EXPLICIT BOUNDS FOR EVEN MOMENTS OF BERNSTEIN'S POLYNOMIALS

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Abstract. We prove explicit and optimal lower and upper bounds for even-order moments of Bernstein polynomials.

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1. Introduction

Let

$$B_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

denote the k, n Bernstein's polynomial, and let

$$M_{n,s}(x) := \sum_{k=0}^{n} \Big|\frac{k}{n} - x\Big|^s B_{k,n}(x)$$

be its absolute moment of order $s > 0$. Lorentz [2, p. 15, Eq. (9)] described an easy argument proving that $M_{n,s}(x) \leq A_s n^{-s/2}$ for a suitable constant A_s, uniformly for $x \in [0,1]$. This argument does not provide an explicit value for the constant A_s. Adell, Bustamante and Quesada [1, Thm. 1] proved an explicit inequality for moments, namely that

$$\sup_{x\in[0,1]} M_{n,s}(x) \leq 2\,\Gamma\Big(\frac{s}{2}+1\Big)\, n^{-s/2}$$

for every $n \in \mathbb{N}$ and all $s > 0$. This is a strong improvement on previous result, but the authors note that the constant is not optimal in its dependence on s.
Recently, Jim Xiang [5, Thm. 5] gave an asymptotic expansion in powers of n^{-1}, which starts as

$$M_{n,s}(x) = \Big(\frac{2x(1-x)}{n}\Big)^{s/2} \frac{1}{\sqrt{\pi}} \Gamma\Big(\frac{s+1}{2}\Big) + \text{lower order terms}. \tag{1}$$

In this paper we prove that the sup in x of the leading term of the expansion is sufficient to bound the moment when its order is an even integer, under a mild restriction for the minimum value of n which is allowed for each choice of s. This fact is not transparent from the asymptotic expansion, whose lower orders are expressed as certain suitable integrals giving the constants multiplying the terms $n^{-s/2-j}$ for $j = 1, \ldots, \lceil s/2\rceil$ plus a tail which is only estimated as $O(n^{-s-1/2})$.

In detail, our result states the following.

Theorem. *Let $r \in \mathbb{N}$, $c_r := \frac{(2r)!}{r!8^r}$ and $c'_r := c_r \frac{r(r-1)}{3}$. Let $\omega := 1.05308\ldots$ be a certain absolute constant which is defined in next section. Pick any $\eta \in [0, 3/4)$ and assume that $n \geq 260{\cdot}\omega^r/(1-4\eta/3)$, then*

$$\sup_{x\in[0,1]} M_{n,2r}(x) \leq \frac{1}{n^r}\Big(c_r - \eta \frac{c'_r}{n}\Big). \tag{2}$$

On the other hands, assume that $n \geq r^2$, then

$$\sup_{x\in[0,1]} M_{n,2r}(x) \geq \frac{1}{n^r}\Big(c_r - \frac{c'_r}{n}\Big). \tag{3}$$

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Note that the constant c_r matches the value of the main constant appearing in (1) for $s = 2r$, once the factor $x(1-x)$ appearing there is estimated with $1/4$.

The upper and lower bounds are essentially optimal, at least in the common validity range for n, apart the fact that η in (2) is restricted to $[0, 3/4)$. However, the theorem has some room for improvement as regards the range of allowed values for n. For example, we strongly believe that the case $\eta = 0$ in (2) holds actually also when $n < 260{\cdot}\omega^r$, i.e., without any restriction for n in terms of r. We have two main arguments supporting this idea: first, a variation of our proof already reaches the conclusion with $\eta = 0$ under the weaker assumption $n \gg (r^2\omega^r)^{1/3}$. Secondly, a different and direct argument we have tested with every $r \leq 22$ proves the claim with $\eta = 0$ without any restriction for n. For $\eta > 0$ the computations show that the range of n has to be restricted in some way in order to have a valid inequality, but probably also in this case the assumption $n \geq 260{\cdot}\omega^r/(1-4\eta/3)$ is more stringent than what is necessary. Also the range $n \geq r^2$ for (3) should be improved, but it is already very close to its best which is $n > r(r-1)/3$, being the claim trivial when $n \leq r(r-1)/3$. We pospone a deeper discussion about these possible improvements to the end of the next section.

Our proof of the theorem is based essentially on Faà di Bruno's formula to decompose the moment as sum of certain terms which we estimate with some care but using essentially only elementary ideas. This is a considerable simplification with respect to the deep tools which are needed to prove the asymptotic formula (1), but this is made possible by the circumstance that $M_{n,s}$ is actually a polynomial both in x and $1/n$ when s is an even integer. In particular, there is no simple way to adapt these arguments to get the analogous result for the non-even moments, in spite of our believe that at least the case $\eta = 0$ of (2) should be true for every positive s when formulated as

$$n^{s/2} \sup_{x\in[0,1]} M_{n,s}(x) \leq 2^{-s/2}\frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{s+1}{2}\Big) \qquad \forall n.$$

However, explicit, thought non-optimal, bounds for $n^{s/2}M_{n,s}$ can still be obtained also for not even integer values of s by exploiting its convex dependence on the order s. The resulting upper bound is a suitable convex combination of $n^{\lfloor s/2\rfloor}M_{n,2\lfloor s/2\rfloor}$ and $n^{\lceil s/2\rceil}M_{n,2\lceil s/2\rceil}$, that we can bound with our theorem. For example, with this procedure we get the following result.

Corollary. *Let $s \in (0, +\infty)$, $s \notin 2\mathbb{N}$, and $n \geq 260{\cdot}\omega^{\lceil s/2\rceil}$. Then*

$$\sup_{x\in[0,1]} M_{n,s}(x) \leq \Big(\frac{\lceil s/2\rceil - s/2}{\lceil s/2\rceil - \lfloor s/2\rfloor}c_{\lfloor s/2\rfloor} + \frac{s/2 - \lfloor s/2\rfloor}{\lceil s/2\rceil - \lfloor s/2\rfloor}c_{\lceil s/2\rceil}\Big)n^{-s/2}.$$

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2. Proof of the theorem

The polynomial $G_r(n;x) := \sum_{k=0}^n (k-nx)^{2r}B_{k,n}(x)$ shows the symmetry $G_r(n;1-x) = G_r(n;x)$, hence its maximum in $[0,1]$ is already in $[0,1/2]$ and the point $x = 1/2$ is an extremal point for G_r. Computations indicate that this is the point where the polynomial attains its maximum as soon as n is large enough with respect to r, and that $n \geq r^2$ suffices for this. It would be useful to have a proof for this claim, because it would simplify the general argument. Lacking this point, our proof needs a detour. Firstly we note that

$$\begin{aligned}\sum_{k=0}^n (k-nx)^{2r}B_{k,n}(x) &= \Big[\partial_y^{2r}\Big(\sum_{k=0}^n e^{(k-nx)y}B_{k,n}(x)\Big)\Big]_{y=0} = \Big[\partial_y^{2r}\Big(e^{-nxy}\sum_{k=0}^n \binom{n}{k}e^{ky}x^k(1-x)^{n-k}\Big)\Big]_{y=0}\\ &= \Big[\partial_y^{2r}\Big(e^{-nxy}(xe^y+1-x)^n\Big)\Big]_{y=0} = \Big[\partial_y^{2r}\Big(g(x,y)^n\Big)\Big]_{y=0}\end{aligned}$$

where

$$g(x,y) := xe^{(1-x)y} + (1-x)e^{-xy}.$$

The derivative may be computed via Faà di Bruno's formula, yielding the equality

$$\begin{aligned} G_r(n;x) &= \sum_{k=0}^{n}(k-nx)^{2r}B_{k,n}(x) \\ &= (2r)! \sum_{\substack{a_1,a_2,\dots\ge 0\\ a_1+2a_2+3a_3+\dots=2r}} \frac{h^{(a_1+a_2+a_3+\cdots)}(g(x,0))}{a_1!a_2!a_3!\cdots}\Big(\frac{\partial_y g(x,0)}{1!}\Big)^{a_1}\Big(\frac{\partial_y^2 g(x,0)}{2!}\Big)^{a_2}\Big(\frac{\partial_y^3 g(x,0)}{3!}\Big)^{a_3}\cdots, \end{aligned}$$

where $h(w):=w^n$ and $h^{(k)}$ denotes its k derivative. Induction on k shows that

$$\partial_y^k g(x,0) = x(1-x)^k+(-x)^k(1-x) \qquad \forall k\ge 0.$$

In particular $g(x,0)=1$ and $\partial_y g(x,0)=0$ for every x, hence no contribution comes in previous formula from terms with $a_1\neq 0$. Moreover, the polynomial $x(1-x)$ factors $\partial_y^k g(x,0)$ when $k\ge 1$, and in order to simplify the computations it is useful to introduce $z:=1-2x$ (so that z ranges in $[0,1]$ when x ranges in $[0,1/2]$). In this way we get

$$\partial_y^k g\Big(\frac{1}{2}(1-z),0\Big) = \frac{1}{2^{k+1}}(1-z^2)[(z+1)^{k-1}-(z-1)^{k-1}] =: \frac{1}{4}(1-z^2)P_k(z)$$

where $P_k(z):=\frac{1}{2^{k-1}}[(z+1)^{k-1}-(z-1)^{k-1}]$. In particular $P_2(z)=1$, $P_3(z)=z$, $P_4(z)=(1+3z^2)/4$. This formula makes evident the symmetry $P_k(-z)=(-1)^kP_k(z)$ and that $P_k(z)\in[0,1]$ when $z\in[0,1]$. These computations show that

$$G_r(n;x) = (2r)!\sum_{\ell=1}^{r} b_\ell(z)n_{[\ell]} \tag{4}$$

where

$$b_\ell(z) := \Big(\frac{1-z^2}{4}\Big)^{\ell} \sum_{\substack{a_2,\dots\ge 0\\ 2a_2+3a_3+\dots=2r\\ a_2+a_3+\dots=\ell}} \frac{1}{a_2!a_3!\cdots}\Big(\frac{P_2(z)}{2!}\Big)^{a_2}\Big(\frac{P_3(z)}{3!}\Big)^{a_3}\cdots \tag{5}$$

and $n_{[\ell]}$ is the falling Pochhammer symbol (i.e., $n_{[0]}:=1$, $n_{[\ell]}=\prod_{j=0}^{\ell-1}(n-j)$ when $\ell\ge 1$). Note that the condition $2a_2+3a_3+\cdots=2r$ forces the inequality $a_2+a_3+\cdots\le r$. Thus, only terms with $\ell\le r$ appear in (4). We estimate separately b_r, b_{r-1} and b_ℓ with $\ell\le r-2$.
The sum $a_2+a_3+\cdots$ equals r only for $a_2=r$ and $a_3=a_4=\cdots=0$. Hence

$$b_r(z) = \frac{(1-z^2)^r}{r!8^r} \le \frac{1}{r!8^r} \qquad \forall z\in[0,1]. \tag{6}$$

The sum producing b_{r-1} contains only terms coming from solutions of $2a_2+3a_3+\cdots=2r$ with $a_2+a_3+\cdots=r-1$. This happens only when (a_2,a_3,a_4) is $(r-3,2,0)$ or $(r-2,0,1)$, and every other a_j is zero. Hence

$$\begin{aligned} b_{r-1}(z) &= \Big(\frac{1-z^2}{4}\Big)^{r-1}\Big[\frac{1}{(r-3)!2!}\Big(\frac{1}{2!}\Big)^{r-3}\Big(\frac{z}{3!}\Big)^2+\frac{1}{(r-2)!1!}\Big(\frac{1}{2!}\Big)^{r-2}\Big(\frac{1+3z^2}{4\cdot 4!}\Big)\Big] \\ &= \frac{(1-z^2)^{r-1}}{(r-3)!8^r}\Big[\frac{4}{9}z^2+\frac{1+3z^2}{6(r-2)}\Big]. \end{aligned} \tag{7}$$

The following lemma gives a bound for this quantity.

Lemma 1. *Let $r\ge 3$. Then*

$$b_{r-1}(z) \le \frac{0.25}{(r-2)!8^r} \qquad \forall z\in[0,1]. \tag{8}$$

Proof. According to (7), we have to prove that

$$R_r(z^2) := \frac{1}{18}(1-z^2)^{r-1}\big[z^2(8r-7)+3\big] \le 0.25 \qquad \forall z\in[0,1].$$

Function R_r is positive in $[0,1)$, equal to $1/6$ for $z=0$ and to zero for $z^2=1$. Its derivative (with respect to z^2) is $\frac{1}{18}(11r-10)>0$ at 0, hence the maximum for R_r is attained at the unique zero in $(0,1)$ of the derivative. This point is $z_r^2:=\frac{5r-4}{8r^2-7r}$, and the maximum is $R_r(z_r^2)=\frac{4}{9}(1-\frac{5r-4}{8r^2-7r})^{r-1}(1-\frac{1}{2r})$. When

$r=3$ we use directly this formula to prove that $R_3(z_3^2)\le 0.25$. Assume $r\ge 4$. Since $1-\frac{5r-4}{8r^2-7r}\le 1-\frac{5}{8r}$ for every $r\ge 1$ and $(1-1/y)^y\le e^{-1}$ for every $y\ge 1$, we have $(1-\frac{5r-4}{8r^2-7})^{r-1}\le(1-\frac{5}{8r})^{r-1}\le e^{-5/8}/(1-\frac{5}{8r})$, so that

$$R_r(z_r^2)\le\frac{4}{9}e^{-5/8}\frac{1-\frac{1}{2r}}{1-\frac{5}{8r}}.$$

The constant appearing to the right is smaller than 0.25 as soon as $r\ge 4$. $\square$

We now need a bound for b_ℓ for all remaining $\ell=1,\dots,r-2$. In (5) we set a_2 from equality $a_2+a_3+a_4+\dots=\ell$. Recalling that $P_2(z)=1$, we get

$$b_\ell(z)=\Big(\frac{1-z^2}{8}\Big)^\ell\sum_{\substack{a_3,\dots\ge 0\\ a_3+a_4+\dots\le\ell\\ a_3+2a_4+3a_5+\dots=2r-2\ell}}\frac{1}{(\ell-a_3-a_4-\cdots)!a_3!a_4!\cdots}\Big(\frac{2P_3(z)}{3!}\Big)^{a_3}\Big(\frac{2P_4(z)}{4!}\Big)^{a_4}\cdots.$$

The inequality $\frac{1}{(\ell-A)!}\le\frac{\ell^A}{\ell!}$ holds for every number $0\le A\le\ell$, so that b_ℓ can be bounded as

$$b_\ell(z)\le\Big(\frac{1-z^2}{8}\Big)^\ell\frac{1}{\ell!}\sum_{\substack{a_3,\dots\ge 0\\ a_3+a_4+\dots\le\ell\\ a_3+2a_4+3a_5+\dots=2r-2\ell}}\frac{\ell^{a_3+a_4+\cdots}}{a_3!a_4!\cdots}\Big(\frac{2P_3(z)}{3!}\Big)^{a_3}\Big(\frac{2P_4(z)}{4!}\Big)^{a_4}\cdots.$$

In this formula only a_j with $j\le 2r-2\ell+2$ may contribute; the right hand side may be estimated with

$$\begin{aligned}&\Big(\frac{1-z^2}{8}\Big)^\ell\frac{1}{\ell!}\sum_{w=1}^{2r-2\ell}\sum_{\substack{a_3,\dots,a_{2r-2\ell+2}\ge 0\\ a_3+\dots+a_{2r-2\ell+2}=w}}\frac{\ell^{a_3+a_4+\dots+a_{2r-2\ell+2}}}{a_3!\cdots a_{2r-2\ell+2}!}\Big(\frac{2P_3(z)}{3!}\Big)^{a_3}\cdots\Big(\frac{2P_{2r-2\ell+2}(z)}{(2r-2\ell+2)!}\Big)^{a_{2r-2\ell+2}}\\ &=\Big(\frac{1-z^2}{8}\Big)^\ell\frac{1}{\ell!}\sum_{w=1}^{2r-2\ell}\frac{1}{w!}\Big(\frac{2\ell P_3(z)}{3!}+\dots+\frac{2\ell P_{2r-2\ell+2}(z)}{(2r-2\ell+2)!}\Big)^w,\end{aligned}$$

because every set of $a_3,\dots,a_{2r-2\ell+2}$ with $a_3+2a_4+3a_5+\dots=2r-2\ell$ satisfies also the equality $a_3+\dots+a_{2r-2\ell+2}=w$ for some $w\le 2r-2\ell$; for the second step we have used the multinomial formula. Each $P_k(z)$ equals $\frac{2}{2^k}[(z+1)^{k-1}-(z-1)^{k-1}]$ and is in $[0,1]$, hence we can extend the inner sum to all P_k's. In this way we get the bound

$$\begin{aligned}b_\ell(z)&\le\Big(\frac{1-z^2}{8}\Big)^\ell\frac{1}{\ell!}\sum_{w=1}^{2r-2\ell}\frac{(2\ell)^w}{w!}\Big(\sum_{k=3}^\infty\frac{1}{k!}\Big[\Big(\frac{z+1}{2}\Big)^{k-1}-\Big(\frac{z-1}{2}\Big)^{k-1}\Big]\Big)^w\\ &=\Big(\frac{1-z^2}{8}\Big)^\ell\frac{1}{\ell!}\sum_{w=1}^{2r-2\ell}\frac{(2\ell)^w}{w!}\Big(g\Big(\frac{z+1}{2}\Big)-g\Big(\frac{z-1}{2}\Big)\Big)^w\end{aligned}$$

where $g(y):=\frac{1}{y}(e^y-1-y-y^2/2)$ for $y\in\mathbb{C}$ (the singularity at 0 is removed by analyticity: $g(0)=0$). Let $h(z):=g\big(\frac{z+1}{2}\big)-g\big(\frac{z-1}{2}\big)$. Polynomials $P_k(z)$ have nonnegative coefficients and g is a power series with nonnegative coefficients, hence h does the same. Thus, $0<h(0)\le h(z)\le h(1)=g(1)=e-5/2=0.21\dots$ for $z\in[0,1]$. In particular, $h(z)$ is always positive and quite small. Thus, extending the range of w to all integers we get the bound

$$b_\ell(z)\le\Big(\frac{1-z^2}{8}\Big)^\ell\frac{e^{2\ell h(z)}}{\ell!},$$

which should be quite close to the true value. In particular we get that

$$b_\ell(z)\le\frac{\omega^\ell}{8^\ell\ell!}\qquad\forall\ell=1,\dots,r-2,\qquad\forall z\in[0,1],\tag{9}$$

where $\omega:=\sup_{z\in[0,1]}(1-z^2)\exp(2h(z))=1.053088\dots$. Note that the bound in (9) is proved for every $\ell\le r$, in particular also for $\ell=r$ and $\ell=r-1$, but in these cases it is weaker than what we have in (6)

and (8), as effect of the extra exponential factor ω^ℓ.
We plug (6), (8) and (9) into (4), getting the bound

$$\frac{1}{(2r)!}G_r(n;x) \le \frac{n_{[r]}}{r!8^r}+0.25\frac{n_{[r-1]}}{(r-2)!8^r}+\sum_{\ell=1}^{r-2}\omega^\ell\frac{n_{[\ell]}}{8^\ell \ell!} \qquad x\in[0,1], \tag{10}$$

and we conclude the proof of (2) showing that this quantity is bounded by $\frac{n^r}{r!8^r}(1-\eta\frac{r(r-1)}{3n})$ when $\eta<3/4$. For this purpose we use a couple of lemmas involving Pochhammer symbols.

Lemma 2. *Let $\ell\ge 0$ and $n\ge 1$, then*

$$n_{[\ell]}\le n^\ell\Big[1-\frac{1}{n}\binom{\ell}{2}+\frac{1}{2n^2}\binom{\ell}{2}^2\Big].$$

Let $\ell\ge 0$ and $n\ge \ell^2$, then

$$n_{[\ell]}\ge n^\ell\Big[1-\frac{1}{n}\binom{\ell}{2}+\frac{3\ell^2-13\ell+5}{15n^2}\binom{\ell}{2}\Big].$$

Proof. Both claims are evident for $\ell=0,1$. Assume $\ell\ge 2$, and write

$$\frac{n_{[\ell]}}{n^\ell}=\prod_{j=0}^{\ell-1}(1-j/n)=\exp\Big(\sum_{j=0}^{\ell-1}\log(1-j/n)\Big).$$

Let $s_1:=\sum_{j=0}^{\ell-1}j=\binom{\ell}{2}$ and $s_2:=\sum_{j=0}^{\ell-1}j^2=\binom{\ell}{2}\frac{2\ell-1}{3}$. Since $\log(1-w)\le -w$ for every $w\in[0,1)$ and $\exp(-w)\le 1-w+w^2/2$ when $w\ge 0$, we deduce that

$$\frac{n_{[\ell]}}{n^\ell}\le\exp\Big(-\frac{s_1}{n}\Big)\le 1-\frac{s_1}{n}+\frac{s_1^2}{2n^2},$$

which is the first claim. For the second one we note that $\log(1-w)\ge -w-w^2$ for every $w\in[0,0.68\ldots]$, and that $\exp(-w)\ge 1-w+2w^2/5$ when $w\in[0,1/2]$. Thus, assuming that $n\ge\ell^2$ we get that

$$\frac{n_{[\ell]}}{n^\ell}\ge\exp\Big(-\frac{s_1}{n}-\frac{s_2}{n^2}\Big)\ge 1-\frac{s_1}{n}-\frac{s_2}{n^2}+\frac{2}{5}\Big(\frac{s_1}{n}+\frac{s_2}{n^2}\Big)^2,$$

since the assumption $n\ge\ell^2$ assures that $\frac{s_1}{n}+\frac{s_2}{n^2}\le\frac{1}{2}$. To conclude it is sufficient to remark that the right hand side is larger than $1-\frac{s_1}{n}+\frac{2s_1^2-5s_2}{5n^2}$ and that $2s_1^2-5s_2=\binom{\ell}{2}\frac{3\ell^2-13\ell+5}{3}$. □

Lemma 3. *Let $r\ge 21$ and $n\ge r^2$. Then*

$$\sum_{\ell=1}^{r}\frac{n_{[\ell]}\omega^\ell}{8^\ell\ell!}\le\frac{n^r\omega^r}{8^r r!}.$$

The range for r cannot be extended: the claim is false when $r\le 20$. The proof will make clear that the claim holds under the weaker assumption $n\ge cr$ for a certain c independent on r when r is large enough, but we will apply this inequality for $n\gg\omega^r$, so that the range $n\ge r^2$ suffices for this purpose and allows us to have a result which excludes only a few values for r.

Proof. The claim is proved differently when $r\le 32$ and when $r\ge 33$. Actually, the strategy we apply for small r is general, but it needs the computation of the coefficients of a certain polynomial depending on r, so that it can be used only for given r. In fact, let

$$Q_r(n):=n^r-\sum_{\ell=1}^{r}n_{[\ell]}\Big(\frac{\omega}{8}\Big)^{\ell-r}\frac{r!}{\ell!}.$$

We have to prove that this polynomial is positive when $n\ge r^2$. For this it is sufficient to prove that the leading coefficient is positive and that the absolute value of each complex root is lower than r^2. Each polynomial Q_r has degree $r-1$ and its leading coefficient is $\binom{r}{2}-\frac{8r}{\omega}$, which is positive when $r\ge 17$. Let $Q_r(n)=\sum_{j=0}^{r-1}d_jn^j$ be the standard presentation of this polynomial, and let $\bar n_r$ denote any of its complex roots with maximum modulus. Then, isolating the leading term in $Q_r(\bar{n}_r)=0$ and dividing by $|d_{r-1}||\bar{n}_r|^{r-2}$ we deduce that

$$|\bar{n}_r| \leq \sum_{j=0}^{r-2} \frac{|d_j|}{|d_{r-1}|} \frac{1}{|\bar{n}_r|^{r-2-j}}.$$

Suppose that $|\bar{n}_r| \geq r^2$, then we have the double inequality

$$r^2 \leq |\bar{n}_r| \leq \sum_{j=0}^{r-2} \frac{|d_j|}{|d_{r-1}|} \frac{1}{r^{2(r-2-j)}}.$$

Thus, we can prove that $|\bar{n}_r| \geq r^2$ is impossible by checking that $\sum_{j=0}^{r-2} \frac{|d_j|}{|d_{r-1}|} \frac{1}{r^{2(r-2-j)}}$ is actually smaller than r^2. Table 1 collects all computations, and proves the inequality for $r=21,\dots,32$.

Table 1. Data for $21 \leq r \leq 32$.

r	r^2	$\sum_{j=0}^{r-2} \frac{\lvert d_j\rvert}{\lvert d_{r-1}\rvert} \frac{1}{r^{2(r-2-j)}}$	r	r^2	$\sum_{j=0}^{r-2} \frac{\lvert d_j\rvert}{\lvert d_{r-1}\rvert} \frac{1}{r^{2(r-2-j)}}$
21	441	348.617	27	729	232.430
22	484	302.935	28	784	236.604
23	529	273.033	29	841	242.388
24	576	252.996	30	900	249.554
25	625	239.595	31	961	257.935
26	676	230.922	32	1024	267.409

Assume $r \geq 33$ and $n \geq r^2$. Lemma 2 shows that when $n \geq \binom{r}{2}$ and $\ell \leq r$ we get

$$n_{[\ell]} \leq n^\ell \Big[1-\frac{1}{n}\binom{\ell}{2}+\frac{1}{2n^2}\binom{\ell}{2}^2\Big] \leq n^\ell \Big[1-\frac{1}{2n}\binom{\ell}{2}\Big].$$

Hence

$$\sum_{\ell=1}^{r} \frac{n_{[\ell]}\omega^\ell}{8^\ell \ell!} \leq \sum_{\ell=1}^{r} \frac{n^\ell\omega^\ell}{8^\ell \ell!}\Big[1-\frac{1}{2n}\binom{\ell}{2}\Big] = \frac{n^r\omega^r}{8^r r!} - \sum_{\ell=1}^{r-1} \frac{n^\ell\omega^\ell}{8^\ell \ell!}\Big[\frac{\omega\ell}{32}-1\Big].$$

The contribution of terms $\ell \geq 31$ is negative (because $\omega\ell-32$ is positive here), therefore the inequality is proved as soon as the total contribution of terms for $\ell=1,\dots,30$ and the one for $\ell=r-1$ (which is another one, since $r \geq 33$) is negative, i.e., as soon as

$$\sum_{\ell=1}^{30} \frac{n^\ell\omega^\ell}{8^\ell \ell!}\Big[1-\frac{\omega\ell}{32}\Big] \leq \frac{n^{r-1}\omega^{r-1}}{8^{r-1}(r-1)!}\Big[\frac{\omega(r-1)}{32}-1\Big].$$

The term appearing to the left hand side is smaller than $5\cdot 10^{-61}\cdot n^{30}$ when $n \geq 900$ (use the assumption to get $n^\ell \leq n^{30}/900^{30-\ell}$, and then compute the constant), and $\frac{\omega(r-1)}{32}-1 \geq 5/100$ (because $r \geq 33$, again). Hence it is sufficient to prove that

$$10^{-59}(8/\omega)^{r-1}(r-1)! \leq n^{r-31}.$$

This is true when $n \geq r^2$ and $r \geq 33$. □

We use Lemma 2 to deduce that

$$\frac{n_{[r]}}{r!8^r}+0.25\frac{n_{[r-1]}}{(r-2)!8^r} \leq \frac{n^r}{r!8^r} - \frac{n^{r-1}}{4(r-2)!8^r} + \frac{n^{r-2}}{4(r-2)!8^r}\Big(r+\frac{r^4}{8n}\Big),$$

and Lemma 3 to deduce that

$$\sum_{\ell=1}^{r-2} \frac{n_{[\ell]}\omega^\ell}{8^\ell \ell!} \leq \frac{n^{r-2}\omega^{r-2}}{8^{r-2}(r-2)!},$$

at least when $r \geq 23$ and $n \geq r^2$. With these bounds for the terms in (10), the claim will be proved as soon as

$$\frac{n^{r-2}}{4(r-2)!8^r}\Big(r+\frac{r^4}{8n}\Big)+\frac{n^{r-2}\omega^{r-2}}{8^{r-2}(r-2)!} \leq \frac{n^{r-1}}{4(r-2)!8^r}-\frac{\eta}{3}\frac{n^{r-1}}{(r-2)!8^r},$$

i.e.,

$$r+\frac{r^4}{8n}+256{\cdot}\omega^{r-2} \leq n\Big(1-\frac{4\eta}{3}\Big).$$

When η is in $[0,3/4)$ this inequality holds as soon as $n \geq 260{\cdot}\omega^r/(1-4\eta/3)$ for every r, but it has been deduced under the assumption that $r \geq 23$ (and that $n \geq r^2$, but this is always true when $n \geq 260{\cdot}\omega^r$). Thus, this argument proves (2) under the extra assumption that $r \geq 23$.

For the range $r \leq 22$ we need a different approach, still based upon the formula (4) for $G_r(n;x)$, but needing the assistance of a software for some computations. In Appendix we describe the main parts of this computation, while a full description and the PARI-gp [4] code realizing the computation are available to the interested reader in [3].

The lower-bound (3) has a simpler proof. In fact, we notice that

$$n^r \sup_{x\in[0,1]} M_{n,2r} = n^{-r} \sup_{x\in[0,1]} G_r(n,x) \geq n^{-r}G_r(n,1/2) \geq (2r)!\Big[b_r(0)\frac{n_{[r]}}{n^r}+b_{r-1}(0)\frac{n_{[r-1]}}{n^r}\Big],$$

because in (4) all b_ℓ's are positive. By (6) and (7) we get the values $b_r(0)=\frac{1}{r!8^r}$ and $b_{r-1}(0)=\frac{1/6}{(r-2)!8^r}$, and from the second inequality in Lemma 2 we conclude that for $n \geq r^2$ and $r \geq 5$ (so that $\frac{n_{[r-1]}}{n^{r-1}} \geq 1-\frac{1}{n}\binom{r-1}{2}$)

$$\begin{aligned} n^r \sup_{x\in[0,1]} M_{n,2r} &\geq \frac{(2r)!}{r!8^r}\Big[1-\frac{1}{n}\binom{r}{2}+\frac{3r^2-13r+5}{15n^2}\binom{r}{2}\Big]+\frac{(2r)!/6}{(r-2)!8^r}\Big[\frac{1}{n}-\frac{1}{n^2}\binom{r-1}{2}\Big] \\ &= c_r-\frac{c'_r}{n}+c'_r\frac{r(r-11)}{20n^2}, \end{aligned}$$

which is larger than $c_r-\frac{c'_r}{n}$ as soon as $r \geq 11$. Once again, the proof of the claim for the remaining range $r \leq 10$ is described in Appendix in its general ideas and in [3] in detail.

The assumption $n \geq 260{\cdot}\omega^r$ we have made for (2) is not natural and actually can be relaxed considerably, at least for the case $\eta=0$. For example, with some extra work we can compute the exact formulas for the coefficients b_{r-2} and b_{r-3} and deduce the bounds

$$b_{r-2}(z) \leq \frac{0.2}{(r-4)!8^r} \quad\text{and}\quad b_{r-3}(z) \leq \frac{0.04}{(r-6)!8^r} \qquad \forall z\in[0,1]. \tag{11}$$

These bounds improve exponentially with respect to what we got in (9) for the same b_ℓ's. Plugging these new bounds in the general formula (4) and retaining the general bound (9) only for coefficients b_ℓ with $\ell \leq r-4$ we produce an estimation which is similar to (10), but that allows to prove (2) under the weaker hypothesis $n \gg (r^2\omega^r)^{1/3}$. Bounds (11) suggest that probably b_ℓ is smaller than $\frac{1}{(2\ell-r)!8^r}$ when $\ell \geq r/2$. If true, this fact implies that $n \gg r^c$ for some constant c suffices for (2). However, while it is possible to prove it also for b_{r-4}, b_{r-5} and in some other case, it is not clear how to produce these improvements for all $\ell \geq r/2$.

As announced in the previous section, we repeat here that the explicit computations for $r \leq 22$ in [3] show also that for these r the case $\eta=0$ of Inequality (2) holds for every n, i.e. without the restriction $n \geq 260{\cdot}\omega^r$. We believe that this fact should be true for every r: if true, its proof probably needs a different approach.

3. Appendix

Here we describe the procedure we used in [3] to prove the claims of the theorem also in the remaining range, i.e. $r \leq 22$ for (2) and $r \leq 10$ for (3).

Firstly, we compute explicitly the polynomial $G_r(n;x)$. This is done using the formula in (4), since its definition as $\sum_{k=0}^{n}(k-nx)^{2r}B_{k,n}(x)$ is not suitable for this purpose. The polynomial is in $\mathbb{Q}[n,x]$ and PARI-gp [4] computes with full precision for this class of polynomials, so that the code produces the correct answer without approximations. The computations show that $G_r(n;x)$ has degree r in n and

depends on x via $(1-2x)^2$: its degree as a polynomial in $(1-2x)^2$ is r again, at least for $r \leq 22$ (but this fact is actually provable for every r). To simplify the following steps we change $(1-2x)^2$ with y and we call $A_r(n,y)$ the resulting polynomial. In terms of A_r, the first claim we have to prove states that $A_r(n,y) \leq c_r n^r - \eta c'_r n^{r-1}$ for every $y \in [0,1]$, for every $\eta \in [0,3/4)$ and for every $n \geq 260\cdot\omega^r/(1-4\eta/3)$ and $r \leq 22$. We actually prove a stronger result, namely that $A_r(n,y) \leq c_r n^r - \frac{3c'_r}{4} n^{r-1}$ for every $y \in [0,1]$ for every $n \geq 260\cdot\omega^r$ and $r \leq 22$.
A close analysis of these polynomials show that A_r can be written as

$$A_r(n,y) = \sum_{k=1}^{r} (1-y)^k P_{r,k}(y) n^k,$$

where $P_{r,k}(y)$ is a polynomial in y with degree $r-k$, and that

$$P_{r,k}(y) = (-1)^{r-k} \sum_{j=0}^{r-k} c_{r,k,j}(-y)^j$$

with $c_{r,k,j} > 0$ for every r,k,j (we believe that this fact is true for every r, but we have not a general proof. However, its truth for $r \leq 22$ is proved by inspection of the polynomial that we have computed before). Moreover, we see that $c_{r,r,0} = c_r$, and that $c_{r,r-1,0} = c'_r$.

The following steps depend on a parameter y_r with a different value for each $r = 1,\dots,22$: any value in $(0,1)$ for y_r is admissible, but certain values produce better conclusions. With a try and error procedure we have selected a convenient value for y_r for each $r \leq 22$.
STEP 1: we prove that there exists an integer $N_{1,r}$ such that

$$A_r(n,y) \leq c_r n^r - \frac{3c'_r}{4} n^{r-1} \quad \text{for every } y \in [y_r,1], \text{ when } n \geq N_{1,r}$$

and we find an explicit value for $N_{1,r}$.
In fact, for $y \in [y_r,1]$ one has

$$A_r(n,y) = (1-y)^r c_r n^r + \sum_{k=1}^{r-1} (1-y)^k P_{r,k}(y) n^k \leq (1-y_r)^r c_r n^r + \sum_{k=1}^{r-1} \|(1-y)^k P_{r,k}(y)\|_{[y_r,1]} n^k,$$

where $\|\cdot\|_{[y_r,1]}$ denotes the sup norm in $[y_r,1]$. Thus, it is lower than $c_r n^r - \frac{3c'_r}{4} n^{r-1}$ as soon as

$$-(1-(1-y_r)^r) c_r n^r + \frac{3c'_r}{4} n^{r-1} + \sum_{k=1}^{r-1} \|(1-y)^k P_{r,k}(y)\|_{[y_r,1]} n^k \leq 0$$

and for $N_{1,r}$ we take the least integer which is larger than the greatest real root of this polynomial. This proves the existence of $N_{1,r}$ for every r, but the formula produces an explicit value for $N_{1,r}$ only when r is fixed, because in this case we can compute each $\|(1-y)^k P_{r,k}(y)\|_{[y_r,1]}$ exactly from its explicit description.
STEP 2: we prove that there exists $N_{2,r}$ such that

$$\partial_y A_r(n,y) \leq 0 \quad \text{for every } y \in [0,y_r], \text{ when } n \geq N_{2,r}$$

and we find an explicit value for $N_{2,r}$.
In fact, for $y \in [0,y_r]$ one has

$$\begin{aligned}\partial_y A_r(n,y) &= -r(1-y)^{r-1} c_r n^r + \sum_{k=1}^{r-1} [(1-y)^k P_{r,k}(y)]' n^k \\ &\leq -r(1-y_r)^{r-1} c_r n^r + \sum_{k=1}^{r-1} \|[(1-y)^k P_{r,k}(y)]'\|_{[0,y_r]} n^k\end{aligned}$$

and for $N_{2,r}$ we take the least integer which is larger than the greatest real root of this polynomial. As for Step 1, this proves the existence of $N_{2,r}$ for every r, but it produces an explicit value for $N_{2,r}$ only when r is fixed.
STEP 3: we prove that there exists $N_{3,r}$ such that

$$A_r(n,0) \leq c_r n^r - \frac{3c'_r}{4} n^{r-1} \qquad \text{for every } n \geq N_{3,r}$$

and we find an explicit value for $N_{3,r}$.
In fact, we see that

$$\begin{aligned} A_r(n,0) &= \sum_{k=1}^{r} P_{r,k}(0) n^k = \sum_{k=1}^{r} (-1)^{r-k} c_{r,k,0} n^k \\ &= c_{r,r,0} n^r - n^{r-2}(c_{r,r-1,0} n - c_{r,r-2,0}) - n^{r-4}(c_{r,r-3,0} n - c_{r,r-4,0}) - \cdots \end{aligned}$$

so that it is $\leq c_r n^r - \frac{3c'_r}{4} n^{r-1}$ as soon as

$$n \geq N_{3,r} := \max\Big\{ 4\frac{c_{r,r-2,0}}{c_{r,r-1,0}}, \frac{c_{r,r-4,0}}{c_{r,r-3,0}}, \dots \Big\},$$

because $c_{r,r,0} = c_r$, $c_{r,r-1,0} = c'_r$, and all coefficients $c_{r,k,j}$ are positive.
Steps 1, 2 and 3 prove that

$$\sup_{x\in[0,1]} G_r(n;x) \leq c_r n^r - \frac{3c'_r}{4} n^{r-1} \qquad \forall x \in [0,1]$$

when $r \leq 22$ and $n \geq N_r := \max\{N_{r,1}, N_{r,2}, N_{r,3}\}$. In all cases $N_r \leq 260 \cdot \omega^r$, hence the upper bound of the theorem is now proved also for $r \leq 22$, with $n \geq 260 \cdot \omega^r$.
As a final bonus, we show that when $r \leq 22$ one has

$$\sup_{x\in[0,1]} G_r(n;x) \leq c_r n^r$$

also when $n < N_r$ by proving that for every choice of n in the given range the resulting polynomial (in x) is bounded by $c_r n^r$. This procedure is possible since we have a finite (and relatively small) set of r's and n's to test. This proves that, at least when $r \leq 22$, the case $\eta = 0$ of the upper bound holds without the restriction to $n \geq 260 \cdot \omega^r$.

For the lower bound we act in similar way, since $\sup_{x\in[0,1]} G_r(n;x) \geq G_r(n;1/2) = A_r(n,0)$. Thus, we can test (3) by checking that

$$A_r(n,0) - c_r n^r + c'_r n^{r-1} = n^{r-3}(c_{r,r-2,0} n - c_{r,r-3,0}) + n^{r-5}(c_{r,r-4,0} n - c_{r,r-5,0}) - \cdots$$

is positive. For this it is sufficient to have

$$n \geq \max\Big\{ \frac{c_{r,r-3,0}}{c_{r,r-2,0}}, \frac{c_{r,r-5,0}}{c_{r,r-4,0}}, \dots \Big\},$$

and the explicit computation shows that this maximum is lower than r^2 when $r \leq 10$.

References

[1] J. A. Adell, J. Bustamante, and J. M. Quesada, *Estimates for the moments of Bernstein polynomials*, J. Math. Anal. Appl. **432** (2015), no. 1, 114–128.
[2] G. G. Lorentz, *Bernstein polynomials*, second ed., Chelsea Publishing Co., New York, 1986.
[3] G. Molteni, *Explicit bounds for even moments of Bernstein's polynomials: computations*, `http://users.mat.unimi.it/users/molteni/research/even-moments/computations.pdf`, preprint 2021.
[4] The PARI Group, Bordeaux, *PARI/GP, version* `2.6.0`, 2013, from `http://pari.math.u-bordeaux.fr/`.
[5] Jim X. Xiang, *Expansion of moments of Bernstein polynomials*, J. Math. Anal. Appl. **476** (2019), no. 2, 585–594.

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