Bi-rational maps in four dimensions with two invariants

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Abstract. In this paper we present a class of four-dimensional bi-rational maps with two polynomial invariants of particular degree patterns. We discuss the integrability properties of these maps from the point of view of degree growth and Liouville integrability.

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1. Introduction

In this paper we classify, under certain conditions, maps of $\mathbb{CP}^4$ to itself, which possess two polynomial invariants. The outcomes of our classification include eight new maps which have interesting properties. Despite the existence of two invariants, there turn out to be non-integrable cases, with exponential growth. Other cases are integrable, with cubic and quadratic growth. The cases of cubic growth are only possible in dimension greater than two [1, 2]. We discuss the geometric properties of these systems [3].

It is well-known that two-dimensional integrable bi-rational maps can be characterized by the existence of a rational invariant. For instance most of the integrable maps on the plane fall in the class of QRT maps [4, 5], even though there are some notable exceptions [6–8]. The integrability of these maps can be explained geometrically and has led to many interesting developments [7, 9, 10].

In higher dimensions an analogous general framework does not exist. In particular, for mappings in four dimensions, a generalisation of the QRT class [4, 5] was given in [11]. However, this generalisation does not cover all possible integrable maps in four dimensions. Indeed, some of the new maps obtained in [11] turn out to be autonomous versions of Painlevé hierarchies [12] which are *multiplicative* equations in Sakai's scheme [9]. On the other hand, there exist hierarchies of *additive* discrete Painlevé equations too [13]. Equations coming from the hierarchies of additive Painlevé equations lie outside the framework of [11]. Other examples of four-dimensional maps falling outside the class presented in [11] are given in [14–19].

Our starting point is [14], where the authors considered the *autonomous limit* of the second member of the $\mathrm{d}P_{\mathrm{I}}$ and $\mathrm{d}P_{\mathrm{II}}$ hierarchies [13]. We will denote these equations as $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations. These $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations are given by recurrence relations of order four, and shown to be integrable according to the algebraic entropy approach. Therein the authors showed that both maps possess two polynomial invariants. Using these invariants, they produced the dual maps of the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations in the sense of [20]. Moreover, they showed that these dual maps are integrable according to the algebraic entropy test and also possess invariants. In fact, the number of invariants showed that the dual maps are actually *superintegrable*. Finally they gave a scheme to construct recurrence relations of an assigned form. Using this scheme, some new examples were presented in [14], but no classification was attempted. Starting from these considerations we consider and solve the problem of finding all fourth-order bi-rational maps possessing two polynomial invariants of general enough form to contain those of the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations in the present paper. Our main result is stated as Theorem 19.

The structure of the paper is the following: in section 2, we give a concise explanation of the background material. In particular, we discuss various definitions of integrability for mappings we will use throughout the paper. In section 3, we present motivations for our search and our search method. Moreover, we state the general result. In section 4, we give the explicit form of the maps and discuss their integrability properties. Finally, in section 5, we make some general comments on the maps we obtained, and possible future development.

2. Setting

In this section we provide the definitions we need to explain how our list of equations is found and integrability.

2.1. Bi-rational maps and invariants

The main subjects of this paper are *bi-rational maps*:

$$\varphi\colon [\mathbf{x}] \in \mathbb{CP}^n \to [\mathbf{x}'] \in \mathbb{CP}^n, \tag{1}$$

where $n > 1$ and $[\mathbf{x}] = [x_1 : x_2 : \cdots : x_{n+1}]$, $[\mathbf{x}'] = \left[x'_1 : x'_2 : \cdots : x'_{n+1}\right]$ are homogeneous coordinates on $\mathbb{CP}^n$. We exclude $n = 1$, since bi-rational maps in $\mathbb{CP}^1$ are just Möbius transformations, and so are trivial. We recall that a bi-rational map is a rational map $\varphi\colon V \to W$ of algebraic varieties V and W such that there exists a rational map $\psi\colon W \to V$, which is the *inverse* of φ in the dense subset where both maps are defined [21].

Bi-rational maps (1) are the natural mathematical object needed to study *autonomous single-valued invertible rational recurrence relations*. An autonomous recurrence relation is a functional equation whose unknown is a sequence $\{w_k\}_{k\in\mathbb{Z}}$ such that:

$$w_{k+n} = f\left(w_k, \dots, w_{k+n-1}\right), \tag{2}$$

and the function f does not depend explicitly on k. Moreover, we say that a recurrence relation is *rational* if the function f in (2) is a rational function. Finally, the recurrence is invertible and single-valued if equation (2) is solvable uniquely with respect to w_k. All the terms of the sequence w_k for $k > n$ are then obtained by iterated substitution of the previous one into equation (2). For this reasons it is possible to interpret the recurrence relation (2) as a map of the complex space of dimension n into itself as:

$$\varpi\colon \mathbf{w} \in \mathbb{C}^n \to \mathbf{w}' \in \mathbb{C}^n, \tag{3}$$

where $\mathbf{w} = (w_{n-1}, \dots, w_0)$ are the initial conditions and the map acts as:

$$\mathbf{w}' = \left(f\left(\mathbf{w}\right), w_{n-1}, \dots, w_1\right). \tag{4}$$

The recurrence relation (2) is then given by the repeated application of the map ϖ, namely w_{n+k} is the first component of ϖ^k. Interpreting the coordinates $\mathbf{w} \in \mathbb{C}^n$ as an affine chart in $\mathbb{CP}^n$, i.e. assuming that $(w_{n-1}, \dots, w_0) = [w_{n-1} : \cdots : w_0 : 1]$ we have that the map (3) can be brought to a bi-rational map of $\mathbb{CP}^n$ into itself (1).

Throughout the paper we will often make use of the correspondence between bi-rational maps and recurrence relations. This is due to the fact that some definitions are easier to state and use in the projective setting, while others are easier to state and use in the affine one. In any case for us "bi-rational map" and "recurrence relation" will be equivalent.

The basic definition we will need in this paper is the following:

Definition 1. An *invariant* of a bi-rational map $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is a homogeneous function $I\colon \mathbb{CP}^n \to \mathbb{C}$ such that it is preserved under the action of the map, i.e.

$$\varphi^*\left(I\right) = I, \tag{5}$$

where $\varphi^*\left(I\right)$ means the pullback of I through the map φ, i.e. $\varphi^*\left(I\right) = I\left(\varphi\left([\mathbf{x}]\right)\right)$.

In what follows we will concentrate on a particular class of invariants:

Definition 2. An invariant I is said to be *polynomial*, if in the affine chart $[x_1 : \cdots : x_n : 1]$ the function I is a polynomial function.

In definition 2 we use x_{n+1} as a homogenising variable to go from an affine (polynomial) form to a projective (rational) form of the invariants. A polynomial invariant in the sense of definition 2 written in homogeneous variables is always a rational function homogeneous of degree 0. The form of the polynomial invariant in homogeneous coordinates is then given by:

$$I\left([\mathbf{x}]\right) = \frac{I'\left([\mathbf{x}]\right)}{x_{n+1}^d}, \quad d = \deg I'\left([\mathbf{x}]\right), \tag{6}$$

where deg is the total degree and I' is a homogeneous polynomial.

For $n > 1$, we say that a polynomial invariant is *non-degenerate* if:

$$\frac{\partial I}{\partial x_1}\frac{\partial I}{\partial x_n} \neq 0. \tag{7}$$

Otherwise an invariant is said to be *degenerate*.

To better characterize the properties of these invariants we introduce the following:

Definition 3. Given a polynomial function $F\colon \mathbb{CP}^n \to V$, where V can be either $\mathbb{CP}^n$ or $\mathbb{C}$, we define the *degree pattern* of F to be:

$$\operatorname{dp} F = \left(\deg_{x_1} F, \deg_{x_2} F, \ldots, \deg_{x_n} F\right). \tag{8}$$

Finally we will consider invariants satisfying the following condition:

Definition 4. We say that a function $I\colon \mathbb{CP}^n \to \mathbb{C}$ is *symmetric* if it is invariant under the following involution:

$$\iota\colon\ [x_1 : x_2 : \cdots : x_n : x_{n+1}] \to [x_n : x_{n-1} : \cdots : x_1 : x_{n+1}], \tag{9}$$

i.e. $\iota^*(I) = I$.

2.2. Integrability of bi-rational maps

Integrability both for continuous and discrete systems can be defined in different ways, see [22, 23] for a complete discussion of the continuous and the discrete case. Different ways of defining integrability do not always necessarily agree. We underline that the list we are going to make is not meant to be completely exhaustive of all the possible definitions of integrability. We will discuss only the definitions for autonomous recurrence relations we will need throughout the rest of the paper. We mention that additional definitions of integrability have been proposed for non-autonomous systems.

In general the solution of a recurrence relation of order n will depend on n arbitrary constants. This means that if a recurrence relation defined by the map $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ possesses $n-1$ invariants I_j, $j = 1, \ldots, n-1$, then, in principle, it is possible to reduce it to a map $\hat{\varphi}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by solving the relations:

$$I_j = \kappa_j, \tag{10}$$

where κ_j are the value of the invariants on a set of initial data. This stimulates the simplest and most natural definition of integrability for maps:

Definition 5 (Existence of invariants)**.** An n-dimensional map is (super)integrable if *it admits $n-1$ functionally independent invariants.*

Remark 6. We underline that, in general, the reduction to a lower-dimensional map solving the system of equations (10) can break the bi-rationality.

Definition 5 is very general, and works for arbitrary maps. If some additional structure are present, then the number of invariants needed for integrability can be significantly reduced. A special, but relevant case is the one of Poisson maps.

Definition 7 (Poisson structures and Poisson maps [11, 24])**.** In affine coordinates $\mathbf{w}$ a *Poisson structure of rank* $2r$ is a skew-symmetric matrix $J = J(\mathbf{w})$ of constant rank $2r$ such that the *Jacobi identity holds*:

$$\sum_{l=1}^{n}\left(J_{li}\frac{\partial J_{jk}}{\partial w_{l-1}} + J_{lj}\frac{\partial J_{ki}}{\partial w_{l-1}} + J_{lk}\frac{\partial J_{ij}}{\partial w_{l-1}}\right) = 0, \quad \forall i,j,k. \tag{11}$$

A Poisson structure defines a *Poisson bracket* through the identity:

$$\{f,g\} = \nabla f J(\mathbf{w}) \nabla g^T, \tag{12}$$

where ∇f is the gradient of f. Two functions f and g are said to be *in involution* with respect to the Poisson structure $J(\mathbf{w})$ if $\{f,g\} = 0$. We can easily see that $\{w_{i-1}, w_{j-1}\} = J_{ij}$. A map of the affine coordinates $\varphi\colon \mathbf{w} \mapsto \mathbf{w}'$ is a *Poisson map* if it preserves the Poisson structure $J(\mathbf{w})$, i.e. if:

$$\mathrm{d}\,\varphi J(\mathbf{w})\,\mathrm{d}\,\varphi^T = J(\mathbf{w}'), \tag{13}$$

where $\mathrm{d}\,\varphi$ is the Jacobian matrix of the map φ.

Then we have the following characterisation of integrability for Poisson maps:

Definition 8 (Liouville integrability [25–27])**.** An n-dimensional Poisson map of rank $2r$ is integrable if *it possesses $n-r$ functionally independent invariants in involution with respect to this Poisson structure.*

Remark 9. A Poisson structure can have rank $2r$ such that $1 \leq r \leq \lfloor n/2 \rfloor$. Therefore, there are two extremal cases: *minimal rank* $r = 1$ and *maximal rank* $r = \lfloor n/2 \rfloor$. From Definition 8 when the rank is minimal the number of invariants needed for integrability form is $n-1$. Therefore Liouville integrability in the case of minimal rank is equivalent to Definition 5. On the other hands when the rank is maximal the number of invariants needed is *minimal* $n - \lfloor n/2 \rfloor$. A very relevant case is when the rank is maximal and the dimension is even, that is $n = 2r$. In such case, we say that the rank is *full* and the Poisson structure is invertible. The inverse of the matrix $J(\mathbf{w})$, i.e. $\Omega(\mathbf{w}) = J^{-1}(\mathbf{w})$ is said to be *a symplectic structure*. In the full rank case the number of invariants needed is exactly half of the dimension of the base space $n/2$.

Symplectic structures are important in the theory of integrable maps. For instance, the classification made in [11] was carried out assuming of the existence of *linear* Poisson structure and of two invariants.

A difficult problem is, given a map, to decide whether or not there exists a symplectic structure compatible with it. In [28] it was proved that there exists a *pre-symplectic structure* (a degenerate sympectic structure) for any n-dimensional volume-preserving map possessing $n-2$ invariants. The rank of the obtained pre-symplectic structure is $n-2$ which implies that to claim integrability in the sense of Liouville one must be able to find another invariant. On the other hand, when the map comes from a *discrete variational principle*, i.e. it is *variational*, to find a symplectic structure is straightforward. We recall that an even-order recurrence relation (2) is said to be variational if there exists a function, called *Lagrangian*, $L = L\left(w_{k+N},\dots,w_n\right)$ such that the recurrence relation (2) is equivalent to the *Euler-Lagrange* equations:

$$\sum_{i=0}^{N} \frac{\partial L}{\partial w_k}\left(w_{k+N-i},\dots,w_{k-i}\right) = 0. \tag{14}$$

Here $N = n/2$ in the recurrence (2). A Lagrangian is called *normal* if

$$\frac{\partial^2 L}{\partial w_k\,\partial w_{k+N}} \neq 0. \tag{15}$$

Let T be a shift operator, i.e $T^j\left(w_{k+i}\right) = w_{k+i+j}$. Then due to the normality condition the *discrete Ostrogradsky transformation* [29]:

$$\mathcal{O}\colon \mathbf{w} \to (\mathbf{q},\mathbf{p}) \tag{16}$$

where the new coordinates $(\mathbf{q},\mathbf{p}) = (q_1,\dots,q_N,p_1,\dots,p_N)$ are defined through the formula:

$$q_i = w_{k+i-1}, \quad i = 1,\dots,N, \tag{17a}$$

$$p_i = T^{-1}\sum_{j=0}^{N-i} T^{-j}\frac{\partial L}{\partial w_{j+i}}, \quad i = 1,\dots,N, \tag{17b}$$

is well defined and invertible. Then the following result holds true [26]:

Lemma 10. *The map given by* $\Phi = \mathcal{O}\circ\varpi\circ\mathcal{O}^{-1}\colon (\mathbf{q},\mathbf{p}) \to (\mathbf{q}',\mathbf{p}')$, *where* ϖ *is the map corresponding to the Euler-Lagrange equations* (14) *has the following form*

$$q_i' = q_{i+1}, \quad i = 1,2,\dots,N-1, \tag{18a}$$

$$q_N' = \alpha(\mathbf{q},p_1), \tag{18b}$$

$$p_i' = p_{i+1} + \frac{\partial\widetilde{L}}{\partial q_{i+1}}(\mathbf{q},p_1), \; i = 1,2,\dots,N-1, \tag{18c}$$

$$p_N' = \left.\frac{\partial\widetilde{L}}{\partial\tau}(q,p_1)\right|_{\tau=\alpha(\mathbf{q},p_1)}, \tag{18d}$$

where $\alpha(\mathbf{q},p_1)$ *is the solution with respect to* $q^{N'}$ *of the equation:*

$$p_1 = -\frac{\partial L}{\partial q_1}(\mathbf{q},q_N'), \tag{19}$$

and $\widetilde{L}(\mathbf{q},p_1) = L(\mathbf{q},\alpha(\mathbf{q},p_1))$. *Moreover, the map* (18) *is symplectic with respect to the canonical symplectic structure:*

$$\Omega = \begin{pmatrix} \mathbb{O}_N & \mathbb{I}_N \\ -\mathbb{I}_N & \mathbb{O}_N \end{pmatrix}, \tag{20}$$

where $\mathbb{O}_N$ *is the zero* $N\times N$ *matrix and* $\mathbb{I}_N$ *the* $N\times N$ *identity matrix.*

Lemma 10 has the following corollary:

Corollary 11. *The Euler–Lagrange equations* (14) *admit the following non-degenerate Poisson bracket:*

$$J(\mathbf{w}) = \mathrm{d}\,\mathcal{O}^{-1}\Omega^{-1}(\mathrm{d}\,\mathcal{O}^{-1})^T, \tag{21}$$

where the differential of the Ostrogradsky transformation $\mathcal{O}$ must be evaluated on the original coordinates.

Therefore Corollary 11 allows us to construct a non-degenerate Poisson structure, and hence a symplectic structure, *for every variational map.*

Lagrangians for $2N$-order recurrence relations can be found following [30] or [31] for $N > 1$. The method presented in [31] allows also to disprove the existence of a Lagragian for a given $2N$-order recurrence relation for $N > 1$.

Moreover, bi-rational maps possess another definition of integrability: the *low growth condition* [32–34]. To be more precise we say that an n-dimensional *bi-rational* map is integrable if *the degree of growth of the iterated map φ^k is polynomial with respect to the initial conditions* $[\mathbf{x_0}]$. Therefore we have the following characterisation of integrability:

Definition 12 (Algebraic entropy [34])**.** An n-dimensional *bi-rational* map is *integrable in the sense of the algebraic entropy* if the following limit

$$\varepsilon = \lim_{k\to\infty} \frac{1}{k} \log \deg_{[\mathbf{x_0}]} \varphi^k, \tag{22}$$

called the *algebraic entropy* is zero for every initial condition $[\mathbf{x_0}] \in \mathbb{CP}^n$.

Algebraic entropy is an *invariant* of bi-rational maps, meaning that its value is unchanged up to bi-rational equivalence. Practically algebraic entropy is a measure of the *complexity* of a map, analogous to the one introduced by Arnol'd [35] for diffeomorphisms. In this sense growth is given by computing the number of intersections of the successive images of a straight line with a generic hyperplane in complex projective space [32].

The value of the degree of the iterates of the map is conditioned by its *singularity structure*. Some hypersurfaces are blown down by the map. If one of the successive images of these hypersurfaces coincide with a singular variety, there is a drop in the degree [34, 36, 37]. Therefore, from a heuristic point of view we can say that the singularity makes the entropy. This actually also applies to non-autonomous cases like the discrete Painlevé equations [9].

In principle, the definition of algebraic entropy in equation (22) requires us to compute all the iterates of a bi-rational map φ to obtain the sequence

$$d_k = \deg_{[\mathbf{x_0}]} \varphi^k, \quad k \in \mathbb{N}. \tag{23}$$

Fortunately, for the majority of applications, the form of the sequence can be inferred by using generating functions [38]:

$$g(z) = \sum_{n=0}^{\infty} d_k z^k. \tag{24}$$

A generating function is a predictive tool which can be used to test the successive members of a finite sequence. It follows that the algebraic entropy is given by the logarithm of the smallest pole of the generating function, see [39, 40].

Several results are known about the relationship of the above definitions of integrability. First of all, the low growth condition means that the complexity of the map is very low, and it is known that invariants help in reducing the complexity of a map. Indeed the growth of a map possessing invariants cannot be generic since the motion is constrained to take place on the intersection of hypersurfaces defined by the invariants. However, the drop in complexity must be big enough to reduce the growth to a polynomial one. On the other hand it is known that the existence of invariants can give some bounds on the growth of bi-rational maps. Indeed, it is known that the orbits of superintegrable maps with rational invariant are confined to *elliptic curves* and the growth is at most *quadratic* [3, 41]. In low dimension some explicit results on the growth of bi-rational maps are known. For maps in $\mathbb{CP}^2$, it was proved in [2] that the growth can be only bounded, linear, quadratic or exponential. Linear cases are trivially integrable in the sense of invariants. We note that for polynomial maps in $\mathbb{C}^2$, it was already known from [32] that the growth can be only linear or exponential. It is known that QRT mappings and other maps with invariants in $\mathbb{CP}^2$ possess quadratic growth [7], so the two notions are actually equivalent for a large class of integrable systems.

2.3. Duality

Now we discuss briefly the concept of *duality* for rational maps, which was introduced in [20]. Let us assume that our map φ possesses L invariants, i.e.I_j for $j \in \{1, \dots, L\}$. Then we can form the linear combination:

$$H = \alpha_1 I_1 + \cdots + \alpha_L I_L. \tag{25}$$

Being a function of invariants it follows that H defined by (25) is itself an invariant of the map.

Remark 13. We note that in principle more general combinations of invariants can be considered:

$$H = P_d \left(I_1, I_2, \dots, I_L\right) \tag{26}$$

where P_d is a homogeneous polynomial of total degree d in L variables. Again even in this generalized case H defined by (26) is an invariant of the map. However, in this paper, following [20], we do not consider this case.

For an unspecified recurrence relation

$$[x_1 : x_2 : \cdots : x_{n+1}] \mapsto \left[x_1' : x_2' : \cdots : x_{n+1}'\right] = [x_1' : x_1 : \cdots : x_{n+1}] , \tag{27}$$

we can write down the invariant condition for H (25):

$$\widehat{H}(x_1', [\mathbf{x}]) = H\left([\mathbf{x}']\right) - H\left([\mathbf{x}]\right) = 0. \tag{28}$$

Since we know that $[\mathbf{x}'] = \varphi\left([\mathbf{x}]\right)$ is a solution of (28) we have the following factorization:

$$\widehat{H}(x_1', [\mathbf{x}]) = A\left(x_1', [\mathbf{x}]\right) B\left(x_1', [\mathbf{x}]\right). \tag{29}$$

We can assume without loss of generality that the map φ corresponds to the annihilation of A in (29). Now since $\deg_{x_1'} \widehat{H} = \deg_{x_1} H$ and $\deg_{x_n} \widehat{H} = \deg_{x_n} H$ we have that if $\deg_{x_1} H, \deg_{x_n} H > 1$ the factor B in (29) is non constant. This assertion follows from the fact that we are assuming that all the invariants are non-degenerate. Degenerate invariants can violate this property. In general, since the map φ is bi-rational, we have the following equalities:

$$\deg B_{x_1'} = \deg_{x_1'} \widehat{H} - \deg_{x_1'} A = \deg_{x_1} H - 1, \tag{30a}$$

$$\deg B_{x_n} = \deg_{x_n} \widehat{H} - \deg_{x_n} A = \deg_{x_n} H - 1. \tag{30b}$$

Therefore we have that, in general, if $\deg_{x_1} H, \deg_{x_n} H > 2$, the annihilation of B does not define a bi-rational map, but an algebraic one. However when $\deg_{x_1} H, \deg_{x_n} H = 2$ the annihilation of B defines a bi-rational projective map. We call this map the *dual map* and we denote it by $\varphi^{\vee}$.

Remark 14. We note that in principle for $\deg_{x_1} H = \deg_{x_n} H = d > 2$, more general factorizations can be considered:

$$\widehat{H}\left(x_1', [\mathbf{x}]\right) = \prod_{i=1}^{d} A_i\left(x_1', [\mathbf{x}]\right), \tag{31}$$

but in this paper we do not consider this case.

Now assume that the invariants (and hence the map φ) depends on some *arbitrary constants* $I_i = I_i\left([\mathbf{x}]; a_i\right)$, for $i = 1, \ldots, K$. Choosing some of the a_i in such a way that there remains M arbitrary constants and such that for a subset a_{i_k} we can write equation (25) in the following way:

$$H = a_{i_1} J_1 + a_{i_2} J_2 + \cdots + a_{i_K} J_{a_{i_K}}, \tag{32}$$

where $J_i = J_i\left([\mathbf{x}]\right)$, $i = 1, 2, \ldots, K$ are new functions. Then using the factorization (29) we have that the J_i functions are invariants for the dual maps.

Remark 15. It is clear from equation (32) that even though the dual map is naturally equipped with some invariants, it is not *necessarily* equipped with a sufficient number of invariants to claim integrability. In fact there exists examples of dual maps with any possible behaviour, integrable, superintegrable and non-integrable [14, 15].

3. Derivation of the class of 4D maps

In this section, we explain how to derive the desired class of 4D maps.

Our starting points are the maps corresponding to autonomous $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations and their invariants as presented in [14]. These are maps of $\mathbb{CP}^4$ into itself with coordinates $[x : y : z : u : t]$ and components given by:

$$\begin{aligned} x' = & -ay\left(x^2 + y^2 + z^2 + 2yz + 2xy + xz + zu\right) \\ & - bty\left(y + z + x\right) - cyt^2 + dt^3, \\ y' = & \, ayx^2, \quad z' = axy^2, \quad u' = axyz, \quad t' = axyt, \end{aligned} \tag{$\mathrm{d}P_{\mathrm{I}}^{(2)}$}$$

and:

$$\begin{aligned} x' &= dt^5 - a\left(t - y\right)\left(t + y\right)\left(ut^2 - yz^2 - uz^2 - 2yxz - x^2 y\right) - cyt^4 \\ &\quad - bt^2\left(t - y\right)\left(t + y\right)\left(z + x\right), \\ y' &= ax\left(t^2 - y^2\right)\left(t^2 - x^2\right), \quad z' = ay\left(t^2 - y^2\right)\left(t^2 - x^2\right), \\ u' &= az\left(t^2 - y^2\right)\left(t^2 - x^2\right), \quad t' = at\left(t^2 - y^2\right)\left(t^2 - x^2\right), \end{aligned} \tag{$\mathrm{d}P_{\mathrm{II}}^{(2)}$}$$

respectively. The map $\mathrm{d}P_{\mathrm{I}}^{(2)}$ has two invariants $I_4^{(\mathrm{I})}$ and $I_5^{(\mathrm{I})}$:

$$\begin{aligned} t^4 I_4^{(\mathrm{I})} &= ayz\left(-y^2 - 2\,yz - xy - z^2 - zu + xu\right) \\ &\quad - btyz\,(z+y) - cyzt^2 + dt^3\,(z+y)\,, \end{aligned} \tag{33a}$$

$$\begin{aligned} t^5 I_5^{(I)} &= ayz\left(zu + xy + y^2 + 2\,yz + z^2\right)(z+u+y+x) \\ &\quad + cyz\,(z+u+y+x)\,t^2 - d\left(zu + xy + y^2 + 2\,yz + z^2\right)t^3 \\ &\quad + byz\,(y+z+x)\,(u+y+z)\,t, \end{aligned} \tag{33b}$$

while the map $\mathrm{d}P_{\mathrm{II}}^{(2)}$ possesses two invariants $I_6^{(\mathrm{II})}$ and $I_8^{(\mathrm{II})}$:

$$\begin{aligned} t^6 I_6^{(\mathrm{II})} &= a\,(t-z)\,(t+z)\,(t-y)\,(t+y)\,(ux - uz - xy - yz) \\ &\quad - bt^2\left(z^2t^2 + t^2y^2 - z^2y^2\right) - ct^4yz + dt^5\,(z+y)\,, \end{aligned} \tag{34a}$$

$$\begin{aligned} t^8 I_8^{(\mathrm{II})} &= a\big[\left(u^2 + z^2 + y^2 + x^2\right)t^6 - z^2y^2\,(uz + xy + yz)^2 \\ &\quad - (2yuz^2 + 2uzxy + x^2z^2 + 2xzy^2 + 2x^2y^2 \\ &\quad + u^2y^2 + 2z^2y^2 + 2u^2z^2)t^4 \\ &\quad + (2x^2y^2z^2 + 2uy^3z^2 + 2xy^2z^3 + 2yuz^4 \\ &\qquad + z^2y^4 + y^2z^4 + 2u^2y^2z^2 + x^2y^4 \\ &\qquad + 2uxy^3z + 2uxyz^3 + 2xzy^4 + z^4u^2)t^2\big] \\ &\quad + bt^2\,(t-z)\,(t+z)\,(t-y)\,(t+y)\,(z+x)\,(u+y) \\ &\quad + ct^4\left(xzt^2 - z^2y^2 + yut^2 - yuz^2 - xzy^2\right) \\ &\quad - dt^5\left(xt^2 + zt^2 - zy^2 - xy^2 - uz^2 + ut^2 - yz^2 + yt^2\right). \end{aligned} \tag{34b}$$

These invariants have the following properties:

Property A: The invariants are symmetric in the sense of Definition 4.

Property B: The lower degree invariants (33a) and (34a) have degree pattern $(1,3,3,1)$ and are particular instances of the following homogeneous polynomial in $\mathbb{C}[x,y,z,u,t]$:

$$\begin{aligned} t^6 I_{\mathrm{low}} &= t^5(y+z)s_1 - t^4(ux - uz - xy)s_2 + s_3t^4yz \\ &\quad + t^4(y^2+z^2)s_4 + t^3yz(y+z)s_5 \\ &\quad + t^2(y^2+z^2)(ux - uz - xy)s_6 \\ &\quad - t^2yz(ux - uz - xy)s_7 + s_8t^2y^2z^2 + t^2yz(y^2+z^2)s_9 \\ &\quad - y^2z^2(ux - uz - xy)s_{10} + s_{11}y^3z^3, \end{aligned} \tag{35}$$

depending parametrically on 11 coefficients, namely s_i, $i = 1, \ldots, 11$.

Property C: The higher degree invariants (33b) and (34b) have degree pattern $(2,4,4,2)$. The most general homogeneous polynomial with degree pattern in $\mathbb{C}[x,y,z,u,t]$ depends parametrically on 1820 coefficients. Taking into account the symmetry with respect to the involution (9) the number of coefficients reduces to 121. Since one of this coefficients is just an additional constant, we can lower the number of independent coefficient to 120. We denote this invariant by I_{high}, but for sake of conciseness we do not present its general form.

Based on the above consideration we address the following problem:

Problem 16. Find all the bi-rational maps $\varphi\colon \mathbb{CP}^4 \to \mathbb{CP}^4$ and their dual maps $\varphi^\vee\colon \mathbb{CP}^4 \to \mathbb{CP}^4$ having two non-degenerate, functionally independent invariants, where the first one has properties A, B and the second one has properties A and C.

Solving this problem amounts to finding a list of equations which are expected to behave like the two fourth-order Painlevé equations $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$. Before going to the solution of this problem, we state the following general result on the dual map of a map with two invariants satisfying properties A, B and properties A and C respectively:

Lemma 17. *Assume that a bi-rational map $\varphi\colon \mathbb{CP}^4 \to \mathbb{CP}^4$ possesses two invariants, where the first one has properties A, B and the second one has properties A and C. Then it follows that the map φ has degree pattern $\mathrm{dp}\,\varphi = (2, 3, 2, 1)$ and the maximal degree pattern of the dual $\varphi^\vee\colon \mathbb{CP}^4 \to \mathbb{CP}^4$ is $\mathrm{dp}\,\varphi^\vee = (2, 1, 2, 1)$.*

Proof. By direct computation is it possible to check that if an invariant I_{low} has the form (35) then the invariant condition (5) implies the following factorisation:

$$I_{\mathrm{low}}\left([\mathbf{x}']\right) - I_{\mathrm{low}}\left([\mathbf{x}]\right) = (x - z)A\left(x', [\mathbf{x}]\right). \tag{36}$$

Equation (36) means that we have the following degree distribution:

	$\deg_{x'}$	$\deg_x$	$\deg_y$	$\deg_z$	$\deg_u$
$\varphi^*\left(I_{\mathrm{low}}\right)$	1	3	3	1	0
I_{low}	0	1	3	3	1
A	1	2	3	2	1

(37)

The second part of the statement comes from an analogous consideration applied to equation (29). Since the degree pattern of A is fixed, the degree pattern of B is maximal when there are no factors depending only on $[\mathbf{x}]$. Under this assumptions we find the following distribution of the degrees:

	$\deg_{x'}$	$\deg_x$	$\deg_y$	$\deg_z$	$\deg_u$
$\varphi^*\left(H\right)$	2	4	4	2	0
H	0	2	4	4	2
A	1	2	3	2	1
B	1	2	1	2	1

(38)

The last rows of tables (37) and (38) provide the desired result. □

Corollary 18. *Bi-rational maps possessing two two invariants, where the first one has properties A, B and the second one has properties A and C are not necessarily* self-dual.

Proof. From Lemma 17 we have that $\mathrm{dp}\,\varphi = (2, 3, 2, 1)$ and the maximal degree pattern of the dual $\varphi^\vee\colon \mathbb{CP}^4 \to \mathbb{CP}^4$ is $\mathrm{dp}\,\varphi^\vee = (2, 1, 2, 1)$. Therefore, the maps φ and $\varphi^\vee$ can be the same if and only their degree pattern is is not maximal and equals $(2, 1, 2, 1)$. □

We now sketch the procedure to solve problem 16. We note that this procedure is based on the one proposed in [14] to find bi-rational maps with invariants of assigned degree pattern.

(i) Find the value of x' from (36) where I_{low} is given by equation (35).

(ii) Substitute the obtained form of x' into the invariant condition (5) for I_{high}. Geometrically this describes the intersection of the two hypersurfaces given by $I_{\text{low}} = I^{(0)}_{\text{low}}$ and $I_{\text{high}} = I^{(0)}_{\text{hight}}$, where $I^{(0)}_{\text{low}}$ and $I^{(0)}_{\text{hight}}$ are arbitrary constants.

(iii) We can take coefficients with respect to the independent variables. This yield a system of nonlinear homogeneous equations. We put this system in a collection of systems that we call $\mathcal{S}$.

(iv) For each system in $\mathcal{S}$ we solve the monomial equations and form a new list of systems $\mathcal{S}'$. We exclude from the list all the combinations of coefficients such that the corresponding I_{low} and I_{high} do not satisfy properties A, B and A, C respectively.

(v) We iterate step (iv) until all the monomial equations are solved.

(vi) This yields 117 different systems.

(vii) Solving these systems we found 25 solutions respecting the properties A, B and C.

Through a degeneration scheme the 25 solutions we obtain can be cast into six different maps along with their duals. We proved the following:

Theorem 19. *The solutions of problem 16, up to degeneration and identification of the free parameters, is given by* six pairs *of main/dual maps which we denote by (P.x) with x small roman number for the main maps and by (Q.x) for the dual map.*

We call this class of maps the *(P,Q)-class*. In the next section we present the explicit form of these maps and we discuss their integrability properties.

4. Maps of the (P,Q)-class and their integrability properties

In this section we show the explicit form of the maps of the class (P,Q). We denote the pairs of main/dual maps by (x) where x is a small Roman number ranging from (i) to (vi). Moreover we discuss their integrability properties from the point of view of the existence of invariants, the degree growth of their iterates and the existence of Lagrangians. For the cases admitting Lagrangian following corollary 11 we present the form of their symplectic structure.

4.1. Maps (i)

The main map $[\mathbf{x}] \mapsto \varphi_{\text{i}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= -\{[\nu t^2(x+z) + uz^2]y + t^2\mu uz + (x+z)^2y^2\}d - at^4, \\ y' &= x^2 d(t^2\mu + xy), \quad z' = yxd(t^2\mu + xy), \\ u' &= zxd(t^2\mu + xy), \quad t' = txd(t^2\mu + xy). \end{aligned} \tag{P.i}$$

This map depends on four parameters a, d and μ, ν. Its dual map $[\mathbf{x}] \mapsto \varphi_{\text{i}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= [\beta(2xy - 2yz + uz)\mu + (\beta\nu - \alpha)y(x-z)]\, t^2 \\ &\quad + \beta y(z^2y - x^2y + uz^2) \\ y' &= x^2\beta(t^2\mu + xy), \quad z' = yx\beta(t^2\mu + xy), \\ u' &= zx\beta(t^2\mu + xy), \quad t' = tx\beta(t^2\mu + xy). \end{aligned} \tag{Q.i}$$

This map depends on three parameters α, β, and μ, ν. The parameters μ and ν are shared with the main map (P.i).

These two maps were shown in the preliminary paper [15], where it was shown that their growth is asymptotically *cubic*, and that the maps were deflatable to three dimensional Liouville integrable maps through the transformation $v_k = w_k w_{k+1}$. The reader will find a summary table with all the relevant properties in section 5. For explicit details we refer to [15].

4.2. Maps (ii)

The main map $[\mathbf{x}] \mapsto \varphi_{\mathrm{ii}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= \left[(x^2+z^2)y - uz^2\right]\mu - t^2(u-2y), \\ y' &= x(t^2+\mu x^2), \quad z' = y(t^2+\mu x^2), \\ u' &= z(t^2+\mu x^2), \quad t' = t(t^2+\mu x^2). \end{aligned} \tag{P.ii}$$

This map depends on the parameter μ. Its dual map $[\mathbf{x}] \mapsto \varphi_{\mathrm{ii}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ is given by the following components:

$$\begin{aligned} x' &= \alpha\left[\left(x^2-z^2\right)y + uz^2\right]\mu^2 + t^2\alpha u + \beta y^2\left(x-z\right)\mu + t^2\beta\left(x-z\right), \\ y' &= \alpha x\left(t^2+\mu x^2\right), \quad z' = \alpha y\left(t^2+\mu x^2\right), \\ u' &= \alpha z\left(t^2+\mu x^2\right), \quad t' = \alpha t\left(t^2+\mu x^2\right). \end{aligned} \tag{Q.ii}$$

This map depends on three parameters α, β and μ. The parameter μ is shared with the main map (P.ii).

These two maps were shown in the preliminary paper [15] where it was shown that their growth is exponential. A peculiarity of map (Q.ii) is that from the construction of duality we can get *only one invariant*. We provide a summary table with all the relevant properties in section 5. For explicit details we refer to [15].

4.3. Maps (iii)

The main map $[\mathbf{x}] \mapsto \varphi_{\mathrm{iii}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= -2at^4 - 2\mu d(z+x+y)t^3 \\ &\quad + \nu d\left[2(x+z)y + uz\right]t^2 + d(2yz^2u + 2y^2zx), \\ y' &= x^2 d(\nu t^2 + 2xy), \quad z' = yxd(\nu t^2 + 2xy), \\ u' &= zxd(\nu t^2 + 2xy), \quad t' = txd(\nu t^2 + 2xy). \end{aligned} \tag{P.iii}$$

This map depends on four parameters a, d and μ, ν. The map (P.iii) has the following degrees of iterates:

$$\begin{aligned} \{d_n\}_{\mathrm{P.iii}} = {} & 1, 4, 12, 26, 49, 79, 113, 153, 199, 250, 310, 378, 449, \\ & 526, 610, 698, 795, 901, 1009, 1123, 1245\ldots \end{aligned} \tag{39}$$

with generating function:

$$g_{\mathrm{P.iii}}(s) = -\frac{4s^8+4s^7+10s^6+9s^5+13s^4+7s^3+6s^2+2s+1}{(s^2-s+1)(s^2+s+1)^2(s-1)^3}. \tag{40}$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth.

The main map (P.iii) has the following invariants:

$$
\begin{aligned}
t^6 I_{\text{low}}^{\text{P.iii}} &= 2at^4yz + 2yz\mu d(y+z)t^3 - yzd\nu(-2yz - xy - uz + ux)t^2 \qquad (41a)\\
&\quad - 2y^2z^2(ux - uz - xy)d,\\
t^8 I_{\text{high}}^{\text{P.iii}} &= 4\mu a(y+z)t^7 \qquad (41b)\\
&\quad + (4dyz\mu^2 - 2ayz\nu + 4dz^2\mu^2 + 4dy^2\mu^2 + 2az\nu u + 2ay\nu x)t^6\\
&\quad + 2\mu\nu d(2uz^2 + 2xy^2 + yzx + zuy)t^5\\
&\quad + (d\nu^2x^2y^2 - 3d\nu^2y^2z^2 + 4d\nu^2uxyz\\
&\qquad + d\nu^2u^2z^2 + 4ay^2zx + 4ayz^2u)t^4\\
&\quad + 4\mu dyz(2uz^2 + 2xy^2 + yzx + zuy)t^3\\
&\quad + 2yd\nu z(uz + 2xy)(2uz + xy)t^2\\
&\quad + 4y^2z^2d(u^2z^2 + x^2y^2 + uxyz).
\end{aligned}
$$

Moreover, we note that according to the test in [31] the main map (P.iii) does not possess a Lagrangian. However, by direct search, we can prove that this map has an additional functionally independent invariant of degree pattern $(2,5,5,2)$. This means that the low growth of the main map (P.iii) is explained in terms of integrability as existence of invariants, as given by Definition 5. More specifically, the quadratic growth it explained by the fact that if a map in $\mathbb{CP}^4$ has three rational invariants, the orbits are confined to elliptic curves and the growth is at most quadratic [3].

The dual map $[\mathbf{x}] \mapsto \varphi_{\text{iii}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$
\begin{aligned}
x' &= 2\mu\beta(z-x)t^3 + \{\beta\nu[zu + 2y(x-z)] + 2\alpha y(z-x)\}t^2 + 2\beta z^2yu\\
y' &= x^2\beta(\nu t^2 + 2xy), \quad z' = yx\beta(\nu t^2 + 2xy), \qquad (\text{Q.iii})\\
u' &= zx\beta(\nu t^2 + 2xy), \quad t' = tx\beta(\nu t^2 + 2xy).
\end{aligned}
$$

This map depends on four parameters α, β and μ, ν. The parameters μ, ν are shared with the main map (P.iii). The dual map (Q.iii) has the following degrees of iterates:

$$
\begin{aligned}
\{d_n\}_{\text{Q.iii}} = 1, &4, 12, 28, 62, 131, 272, 554, 1120, 2253, 4528, 9092,\\
&18244, 36601, 73420, 147270, 295392, 592487,\\
&1188378, 2383576, 4780824, 9589061, 19233098,\\
&38576452, 77374040, 155191611, 311272822, 624329930\dots,
\end{aligned} \tag{42}
$$

with generating function:

$$
g_{\text{Q.iii}}(s) = \frac{P_{\text{Q.iii}}(s)}{Q_{\text{Q.iii}}(s)}, \tag{43}
$$

where

$$
\begin{aligned}
P_{\text{Q.iii}}(s) &= s^{14} + 2s^{13} + 4s^{12} + 6s^{10} - s^9 + 5s^8 \qquad (44a)\\
&\quad + s^7 + 5s^6 + s^5 + 4s^4 + s^3 + 3s^2 + s + 1\\
Q_{\text{Q.iii}}(s) &= (1-s)\left(s^2+1\right)\left(s^{10} - 2s^9 - 2s + 1\right). \qquad (44b)
\end{aligned}
$$

The algebraic entropy of the dual map (Q.iii) is given by the logarithm of the inverse of the smallest zero of the function $Q_{\text{Q.iii}}(s)$ in (44b). As the zeroes of $1-s$ and s^2+1, lie on the unit circle, therefore we have to look at the location of the zeroes of the polynomial:

$$q(s) = s^{10} - 2s^9 - 2s + 1. \tag{45}$$

Defining $q_1(s) = -2s+1$ and $q_2(s) = s^{10} - s^9$ we have that on the circle $C_\rho := \{s \in \mathbb{C} || s| = \rho\}$ with $\rho \in (1/2, 1)$ the following inequality holds:

$$|q_2(s)| < |q_1(s)|. \tag{46}$$

By Rouche's theorem [42] this implies that $q_1(s)$ and $q_1(s) + q_2(s) = q(s)$ have the same number of zeroes inside the circle C_ρ, i.e. the polynomial $Q_{\text{Q.iii}}(s)$ has a unique zero inside the circle C_ρ. This zero is the smallest one of $Q_{\text{Q.iii}}(s)$ and due to the fact that $Q_{\text{Q.iii}}(s)$ has real coefficients this zero is real. This implies the growth of the dual map (Q.iii) is *exponential*. The approximate value of the zero of $Q_{\text{Q.iii}}(s)$ inside C_ρ is $s_0 = 0.49857104591719819\ldots$. This implies that the algebraic entropy of the dual map (Q.iii) is:

$$\varepsilon_{\text{Q.iii}} = \log(2.0057321984279013\ldots). \tag{47}$$

The growth of the sequence of degrees of equation (Q.iii) is then slightly greater than 2^n.

Since the main map (P.iii) possesses two invariants and depends on a and d whereas the dual map (Q.iii) does not depend on them according to (32) we can write down the invariants for the dual map (Q.iii) as:

$$\alpha I_{\text{low}}^{\text{P.iii}} + \beta I_{\text{high}}^{\text{P.iii}} = a I_{\text{low}}^{\text{Q.iii}} + d I_{\text{high}}^{\text{Q.iii}}. \tag{48}$$

Therefore we obtain the following expressions:

$$\begin{aligned} t^4 I_{\text{low}}^{\text{Q.iii}} &= 2\mu(y+z)\beta t^3 + \left[2\alpha yz + \nu(uz+xy-yz)\beta\right] t^2 \\ &\quad + 2yz(uz+xy)\beta, \end{aligned} \tag{49a}$$

$$\begin{aligned} t^8 I_{\text{high}}^{\text{Q.iii}} &= 2\mu^2(yz+z^2+y^2)\beta t^6 \\ &\quad + \left[y\mu z(y+z)\alpha + \mu\nu(2uz^2+2xy^2+yzx+zuy)\beta\right] t^5 \\ &\quad + \frac{1}{2}[\nu yz(2yz+xy+uz-ux)\alpha \\ &\qquad + \nu^2(4uxyz+u^2z^2-3z^2y^2+x^2y^2)\beta]t^4 \\ &\quad + 2\mu yz(2uz^2+2xy^2+yzx+zuy)\beta t^3 \\ &\quad + (z^2y^2(uz+xy-ux)\alpha + \nu yz(uz+2xy)(2uz+xy)\beta)t^2 \\ &\quad + 2y^2z^2(u^2z^2+x^2y^2+uxyz)\beta. \end{aligned} \tag{49b}$$

The first invariant (49a) has degree pattern $(1, 2, 2, 1)$ and the second invariant has degree pattern $(2, 4, 4, 2)$. However, the degree pattern of the second invariant is not *minimal*: we can reduce the degree pattern of the second invariant to $(1, 3, 3, 1)$ by replacing I_{high} with $2\beta I_{\text{high}} - I_{\text{low}}^2$. Moreover, we can see that the existence of these two invariants is not sufficient to ensure the low growth of the dual map (Q.iii). Finally, we note that that the dual map (Q.iii) according to the test in [31] does not possess a Lagrangian.

4.4. Maps (iv)

The main map $[\mathbf{x}] \mapsto \varphi_{\text{iv}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= -t^3 a - bt^2 y - d\nu y(x+y+z)t \\ &\quad - dy(y^2 + 2xy + 2yz + x^2 + xz + uz + z^2), \\ y' &= dyx^2, \quad z' = dxy^2, \quad u' = dzxy, \quad t' = dtxy. \end{aligned} \tag{P.iv}$$

This map depends on four parameters a, b, d and ν. We note that the map (P.iv) is the autonomous $\mathrm{d}P_{\mathrm{I}}^{(2)}$, derived in [13] and whose invariants, duality and growth properties where studied in [14]. For sake of completeness we repeat these properties here. The map (P.iv) has the following degrees of iterates:

$$\begin{aligned} \{d_n\}_{\text{P.iv}} = 1, 3, 6, 12, 21, 33, 47, 64, 83, \\ 104, 128, 154, 183, 214, 248, 284 \ldots \end{aligned} \tag{50}$$

with generating function:

$$g_{\text{P.iv}}(s) = -\frac{s^{10} - s^9 - s^6 + 2s^4 + 2s^3 + s + 1}{(s+1)(s-1)^3}. \tag{51}$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth.

The map (P.iv) has the following invariants:

$$\begin{aligned} t^4 I_{\text{low}}^{\text{P.iv}} &= t^3(y+z)a + zbt^2 y + d\nu yz(y+z)t \\ &\quad - dyz(ux - xy - 2yz - uz - y^2 - z^2), \end{aligned} \tag{52a}$$

$$\begin{aligned} t^5 I_{\text{high}}^{\text{P.iv}} &= -\nu a(y+z)t^4 + \left[\left(y^2 + z^2 + 2yz + uz + xy\right) a - yzb\nu \right] t^3 \\ &\quad - yz\left[\nu^2 d(y+z) - b(y+z+u+x)\right] t^2 + d\nu yz(uy + xz + 2ux)t \\ &\quad + dzy(x+u+z+y)(y^2 + z^2 + 2yz + uz + xy). \end{aligned} \tag{52b}$$

Using the methods of [31] we have that the map (P.iv) is a variational maps arising from the Lagrangian:

$$\begin{aligned} L_{\text{P.iv}} &= w_n w_{n+1} w_{n+2} + \frac{w_n^3}{3} + w_{n+1} w_n^2 + w_{n+1}^2 w_n \\ &\quad + \nu\left(\frac{w_n^2}{2} + w_{n+1} w_n\right) + \frac{a}{d}\log(w_n) + \frac{b}{d} w_n m \end{aligned} \tag{53}$$

in affine coordinates. Using Corollary 11, we obtain the following non-degenerate Poisson bracket:

$$J_{\text{P.iv}} = \begin{bmatrix} 0 & 0 & \dfrac{1}{dw_{n-1}} & -\dfrac{\mu + w_{n-2} + 2(w_n + w_{n-1}) + w_{n+1}}{dw_{n-1} w_n} \\ 0 & 0 & 0 & \dfrac{1}{dw_n} \\ -* & 0 & 0 & 0 \\ -* & -* & 0 & 0 \end{bmatrix}. \tag{54}$$

Note that a Poisson bracket is skew-symmetric $J_{i,j} = -J_{j,i}$ and the asterisks in formula (54) denote the corresponding entries on the other side of the main diagonal. One can check that the invariants (52) are in involution with respect to the Poisson bracket (54). Therefore, the map (P.iv) is Liouville integrable.

The dual map $[\mathbf{x}] \mapsto \varphi_{\text{iv}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= \left[z^2 + (y+u-\nu t)z + x(\nu t - x - y)\right]\beta + t\alpha(z-x), \\ y' &= x^2\beta, \quad z' = xy\beta, \quad u' = x\beta z, \quad t' = t\beta x. \end{aligned} \tag{Q.iv}$$

This map depends on three parameters α, β, and ν. The parameter ν is shared with the main map (P.iv). The dual map (Q.iv) has the following degrees of iterates:

$$\begin{aligned} \{d_n\}_{\text{Q.iv}} = 1, 2, 4, 7, 11, 17, 24, 32, 41, 52, 64, 77, 91, 107, 124, \\ 142, 161, 182, 204, 227, 251 \dots \end{aligned} \tag{55}$$

with generating function:

$$g_{\text{Q.iv}}(s) = -\frac{2s^5 + s^3 + s^2 + 1}{(s+1)(s^2+1)(s-1)^3}. \tag{56}$$

This means that the dual map is integrable according to the algebraic entropy test with *quadratic* growth, just like the main map.

Since the main map (P.iv) possesses two invariants and depends on a, b and d whereas the dual map (Q.iv) does not, according to (32) we can write down the invariants for the dual map (Q.iv) as:

$$\alpha I_{\text{low}}^{\text{P.iv}} + \beta I_{\text{high}}^{\text{P.iv}} = a I_1^{\text{Q.iv}} + d I_2^{\text{Q.iv}} + b I_3^{\text{Q.iv}}. \tag{57}$$

Therefore we obtain the following expressions:

$$t^2 I_1^{\text{Q.iv}} = (y+z)(\alpha - \nu\beta)t + (y^2 + z^2 + 2yz + uz + xy)\beta, \tag{58a}$$

$$\begin{aligned} t^5 I_2^{\text{Q.iv}} &= \nu yz(y+z)(\alpha - \nu\beta)t^2 \\ &\quad + \left[yz(uy + xz + 2ux)\beta\nu - yz(ux - xy - 2yz - uz - y^2 - z^2)\alpha\right]t \\ &\quad + yz(x+u+z+y)(y^2 + z^2 + 2yz + uz + xy)\beta, \end{aligned} \tag{58b}$$

$$t^3 I_3^{\text{Q.iv}} = yz(\alpha - \nu\beta)t + yz(x+u+z+y)\beta. \tag{58c}$$

The invariants (58a) and (58c) both have degree pattern $(1, 2, 2, 1)$. However, the second invariant is not minimal and it can be replaced with an invariant of degree pattern $(1, 3, 3, 1)$. Moreover, using the test of [31], we find that the map (Q.iv) is not variational. Therefore, we conclude that the dual map (Q.iv) is integrable according to Definition 5.

4.5. Maps (v)

The main map $[\mathbf{x}] \mapsto \varphi_{\text{v}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= -d(x+z)^2 y^3 - \left[\nu(x+z)t^2 + uz^2\right]dy^2 - ct^4 y - t^5 a, \\ y' &= dx^3y^2, \quad z' = dy^3x^2, \quad u' = dzx^2y^2, \quad t' = dtx^2y^2. \end{aligned} \tag{P.v}$$

This map depends on the parameters a, c, d and ν. The map (P.v) has the following degrees of iterates:

$$\{d_n\}_{\text{P.v}} = 1, 5, 15, 35, 65, 103, 149, 201, 261, 329, 405, 489, 581, 681 \dots \tag{59}$$

with generating function:

$$g_{\mathrm{P.v}}(s) = -\frac{1+2s+3s^2+4s^3-2s^5-2s^7+2s^8}{(s-1)^3}. \tag{60}$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth.

The map (P.v) has the following invariants:

$$\begin{aligned} t^6 I_{\mathrm{low}}^{\mathrm{P.v}} &= z^2 d\,(x+z)\,y^3 + z^2 d\left(\nu t^2 + zu - ux\right) y^2 \\ &\quad + t^4\,(cz+at)\,y + t^5 az \end{aligned} \tag{61a}$$

$$\begin{aligned} t^8 I_{\mathrm{high}}^{\mathrm{P.v}} &= z^2 d\,(x+z)^2\, y^4 + 2z^3 ud\,(x+z)\,y^3 \\ &\quad + \left\{z^4 du^2 + \left[\left(c-\nu^2 d\right)t^2 + 2xu\nu d\right] t^2 z^2 \right. \\ &\qquad \left. + (at+xc)\,t^4 z + t^5 ax\right\} y^2 \\ &\quad + \left[(at+uc)\,z^2 - t^2\nu cz - a\nu t^3\right] t^4 y \\ &\quad + zt^5 a\left(zu - \nu t^2\right). \end{aligned} \tag{61b}$$

Using the methods of [31] we have that the map (P.v) is a variational maps arising from the Lagrangian:

$$L_{\mathrm{P.v}} = w_n w_{n+1}^2 w_{n+2} + \frac{w_{n+1}^2 w_n^2}{2} + \nu w_{n+1} w_n - \frac{a}{d}\frac{1}{w_n} + \frac{c}{d}\log\left(w_n\right) \tag{62}$$

in affine coordinates. Using Corollary 11 we obtain the following non-degenerate Poisson structure

$$J_{\mathrm{P.v}} = \begin{bmatrix} 0 & 0 & \dfrac{1}{w_{n-1}^2} & -\dfrac{2\left(w_n w_{n-1} + w_n w_{n+1} + w_{n-2} w_{n-1}\right) + \nu}{w_{n-1}^2 w_n^2} \\ 0 & 0 & 0 & \dfrac{1}{w_n^2} \\ -* & 0 & 0 & 0 \\ -* & -* & 0 & 0 \end{bmatrix}. \tag{63}$$

One can check that the invariants (61) are in involution with respect to the Poisson bracket (63). Hence, the map (P.v) is Liouville integrable.

The dual map $[\mathbf{x}] \mapsto \varphi_{\mathrm{v}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= \left[\nu\,(x-z)\,t^2 + (u+y)\,z^2 - x^2 y\right]\beta - t^2\alpha\,(x-z)\,, \\ y' &= \beta x^3, \quad z' = \beta x^2 y, \quad u' = \beta x^2 z, \quad t' = \beta x^2 t. \end{aligned} \tag{Q.v}$$

This map depends on three parameters α, β and ν. The parameter ν is shared with the main map (P.v). The dual map given by (Q.v) has the following degrees of iterates:

$$\{d_n\}_{\mathrm{Q.v}} = 1, 3, 9, 19, 33, 51, 73, 99, 129, 163\dots \tag{64}$$

with generating function:

$$g_{\mathrm{Q.v}}(s) = -\frac{3s^2+1}{(s-1)^3}. \tag{65}$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth like the main map.

Since the main map (P.v) possesses two invariants and depends on a, c and d whereas the dual map (Q.v) does not, according to (32) we can write down the invariants for the dual map (Q.v) as:

$$\alpha I_{\text{low}}^{\text{P.v}} + \beta I_{\text{high}}^{\text{P.v}} = a I_1^{\text{Q.v}} + c I_2^{\text{Q.v}} + d I_3^{\text{Q.v}}, \tag{66}$$

where:

$$t^4 I_1^{\text{Q.v}} = \beta \left[(x+z)\, y^2 + yz^2 + uz^2 \right] t + (\alpha - \beta\nu)(y+z) t^3, \tag{67a}$$

$$t^4 I_2^{\text{Q.v}} = \left\{ \left[(z+x)\, y + zu - \nu t^2 \right] \beta + t^2 \alpha \right\} zy, \tag{67b}$$

$$\begin{aligned} t^8 I_3^{\text{Q.v}} = y^2 z^2 \Big\{ &\left[(x+z)^2 y^2 + 2uz\,(x+z)\, y - \nu^2 t^4 + 2\nu t^2 ux + u^2 z^2 \right] \beta \\ &+ \left[(x+z)\, y + \nu t^2 - ux + zu \right] t^2 \alpha \Big\}. \end{aligned} \tag{67c}$$

We note that the degree pattern of these invariants is $(1,2,2,1)$, $(1,2,2,1)$ and $(2,4,4,2)$ respectively. Finally, using the test of [31], we deduce that the map (Q.v) is not variational. Therefore, we conclude that the dual map (Q.v) is integrable according to Definition 5.

4.6. Maps (vi)

The main map $[\mathbf{x}] \mapsto \varphi_{\text{vi}}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$\begin{aligned} x' &= -\delta a t^5 - \delta \left[(u-y)\, \mu a \delta + cy + d\,(x+z) \right] t^4 \\ &\quad + \left\{ a\mu \left[uy^2 + (x+z)^2 y + uz^2 \right] \delta + d\,(x+z)\, y^2 \right\} t^2 \\ &\quad - \mu \left[(x+z)^2 y + uz^2 \right] a y^2, \\ y' &= a\mu x \left(\delta t^2 - y^2 \right) \left(\delta t^2 - x^2 \right), \quad z' = a\mu y \left(\delta t^2 - y^2 \right) \left(\delta t^2 - x^2 \right), \\ u' &= a\mu z \left(\delta t^2 - y^2 \right) \left(\delta t^2 - x^2 \right), \quad t' = a\mu t \left(\delta t^2 - y^2 \right) \left(\delta t^2 - x^2 \right). \end{aligned} \tag{P.vi}$$

This map depends on the five parameters a, c, d and μ, δ. The maps (P.vi) it is a slight generalization of autonomous $\mathrm{d}P_{\text{II}}^{(2)}$ equation discussed in [14]. Here, we recall its properties and we discuss its duality in the parameter space. First, the map (P.vi) has the following degrees of iterates:

$$\{d_n\}_{\text{P.vi}} = 1, 5, 15, 35, 65, 103, 149, 201, 261, 329, 405, 489, 581, 681 \ldots \tag{68}$$

with generating function:

$$g_{\text{P.vi}}(s) = -\frac{1 + 2s + 3s^2 + 4s^3 - 2s^5 - 2s^7 + 2s^8}{(s-1)^3}. \tag{69}$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth.

The map (P.vi) has the following invariants:

$$\begin{aligned} t^6 I_{\text{low}}^{\text{P.vi}} = a\delta\,(y+z)\, t^5 &- \left[(u\,(x-z) - xy)\, \mu a \delta - cyz - d\left(y^2 + z^2 \right) \right] \delta t^4 \\ &- \left\{ dy^2 z^2 + a\delta\mu \left(y^2 + z^2 \right) \left[(x+z)\, y - (x-z)\, u \right] \right\} t^2 \\ &+ a\mu y^2 z^2 \left[(x+z) y - (x-z) u \right], \end{aligned} \tag{70a}$$

$$
\begin{aligned}
t^8 I_{\mathrm{high}}^{\mathrm{P.vi}} = & \, \delta^2 a \left(u+x+y+z\right) t^7 \\
& + \delta^2 \left[a\delta\mu \left(u^2-uy+x^2-xz+y^2+z^2\right)\right. \\
& \quad \left. + \left(cu+dx+dz\right) y + \left(cx+du\right) z + xdu\right] t^6 \\
& - \delta a \left[\left(x+z\right) y^2 + yz^2 + uz^2\right] t^5 \\
& - \delta \left\{ \left[\left(u^2+2x^2+xz+z^2\right) y^2 + z\left(2x+z\right) uy + z^2 \left(2u^2+x^2\right)\right] \mu\delta a \right. \\
& \quad + d\left(x+z\right) y^3 + \left(x+z\right)\left(cz+du\right) y^2 \\
& \quad \left. + z^2 \left(cu+dx+dz\right) y + duz^2 \left(x+z\right) \right\} t^4 \\
& + \left\{ 2\mu a\delta \left[\frac{1}{2}\left(x+z\right)^2 y^4 + uz\left(x+z\right) y^3 + \frac{u^2 z^4}{2} \right.\right. \\
& \quad \left. + z^2 \left(u^2+x^2+xz+\frac{z^2}{2}\right) y^2 + uz^3 \left(x+z\right) y \right] \\
& \quad \left. + dy^2 z^2 \left(x+z\right)\left(u+y\right) \right\} t^2 - \left[\left(x+z\right) y + uz\right]^2 \mu a z^2 y^2 .
\end{aligned}
\tag{70b}
$$

Using the methods of [31] we have that the map (P.vi) is a variational maps arising from the Lagrangian:

$$
\begin{aligned}
L_{\mathrm{P.vi}} = & \left(w_{n+1}^2 - \delta\right) w_n w_{n+2} + \frac{w_{n+1}^2 w_n^2}{2} - \frac{d}{a\mu} w_{n+1} w_n \\
& - \frac{1}{2a\mu} \left[\delta\left(\delta a\mu - c\right) \log\left(w_n^2 - \delta\right) + 2a\sqrt{\delta}\, \operatorname{arctanh}\left(\frac{w_n}{\sqrt{\delta}}\right)\right]
\end{aligned}
\tag{71}
$$

in affine coordinates. Using Corollary 11, we obtain the following non-degenerate Poisson structure for the map (P.vi)

$$
J_{\mathrm{P.vi}} = \begin{bmatrix}
0 & 0 & \dfrac{1}{w_{n-1}^2 - \delta} & -\dfrac{2a\mu\ \left(w_n w_{n-1} + w_n w_{n+1} w_{n-1} w_{n-2}\right) - d}{a\mu\left(\delta - w_{n-1}^2\right)\left(\delta - w_n^2\right)} \\
0 & 0 & 0 & \dfrac{1}{w_n^2 - \delta} \\
-* & 0 & 0 & 0 \\
-* & -* & 0 & 0
\end{bmatrix}.
\tag{72}
$$

One can check that the invariants (70) are in involution with respect to the Poisson bracket (72). Therefore, the map (P.vi) is Liouville integrable.

The dual map $[\mathbf{x}] \mapsto \varphi_{\mathrm{vi}}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ has the following components:

$$
\begin{aligned}
& x' = \left[\delta\, t^2 u - \left(y+u\right) z^2 + x^2 y\right] \beta + \alpha\, t^2 \left(x-z\right) \\
& y' = \beta x \left(\delta\, t^2 - x^2\right), \quad z' = \beta y \left(\delta\, t^2 - x^2\right), \\
& u' = \beta z \left(\delta\, t^2 - x^2\right), \quad t' = \beta t \left(\delta\, t^2 - x^2\right).
\end{aligned}
\tag{Q.vi}
$$

This map depends on three parameters α, β and δ. The parameter δ is shared with the main map (P.vi). The map given by (Q.vi) has the following degrees of iterates:

$$
\{d_n\}_{\mathrm{Q.vi}} = 1, 3, 9, 19, 33, 51, 73, 99 \ldots \tag{73}
$$

with generating function:

$$
g_{\mathrm{Q.vi}}(s) = -\frac{3s^2+1}{(s-1)^3}. \tag{74}
$$

This means that the main map is integrable according to the algebraic entropy test with *quadratic* growth like the main map.

Since the main map (P.vi) possesses two invariants and depends on a, c and d whereas the dual map (Q.vi) does not, according to (32) we can write down the invariants for the dual map (Q.vi) as:

$$\alpha I_{\text{low}}^{\text{P.vi}} + \beta I_{\text{high}}^{\text{P.vi}} = a I_1^{\text{Q.vi}} + c I_2^{\text{Q.vi}} + d I_3^{\text{Q.vi}}, \tag{75}$$

where:

$$\begin{aligned}
t^8 I_1^{\text{Q.vi}} &= \delta \left[\beta\left(u+x+y+z\right)\delta - \alpha\left(y+z\right)\right] t^7 \\
&+ \delta^2 \mu \left\{\beta\left(u^2 - uy + x^2 - xz + y^2 + z^2\right)\delta \right.\\
&\quad \left. + \left[u\left(x-z\right) - xy\right]\alpha\right\} t^6 \\
&- \beta\delta\left[\left(x+z\right)y^2 + yz^2 + uz^2\right] t^5 \\
&- \delta\mu \left\{\beta\delta\left[\left(u^2 + 2x^2 + xz + z^2\right)y^2 \right.\right.\\
&\qquad \left. + z\left(2x+z\right)uy + z^2\left(2u^2 + x^2\right)\right] \\
&\quad \left. + \alpha\left(y^2+z^2\right)\left[u\left(x-z\right) - \left(x+z\right)y\right]\right\} t^4 \\
&+ 2\mu \left\{\beta\delta\left[\frac{y^4}{2}\left(x+z\right)^2 + uz\left(x+z\right)y^3 + uz^3\left(x+z\right)y \right.\right.\\
&\qquad \left. + z^2\left(u^2 + x^2 + xz + \frac{z^2}{2}\right)y^2 + \frac{u^2 z^4}{2}\right] \\
&\qquad \left. + \frac{\alpha y^2 z^2}{2}\left[u\left(x-z\right) - \left(x+z\right)y\right]\right\} t^2 \\
&- \left[\left(x+z\right)y + uz\right]^2 \beta z^2 \mu y^2,
\end{aligned} \tag{76a}$$

$$t^4 I_2^{\text{Q.vi}} = \beta\left[u\left(\delta t^2 - z^2\right)y + t^2 xz\delta - z\left(x+z\right)y^2\right] - \alpha yz t^2, \tag{76b}$$

$$\begin{aligned}
t^6 I_3^{\text{Q.vi}} &= \beta\left(\delta t^2 - z^2\right)\left(\delta t^2 - y^2\right)\left(x+z\right)\left(u+y\right) \\
&- \alpha\left[\delta\left(y^2+z^2\right)t^2 - y^2 z^2\right] t^2.
\end{aligned} \tag{76c}$$

We note that the degree pattern of these invariants is $(2,4,4,2)$, $(1,2,2,1)$ and $(1,2,2,1)$ respectively. Finally, using the test of [31], we obtain that the map (Q.vi) is not variational. Therefore, we conclude that the dual map (Q.vi) is integrable according to Definition 5.

5. Summary and outlook

In this paper we presented six families of four-dimensional maps, forming the (P,Q)-class. These maps are the result of a search problem, based on the existence of two invariants satisfying properties A, B and A and C respectively. In section 4 we discussed the integrability properties of these maps.

Integrability in the (P,Q)-class can arise in different ways depending whether the map is variational or not. Variational maps are all Liouville integrable, as remarked in section 2. The only additional structure needed for integrability was then the Lagrangian, constructed using the method in [31].

On the other hands integrability in the non-variational maps can arise in two different ways. The pair of maps (P.i) and (Q.i) possessing cubic growth is *deflatable*.

This means that the two maps arise as non-invertible non-local transformations from two lower-dimensional maps. In [15] we proved that the invariants are preserved in this process and that the integrability of the three-dimensional maps can be understood using the definition of Liouville integrability with a rank two Poisson structure. All the other maps possess quadratic growth and possess a third invariant of motion. In the case of the map (P.iii) the third invariant was found by direct inspection, while in all the other cases it was produced directly from the duality approach. As a last remark, we note that the maps with three invariants admit three different degenerate Poisson structures constructed using the method of [28], henceforth are also Liouville integrable in the sense of definition 8. However, as discussed in remark 9 in this case Liouville integrability is equivalent to Definition 5.

All the remaining maps have exponential growth and are therefore not integrable in the sense of algebraic entropy. Direct search of invariants for these maps excluded their existence up to degree 14. Moreover, using the test of [31], we proved that these exponentially-growing maps are not variational. Therefore we have strong evidence of the fact that these maps do not possess any non-degenerate Poisson structure, and therefore these cannot be Liouville integrable. Unfortunately, this result is not sufficient for a complete proof of the non-existence of non-degenerate Poisson structure. This is because, in principle, a fourth-order recurrence relation can be cast into a system of two second-order recurrence relations which can be variational. Therefore, as we did in [15], we conjecture that the maps (P.ii) and (Q.iii) either do not admit *any* full-rank Poisson structure, or for all full-rank Poisson structure they admit their invariants do not commute.

Table 1 provides a summary of the above considerations.

Equation	Degree pattern of invariants	Degree of growth	Variational
(P.i)*	(1,3,3,1), (2,4,4,2)	cubic	no
(Q.i)*	(1,2,2,1), (2,4,4,2)	cubic	no
(P.ii)	(1,2,2,1), (1,3,3,1)	exponential	no
(Q.ii)	(2,4,4,2)	exponential	no
(P.iii)	(1,3,3,1), (2,4,4,2), (2,5,5,2)	quadratic	no
(Q.iii)	(1,2,2,1), (2,4,4,2)	exponential	no
(P.iv) ($\mathrm{d}P_{\mathrm{I}}^{(2)}$)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.iv)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no
(P.v)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.v)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no
(P.vi) ($\mathrm{d}P_{\mathrm{II}}^{(2)}$)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.vi)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no

* Deflatable to a three-dimensional Liouville integrable map [15].

Table 1: Integrability properties of the (P,Q) maps.

The search procedure carried out in this paper has provided interesting and non-trivial examples of four-dimensional maps. All the resulting maps, except for four are new. Particularly interesting is the variety of behaviours we encountered in the maps of the class (P,Q). Work is in progress to characterize the surfaces generated by the invariants in both integrable and non-integrable cases. We expect this to give some hints on how integrability arises from purely geometrical considerations. This is well known for maps in two dimension with the theory of elliptic fibrations applied to the QRT mapping [7]. However, it was discussed in [14, 15] how examples with cubic growth can go beyond the existence of elliptic fibrations making the underlying geometrical structure more complex and richer. We recall that cubic growth already appeared in the literature in [43, 44].

Finally, we believe that more integrable maps may arise from the direct search of maps with invariants alongside other algorithmic tests available in the discrete setting. Analogous procedures in the continuous case still yield many new results more than fifty years after their introduction [45–47]. Work is in progress to extend the present class by considering invariants of more general form.

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