito-Miura-Okawa-Ueda [IMOU], we show that in this case suitable motivic classes attached to a pair of derived equivalent varieties X and Y are equal in suitable variants of the Grothendieck ring of varieties *modulo isogeneous abelian varieties* (see §6 and §7). More precisely, we consider the relative canonical model ([BCHM]) with respect to the Albanese map

$$X_a^{\text{can}} = \text{Proj } \mathcal{R}(a_X) \rightarrow \text{Alb } X.$$

Theorem D. *Let X and Y be smooth projective varieties over \mathbb{C} such that $D(X) \cong D(Y)$. Then:*

- (a) $[X_a^{\text{can}}] = [Y_a^{\text{can}}]$ in the Grothendieck ring $K_0^!(\text{Var}/\mathbb{C})$ of varieties over \mathbb{C} modulo isogeneous abelian varieties (see Theorem 6.1.1).
- (b) If ω_X is a_X -big and nef, then $[X] = [Y]$ in the localized and completed Grothendieck ring modulo isogeneous abelian varieties $\widehat{\mathcal{M}}_{\mathbb{C}}'$ (see Theorem 7.0.2).

In [IMOU, Problem 7.2] the authors ask whether derived equivalent varieties have the same class in $\widehat{\mathcal{M}}_{\mathbb{C}}'$. Theorem D(b) answers affirmatively their question when ω_X is a_X -big and nef. We

also refer the reader to the formulation under more general hypotheses provided by Theorem 7.5.1 below.

By taking virtual Hodge realizations and Hodge-Deligne polynomials, this has the following consequence about the invariance of Hodge numbers (Theorems 6.2.1 and 7.0.1).

Theorem E. *In the setting of Theorem D:*

- (a) *If ω_X is a_X -big, then $h^{0,j}(X) = h^{0,j}(Y)$ for all j .*
- (b) *If ω_X is nef and a_X -big, then the rational Hodge structures of X and Y are isomorphic for all k . In particular, $h^{i,j}(X) = h^{i,j}(Y)$ for all i and j .*

Chen-Jiang decompositions. In answer to a question of M. Popa, we prove the derived invariance of the Chen-Jiang decomposition of pushforwards of pluricanonical sheaves under the Albanese map (Proposition 5.3.2). This decomposition is a more refined version than the one appearing in Theorem C(a), which is often very useful in the study of the birational geometry of irregular varieties (see e.g. [CJ], [LoPoS], [MPo]).

Finally, variants and further applications are provided in Theorem 7.5.1 and in §8. The content of Appendix in §9 is a property of independent interest concerning Chen-Jiang decompositions (used in the proof of Proposition 5.3.2).

Let us briefly indicate the methods of the proofs. Crucial background results are recalled in §2. These include: (1) an explicit description of the Rouquier isomorphism ([PoS1]), especially when applied to the non-vanishing loci of pluricanonical bundles ([Lo1]); (2) the relation, based on generic vanishing theory, between the non-vanishing loci of pluricanonical bundles and the Albanese-Iitaka variety $\text{Alb } Z_X$ ([CH1] and [HaPoS]); (3) basic properties of the Fourier-Mukai-Poincaré (FMP) derived equivalence for an abelian variety and its dual. In some points we also make use of the recent symmetric version of such equivalence, due to Schnell ([Sch]).

Theorem A occupies §3. The essential ingredient is an adaptation of the classical results of Kawamata on the geometric properties of the support of the kernel of a derived equivalence ([Ka]). In doing this the above-mentioned preliminary results (1) and (2) play a crucial role.

The arguments of Theorems B and C (§4 and §5) consist of several steps, hopefully of independent interest. In particular, we make essential use of the FMP derived equivalence. The main step is Theorem 4.4.1, which in turn makes use of Theorem 4.3.1. Briefly, we first prove the isomorphisms between the *FMP transforms* of the two sides of the isomorphisms of Theorems B and C, and then apply the inverse FMP transform in order to obtain the desired isomorphisms. The same principle is at the base of the proof of the multiplicativity of the isomorphism of Theorem B(a). An antecedent of this argument, interesting for its own sake, is the derived invariance of the *paracanonical ring*, proved in the section of background results (Theorem 2.4.1). We refer to the introduction of §4 for a more detailed presentation of the line of the argument. Another essential ingredient in the proof of Theorem C(a) is a direct-sum decomposition of direct images of pluricanonical bundles under the Albanese map (see Proposition 5.1.1). This in turn is proved using the previously mentioned Chen-Jiang decomposition ([CJ], [LoPoS]), a further result of generic vanishing theory.

The applications provided by Theorems D and E are proved in §6 and §7. They make essential use of Theorem C and, in particular, of the scheme

$$\text{Proj } (\mathcal{U}_X) \rightarrow \text{Alb } Z_X$$

where \mathcal{U}_X is the sheaf of rings appearing in the statement. The argument of Theorem D(b) first proves that Theorem D(a) implies the equality of the *Gorenstein volumes* of X_a^{can} and Y_a^{can} in the ring $\widehat{\mathcal{M}}_{\mathbb{C}}$. In order to pass from the Gorenstein volumes to the classes of the varieties X and Y themselves, we make use of well-known results of the minimal model program.

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2. BACKGROUND MATERIAL

In this section we establish the notation and recall known results which will be used throughout the paper. The only new result is Theorem 2.4.1 regarding the derived invariance of the paracanonical ring.

2.1. Derived equivalences. A variety is a smooth and irreducible projective variety defined over \mathbb{C} , unless otherwise specified. If X is a variety, we denote by $D(X) = D^b(\mathcal{Coh}(X))$ its bounded derived category of coherent sheaves, endowed with its natural structure of triangulated category.

Let X and Y be varieties. An exact equivalence $\Phi: D(X) \rightarrow D(Y)$ will be referred to as a *derived equivalence*, or simply, *equivalence*. By Orlov's representability theorem, any equivalence is a Fourier-Mukai functor. Namely there exists a complex \mathcal{E} in $D(X \times Y)$, unique up to isomorphism, such that Φ is isomorphic to the functor

$$\Phi_{\mathcal{E}}(-) := \mathbf{R}q_*(p^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{E}),$$

where p and q are the projections from $X \times Y$ onto X and Y , respectively. A quasi-inverse of $\Phi_{\mathcal{E}}$ is given by the equivalence $\Phi_{\text{ad } \mathcal{E}}$ associated to the kernel

$$\text{ad } \mathcal{E} := \mathcal{E}^{\vee} \otimes p^* \omega_X[n] \cong \mathcal{E}^{\vee} \otimes q^* \omega_Y[n],$$

where $n = \dim X = \dim Y$ (recall that the dimension of a variety is a derived invariant), ω_X (*resp.* ω_Y) denotes the canonical line bundle of X (*resp.* Y), and $\mathcal{E}^{\vee} := \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X \times Y})$ is the derived dual. Denote now by

$$\delta_X: X \hookrightarrow X \times X \quad \text{and} \quad \delta_Y: Y \hookrightarrow Y \times Y$$

the diagonal embeddings of X and Y , respectively. Moreover, denote by p_{ij} the projections from the product $X \times X \times Y \times Y$ onto the i -th and j -th factor, respectively. By a further result of Orlov [O, p. 535], the object

$$\mathcal{E} \boxtimes \text{ad } \mathcal{E} := p_{13}^* \mathcal{E} \otimes p_{24}^* \text{ad } \mathcal{E} \quad \text{in} \quad D(X \times X \times Y \times Y)$$

defines a new equivalence $\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}}: D(X \times X) \rightarrow D(Y \times Y)$ which satisfies

$$\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}}(\delta_{X*} \omega_X^{\otimes m}) \cong \delta_{Y*} \omega_Y^{\otimes m} \quad \forall m \in \mathbb{Z}. \quad (2.1)$$

2.2. The Rouquier isomorphism and Rouquier-stable line bundles. We will adopt the following notation. After having fixed Poincaré line bundles \mathcal{P} and \mathcal{Q} on $X \times \text{Pic}^0 X$ and $Y \times \text{Pic}^0 Y$, respectively, we denote by $P_{\alpha} := \mathcal{P}|_{X \times \{\alpha\}}$ (*resp.* $Q_{\beta} := \mathcal{Q}|_{Y \times \{\beta\}}$) the corresponding line bundle on X (*resp.* Y) parametrized by the point $\alpha \in \text{Pic}^0 X$ (*resp.* $\beta \in \text{Pic}^0 Y$). Moreover, given an automorphism $f: X \rightarrow X$, we denote by $\gamma_f: X \hookrightarrow X \times X$ the graph-embedding $x \mapsto (x, f(x))$. With this terminology, we point out that $\gamma_{\text{id}_X} = \delta_X$ is the diagonal embedding.

In [R] (see also [PoS1] for the present formulation), Rouquier shows that the equivalence $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ induces a functorial isomorphism of algebraic groups

$$\varphi_{\mathcal{E}}: \mathrm{Aut}^0 X \times \mathrm{Pic}^0 X \xrightarrow{\sim} \mathrm{Aut}^0 Y \times \mathrm{Pic}^0 Y \quad (2.2)$$

which will be referred to as the *Rouquier isomorphism*. As pointed out by Popa and Schnell in [PoS1, Lemma 3.1], this can be explicitly described as follows: given pairs $(f, \alpha) \in \mathrm{Aut}^0 X \times \mathrm{Pic}^0 X$ and $(h, \beta) \in \mathrm{Aut}^0 Y \times \mathrm{Pic}^0 Y$, then $\varphi_{\mathcal{E}}(f, \alpha) = (h, \beta)$ if and only if

$$p^* P_{\alpha} \otimes (f \times \mathrm{id}_Y)^* \mathcal{E} \cong q^* Q_{\beta} \otimes (\mathrm{id}_X \times h)_* \mathcal{E}. \quad (2.3)$$

By a result of the second author [Lo1, Lemma 2.1], the following generalization of (2.1) holds: if $\varphi_{\mathcal{E}}(f, \alpha) = (h, \beta)$, then

$$\Phi_{\mathcal{E} \boxtimes \mathrm{ad} \mathcal{E}}(\gamma_{f*}(\omega_X^{\otimes m} \otimes P_{\alpha})) \cong \gamma_{h*}(\omega_Y^{\otimes m} \otimes Q_{\beta}) \quad (2.4)$$

for all $m \in \mathbb{Z}$.

A closed point $\alpha \in \mathrm{Pic}^0 X$, or, equivalently, the corresponding line bundle P_{α} , is said to be *Rouquier-stable with respect to the equivalence $\Phi_{\mathcal{E}}$* (or simply *R-stable* if this will not cause any confusion) if

$$\varphi_{\mathcal{E}}(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$$

for some $\beta \in \mathrm{Pic}^0 Y$. If this is the case, with a slight abuse of notation we will set

$$\beta = \varphi_{\mathcal{E}}(\alpha). \quad (2.5)$$

Hence, for *R-stable* line bundles, (2.4) reduces to

$$\Phi_{\mathcal{E} \boxtimes \mathrm{ad} \mathcal{E}}(\delta_{X*}(\omega_X^{\otimes m} \otimes P_{\alpha})) \cong \delta_{Y*}(\omega_Y^{\otimes m} \otimes Q_{\varphi_{\mathcal{E}}(\alpha)}). \quad (2.6)$$

2.3. Non-vanishing loci. Given a coherent sheaf \mathcal{F} on X , the following Zariski-closed subset of $\mathrm{Pic}^0 X$

$$V^0(X, \mathcal{F}) = V^0(\mathcal{F}) = \{ \alpha \in \mathrm{Pic}^0 X \mid h^0(X, \mathcal{F} \otimes P_{\alpha}) > 0 \} \quad (2.7)$$

will be referred to as *the non-vanishing locus of \mathcal{F}* . Note that

$$V^0(X, \mathcal{F}) = V^0(\mathrm{Alb} X, a_{X*} \mathcal{F}).$$

In this paper the non-vanishing loci $V^0(X, \omega_X^{\otimes m})$ will play an important role. By seminal results in generic vanishing theory, they are finite unions of torsion translates of abelian subvarieties of $\mathrm{Pic}^0 X$. A more precise description will be given in §2.5 below. Their importance in the present paper stems from another result of the second author ([Lo1, Proposition 3.1]) stating, in the terminology of §2.2, that: *if $\alpha \in V^0(X, \omega_X^{\otimes m})$ for some $m \in \mathbb{Z}$, then α is R-stable, and the equivalence $\Phi_{\mathcal{E} \boxtimes \mathrm{ad} \mathcal{E}}$ induces an isomorphism*

$$H^0(X, \omega_X^{\otimes m} \otimes P_{\alpha}) \cong H^0(Y, \omega_Y^{\otimes m} \otimes Q_{\varphi_{\mathcal{E}}(\alpha)}) \quad (2.8)$$

for all $m \in \mathbb{Z}$.

In particular, it follows that the Rouquier isomorphism acts as

$$\varphi_{\mathcal{E}}(\mathrm{id}_X, V^0(X, \omega_X^{\otimes m})) = (\mathrm{id}_Y, V^0(Y, \omega_Y^{\otimes m})) \quad (2.9)$$

for all $m \in \mathbb{Z}$. Hence, for any $m \in \mathbb{Z}$, the Rouquier isomorphism induces an isomorphism of algebraic closed sets $V^0(X, \omega_X^{\otimes m}) \cong V^0(Y, \omega_Y^{\otimes m})$.

2.4. The paracanonical ring. The *paracanonical ring* of a smooth projective variety is, by definition, the bigraded ring

$$P(X, \omega_X) = \bigoplus_{m \geq 0, \alpha \in \text{Pic}^0 X} H^0(X, \omega_X^{\otimes m} \otimes P_\alpha).$$

We prove that this ring is a derived invariant.

Theorem 2.4.1. *Given an equivalence $\Phi_\varepsilon: D(X) \rightarrow D(Y)$ of smooth projective varieties, the equivalence $\Phi_{\varepsilon \boxtimes \text{ad} \varepsilon}$ induces an isomorphism of algebras*

$$P(X, \omega_X) \cong P(Y, \omega_Y).$$

Proof. The isomorphism $P(X, \omega_X) \cong P(Y, \omega_Y)$ follows from (2.8) and (2.9) because the (m, α) -graded component of the paracanonical ring is non-zero if and only if $\alpha \in V^0(X, \omega_X^{\otimes m})$. As usual, the multiplicativity holds because the multiplication maps

$$H^0(X, \omega_X^{\otimes m} \otimes P_\alpha) \otimes H^0(X, \omega_X^{\otimes k} \otimes P_{\alpha'}) \longrightarrow H^0(X, \omega_X^{\otimes (m+k)} \otimes P_{\alpha+\alpha'}) \quad (2.10)$$

are given by composition (see e.g. [O, p. 535] or [Huy, Proposition 6.1]) for all $\alpha, \alpha' \in \text{Pic}^0 X$ and $m, k \in \mathbb{Z}$. \square

Note that that the canonical ring is a subring of the paracanonical ring. Hence the above theorem includes Orlov's derived invariance of the canonical ring ([O, Corollary 2.1.9]).

2.5. The Albanese-Iitaka morphism. Let X be a variety with non-negative Kodaira dimension. In this subsection we recall work of Chen, Hacon, Pardini, Popa and Schnell, among others, showing that for $m \geq 1$ the geometry of the non-vanishing loci $V^0(X, \omega_X^{\otimes m})$ is closely related to the geometry of the Iitaka fibration.

To set up the notation, let us choose a smooth birational modification $\tilde{X} \rightarrow X$ such that the Iitaka fibration is represented by a morphism $f_{\tilde{X}}: \tilde{X} \rightarrow Z_X$ with Z_X smooth. There is a commutative diagram

$$\begin{array}{ccccc} & & a_{\tilde{X}} & & \\ & & \curvearrowright & & \\ \tilde{X} & \longrightarrow & X & \xrightarrow{a_X} & \text{Alb}(X) \\ & & \downarrow & \searrow c_X & \downarrow p_X \\ & & Z_X & \xrightarrow{a_{Z_X}} & \text{Alb}(Z_X) \\ & \searrow f_{\tilde{X}} & & & \end{array} \quad (2.11)$$

where $a_X, a_{\tilde{X}}$ and a_{Z_X} are Albanese maps, and p_X is a fibration (*i.e.* a surjective morphism with connected fibers) of abelian varieties induced by $f_{\tilde{X}}$ (see [HaPa, Proposition 2.1.b]¹). Note that the abelian variety $\text{Alb } Z_X$ and the morphisms p_X and $c_X = p_X \circ a_X$ only depend on X , and not on the modification \tilde{X} . We call

$$c_X: X \rightarrow \text{Alb } Z_X$$

the *Albanese-Iitaka morphism* of X . We recall that the Albanese morphism a_X is defined up to translation in $\text{Alb } X$, and similarly the morphism c_X is defined up to translation in $\text{Alb } Z_X$. The choice of a morphism a_X determines a morphism c_X .

¹in the quoted reference this is stated only for varieties of maximal Albanese dimension, but the proof works in general

As previously mentioned, the irreducible components of the loci $V^0(X, \omega_X^{\otimes m})$ are torsion translates of abelian subvarieties of $\text{Pic}^0 X$. For $m = 1$ this is part of a classical result of Green-Lazarsfeld ([GL]), while for $m \geq 2$ this is a result of Chen-Hacon ([CH2, Theorem 3.2]) see also [Lai, Theorem 3.5] and [HaPoS, Theorem 10.1]). Moreover, by another result of Chen-Hacon ([CH1, Lemma 2.2]; see also [HaPoS, (2) after Lemma 11.1]) for $m = 1$ such subvarieties are contained in torsion translates of the abelian subvariety $p_X^* \text{Pic}^0 Z_X$, namely

$$V^0(\text{Alb } X, a_{X*} \omega_X) \subseteq \bigsqcup_{i=1}^{N_X(1)} p_X^* \text{Pic}^0 Z_X - \alpha_{1,i} \quad (2.12)$$

where $\alpha_{1,i}$ are torsion points of $\text{Pic}^0 X$ and the translates $p_X^* \text{Pic}^0 Z_X - \alpha_{1,i}$ are distinct translates intersecting $V^0(\text{Alb } X, a_{X*} \omega_X)$ non-trivially. For $m > 1$, by a theorem of Hacon-Popa-Schnell ([HaPoS, Theorem 11.2(b)]), the above mentioned abelian subvarieties *coincide* with $p_X^* \text{Pic}^0 Z_X$, namely

$$V^0(\text{Alb } X, a_{X*} \omega_X^{\otimes m}) = \bigsqcup_{i=1}^{N_X(m)} p_X^* \text{Pic}^0 Z_X - \alpha_{m,i}. \quad (2.13)$$

Let G_X be the union of the loci $V^0(X, \omega_X^{\otimes m})$ for all $m \geq 0$. As observed in [CH2, Lemma 3.3], G_X is a subgroup of $\text{Pic}^0 X$. Indeed, it is a semigroup because if $\alpha_1 \in V^0(X, \omega_X^{\otimes m})$ and $\alpha_2 \in V^0(X, \omega_X^{\otimes n})$ then clearly $\alpha_1 + \alpha_2 \in V^0(X, \omega_X^{\otimes(m+n)})$. The fact that the various components are *torsion* translates of abelian subvarieties make G_X automatically a group, containing $\text{Pic}^0 Z_X$ as a subgroup. Although not strictly necessary for the results of the present paper, it is worth to remark that $\text{Pic}^0 Z_X$ has finite index in G_X . This was proved, using generic vanishing methods, in [CH2, p. 204 and Corollary 3.4], and follows also by the general finite generation result of [BCHM] (see Remark 5.1.4 below).

Now, let Y be another smooth projective variety. There are similar decompositions on Y :

$$V^0(\text{Alb } Y, a_{Y*} \omega_Y) \subseteq \bigsqcup_{i=1}^{N_Y(1)} p_Y^* \text{Pic}^0 Z_Y - \beta_{1,i} \quad (2.14)$$

$$V^0(\text{Alb } Y, a_{Y*} \omega_Y^{\otimes m}) = \bigsqcup_{i=1}^{N_Y(m)} p_Y^* \text{Pic}^0 Z_Y - \beta_{m,i} \quad (2.15)$$

for some torsion points $\beta_{m,i} \in \text{Pic}^0 Y$. From (2.9), it follows that if X and Y are derived equivalent, then the Rouquier isomorphism induces isomorphisms of algebraic groups

$$\varphi_{\mathcal{E}}: G_X \xrightarrow{\sim} G_Y, \quad (2.16)$$

$$\varphi_{\mathcal{E}}: \text{Pic}^0 Z_X \xrightarrow{\sim} \text{Pic}^0 Z_Y \quad (2.17)$$

and, moreover, equalities

$$N_X(m) = N_Y(m) \quad (2.18)$$

for all $m \geq 1$. Finally, up to reordering, one can arrange

$$\beta_{m,i} = \varphi_{\mathcal{E}}(\alpha_{m,i}).$$

Crucial to our main results will be a sheaf-theoretic version of the decompositions (2.12) and (2.13). This will be the content of Proposition 5.1.1 in the sequel.

2.6. (Symmetric) FMP transform on abelian varieties and generic vanishing. Given an abelian variety A , the normalized Poincaré line bundle \mathcal{P} on $A \times \text{Pic}^0 A$ may be regarded as the kernel of an equivalence, introduced in [Mu]:

$$\Phi_{\mathcal{P}}: D(A) \rightarrow D(\text{Pic}^0 A). \quad (2.19)$$

We will refer to it as the Fourier-Mukai-Poincaré (FMP) transform. In the recent paper [Sch], Schnell introduced a useful variant of it, the *symmetric FMP transform*, which carries the same information, but renders the notation and several arguments more transparent. It is defined as follows:

$$\text{FM}_A := \Phi_{\mathcal{P}} \circ \Delta_A: D(A) \rightarrow D(\text{Pic}^0 A)^{op}$$

where $\Delta_A(-) = \mathbf{R}\mathcal{H}om(-, \mathcal{O}_A[g])$ and $g = \dim A$.

In this language, a coherent sheaf \mathcal{F} on A is said to be a *GV-sheaf* (generic vanishing sheaf) if $\text{FM}_A(\mathcal{F})$ is a sheaf (in degree 0). If this is the case, the usual notation is

$$\text{FM}_A(\mathcal{F}) \cong \widehat{\mathcal{F}^\vee}.$$

If the sheaf $\widehat{\mathcal{F}^\vee}$ is, in addition, torsion-free, then the sheaf \mathcal{F} is said to be *M-regular*. References concerning these notions are for instance contained in [Pa1], [PaPo2], [PaPoS] and [Sch]. It suffices here to say that, by base change and Serre duality, if \mathcal{F} is a *GV-sheaf*, then the set-theoretic support of the sheaf $\widehat{\mathcal{F}^\vee}$ is the subvariety $-V^0(A, \mathcal{F})$, and the fiber $\widehat{\mathcal{F}^\vee} \otimes \mathbb{C}(\alpha)$ at a point $\alpha \in \text{Pic}^0 A$ is naturally identified to the linear space $H^0(A, \mathcal{F} \otimes P_{-\alpha})^\vee$ (see §9 especially (9.5) below for the scheme-theoretic support). If \mathcal{F} is a nonzero *M-regular* sheaf, then of course the support of $\widehat{\mathcal{F}^\vee}$ is the full $\text{Pic}^0 A$ and

$$V^0(A, \mathcal{F}) = \text{Pic}^0 A. \quad (2.20)$$

The generic vanishing theorems of Hacon and Popa-Schnell for morphisms $f: X \rightarrow A$ from smooth complex projective varieties to abelian varieties assert that *for all $m \geq 1$ the direct images $f_* \omega_X^{\otimes m}$ are GV-sheaves* (see Hacon's paper [Ha, Theorem 1.5] for the case $m = 1$, and Popa-Schnell's paper [PoS2, Theorem 1.10] for the other cases $m > 1$). A refinement of these results (the *Chen-Jiang decomposition*) will be recalled in the course of the proof of Proposition 5.1.1.

3. DERIVED INVARIANCE OF THE STEIN FACTORIZATION OF THE ALBANESE-ITAKA MORPHISM

In this section we consider the Stein factorization of the Albanese-Iitaka morphism. Let X and Y be smooth projective varieties and consider the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \text{Alb } X \\ \downarrow s_X & \searrow c_X & \downarrow p_X \\ X' & \xrightarrow{c'_X} & \text{Alb } Z_X \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{a_Y} & \text{Alb } Y \\ \downarrow s_Y & \searrow c_Y & \downarrow p_Y \\ Y' & \xrightarrow{c'_Y} & \text{Alb } Z_Y \end{array} \quad (3.1)$$

where we keep the notation of §2.5, and the left-bottom sides of the diagrams are the Stein factorizations of c_X and c_Y .

In this section we prove that a derived equivalence $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ induces an isomorphism of the bottom arrows of the above diagrams, *i.e.* of the finite morphisms appearing in the Stein factorizations of the Albanese-Iitaka morphisms. To introduce the precise statement, given a morphism of abelian varieties $f: A \rightarrow B$, we denote by $\widehat{f}: \text{Pic}^0 B \rightarrow \text{Pic}^0 A$ the dual

morphism. By dualizing the induced Rouquier isomorphism (2.17), we obtain the isomorphism $\widehat{\varphi}_{\mathcal{E}}: \text{Alb } Z_Y \rightarrow \text{Alb } Z_X$.

Theorem 3.0.1. *An equivalence $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ induces an isomorphism $\psi: Y' \rightarrow X'$ such that the following diagram*

$$\begin{array}{ccc} X' & \xleftarrow[\psi]{\sim} & Y' \\ \downarrow c'_X & & \downarrow c'_Y \\ \text{Alb } Z_X & \xleftarrow[\widehat{\varphi}_{\mathcal{E}}]{\sim} & \text{Alb } Z_Y \end{array} \quad (3.2)$$

is commutative.

Remark 3.0.2. By recalling that the Albanese-Iitaka morphisms c_X and c_Y are defined only up to translations, the meaning of the above statement is that one can find representatives of c'_X and c'_Y such that diagram (3.2) is commutative.

Proof. We adapt an argument of [Lo2, Theorem 1], inspired by Kawamata's kernel support technique [Ka], to our more general setting. By keeping the notation of diagrams (3.1), we consider the support $\text{Supp}(\mathcal{E}) = \bigcup_j \mathcal{H}^j(\mathcal{E})$ of the kernel \mathcal{E} (with the reduced structure). We will show that the image of $\text{Supp}(\mathcal{E})$ via the morphism

$$(s_X \times s_Y): X \times Y \rightarrow X' \times Y'$$

is the graph of an isomorphism $\psi: Y' \rightarrow X'$ satisfying the conclusion of the statement.

Let p' and q' be the natural projections from $X' \times Y'$ onto the first and second factor, respectively. In the first place we note that:

(*) *the restrictions of the projections p' and q' to $(s_X \times s_Y)(\text{Supp}(\mathcal{E}))$ dominate X' and Y' , respectively, and have connected fibers.*

This follows from the well-known fact that $\text{Supp}(\mathcal{E})$ itself dominates both X and Y , via morphisms with connected fibers ([Huy, Lemma 6.4]).

Next, we claim that the projection from $(s_X \times s_Y)(\text{Supp}(\mathcal{E}))$ onto X' has finite fibers. In order to prove this, firstly we note that, for a sufficiently ample line bundle L on $X \times Y$, there is an equality of closed subsets

$$\text{Supp}(\mathbf{R}(s_X \times s_Y)_*(\mathcal{E} \otimes L)) = (s_X \times s_Y)(\text{Supp}(\mathcal{E})). \quad (3.3)$$

Indeed, since L is sufficiently positive, by the degeneration of the hypercohomology spectral sequence computing $R^i(s_X \times s_Y)_*(\mathcal{E} \otimes L)$, we have that

$$R^i(s_X \times s_Y)_*(\mathcal{E} \otimes L) \cong (s_X \times s_Y)_*\mathcal{H}^i(\mathcal{E} \otimes L) \quad (3.4)$$

This proves (3.3). Now, every line bundle on X parametrized by a point $\alpha \in c_X^* \text{Pic}^0 Z_X$ is of the form $c_X^* P'_\alpha$, where P'_α is a line bundle on $\text{Alb } Z_X$. Similarly, we denote Q'_β line bundles parametrized by $\text{Pic}^0 Z_Y$. Thus the relation (2.3) defining the Rouquier isomorphism $\varphi_{\mathcal{E}}$ can be written as follows

$$p^* c_X^* P'_\alpha \otimes \mathcal{E} \cong q^* c_Y^* Q'_{\varphi_{\mathcal{E}}(\alpha)} \otimes \mathcal{E}.$$

Tensorizing with our sufficiently positive line bundle L on $X \times Y$ we get

$$p^* c_X^* P'_\alpha \otimes \mathcal{E} \otimes L \cong q^* c_Y^* Q'_{\varphi_{\mathcal{E}}(\alpha)} \otimes \mathcal{E} \otimes L. \quad (3.5)$$

Applying $\mathbf{R}(s_X \times s_Y)_*$, by projection formula we get

$$p'^* c'_X{}^* P'_\alpha \otimes \mathbf{R}(s_X \times s_Y)_*(\mathcal{E} \otimes L) \cong q'^* c'_Y{}^* Q'_{\varphi_{\mathcal{E}}(\alpha)} \otimes \mathbf{R}(s_X \times s_Y)_*(\mathcal{E} \otimes L). \quad (3.6)$$

Assume that the projection p' contracts a curve $C \subset \text{Supp}(\mathbf{R}(s_X \times s_Y)_*(\mathcal{E} \otimes L))$. Let i be an integer such that the irreducible curve C is contained in $\text{Supp}(R^i(s_X \times s_Y)_*(\mathcal{E} \otimes L))$. Taking the i -th cohomology \mathcal{H}^i in (3.6) and then restricting to C , it follows that

$$(R^i(s_X \times s_Y)_*(\mathcal{E} \otimes L))|_C \cong q'^* c'_Y{}^* Q'_\beta \otimes (R^i(s_X \times s_Y)_*(\mathcal{E} \otimes L))|_C$$

for all $\beta = \varphi_{\mathcal{E}}(\alpha) \in \text{Pic}^0 Z_Y$. This implies that $(c'_Y{}^* Q'_\beta)|_C$ is of finite order for all $\beta \in \text{Pic}^0 Z_Y$. In turn, this implies that C is contracted by c'_Y . But c'_Y is a map with finite fibers. Hence no curve can be contracted by the projection p' , as claimed.

It follows in particular that $\dim X' \geq \dim Y'$. Since the role of X and Y can be exchanged we have that $\dim X' = \dim Y'$. Moreover, from (*) it follows that $(s_X \times s_Y)(\text{Supp}(\mathcal{E}))$ projects isomorphically onto both X' and Y' , so that the map

$$\psi := p' \circ q'^{-1}: Y' \rightarrow X'$$

is an isomorphism.

In order to prove the last part of the statement, it is enough to show that the isomorphism $\psi: Y' \rightarrow X'$ induces an isomorphism $\psi^*: c'_X{}^* \text{Pic}^0 Z_X \rightarrow c'_Y{}^* \text{Pic}^0 Z_Y$ which agrees with the Rouquier isomorphism $\varphi_{\mathcal{E}}$ of (2.17). By restricting (3.6) to the normal variety $\text{Graph}(\psi)$, and by taking determinants, we find a positive integer r such that

$$\psi^* c'_X{}^* (P'_\alpha)^{\otimes r} \cong c'_Y{}^* (Q'_{\varphi_{\mathcal{E}}(\alpha)})^{\otimes r}$$

for all $\alpha \in \text{Pic}^0 Z_X$. From this we deduce that, for all $\alpha \in \text{Pic}^0 Z_X$,

$$\psi^* c'_X{}^* P'_\alpha \cong c'_Y{}^* Q'_{\varphi_{\mathcal{E}}(\alpha)}.$$

This is exactly what needed. \square

4. RELATIVE CANONICAL RING AND HOCHSCHILD STRUCTURE OF THE ALBANESE-IITAKA MORPHISM

This section is concerned with the derived invariance of the relative canonical ring under the Albanese-Iitaka morphism (Theorem 4.6.1). This will be crucial for the result concerning the relative canonical ring under the Albanese morphism (Theorem 5.2.2). Along the way, in Subsection 4.5, we will consider a relative version of the algebras in [O, Corollary 2.1.10] and show its derived invariance (as graded coherent sheaf, see Corollary 4.5.1).

The argument involves several steps as follows: we first globalize the isomorphisms (2.6) to an isomorphism involving the Poincaré line bundles (Theorem 4.3.1 below). This allows the use of the FMP transform, yielding the isomorphisms (4.13) for FMP transforms, globalizing and generalizing the isomorphisms of linear spaces (2.8). Applying the inverse FMP transform we get the desired isomorphisms for the relative canonical rings, and, more generally, for the previously mentioned relative algebras (Theorem 4.4.1 and Corollaries 4.5.1, 4.5.2). Finally, we prove the isomorphism of the multiplicative structure of the relative canonical rings by reducing it to the isomorphism of the paracanonical rings, again using the FMP transform (Theorem 4.6.1).

4.1. Preliminary lemma. We will need the following preliminary standard Lemma. Let R, T, A, B be smooth projective varieties. Let \mathcal{F} be an object in $D(R \times T)$ and let $f: A \rightarrow B$ be an isomorphism. We consider the functor

$$\Phi_{\mathcal{F}} \boxtimes f_* = \Phi_{\mathcal{F} \boxtimes \mathcal{O}_{\Gamma_f}}: D(R \times A) \rightarrow D(T \times B),$$

where $\Gamma_f \subset A \times B$ denotes the graph of f . We denote by $p_A: R \times A \rightarrow A$ and $p_B: T \times B \rightarrow B$ the natural projections.

Lemma 4.1.1. *Let $U \in D(R \times A)$ and $V \in D(A)$. Then there is an isomorphism*

$$(\Phi_{\mathcal{F}} \boxtimes f_*)(U \otimes^{\mathbf{L}} p_A^* V) \cong ((\Phi_{\mathcal{F}} \boxtimes f_*)(U)) \otimes^{\mathbf{L}} p_B^*(f_* V).$$

Proof. From the basic relation on $A \times B$

$$q_A^* V \otimes^{\mathbf{L}} \mathcal{O}_{\Gamma_f} \cong q_B^*(f_* V) \otimes^{\mathbf{L}} \mathcal{O}_{\Gamma_f}, \quad (4.1)$$

we get the relation on $(R \times A) \times (T \times B)$

$$\bar{p}_A^* V \otimes^{\mathbf{L}} p_{A \times B}^* \mathcal{O}_{\Gamma_f} \cong \bar{p}_B^*(f_* V) \otimes^{\mathbf{L}} p_{A \times B}^* \mathcal{O}_{\Gamma_f}, \quad (4.2)$$

where in (4.1) q_A and q_B are the projections from $A \times B$ onto A and B , respectively, while \bar{p}_A and \bar{p}_B in (4.2) are the projections from $R \times A \times T \times B$ onto A and B , respectively. Therefore there are isomorphisms

$$\begin{aligned} \mathbf{R}p_{\mathcal{F} \boxtimes \mathcal{O}_{\Gamma_f}}(U \otimes^{\mathbf{L}} p_A^* V) &\cong \mathbf{R}p_{T \times B*}(p_{R \times A}^*(U \otimes^{\mathbf{L}} p_A^* V) \otimes^{\mathbf{L}} (\mathcal{F} \boxtimes \mathcal{O}_{\Gamma_f})) \\ &\cong \mathbf{R}p_{T \times B*}(p_{R \times A}^* U \otimes^{\mathbf{L}} \bar{p}_A^* V \otimes^{\mathbf{L}} (\mathcal{F} \boxtimes \mathcal{O}_{\Gamma_f})) \\ &\stackrel{(4.2)}{\cong} \mathbf{R}p_{T \times B*}(p_{R \times A}^* U \otimes^{\mathbf{L}} \bar{p}_B^*(f_* V) \otimes^{\mathbf{L}} (\mathcal{F} \boxtimes \mathcal{O}_{\Gamma_f})) \\ &\cong ((\Phi_{\mathcal{F}} \boxtimes f_*)(U)) \otimes^{\mathbf{L}} p_B^*(f_* V) \end{aligned}$$

where the last isomorphism follows by the projection formula. \square

4.2. Setting and notation. The next step will be the proof of Theorem 4.3.1 here below, which is in fact a global version of (2.6). As usual we assume that there is an equivalence $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$. To establish the notation, recall that the Rouquier isomorphism induces an isomorphism $\varphi_{\mathcal{E}}: \text{Pic}^0 Z_X \rightarrow \text{Pic}^0 Z_Y$ (see (2.17)). We choose Poincaré line bundles \mathcal{P}_Z and \mathcal{Q}_Z respectively on $\text{Alb } Z_X \times \text{Pic}^0 Z_X$ and $\text{Alb } Z_Y \times \text{Pic}^0 Z_Y$, normalized so that $(\mathcal{P}_Z)|_{\{e_X\} \times \text{Pic}^0 Z_X}$ is trivial where e_X is the identity element of $\text{Alb } Z_X$, and similarly for \mathcal{Q}_Z . From the universal property of the Poincaré line bundle, it follows that $(\widehat{\varphi}_{\mathcal{E}}^{-1} \times \varphi_{\mathcal{E}})^* \mathcal{Q}_Z$ is a Poincaré line bundle on $\text{Alb } Z_X$. Since it satisfies the above normalization, it follows that

$$(\widehat{\varphi}_{\mathcal{E}}^{-1} \times \varphi_{\mathcal{E}})^* \mathcal{Q}_Z \cong \mathcal{P}_Z.$$

Let

$$\mathcal{P}_{Z,X} := (c_X \times \text{id}_{\text{Pic}^0 Z_X})^* \mathcal{P}_Z \quad \text{and} \quad \mathcal{Q}_{Z,Y} := (c_Y \times \text{id}_{\text{Pic}^0 Z_Y})^* \mathcal{Q}_Z$$

be the induced Poincaré line bundles on $X \times \text{Pic}^0 Z_X$ and $Y \times \text{Pic}^0 Z_Y$, respectively. We consider the fibered diagonal embedding

$$\widetilde{\delta}_X = (\delta_X \times \text{id}_{\text{Pic}^0 Z_X}): X \times \text{Pic}^0 Z_X \rightarrow X \times X \times \text{Pic}^0 Z_X, \quad (x, \alpha) \mapsto (x, x, \alpha) \quad (4.3)$$

and similarly for Y . Finally, we consider the equivalence

$$\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*} = \Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E} \boxtimes \mathcal{O}_{\Gamma_{\varphi_{\mathcal{E}}}}} : D(X \times X \times \text{Pic}^0 Z_X) \rightarrow D(Y \times Y \times \text{Pic}^0 Z_Y) \quad (4.4)$$

where we are adopting the notation of §§2.1, 2.2 and 2.5. We also denote by $q_X: X \times \text{Pic}^0 Z_X \rightarrow X$ and $q_Y: Y \times \text{Pic}^0 Z_Y \rightarrow Y$ the natural projections onto the first factors.

4.3. Global version of (2.6).

Theorem 4.3.1. *In the setting of §4.2 there are isomorphisms*

$$(\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \cong \widetilde{\delta}_{Y*} (q_Y^* \omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y})$$

for all $m \in \mathbb{Z}$.

Proof. As usual, given a closed point $\alpha \in \text{Pic}^0 Z_X$, we denote by $P_{Z,\alpha}$ the corresponding line bundle on X (i.e. $c_X^*(\mathcal{P}_Z|_{\text{Alb } Z_X \times \{\alpha\}})$), and we will adopt the notation $Q_{Z,\beta}$ for line bundles on Y parametrized by points in $\text{Pic}^0 Z_Y$. To begin with, we claim that, for all closed points β of $\text{Pic}^0 Z_Y$, there are isomorphisms

$$(\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \otimes \mathcal{O}_{Y \times Y \times \{\beta\}} \cong \delta_{Y*} (\omega_Y^{\otimes m} \otimes Q_{Z,\beta}). \quad (4.5)$$

Indeed, for $\alpha = \varphi_{\mathcal{E}}^{-1}(\beta)$, we have that

$$\begin{aligned} \Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} (\delta_{X*} (\omega_X^{\otimes m} \otimes P_{Z,\alpha})) &\cong (\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X}) \otimes \mathcal{O}_{X \times X \times \{\alpha\}}) \\ &\cong (\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \otimes^{\mathbf{L}} \mathcal{O}_{Y \times Y \times \{\beta\}} \end{aligned}$$

where the last isomorphism follows from Lemma 4.1.1 (note that, because of the flatness of $\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})$, the second underived tensor product in the right-hand side of the first line coincides with the derived tensor product). Therefore, from the above quoted (2.6), it follows that

$$(\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \otimes^{\mathbf{L}} \mathcal{O}_{Y \times Y \times \{\beta\}} \cong \delta_{Y*} (\omega_Y^{\otimes m} \otimes Q_{Z,\beta}).$$

Note that this implies that the derived tensor product in the left-hand side is in fact concentrated in degree zero. In conclusion, we have proved (4.5).

By the see-saw principle, (4.5) implies that

$$(\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \cong \widetilde{\delta}_{Y*} (q_Y^* \omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y}) \otimes p_{\text{Pic}^0 Z_Y}^* L \quad (4.6)$$

where L is a line bundle on $\text{Pic}^0 Z_Y$ and $p_{\text{Pic}^0 Z_Y} : Y \times Y \times \text{Pic}^0 Z_Y \rightarrow \text{Pic}^0 Z_Y$ is the projection onto the third factor.

Let Δ_Y be the diagonal in $Y \times Y$. Finally, we prove that the line bundle L is trivial by proving that the restriction to $\{(y_0, y_0)\} \times \text{Pic}^0 Z_Y$ of the line bundle on $\Delta_Y \times \text{Pic}^0 Z_Y$ appearing in the left-hand side of (4.6) is the trivial line bundle, where $y_0 \in Y$ is a point such that $c_Y(y_0) = e_Y$ (the identity element of $\text{Alb } Z_Y$). Such restricted line bundle is the complex (concentrated in degree 0)

$$(\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*}) (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \otimes^{\mathbf{L}} \mathcal{O}_{\{(y_0, y_0)\} \times \text{Pic}^0 Z_Y}. \quad (4.7)$$

By the projection formula, such line bundle can be described as

$$\mathbf{R}\tilde{q}_* (\tilde{p}^* (\widetilde{\delta}_{X*} (q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \otimes^{\mathbf{L}} (\mathcal{E} \boxtimes \text{ad } \mathcal{E} \boxtimes \mathcal{O}_{\Gamma_{\varphi_{\mathcal{E}}}}) \otimes^{\mathbf{L}} \mathcal{O}_{X \times X \times \text{Pic}^0 Z_X \times \{(y_0, y_0)\} \times \text{Pic}^0 Z_Y}), \quad (4.8)$$

where \tilde{p} and \tilde{q} are the projections from $(X \times X \times \text{Pic}^0 Z_X) \times (Y \times Y \times \text{Pic}^0 Z_Y)$ onto the first and second factor, respectively. By means of Theorem 3.0.1 one can prove that $(\mathcal{E} \otimes^{\mathbf{L}} \text{ad } \mathcal{E}) \otimes^{\mathbf{L}} \mathcal{O}_{X \times \{y_0\}}$ is set-theoretically supported on the fiber of c_X over the identity point $e_X \in \text{Alb } Z_X$.² By using this,

²This goes as follows. By construction $(s_X \times s_Y)(\text{Supp}(\mathcal{E})) = \text{Graph}(\psi)$ (see the proof of Theorem 3.0.1), hence $\text{Supp}(\mathcal{E})$ is contained in $(s_X \times s_Y)^{-1}(\text{Graph}(\psi))$. So

$$\text{Supp}(\mathcal{E} \otimes^{\mathbf{L}} \text{ad } \mathcal{E} \otimes^{\mathbf{L}} \mathcal{O}_{X \times \{y_0\}}) \subseteq \text{Supp}(\mathcal{E} \otimes^{\mathbf{L}} \mathcal{O}_{X \times \{y_0\}}) = \text{Supp}(\mathcal{E}) \cap (X \times \{y_0\}) \subseteq s_X^{-1}(\psi(s_Y(y_0))),$$

together with some calculations, it follows that up to tensorizing with a trivial locally free sheaf of rank one, the line bundle (4.8) (on $\{(y_0, y_0)\} \times \text{Pic}^0 Z_Y$) is isomorphic to $\varphi_{\mathcal{E}*}(\mathcal{P}_Z \otimes \mathcal{O}_{\{e_X\} \times \text{Pic}^0 Z_X})$, which is trivial. This concludes the proof. \square

The following variant will be useful in the next section.

Variant 4.3.2. *Let $\bar{\alpha} \in \text{Pic}^0 X$ be a point parametrizing an R -stable line bundle $P_{\bar{\alpha}}$ (§2.2). Then for all $m \in \mathbb{Z}$ there are isomorphisms*

$$(\Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*})(\widetilde{\delta}_{X*}(q_X^*(\omega_X^{\otimes m} \otimes P_{\bar{\alpha}}) \otimes \mathcal{P}_{Z,X})) \cong \widetilde{\delta}_{Y*}(q_Y^*(\omega_Y^{\otimes m} \otimes Q_{\varphi_{\mathcal{E}}(\bar{\alpha})}) \otimes \mathcal{Q}_{Z,Y}).$$

The proof is exactly the same as that of Theorem 4.3.1.

4.4. Main result.

Theorem 4.4.1. *By keeping the previous notation and setting, there is an isomorphism of functors*

$$\mathbf{R}c_{Y*} \circ \mathbf{L}\delta_Y^* \cong \widehat{\varphi}_{\mathcal{E}}^* \circ \mathbf{R}c_{X*} \circ \mathbf{L}\delta_X^* \circ \Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}},$$

where $\widehat{\varphi}_{\mathcal{E}}$ denotes the dual isomorphism of the Rouquier isomorphism and the equivalence $\Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}}$ is taken in the opposite direction:

$$\Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}}: D(Y \times Y) \rightarrow D(X \times X).$$

Proof. To begin with, we claim that there are isomorphisms

$$\Phi_{\widetilde{\delta}_{X*}(\mathcal{P}_{Z,X})} \cong \Phi_{\mathcal{P}_Z} \circ \mathbf{R}c_{X*} \circ \mathbf{L}\delta_X^* \quad \text{and} \quad \Phi_{\widetilde{\delta}_{Y*}(\mathcal{Q}_{Z,Y})} \cong \Phi_{\mathcal{Q}_Z} \circ \mathbf{R}c_{Y*} \circ \mathbf{L}\delta_Y^*. \quad (4.9)$$

In order to prove this, we first note that, by employing [O, Proposition 2.1.2] describing the kernel of the composition of integral functors, the functors

$$\Phi_{\widetilde{\delta}_{X*}(q_X^*\omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})}: D(X \times X) \rightarrow D(\text{Pic}^0 Z_X), \quad \Phi_{\widetilde{\delta}_{Y*}(q_Y^*\omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y})}: D(Y \times Y) \rightarrow D(\text{Pic}^0 Z_Y)$$

verify the isomorphisms:

$$\begin{aligned} \Phi_{\widetilde{\delta}_{X*}(q_X^*\omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})}(-) &\cong \Phi_{\mathcal{P}_{Z,X}}(\mathbf{L}\delta_X^*(-) \otimes \omega_X^{\otimes m}) \\ \text{and} \quad \Phi_{\widetilde{\delta}_{Y*}(q_Y^*\omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y})}(-) &\cong \Phi_{\mathcal{Q}_{Z,Y}}(\mathbf{L}\delta_Y^*(-) \otimes \omega_Y^{\otimes m}). \end{aligned} \quad (4.10)$$

For $m = 0$ we get

$$\Phi_{\widetilde{\delta}_{X*}(\mathcal{P}_{Z,X})} \cong \Phi_{\mathcal{P}_{Z,X}} \circ \mathbf{L}\delta_X^* \quad \text{and} \quad \Phi_{\widetilde{\delta}_{Y*}(\mathcal{Q}_{Z,Y})} \cong \Phi_{\mathcal{Q}_{Z,Y}} \circ \mathbf{L}\delta_Y^*. \quad (4.11)$$

On the other hand, since $\mathcal{P}_{Z,X} = (c_X \times \text{id})^* \mathcal{P}_Z$ and $\mathcal{Q}_{Z,Y} = (c_Y \times \text{id})^* \mathcal{Q}_Z$, from projection formula it follows that

$$\Phi_{\mathcal{P}_{Z,X}} \cong \Phi_{\mathcal{P}_Z} \circ \mathbf{R}c_{X*} \quad \Phi_{\mathcal{Q}_{Z,Y}} \cong \Phi_{\mathcal{Q}_Z} \circ \mathbf{R}c_{Y*}. \quad (4.12)$$

By plugging (4.12) into (4.11) we get what claimed, namely (4.9).

The next step consists in showing that Theorem 4.3.1 yields the following isomorphism of functors, involving the Fourier-Mukai-Poincaré equivalences $\Phi_{\mathcal{P}_Z}: D(\text{Alb } Z_X) \rightarrow D(\text{Pic}^0 Z_X)$ and $\Phi_{\mathcal{Q}_Z}: D(\text{Alb } Z_Y) \rightarrow D(\text{Pic}^0 Z_Y)$:

$$\Phi_{\mathcal{Q}_Z} \circ \mathbf{R}c_{Y*} \circ \mathbf{L}\delta_Y^* \cong \varphi_{\mathcal{E}*} \circ \Phi_{\mathcal{P}_Z} \circ \mathbf{R}c_{X*} \circ \mathbf{L}\delta_X^* \circ \Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}}, \quad (4.13)$$

where $\Phi_{\mathcal{E} \boxtimes_{\text{ad}} \mathcal{E}}$ is taken in the opposite direction.

where the first inclusion comes from the spectral sequence in [Huy, (3.9)]. Now $c'_X(\psi(s_Y(y_0))) = e_X$, because of $c_Y(y_0) = e_Y$ and the commutativity of (3.2).

In order to prove this fact, we recall first Orlov's relation [O, Proposition 2.1.6] (or [Huy, Exercise 5.13(ii)]): given two equivalences $\Phi_{\mathcal{F}_i} : D(X_i) \rightarrow D(Y_i)$ for $i = 1, 2$, and an object $\mathcal{R} \in D(X_1 \times X_2)$, then the image $\mathcal{S} := \Phi_{\mathcal{F}_1 \boxtimes \mathcal{F}_2}(\mathcal{R})$ satisfies the relation

$$\Phi_{\mathcal{S}} \cong \Phi_{\mathcal{F}_2} \circ \Phi_{\mathcal{R}} \circ \Phi_{\mathcal{F}_1},$$

where the kernel \mathcal{F}_1 is used in the opposite direction, *i.e.* $\Phi_{\mathcal{F}_1} : D(Y_1) \rightarrow D(X_1)$. Now we apply this to $\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}} \boxtimes \varphi_{\mathcal{E}*} \cong \Phi_{(\mathcal{E} \boxtimes \text{ad } \mathcal{E}) \boxtimes \mathcal{O}_{\Gamma \varphi_{\mathcal{E}}}}$ and the object $\mathcal{R} = \delta_{X*}(p_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})$. By combining with Theorem 4.3.1, we get the relation

$$\Phi_{\delta_{Y*}(q_Y^* \omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y})} \cong \varphi_{\mathcal{E}*} \circ \Phi_{\delta_{X*}(q_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})} \circ \Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}}.$$

For $m = 0$ we get

$$\Phi_{\delta_{Y*}(\mathcal{Q}_{Z,Y})} \cong \varphi_{\mathcal{E}*} \circ \Phi_{\delta_{X*}(\mathcal{P}_{Z,X})} \circ \Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}}. \quad (4.14)$$

By plugging (4.9) into (4.14), we get (4.13).

Finally, let us recall that a quasi-inverse of the FMP equivalence $\Phi_{\mathcal{Q}_Z} : D(\text{Alb } Z_Y) \rightarrow D(\text{Pic}^0 Z_Y)$ is $(-1_{\text{Alb } Z_Y})^* \circ \Phi_{\mathcal{Q}_Z}[q]$, where now $\Phi_{\mathcal{Q}_Z}$ goes in the opposite direction and $q = \dim \text{Alb } Z_Y$. We recall the commutativity relation between isogenies of abelian varieties (in particular isomorphisms) and the FMP equivalence:

$$\Phi_{\mathcal{Q}_Z} \circ \varphi_{\mathcal{E}*} \cong \widehat{\varphi}_{\mathcal{E}}^* \circ \Phi_{\mathcal{P}_Z}$$

([Mu, (3.4)]). Therefore, by applying the equivalence $(-1_{\text{Alb } Z_Y})^* \circ \Phi_{\mathcal{Q}_Z}[q]$ to both sides of (4.13) we get the statement. \square

4.5. Relative structures. We recall the Kostant-Hochschild-Rosenberg quasi-isomorphism (see e.g. [Cal]):

$$I : \mathbf{L}\delta_X^*(\delta_{X*} \omega_X^{\otimes m}) \xrightarrow{\sim} \bigoplus_{i=0}^{\dim X} L^{-i} \delta_X^*(\delta_{X*} \omega_X^{\otimes m})[i] = \bigoplus_{i=0}^{\dim X} \Omega_X^i \otimes \omega_X^{\otimes m}[i]. \quad (4.15)$$

By combining this with Theorem 4.4.1, we get the derived invariance of the relative structure (1.3) of the Introduction.

Corollary 4.5.1. *For any $m \in \mathbb{Z}$ we have isomorphisms*

$$\bigoplus_{q=0}^{\dim X} \widehat{\varphi}_{\mathcal{E}}^* \mathbf{R}c_{X*}(\Omega_X^q \otimes \omega_X^{\otimes m})[q] \cong \bigoplus_{q=0}^{\dim Y} \mathbf{R}c_{Y*}(\Omega_Y^q \otimes \omega_Y^{\otimes m})[q].$$

Proof. From (2.1) it follows that

$$\Phi_{\mathcal{E} \boxtimes \text{ad } \mathcal{E}}(\delta_{Y*} \omega_Y^{\otimes m}) \cong \delta_{X*} \omega_X^{\otimes m} \quad \forall m \in \mathbb{Z} \quad (4.16)$$

(note: in (2.1) this is stated “in the other direction”, *i.e.* for the functor $D(X \times X) \rightarrow D(Y \times Y)$ but the statement of (2.1) recovers also (4.16)). Hence by plugging $\delta_{Y*} \omega_Y^{\otimes m}$ into Theorem 4.4.1 we get isomorphisms

$$\mathbf{R}c_{Y*} \circ \mathbf{L}\delta_Y^*(\delta_{Y*} \omega_Y^{\otimes m}) \cong \widehat{\varphi}_{\mathcal{E}}^* \circ \mathbf{R}c_{X*} \circ \mathbf{L}\delta_X^*(\delta_{X*} \omega_X^{\otimes m}) \quad \forall m \in \mathbb{Z}.$$

Finally, by invoking the HKR isomorphisms (4.15) we get the corollary. \square

By taking k -th cohomology in the isomorphisms of the previous corollary, we obtain the following result.

Corollary 4.5.2. *For any m and k there are isomorphisms*

$$\bigoplus_{p-q=k} \widehat{\varphi}_{\mathcal{E}}^*(R^p c_{X*}(\Omega_X^q \otimes \omega_X^{\otimes m})) \cong \bigoplus_{p-q=k} R^p c_{Y*}(\Omega_Y^q \otimes \omega_Y^{\otimes m}).$$

In particular, there are isomorphisms

$$\widehat{\varphi}_{\mathcal{E}}^*(c_{X*}\omega_X^{\otimes m}) \cong c_{Y*}\omega_Y^{\otimes m} \quad \forall m \in \mathbb{Z}.$$

Proof. The last statement is obtained for $k = -\dim X$. \square

In analogy with Variant 4.3.2, we have the following variant of Corollary 4.5.2.

Variante 4.5.3. *Let $\bar{\alpha} \in \text{Pic}^0 X$ be a point parametrizing an R -stable line bundle on X (see §2.2). For all m and k there are isomorphisms*

$$\bigoplus_{p-q=k} \widehat{\varphi}_{\mathcal{E}}^*(R^p c_{X*}(\Omega_X^q \otimes \omega_X^{\otimes m} \otimes P_{\bar{\alpha}})) \cong \bigoplus_{p-q=k} R^p c_{Y*}(\Omega_Y^q \otimes \omega_Y^{\otimes m} \otimes Q_{\varphi_{\mathcal{E}}(\bar{\alpha})}).$$

In particular, we obtain isomorphisms

$$\widehat{\varphi}_{\mathcal{E}}^*(c_{X*}(\omega_X^{\otimes m} \otimes P_{\bar{\alpha}})) \cong c_{Y*}(\omega_Y^{\otimes m} \otimes Q_{\varphi_{\mathcal{E}}(\bar{\alpha})}) \quad \forall m \in \mathbb{Z}.$$

4.6. The relative canonical ring. Given a morphism of projective varieties $f: X \rightarrow A$ we denote by

$$\mathcal{R}(f) := \bigoplus_{m \geq 0} f_* \omega_X^{\otimes m}$$

the associated relative canonical algebra. We will show the derived invariance of the multiplicative structure of $\mathcal{R}(c_X)$ (in this paper we will not address the problem of the derived invariance of the multiplicative structure of $\mathcal{H}\mathcal{A}(c_X)$).

Theorem 4.6.1. *With the notation as in the previous setting there is an isomorphism of $\mathcal{O}_{\text{Alb } Z_Y}$ -algebras*

$$\widehat{\varphi}_{\mathcal{E}}^* \mathcal{R}(c_X) \cong \mathcal{R}(c_Y).$$

Proof. The isomorphism as graded $\mathcal{O}_{\text{Alb } Z_Y}$ -modules follows from Corollary 4.5.2. What is left to prove is the multiplicativity of the isomorphism. This does not follow as easily as in the absolute case. Our argument will appeal to generic vanishing theorems of Hacon, Popa, Schnell and others, although we expect that there should be a self-contained argument. Specifically let

$$M_{k,n}(X): (c_{X*} \boxtimes c_{X*})(\omega_X^{\otimes k} \boxtimes \omega_X^{\otimes n}) \rightarrow (c_{X*} \boxtimes c_{X*})(\delta_{X*} \omega_X^{\otimes(k+n)}) \cong \delta_{\text{Alb } Z_X} c_{X*} \omega_X^{\otimes(k+n)}$$

be the multiplication map (the map $\delta_{\text{Alb } Z_X}$ is the diagonal embedding of $\text{Alb } Z_X$). We need to prove that

$$(\widehat{\varphi}_{\mathcal{E}}^* \boxtimes \widehat{\varphi}_{\mathcal{E}}^*)(M_{k,n}(X)) \cong M_{k,n}(Y). \quad (4.17)$$

We apply the symmetric FMP transform $\text{FM}_{\text{Alb } Z_Y}$ (§2.6) to (4.17). Recall the usual commutation formulas between FMP transforms and morphisms of abelian varieties, say $f: A \rightarrow B$:

$$\text{FM}_B \circ \mathbf{R}f_* \cong \mathbf{L}\hat{f}^* \circ \text{FM}_A, \quad \text{FM}_A \circ \mathbf{L}f^* \cong \mathbf{R}\hat{f}_* \circ \text{FM}_B \quad (4.18)$$

([Sch, Proposition 4.1]). We have that

$$\text{FM}_{\text{Alb } Z_Y} \circ \widehat{\varphi}_{\mathcal{E}}^* \cong \varphi_{\mathcal{E}*} \circ \text{FM}_{\text{Alb } Z_X}.$$

Therefore (4.17) is equivalent to

$$\varphi_{\mathcal{E}*}((\text{FM}_{\text{Alb } Z_X} \boxtimes \text{FM}_{\text{Alb } Z_X})(M_{k,n}(X))) \cong (\text{FM}_{\text{Alb } Z_Y} \boxtimes \text{FM}_{\text{Alb } Z_Y})(M_{k,n}(Y)). \quad (4.19)$$

Now $(\mathrm{FM}_{\mathrm{Alb} Z_X} \boxtimes \mathrm{FM}_{\mathrm{Alb} Z_X})(M_{k,n}(X))$ coincides with the following natural morphism in $D(\mathrm{Pic}^0 Z_X \times \mathrm{Pic}^0 Z_X)$:

$$(\mathrm{FM}_{\mathrm{Alb} Z_X} \boxtimes \mathrm{FM}_{\mathrm{Alb} Z_X})(\delta_{\mathrm{Alb} Z_X} c_{X*} \omega_X^{\otimes(k+n)}) \rightarrow \mathrm{FM}_{\mathrm{Alb} Z_X}(c_{X*} \omega_X^{\otimes k}) \boxtimes \mathrm{FM}_{\mathrm{Alb} Z_X}(c_{X*} \omega_X^{\otimes n}). \quad (4.20)$$

By the generic vanishing theorems of Hacon and Popa-Schnell quoted in §2.5, the complexes $\mathrm{FM}_{\mathrm{Alb} Z_X}(c_{X*} \omega_X^{\otimes h})$ are *sheaves concentrated in degree 0* for every $h > 0$, which we denote by $\overline{(c_{X*} \omega_X^{\otimes h})}^\vee$. Therefore (4.20) is naturally identified to a morphism

$$m_{\mathrm{Pic}^0 Z_X}^* \overline{(c_{X*} \omega_X^{\otimes(k+n)})}^\vee \longrightarrow \overline{(c_{X*} \omega_X^{\otimes k})}^\vee \boxtimes \overline{(c_{X*} \omega_X^{\otimes n})}^\vee, \quad (4.21)$$

where $m_{\mathrm{Pic}^0 Z_X} : \mathrm{Pic}^0 Z_X \times \mathrm{Pic}^0 Z_X \rightarrow \mathrm{Pic}^0 Z_X$ is the group law in $\mathrm{Pic}^0 Z_X$, as by (4.18) there is an isomorphism

$$(\mathrm{FM}_{\mathrm{Alb} Z_X} \boxtimes \mathrm{FM}_{\mathrm{Alb} Z_X}) \circ \delta_{\mathrm{Alb} Z_X} \cong m_{\mathrm{Pic}^0 Z_X}^* \circ \mathrm{FM}_{\mathrm{Alb} Z_X}$$

(the dual of the diagonal embedding of an abelian variety is the group law of the dual variety). By base change and Serre duality, the fiber of the map (4.21) over any point $(\alpha, \alpha') \in \mathrm{Pic}^0 Z_X \times \mathrm{Pic}^0 Z_X$ is a linear map

$$H^0(\mathrm{Alb} Z_X, c_{X*} \omega_X^{\otimes(k+n)} \otimes P_{\alpha+\alpha'})^\vee \rightarrow H^0(\mathrm{Alb} Z_X, c_{X*} \omega_X^{\otimes k} \otimes P_\alpha)^\vee \otimes H^0(\mathrm{Alb} Z_X, c_{X*} \omega_X^{\otimes n} \otimes P_{\alpha'})^\vee$$

i.e.

$$H^0(X, \omega_X^{\otimes(k+n)} \otimes P_{\alpha+\alpha'})^\vee \rightarrow H^0(X, \omega_X^{\otimes k} \otimes P_\alpha)^\vee \otimes H^0(X, \omega_X^{\otimes n} \otimes P_{\alpha'})^\vee.$$

These maps are dual to the usual multiplication maps (2.10). The same holds for the fibers of the map $(\mathrm{FM}_{\mathrm{Alb} Z_Y} \boxtimes \mathrm{FM}_{\mathrm{Alb} Z_Y})(M_{k,n}(Y))$.

Having said that, Theorem 2.4.1 and its proof ensure that (4.19) holds *at the fiber of every point* $(\varphi_{\mathcal{E}}(\alpha), \varphi_{\mathcal{E}}(\alpha')) \in \mathrm{Pic}^0 Z_Y \times \mathrm{Pic}^0 Z_Y$. By Nakayama's Lemma, this implies that (4.19) holds, since it is known that, for k and n positive, the scheme-theoretic support of both the source and target of (4.19) are the various components of the loci $-V^0(X, \omega_X^{\otimes h})$ (*i.e.* translates of abelian subvarieties), with their reduced scheme structure (see the Appendix below).

In the case where either k or n is zero the desired multiplicativity follows automatically because the map

$$c_{X*} \mathcal{O}_X \otimes c_{X*} \omega_X^{\otimes n} \rightarrow c_{X*} \omega_X^{\otimes n}$$

(*i.e.* the $(c_{X*} \mathcal{O}_X)$ -module structure of $c_{X*} \omega_X^{\otimes n}$) is induced canonically from the identity $\mathrm{id} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Alb} Z_X}}(c_{X*} \omega_X^{\otimes n}, c_{X*} \omega_X^{\otimes n})$, via the following composition of homomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathrm{Alb} Z_X}}(c_{X*} \omega_X^{\otimes n}, c_{X*} \omega_X^{\otimes n}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_X}(c_X^* c_{X*} \omega_X^{\otimes n}, \omega_X^{\otimes n}) \\ &\downarrow \\ \mathrm{Hom}_{\mathcal{O}_{\mathrm{Alb} Z_X}}(c_{X*} c_X^* c_{X*} \omega_X^{\otimes n}, c_{X*} \omega_X^{\otimes n}) & \\ &\downarrow \cong \\ \mathrm{Hom}_{\mathcal{O}_{\mathrm{Alb} Z_X}}(c_{X*} \mathcal{O}_X \otimes c_{X*} \omega_X^{\otimes n}, c_{X*} \omega_X^{\otimes n}). & \end{aligned}$$

□

5. THE RELATIVE CANONICAL ALGEBRA OF THE ALBANESE MORPHISM

The aim of this section is to extend Theorem 4.6.1 to the Albanese morphism itself. The main difference with the setting of Theorem 4.6.1 is that Albanese varieties are in general not stable under the Rouquier isomorphism. A universally known example of this phenomenon is given by the Fourier-Mukai-Poincaré equivalence between an abelian variety A and its dual $\text{Pic}^0 A$. Its Rouquier isomorphism exchanges $\text{Aut}^0 A$ (*i.e.* A itself) with $\text{Pic}^0 A$. Therefore there are no non-trivial R -stable subvarieties of $\text{Pic}^0 A$ in this case (see also §8.1 below). Therefore, in general, there is no hope to dually recover from the Rouquier isomorphism any non trivial map between $\text{Alb } Y$ and $\text{Alb } X$, not even between quotients of them.

However it turns out that the relative canonical ring of the Albanese morphism is still preserved by the Rouquier isomorphism, although in a weaker form. This is the content of Theorem 5.2.2 below. Roughly speaking, we construct a finitely generated graded algebra \mathcal{U}_X of coherent sheaves on the Albanese-Iitaka variety $\text{Alb } Z_X$. Such algebra is preserved by the Rouquier isomorphism and its graded components carry an essentially canonical direct-sum decomposition. The relative canonical algebra $\mathcal{R}(a_X)$ is obtained from the algebra $p_X^* \mathcal{U}_X$ by twisting the pullbacks of the various summands by certain torsion line bundles in $\text{Pic}^0 X$.

5.1. Decomposition of direct images. The main step towards the proof of Theorem 5.2.2 is the following fact, already observed for $m = 1$ by the first and third author in [CP, Step 2 p.737], and in [LoPoS, Theorem D] for $m \geq 2$.

Proposition 5.1.1. *Let X be a smooth projective complex variety and let $a_X: X \rightarrow \text{Alb } X$ be its Albanese morphism. Moreover, let $c_X: X \rightarrow \text{Alb } Z_X$ be the Albanese-Iitaka morphism and $p_X: \text{Alb } X \rightarrow \text{Alb } Z_X$ the natural quotient induced by c_X .*

(a) *For any $m \geq 1$ there is a decomposition of the form*

$$a_{X*} \omega_X^{\otimes m} \cong \bigoplus_{j=1}^{N_X(m)} (p_X^* \mathcal{G}_{X,m,j}) \otimes P_{\alpha_{m,j}}, \quad (5.1)$$

where:

- (i) *the positive integer $N_X(m)$ is the one defined in (2.12) and (2.13);*
- (ii) *the sheaves $\mathcal{G}_{X,m,j}$ are nonzero and GV on $\text{Alb } Z_X$ for all m and j (see §2.6);*
- (iii) *the $\alpha_{m,j}$'s are torsion points in $\text{Pic}^0 X$ for all m and j .*

(b) *There are isomorphisms*

$$c_{X*}(\omega_X^{\otimes m} \otimes P_{-\alpha_{m,j}}) \cong \mathcal{G}_{X,m,j} \quad (5.2)$$

for all m and j .

Proof. Point (a) follows from the existence of the *Chen-Jiang decomposition* for direct images of pluricanonical sheaves under morphisms to abelian varieties. This is due to J. Chen and Z. Jiang in [CJ] for canonical sheaves and generically finite morphisms, and for arbitrary morphisms to abelian varieties to [PaPoS] (another proof is found in [V]). It was extended to pluricanonical sheaves in

[LoPoS]. In the case of the Albanese morphism, the Chen-Jiang decomposition is as follows:

$$a_{X*}\omega_X^{\otimes m} \cong \bigoplus_i \rho_{m,i}^* \mathcal{F}_{m,i} \otimes P_{\gamma_{m,i}}, \quad (5.3)$$

where:

- (i) each $\rho_{m,i}: \text{Alb } X \rightarrow B_{m,i}$ is a quotient morphism of abelian varieties with connected fibers;
- (ii) each $\mathcal{F}_{m,i}$ is a nonzero M -regular sheaf on $B_{m,i}$ (see §2.6);
- (iii) each $\gamma_{m,i} \in \text{Pic}^0 X$ is a torsion point;
- (iv) $\rho_{m,i}^* \text{Pic}^0 B_{m,i} - \gamma_{m,i} \neq \rho_{m,j}^* \text{Pic}^0 B_{m,j} - \gamma_{m,j}$ for $i \neq j$.

These decompositions are essentially canonical (see [Pa2, Remark 3.5]), and they are discussed at length in the references above and also in [Pa2]. We also refer to §9 below where it is shown that the Chen-Jiang decompositions essentially carry the same information of the torsion filtration of the FMP transform, which shows once again that they are essentially canonical. It follows that

$$V^0(\text{Alb } X, a_{X*}\omega_X^{\otimes m}) = \bigcup_i \rho_{m,i}^* V^0(B_{m,i}, \mathcal{F}_{m,i} \otimes P_{\gamma_{m,i}}) = \bigcup_i (\rho_{m,i}^* (\text{Pic}^0 B_{m,i}) - \gamma_{m,i}), \quad (5.4)$$

where the last equality holds because the sheaves $\mathcal{F}_{m,i}$ are M -regular, hence (2.20) holds for them. Note that, by condition (iv), the translates of abelian subvarieties appearing in the decomposition (5.4) are different from each other (but some of them can be strictly contained in others). By comparing (5.4) with (2.12) in the case $m = 1$, and (2.13) otherwise, it follows that $\rho_{m,i}^* \text{Pic}^0 B_{m,i}$ is contained in $p_X^* \text{Pic}^0 Z_X$ for all m and i , and therefore all quotient morphisms $\rho_{m,i}$ factor through $p_X: \text{Alb } X \rightarrow \text{Alb } Z_X$.

Next, we observe that, still comparing (5.4) with (2.12) in the case $m = 1$, and with (2.13) in the case $m > 1$, we can gather those sheaves $\rho_{m,i}^* \mathcal{F}_{m,i} \otimes P_{\gamma_{m,i}}$ appearing in the Chen-Jiang decomposition of $a_{X*}\omega_X^{\otimes m}$ such that the corresponding loci $V^0(\text{Alb } X, \rho_{m,i}^* \mathcal{F}_{m,i} \otimes P_{\gamma_{m,i}})$ are contained in *the same translate of $p_X^* \text{Pic}^0 Z_X$* . This produces the decomposition (5.1).

(b) Since $c_X = p_X \circ a_X$, by applying p_{X*} to the right hand side of (5.1) twisted by $P_{-\alpha_{m,j}}$, the projection formula yields

$$c_{X*}(\omega_X^{\otimes m} \otimes P_{-\alpha_{m,j}}) \cong \left(\bigoplus_{i \neq j} \mathcal{G}_{X,m,j} \otimes p_{X*}(P_{\alpha_{m,i} - \alpha_{m,j}}) \right) \oplus \mathcal{G}_{X,m,j}.$$

The summands in the right-hand side vanish except for the last one, because $H^0((P_{\alpha_{m,i} - \alpha_{m,j}})|_{\ker(p_X)}) = 0$ for $i \neq j$. This is because, by construction, for $i \neq j$ the line bundle $P_{\alpha_{m,i} - \alpha_{m,j}}$ does not belong to $\text{Pic}^0 Z_X = \ker(\text{Pic}^0(\text{Alb } X) \rightarrow \text{Pic}^0(\ker(p_X)))$ (recall from §2.5 that $\ker p_X$ is connected). \square

Note that, as mentioned in Subsection 2.5, the decompositions (5.1) are the sheaf-theoretic versions of the decompositions of non-vanishing loci (2.12) and (2.13). In fact

$$V^0(\text{Alb } X, a_{X*}\omega_X) = \bigsqcup_{i=1}^{N_X(1)} V^0(\text{Alb } X, (p_X^* \mathcal{G}_{X,1,j}) \otimes P_{\alpha_{1,j}}) \subseteq \bigsqcup_{i=1}^{N_X(1)} p_X^* \text{Pic}^0 Z_X - \alpha_{1,i} \quad (5.5)$$

and, for $m \geq 2$,

$$V^0(\mathrm{Alb} X, a_{X*}\omega_X^{\otimes m}) = \bigsqcup_{i=1}^{N_X(m)} V^0(\mathrm{Alb} X, (p_X^*\mathcal{G}_{X,m,j}) \otimes P_{\alpha_{m,j}}) = \bigsqcup_{i=1}^{N_X(m)} p_X^*\mathrm{Pic}^0 Z_X - \alpha_{m,i}. \quad (5.6)$$

Remark 5.1.2. The decompositions of Proposition 5.1.1 are unique in the following sense: for any $m > 0$ the integer $N_X(m)$ (the number of translates of $p_X^*\mathrm{Pic}^0 Z_X$ that contain at least one component of $V^0(\mathrm{Alb} X, a_{X*}\omega_X^{\otimes m})$) is obviously unique. Since (2.12) and (2.13) hold for the torsion points $\alpha_{m,j}$ of Proposition 5.1.1, it follows that the $\alpha_{m,j}$'s are unique modulo torsion points in $p_X^*\mathrm{Pic}^0 Z_X$. Consequently, by (b) of the proposition above, also the sheaves $\mathcal{G}_{X,m,j}$ are unique up to tensorization with torsion line bundles parametrized by $\mathrm{Pic}^0 Z_X$.

Remark 5.1.3. For $m \geq 2$ and all j actually the sheaves $\mathcal{G}_{X,m,j}$ satisfy a stronger vanishing condition called index theorem with index zero. Namely we have that $H^p(\mathrm{Alb} Z_X, \mathcal{G}_{X,m,j} \otimes P_\alpha) = 0$ for all $p \geq 1$ and $\alpha \in \mathrm{Pic}^0 Z_X$.

Remark 5.1.4. As mentioned in Subsection 2.5, after (2.13), the fact that $\mathrm{Pic}^0 Z_X$ is a subgroup of finite index of the group G_X defined there follows also from the general finite generation result [BCHM, Theorem 1.2]. To see this, we first recall that $V^0(X, \omega_X^{\otimes m}) = V^0(\mathrm{Alb} X, a_{X*}\omega_X^{\otimes m})$. Let us consider a component $p_X^*\mathrm{Pic}^0 Z_X - \alpha_{m,i}$ of $V^0(\mathrm{Alb} X, a_{X*}\omega_X^{\otimes m})$ and a component $p_X^*\mathrm{Pic}^0 Z_X - \alpha_{k,h}$ of $V^0(\mathrm{Alb} X, a_{X*}\omega_X^{\otimes k})$. Via multiplication of global sections, their product must map to a component of the form $p_X^*\mathrm{Pic}^0 Z_X - (\alpha_{m,i} + \alpha_{k,h})$. By (5.5) and (5.6) such components correspond to the sheaves $(p_X^*\mathcal{G}_{X,m,j}) \otimes P_{\alpha_{m,j}}$ and $(p_X^*\mathcal{G}_{X,k,h}) \otimes P_{\alpha_{k,h}}$ appearing in the decompositions (5.1). Therefore, in the multiplicative structure of the relative canonical algebra $\mathcal{R}(a_X)$, their tensor product must map to a sheaf $(p_X^*\mathcal{G}_{X,m+k,i}) \otimes P_{\alpha_{m+k,i}}$ in the decomposition (5.1) of $a_{X*}\omega_X^{\otimes(m+k)}$. Hence we have that, modulo $\mathrm{Pic}^0 Z_X$,

$$\alpha_{m+k,i} = \alpha_{m,j} + \alpha_{k,h}.$$

It follows that the torsion group $G_X/\mathrm{Pic}^0 Z_X$ is finitely generated, hence finite, as soon as there is a finite set \mathcal{X} of sheaves $(p_X^*\mathcal{G}_{X,m,j}) \otimes P_{\alpha_{m,j}}$ appearing in the decompositions (5.1) such that, for any other sheaf $\mathcal{H}_{r,s} := (p_X^*\mathcal{G}_{X,r,s}) \otimes P_{\alpha_{r,s}}$, there is a tensor product of sheaves in the set \mathcal{X} mapping, via the multiplicative structure of $\mathcal{R}(a_X)$, to $\mathcal{H}_{r,s}$. But this follows from the decompositions (5.1) and the finite generation of $\mathcal{R}(a_X)$.

5.2. Main theorem.

Setting and Notation 5.2.1. The precise statement of Theorem 5.2.2 will involve the decomposition of Proposition 5.1.1. Keeping the setting of the previous Remark, we choose a section s of the quotient of abelian groups $G_X \rightarrow G_X/\mathrm{Pic}^0 Z_X$, and we denote H_X the image of s . This choice determines consistent representatives for the points $\alpha_{m,j} \in \mathrm{Pic}^0 X$ appearing in the decompositions (5.1) and therefore consistent representatives for the sheaves $\mathcal{G}_{X,m,j}$ in such a way that the sheaf $\mathcal{G}_{X,m,j} \otimes \mathcal{G}_{X,k,h}$ naturally maps to a sheaf $\mathcal{G}_{X,m+k,i}$ appearing in the decomposition of $a_{X*}\omega_X^{\otimes(m+k)}$. This defines a finitely generated graded algebra

$$\mathcal{U}_X = \bigoplus_{m \geq 0} \mathcal{U}_{X,m} \quad (5.7)$$

where $\mathcal{U}_{X,0} := c_{X*}\mathcal{O}_X$ and

$$\mathcal{U}_{X,m} := \bigoplus_{j=1}^{N_X(m)} \mathcal{G}_{X,m,j} \quad \text{for } m \geq 1.$$

Theorem 5.2.2. *Let $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ be an equivalence and let us consider decompositions as in Proposition 5.1.1:*

$$a_{X*}\omega_X^{\otimes m} \cong \bigoplus_{j=1}^{N_X(m)} (p_X^*\mathcal{G}_{X,m,j}) \otimes P_{\alpha_{m,j}} \quad \text{and} \quad a_{Y*}\omega_Y^{\otimes m} \cong \bigoplus_{j=1}^{N_Y(m)} (p_Y^*\mathcal{G}_{Y,m,j}) \otimes Q_{\beta_{m,j}}.$$

Then:

- (1) *Given a subgroup H_X of $\text{Pic}^0 X$ as in Setting and Notation 5.2.1, the subgroup $H_Y := \varphi_{\mathcal{E}}(H_X)$ of $\text{Pic}^0 Y$ is the image of a section of $G_Y/\text{Pic}^0 Z_Y$, determining consistent representatives $\mathcal{G}_{Y,m,j}$ such that, up to reordering, for all $m \geq 1$ and j we have*

$$\widehat{\varphi}_{\mathcal{E}}^*\mathcal{G}_{X,m,j} \cong \mathcal{G}_{Y,m,j}$$

(here, as usual, $\widehat{\varphi}_{\mathcal{E}}: \text{Alb } Z_Y \rightarrow \text{Alb } Z_X$ is the dual isomorphism of the induced Rouquier isomorphism (2.17), and $\varphi_{\mathcal{E}}: G_X \rightarrow G_Y$ is the isomorphism (2.16)). *In particular, $N_X(m) = N_Y(m)$.*

- (2) *There is an isomorphism*

$$\widehat{\varphi}_{\mathcal{E}}^*\mathcal{U}_X \cong \mathcal{U}_Y$$

as graded algebras.

Proof. (1) By construction H_X is a subgroup of the group G_X hence it is R -stable. Therefore point (1) follows from Variant 4.5.3 and Proposition 5.1.1(b). This provides an isomorphism of graded $\mathcal{O}_{\text{Alb } Z_Y}$ -modules $\widehat{\varphi}_{\mathcal{E}}^*\mathcal{U}_X \cong \mathcal{U}_Y$. Note that the last assertion was already proved in (2.18).

To prove (2) it remains to prove the invariance of the multiplicative structure. This is done exactly as in the proof of Theorem 4.6.1 by considering, rather than the multiplication maps $M_{k,n}(X)$, the multiplication

$$(c_{X*} \boxtimes c_{X*})((\omega_X^{\otimes k} \otimes P_{-\alpha_{k,j}}) \boxtimes (\omega_X^{\otimes n} \otimes P_{-\alpha_{n,h}})) \rightarrow (c_{X*} \boxtimes c_{X*})(\delta_{X*}(\omega_X^{\otimes(k+n)} \otimes P_{-\alpha_{k,j}-\alpha_{n,h}}))$$

and the correspondent multiplication of coherent sheaves on $\text{Alb } Z_Y$. (To this purpose we recall that the sheaves $c_{X*}(\omega_X^{\otimes m} \otimes P_{\alpha_{m,j}})$ are GV as well for $m \geq 1$ and j , as it can be seen by reducing to untwisted pluricanonical bundles on the étale cover of X induced by the torsion line bundle $P_{-\alpha_{m,j}}$). \square

It would be interesting to know whether a similar invariance holds true for relative structures, as in Corollary 4.5.1.

5.3. Derived invariance of the Chen-Jiang decomposition. In answer to a question of Mihaea Popa, we show that not only the direct images of pluricanonical sheaves under the Albanese map are derived invariant, but also their Chen-Jiang decompositions are so.

In the first place we claim that the sheaves $\mathcal{G}_{X,m,j}$ have a Chen-Jiang decomposition (see the Appendix). This because, for a torsion point $\alpha \in \text{Pic}^0 X$, the sheaf $c_{X*}(\omega_X^{\otimes m} \otimes P_{\alpha})$ is a direct summand of the pushforward of the m -th pluricanonical sheaf of the cyclic étale cover of X , say X' under the morphism $X' \rightarrow \text{Alb } Z_X$, and as such it has its own Chen-Jiang decomposition (this follows from [LoPoS, Proposition 3.6]). Therefore the claim follows from Proposition 5.1.1(b).

Lemma 5.3.1. *For all $m > 0$ the isomorphisms $\widehat{\varphi}_{\mathcal{E}}^*\mathcal{G}_{X,m,j} \cong \mathcal{G}_{Y,m,j}$ of Theorem 5.2.2(1) are such that $\widehat{\varphi}_{\mathcal{E}}^*$ transforms a Chen-Jiang decomposition of $\mathcal{G}_{X,m,j}$ to a Chen-Jiang decomposition of $\mathcal{G}_{Y,m,j}$.*

Proof. We claim that, by applying the symmetric FMP transform to the isomorphisms of the statement, there are isomorphisms of sheaves

$$\varphi_{\mathcal{E}*}\widehat{\mathcal{G}_{X,m,j}^{\vee}} \cong \widehat{\mathcal{G}_{Y,m,j}^{\vee}}.$$

Indeed we know that such isomorphisms hold at the level of the usual FMP transforms (*i.e.* (4.13)). Hence they also hold for the symmetric FMP transform: just apply the dualizing functor to both sides of (4.13) and use the fact that, by Grothendieck duality, $\Delta_{\text{Pic}^0 Z_Y} \circ \Phi_{\mathcal{Q}_Y} \cong ((-1)^* \circ \Phi_{\mathcal{Q}_Y} \circ \Delta_{\text{Alb } Z_Y})[\dim \text{Pic}^0 Z_Y]$ ([Mu, (3.8)]; the details are left to the reader). Obviously such isomorphisms preserve the torsion filtrations of §9. Therefore, by the conclusion of §9, the isomorphisms of the statement preserve the Chen-Jiang decompositions. \square

Next we observe that by its very definition, given a Chen-Jiang decomposition

$$\mathcal{G}_{X,m,j} \cong \bigoplus_i \rho_{m,j,i}^* \mathcal{F}_{m,j,i} \otimes P_{\gamma_{m,j,i}},$$

then

$$p_X^* \mathcal{G}_{X,m,j} \cong \bigoplus_i (\rho_{m,j,i} \circ p_X)^* \mathcal{F}_{m,j,i} \otimes p_X^* P_{\gamma_{m,j,i}},$$

is a Chen-Jiang decomposition. Moreover, from the definition of the decompositions of Proposition 5.1.1, it follows that

$$a_{X*} \omega_X^{\otimes m} \cong \bigoplus_{i,j} (\rho_{m,j,i} \circ p_X)^* \mathcal{F}_{m,j,i} \otimes p_X^* P_{\gamma_{m,j,i}} \otimes P_{\alpha_{m,j}} \quad (5.8)$$

is a Chen-Jiang decomposition.

Let us place ourselves in the setting of Theorem 5.2.2(1). Recalling the essential uniqueness of Chen-Jiang decompositions, it follows from Lemma 5.3.1 and (5.8) that Chen-Jiang decompositions of pushforwards of pluricanonical sheaves under the Albanese map are derived invariants:

Proposition 5.3.2. *For all $m \geq 1$*

$$a_{Y*} \omega_Y^{\otimes m} \cong \bigoplus_{i,j} (\rho_{m,j,i} \circ \widehat{\varphi}_{\mathcal{E}} \circ p_Y)^* \mathcal{F}_{m,j,i} \otimes p_Y^* Q_{\varphi_{\mathcal{E}}(\gamma_{m,j,i})} \otimes Q_{\beta_{m,j}}$$

is a Chen-Jiang decomposition of $a_{Y} \omega_Y^{\otimes m}$.*

Notice that, in particular, it follows from the above Proposition that, for all $m > 0$, the quotient abelian varieties of $\text{Alb } X$ and $\text{Alb } Y$ involved in the Chen-Jiang decompositions of $a_{X*} \omega_X^{\otimes m}$ and $a_{Y*} \omega_Y^{\otimes m}$, respectively, are isomorphic.

5.4. Derived invariance of the canonical ring of the general fiber. Starting from the present subsection, in the rest of the paper we will draw some applications of Theorem 5.2.2. We begin with the derived invariance of the canonical ring of the general fiber of the Albanese map.

Given a quasi-coherent sheaf \mathcal{F} on a complex variety Z and a closed point $z \in Z$, we denote by $\mathcal{F}(z) := \mathcal{F} \otimes \mathbb{C}(z)$ the fiber of \mathcal{F} at z equipped with its structure of \mathbb{C} -vector space. Given a smooth projective variety X , we denote, as above, by $\mathcal{R}(a_X)$ the relative canonical algebra under the Albanese morphism. For $x \in X$, we let $z = a_X(x)$. We consider the fiber of $\mathcal{R}(a_X)$ at z :

$$\mathcal{R}(a_X)(z) \cong \left(\bigoplus_{m \geq 0} a_{X*} \omega_X^{\otimes m} \right)(z).$$

We denote by $F_{X,z}$ the fiber of the Albanese map at z (note that fibers may be disconnected, in which case a general fiber is the disjoint union of general fibers of the Stein factorization). Let

$$R(F_{X,z}) = \bigoplus_{m \geq 0} H^0(F_{X,z}, \omega_{F_{X,z}}^{\otimes m})$$

be the canonical ring of $F_{X,z}$, and let

$$(F_{X,z})_{can} = \text{Proj}(R(F_{X,z}))$$

be the canonical model.

Corollary 5.4.1. *In the setting of Theorem 5.2.2, let $x \in X$ and $y \in Y$ be such that $\widehat{\varphi}_{\mathcal{E}}(c_X(x)) = c_Y(y)$. Let also $z = a_X(x)$ and $t = a_Y(y)$. Then, for general $x \in X$, one has that*

$$R(F_{X,z}) \cong R(F_{Y,t})$$

as \mathbb{C} -algebras. In particular, $(F_{X,z})_{can} \cong (F_{Y,t})_{can}$. Hence the Kodaira dimension of the general fiber of the Albanese map is invariant under derived equivalence.

Proof. It follows from Theorem 5.2.2 that, for $x \in X$ and $y \in Y$ as in the hypothesis, we have \mathbb{C} -algebra isomorphisms

$$\begin{aligned} \bigoplus_{m \geq 0} (a_{X*} \omega_X^{\otimes m})(z) &\cong \bigoplus_{m \geq 0} \left(\bigoplus_{j=1}^{N_X(m)} (p_X^* \mathcal{G}_{X,m,j})(z) \otimes P_{\alpha_{m,j}}(z) \right) \\ &\cong \bigoplus_{m \geq 0} (p_X^* \left(\bigoplus_{j=1}^{N_X(m)} (\mathcal{G}_{X,m,j}) \right))(z) \\ &\cong \bigoplus_{m \geq 0} (p_Y^* \left(\bigoplus_{j=1}^{N_Y(m)} (\mathcal{G}_{Y,m,j}) \right))(t) \\ &\cong \bigoplus_{m \geq 0} \left(\bigoplus_{j=1}^{N_Y(m)} (p_Y^* \mathcal{G}_{Y,m,j})(t) \otimes Q_{\beta_{m,j}}(t) \right) \\ &\cong \bigoplus_{m \geq 0} (a_{Y*} \omega_Y^{\otimes m})(t) \end{aligned}$$

where the third isomorphism is given by (2) of Theorem 5.2.2.

The asserted isomorphism follows from this because, by base-change, the fiber at a general point of the relative canonical ring is canonically isomorphic to the canonical ring of the fiber. \square

6. THE RELATIVE CANONICAL MODEL OF THE ALBANESE MAP

In this section we apply the results of the previous section to study the invariance under derived equivalence of

$$X_a^{can} := \text{Proj}(\mathcal{R}(a_X)) \longrightarrow \text{Alb } X,$$

the relative canonical model of the Albanese morphism. In the spirit of the recent work [IMOU] of Ito-Miura-Okawa-Ueda we show that the relative Albanese canonical models of two derived-equivalent varieties have the same class in the Grothendieck ring modulo isogenies of abelian varieties. In the final subsection we draw an application to the derived invariance of the Hodge numbers $h^{0,j}$ for varieties relatively of general type with respect to the Albanese morphism.

6.1. Main result. We will consider the Grothendieck ring $K_0(\text{Var}/\mathbb{C})$ of algebraic varieties over \mathbb{C} . Actually, for our purposes, it is useful to take a certain quotient of $K_0(\text{Var}/\mathbb{C})$ introduced in [IMOU]:

$$K'_0(\text{Var}/\mathbb{C}) \quad (6.1)$$

is the quotient of $K_0(\text{Var}/\mathbb{C})$ by the ideal generated by $[A] - [B]$, where A and B are isogenous abelian varieties. Motivated by a result of Orlov asserting that derived equivalent abelian varieties are isogenous, the authors of [IMOU] ask whether derived equivalent varieties have the same class in the localized Grothendieck ring $K'_0(\text{Var}/\mathbb{C})[\mathbb{L}^{-1}]$, with $\mathbb{L} := [\mathbb{A}^1]$ be the class of the affine line (*op.cit.*, Problem 7.2). We prove that in any case – even without localizing – the relative Albanese canonical models of derived equivalent varieties do:

Theorem 6.1.1. *Let X and Y be derived equivalent varieties. Then the equality*

$$[X_a^{can}] = [Y_a^{can}] \quad (6.2)$$

holds in $K'_0(\text{Var}/\mathbb{C})$.

Proof. We go back to the setting of §5. Let \mathcal{U}_X be the algebra of sheaves over $\text{Alb } Z_X$ introduced in §5.2. It is a derived invariant by Theorem 5.2.2(2). In particular, the structure map $\text{Proj}(\mathcal{U}_X) \rightarrow \text{Alb } Z_X$ is a derived invariant. First, we claim that the equality

$$[\text{Proj}(\mathcal{R}(a_X))] = [\text{Proj}(p_X^*\mathcal{U}_X)] \quad (6.3)$$

holds in the Grothendieck ring of varieties $K_0(\text{Var}/\mathbb{C})$. To prove this, recall that the two algebras $\mathcal{R}(a_X)$ and $p_X^*\mathcal{U}_X$ only differ for the twisting by the torsion line bundles appearing in the finite semigroup $\{P_{\alpha_{m,j}}\}_{m \geq 1, j}$ (Theorem 5.2.2). By choosing a finite open covering of the quasi-compact variety $\text{Alb } X$ given by a common local trivialization of the $P_{\alpha_{m,j}}$'s, we see that there exist finite decompositions

$$\text{Proj}(\mathcal{R}(a_X)) = \bigsqcup_{i=1}^s V_i \quad \text{and} \quad \text{Proj}(p_X^*\mathcal{U}_X) = \bigsqcup_{i=1}^s W_i,$$

where $V_1, \dots, V_s, W_1, \dots, W_s$ are locally closed subvarieties, and V_i is isomorphic to W_i for all $i = 1, \dots, s$. Namely, $\text{Proj}(\mathcal{R}(a_X))$ and $\text{Proj}(p_X^*\mathcal{U}_X)$ are *piecewise isomorphic*. Therefore, (6.3) holds in $K_0(\text{Var}/\mathbb{C})$ by definition.

Note that the same happens for Y . Now, thanks to the functoriality of the Proj ([EGA II, Prop. 3.5.3]), we have that

$$\text{Proj}(p_X^*\mathcal{U}_X) \simeq \text{Proj}(\mathcal{U}_X) \times_{\text{Alb } Z_X} \text{Alb } X,$$

and similarly for Y . This is expressed by the cartesian diagrams

$$\begin{array}{ccc} \text{Proj}(p_X^*\mathcal{U}_X) & \longrightarrow & \text{Alb } X \\ \downarrow & & \downarrow p_X \\ \text{Proj}(\mathcal{U}_X) & \longrightarrow & \text{Alb } Z_X \end{array} \quad \begin{array}{ccc} \text{Proj}(p_Y^*\mathcal{U}_Y) & \longrightarrow & \text{Alb } Y \\ \downarrow & & \downarrow p_Y \\ \text{Proj}(\mathcal{U}_Y) & \longrightarrow & \text{Alb } Z_Y \end{array} \quad (6.4)$$

Since p_X is a surjective morphism of abelian varieties with connected fibers, it follows from a standard property on the behaviour of Grothendieck rings (see, e.g., [C-LNS, Propositions 2.3.3 and 2.3.4]) that

$$[\text{Proj}(p_X^*\mathcal{U}_X)] = [\ker(p_X)] [\text{Proj}(\mathcal{U}_X)] \quad (6.5)$$

in $K_0(\text{Var}/\mathbb{C})$, and the same holds for Y .

Claim 6.1.2. *The abelian varieties $\ker(p_X)$ and $\ker(p_Y)$ are isogenous.*

The Theorem follows from this since $\text{Proj}(\mathcal{U}_X)$ and $\text{Proj}(\mathcal{U}_Y)$ are isomorphic by Theorem 5.2.2(2).

Proof of Claim 6.1.2. By a result of Popa and Schnell ([PoS1, Theorem A (1)]) the abelian varieties $\text{Alb } X$ and $\text{Alb } Y$ are isogenous. Therefore, by Poincaré's complete reducibility $\ker p_X \times \text{Alb } Z_X$ is isogenous to $\ker p_Y \times \text{Alb } Z_Y$, and we know that $\text{Alb } Z_X$ and $\text{Alb } Z_Y$ are isomorphic. This implies that $\ker p_X$ and $\ker p_Y$ are isogenous by the cancellation theorem of Fujita ([F, Proposition 9]). \square

\square

6.2. Derived invariance of Hodge numbers $h^{0,j}$. As observed in [IMOU], the usefulness of the quotient ring $K'_0(\text{Var}/\mathbb{C})$ is that the Hodge-Deligne polynomial factors through it. Namely, let

$$\text{HD}: K_0(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}[u, v]$$

be the ring homomorphism that associates to each class in $K_0(\text{Var}/\mathbb{C})$ its Hodge-Deligne polynomial. For the class of a smooth and projective variety Z , one has

$$\text{HD}([Z]) = \sum_{i,j \geq 0} (-1)^{i+j} h^{i,j}(Z) u^i v^j. \quad (6.6)$$

Since isogenous abelian varieties have the same Hodge numbers, HD passes to the quotient and we obtain an induced ring homomorphism

$$\text{HD}': K'_0(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}[u, v]$$

such that (6.6) continues to hold.

Theorem 6.2.1. *Let X and Y be derived equivalent varieties. Assume that the generic fiber of the Albanese morphism of X is of general type, that is ω_X is a_X -big. Then we have*

$$h^{0,j}(X) = h^{0,j}(Y),$$

for all j .

Proof. We consider again diagrams (6.4), keeping in mind that the two bottom rows are isomorphic by Theorem 5.2.2(2). Let W be a resolution of singularities of $\text{Proj}(\mathcal{U}_X)$, and let $T_X \rightarrow W$ and $T_Y \rightarrow W$ be the corresponding base extensions of $\text{Proj}(p_X^* \mathcal{U}_X)$ and $\text{Proj}(p_Y^* \mathcal{U}_Y)$, respectively. We have that T_X and T_Y are smooth and, repeating the argument of Theorem 6.1.1 (as in (6.5)), the equality $[T_X] = [T_Y]$ holds true in $K'_0(\text{Var}/\mathbb{C})$. Hence

$$\text{HD}'([T_X]) = \text{HD}'([T_Y]),$$

so that all Hodge numbers of T_X and T_Y are equal. As already noted in the proof of Theorem 6.1.1, there are isomorphic non-empty open subsets of $\text{Proj}(p_X^* \mathcal{U}_X)$ and $\text{Proj}(\mathcal{R}(a_X))$. Since $\text{Proj}(p_X^* \mathcal{U}_X)$ and $\text{Proj}(\mathcal{R}(a_X)) = X_a^{\text{can}}$ are irreducible,³ they are indeed birational.⁴ Now the hypothesis that the generic fiber of a_X (and hence of a_Y , by Corollary 5.4.1) is of general type yields that X is birational to X_a^{can} and in turn Y is birational to Y_a^{can} . Hence X (*resp.* Y) is birational to T_X (*resp.* to T_Y). Hence $h^{0,j}(X) = h^{0,j}(T_X) = h^{0,j}(T_Y) = h^{0,j}(Y)$. \square

³Thanks to [EGA IV₂, Cor. 2.3.5 (iii)] and (6.4), $\text{Proj}(p_X^* \mathcal{U}_X)$ is irreducible because $\text{Proj}(\mathcal{U}_X)$ is so by [EGA II, Prop. 3.1.14 (i)]. The irreducibility of $\text{Proj}(\mathcal{R}(a_X))$ is clear.

⁴In the next §7 we will prove that X_a^{can} and $\text{Proj}(p_X^* \mathcal{U}_X)$ are isomorphic under the assumption that ω_X is a_X -big (see Lemma 7.4.2 below). However, at this stage, this result is not necessary.

7. DERIVED INVARIANCE OF HODGE NUMBERS FOR MINIMAL VARIETIES

In this section we prove the invariance of the Hodge numbers of smooth minimal varieties relatively of general type with respect to their Albanese morphism.

Theorem 7.0.1. *Let X and Y be derived equivalent varieties. Assume that ω_X is nef and a_X -big.⁵ Then the rational Hodge structures of X and Y are isomorphic. In particular, we have*

$$h^{i,j}(X) = h^{i,j}(Y)$$

for all i and j .

This will be the consequence of a result involving a variant of the Grothendieck ring of varieties, for which we recall and introduce some terminology. Let

$$\mathcal{M}'_{\mathbb{C}} := K'_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$$

be the localization of the Grothendieck ring $K'_0(\mathrm{Var}/\mathbb{C})$ by the class of the affine line \mathbb{L} introduced in [IMOU] (see §6.1). Then we have the completion

$$\widehat{\mathcal{M}'_{\mathbb{C}}} := \varprojlim_{d \in \mathbb{Z}} \frac{\mathcal{M}'_{\mathbb{C}}}{F_d \mathcal{M}'_{\mathbb{C}}}$$

of $\mathcal{M}'_{\mathbb{C}}$ with respect to the filtration induced by dimension ($F_{\bullet} \mathcal{M}'_{\mathbb{C}}$). This is defined as follows. For every integer d , $F_d \mathcal{M}'_{\mathbb{C}}$ is the subgroup of $\mathcal{M}'_{\mathbb{C}}$ generated by the elements of the form $[Z]\mathbb{L}^{-m}$, where Z is an algebraic variety such that $\dim Z - m \leq d$. This is a ring filtration of $\mathcal{M}'_{\mathbb{C}}$, in the sense that $F_d \mathcal{M}'_{\mathbb{C}} \cdot F_k \mathcal{M}'_{\mathbb{C}} \subseteq F_{d+k} \mathcal{M}'_{\mathbb{C}}$, so $\widehat{\mathcal{M}'_{\mathbb{C}}}$ is also a ring.

The ring $\widehat{\mathcal{M}'_{\mathbb{C}}}$ was defined analogously to the completed Grothendieck ring of varieties $\widehat{\mathcal{M}}_{\mathbb{C}}$ (the completion of the localized Grothendieck ring $\mathcal{M}_{\mathbb{C}} := K_0(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$ with respect to the same filtration by dimension). There is the natural ring homomorphism

$$\widehat{\mathcal{M}}_{\mathbb{C}} \rightarrow \widehat{\mathcal{M}'_{\mathbb{C}}}. \quad (7.1)$$

Theorem 7.0.2. *Under the assumptions of Theorem 7.0.1 the equality $[X] = [Y]$ holds in $\widehat{\mathcal{M}'_{\mathbb{C}}}$.*

The proof of Theorem 7.0.2 will be the content of the next subsections, starting from 7.1.

Proof of Theorem 7.0.1. The result follows from Theorem 7.0.2 in the usual way (see, e.g., [C-LNS, Chapter 2, Corollary 4.3.8]). We include the details for the reader's benefit. From the mixed Hodge theory of Deligne, it follows the existence of a ring homomorphism

$$\chi_{\mathrm{Hdg}}: K_0(\mathrm{Var}/\mathbb{C}) \rightarrow K_0(\mathbf{pHS})$$

from $K_0(\mathrm{Var}/\mathbb{C})$ to the Grothendieck ring of the *semisimple* category \mathbf{pHS} of polarizable Hodge structures [C-LNS, Chapter 2, §3.2]. If a variety X is smooth and proper, then

$$\chi_{\mathrm{Hdg}}([X]) = \sum_{k \geq 0} (-1)^k [H^k(X, \mathbb{Q})]$$

is the class of its rational Hodge structure in $K_0(\mathbf{pHS})$. Isogenous abelian varieties have isomorphic rational Hodge structures, hence χ_{Hdg} passes to the quotient $K'_0(\mathrm{Var}/\mathbb{C})$

$$\chi'_{\mathrm{Hdg}}: K'_0(\mathrm{Var}/\mathbb{C}) \rightarrow K_0(\mathbf{pHS}).$$

⁵Actually, since an abelian variety contains no rational curves, the nefness of ω_X is equivalent to its a_X -nefness (see, e.g., [BC, §3.2]).

Moreover, $\chi_{\text{Hdg}}(\mathbb{L})$ is invertible in $K_0(\mathbf{pHS})$ (indeed, it coincides with the inverse of the class of the Tate object $\mathbb{Q}(1)$ in $K_0(\mathbf{pHS})$). Therefore, we get an induced ring homomorphism

$$\chi'_{\text{Hdg}}: \mathcal{M}'_{\mathbb{C}} \rightarrow K_0(\mathbf{pHS}). \quad (7.2)$$

Let $\widehat{\mathcal{M}'_{\mathbb{C}}}$ be the image of the map $\mathcal{M}'_{\mathbb{C}} \rightarrow \widehat{\mathcal{M}'_{\mathbb{C}}}$. By definition, it is isomorphic to $\mathcal{M}'_{\mathbb{C}}/F_{\infty}\mathcal{M}'_{\mathbb{C}}$, where $F_{\infty}\mathcal{M}'_{\mathbb{C}} := \bigcap_{d \in \mathbb{Z}} F_d\mathcal{M}'_{\mathbb{C}}$. We want to prove that (7.2) factorizes through $\widehat{\mathcal{M}'_{\mathbb{C}}}$. In order to achieve this, one needs that $\chi'_{\text{Hdg}}(F_{\infty}\mathcal{M}'_{\mathbb{C}}) = 0$. For every integer d , let $W_d K_0(\mathbf{pHS})$ be the subgroup of $K_0(\mathbf{pHS})$ generated by the polarizable Hodge structures of weight $\leq d$. Since the subgroup $F_d\mathcal{M}'_{\mathbb{C}}$ is generated by the elements of the form $[Z]\mathbb{L}^{-m}$ with $\dim Z - m \leq d$, we have

$$\chi'_{\text{Hdg}}([Z]\mathbb{L}^{-m}) = \chi'_{\text{Hdg}}([Z])\chi'_{\text{Hdg}}(\mathbb{L})^{-m} \in W_{2(\dim Z - m)}K_0(\mathbf{pHS}) \subseteq W_{2d}K_0(\mathbf{pHS}).$$

So, $\chi'_{\text{Hdg}}(F_{\infty}\mathcal{M}'_{\mathbb{C}}) \subseteq \bigcap_{d \in \mathbb{Z}} W_d K_0(\mathbf{pHS})$. But $\bigcap_{d \in \mathbb{Z}} W_d K_0(\mathbf{pHS}) = 0$ and hence there exists a ring homomorphism

$$\widehat{\mathcal{M}'_{\mathbb{C}}} \rightarrow K_0(\mathbf{pHS})$$

through which (7.2) factorizes. If now X and Y are smooth projective such that $[X] = [Y]$ holds in $\widehat{\mathcal{M}'_{\mathbb{C}}}$, then the rational Hodge structures of X and Y have the same class in $K_0(\mathbf{pHS})$. Finally, since the category \mathbf{pHS} is semisimple, one has that these two Hodge structures are indeed isomorphic.

The equality of the Hodge numbers follows directly from Theorem 7.0.2 much more easily. Indeed, the Hodge-Deligne polynomial $\text{HD}' : K'_0(\text{Var}/\mathbb{C}) \rightarrow \mathbb{Z}[u, v]$ appearing in §6.2 extends to a ring homomorphism

$$\widehat{\mathcal{M}'_{\mathbb{C}}} \rightarrow \mathbb{Z}[u, v, (uv)^{-1}].$$

So, from Theorem 7.0.2, it follows that

$$\text{HD}'([X]) = \text{HD}'([Y]),$$

and therefore all Hodge numbers of X and Y are equal. \square

7.1. Gorenstein volume of a log terminal variety. In this subsection and in the next one we recall some background involved in the proof of Theorems 7.0.2 and 7.5.1. Note that the former is a particular case of the latter.

Let W be an algebraic variety with log terminal singularities, and let $\widetilde{W} \rightarrow W$ be a log resolution of W of relative canonical divisor

$$K_{\widetilde{W}/W} = \sum_{i \in I} a_i F_i.$$

Let m be a positive integer such that mK_W is Cartier. The *Gorenstein volume* of W is defined by the formula

$$\mu_{\text{Gor}}(W) := \mathbb{L}^{-\dim W} \sum_{J \subseteq I} [F_J] \prod_{j \in J} \left(\frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} - 1 \right) \in \widehat{\mathcal{M}'_{\mathbb{C}}}[\mathbb{L}^{\frac{1}{m}}] := \widehat{\mathcal{M}'_{\mathbb{C}}}[T]/(T^m - \mathbb{L}),$$

where, for every subset $J \subseteq I$, we set $F_J := \bigcap_{j \in J} F_j$, and $[F_J]$ is its class in $K_0(\text{Var}/\mathbb{C})$. (See also [C-LNS, §7.3.4] for an equivalent definition involving motivic Igusa zeta functions.) It may be checked that the definition does not depend on the chosen log resolution. Moreover, in order to compute $\mu_{\text{Gor}}(W)$, one can take any resolution of W satisfying the only requirement that its exceptional locus is a divisor such that all irreducible components F_i with non-zero discrepancies a_i are simple normal crossing (see [Ba1, Theorem 3.4], [Ba2, Definition 2.1] and [Cr]). Note that if W admits a *crepant* resolution $T \rightarrow W$, then

$$\mathbb{L}^{-\dim T}[T] = \mu_{\text{Gor}}(T) = \mu_{\text{Gor}}(W), \quad (7.3)$$

where the first equality follows from the smoothness of T . Moreover, if K_W is a Cartier divisor, i.e., W is Gorenstein, then $\mu_{\text{Gor}}(W) \in \widehat{\mathcal{M}}_{\mathbb{C}}$ by definition.

Finally, we denote by $\mu'_{\text{Gor}}(W)$ the image of the Gorenstein volume of W via the ring homomorphism

$$\widehat{\mathcal{M}}_{\mathbb{C}}[\mathbb{L}^{\frac{1}{m}}] \rightarrow \widehat{\mathcal{M}}'_{\mathbb{C}}[\mathbb{L}^{\frac{1}{m}}] \quad (7.4)$$

induced by (7.1).

7.2. Relative abundance conjecture. Next, we address the proof of Theorem 7.0.2. We first recall that, under the assumption of Theorem 7.0.2, the abundance conjecture is known to hold true. Indeed, since ω_X is a_X -big and nef, it is a_X -semiample by the relative basepoint-free theorem [KM, Theorem 3.24].⁶ Therefore, there exists a natural birational *morphism*

$$\vartheta: X \rightarrow X_a^{\text{can}}.$$

Since X is smooth (hence Gorenstein), it follows that the canonical divisor of X_a^{can} is Cartier.⁷ Indeed the canonical divisor K_X of X is Cartier, ϑ -nef and ϑ -big (as ϑ is birational). Hence, by the relative basepoint-free theorem [KM, Theorem 3.24], dK_X is ϑ -free for all $d \gg 0$. In particular, $K_X = (d+1)K_X - dK_X \sim \vartheta^*D$ for a Cartier divisor D on the relative canonical model of X . So $K_{X_a^{\text{can}}} = \vartheta_*K_X \sim D$ and $K_{X_a^{\text{can}}}$ is Cartier.

Moreover, ϑ is a *crepant* resolution. This is a consequence of the nefness of ω_X together with the fact that X_a^{can} has canonical singularities, i.e., the relative canonical divisor $K_{X/X_a^{\text{can}}} \geq 0$ (see [KMM, Theorem 0-3-12]). Indeed, if $K_{X/X_a^{\text{can}}} > 0$, then by the Hodge index theorem (see, e.g., [BCHM, Lemma 3.6.2 (1)]) there exists a curve $C \subseteq X$ contracted by ϑ such that $(K_X \cdot C) = (K_{X/X_a^{\text{can}}} \cdot C) < 0$. This gives a contradiction. Therefore, by (7.3),

$$\mu_{\text{Gor}}(X_a^{\text{can}}) = \mathbb{L}^{-\dim X}[X] \quad (7.5)$$

holds in $\widehat{\mathcal{M}}_{\mathbb{C}}$. Note that the same holds for Y . Indeed, as already noted, ω_Y is a_Y -big, and it is also nef by [Ka, Theorem 1.4].

7.3. Proof of Theorem 7.0.2. By (7.5) and (7.4), we have that

$$\mu'_{\text{Gor}}(X_a^{\text{can}}) = \mathbb{L}^{-\dim X}[X] \quad (7.6)$$

in the ring $\widehat{\mathcal{M}}'_{\mathbb{C}}$ and the same happens for Y . In order to conclude the proof of Theorem 7.0.2 it remains to verify that

$$\mu'_{\text{Gor}}(X_a^{\text{can}}) = \mu'_{\text{Gor}}(Y_a^{\text{can}}) \quad (7.7)$$

holds in $\widehat{\mathcal{M}}'_{\mathbb{C}}$. This will be the content of the next subsection.

7.4. Proof of (7.7). We prove, more generally, the following theorem.

Theorem 7.4.1. *Let X and Y be derived equivalent varieties. If the general fiber of the Albanese morphism of X is of general type, then (7.7) holds true in $\widehat{\mathcal{M}}'_{\mathbb{C}}[\mathbb{L}^{\frac{1}{m}}]$, where $m \gg 0$ is such that $mK_{X_a^{\text{can}}}$ and $mK_{Y_a^{\text{can}}}$ are both Cartier divisors.*⁸

⁶By [Hu, Corollary 3.8], ω_X is indeed semiample.

⁷We thank Fabio Bernasconi for having pointed out this fact (and its proof) to us.

⁸Note that, without the nefness assumption on ω_X , it may happen that X_a^{can} is \mathbb{Q} -Gorenstein but not Gorenstein.

This can be done quite similarly to the proof of Theorem 6.1.1. Keeping the notation of §6, we begin with some preparatory lemmas.

Lemma 7.4.2. *If ω_X is a_X -big, then*

$$X_a^{can} \simeq \text{Proj}(p_X^* \mathcal{U}_X)$$

over $\text{Alb } X$.

Proof. By going back to the proof of Theorem 6.1.1, we have already observed that the two algebras $\mathcal{R}(a_X)$ and $p_X^* \mathcal{U}_X$ only differ for the twisting of the torsion line bundles appearing in the finite semigroup $\{P_{\alpha_{m,j}}\}_{m \geq 1, j}$. Since the $\alpha_{m,j}$'s are torsion points of $\text{Alb } X$, there exists an integer k_X big enough such that $\mu_{k_X}^* P_{\alpha_{m,j}} \cong \mathcal{O}_{\text{Alb } X}$ for all m and j , where $\mu_{k_X}: \text{Alb } X \rightarrow \text{Alb } X$ is the multiplication-by- k_X isogeny of $\text{Alb } X$. Therefore, $\mu_{k_X}^* \mathcal{R}(a_X)$ and $\mu_{k_X}^* p_X^* \mathcal{U}_X$ are isomorphic in degrees ≥ 1 . Then, from [EGA II, Prop. 3.1.8 (ii)] and the functoriality of the Proj construction [EGA II, Prop. 3.5.3], it follows that $X_a^{can} \rightarrow \text{Alb } X$ and $\text{Proj}(p_X^* \mathcal{U}_X) \rightarrow \text{Alb } X$ become isomorphic to the same variety Z after base change with the isogeny μ_{k_X} . Namely, we have cartesian commutative diagrams

$$\begin{array}{ccccc} Z & \longrightarrow & \text{Alb } X & \longleftarrow & Z \\ \mu'_{k_X} \downarrow & & \downarrow \mu_{k_X} & & \downarrow \mu''_{k_X} \\ X_a^{can} & \longrightarrow & \text{Alb } X & \longleftarrow & \text{Proj}(p_X^* \mathcal{U}_X) \end{array}$$

In particular, since X_a^{can} has canonical singularities, $\text{Proj}(p_X^* \mathcal{U}_X)$ has canonical singularities as well (see [Ko, Proposition 2.15]). Moreover, since $K_{X_a^{can}}$ is relatively ample with respect to $X_a^{can} \rightarrow \text{Alb } X$, one has that the canonical divisor of $\text{Proj}(p_X^* \mathcal{U}_X)$ is relatively ample with respect to its structure morphism $\text{Proj}(p_X^* \mathcal{U}_X) \rightarrow \text{Alb } X$ because μ_{k_X} (and hence μ'_{k_X} and μ''_{k_X}) is an étale finite surjective morphism. By the uniqueness of the relative canonical model [KMM, Theorem 0-3-12], X_a^{can} and $\text{Proj}(p_X^* \mathcal{U}_X)$ are therefore isomorphic over $\text{Alb } X$. \square

We get the cartesian diagram

$$\begin{array}{ccc} X_a^{can} & \xrightarrow{\tau_X} & \text{Alb } X \\ \downarrow p'_X & & \downarrow p_X \\ \text{Proj}(\mathcal{U}_X) & \xrightarrow{\sigma_X} & \text{Alb } Z_X \end{array} \quad (7.8)$$

Lemma 7.4.3. *If T is a log resolution of $\text{Proj}(\mathcal{U}_X)$, then the corresponding base change via the morphism p'_X defined in (7.8) gives a resolution \tilde{T} of X_a^{can} such that the exceptional divisors with non-zero discrepancies are simple normal crossing.⁹*

Proof. We have the following cartesian diagrams by construction

$$\begin{array}{ccccc} \tilde{T} & \xrightarrow{\tilde{\rho}} & X_a^{can} & \xrightarrow{\tau_X} & \text{Alb } X \\ \downarrow \nu_X & & \downarrow p'_X & & \downarrow p_X \\ T & \xrightarrow{\rho} & \text{Proj}(\mathcal{U}_X) & \xrightarrow{\sigma_X} & \text{Alb } Z_X \end{array}$$

⁹Actually, it can be proved that \tilde{T} is a log resolution of X_a^{can} . However this is unnecessary for our purposes.

and \tilde{T} is a resolution of X_a^{can} . We only need to prove that the exceptional divisors of $\tilde{\rho}$ with non-zero discrepancies are simple normal crossing. Note that

$$\nu_X^* K_\rho = K_{\tilde{\rho}}, \quad (7.9)$$

where K_ρ and $K_{\tilde{\rho}}$ are the relative canonical divisors of ρ and $\tilde{\rho}$, respectively (see [Ch, Proof of Proposition 2.7] for an even more general formula based on [dFH]). Therefore, every $\tilde{\rho}$ -exceptional divisor that is not a component of $\nu_X^* F$ for any ρ -exceptional divisor F has discrepancy equal to 0. The Lemma follows because by assumption ρ is a log resolution, and the pullback of a normal crossing divisor is a normal crossing divisor ([S-P, Tag 0CBP]). \square

Now we are ready to prove (7.7). Take $\nu_X: \tilde{T} \rightarrow T$ as in Lemma 7.4.3 and write

$$K_\rho = \sum_{i \in I} a_i F_i \quad \text{and} \quad K_{\tilde{\rho}} = \sum_{i \in I} a_i F_{X,i},$$

where $F_{X,i} := \nu_X^* F_i = \nu_X^{-1}(F_i)$ (see (7.9)). Note that, since ν_X is a smooth morphism, the $F_{X,i}$'s are reduced for all i . The same holds for Y , with similar notations. Indeed, recall that $\text{Proj}(\mathcal{U}_X) \rightarrow \text{Alb } Z_X$ is a derived invariant thanks to Theorem 5.2.2(2). Therefore,

$$\mu_{\text{Gor}}(X_a^{can}) = \mathbb{L}^{-n} \sum_{J \subseteq I} [F_{X,J}] \prod_{j \in J} \left(\frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} - 1 \right),$$

and

$$\mu_{\text{Gor}}(Y_a^{can}) = \mathbb{L}^{-n} \sum_{J \subseteq I} [F_{Y,J}] \prod_{j \in J} \left(\frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} - 1 \right),$$

where $n = \dim X_a^{can} = \dim Y_a^{can}$. As in (6.5), we get $[F_{X,J}] = [\ker(p_X)][F_J]$ and $[F_{Y,J}] = [\ker(p_Y)][F_J]$. Hence the equality $[F_{X,J}] = [F_{Y,J}]$ in $K'_0(\text{Var}/\mathbb{C})$ follows from Claim 6.1.2 as in the proof of Theorem 6.1.1. In particular, $\mu'_{\text{Gor}}(X_a^{can}) = \mu'_{\text{Gor}}(Y_a^{can})$. \square

7.5. Relative minimal models. Given a variety X such that ω_X is a_X -big, let X_a^{min} be a relative minimal model of X with respect to a_X , whose existence is guaranteed by [BCHM, Theorem 1.2]. Contrary to the relative canonical model, X_a^{min} is not necessarily unique. However, two a_X -relative minimal models of X are K -equivalent by [Wa, Variant 1.11 of Theorem 1.4] and, therefore, they have the same Gorenstein volume [Ya, Proposition 1.2]. We use a similar notation for Y . The above argument proves the following result, valid under hypotheses more general than Theorem 7.0.2:

Theorem 7.5.1. *Let X and Y be derived equivalent varieties. If ω_X is a_X -big, then the equality*

$$\mu'_{\text{Gor}}(X_a^{min}) = \mu'_{\text{Gor}}(Y_a^{min}) \quad (7.10)$$

holds in $\widehat{\mathcal{M}}'_\mathbb{C}[\mathbb{L}^{\frac{1}{m}}]$, for $m \gg 0$.

Indeed, as in §7.2, we have a birational morphism $f: X_a^{min} \rightarrow X_a^{can}$ such that $K_{X_a^{min}} = f^* K_{X_a^{can}}$. Hence $\mu_{\text{Gor}}(X_a^{min}) = \mu_{\text{Gor}}(X_a^{can})$, and the same happens for Y . Therefore, (7.10) follows from Theorem 7.4.1.

8. FURTHER RESULTS, VARIANTS, QUESTIONS

8.1. Rouquier-stable morphisms. Going back to the terminology of Subsection 2.2, given an equivalence $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$, an abelian subvariety \widehat{B}_X of $\text{Pic}^0 X$ is said to be *R-stable* if its points are *R-stable* with respect to the Rouquier isomorphism $\varphi_{\mathcal{E}}$. Therefore the Rouquier isomorphism induces an isomorphism of abelian varieties

$$\varphi_{\mathcal{E}}: \widehat{B}_X \xrightarrow{\sim} \widehat{B}_Y$$

where \widehat{B}_Y is the abelian subvariety of $\text{Pic}^0 Y$ such that $\varphi(\text{id}_X, \widehat{B}_X) = (\text{id}_Y, \widehat{B}_Y)$. Note that, in this terminology, the Albanese-Iitaka variety is the dual of an *R-stable* subvariety of $\text{Pic}^0 X$ (Subsection 2.5). It turns out that Theorem 3.0.1 holds more generally, with the same proof, for any morphism to an abelian variety which is the dual of an *R-stable* abelian subvariety of $\text{Pic}^0 X$. To be precise, as in (3.1), an *R-stable* subvariety \widehat{B}_X gives commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \text{Alb } X \\ \downarrow s_X & \searrow b_X & \downarrow p_X \\ X' & \xrightarrow{b'_X} & B_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{a_Y} & \text{Alb } Y \\ \downarrow s_Y & \searrow b_Y & \downarrow p_Y \\ Y' & \xrightarrow{b'_Y} & B_Y \end{array} \quad (8.1)$$

where p_X (*resp.* p_Y) is the dual morphism of the inclusion $\widehat{B}_X \subseteq \text{Pic}^0 X$ (*resp.* $\widehat{B}_Y \subseteq \text{Pic}^0 Y$) and the left-bottom sides of the diagrams are the Stein factorizations of the composed morphisms $b_X := p_X \circ a_X$ and $b_Y := p_Y \circ a_Y$.

Theorem 8.1.1. *In the above notation, given an R-stable subvariety $\widehat{B}_X \subseteq \text{Pic}^0 X$ (with respect to the equivalence $\Phi_{\mathcal{E}}$), there is an induced isomorphism $\psi: Y' \rightarrow X'$ such that following diagram*

$$\begin{array}{ccc} X' & \xleftarrow{\sim} & Y' \\ \downarrow b'_X & \psi & \downarrow b'_Y \\ B_X & \xleftarrow{\sim} & B_Y \\ & \varphi_{\mathcal{E}} & \end{array} \quad (8.2)$$

is commutative.

Also, all the results of §4 hold more generally, with the same proof, replacing the Albanese-Iitaka morphism with any morphism $b_X: X \rightarrow B_X$ to the dual of an *R-stable* abelian subvariety. In particular, we have the following generalization of Theorem 4.6.1.

Theorem 8.1.2. *There is an isomorphism of relative canonical algebras*

$$\widehat{\varphi}_{\mathcal{E}}^* \mathcal{R}(b_X) \cong \mathcal{R}(b_Y).$$

An interesting problem is to determine the maximal derived invariant subvariety of $\text{Pic}^0 X$, namely the maximal *R-stable* abelian subvariety with respect to any derived equivalence of X (by §2.5 this contains the Albanese-Iitaka variety $\text{Alb } Z_X$). The following lemma provides a sufficient condition to ensure that such maximal variety is the full $\text{Pic}^0 X$.

Lemma 8.1.3. *If $\text{Aut}^0 X$ is an affine algebraic group, then $\text{Pic}^0 X$ is R-stable.*

Proof. If $\text{Aut}^0 X$ is affine, then $\text{Aut}^0 Y$ is affine too (see [PoS1, p. 534]), and, since $\text{Pic}^0 X$ is projective, the composed morphism

$$\{\text{id}_X\} \times \text{Pic}^0 X \xrightarrow{\varphi_{\mathcal{E}}} \text{Aut}^0 Y \times \text{Pic}^0 Y \xrightarrow{p_1} \text{Aut}^0 Y$$

is constant, where p_1 is the projection onto the first factor. \square

For instance, $\text{Aut}^0 X$ is affine when $\chi(X, \mathcal{O}_X) \neq 0$ by [PoS1, Corollary 2.6]. In conclusion, in this case the Albanese map itself satisfies the hypotheses of Theorem 8.1.2, hence Theorem C in the Introduction is greatly simplified, as we have that the Albanese relative canonical ring is preserved by the Rouquier isomorphism

Corollary 8.1.4. *If $\chi(X, \mathcal{O}_X) \neq 0$, then*

$$\widehat{\varphi}_{\mathcal{E}}^* \mathcal{R}(a_X) \cong \mathcal{R}(a_Y).$$

Connected to this, another related interesting line of research, introduced in the paper [LO] in a wider context, consists in a systematic study of those derived equivalences $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ such that the full $\text{Pic}^0 X$ is R -stable. In the terminology of [LO], these are *strongly filtered equivalences*. An in-depth study of such derived equivalences and of invariant properties for morphisms to abelian varieties dual to R -stable abelian subvarieties of Pic^0 will be carried out in the forthcoming paper [CL].

8.2. Derived invariance of refined non-vanishing canonical loci. For each $k \geq 1$, let

$$V_k^i(\omega_X) := \{\alpha \in \text{Pic}^0 X \mid h^i(X, \omega_X \otimes P_{\alpha}) \geq k\}$$

and denote by $V_k^i(\omega_X)_0$ the union of the irreducible components of $V_k^i(\omega_X)$ passing through the origin (such irreducible components are in fact abelian subvarieties). Conjectures of Popa and the second author ([Po], [LoPo]) predict, in the present language, that such abelian subvarieties are R -stable with respect to any derived equivalence and, moreover $h^i(X, \omega_X \otimes P_{\alpha}) = h^i(Y, \omega_Y \otimes Q_{\varphi_{\mathcal{E}}(\alpha)})$ for all $\alpha \in V_1^i(\omega_X)_0$. Via methods introduced in [LoPo], when the canonical bundle is a_X -big this follows from Theorem 6.2.1.

Theorem 8.2.1. *Suppose $\Phi_{\mathcal{E}}: D(X) \rightarrow D(Y)$ is an equivalence and that ω_X is a_X -big. Then, for all integers $i \geq 0$ and $k \geq 1$, the varieties $V_k^i(\omega_X)_0$ are R -stable with respect to $\Phi_{\mathcal{E}}$. In other words, the Rouquier isomorphism $\varphi_{\mathcal{E}}$ acts on $V_k^i(\omega_X)_0$ as*

$$\varphi_{\mathcal{E}}(\text{id}_X, V_k^i(\omega_X)_0) = (\text{id}_Y, V_k^i(\omega_Y)_0).$$

Proof. The proof follows from the proof of [LoPo, Theorem 3.1], where it is shown that the assertion follows from the invariance of the Hodge numbers $h^{0,i}$ of all derived equivalent varieties X_{α} and Y_{β} , which are étale covers of X (resp. Y), induced by torsion points $\alpha \in \text{Pic}^0 X$ (resp. $\beta \in \text{Pic}^0 Y$). But the a_X -bigness of the canonical bundle of X implies the same property for X_{α} . Hence, by Theorem 6.2.1, $h^{0,i}(X_{\alpha}) = h^{0,i}(Y_{\beta})$ as soon as there is an equivalence $D(X_{\alpha}) \rightarrow D(Y_{\beta})$. \square

9. APPENDIX: THE CHEN-JIANG DECOMPOSITION AND THE TORSION FILTRATION

Chen-Jiang decompositions as (5.3) hold more generally for pushforwards of pluricanonical sheaves under any morphism $f: X \rightarrow A$ from a smooth projective complex variety to an abelian variety ([LoPoS, Theorem C]). They appear to be quite useful in the study of the birational geometry of irregular varieties. Here we point out a useful feature of them, used in the proof of Proposition 5.3.2.

More generally, let \mathcal{F} be a coherent sheaf on an abelian variety A having *the Chen-Jiang decomposition property* (this notion is introduced in [LoPoS, §3]), *i.e.* admitting a direct-sum decomposition as in (5.3), namely

$$\mathcal{F} \cong \bigoplus_i \rho_i^* \mathcal{F}_i \otimes P_{\gamma_i}, \quad (9.1)$$

satisfying the four properties listed in the proof of Proposition 5.1.1, namely:

- (i) each $\rho_i: A \rightarrow B_i$ is a quotient morphism of abelian varieties with connected fibers;
- (ii) each \mathcal{F}_i is a nonzero M -regular sheaf on B_i (see §2.6);
- (iii) each $\gamma_i \in \text{Pic}^0 X$ is a torsion point;
- (iv) $\rho_i^* \text{Pic}^0 B_i - \gamma_i \neq \rho_j^* \text{Pic}^0 B_j - \gamma_j$ for $i \neq j$.

Applying FM_A to (9.1) we get that

$$\widehat{\mathcal{F}^\vee} \cong \bigoplus_i t_{\gamma_i^*}(\widehat{\rho_{i*} \mathcal{F}_i^\vee}). \quad (9.2)$$

This follows from the commutativity relation between FMP transforms and morphisms of abelian varieties $\pi: A \rightarrow B$

$$\text{FM}_A \circ \mathbf{L}\pi^* \cong \mathbf{R}\widehat{\pi}_* \circ \text{FM}_B \quad (9.3)$$

([Sch, Proposition 4.1]), and the formula

$$\text{FM}_A(- \otimes P_\beta) \cong t_{\beta^*} \circ \text{FM}_A \quad \forall \beta \in \text{Pic}^0 A \quad (9.4)$$

([Sch, Proposition 5.1]). Note that, since the sheaves \mathcal{F}_i are M -regular, the sheaves $t_{\gamma_i^*}(\widehat{\rho_{i*} \mathcal{F}_i^\vee})$ are of pure dimension equal to $\dim B_i$ and their scheme-theoretic support are the translated subtori

$$-V^0(t_{\gamma_i^*}(\rho_i^* \mathcal{F}_i)) = \rho_i^* \text{Pic}^0 B_i + \gamma_i \quad (9.5)$$

with the reduced scheme structure.

We consider the torsion filtration of the sheaf $\widehat{\mathcal{F}^\vee}$

$$0 \subset T_0(\widehat{\mathcal{F}^\vee}) \subset \dots \subset T_d(\widehat{\mathcal{F}^\vee}) = \widehat{\mathcal{F}^\vee} \quad (9.6)$$

where $d = \dim \widehat{\mathcal{F}^\vee}$ and $T_i(\widehat{\mathcal{F}^\vee})$ is the maximal subsheaf of $\widehat{\mathcal{F}^\vee}$ of dimension $\leq i$ (see [HuLe, §1.1]). It follows that d is equal to the maximal dimension of the tori B_i . In particular $d = \dim A$ if and only if there is a (necessarily unique, by condition (iv)) i such that $B_i = A$.

Proposition 9.0.1. *For all $k \leq d$ we have that*

$$T_k(\widehat{\mathcal{F}^\vee}) \cong \bigoplus_{\dim B_i \leq k} t_{\gamma_i^*}(\widehat{\rho_{i*} \mathcal{F}_i^\vee}) \quad (9.7)$$

and

$$T_k(\widehat{\mathcal{F}^\vee}) \cong (T_{k-1}(\widehat{\mathcal{F}^\vee})) \oplus (T_k(\widehat{\mathcal{F}^\vee})/T_{k-1}(\widehat{\mathcal{F}^\vee})) \cong \left(\bigoplus_{\dim B_i \leq k-1} t_{\gamma_i^*}(\widehat{\rho_{i*} \mathcal{F}_i^\vee}) \right) \oplus \left(\bigoplus_{\dim B_i = k} t_{\gamma_i^*}(\widehat{\rho_{i*} \mathcal{F}_i^\vee}) \right). \quad (9.8)$$

Proof. For $k = d$ (9.7) is obvious while (9.8) holds because there is an inclusion of sheaves

$$\bigoplus_{\dim B_i \leq d-1} t_{\gamma_i^*}(\widehat{\rho}_{i^*} \widehat{\mathcal{F}}_i^\vee) \subset T_{d-1}(\widehat{\mathcal{F}}^\vee)$$

and the natural map

$$\bigoplus_{\dim B_i = d} t_{\gamma_i^*}(\widehat{\rho}_{i^*} \widehat{\mathcal{F}}_i^\vee) \rightarrow T_d(\widehat{\mathcal{F}}^\vee)/T_{d-1}(\widehat{\mathcal{F}}^\vee)$$

is injective. For $k < d$, (9.7) holds by induction and (9.8) follows as above. \square

In particular, it follows that, via the symmetric FMP transform FM_A , the Chen-Jiang decomposition of a sheaf \mathcal{F} is recovered by the torsion filtration of $\widehat{\mathcal{F}}^\vee$, and conversely.

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