Spectral asymptotics of all the eigenvalues of Schrödinger operators on flat tori

Dario Bambusi[*], Beatrice Langella[†], Riccardo Montalto [‡]

Abstract

We study Schrödinger operators with Floquet boundary conditions on flat tori obtaining a spectral result giving an asymptotic expansion of all the eigenvalues. The expansion is in $\lambda^{-\delta}$ with $\delta \in (0,1)$ for most of the eigenvalues λ (stable eigenvalues), while it is a "directional expansion" for the remaining eigenvalues (unstable eigenvalues). The proof is based on a structure theorem and on a new iterative quasi-mode argument.

Keywords: Schrödinger operator, spectral asymptotics, normal form, pseudo differential operators, Nekhoroshev theorem

MSC 2010: 37K10, 35Q55

Contents

[*]Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano. *Email:* `dario.bambusi@unimi.it`

[†]Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano. *Email:* `beatrice.langella@unimi.it`

[‡]Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano. *Email:* `riccardo.montalto@unimi.it`

1 Introduction

The spectrum of periodic Schrödinger operators has been extensively studied in the last decades and it is essentially fully understood in dimension one. In particular a full asymptotic expansion of the eigenvalues $(\lambda_j)_{j\in\mathbb{Z}}$ in the parameter $1/|j|^2$ has been given by Marchenko [Mar86]. In higher dimension the situation is considerably more complicated. Consider the Laplacian with periodic boundary conditions on a general torus $\mathbb{T}^d_\Gamma := \mathbb{R}^d/\Gamma$, with Γ a maximal dimensional lattice. Its eigenvalues are given by $\{\|\xi\|^2\}_{\xi\in\Gamma^*}$ with Γ^* the dual lattice[1] to Γ. For generic lattices the differences between couples of eigenvalues accumulate at zero and this makes difficult to use standard resolvent expansions in order to obtain properties of the eigenvalues.

A milestone of the higher dimensional theory is the result of [FKT90], [Fri90] (see also [Wei77]) who proved that, provided $\mathcal{V}$ is a sufficiently smooth potential with zero average, and Γ a generic lattice, most of the eigenvalues of the Laplace operator $-\Delta$ are stable under the perturbation given by the potential $\mathcal{V}$, in the sense that there are two eigenvalues $\lambda_{\pm\xi}$ of

$$-\Delta + \mathcal{V}(x) \ , \tag{1.1}$$

[1] We recall that the dual lattice is defined as the set of ξ's s.t. $\xi\cdot\gamma\in 2\pi\mathbb{Z}\ \forall\gamma\in\Gamma$

in the interval

$$\left[\|\xi\|^2 - \frac{1}{\|\xi\|^{2\delta}}\,,\,\|\xi\|^2 + \frac{1}{\|\xi\|^{2\delta}}\right]$$

with $\delta \in (0,1)$ a parameter. However, it was shown in [FKT91] (developing an argument by [ERT84]), that there are also eigenvalues which behave differently and are not stable.

The stable eigenvalues also admit a full asymptotic expansion in $\lambda^{-\delta} \sim \|\xi\|^{-2\delta}$, which can be obtained as a byproduct of the works [Par08, PS10, PS12] (see also [Kar97] and [Vel15] for some partial previous results) and is explicitly given in [BLM20]. For the unstable eigenvalues such an asymptotic expansion is simply false. Here we address the problem of understanding the kind of asymptotic expansion valid for unstable eigenvalues.

To present our approach we first recall the method developed in [PS10] (see also [PS12]). In the paper [PS10] the authors developed a technique to construct a unitary transformation which conjugates the operator (1.1) to a new operator which is the sum of a "normal form operator" and a remainder. Such a technique can be interpreted as a quantization of the classical normal form algorithm usually employed to study the dynamics of the Hamiltonian system $h(x,\xi) := \|\xi\|^2 + V(x)$ whose quantization is (1.1). On the classical side, it is well known that the dynamics (and thus the normal form) of a Hamiltonian system is completely different in the resonant and in the nonresonant regions: it turns out that stable eigenvalues correspond to the nonresonant regions, while the unstable eigenvalues correspond to resonant regions. In particular, in [PS10] a precise definition of resonant/nonresonant regions was given and it was shown that the normal form operator is a block diagonal operator which is just a Fourier multiplier if one localizes it in the nonresonant region; then the blocks corresponding to resonant regions turn out to be finite dimensional, but their dimension is not bounded. When adding the remainder, a quasimode argument can be used to get the asymptotics of the eigenvalues corresponding to the nonresonant region, but almost nothing is known on the eigenvalues of the other blocks.

Here we want to obtain precise asymptotics also of all the eigenvalues corresponding to the resonant regions. The asymptotic expansions we get are not in the parameter $\|\xi\|^{-2\delta}$: instead, they are directional asymptotics. To explain this point, label the eigenvalues of (1.1) using the points $\xi \in \Gamma^*$, then, roughly speaking, the result is the following: consider a submodule M of Γ^* and assume that the vector $\xi \in \Gamma^*$ is resonant with the vectors of a basis of M, but with no other vectors in Γ^*, then the corresponding eigenvalue λ_ξ admits an asymptotic expansion in the parameter $\|(\xi)_M\|^{-2\delta}$, the lower index M denoting orthogonal projection on M.

The main point in order to get such an expansion consists in first proving a structure theorem which is a variant of the block diagonal decomposition of [PS10, PS12], but which is suitable for iteration. This is needed in order to further decompose the resonant blocks in sub-blocks which at the end of the procedure will be just isolated points or finite dimensional, but with *uniformly bounded dimension.*

More precisely, (as in [PS10, PS12]) as a first step we conjugate the operator in (1.1) to $\widetilde{H}+\mathcal{R}$ with $\mathcal{R}$ a smoothing pseudodifferential operator and $\widetilde{H}$ a block diagonal operator. The blocks corresponding to the non resonant zone are just isolated points. The main novelty of our structure theorem is that we prove that in the nontrivial blocks $\widetilde{H}$ is still a periodic Schrödinger operator, but on a lower dimensional torus: essentially it contains only the angles in the resonant directions. We point out that a similar, but less precise property, was proved in [PS09] just for the 2-d case. Then, since in each block one has the same structure as that of (1.1), one can apply again the normal form procedure and iterate until one is left with trivial blocks and blocks with uniformly bounded dimension. However, since the new operator only depends on the resonant angles one gets that the new normal form is only up to a remainder which is smoothing in the resonant directions. This is the source of the directional decay.

One further difficulty is that, since there are infinitely many blocks, one must have a uniform control of all the constants of the restricted operators. We will achieve this goal by performing the whole construction in an intrinsic way: we define the resonant regions, the blocks, and the seminorms of the pseudodifferential operators in terms of the natural metric of the torus. This allows a control of all the constants of the restricted operators in terms of the constants of the original operator.

The final step of the proof consists in reconstructing the eigenvalues of the original operator. This is obtained through an iterative quasimode argument that, as far as we know, is new. To explain it consider the case $d=2$; in this case, when restricting to the blocks, $\widetilde{H}$ turns out to be either a Fourier multiplier or a 1 dimensional Schrödinger operator. So essentially everything is known on the spectrum of each block operator. However in order to establish a correspondence between the spectrum of $\widetilde{H}$ and the spectrum of $\widetilde{H}+\mathcal{R}$ one must have some information on how the eigenvalues of $\widetilde{H}$ are distributed on the real line and to know something on the eigenfunctions. The information that we use on the eigenvalues is just Weyl law, which allows to partition of the spectrum in clusters. Then to each cluster we apply a new quasimode argument (which is a development of that used in [BKP15]) which allows to describe how the perturbation changes the eigenvalues. Concerning the eigenfunctions, the information that we extract is that their negative Sobolev norms decay fast with $|\xi|^{-1}$ (see Equation (3.15) for a precise statement). We point out that it could be interesting to extract more information on the eigenfunctions.

We point out that we think that our formulation of the Structure theorem could be useful also for more applications, for example we think that one could get a detailed description of the semiclassical measures [AFKM15] or (following [Roy07]) a precise semiclassical expansion in $\hbar$ of the eigenvalues.

Finally, we emphasize that our motivation for this research comes from our work in KAM theory for PDEs: the construction of quasiperiodic solutions of a Hamiltonian PDE requires a full understanding of the dynamics of the operator obtained by linearizing the PDE at any approximate solution. A good model problem is the time dependent Schrödinger equation $-i\partial_t u = -\Delta u + \mathcal{V}(t,x)u$, where $\mathcal{V}$ is a smooth potential depending in a quasiperiodic way on time and an efficient way to completely characterize its dynamics consists in conjugating such an operator to a time independent equation (reducibility problem). This can be done using a general strategy developed in [BBM14, Bam17, Bam18, BM16, BBHM18] for the study of quasilinear 1-d problems and extended to some very particular higher dimensional cases in [BGMR18, FGMP19, BLM19, Mon19, BGMR17, FGN19]. The first step of this approach requires a very precise knowledge of the eigenvalues of the problem in which time is frozen, and that's why we attack here this problem. The final aim of this line of research is to bypass the limitation of the results of [Bou98, Bou04, EK10, PX13, BM19] and to get a KAM theory applicable to equations on manifolds or domains which are as general as possible.

The paper is split in two parts: Part I, containing Sections 2 and 3, in which we give our main results, and Part II containing the proofs. In Section 2, we give a statement of the Structure Theorem, recalling also the main notions needed to give a precise statement. In Section 3, we start by describing in detail the partition of $L^2(\mathbb{T}^d_\Gamma)$ in invariant subspaces. This is the quantum analogue of the construction of the geometric part of Nekhoroshev theorem. In particular this is needed in order to give a precise statement of our spectral result (see Theorem 3.16): the kind of asymptotics that we give depends on the block to which the eigenvalue belongs (in a sense that will be made precise).

Part II is devoted to the proof of the main results. In Sect. 4 we give our normal form lemma conjugating up to a smoothing operator (1.1) to a normal form operator. This corresponds to the analytic part of Nekhoroshev's theorem. In Section 5 we study the partition of Subsect. 3.1 in order to show that it is actually a partition and is left invariant by an operator in normal form. This corresponds to the geometric part of Nekhoroshev's theorem. Finally in Sect. 6 we give a quasimode argument adapted to our situation and prove our spectral result.

The paper contains also three appendixes: in Appendix A we adapt some standard results on pseudodifferential calculus to our context, in Appendix B we prove some very technical lemmas which are used in the core of the paper, finally in Appendix C we prove a couple of results on spectral problems needed in Section. 6.

ACKNOWLEDGMENTS. While working on this project we had many discussions with several people. In particular we had several enlightening discussions with Antonio Giorgilli who explained to us the details of the geometric part of the proof of Nekhoroshev's theorem: its understanding has been a key step for the completion of this work. A particular thank also goes to Thomas Kappeler who introduced us to the literature on higher dimensional periodic Schrödinger operators. We also thank Emanuele Haus, Fabricio Macia, Alberto Maspero, Michela Procesi and Didier Robert for some very stimulating discussions.

This work is partially supported by GNFM.

PART I: Statements

2 The structure theorem

2.1 Preliminaries

Let Γ be a lattice of dimension d in $\mathbb{R}^d$, with basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$, namely

$$\Gamma := \left\{ \sum_{i=1}^{d} k_i \mathbf{e}_i : k_1, \dots, k_d \in \mathbb{Z} \right\} , \tag{2.1}$$

and define

$$\mathbb{T}^d_\Gamma := \mathbb{R}^d / \Gamma \, . \tag{2.2}$$

Consider the Schrödinger operator

$$-\Delta + \mathcal{V} \, , \tag{2.3}$$

with Floquet boundary conditions on $\mathbb{T}^d_\Gamma$, namely acting on functions u which fulfill (together with their first derivatives) the boundary conditions

$$u(x+\gamma) = e^{i\gamma\cdot\kappa}u(x) \ , \quad \forall \gamma \in \Gamma; \tag{2.4}$$

$\kappa \in \mathbb{T}^d_{\Gamma^*}$ is a parameter. Here $\mathcal{V}$ is either a potential, or more generally a pseudodifferential operator of order zero on $T^*\mathbb{T}^d_\Gamma$ (see Definition 2.2 below for a precise definition).

By making the Gauge transformation $u = e^{i\kappa\cdot x}\tilde{u}$ the operator (2.3) is conjugated to the operator

$$H = \sum_j (D_j + \kappa_j)^2 + \mathcal{V} \ , \quad D_j := -\mathrm{i}\partial_j \tag{2.5}$$

with periodic boundary conditions (p.b.c.) on $\mathbb{T}^d_\Gamma$; from now on we will only use the variable $\tilde{u}$ and omit the tilde. If $\mathcal{V} \equiv 0$, then the eigenvalues of H are simply given by

$$\lambda^{(0)}_\xi := \|\xi + \kappa\|^2 \ , \quad \xi \in \Gamma^* \ . \tag{2.6}$$

By introducing in $\mathbb{T}^d_\Gamma$ the basis of the vectors $\mathbf{e}_i$, H is reduced to

$$H = -\Delta_{g,\kappa} + \mathcal{V} \ , \tag{2.7}$$

$$-\Delta_{g,\kappa} := g^{AB}(D_A + \kappa_A)(D_B + \kappa_B) \tag{2.8}$$

with p.b.c. on the standard torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. Note that in formula (2.8), we use the standard Einstein notation, namely

$$g^{AB}(D_A + \kappa_A)(D_B + \kappa_B) := \sum_{A,B=1}^{d} g^{AB}(D_A + \kappa_A)(D_B + \kappa_B)$$

where

$$g_{AB} := \mathbf{e}_A \cdot \mathbf{e}_B, \tag{2.9}$$

and the matrix with upper indexes is the inverse of the matrix with lower indexes, namely it is defined by

$$g_{AB}g^{BC} = \delta^C_A \ .$$

Conversely, given an operator of the form (2.7), by introducing a basis which is orthonormal with respect to the metric $g := (g_{AB})$ and making a Gauge transformation, one is reduced to a standard Schrödinger operator with Floquet boundary conditions on a suitable torus $\mathbb{T}^d_\Gamma$. For this reason, from now we will restrict our study to the operator (2.7) and we will call it a *Schrödinger operator of dimension d with Floquet boundary conditions*. Furthermore, with a slight abuse of language, we will use the same name for operators which are the restriction of an operator of the form (2.7) to a subspace of L^2.

In the following we will only deal with scalar products and norms with respect to the metric g. We will denote

$$(x;y)_g := g_{AB}x^A y^B \ , \quad (\xi;\eta)_{g*} := g^{AB}\xi_A \eta_B \tag{2.10}$$

the scalar product with respect to this metric of two vector x, y or two covectors ξ, η. Correspondingly we will denote

$$\|x\|_g^2 := (x;x)_g \ , \quad \|\xi\|_{g*}^2 := (\xi;\xi)_{g*} \ . \tag{2.11}$$

Finally we will denote by $d\mu_g(x)$ the volume form corresponding to g. The following constant play a relevant role in our construction:

$$\mathfrak{c} := \inf_{k\in\mathbb{Z}^d\setminus\{0\}} \|k\|_{g*}^2 \ . \tag{2.12}$$

Given s linearly independent vectors $\{u_1, \ldots, u_s\}$ in $\mathbb{Z}^d$, denote by $\mathrm{Vol}_{g^*}\{u_1|\cdots|u_s\}$ the $s-$ dimensional volume, calculated with respect to the metric g^*, of the parallelepiped in $\mathbb{R}^d$ with edges given by $\{u_1, \ldots, u_s\}$. The second relevant constant is

$$\mathfrak{C} := \min_{1\le s\le d} \ \min_{u_1,\ldots,u_s\in\mathbb{Z}^d} \mathrm{Vol}_{g^*}\{u_1|\cdots|u_s\} \, . \tag{2.13}$$

Remark 2.1. *In Lemma B.2 of the Appendix B, we will prove that $\mathfrak{C}$ is strictly positive.*

In the following we will often refer to the constants $\mathfrak{c}$, $\mathfrak{C}$ as the constants of the metric.

2.2 Pseudodifferential calculus

Given $u \in L^2(\mathbb{T}^d)$, we define as usual its Fourier series by

$$u(x) = \sum_{\xi\in\mathbb{Z}^d} \hat{u}_\xi \ e^{i\xi\cdot k}$$

where $\xi \cdot k = \xi_A x^A$ is the usual pairing between a vector and a covector.
Fix $\kappa \in \mathbb{R}^d/\mathbb{Z}^d$, then we define $H^s(\mathbb{T}^d)$ to be the completion of $\mathcal{C}^\infty(\mathbb{T}^d)$ in the norm

$$\|u\|_{H^s}^2 = \sum_{\xi\in\mathbb{Z}^d} \|\xi+\kappa\|_{g*}^{2s} |\hat{u}_\xi|^2 \ . \tag{2.14}$$

Given a function $a \in C^\infty(T^*\mathbb{T}^d)$, we define (exploiting the equivalence $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$),

$$\|d_x^M d_\xi^N a(x,\xi)\| = \sup_{\substack{\|h^{(i)}\|_g = 1 \\ \|k^{(j)}\|_{g*}=1}} \left| d_x^M d_\xi^N a(x,\xi) \left[h^{(1)}, \dots, h^{(M)}, k^{(1)}, \dots, k^{(N)}\right] \right| . \tag{2.15}$$

Definition 2.2. *Let* $a \in \mathcal{C}^\infty\left(T^*\mathbb{T}^d\right)$ *and* $m \in \mathbb{R}$, $\delta > 0$ *and* $\kappa \in \mathbb{R}^d/\mathbb{Z}^d$. *We say that* $a \in S^{m,\delta}$ *is a symbol of order* m, *if* $\forall\ N_1, N_2 \in \mathbb{N}$, *there exists a constant* $C_{N_1,N_2} > 0$ *such that*

$$\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \le C_{N_1,N_2} \langle \xi + \kappa\rangle_g^{m-\delta|N_2|} \quad \forall x \in \mathbb{T}^d\,,\ \xi \in \mathbb{R}^d$$

where $\langle\xi\rangle_g := \left(1 + \|\xi\|_{g*}^2\right)^{1/2}$.

We also define $S^{-\infty,\delta} := \cap_m S^{m,\delta}$.

Remark 2.3. *The parameter* κ *which appears in the definition of symbol and as a weight in the Sobolev norms* (2.14) *has been introduced in order to get uniform estimates suitable for the iteration of Theorem 2.18.*

Definition 2.4. *Let* $a \in S^{m,\delta}$, *its* Weyl quantization *is the linear operator* $A \equiv Op^W(a)$ *defined by*

$$\left(Op^W(a)[u]\right)(x) = \sum_{\xi\in\mathbb{Z}^d}\sum_{h\in\mathbb{Z}^d} \hat{a}_h\left(\xi + \frac{h}{2}\right)\hat{u}_\xi\ e^{i(\xi+h)\cdot x}\,, \tag{2.16}$$

where $\forall k \in \mathbb{Z}^d$ *and* $\forall \xi \in \mathbb{R}^d$

$$\hat{a}_k(\xi) = \frac{1}{\mu_g(\mathbb{T}^d)} \int_{\mathbb{T}^d} a(x,\xi) e^{-i\xi\cdot x} d\mu_g\,.$$

Definition 2.5. *Let* A *be a linear operator on* $L^2(\mathbb{T}^d)$, *we say that it is a pseudodifferential operator of class* $OPS^{m,\delta}$ *if there exists* $a \in S^{m,\delta}$, *such that* $A = Op^W(a)$. *Operators of class* $OPS^{-\infty,\delta}$ *will be called* smoothing.

Definition 2.6 (Seminorms). *Let* $a \in S^{m,\delta}$ *and* $N_1, N_2 \in \mathbb{N}$. *We define*

$$C_{N_1,N_2}(a) := \sup_{(x,\xi)\in\mathbb{T}^d\times\mathbb{R}^d} \langle\xi+\kappa\rangle_g^{\delta N_2 - m} \|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\|\,.$$

Equivalently, if $A = Op^W(a)$, *we set* $C_{N_1,N_2}(A) = C_{N_1,N_2}(a)$.

Remark 2.7. *$\{C_{N_1,N_2}(\cdot)\}_{N_1,N_2\in\mathbb{N}}$ is a family of seminorms on $S^{m,\delta}$, and we will refer to $\{C_{N_1,N_2}(A)\}_{N_1,N_2\in\mathbb{N}}$ as the family of seminorms of the operator A. All the definitions are given in such a way that the seminorms do not depend on the coordinates that one uses in $\mathbb{T}^d$, namely, if one changes the basis $\{\mathbf{e}_i\}$ by means of a unimodular transformation A (i.e. a unimodular matrix with integer coefficients), then this does not change the value of the seminorms. This is crucial for our procedure.*

We refer to the Appendix A for some basic properties of pseudo-differential calculus in the intrisic formulation. In particular, we emphasize that all the constants controlling the seminorms of the composition, commutators, and exponentiation of pseudo-differential operators depend only on the constants of the metric. This is is needed for iterating the structure theorem.

2.3 Submoduli, subspaces and statement of the Structure Theorem

Definition 2.8. *Given $E \subseteq \mathbb{Z}^d$, we denote*

$$\mathcal{E} = \overline{\mathrm{span}\{e^{i\xi\cdot x} \mid \xi \in E\}}\,, \tag{2.17}$$

where the bar denotes the closure in L^2. We will call such a subspace subspace generated by E.

Definition 2.9. *We will denote by $\Pi_{\mathcal{E}} : L^2(\mathbb{T}^d) \to \mathcal{E}$ the orthogonal projector on $\mathcal{E}$ and, given a linear (pseudodifferential) operator F, we will write*

$$F_{\mathcal{E}} := \Pi_{\mathcal{E}} F \Pi_{\mathcal{E}}\,. \tag{2.18}$$

The block decomposition as well as the spectral asymptotics of the Schrödinger operator are related to the submoduli of $\mathbb{Z}^d$, for this reason we recall some properties of the bases of the moduli. The systematic use of the properties of discrete submoduli is one of the differences with the construction of [Par08, PS10, PS12]. This plays a crucial role in order to show that the operator one obtains in each invariant block still has the structure of a Laplacian plus a potential plus a more regularizing pseudodifferential operator.

Definition 2.10. *A subgroup M of $\mathbb{Z}^d$ is called a submodule if $\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}} M = M$. Here and below,* $\mathrm{span}_{\mathbb{R}} M$ *is the subspace generated by taking linear combinations with real coefficients of elements of M.*

Given a discrete submodule M of $\mathbb{Z}^d$ it is well known that it admits a basis, namely that there exist d' independent vectors $\mathbf{v}^1, ..., \mathbf{v}^{d'}$ such that

$$M = \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}^1, ..., \mathbf{v}^{d'}\} := \left\{ w \in \mathbb{Z}^d \ : \ w = \sum_{k=1}^{d'} n_k \mathbf{v}^k, \quad n_1, \ldots, n_{d'} \in \mathbb{Z} \right\} . \tag{2.19}$$

Definition 2.11. *Given a basis $\{\mathbf{v}^k\}_{k=1,\ldots,d}$ of M and a vector $m = m_k \mathbf{v}^k \in$span$_{\mathbb{R}} M$, we denote*

$$\lfloor m \rfloor := \lfloor m_k \rfloor \mathbf{v}^k \ ,$$

with $\lfloor m_k \rfloor$ the integer part of m_k, and

$$\{m\} := \{m_k\} \mathbf{v}^k \ ,$$

where $\{m_k\}$ is the fractional part of m_k.

Given a covector $\xi \in \mathbb{Z}^d$, a Floquet parameter κ, and a module M, we will have to decompose the covector $w = \xi + \kappa \in \mathbb{R}^d$ in a component along M and a component in the orthogonal direction, and this has to be done in a way compatible with the lattice structure of $\mathbb{Z}^d$ and with the Floquet parameter.
We consider the orthogonal decomposition $\mathbb{R}^d = \mathrm{span}_{\mathbb{R}} M \oplus (\mathrm{span}_{\mathbb{R}} M)^\perp$. Correspondingly, given a vector $w \in \mathbb{R}^d$, we decompose it as

$$w = w_M + w_{M^\perp} \ , \quad w_M \in \mathrm{span}_{\mathbb{R}} M \ , \quad w_{M^\perp} \in (\mathrm{span}_{\mathbb{R}} M)^\perp \ .$$

Definition 2.12. *Given a vector $\xi \in \mathbb{Z}^d$, a module M and a Floquet parameter κ, we define the following two objects:*

$$\begin{aligned} \widetilde{\xi} &:= \xi - \lfloor (\xi + \kappa)_M \rfloor \ , \\ \kappa' &:= \{(\xi + \kappa)_M\} \ . \end{aligned} \tag{2.20}$$

Remark 2.13. *If we denote $\zeta := \lfloor (\xi + \kappa)_M \rfloor$, one has*

$$(\xi + \kappa)_M = \zeta + \kappa' \ , \quad (\xi + \kappa)_{M^\perp} = (\widetilde{\xi} + \kappa)_{M^\perp} \ . \tag{2.21}$$

Given a vector $\beta \in \mathbb{Z}^d$, we will have to consider the space

$$M + \beta := \left\{ \xi \in \mathbb{Z}^d \ : \ \exists v \in M \ : \ \xi = v + \beta \right\} \ . \tag{2.22}$$

Remark 2.14. *Notice that, for any $\xi \in M + \beta$, one has*

$$\tilde{\xi} = \tilde{\beta} , \quad \{(\xi + \kappa)_M\} = \{(\beta + \kappa)_M\} ,$$

thus the quantities $\widetilde{\xi}$ and κ' defined in (2.20) *are constant on $M + \beta$.*

Remark 2.15. *The set $M+\beta$ defined as in* (2.22) *is clearly an affine module isomorphic to M. A convenient way to identify the two spaces $M+\beta$ and M is to subtract $\widetilde{\beta}$ to a vector $w \in M+\beta$.*
Correspondingly, the subspace of $L^2(\mathbb{T}^d)$ generated by $M+\beta$ (in the sense of Definition 2.17) is isomorphic to the subspace generated by M. Explicitly, the isomorphism can be realized by using the Gauge transformation U_β defined by

$$U_\beta u := e^{-ix\cdot\tilde{\beta}} u \ . \tag{2.23}$$

Definition 2.16. *Given a module M, a vector $\beta \in \mathbb{Z}^d$ and a set $W \subset M+\beta$, we denote $W^t := W - \widetilde{\beta}$ so that $\mathcal{W}^t := U_\beta \mathcal{W} \subset L^2(\mathbb{T}^d)$.*

As a last step, we introduce the definitions of *coordinates adapted to a module.* If $\mathbf{v}^1, ..., \mathbf{v}^{d'}$ ($d' < d$) is a basis of $M \subset \mathbb{Z}^d$, then it can be completed to a basis of $\mathbb{Z}^d$, namely there exist $\mathbf{v}^{d'+1}, ..., \mathbf{v}^d$ such that the whole collection $\mathbf{v}^1, ..., \mathbf{v}^d$ generates $\mathbb{Z}^d$. Such a basis will be called a basis *adapted to M*. In what follows, given a collection of such vectors $\{\mathbf{v}^{d'+1}, \dots, \mathbf{v}^d\}$, we will denote

$$M^{(c)} := \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}^{d'+1}, ..., \mathbf{v}^d\} \ ; \tag{2.24}$$

if $M = \mathbb{Z}^d$ then $M^{(c)} = \{0\}$ and if $M = \{0\}$ then $M^{(c)} = \mathbb{Z}^d$. Of course, in general $M^{(c)}$ is not unique, but this will not affect our construction. Consider now the basis $\{\mathbf{u}_j\}_{j=1,\dots,d}$ of $\mathbb{R}^d$ dual to $\{\mathbf{v}^j\}_{j=1,\dots,d}$.

Definition 2.17. *The coordinates z^j introduced by*

$$x = z^j \mathbf{u}_j \tag{2.25}$$

are good coordinates on $\mathbb{T}^d$ (in the sense that they respect the 2π periodicity of the torus). These coordinates will be called coordinates adapted to M.

The main result of this section is the following theorem.

Theorem 2.18. *[Structure Theorem] Given $\epsilon, \delta \in \mathbb{R}^+$ and $\tau > d-1$ fulfilling*

$$\delta + d(d+\tau+1)\epsilon < 1 \,, \quad \epsilon(\tau+1) \le \delta \,, \tag{2.26}$$

a Floquet parameter κ and a flat metric g, there exists a partition of $\mathbb{Z}^d$:

$$\mathbb{Z}^d = \bigcup_{M \subseteq \mathbb{Z}^d} \bigcup_{\beta \in \widetilde{M}} W_{M,\beta} \tag{2.27}$$

where M runs over the submoduli of $\mathbb{Z}^d$ and $\widetilde{M}$ is a subset of $M^{(c)}$. All the sets $W_{M,\beta}$ have finite cardinality, the set $E_{\{0\}} := \bigcup_\beta W_{\{0\},\beta}$ has density one at *infinity, and $W_{\mathbb{Z}^d,\{0\}}$ has cardinality bounded by an integer n_* which depends on the constants of the metric and on d, δ, ϵ, τ only.*

Consider the operator (2.7) *and assume that $\mathcal{V} \in OPS^{0,\delta}$, then $\forall \mathtt{N} > 0$ there exists a unitary transformation U which depends smoothly on $\mathcal{V}$, which fulfills*

$$U - \mathrm{Id}\,,\ U^{-1} - \mathrm{Id} \in OPS^{-\delta,\delta} \tag{2.28}$$

and is s.t.

$$UHU^{-1} = \widetilde{H} + \mathcal{R}\ , \tag{2.29}$$

with

1. $\mathcal{R} \in OPS^{-2\mathtt{N}\delta,\delta}$
2. $\widetilde{H}$ *leaves invariant the subspaces generated by $W_{M,\beta}$ (according to Definition 2.8) for all M and $\beta \in \widetilde{M}$. Furthermore*

   2.1 $\forall\beta$, $\widetilde{H}_{\mathcal{W}_{\{0\},\beta}} \equiv \widetilde{H}\big|_{\mathcal{W}_{\{0\},\beta}}$ *is a Fourier multiplier*

   2.2 *$\forall M$ proper submodule and $\forall\beta \in \widetilde{M}$, one has that $H^{(1)}_{M,\beta} := U^*_\beta \tilde{H}_{\mathcal{W}_{M,\beta}} U_\beta$ is a* Schrödinger operator of dimension $d' = dim M$, *in the sense that introducing coordinates adapted to M, it takes the form*

   $$H^{(1)}_{M,\beta} = \Pi_{\mathcal{W}^t_{M,\beta}} \left(-\Delta_{g,\kappa'} + \mathcal{V}_{M,\beta} + \left\| (\widetilde{\beta} + \kappa)_{M^\perp} \right\|^2_{g*} \right) \Pi_{\mathcal{W}^t_{M,\beta}} \tag{2.30}$$

   here $-\Delta_{g,\kappa'}$ is the d' dimensional Laplacian computed with respect to the restriction of the metric g^* to $\mathrm{span}_{\mathbb{R}} M$ and with Floquet parameter $\kappa' = \{(\beta + \kappa)_M\}$. $\mathcal{V}_{M,\beta}$ is a periodic pseudo-differential operator of order 0 *(in d' dimensions).*

Furthermore, the seminorms of the operators $U, \mathcal{R}$ and $\mathcal{V}_{M,\beta}$ only depend on the constants of the metric $\mathfrak{c}, \mathfrak{C}$, and on the seminorms of $\mathcal{V}$.

Remark 2.19. *The partition of $\mathbb{Z}^d$ does not depend on the operator* (2.7), *but only on the properties of the metrics, and on κ.*

Remark 2.20. *The theorem holds also if the initial operator* (2.7) *is replaced by the restriction of a Schrödinger operator to the subspace generated by any finite subset E of $\mathbb{Z}^d$, with the only exception that in such a case the set $E_{\{0\}}$ does not have, of course, density one at infinity. This is useful for iterating the construction.*

Remark 2.21. *The restriction of the metric g to a module M has new constants which are controlled by the constants $\mathfrak{c}$ and $\mathfrak{C}$ of the initial metric g. This is useful for the iteration of the construction.*

Theorems similar to Theorem 2.18 were proved in [Par08, PS10, PS12] (see also [PS09]). The main differences with the theorems proved in those papers are the following:

1. The remainder $\mathcal{R}$: the kind of remainders obtained in those papers are not smoothing operators, but operators which are small when localized in an annulus in the action space ξ. We will comment more on it in Sect. 4 when we will discuss the normal form theorem (Theorem 4.6); see Remark **??**. The fact of getting a remainder which is a smoothing operator is one of the points needed in order to get an iterable version of the Theorem, which in turn is needed in order to get detailed informationson the resonant eigenvalues.

2. The presence of Item 2.2, namely the dimensional reduction: this is the main new point contained in Theorem 2.18. In particular the presence of this point allows to apply Theorem 2.18 in an iterative way.

3. The last statement of the theorem, namely the uniformity of the constants with respect to the block: again this is needed in order to get an asymptotic expansion useful in order to study the time dependent case.

3 The partition and the spectral theorem

3.1 Construction of the Partition

We are now giving the explicit construction of the sets $W_{M,\beta}$. This is a quantum analogue of the classical geometrical construction of the Nekhoroshev theorem [Nek77, Nek79] (see also [Gio03]). A direct classical counterpart can be found in [BL21]. We found very striking the fact that there is a so close connection between classical and quantum dynamics. We remark that the construction of this section can also be considered as a variant of the construction of [PS10, PS12]: the differences will be pointed out in the following.

Roughly speaking, given a submodule $M \subseteq \mathbb{Z}^d$ of dimension s, the sets

$$E_M^{(s)} := \bigcup_{\beta \in \widetilde{M}} W_{M,\beta} \tag{3.1}$$

are the points $\xi \in \mathbb{Z}^d$ which are resonant only with the integer vectors of M. For this reason, in the following we will often refer to a submodule $M \subseteq \mathbb{Z}^d$ ad a *resonance module*. In order to make the construction precise, consider the classical symbol of $-\Delta_{g,\kappa}$, namely

$$h_0(\xi) = \|\xi + \kappa\|_{g^*}^2 \ ; \tag{3.2}$$

the frequencies of the corresponding classical motion are

$$\omega_j = \xi_j + \kappa_j \ ,$$

so that a point ξ is (exactly) resonant with some integer k if

$$\left((\xi + \kappa); k\right)_{g*} = 0 \ .$$

Actually, the theory developed in [BLM20] shows that, in a quantum context a possible definition of point resonant with a vector k is

$$\left|\left((\xi + \kappa); k\right)_{g*}\right| < \frac{\langle \xi + \kappa \rangle_g^\delta}{\|k\|_{g^*}^\tau} \ , \tag{3.3}$$

furthermore, due to the decay of the Fourier coefficients of a smooth function, it is enough to consider the k's s.t.

$$\|k\|_{g^*} \leq \langle \xi \rangle_g^\epsilon$$

for some positive small ϵ. So, in principle $E_M^{(s)}$ should be the set of the ξ's which are in resonance with the k's belonging to M and having a not too large module. However this has to be modified due to the translation by $k/2$ present in the definition of Weyl quantization. Furthermore, one has to modify the construction both in order to get that the sets $E_M^{(s)}$ do not overlap and in order to obtain invariant sets.

To start with we define the resonance zones, in which the following notation will be used:

Definition 3.1. *Given $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$, and a Floquet parameter κ, we define*

$$\xi^\kappa := \xi + \kappa \ , \tag{3.4}$$

$$\xi_k := \xi + \kappa + \frac{k}{2} \equiv \xi^\kappa + \frac{k}{2} \, . \tag{3.5}$$

Definition 3.2 (Resonant zones)**.** *Fix δ, ϵ, τ as in the statement of Theorem 2.18; fix also constants fulfilling:*

$$\begin{gathered} \delta_0 = \delta \, , \\ \delta_{s+1} = \delta_s + (d+\tau+1)\epsilon \quad \forall s = 0, \dots, d-1 \, , \\ 1 = D_0 < D_1 < \cdots D_{d-1} \, , \\ 1 = C_0 < C_1 < \cdots C_{d-1} \, , \end{gathered}$$

then we define the following sets:

1. $Z^{(0)} = \left\{ \xi \in \mathbb{Z}^d \;\middle|\; |\left(\xi_k; k\right)_{g*}| > \langle \xi_k \rangle_g^{\delta} \|k\|_{g*}^{-\tau} \quad \forall k \in \mathbb{Z}^d \;\; s.\ t.\ \; \|k\|_{g*} \le \langle \xi_k \rangle_g^{\epsilon} \right\}$

2. *given $M \subseteq \mathbb{Z}^d$ a resonance module of dimension $s \ge 1$ and s linearly independent vectors $\{k_1, k_2, \dots, k_s\} \subset M$, we define*

$$Z_{k_1,\dots,k_s} = \left\{ \xi \in \mathbb{Z}^d \;\middle|\; \forall j = 1, \dots s \quad |\left(\xi_{k_1}; k_j\right)_{g*}| \le C_{j-1} \langle \xi_{k_1} \rangle_g^{\delta_{j-1}} \|k_j\|_{g*}^{-\tau} \right. \\ \left. \text{and } \|k_j\|_{g*} \le D_{j-1} \langle \xi_{k_1} \rangle_g^{\epsilon} \right\} \quad (3.6)$$

   and

$$Z_M^{(s)} = \bigcup_{\substack{\{k_1,\dots,k_s\} \subset M \\ \text{lin. ind.}}} Z_{k_1,\dots,k_s} \, . \quad (3.7)$$

Remark 3.3. *By* (3.6)*, $\forall s \ge 1$ and $\forall M$, one has $Z_M^{(s)} \cap Z^{(0)} = \emptyset$.*

Remark 3.4. *If $1 \le r < s$, then for any M with dim $M = s$, one has*

$$Z_M^{(s)} \subseteq \bigcup_{\substack{M' \subset M \\ \dim . M' = r}} Z_{M'}^{(r)} \, .$$

Remark 3.5. *The fact that the zones $Z^{(0)}$ and $Z_{k_1,\dots,k_s}$ are defined only in terms of the metric is one of the key ingredients allowing to iterate the structure theorem.*

The regions $Z_M^{(s)}$ contain points $\xi \in \mathbb{Z}^d$ which are in resonance with *at least* s linearly independent vectors in M. Thus such regions are clearly not reciprocally disjoint. We identify now the points $\xi \in \mathbb{Z}^d$ which admit *exactly* s linearly independent resonance relations.

Definition 3.6 (Resonant blocks)**.** *Consider the following sets:*

1. $$B^{(d)} := Z_{\mathbb{Z}^d}^{(d)} \, .$$

2. *Given $M \subset \mathbb{Z}^d$ a resonance module of dimension $s \in \{1, \dots, d-1\}$,*

$$B_M^{(s)} := Z_M^{(s)} \setminus \left\{ \bigcup_{M' s.t. \dim M' = s+1} Z_{M'}^{(s+1)} \right\}$$

3. $$B^{(0)} := Z^{(0)}$$

We say that B is a resonant block if $B = B^{(d)}, B = B^{(0)}$ or $B = B_M^{(s)}$ for some module M of dimension s.

Remark 3.7. *The resonant blocks form a covering of $\mathbb{Z}^d$.*

As proven below in Lemma 5.6 there exists a suitable choice of the constants $C_s,\ D_s,\ \delta_s$ such that two blocks $B_M^{(s)},\ B_{M'}^{(s)}$ are disjoint if $M,\ M'$ are two distinct subspaces of equal dimension.

Still the blocks defined in Definition 3.6 do not provide a suitable partition of $\mathbb{Z}^d$, since they are not left invariant by the operator $\widetilde{H}$ of eq. (2.29).

Recall now that, given two sets A and B, their Minkowski sum $A + B$ is defined by:

$$A + B := \{a + b \quad | \quad a \in A\,,\ b \in B\}\ .$$

Definition 3.8 (Extended blocks)**.**

1. $E^{(0)} := B^{(0)} \equiv Z^{(0)}$

2. *Given a resonance module M of dimension 1, we define*

$$E_M^{(1)} := \left\{ B_M^{(1)} + M \right\} \cap Z_M^{(1)}\,,$$

$$E^{(1)} := \bigcup_{M \text{ of dim. } 1} E_M^{(1)}$$

3. *Given a resonance module M of dimension s, with $2 \le s \le d$, we define*

$$E_M^{(s)} := \left\{ B_M^{(s)} + M \right\} \cap Z_M^{(s)} \cap \bigcap_{j=1}^{s-1} \left(E^{(s-j)} \right)^c\,,$$

where, given $E \subseteq \mathbb{Z}^d$, E^c is the complementary set of E in $\mathbb{Z}^d$. Correspondingly we define

$$E^{(s)} := \bigcup_{M \text{ of dim. } s} E_M^{(s)}\,.$$

Remark 3.9. *The blocks* $\{E_M^{(s)}\}_{M,s}$, $E^{(0)}$, $E^{(d)}$ *form a covering of* $\mathbb{Z}^d$*. Actually, as shown in Theorem 5.8 below, they form a partition of* $\mathbb{Z}^d$*.*

It turns out that the decomposition $\mathbb{Z}^d = \bigcup_M E_M^{(s)}$ is invariant for the operator $\widetilde{H}$ of Theorem 2.18. Furthermore the sets $E_M^{(s)}$ can still be decomposed in invariant subsets which are given by

$$W_{M,\beta} := E_M^{(s)} \cap (\beta + M) \ , \tag{3.8}$$

Definition 3.10. *The set of the* $\beta \in M^{(c)}$ *s.t. the set* (3.8) *is not empty is denoted by* $\widetilde{M}$*.*

Theorem 3.11. *The sets* $W_{M,\beta}$ *of Theorem 2.18 are the sets defined by equation* (3.8) *.*

3.2 Iteration and Spectral Theorem

Theorem 2.18, allows to conjugate, up to smoothing operators, the operator H to a sequence of lower dimensional Schrödinger operators, the majority of which is trivial (there are infinitely many Fourier multipliers and one finite dimensional operator). In order to study the nontrivial Schrödinger operators one can apply again Theorem 2.18 to the operators of eq. (2.30). In this way one can conjugate each of these operators to Schrödinger operators of lower dimension. Iterating further and further, one is finally reduced to either finite dimensional operators or Fourier multipliers.

Remark 3.12. *The Schrödinger operators of eq.* (2.30) *act on* $\mathbb{T}^{d'}$ *and the corresponding symbols, written in coordinates adapted to* M *depend only on the first* d' *variables (both* x *and* ξ*). If one looks at such a symbol as the symbol of an operator on the original torus, namely as a function in* $\mathcal{C}^\infty(T^*\mathbb{T}^d)$*, then one has that taking derivatives with respect to the* ξ *variables does not improve the decay in the directions of the variables which are not present in the symbol, namely* $(\xi^{d'+1}, ..., \xi^d)$*. For this reason we will get that some eigenvalues (these are the unstable eigenvalues of [FKT91]) have asymptotics with only a directional decay.*

Directional decay is captured by the following definition which avoids the introduction of adapted coordinates.

Definition 3.13. *Let* $m \leq 0$*, and let* $M \subset \mathbb{Z}^d$ *be a proper submodule, we say that* $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$ *is a symbol of order* m *in the direction* M *if* $\forall N_1, N_2 \in \mathbb{N}^d$ *there exists a constant* $C_{N_1,N_2} > 0$ *such that*

$$\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \leq C_{N_1,N_2} \langle (\xi+\kappa)_M \rangle_g^{m-\delta N_2} \quad \forall x \in \mathbb{T}^d, \ \forall \xi \in \mathbb{R}^d \,. \tag{3.9}$$

In this case we will write $a \in S_M^{m,\delta}$*.*

Definition 3.14. *Given a module $M \subset \mathbb{Z}^d$, a sequence of symbols $m_j \in S_M^{-2j\delta,\delta}$, $j \geq 0$, depending only on ξ and a function $m(\xi)$, possibly defined only on $\mathbb{Z}^d$ or on a subset E of $\mathbb{Z}^d$, we write*

$$m \overset{M}{\sim} \sum_j m_j \ , \tag{3.10}$$

if for any N there exists C_N s.t.

$$\left| m(\xi) - \sum_{j=0}^{N} m_j(\xi) \right| \leq \frac{C_N}{\langle (\xi+\kappa)_M \rangle_g^{(N+1)2\delta}} \ . \tag{3.11}$$

Definition 3.15. *A sequence of moduli*

$$\mathbb{Z}^d \supset M^{(1)} \supset ... \supset M^{(r)} \ , \quad \dim M^{(j)} = d_j \ , \tag{3.12}$$

will be said to be admissible if

$$d_r \leq d_{r-1} < d_{r-2} < ... < d_1 \leq d \ , \tag{3.13}$$

and either $d_r = d_{r-1}$ or $d_r = 0$ (namely the sequence ends when either the last module coincides with the previous one or it consists of $\{0\}$).

The number r will be called the length of the sequence.

We will denote by $\mathcal{M}od$ the set of all admissible sequences of moduli.

We also denote $\vec{M} := (M^{(1)}, ..., M^{(r)})$.

Let now $\vec{M} \in \mathcal{M}od$, then for any j consider a module $\left(M^{(j)}\right)^{(c)}$ complementary to $M^{(j)}$ in $M^{(j-1)}$, namely a module such that

$$M^{(j)} + \left(M^{(j)}\right)^{(c)} = M^{(j-1)} \ , \quad M^{(j)} \cap \left(M^{(j)}\right)^{(c)} = \{0\} \ ,$$

then the above construction forces to use also subsets

$$\widetilde{M}^{(j)} \subset \left(M^{(j)}\right)^{(c)} \ .$$

We denote

$$\vec{\widetilde{M}} := (\widetilde{M}^{(1)}, ..., \widetilde{M}^{(k)}) \ ,$$

then the sequence of normalizations that one performs is determined by the couple $(\vec{M}, \vec{\beta})$ with $\vec{\beta} \equiv (\beta_1, ..., \beta_k) \in \vec{\widetilde{M}}$.

Theorem 3.16. *There exists a bijective map*

$$\mathbb{Z}^d \ni \xi \mapsto \lambda_\xi \in \sigma(H) \ , \tag{3.14}$$

(the eigenvalues being counted with multiplicity) with the following properties:

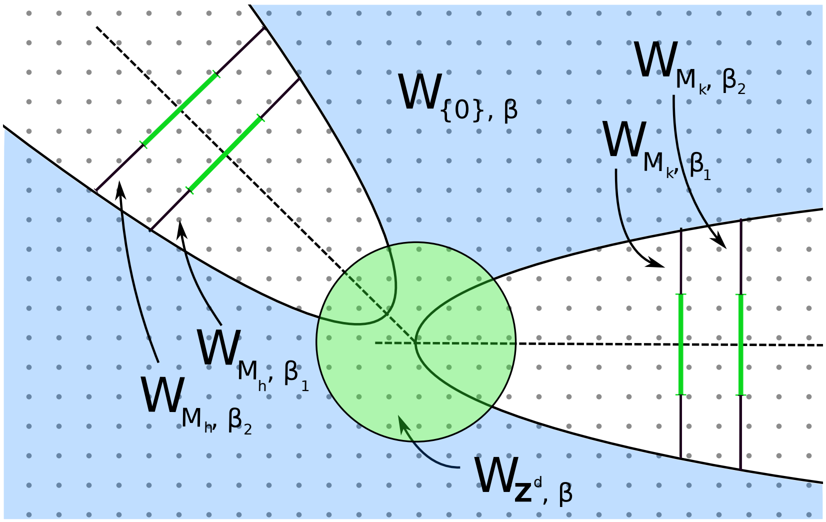

Figure 1: A cartoon of the block decomposition described in Theorem 2.18, in the case $d = 2$, with $\kappa = 0$ and g the Euclidean metric.

(i) there exists a constant $a \in (0, 1)$, s.t. $\forall \mathtt{N} \in \mathbb{N}$ there exists a constant $C_{\mathtt{N}}$ s.t. the eigenfunction ϕ_ξ corresponding to λ_ξ fulfills

$$\|\phi_\xi\|_{H^{-\mathtt{N}}} \leq \frac{C_{\mathtt{N}}}{\lambda_\xi^{a\mathtt{N}}} , \tag{3.15}$$

(ii) There exists a partition

$$\mathbb{Z}^d = \bigcup_{\vec{M}\in\mathcal{M}ad} \bigcup_{\vec{\beta}\in\vec{M}^{\sim}} W_{\vec{M},\vec{\beta}} ,$$

and for any $(\vec{M}, \vec{\beta})$ with $\vec{M} \in \mathcal{M}ad$ and $\vec{\beta} \in \vec{M}^{\sim}$ there exists a sequence of x independent symbols $\{m^{(j)}_{\vec{M},\vec{\beta}}\}_{j\in\mathbb{N}}$, $m^{(j)}_{\vec{M},\vec{\beta}} \in S^{-2\delta j,\delta}_{M^{(r-1)}}$ $\forall j$, with the following property. If $\xi \in W_{\vec{M},\vec{\beta}}$, then λ_ξ which admits the asymptotic expansion

$$\lambda_\xi \stackrel{M^{(r-1)}}{\sim} \|\xi + \kappa\|^2_{g*} + \sum_{j\in\mathbb{N}} m^{(j)}_{\vec{M},\vec{\beta}}(\xi) , \tag{3.16}$$

where r is the length of the sequence $\vec{M}$. The operator H does not have other eigenvalues. Furthermore, the constants C_N of (3.11) are uniform with respect to the choice of the pair $(\vec{M}, \vec{\beta})$.

The situation is illustrated in Figure 1 in the case $d = 2$, $\kappa = 0$ and g the Euclidean metric. In such a case either $M = \mathbb{Z}^2$, or $M = \{0\}$ or

$M = M_k = \text{span}_{\mathbb{Z}}\{k\}$ for some $k \in \mathbb{Z}^2$. In the figure only the resonant zones corresponding to $k = (0,1)$ and $k = (1,1)$ are plotted. In the blocks corresponding to $M = \{0\}$ (in blue) the normal form operator is a Fourier multiplier and here one gets the standard asymptotic expansion. The block corresponding to $M = \mathbb{Z}^2$ (in green) has finite dimension. In the other blocks (white region) one can apply again Theorem 2.18 getting finite dimensional sub-blocks (green segments) with uniformly bounded dimension plus trivial blocks (in the black segments). When adding the remainder one gets the directional asymptotics with decay in the direction of the segments.

PART II: Proofs

4 Quantum normal form

In this section we give a variant of the normal form construction of [BLM20] and [Par08, PS10, PS12] suitable for our goal. From now on we will drop the index g or g^* from the notation of the scalar products and of the norms. Of course the scalar products of vectors will be computed using the metric g_{AB} and those of covectors using the metric g^{AB}. In particular the scalar products involved in the definitions of resonance are always between covectors.

Definition 4.1. *A symbol* $N(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{N}_k(\xi)e^{ik\cdot x} \in S^{0,\delta}$ *is said to be in (resonant) normal form if,* $\forall k \in \mathbb{Z}^d\backslash\{0\}$,

$$supp(\hat{N}_k) \subseteq \left\{ \xi \in \mathbb{R}^d \;\middle|\; |(\xi+\kappa,k)| \le \langle \xi+\kappa\rangle^{\delta} \|k\|^{-\tau} \text{ and } \|k\| \le \langle \xi+\kappa\rangle^{\epsilon} \right\}. \tag{4.1}$$

Definition 4.2. *Let* $M \subset \mathbb{Z}^d$ *be a module, then a symbol* $N \in S^{0,\delta}$ *is said to be in* normal form with respect to M *if it is in normal form and furthermore its Fourier transform is given by*

$$N(x,\xi) = \sum_{k\in M} \hat{N}_k(\xi)e^{ik\cdot x}. \tag{4.2}$$

Definition 4.3. *A pseudodifferential operator will be said to be in normal form (resp. normal form with respect to a module M) if the corresponding symbol is in normal form (resp. normal form with respct to a module M).*

Lemma 4.4. *In dimension one (namely if $d = 1$) operators in normal form are smoothing, namely of class $OPS^{-\infty,\delta}$.*

Proof. Let $\mathcal{N}$ be in normal form then, $\forall k \in \mathbb{Z}$

$$\hat{N}_k(\xi) \neq 0 \Rightarrow |\,(\xi+\kappa,k)\,| \leq \langle \xi+\kappa\rangle^{\delta}\|k\|^{-\tau} \leq \langle \xi+\kappa\rangle^{\delta}\mathfrak{c}^{-\frac{\tau}{2}}\,.$$

Since $\xi+\kappa \parallel k$, it follows that

$$\begin{aligned}\|\xi+\kappa\| &\leq \mathfrak{c}^{-\frac{1}{2}}\|k\|\|\xi+\kappa\| \\ &= \mathfrak{c}^{-\frac{1}{2}}|\,(\xi+\kappa,k)\,| \\ &\leq \mathfrak{c}^{-\frac{(1+\tau)}{2}}\langle \xi+\kappa\rangle^{\delta}\end{aligned}$$

which, by $\delta<1$, implies the existence of a constant C such that $\|\xi+\kappa\| \leq C$. □

Definition 4.5. *Given a symbol a we define its average by*

$$\langle a\rangle(\xi) = \frac{1}{\mu_g(\mathbb{T}^d)}\int_{\mathbb{T}^d} a(x,\xi)\ d\mu_g(x)\,.$$

If $A = Op^W(a)$, we denote $\langle A\rangle(D) = Op^W(\langle a\rangle(\xi))$.

The following result is just a small modification of Theorem 5.1 of [BLM20] (which in turn is a variant of Theorem 4.3 of [PS10] and Theorem 9.2 of [PS12].).

Theorem 4.6. *Consider the operator $H = -\Delta_{g,\kappa} + \mathcal{V}$, with $\mathcal{V} = Op^W(V) \in OPS^{\,0,\delta}$. For all $\mathtt{N} > 0$ there exists a unitary transformation $U_{\mathtt{N}}$ such that*

1)

$$U_{\mathtt{N}} - \mathrm{Id}\,,\ U_{\mathtt{N}}^{-1} - \mathrm{Id} \in OPS^{\,-\delta,\delta} \tag{4.3}$$

$$U_{\mathtt{N}} H U_{\mathtt{N}}^{-1} = \mathcal{L}^{(\mathtt{N})} = \tilde{H}^{(\mathtt{N})} + \mathcal{R}^{(\mathtt{N})} \tag{4.4}$$

with $\mathcal{R}^{(\mathtt{N})} \in OPS^{-\mathtt{N},\delta}$, and

$$\tilde{H}^{(\mathtt{N})} = -\Delta_{g,\kappa} + \mathcal{N}^{(\mathtt{N})}\,, \tag{4.5}$$

where $\mathcal{N}^{(\mathtt{N})} \in OPS^{\,0,\delta}$ is in resonant normal form.
Furthermore, the families of seminorms of the operators $\mathcal{N}^{(\mathtt{N})}, \mathcal{R}^{(\mathtt{N})}, U_{\mathtt{N}}$ only depend on the family of seminorms of the operator $\mathcal{V}$ and on the constants of the metric, as well as on $\mathtt{N}$, on d and on the parameters δ, ϵ, τ.

2) Let $E \subset \mathbb{Z}^d$ be a subset and let $\mathcal{E}$ be the space it generates according to (2.17). If $\mathcal{V}$ leaves $\mathcal{E}$ invariant, namely $[\mathcal{V}, \Pi_{\mathcal{E}}] = 0$, then one has

$$[U_{\mathtt{N}}\,, \Pi_{\mathcal{E}}] = 0\,. \tag{4.6}$$

Remark 4.7. *Assume that $\mathcal{V}$ leaves invariant a subspace $\mathcal{E}$ of the form* (2.17)*. Then, by Item* 2 *of Theorem 4.6, one also has*

$$U_{\mathtt{N}}\Pi_{\mathcal{E}}H\Pi_{\mathcal{E}}U_{\mathtt{N}}^{-1} = \Pi_{\mathcal{E}}\mathcal{L}^{(\mathtt{N})}\Pi_{\mathcal{E}}\,.$$

The proof of Theorem 4.6 is a small variant of the proof of Theorem 5.1 of [BLM20] (and of Theorem 4.3 of [PS10]), so here it will only be sketched.

The proof is obtained working at the level of the symbols and is based on a decomposition that we now recall. First consider an even function $\chi : \mathbb{R} \to [0,1]$ with the property that $\chi(t) = 1$ for all t with $|t| \leq \frac{1}{2}$ and $\chi(t) = 0$ for all t with $|t| \geq 1$.

Definition 4.8. *Given $\epsilon, \delta > 0$ and $\tau > d-1$ as in Theorem 2.18, define the following functions:*

$$\chi_k(\xi) = \chi\left(\frac{2\|k\|^{\tau}(\xi+\kappa, k)}{\langle \xi+\kappa\rangle^{\delta}}\right)\ , \quad k \in \mathbb{Z}^d\backslash\{0\},$$
$$\tilde{\chi}_k(\xi) = \chi\left(\frac{\|k\|}{\langle \xi+\kappa\rangle^{\epsilon}}\right)\ , \quad k \in \mathbb{Z}^d\backslash\{0\}\ .$$

Correspondingly, given a symbol $w \in S^{0,\delta}$, we decompose it as follows:

$$w = \langle w\rangle + w^{(\mathrm{nr})} + w^{(\mathrm{res})} + w^{(S)}\,, \tag{4.7}$$

where $\langle w\rangle$ is the average symbol of w, and

$$w^{(\mathrm{res})}(x,\xi) = \sum_{k\neq 0} \hat{w}_k(\xi)\chi_k(\xi)\tilde{\chi}_k(\xi)e^{ik\cdot x}\,,$$
$$w^{(\mathrm{nr})}(x,\xi) = \sum_{k\neq 0} \hat{w}_k(\xi)\left(1-\chi_k(\xi)\right)\tilde{\chi}_k(\xi)e^{ik\cdot x}\,,$$
$$w^{(S)}(x,\xi) = \sum_{k\neq 0} \hat{w}_k(\xi)\left(1-\tilde{\chi}_k(\xi)\right)e^{ik\cdot x}\,.$$

As proved in Lemma 5.6 of [BLM20], the functions $\langle w\rangle\,, w^{(\mathrm{nr})}\,, w^{(\mathrm{res})}$ are symbols of the same order of w while $w^{(S)} \in S^{-\infty,\delta}$.

Sketch of the proof of Thm 4.6. Following [BLM20], Part 1 is proved iteratively: consider an operator of the form

$$H_j = -\Delta_{g,\kappa} + \mathcal{N}^{(j)} + \mathcal{R}^{(j)}\,,$$

with $\mathcal{N}^{(j)} \in OPS^{0,\delta}$ in resonant normal form and $\mathcal{R}^{(j)} \in OPS^{-2j\delta,\delta}$. We look for a pseudodifferential operator $\mathcal{G}_j$ such that

$$e^{i\mathcal{G}_j}H_j e^{-i\mathcal{G}_j} = H_{j+1}.$$

To this end remark that, by standard pseudodifferential calculus, one has

$$e^{i\mathcal{G}_j} H_j e^{-i\mathcal{G}_j} = -\Delta_{g,\kappa} - \mathrm{i}[-\Delta_{g,\kappa}; \mathcal{G}_j] + \mathcal{N}^{(j)} + \mathcal{R}^{(j)} + \text{lower order terms} .$$

If G_j is the symbol of $\mathcal{G}_j$ and R_j the symbol of $\mathcal{R}_j$, then the symbol of $\mathrm{i}[-\Delta_{g,\kappa}; \mathcal{G}_j] + \mathcal{R}^{(j)}$ is, using the decomposition of definition 4.8,

$$\left\{ \|\xi+\kappa\|^2; G_j \right\} + \langle R_j \rangle + R_j^{(\mathrm{nr})} + R_j^{(\mathrm{res})} + R_j^{(S)} .$$

This is in normal form up to smoothing terms if G_j is chosen in such a way that the following equation is fulfilled

$$\{ \|\xi+\kappa\|^2 , \ G_j \} + R_j^{(\mathrm{nr})} = 0 \ ;$$

this equation is fulfilled if G_j is defined by

$$G_j(\xi, x) = -\sum_{k\neq 0} \frac{\widehat{(R_j^{(\mathrm{nr})})}_k(\xi)}{2i(\xi+\kappa)\cdot k} e^{ik\cdot x} . \tag{4.8}$$

Using such a G_j to generate the corresponding unitary transformation and iterating, one gets the proof of part 1 of the Theorem.

To prove part 2, namely the commutation relation (4.6), we proceed inductively. First of all we observe that, in the case $j = 0$, $\mathcal{N}^{(0)} = 0$ and

$$[\mathcal{R}^{(0)}, \Pi_{\mathcal{E}}] = [\mathcal{V}, \Pi_{\mathcal{E}}] = 0 .$$

Let us now fix some $j \geq 0$ and suppose that $\mathcal{N}^{(j)}$ and $\mathcal{R}^{(j)}$ commute with $\Pi_{\mathcal{E}}$.

Given a self-adjoint operator A, since $\mathcal{E}$ has the form $\mathcal{E} = \overline{\mathrm{span}\{e^{ik\cdot x} \mid k \in E\}}$, the condition $[A, \Pi_{\mathcal{E}}] = 0$ holds if and only if

$$\left(k \in E \ , \quad A_k^{k'} \neq 0 \right) \Rightarrow k' \in E , \tag{4.9}$$

where $A_k^{k'} = \frac{1}{\mu_g(\mathbb{T}^d)} \langle A e^{ik\cdot x}, \ e^{ik'\cdot x} \rangle$ are the matrix elements of A with respect to the basis of the Fourier modes. Furthermore, by definition of Weyl quantization one has that, if $A = Op^W(a)$,

$$A_k^{k'} = \hat{a}_{k'-k} \left(\frac{k+k'}{2} \right) . \tag{4.10}$$

Due to definitions (4.7) of the symbols $R_j^{(\mathrm{nr})}$ and $R_j^{(\mathrm{res})}$, equation (4.10) immediately implies that

$$\left(Op^W(R_j^{(\mathrm{nr})})\right)_k^{k'} \neq 0\,, \text{ or } \left(Op^W(R_j^{(\mathrm{res})})\right)_k^{k'} \neq 0 \text{ for some } k, k' \in \mathbb{Z}^d\,, \\ \Longrightarrow (\mathcal{R}^{(j)})_k^{k'} \neq 0\,,$$

Similarly,

$$(\mathcal{G}_j)_k^{k'} \neq 0 \quad \Longrightarrow (\mathcal{R}^{(j)})_k^{k'} \neq 0\,.$$

This, together with condition (4.9), enables to conclude that $\mathcal{G}_j$ commutes with $\Pi_{\mathcal{E}}$, and so do $Op^W(R_j^{(\mathrm{res})})$ and $Op^W(R_j^{(\mathrm{nr})})$. Hence $e^{-i\mathcal{G}_j}$ commutes with $\Pi_{\mathcal{E}}$, since $\mathcal{G}_j$ does. The same holds for $\mathcal{N}^{(j+1)}$, since

$$[\mathcal{N}^{(j+1)}, \Pi_{\mathcal{E}}] = [\mathcal{N}^{(j)}, \Pi_{\mathcal{E}}] + [Op^W(\langle R_j \rangle), \Pi_{\mathcal{E}}] + [Op^W(R_j^{(\mathrm{res})}), \Pi_{\mathcal{E}}] = 0\,,$$

and

$$\mathcal{R}^{(j+1)} = e^{i\mathcal{G}_j} H_j e^{-i\mathcal{G}_j} - \left(-\Delta_{g,\kappa} + \mathcal{N}^{(j+1)}\right)\,.$$

□

5 Geometric part

In order to iterate Theorem 2.18 we will have to work in a subspace of L^2 generated by some subset $E \subset \mathbb{Z}^d$. From now on we will fix $E \subseteq \mathbb{Z}^d$ and develop all the proofs taking ξ as a variable in E. Accordingly to this, we will replace the extended blocks $E_M^{(s)}$ of Definition (3.8) with $E_M^{(s)} \cap E$, which we still denote by $E_M^{(s)}$. We will do the same for the blocks $B_M^{(s)}$ and for the zones $Z_M^{(s)}$ of Definitions 3.2, 3.6.

5.1 Properties of the extended blocks $E_M^{(s)}$, non overlapping of resonances.

We show here that the extended blocks $E_M^{(s)}$ form a partition of E and prove some properties which are needed in order to show that they are left invariant by an operator in normal form. As in the proof of the classical Nekhoroshev Theorem, the following Lemma plays a fundamental role.

Lemma 5.1. *Let $s \in \{1, \dots, d\}$ and let $\{u_1, \dots u_s\}$ be linearly independent vectors in $\mathbb{R}^d$. Let $w \in \mathrm{span}\{u_1, \dots u_s\}$ be any vector. If α, N are such that*

$$\|u_j\| \leq N \quad \forall j = 1, \dots s\,, \\ |(w; u_j)| \leq \alpha \quad \forall j = 1, \dots s\,,$$

then

$$\|w\| \le \frac{sN^{s-1}\alpha}{Vol_g\{u_1\,|\cdots|u_s\}}\,.$$

This is just a coordinate free formulation of Lemma 5.7 of [Gio03], which is recalled in the appendix as Lemma B.1. By (2.13), one also has that, if $u_j \in \mathbb{Z}^d$, $\forall j = 1, ..., s$, then

$$\|w\| \le sN^{s-1}\alpha\mathfrak{C}^{-1}\,. \tag{5.1}$$

We state now a couple of simple properties of the extended blocks.

Lemma 5.2. *The extended block $E^{(d)}$ is finite dimensional; in particular, there exists a positive $n_* = n_*(\mathfrak{c}, \mathfrak{C}, \epsilon, \tau, \delta_{d-1}, C_{d-1}, D_{d-1})$ such that*

$$E^{(d)} \subseteq \left\{\xi \in \mathbb{R}^d \ \mid \|\xi + \kappa\| \le n_*\right\}\,.$$

Proof. If $\xi \in E^{(d)}$, in particular there exist $\{k_1, \dots, k_d\} \subset \mathbb{Z}^d$ linear independent vectors such that

$$\begin{aligned} \|k_1\| &\le D_0\langle\xi_{k_1}\rangle^\epsilon\,,\\ \|k_j\| &\le D_{j-1}\langle\xi_{k_1}\rangle^\epsilon \le D_{d-1}\langle\xi_{k_1}\rangle^\epsilon\,,\\ |\,(\xi_{k_1}, k_j)\,| &\le C_{d-1}\langle\xi_{k_1}\rangle^{\delta_{d-1}}\|k_j\|^{-\tau}\,. \end{aligned}$$

In order to eliminate the indexes k_1 from ξ, we apply Lemma B.6, with $\varsigma = \eta = \xi^\kappa$, $l = 0$, $h = k_j$ and $k = \frac{k_1}{2}$ to deduce that there exist constants

$$C' = C'(\mathfrak{c}, \epsilon, \tau, \delta_{d-1}, D_{d-1}, C_{d-1})\,, \quad D' = D'(\mathfrak{c}, \epsilon, \tau, \delta_{d-1}, D_{d-1}, C_{d-1})$$

such that

$$|\,(\xi^\kappa, k_j)\,| \le C'\langle\xi^\kappa\rangle^\delta\|k_j\|^{-\tau}\,, \quad \|k_j\| \le D'\langle\xi^\kappa\rangle^\epsilon\,.$$

Recalling that $\mathfrak{c}$ is such that, for all $h \in \mathbb{Z}^d$, $\|h\|^2 \ge \mathfrak{c}$ and using Lemma 5.1, and Eq. (5.1) we have

$$\|\xi^\kappa\| \le d\mathfrak{c}^{-\tau/2}\mathfrak{C}^{-1}C'\,(D')^{d-1}\langle\xi^\kappa\rangle^{d\epsilon+\delta}\,,$$

which, applying Remark B.4 with $a = d\epsilon + \delta < 1$, implies the existence of a constant $n_* = n_*(\delta, \epsilon, \tau, C', D', \mathfrak{c}, \mathfrak{C})$ such that $\|\xi^\kappa\| < \bar{N}$. □

Lemma 5.3. *If $E = \mathbb{Z}^d$, the set $E^{(0)}$ is of density one at infinity, namely*

$$\lim_{R\to\infty} \frac{\sharp\left(E^{(0)} \cap B_R(0)\right)}{\sharp\,(\mathbb{Z}^d \cap B_R(0))} = 1\,.$$

Proof. We exploit the fact that a set is of density one at infinity if and only if its complementary set is of density zero, and we analyze the complementary set of $E^{(0)}$. Recall that $E^{(0)} = Z^{(0)}$ so that, by Definition 3.2, its complementary set is

$$\begin{aligned}\mathbb{Z}^d \backslash E^{(0)} &= \bigcup_{M \text{ of dim. } 1} Z_M^{(1)} \\ &= \{\xi \in \mathbb{Z}^d \mid \exists k \in \mathbb{Z}^d \text{ s. t. } |(\xi_k, k)| \leq \langle \xi_k \rangle^\delta \|k\|^{-\tau}, \ \|k\| \leq \langle \xi_k \rangle^\epsilon\}.\end{aligned}$$

By Lemma B.6 there exists constants C', D' depending only on $\delta, \epsilon, \tau, \mathfrak{c}, \mathfrak{C}$ such that

$$\mathbb{Z}^d \backslash E^{(0)} \subseteq \{\xi \in \mathbb{Z}^d \mid \exists k \in \mathbb{Z}^d \text{ s. t. } |(\xi, k)| \leq C' \langle \xi \rangle^\delta \|k\|^{-\tau}, \ \|k\| \leq D' \langle \xi \rangle^\epsilon\}.$$

But the latter is the complementary set to

$$\Omega = \{\xi \in \mathbb{Z}^d \mid |(\xi_k, k)| > C' \langle \xi_k \rangle^\delta \|k\|^{-\tau} \quad \forall k \in \mathbb{Z}^d \ \text{ s. t. } \|k\| \leq D' \langle \xi_k \rangle^\epsilon\}.$$

Then Proposition 5.9 of [BLM20] gives the result. □

We now analyze the other blocks.

First remark that, if $s' \neq s$, then two extended blocks $E_M^{(s)}$ and $E_{M'}^{(s')}$ are disjoint. Then we have to prove that two different extended blocks of the same dimension do not intersect. To this end a further geometric analysis is required.

Lemma 5.4. *If* $\xi \in Z_M^{(s)}$ *then there exists a positive constant* K *depending only on* $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$*, such that*

$$\|(\xi^\kappa)_M\| \leq K \langle \xi^\kappa \rangle^{\delta_{s-1} + d\epsilon}. \tag{5.2}$$

Proof. Since $\xi \in Z_M^{(s)}$, there exist $\{k_1, \ldots, k_s\} \subset M$ linearly independent vectors such that for all $j = 1, \ldots, s$

$$|((\xi_{k_1})_M, k_j)| = |(\xi_{k_1}, k_j)| \leq C_{j-1} \langle \xi_{k_1} \rangle^{\delta_{j-1}} \|k_j\|^{-\tau}, \quad \|k_j\| \leq D_{j-1} \langle \xi_{k_1} \rangle^\epsilon. \tag{5.3}$$

Then, by Lemma B.6 one can substitute in the above formulae ξ^κ to ξ_{k_1}; precisely, there exist two positive constants $C', \ D' = C', D'(\mathfrak{c}, \epsilon, \delta_{s-1}, C_{s-1}, D_{s-1})$, such that,

$$\begin{gathered}|((\xi^\kappa)_M, k_j)| = |(\xi^\kappa, k_j)| \leq C' \langle \xi^\kappa \rangle^{\delta_{s-1}} \|k_j\|^{-\tau} \leq C' \mathfrak{c}^{-\tau/2} \langle \xi^\kappa \rangle^{\delta_{s-1}}, \\ \|k_j\| \leq D' \langle \xi^\kappa \rangle^\epsilon.\end{gathered}$$

By Lemma 5.1, there exists $C = C(d)$ such that

$$\|(\xi^\kappa)_M\| \le C(d)\frac{(D')^d\langle\xi^\kappa\rangle^{d\epsilon}}{\mathrm{Vol}_g(k_1|\cdots|k_s)}C'\mathfrak{c}^{-\tau/2}\langle\xi^\kappa\rangle^{\delta_{s-1}},$$

and therefore, recalling that $\mathrm{Vol}_g(k_1|\cdots|k_s) \ge \mathfrak{C}$ (see the definition of $\mathfrak{C}$ as in (2.13)), the thesis holds. □

By definition, the points belonging to a block $B_M^{(s)}$ are resonant only with vectors $k \in M$. A priori, this property does not hold true for points in the extended block $E_M^{(s)}$. So we need an estimate of the distance between $E_M^{(s)}$ and $B_M^{(s)}$.

Lemma 5.5. *Let $\delta_{s-1} + d\epsilon < 1$ and M with dim$M = s$; if $\zeta \in E_M^{(s)}$ then there exists $\xi \in B_M^{(s)}$ and a positive constant F depending only on $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$ such that*

$$\|\xi - \zeta\| \le F\langle\xi^\kappa\rangle^{\delta_{s-1}+\epsilon d}, \quad \|\xi - \zeta\| \le F\langle\zeta^\kappa\rangle^{\delta_{s-1}+\epsilon d} \tag{5.4}$$

Proof. If $\zeta \in E_M^{(s)}$, then in particular $\zeta \in Z_M^{(s)}$ and there exists a point $\xi \in B_M^{(s)}$ such that $\zeta = \xi + \upsilon$, with $\upsilon \in M$. In particular, $(\xi)_M^\perp = (\zeta)_M^\perp$, hence one has

$$\|\xi - \zeta\| = \|(\xi - \zeta)_M\| \le \|(\xi^\kappa)_M\|\| + \|(\zeta^\kappa)_M\|.$$

Since $\xi \in Z_M^{(s)}$ and $\zeta \in Z_M^{(s)}$, due to Lemma 5.4, there exists K, such that

$$\|(\xi^\kappa)_M\| \le K\langle\xi^\kappa\rangle^{d\epsilon+\delta_{s-1}}, \quad \|(\zeta^\kappa)_M\| \le K\langle\zeta^\kappa\rangle^{d\epsilon+\delta_{s-1}}. \tag{5.5}$$

Exploiting Remark B.4 with $a = \delta_{s-1} + \epsilon d$, one gets

$$\langle\zeta^\kappa\rangle^a = \langle\xi + \kappa + \zeta - \xi\rangle^a \le K'(\langle\xi^\kappa\rangle^a + \|\zeta - \xi\|^a)$$

and, exploiting Lemma B.5, we immediately get

$$\|\zeta - \xi\| \le F\langle\xi^\kappa\rangle^a .$$

Inverting the role of ξ and ζ one gets the other estimate. □

The next two lemmata ensure that, if the parameters C_j, D_j are suitably chosen for all j, an extended block $E_M^{(s)}$ is far from every resonant zone associated to a lower dimensional module M' which is not contained in M.

Lemma 5.6. *[Non overlapping of resonances] For all $s = 1, \dots d-1$ there exist positive constants $\bar{C}_s$ and $\bar{D}_s$, depending only on $\mathfrak{c}, \mathfrak{C}, d, C_{s-1}, D_{s-1}, \epsilon, \delta_{s-1}, \tau$, such that the following holds: suppose that M and M' are two distinct resonance modules of respective dimensions s and s' with $s' \le s$ and $M' \nsubseteq M$. If*

$$C_s > \bar{C}_s\,, \quad D_s > \bar{D}_s\,,$$

then

$$E_M^{(s)} \cap Z_{M'}^{(s')} = \emptyset\,.$$

Proof. Assume by contradiction, that there exists $\zeta \in E_M^{(s)} \cap Z_{M'}^{(s')}$ then there exists $\xi \in B_M^{(s)}$ s.t. (5.4) holds.

Since $\zeta \in Z_{M'}^{(s')}$, there exist s' integer vectors, $k_1, ..., k_{s'} \in M'$ *among which at least one does not belong to* M s.t.

$$|(\zeta_{k_1}, k_j)| \le C_{j-1} \langle \zeta_{k_1} \rangle^{\delta_{j-1}} \|k_j\|^{-\tau}\;, \quad \|k_j\| \le D_{j-1} \langle \zeta_{k_1} \rangle^{\epsilon}\,. \tag{5.6}$$

Let $k_{\bar{\jmath}}$ be the vector which does not belong to M; the idea is to show that the resonance relation of ζ with $k_{\bar{\jmath}}$ implies an analogous relation for ξ, but this will be in contradiction with the fact that $\xi \in B_M^{(s)}$ (which contains vectors which are *only* resonant with M).

To start with remark that, since $\xi \in B_M^{(s)} \subset Z_M^{(s)}$, there exist $l_1, ..., l_s \in M$, linearly independent, s.t.

$$|(\xi_{l_1}, l_j)| \le C_{j-1} \langle \xi_{l_1} \rangle^{\delta_{j-1}} \|l_j\|^{-\tau}\;, \quad \|l_j\| \le D_{j-1} \langle \xi_{l_1} \rangle^{\epsilon}\,. \tag{5.7}$$

We now apply Lemma B.6 with $h := k_{\bar{\jmath}}/2$, $\ell := l_1/2$, $\varsigma := \zeta + \kappa$, $\eta := \xi + \kappa$. So, (B.12) implies

$$|(\xi_{l_1}, k_{\bar{\jmath}})| \le K' \langle \xi_{l_1} \rangle^{\delta_{s-1} + \epsilon(d+\tau+1)} \|k_{\bar{\jmath}}\|^{-\tau}\;, \quad \|k_{\bar{\jmath}}\| \le D' \langle \xi_{l_1} \rangle^{\epsilon}\,.$$

But, if $C_s > K'$, $D_s > D'$ and $\delta_s \ge \delta_{s-1} + \epsilon(d + \tau + 1)$, this means that ξ is also resonant with $k_{\bar{\jmath}}$, and thus it belongs to $Z_{M''}^{(s+1)}$ with $M'' := span_{\mathbb{Z}}(M, k_{\bar{\jmath}})$, but this contradicts the fact that $\xi \in B_M^{(s)}$. □

Lemma 5.7. *[Separation of resonances] There exist positive constants $\tilde{C}_s$ and $\tilde{D}_s$ depending only on $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$ such that, if*

$$C_s > \tilde{C}_s\,, \quad D_s > \tilde{D}_s,$$

then the following holds true. Let $\zeta \in E_M^{(s)}$ for some M of dimension $s = 1, \dots, d-1$, and let k' be such that

$$\|k'\| \le \langle \zeta_{k'} \rangle^{\epsilon}\,,$$

then $\forall M' \not\subset M$ *s. t.* $s' := \dim M' \leq s$ *one has*

$$\zeta + k' \notin Z_{M'}^{(s')} \quad .$$

Proof. The proof is very similar to that of Lemma 5.6. Assume by contradiction that $\zeta + k' \in Z_{M'}^{(s')}$ for some $M' \neq M$. It follows that there exist s integer vectors, $k_1, ..., k_{s'} \in M'$ *among which at least one does not belong to* M s.t.

$$|(\zeta_{k_1} + k', k_j)| \leq C_{j-1} \langle \zeta_{k_1} + k' \rangle^{\delta_{j-1}} \|k_j\|^{-\tau} \ , \quad \|k_j\| \leq D_{j-1} \langle \zeta_{k_1} + k' \rangle^{\epsilon} \ . \tag{5.8}$$

Let $k_{\bar{\jmath}}$ be the vector which does not belong to M. By (5.4) there exists $\xi \in B_M^{(s)}$ s.t. $\|\xi - \zeta\| \leq F \langle \xi^{\kappa} \rangle^{\delta_{s-1} + \epsilon d}$. Since in particular $\xi \in Z_M^{(s)}$ there exist $l_1, ..., l_s \in M$, linearly independent, s.t.

$$|(\xi_{l_1}, l_j)| \leq C_{j-1} \langle \xi_{l_1} \rangle^{\delta_{j-1}} \|l_j\|^{-\tau} \ , \quad \|l_j\| \leq D_{j-1} \langle \xi_{l_1} \rangle^{\epsilon} \ . \tag{5.9}$$

We now apply Lemma B.6 with $h := k_{\bar{\jmath}}/2$, $\ell := l_1/2$, $\varsigma := \zeta + \kappa + k'$, $\eta := \xi + \kappa$. The only nontrivial assumption of Lemma B.6 to verify is the first of (B.10). One has

$$\|\xi - \zeta - k'\| \leq \|\xi - \zeta\| + \|k'\| \leq F \|\xi^{\kappa}\|^{\delta_{s-1} + \epsilon d} + \|k'\| \ .$$

To estimate $\|k'\|$ we proceed as follows:

$$\|k'\| \leq D_0 \left\langle \zeta + \kappa + \frac{k'}{2} \right\rangle^{\epsilon} \leq D_0 K \left(\langle \zeta + \kappa \rangle^{\epsilon} + \frac{1}{2^{\epsilon}} \langle k' \rangle^{\epsilon} \right) \ ,$$

where we used eq. (B.4). Using Lemma B.5, we get $\|k'\| \leq K'' \langle \zeta + \kappa \rangle^{\epsilon}$ and therefore

$$\|\xi - \zeta - k'\| \leq K \|\xi^{\kappa}\|^{\delta_{s-1} + \epsilon d}$$

Thus (B.12) implies

$$|(\xi_{l_1}, k_{\bar{\jmath}})| \leq K' \langle \xi_{l_1} \rangle^{\delta_{s-1} + (d + \tau + 1)\epsilon} \|k_{\bar{\jmath}}\|^{-\tau} \ , \quad \|l_1\| \leq D' \langle \xi_{l_1} \rangle^{\epsilon} \ .$$

But, if $C_s > K'$, $D_s > D'$, this means that ξ is also resonant with $k_{\bar{\jmath}}$, and thus it belongs to $Z_{M''}^{(s+1)}$ with $M'' := span_{\mathbb{Z}}(M, k_{\bar{\jmath}})$, and this contradicts the fact that $\xi \in B_M^{(s)}$. □

The following theorem summarizes the result of this subsection

Theorem 5.8. *Under the hypotheses of Theorem 5.10, the blocks* $E^{(0)}$, $E^{(d)}$, $\{E_M^{(s)}\}_{s,M}$ *are a partition of* E. *Furthermore* $E^{(d)}$ *has dimension less then* $n_* < \infty$, *with* n_* *only depending on* $\mathfrak{c}, \mathfrak{C}, \delta, \epsilon, \tau$ *and, if* $E = \mathbb{Z}^d$, $E^{(0)}$ *is of density 1 at infinity.*

Proof. Let M_1 and M_2 be two submoduli of respective dimension s_1 and s_2. If $s_1 > s_2$, by definition of the extended blocks one has $E_{M_1}^{(s_1)} \cap E_{M_2}^{(s_2)} = \emptyset$. Let then $s_1 = s_2$: by Lemma 5.6,

$$E_{M_1}^{(s_1)} \cap Z_{M_2}^{(s_2)} = \emptyset \,,$$

hence, being $E_{M_2}^{(s_2)} \subseteq Z_{M_2}^{(s_2)}$, it follows that $E_{M_1}^{(s_1)}$ and $E_{M_2}^{(s_2)}$ have no intersection. □

5.2 Invariance of the sets $E_M^{(s)}$.

Consider now an operator of the form

$$\mathcal{L} = \tilde{H} + \mathcal{R} \,, \tag{5.10}$$

$$\tilde{H} := -\Delta_{g,\kappa} + \mathcal{N} \,, \quad \mathcal{R} \in OPS^{-\mathtt{N}\delta,\delta} \tag{5.11}$$

with $\mathcal{N}$ in resonant normal form. Since a Fourier multiplier like $-\Delta_{g,\kappa}$, leaves invariant any set of the form (2.17), we focus on $\mathcal{N}$ only.

Remark that, in order to study if a set is invariant, we have to analyze the indices $\xi, \zeta \in E \subset \mathbb{Z}^d$ s.t.

$$\langle \mathcal{N} e^{i\xi\cdot x} \,, e^{i\zeta\cdot x} \rangle \neq 0 \,.$$

Lemma 5.9. *Let $\mathcal{N} = Op^W(N)$, $N(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{N}_k(\xi) e^{ik\cdot x}$, be a normal form operator; let M be a submodule with $dim M \geq 1$, then*

$$\xi \in E_M^{(s)} \quad \Longrightarrow \mathcal{N}[e^{i\xi\cdot x}] = \sum_{k\in M} \hat{N}_k \left(\xi + \frac{k}{2}\right) e^{i(k+\xi)\cdot x} \quad . \tag{5.12}$$

Proof. By the definition of Weyl quantization one has

$$\mathcal{N}[e^{i\xi\cdot x}] = \sum_{k\in\mathbb{Z}^d} \hat{N}_k \left(\xi + \frac{k}{2}\right) e^{i(\xi+k)\cdot x} \,.$$

In particular, given $\xi \in \mathbb{Z}^d$,

$$\langle \mathcal{N}[e^{i\xi\cdot x}] \,, e^{i(\xi+k)\cdot x} \rangle \neq 0$$

implies that, either $k = 0$, or

$$\left(\xi + \frac{k}{2}\right) \in \operatorname{supp}(\hat{N}_k) \,.$$

Assume now by contradiction that $\exists k \notin M$ s.t. $\hat{N}_k\left(\xi + \frac{k}{2}\right) \neq 0$; since N is in normal form this implies in particular

$$|(\xi_k, k)| \leq \langle \xi_k \rangle^{\delta} \ , \quad \|k\| \leq \langle \xi_k \rangle^{\epsilon} \ ,$$

which means that, defining $M' := \text{span}_{\mathbb{Z}} k$, that $\xi \in Z^{(1)}_{M'}$, with $M' \not\subset M$. This conclusion however is in contradiction with the conclusion of Lemma 5.6. $\square$

The main result of this subsection is the following theorem.

Theorem 5.10. *Let $E \subset \mathbb{Z}^d$ and let $\mathcal{E} \subset L^2(\mathbb{T}^d)$ be the corresponding subset of L^2. There exists a choice of the constants $C_1, \dots, C_{d-1}$, $D_1, \dots, D_{d-1}$ in Definition 3.2 and in Equation (4.1) such that $\forall s, M$ the set $\mathcal{E}^{(s)}_M$ is left invariant by an operator $\mathcal{N}$ in normal form, namely: if $\zeta \in E^{(s)}_M$ and $\langle \mathcal{N}[e^{i\zeta\cdot x}], e^{i\xi\cdot x}\rangle \neq 0$, then $\xi \in E^{(s)}_M$. Furthermore, in such a case one has*

$$\zeta - \xi \in M \, . \tag{5.13}$$

Furthermore, the constants $C_1, \dots, C_{d-1}$ and $D_1, \dots, D_{d-1}$ depend on the parameters $d, \epsilon, \delta, \tau, \mathfrak{c}, \mathfrak{C}$ only.

Proof. Take $\zeta \in E^{(s)}_M$, assume that ξ is such that

$$\langle e^{i\xi\cdot x}; \mathcal{N} e^{i\zeta\cdot x}\rangle \neq 0 \ . \tag{5.14}$$

First we remark that, by Lemma 5.9, one has

$$\mathcal{N} e^{i\zeta\cdot x} = \sum_{k\in M} \hat{N}_k\left(\zeta + \frac{k}{2}\right) e^{i(\zeta+k)\cdot x} \ ,$$

so, in particular

$$(5.14) \implies \xi - \zeta \in M$$

and also

$$\xi = \zeta + k \ , \quad \|k\| \leq \langle \zeta_k \rangle^{\epsilon} \ . \tag{5.15}$$

We now proceed in proving that (5.14) also implies $\xi \in E^{(s)}_M$.

First, if $M = \{0\}$, then, by the very definition of normal form, $\mathcal{N}$ acts as a Fourier multiplier on $E^{(0)}$, and thus in particular it is diagonal and leaves it invariant. Furthermore, $E^{(0)}$ decomposes into invariant subspaces. Each one of these subspaces is just a single point of $\mathbb{Z}^d = M^{(c)}$.

In order to prove the result for higher values of s, we first remark that

$$E^{(s)}_M = \left(\left\{B^{(s)}_M + M\right\} \cap Z^{(s)}_M\right) \setminus \left(\bigcup_{r<s} E^{(s)}\right) \ .$$

From (5.15) it follows that $\xi \in E_M^{(s)} + M \subset B_M^{(s)} + M$. We are going to prove by induction on s that $\xi \in Z_M^{(s)}$ and that it also belongs to the complement of $\bigcup_{r<s} E^{(s)}$.

We know the result is true for $s = 0$. By induction we have that if $\zeta \in E_M^{(s-1)}$ then $\xi \in E_M^{(s-1)}$, and therefore also $\xi \in Z_M^{(s-1)}$; we prove now that if $\zeta \in E_M^{(s)}$ then $\xi \in Z_M^{(s)}$. Assume by contradiction that this is not true. Since the sets $\{E_{\tilde{M}}^{(\tilde{s})}\}_{\tilde{s},\tilde{M}}$ form a partition, then there exists s', and $M' \neq M$ s.t. $\xi \in E_{M'}^{(s')} \subset Z_{M'}^{(s')}$.

There are three cases

1) $s' = s$. Then, by (5.15), one can apply Lemma 5.7, which implies

$$\xi \notin Z_{M'}^{(s)} , \quad \text{unless } M = M' .$$

Thus this case is not possible.

2) $s' > s$. By Remark (3.4), and item 1), this implies $\xi \in Z_M^{(s)}$, against the contradiction assumption.

3) $s' < s$. Just remark that (5.14) is equal to

$$\langle e^{i\xi\cdot x}; \mathcal{N} e^{i\zeta\cdot x}\rangle = \langle \mathcal{N} e^{i\xi\cdot x}; e^{i\zeta\cdot x}\rangle \neq 0 , \tag{5.16}$$

but the inductive assumptions says that $E_{M'}^{(s')}$ is invariant for $s' < s$, thus (5.16) implies $\zeta \in E_{M'}^{(s')}$ which is impossible since the extended blocks form a partition.

Thus we have $\zeta \in E_M^{(s)}$ then $\xi \in \left\{ B_M^{(s)} + M \right\} \cap Z_M^{(s)}$. Then by induction, using (5.16), $\xi \in E_{M'}^{(s')}$, $s' < s$, implies $\zeta \in E_{M'}^{(s')}$ and thus $\zeta \in E_M^{(s)}$ implies $\xi \notin E_{M'}^{(s')}$, $\forall s' < s$, and this concludes the proof. □

By equation (5.13), each extended block is foliated in equivalence classes left invariant by an operator in normal form. *We define the sets $W_{M,\beta}$ of Theorem 2.18 to be such equivalence classes.* We are now going to show that they are labeled by β in a subset of $M^{(c)}$. First remark that, if $\xi \in E_M^{(s)}$, there exists $W_{M,\beta}$ s.t. $\xi \in W_{M,\beta}$ and then one has

$$W_{M,\beta} \subset \xi + M .$$

Introduce now a basis adapted to M, then, since $\mathbb{Z}^d = M + M^{(c)}$, for any equivalence class there exists $\beta \in M^{(c)}$ s.t. $W_{M,\beta} \subset \beta + M$. Conversely, given $\beta \in M^{(c)}$ we define

$$W_{M,\beta} := (\beta + M) \cap E_M^{(s)} ,$$

which is possibly empty. Following Definition 3.10, $\widetilde{M}$ is the subset of the β's s.t. $W_{M,\beta}$ is not empty.

We have thus established the following Corollary.

Corollary 5.11. *The partition* $\{W_{M,\beta}\}_{M\subseteq\mathbb{Z}^d,\beta\in M^{(c)}}$ *just defined is left invariant by any operator in normal form.*

5.3 Dimensional reduction

We analyze now the restriction of $\widetilde{H}$ to each invariant set. Thus consider

$$\tilde{H}_{M,\beta} \equiv \Pi_{\mathcal{W}_{M,\beta}} \left(-\Delta_{g,\kappa} + \mathcal{N}_M\right) \Pi_{\mathcal{W}_{M,\beta}} , \tag{5.17}$$

with

$$\mathcal{N}_M = Op^W(N_M) , \quad N_M(x,\xi) = \sum_{k\in M} \hat{N}_k(\xi) e^{ik\cdot x} , \tag{5.18}$$

in normal form.

Given $\xi \in W_{M,\beta}$, let $\tilde{\xi}$ and κ' be defined as in (2.20), namely

$$\tilde{\xi} = \xi - \lfloor(\xi+\kappa)_M\rfloor , \quad \kappa' = \{(\xi+\kappa)_M\} ,$$

and recall that, as pointed out in Remark 2.14, one has $\tilde{\xi} = \tilde{\beta}$. Thus, defining

$$\zeta := \lfloor(\xi+\kappa)_M\rfloor , \quad \ell^2 := \|(\tilde{\beta}+\kappa)_{M^\perp}\|^2 , \tag{5.19}$$

one has

$$\xi = \zeta + \tilde{\beta} , \quad (\xi+\kappa)_M = \zeta + \kappa' , \tag{5.20}$$

$$(\xi+\kappa)_{M^\perp} = (\tilde{\beta}+\kappa)_{M^\perp} , \tag{5.21}$$

$$\|\xi+\kappa\|^2 = \|\zeta+\kappa'\|^2 + \ell^2 . \tag{5.22}$$

Remark 5.12. *Consider the translation* $W_{M,\beta} \ni \xi \mapsto \zeta = \xi - \tilde{\beta} \in W^t_{M,\beta} \subset M$*; as pointed out in Remark 2.15, its quantization is the Gauge transformation* $U_{\tilde{\beta}} = e^{i\tilde{\beta}\cdot x}$*. By standard pseudodifferential calculus, given a symbol* $a(x,\xi)$ *one has that the symbol of* $U_{\tilde{\beta}}^{-1} Op^W(a) U_{\tilde{\beta}}$ *is*

$$a^{trasl}(x,\zeta) := a(x,\zeta+\tilde{\beta}) , \tag{5.23}$$

which, if a is in normal form, is a function on $T^*\mathbb{T}^s$*.*

Precisely, we have the following lemma

Lemma 5.13. *With the above notations, assume that $N_M \in S^{m,\delta}$ with $m \leq 0$, is in normal form with respect to M, then, in coordinates adapted to M, one has*

$$U_{\tilde{\beta}}^{-1} \left(-\Delta_{g,\kappa} + \mathcal{N}_M\right)\big|_{\mathcal{W}_{M,\beta}} U_{\tilde{\beta}} = \left(-\Delta_{g,\kappa'} + \mathcal{N}'_M + \ell^2\right)\big|_{\mathcal{W}^t_{M,\beta}} , \tag{5.24}$$

where $-\Delta_{g,\kappa'}$ is the Laplacian (in s dimensions) with respect to the restriction of the metric to M and

$$N'_M(x,\zeta) = N_M(x, \zeta + \tilde{\beta})$$

is of class $S^{m,\delta}$ (as a symbol on $\mathbb{T}^s$), with seminorms bounded by the seminorms of $\mathcal{N}_M$.

Proof. First remark that, by (5.22) the transformation of the Laplacian is $-\Delta_{g,\kappa'} + \ell^2$.

We come to the transformation of $\mathcal{N}_M$. We observe that, since it is in normal form with respect to M its symbol has the structure

$$N_M(x,\xi) = \sum_{k\in M} \hat{N}_k(\xi) e^{\mathrm{i}k\cdot x} .$$

Furthermore, introducing a basis $\mathbf{v}^A$ adapted to M, and denoting by $\mathbf{u}_A$ its dual basis, one has, for $k \in M$,

$$k\cdot x = \sum_{a=1}^{d'} x^a k_a$$

(since the coordinates k_A, $A = d'+1, ..., d$ of a vector in M vanish). Thus one gets that the symbol N'_M of the transformed operator is

$$N'_M(\zeta, \hat{z}) = \sum_{k\in\mathbb{Z}^{d'}} \hat{N}_{k_a \mathbf{v}^a}(\zeta + \widetilde{\beta}) e^{\mathrm{i}x^a k_a} = N_M(\zeta' + \widetilde{\beta}, \hat{x}) \ , \quad \hat{x} := (x^1, ..., x^{d'})$$

Remark that, denoting $M_R := \mathrm{span}_R(\mathbf{v}_1, ..., \mathbf{v}_{d'})$ and $M_R^* := \mathrm{span}_R(\mathbf{u}_1, ..., \mathbf{u}_{d'})$, one has

$$\begin{aligned}
\|d_{\hat{x}}^{N_2} d_{\zeta'}^{N_1} N'_M(\hat{x},\zeta')\| &= \sup_{\substack{\|h^{(j)}\|=1,\ h^{(j)}\in M_R^* \\ \|k^{(j)}\|=1,\ k^{(j)}\in M_R}} \left| d_{\hat{x}}^{N_2} d_{\zeta'}^{N_1} N'_M(\zeta',\hat{z}) \left[h^{(1)}, \cdots, h^{(M)}, k^{(1)}, \cdots, k^{(N)}\right]\right| \\
&\leq \sup_{\substack{\|h^{(j)}\|=1,\ h^{(j)}\in \mathbb{R}^d \\ \|k^{(j)}\|=1,\ k^{(j)}\in \mathbb{R}^d}} \left| d_{x}^{N_2} d_{\xi}^{N_1} N_M(\zeta' + \widetilde{\beta},\hat{z}) \left[h^{(1)}, \cdots, h^{(M)}, k^{(1)}, \cdots, k^{(N)}\right]\right| \\
&= \|d_{x}^{N_2} d_{\xi}^{N_1} N_M(\hat{x},\zeta' + \widetilde{\beta})\| \leq C\langle \zeta' + \widetilde{\beta} + \kappa\rangle^{m-N_1\delta} \leq C\langle(\zeta' + \widetilde{\beta} + \kappa)_M\rangle^{m-N_1\delta} \\
&= C\langle \zeta' + \kappa'\rangle^{m-N_1\delta} .
\end{aligned}$$

which is the thesis. □

In order to deduce the spectral result, the following corollary will be useful

Corollary 5.14. *Let* $\|\zeta+\kappa'\|^2+m(\zeta)$ *be an eigenvalue of* $(-\Delta_{g,\kappa'}+\mathcal{N}'_M)|_{\mathcal{W}^t_{M,\beta}}$ *with eigenfunction* $\phi^{(\zeta)}$*. Then* $\|\xi+\kappa\|^2+m(\xi-\tilde{\beta})$ *is an eigenvalue of* $(-\Delta_{g,\kappa}+\mathcal{N}_M)|_{\mathcal{W}_{M,\beta}}$ *with eigenfunction* $\psi^{(\xi)} := e^{\mathrm{i}\tilde{\beta}\cdot x}\phi^{(\zeta)}$*.*

Remark 5.15. *By* (5.20)*, in the particular case where* $\phi^{(\zeta)} = e^{\mathrm{i}\zeta\cdot x}$*, one has* $\psi^{(\xi)} = e^{\mathrm{i}\xi\cdot x}$*.*

6 A spectral result by quasi-modes

In this section we prove Theorem 3.16.

The key quasimode argument we are going to use is a variant of that used in [BKP15] (see Proposition 5.1) and is the following one

Lemma 6.1 (Quasi-mode argument)**.** *Let* $H = H_0 + H_1$ *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$ *such that* H *and* H_0 *have pure point spectrum. Suppose that* $\lambda_1^{(0)} \le \cdots \le \lambda_M^{(0)}$ *are* M *eigenvalues of* H_0 *counted with multiplicity, such that* $\exists D > 0$ *with*

$$(\lambda_1^{(0)} - D\,,\ \lambda_1^{(0)})\,, \quad (\lambda_M^{(0)}\,,\ \lambda_M^{(0)} + D) \tag{6.1}$$

which contain no eigenvalues of H_0*. Denote by* $\{\psi_k\}_{k=1}^M$ *the orthonormal eigenfunctions corresponding to* $\{\lambda_k^{(0)}\}_{k=1}^M$*, and let* $\{\varepsilon_k\}_{k=1}^M$ *be such that*

$$\|H_1\psi_k\| \le \varepsilon_k\ , \quad k = 1\,,\dots\,,M\,. \tag{6.2}$$

If $D > 0$ *and* $\delta \in (0,1)$ *are such that*

$$D^2 \ge \frac{16}{\pi\delta^2}M^3\left(\max_k \varepsilon_k\right)\left(|\lambda_M^{(0)} - \lambda_1^{(0)}| + D\right)\,, \tag{6.3}$$

then there are at least M *(not necessarily distinct) eigenvalues of* H *in the interval*

$$(\lambda_1^{(0)} - \delta D\,,\ \lambda_M^{(0)} + \delta D)\,.$$

Proof. By contradiction, assume that there are less than M eigenvalues of H inside the interval $\left(\lambda_1^{(0)} - \delta D, \lambda_M^{(0)} + \delta D\right)$, with $\delta \in (0,1)$. In particular, there are less than M eigenvalues in the intervals

$$I^- = \left(\lambda_1^{(0)} - \delta D,\ \lambda_1^{(0)}\right)\,, \quad I^+ = \left(\lambda_M^{(0)},\ \lambda_M^{(0)} + \delta D\right)\,.$$

Since I^+ has length δD, there exists at least one interval $J^+ \subset I^+$ such that $|J^+| \ge \frac{\delta D}{M}$ which contains no eigenvalues of H, analogously for I^- :

there exists at least an interval $J^- \subset I^-$ containing no eigenvalues of H and having length $|J^-| \geq \frac{\delta D}{M}$. Remark that, by hypothesis (6.1), $J^\pm$ do not contain eigenvalues of H_0 either. Consider then a square closed path γ in the complex plane intersecting the real axis at the middle points of J^+ and J^-. By construction,

$$\text{dist}\,(\gamma, \sigma(H_0)) \geq \frac{\delta D}{2M}\,, \tag{6.4}$$

and

$$\text{dist}\,(\gamma, \sigma(H)) \geq \frac{\delta D}{2M}\,. \tag{6.5}$$

Moreover, the length $\ell(\gamma)$ of γ, fulfills

$$\ell(\gamma) \leq 4\left|\lambda_M^{(0)} - \lambda_1^{(0)} + D\right|\,. \tag{6.6}$$

By the contradiction assumption there are less than M eigenvalues of H inside the path γ. Let M_0 be their number (counted with multiplicity).

Denote by $R(z) = (H - z\mathbb{I})^{-1}$ the resolvent of H, and by R_0 the resolvent of H_0, then, if P denotes the projection operator on the eigenspace corresponding to such eigenvalues of H, one has

$$P = \frac{1}{2\pi i}\int_\gamma R(z)\;dz\,,$$

and, using the resolvent identity

$$R(z) - R_0(z) = R(z)H_1R_0(z)\,,$$

one has

$$P\psi_k = \frac{1}{2\pi i}\int_\gamma R_0(z)\;dz\psi_k + \frac{1}{2\pi 1}\int_\gamma R(z)H_1R_0(z)\;dz\;\psi_k = \psi_k + r_k\,. \tag{6.7}$$

where

$$r_k = \frac{1}{2\pi i}\int_\gamma R(z)H_1R_0(z)\;dz\;\psi_k\,.$$

By (6.4) and (6.5), using that $R_0(z)\psi_k = \frac{1}{\lambda_k^{(0)} - z}\psi_k$, and the hypothesis (6.2), one gets the estimate one has that

$$\|r_k\| \leq \frac{\ell(\gamma)}{2\pi}\Big(\frac{2M}{\delta D}\Big)^2\varepsilon_k\,. \tag{6.8}$$

We are going to show that the vectors (6.7) are independent, against the assumption $M_0 < M$. We prove that

$$\sum_{k=1}^{M}\alpha_k P\psi_k = 0$$

implies $\alpha_k = 0$, $\forall k$. Indeed, one has

$$\sum_{k=1}^{M} \alpha_k P\psi_k = \sum_{k=1}^{M} \alpha_k \left(\psi_k + r_k\right) = 0\,;$$

in particular,

$$\sum_{k=1}^{M} \alpha_k \left(\langle \psi_k\,,\ \psi_j \rangle + \langle r_k\,,\ \psi_j \rangle\right) = 0 \quad \forall\ j\,,$$

namely $(\mathbb{I} + A)\,\alpha = 0$, with A the M dimensional matrix with matrix elements given by $A_{k,j} = \langle \psi_k, r_j \rangle$. Since by (6.8)

$$\|A\| \le M \sup_{i,j}\{|A_{i,j}|\} \le M \frac{\ell(\gamma)}{2\pi}\left(\frac{2M}{\delta D}\right)^2 \varepsilon\ , \quad \varepsilon := \max_k \varepsilon_k\,.$$

Then hypothesis (6.3) ensures $\|A\| < 1$, so that $\mathbb{I} + A$ is invertible and thus $\alpha_k = 0\ \forall k$. This shows that $\{P\psi_k\}_{k=1}^{M}$ form a set of M linearly independent eigenfunctions, which contradicts the hypothesis that there is only a set of multiplicity $M_0 < M$ of eigenvalues of H inside the interval $\left(\lambda_1^{(0)} - \delta D\,, \lambda_M^{(0)} + \delta D\right)$. □

The main tool in order to describe the unperturbed spectrum is a Weyl type estimate for the eigenvalues of an operator $H^{(0)}$ with spectrum given by

$$\begin{aligned} \sigma(H_0) &= \left\{ h_0(\xi) \mid \xi \in E \subset \mathbb{Z}^d \right\}\,, \\ h_0(\xi) &= \|\xi + \kappa\|^2 + m(\xi) \quad \forall \xi \in E\,, \end{aligned} \tag{6.9}$$

with m a bounded function.

Lemma 6.2. *Consider an operator H_0 as above and denote $\mathtt{m} = \sup_{\xi\in E}|m(\xi)|$, let $R > \sqrt{3\mathtt{m}}$, then one has*

$$\#\{\xi\ :\ |h_0(\xi)| \le R^2\} \le \left(\frac{4}{\mathfrak{c}_1}\right)^d R^d\,. \tag{6.10}$$

Proof. An estimate of the quantity (6.10) is the number of points $\xi \in \mathbb{Z}^d$ contained in a ball centered at $-\kappa$ and having radius $\sqrt{R^2 + \mathtt{m}} \le 2R$. Of course the ball is defined in terms of the metric g^*. For any $\xi \in \mathbb{Z}^d$, consider a ball $B_{\mathfrak{c}/2}(\xi)$ of radius $\mathfrak{c}/2$ and center ξ. Then, as ξ varies, such balls do not intersect, thus the "volume occupied" by n points of the lattice is bigger than $n\mathrm{Vol}B_{\mathfrak{c}/2}(\xi) = nC_d(\mathfrak{c}/2)^d$, with C_d the volume of the unitary ball. It follows that for the number n of points in the ball of radius $2R$ (independently of its centrum) the following inequality holds

$$nC_d(\mathtt{c}/2)^d \leq \mathrm{Vol} B_0(2R) = C_d 2^d R^d \ ,$$

from which the thesis follows. □

This allows to prove the existence of gaps in the spectrum; precisely, the following Lemma holds.

Lemma 6.3. *There exists a constant C, depending only on $\mathtt{c}$ and d, with the following properties: for any $\bar{\lambda} > 4\mathtt{m}$ and any $0 < L \leq \mathtt{m}$, there exist $0 < L_1, L_2 < L$ s.t.*

$$\#\left(\sigma(H_0) \cap [\bar{\lambda} - L_1; \bar{\lambda} + L_2]\right) \leq C\bar{\lambda}^{d/2} \tag{6.11}$$

$$\sigma(H_0) \cap \left[\bar{\lambda} - L_1 - \frac{L}{C\bar{\lambda}^{d/2}}; \bar{\lambda} - L_1\right] = \emptyset \tag{6.12}$$

$$\sigma(H_0) \cap \left[\bar{\lambda} + L_2; \bar{\lambda} + L_2 + \frac{L}{C\bar{\lambda}^{d/2}}\right] = \emptyset \tag{6.13}$$

Proof. By Lemma 6.2 the maximal number of eigenvalues smaller than $\bar{\lambda} + L < 2\bar{\lambda}$ is smaller than a constant C (the constant whose existence is claimed in the statement) times $\bar{\lambda}^{d/2}$, so equation (6.11) is true (and very pessimistic) for any choice of $L_1, L_2 < L$. To prove (6.12), consider the interval $[\bar{\lambda} - L, \bar{\lambda}]$; by Lemma 6.2 it contains at most $C\bar{\lambda}^{d/2}$ eigenvalues, so there is at least a gap between two of them of length $L/C\bar{\lambda}^{d/2}$. Its right end determines L_1, and this proves (6.12). Equation (6.13) is proved in the same way. □

Corollary 6.4. *For any $N > 0$ and $0 < L < \mathtt{m}$, there exists a sequence of intervals*

$$E_j = [a_j, b_j] \ , \quad j \in \mathbb{N} \tag{6.14}$$

and a positive constant C, with the following properties:

$$\sigma(H_0) \subset [0, a_1 - \frac{1}{a_1^N}] \bigcup \left(\bigcup_j E_j\right) \ , \tag{6.15}$$

$$|b_j - a_j| \equiv |E_j| \leq 2L \tag{6.16}$$

$$d(E_j, E_{j+1}) \equiv a_{j+1} - b_j \geq \frac{L}{b_j^N} \tag{6.17}$$

$$\#\left(\sigma(H_0) \cap E_j\right) \leq C b_j^{d/2} \ . \tag{6.18}$$

Proof. We use the same notations as in Lemma 6.3. Take $\bar{\lambda} := \min\{\lambda \in \sigma(H_0) : \lambda \geq 4\mathtt{m}\}$. Then the first interval is the one constructed in Lemma

6.3. Let b_1 be the largest point of the spectrum in the interval. Let a_2 be the subsequent point of the spectrum. To determine b_2, consider the subsequent points of the spectrum. By Lemma 6.3 after at most an interval of length $2L$ one finds a gap of width $\frac{L}{a_2^N}$. This gives the second interval. Iterating one gets the result. □

The following Lemma enables to relate the spectrum and the structure of eigenfunctions of the two operators $H^{(1)}_{M,\beta}$ and $\widetilde{H}_{M,\beta}$ of Theorem 2.18, for any $M \subset \mathbb{Z}^d$ and $\beta \in \widetilde{M}$:

Lemma 6.5. *For any M, β, consider the operator $-\Delta_{g,\kappa'} + \mathcal{V}_{M,\beta}$ as in* (2.30) *of Theorem 2.18, and assume that its eigenvalues are given by*

$$\lambda_\zeta = h_{M,\beta}(\zeta) = \|\zeta + \kappa'\|^2 + m_{M,\beta}(\zeta)\,, \quad \zeta \in M, \tag{6.19}$$

with $\sup_{M,\beta} \sup_\zeta |m_{M,\beta}(\zeta)| \leq \mathtt{m}$*. Assume that there exist positive constants* $a < \frac{1}{2}$, $\mathtt{N} \in \mathbb{N}$ *and* C *such that, given any eigenvalue* $\lambda_\zeta \neq 0$, *the corresponding eigenfunction* $\phi^{(\zeta)}$ *fulfills*

$$\|\phi^{(\zeta)}\|_{H^{-\mathtt{N}}} \leq \frac{C}{\lambda_\zeta^{a\mathtt{N}}} \quad \forall \zeta \in M\,. \tag{6.20}$$

Then the eigenvalues of $U_\beta\,(-\Delta_{g,\kappa'} + \mathcal{V}_{M,\beta})\,U_\beta^* + \ell^2$ *are given by*

$$\lambda_\xi = h_0(\zeta) = \|\xi + \kappa\|^2 + m_{M,\beta}(\xi - \tilde{\beta})\ , \quad \xi = \zeta + \tilde{\beta} \tag{6.21}$$

and, if $\lambda_\xi \neq 0$, *there exists* $C' > 0$, *depending only on* $a, \mathtt{m}, \mathtt{N}, C$, *such that the corresponding eigenfunction* $\psi^{(\xi)}$ *fulfills*

$$\|\psi^{(\xi)}\|_{H^{-2\mathtt{N}}} \leq \frac{C'}{\lambda_\xi^{a\mathtt{N}}}\ . \tag{6.22}$$

Proof. The form of the eigenvalues is a direct consequence of eq. (5.22). Concerning the eigenfunctions, the unitary map U_β transforms them in $\psi^{(\xi)} := e^{\mathrm{i}\tilde{\beta}\cdot x}\phi^{(\zeta)}$, which, by Lemma C.2, are estimated by

$$\|\psi^{(\xi)}\|_{H^{-2\mathtt{N}}} \leq \frac{C}{\lambda_\zeta^{a\mathtt{N}}} \frac{1}{\langle(\tilde{\beta} + \kappa)_{M^\perp}\rangle^{\mathtt{N}}}\ . \tag{6.23}$$

Then one has

$$\lambda_\zeta^a \langle(\tilde{\beta} + \kappa)_{M^\perp}\rangle \geq \left(\lambda_\zeta^{1/2}\langle(\tilde{\beta} + \kappa)_{M^\perp}\rangle\right)^{2a}\ ,$$

since $2a < 1$. Then, provided λ_ζ is large enough, $\lambda_\zeta^{1/2} \geq \langle(\zeta+\kappa')\rangle/2$, from which

$$\lambda_\zeta^{1/2}\langle(\tilde{\beta}+\kappa)_{M^\perp}\rangle \geq \frac{1}{2}\langle\zeta+\kappa'\rangle\langle(\tilde{\beta}+\kappa)_{M^\perp}\rangle = \frac{1}{2}\langle(\xi+\kappa)_M\rangle\langle(\xi+\kappa)_{M^\perp}\rangle \geq \frac{1}{2}\langle\xi+\kappa\rangle\ , \tag{6.24}$$

where the last inequality follows from the trivial remark that for any real x, y, one has $(1+x^2)(1+y^2) \geq 1+x^2+y^2$. Collecting the results and remarking that, for λ_ξ large enough, $\lambda_\xi < 2\langle\xi+\kappa\rangle^2$, one gets the thesis for large eigenvalues. In order to cover all the nonvanishing eigenvalues, just remark that the number of eigenvalues smaller than any threshold is finite, so that the claimed estimates trivially hold. □

Lemma 6.6. *Assume that all the operators* (2.30) *fulfill the assumptions of Lemma 6.5, then the properties* (6.21) *and* (6.22) *hold, also for the eigenvalues and the eigenfunctions of the operator* (2.7)*, but with new constants depending only on the seminorms of $\mathcal{V}$ and on the constants of the metric, and with a new function $m'_{M,\beta}$ such that*

$$m'_{M,\beta}(\xi) = m_{M,\beta}(\xi) + r_\xi\ , \quad |r_\xi| \leq C\|\xi+\kappa\|^{-a\mathtt{N}} \quad \forall\xi\ .$$

Proof. First, by Theorem 2.18, for any $\mathtt{N}' \in \mathbb{N}$, the operator $-\Delta_{g,\kappa}+\mathcal{V}$ is unitarily equivalent, through a pseudodifferential operator U of order 0, to $\widetilde{H}_{\mathtt{N}'}+\mathcal{R}_{\mathtt{N}'}$. Fix $\mathtt{N} \in \mathbb{N}$, let $\mathtt{N}' = \frac{\mathtt{N}}{2}$ and from now on drop the dependence on $\mathtt{N}'$ by the operators $\widetilde{H}_{\mathtt{N}'}, \mathcal{R}_{\mathtt{N}'}$. By Lemma 6.5 the eigenvalues of $\widetilde{H}$ fulfill (6.21) and (6.22) with $2\mathtt{N}$ replaced by $\mathtt{N}$, due to the choice of $\mathtt{N}'$. Concerning the eigenfunctions, we observe that, by (6.20), Lemma C.1 of the Appendix ensures that there exists a constant $C'' > 0$ such that any eigenvalue λ_ξ of $\widetilde{H}+\mathcal{R}$ with $\lambda_\xi \neq 0$ has a related normalized eigenfunction ψ_ξ satisfying

$$\|\psi_\xi\|_{H^{-\mathtt{N}}} \leq C''|\lambda_\xi|^{\frac{d}{2}-a\frac{\mathtt{N}}{2}}\ , \tag{6.25}$$

thus (6.22) still holds for the eigenfunctions of $\widetilde{H}+\mathcal{R}$. It remains to prove (6.21). We split $\sigma(\widetilde{H})$ according to Corollary 6.4, choosing $L=1$ and $\mathtt{N} = \mathtt{N}/3$ and in each of the intervals E_j we apply Lemma 6.1. To this end, remark that, for all eigenvalues $\lambda \in E_j$ one has $\lambda/2 < a_j < b_j < 2\lambda$. Let ϕ be the eigenfunction of $\widetilde{H}$ corresponding to λ: then by the Calderon Vaillancourt Theorem and since the eigenfunctions of $\widetilde{H}$ satisfy eq. (6.22), one has

$$\|\mathcal{R}\phi\|_{L^2} \leq \|\mathcal{R}\|_{\mathcal{B}(H^{-\mathtt{N}},H^0)}\frac{2^{\mathtt{N}}C'}{\lambda^{a\frac{\mathtt{N}}{2}}}\ .$$

Thus an application of Lemma 6.1 with $H_0 = \widetilde{H}$, $H_1 = \mathcal{R}$, ensures that, if for all $j \in \mathbb{N}$ one defines $D_j^- = a_j^{-\mathtt{N}/3}$, $D_j^+ = b_j^{-N/3}$ and $M_j = \sharp\left(\sigma(\widetilde{H}) \cap E_j\right)$, then there are $M_j' \geq M_j$ eigenvalues of $\widetilde{H} + \mathcal{R}$ inside the interval

$$\widetilde{E}_j = \left[a_j - \frac{1}{4}D_j^- \ , \ b_j + \frac{1}{4}D_j^+\right] \supset E_j \, .$$

We prove now that there are no eigenvalues of $\widetilde{H}+\mathcal{R}$ outside the intervals $\widetilde{E}_j$. Assume by contradiction that $\bar{\lambda}$ is an eigenvalue of $\widetilde{H} + \mathcal{R}$ with $\bar{\lambda} \notin \bigcup_j \widetilde{E}_j$. Let $\bar{j}$ be the positive integer such that $b_{\bar{j}} < \bar{\lambda} < a_{\bar{j}+1}$. Since the eigenfunction ψ of $\widetilde{H} + \mathcal{R}$ related to $\bar{\lambda}$ satisfies (6.25), one has

$$\|\mathcal{R}\psi\|_{L^2} \lesssim \bar{\lambda}^{\frac{d}{2}-a\frac{\mathtt{N}}{2}} \lesssim a_{\bar{j}+1}{}^{\frac{d}{2}-a\frac{\mathtt{N}}{2}} \, ,$$

which implies that ψ is a quasi-mode for $\widetilde{H}$ with approximated eigenvalue $\bar{\lambda}$. In particular (up to choosing a_1 big enough), this implies that there exists an exact eigenvalue $\lambda = \bar{\lambda} + O(a_{\bar{j}+1}{}^{\frac{d}{2}-a\frac{\mathtt{N}}{2}})$ of $\widetilde{H}$ such that $b_{\bar{j}} < \lambda < a_{\bar{j}+1}$, which is absurd, by definition of the intervals E_j.

We prove now that $M_j' = M_j$ for all $j \in \mathbb{N}$. Arguing as before, we can apply the quasi-mode argument of Lemma 6.1 with $H_0 = \widetilde{H} + \mathcal{R}$, $H_1 = -\mathcal{R}$, $M = M_j'$ and $[\lambda_1^{(0)}, \lambda_{M_j'}^{(0)}] = \widetilde{E}_j$ to deduce that, since all the eigenfunctions of $\widetilde{H}+\mathcal{R}$ related to the eigenvalues contained inside $\widetilde{E}_j$ satisfy (6.25), then there are $M_j'' \geq M_j'$ eigenvalues of $\widetilde{H}$ inside a slight enlargement of the interval $\widetilde{E}_j$. But there are exactly M_j eigenvalues of $\widetilde{H}$ inside $E_j \subset \widetilde{E}_j$, thus $M_j'' = M_j$, which proves that *all* the eigenvalues of $\widetilde{H} + \mathcal{R}$ are of the form (6.21). We finally observe that, since for any eigenvalue λ' of $\widetilde{H} + \mathcal{R}$ the corresponding eigenfunction ψ fulfills again equation (6.22) with updated constants, the corresponding eigenfunction $U\psi$ of $-\Delta_{g,\kappa} + \mathcal{V}$ fulfills again equation (6.22), due to the fact that U is a bounded operator onto $H^{-\mathtt{N}}$, since U is a pseudo-differential operator of order 0. □

By iteratively applying this Lemma one gets the proof of Theorem 3.16.

Remark 6.7. *From the above Lemma it follows in particular that all the eigenvalues and eigenfunctions of $\widetilde{H} + \mathcal{R}$ are constructed through our quasi-mode procedure.*

A Pseudo differential calculus

In this section we recall some standard facts on pseudo-differential calculus with the aim of pointing out that, with our coordinate independent definition of the seminorms, they still hold. In particular the coordinate independent definition is needed in order to perform the dimensional reduction of Subsect. 5.3.

Lemma A.1 (**Calderon Vaillancourt**). *Let $A \in OPS^{m,\delta}$. Then A is a bounded linear operator $H^s \to H^{s-m}$ for any $s \in \mathbb{R}$. In particular, for any s there exist $K > 0$ and $N \in \mathbb{N}$, depending only on the parameters $m, s, d, \mathfrak{c}$, such that $\|A\|_{\mathcal{B}(H^s;H^{s-m})} \leq K \sup_{N' \leq N} C_{N',0}(a)$.*

Since $T^*\mathbb{T}^d$ is a cotangent boundle it carries a natural symplectic structure, and the Poisson Brackets can be computed in any system of coordinate originated by a system of coordinate in $\mathbb{T}^d$. Using such a system of coordinates one can easily show that the the following Lemma holds

Lemma A.2. *Let $a \in S^{m,\delta}$ and $b \in S^{m',\delta}$; then $\{a, b\} \in S^{m+m'-\delta,\delta}$. In particular, for all $N_1, N_2 \in \mathbb{N}$ one has*

$$C_{N_1,N_2}(\{a, b\}) \leq C_{N_1+1,N_2}(a)C_{N_1,N_2+1}(b) + C_{N_1,N_2+1}(a)C_{N_1+1,N_2}(b) .$$

Concerning Moyal brackets, the situation is slightly more delicate, but reproducing the standard proof (see e.g. [Tay], [SVA]) one easily gets the following result.

Lemma A.3. *Let $A = Op^W(a) \in OPS^{m,\delta}$ and $B = Op^W(b) \in OPS^{m',\delta}$; then*

1. *$AB \in OPS^{m+m',\delta}$. Let $a\sharp b$ be its symbol: for any $N_1, N_2 \in \mathbb{N}$, there exist $K > 0$ and $\widetilde{N}_1 > N_1$, depending only on $N_1, N_2, \mathfrak{c}, p, m, m', \delta$, such that*

$$\sup_{N_1' \leq N_1, N_2' \leq N_2} C_{N_1',N_2'}(a\sharp b) \leq K \sup_{N_1' \leq \widetilde{N}_1, N_2' \leq N_2} C_{N_1',N_2'}(a) \sup_{N_1' \leq \widetilde{N}_1, N_2' \leq N_2} C_{N_1',N_2'}(b) . \tag{A.1}$$

2. *$-i[A, B] \in OPS^{m+m'-\delta,\delta}$ and the seminorms of its symbol, denoted by $\{a, b\}_{\mathcal{M}}$, are controlled as follows: for all N_1 and $N_2 \in \mathbb{N}$ there exist $K > 0$ and $\widetilde{N}_1 > N_1$, depending only on $N_1, N_2, p, m, m', \delta, \mathfrak{c}$, such that*

$$\sup_{N_1' \leq N_1, N_2' \leq N_2} C_{N_1',N_2'}(\{a, b\}_{\mathcal{M}}) \leq K \sup_{\substack{N_1' \leq \widetilde{N}_1 \\ N_2' \leq N_2+1}} C_{N_1',N_2'}(a) \sup_{\substack{N_1' \leq \widetilde{N}_1 \\ N_2' \leq N_2+1}} C_{N_1',N_2'+1}(b) . \tag{A.2}$$

3. *If a is a quadratic polynomial in ξ, independent of x, then*

$$\{a, b\}_{\mathcal{M}} = \{a, b\} .$$

Finally, we recall that the following result holds:

Lemma A.4 (**Egorov Theorem**). *Let $\eta \geq 0$, $\delta > 0$, $m \in \mathbb{R}$, $G := \mathrm{Op}^W(g) \in OPS^{-\eta,\delta}$ and $A := \mathrm{Op}^W(a) \in OPS^{m,\delta}$. Then the following holds.*

1. *For any $\tau \in [-1,1]$, $e^{i\tau G} \in OPS^{0,\delta}$. In particular, if σ is its symbol, for all $N_1, N_2 \in \mathbb{N}$ one has that there exist $K_1, K_2 > 0$ and $\widetilde{N}_1 > N_1$, depending only on $N_1, N_2, \mathfrak{c}, \delta, d$, such that*

$$\sup_{N_1' \leq N_1, N_2' \leq N_2} C_{N_1',N_2'}(\sigma) \leq K_1 \sup_{N_1' \leq \widetilde{N}_1, N_2' \leq N_2} e^{K_2 C_{N_1',N_2'}(g)} . \tag{A.3}$$

2. *The linear operator $H := e^{iG} A e^{-iG} \in OPS^{m,\delta}$ and its symbol $h(x,\xi)$ admits an asymptotic expansion of the form*

$$h = a + \{a; g\}_{\mathcal{M}} + S^{m-2(\eta+\delta),\delta} .$$

B Technical Lemmas

We first recall Lemma 5.7 of [Gio03].

Lemma B.1 (Lemma 5.7 of [Gio03]). *Let $s \in \{1, \dots, d\}$ and let $\{u_1, \dots u_s\}$ be linearly independent vectors in $\mathbb{R}^d$ equipped with the euclidean metric $|\cdot|$. Denote by $Vol\{u_1 | \cdots | u_s\}$ the $s-$ dimensional volume of the parallelepiped with sides $u_1, \dots, u_s$. Let moreover $w \in \mathrm{span}\{u_1, \dots u_s\}$ be any vector. If there exists positive constants α, N such that*

$$|u_j| \leq N \quad \forall j = 1, \dots s ,$$
$$|w \cdot u_j| \leq \alpha \quad \forall j = 1, \dots s ,$$

then

$$|w| \leq \frac{sN^{s-1}\alpha}{Vol\{u_1 | \cdots | u_s\}} .$$

We remark that, since all the quantities involved in the statement are coordinate independent, Lemma 5.1 immediately follows from it.

Lemma B.2. *Let $\{e_1, \dots, e_d\}$ be the vectors of the standard basis in $\mathbb{R}^d$. There exists a positive constant $\mathfrak{C}$, depending only on*

$$\mathfrak{c}_2 = \max_{j=1,\dots,d} \|e_j\| \tag{B.1}$$

and

$$\mathfrak{v} = \int_{\mathbb{T}^d} d\mu_g(x) \equiv \mu_g\left(\mathbb{T}^d\right) , \tag{B.2}$$

such that for any $s \in \{1, \dots, d\}$ and for any set $\{u_1, \dots u_s\}$ of linearly independent vectors in $\mathbb{Z}^d$

$$Vol_g\{u_1 | \cdots | u_s\} \geq \mathfrak{C} .$$

Proof. We observe that, if $\{e_1, \dots, e_d\}$ is the canonical basis of $\mathbb{Z}^d$, there exists a subset $\{u'_{s+1}, \dots u'_d\} \subset \{e_1, \dots, e_d\}$ such that

$$\{u_1, \dots, u_s, u'_{s+1}, \dots u'_d\}$$

is a set of linearly independent vectors in $\mathbb{Z}^d$. Hence one has that, if M is the linear subspace generated by $\{u_1, \dots, u_s\}$,

$$\begin{aligned}\mathrm{Vol}_g\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\} &\le \|u'_{s+1}\|\cdots\|u'_d\| \ \mathrm{Vol}_g\{u_1|\dots|u_s\} \\ &\le (\mathfrak{c}_2)^d \mathrm{Vol}_g\left(\{u_1|\dots|u_s\}\right) ,\end{aligned}$$

by the definition of $\mathfrak{c}_2$ as in (B.1). In particular, one has that

$$\mathrm{Vol}_g\left(\{u_1|\dots|u_s\}\right) \ge (\mathfrak{c}_2)^{-d} \ \mathrm{Vol}_g\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\} . \tag{B.3}$$

Write

$$u_j = \sum_{k=1}^{d} n_{j,k} e_k , \quad u'_j = \sum_{k=1}^{d} n_{j,k} e_k ,$$

and if $\forall k = 1, \dots, d$ $\tilde{e}_k$ is the vector of the components of e_k with respect to an orthonormal basis for the inner product $(\cdot,\cdot)_g$, then

$$\begin{aligned}\mathrm{Vol}_g\left(\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\}\right) &= \mathrm{Vol}_g\left(\left\{\sum_{k=1}^{d} n_{1,k} e_k|\dots|\sum_{k=1}^{d} n_{d,k} e_k\right\}\right) \\ &= \mathrm{Vol}\left(\left\{\sum_{k=1}^{d} n_{1,k} \tilde{e}_k|\dots|\sum_{k=1}^{d} n_{d,k} \tilde{e}_k\right\}\right) \\ &\ge \mathrm{Vol}\left(\tilde{e}_1|\cdots|\tilde{e}_d\right) \\ &= \mathrm{Vol}_g\left(e_1|\cdots|e_d\right) = \mathfrak{v} .\end{aligned}$$

Thus (B.3) implies that

$$\mathrm{Vol}\{u_1 \ | \cdots \ | \ u_s\} \ge (\mathfrak{c}_2)^{-d} \mathfrak{v} =: \mathfrak{C} .$$

□

Remark B.3. *By studying the function $(1+x^2)^{a/2}$ it is easy to see that there exists a constant K s.t. $\forall \xi, \eta \in \mathbb{R}^d$ one has*

$$\langle \xi + \eta \rangle^a \le K(\langle \xi \rangle^a + \langle \eta \rangle^a) . \tag{B.4}$$

Furthermore, since, for any $C > 0$

$$\sup_{y > C} \frac{\langle y \rangle}{y} < \infty ,$$

one also has $\exists K' = K'(a, C)$ s.t.

$$\langle \xi + \eta \rangle^a \le K'(\langle \xi \rangle^a + \|\eta\|^a) , \quad \forall \eta \ : \|\eta\| \ge C . \tag{B.5}$$

Remark B.4. *If $\|\xi - \eta\| \leq F\langle\xi\rangle^a$, with $a < 1$, one has*

$$\langle\xi\rangle \leq K(1+F)\langle\eta\rangle \ .$$

A further useful lemma is the following one

Lemma B.5. *Let $N \geq 1$, $a < 1$, $K \geq 2^{-a}$ be positive real numbers, Then*

$$x - Kx^a \leq N \implies x \leq (2K)^{\frac{1}{1-a}} N \ . \tag{B.6}$$

Proof. If $Kx^a \leq \frac{x}{2}$, which is equivalent to

$$x \geq (2K)^{\frac{1}{1-a}} \ , \tag{B.7}$$

then the assumed inequalities implies

$$\frac{1}{2}x \leq x - Kx^a \leq N \implies x < 2N \ ,$$

but, by assumption, the r.h.s is smaller than $(2K)^{\frac{1}{1-a}}$, and therefore the thesis holds in this case. On the contrary, the converse of (B.7), implies

$$x < (2K)^{\frac{1}{1-a}} \leq (2K)^{\frac{1}{1-a}} N \ ,$$

which again implies the thesis. $\square$

Lemma B.6. *Let $1 > a > \epsilon > 0$ and $1 > \delta > 0$ be parameters. Let ς, η, k, ℓ be vectors. Assume that there exist constants C, F, D, D_0 s.t.*

$$|(\varsigma + k, h)| \leq C\langle\varsigma + k\rangle^\delta |h|^{-\tau} \ , \tag{B.8}$$

$$\|k\| \leq D\langle\varsigma + k\rangle^\epsilon \ , \quad \|h\| \leq D_0\langle\varsigma + k\rangle^\epsilon \tag{B.9}$$

$$\|\eta - \varsigma\| \leq F\langle\eta\rangle^a \ , \quad \|\ell\| \leq D\langle\eta + \ell\rangle^\epsilon \ ; \tag{B.10}$$

then there exists K' and D' (which depends on the above constants), s.t.

$$\langle\varsigma + k\rangle \leq D'\langle\eta + \ell\rangle \ , \tag{B.11}$$

$$|(\eta + \ell, h)| \leq K'\langle\eta + \ell\rangle^{\max\{\delta, a+\epsilon(\tau+1)\}} |h|^{-\tau} \ . \tag{B.12}$$

Proof. Start by writing

$$\varsigma + k = \eta + \ell + v \tag{B.13}$$

$$v := k - \ell + \varsigma - \eta \ ; \tag{B.14}$$

then we estimate v (with $\eta+\ell$). One has

$$\begin{aligned}\|v\| &\le D\langle\varsigma+k\rangle^{\epsilon}+D\langle\eta+\ell\rangle^{\epsilon}+F\langle\eta\rangle^{a}\\ &= D\langle\eta+\ell+v\rangle^{\epsilon}+D\langle\eta+\ell\rangle^{\epsilon}+F\langle\eta+\ell-\ell\rangle^{a}\\ &\le DK\left(\langle\eta+\ell\rangle^{\epsilon}+\langle v\rangle^{\epsilon}\right)+D\langle\eta+\ell\rangle^{\epsilon}+FK\left(\langle\eta+\ell\rangle^{a}+\langle\ell\rangle^{a}\right)\\ &\le D(K+1)\langle\eta+\ell\rangle^{\epsilon}+FK\langle\eta+\ell\rangle^{a}+FK(1+D)\langle\eta+\ell\rangle^{a\epsilon}+DK\langle v\rangle^{\epsilon}\ .\end{aligned}$$

Using $a>\epsilon$ and $a>a\epsilon$, (and exploiting $\langle x\rangle\le 1+x$, which holds for all positive x) we get

$$\begin{aligned}\langle v\rangle \le \left(D(K+1)+FK+FK(1+D)+1\right)\langle\eta+\ell\rangle^{a}\\ +DK\langle v\rangle^{\epsilon}\ .\end{aligned}$$

Applying Lemma B.5 with N equal to the first line, we get that there exists a constant K'' (explicitely computable), s.t.

$$\langle v\rangle \le K''\langle\eta+\ell\rangle^{a}\ . \tag{B.15}$$

Exploiting this and using again (B.13), we immediately get (B.11). We are now ready for the final estimate:

$$\begin{aligned}|(\eta+\ell,h)| &\le |(\varsigma+k,h)|+|(v,h)|\,\|h\|^{\tau}\|h\|^{-\tau}\\ &\le C\langle\varsigma+k\rangle^{\delta}\|h\|^{-\tau}+K''\langle\eta+\ell\rangle^{a}D_0\langle\varsigma+k\rangle^{\epsilon}D_0^{\tau}\langle\varsigma+k\rangle^{\epsilon\tau}\|h\|^{-\tau}\\ &\le C(D')^{\delta}\langle\eta+\ell\rangle^{\delta}\|h\|^{-\tau}+K''\langle\eta+\ell\rangle^{a}D_0^{\tau+1}(D')^{\epsilon(\tau+1)}\langle\eta+\ell\rangle^{\epsilon(\tau+1)}\|h\|^{-\tau}\ ,\end{aligned}$$

from which the thesis immediately follows. □

C Properties of eigenfunctions

Lemma C.1. *Consider an operator $H_0+\mathcal{R}$, with $\mathcal{R}\in OPS^{-\mathtt{N}\delta,\delta}$; assume that*

1. *$\exists C$ and d s.t. the spectrum of H_0 satisfies a Weyl's law of the form*

$$\sharp\{\lambda^{(0)}\in\sigma(H_0)\ |\ \lambda^{(0)}\le r\}\le Cr^{\frac{d}{2}}\,. \tag{C.1}$$

2. *There exist $a>0$ and C_1 such that any normalized eigenfunction ψ relative to an eigenvalue $\lambda^{(0)}$ of H_0 fulfills*

$$\|\psi\|_{H^{-\mathtt{N}}}\le C_1|\lambda^{(0)}|^{-a\mathtt{N}}\,. \tag{C.2}$$

Then there exists $\Lambda, C_1' > 0$ *which depend on* $C, C_1, d, \|R\|_{\mathcal{B}(H^{-N},H^0)}$ *only, with the following properties: any normalized eigenfunction* ϕ *of* $H_0 + R$ *which corresponds to an eigenvalue* $\lambda > \Lambda$ *fulfills*

$$\|\phi\|_{H^{-\mathtt{N}}} \le C_1' |\lambda|^{\frac{d}{2} - a\mathtt{N}} . \tag{C.3}$$

Proof. First remark that, by the Calderon Vaillancourt theorem, one has

$$\|R\psi\|_{L^2} \le \|R\|_{\mathcal{B}(H^{-\mathtt{N}},H^0)} \|\psi\|_{H^{-\mathtt{N}}} \le \frac{\|R\|_{\mathcal{B}(H^{-\mathtt{N}},H^0)} C_1}{|\lambda^{(0)}|^{a\mathtt{N}}} . \tag{C.4}$$

Fix $\mathtt{c}_1 < \lambda/2$ and decompose

$$\phi = \phi_0 + \phi_1$$

with

$$\phi_0 \in \mathcal{Q} = \mathrm{span}\left\{\psi \mid H_0\psi = \lambda_\psi \psi , \quad |\lambda_\psi - \lambda| \le \mathtt{c}_1 \right\} ;$$

and $\phi_1 \in \mathcal{Q}^\perp$. We analyze the eigenvalue equation

$$(H_0 + R)\,\phi = \lambda\phi .$$

by using the method of Lyapunov Schmidt decomposition. Denote by $\Pi^\perp$ the orthogonal projector on $\mathcal{Q}^\perp$ and by Π the orthogonal projector on $\mathcal{Q}$. Inserting the decomposition of ϕ in the eigenvalue equation, applying $\Pi^\perp$ and taking into account that the projector commutes with H_0, we get (reorganizing the terms)

$$\left[\left(\Pi^\perp H_0 \Pi^\perp - \lambda\right) + \Pi^\perp R\right] \phi_1 = -\Pi^\perp R \phi_0 .$$

By definition of $\mathcal{Q}^\perp$, the operator in square brackets is invertible and the norm of its inverse is bounded by 2, provided $\mathtt{c}_1 \ge 2\|R\|_{\mathcal{B}(H^{-\mathtt{N}},H^0)}$. It follows that

$$\|\phi_1\|_{L^2} \le 2\|R\phi_0\|_{L^2} .$$

To estimate $\|R\phi_0\|_{L^2}$ we decompose ϕ_0 in eigenfunctions of H_0 and use assumption (C.2). First remark that by construction ϕ_0 has components only on eigenfunctions corresponding to eigenvalues between $\lambda - \mathtt{c}_1 > \lambda/2$ and $\lambda + \mathtt{c}_1 < 2\lambda$. So it has at most $J \le 2^{d/2} C \lambda^{\mathtt{d}/2}$ components:

$$\phi_0 = \sum_{j=1}^{J} \alpha_j \psi_j .$$

It follows that the $H^{-\mathtt{N}}$ norm of ϕ_0 is bounded by $2^{\mathtt{N}a}J/\lambda^{a\mathtt{N}}$. Concerning ϕ_1, we show that its L^2 norm, which bounds all the negative Sobolev norms, is small. One has

$$\|R\phi_0\|_{L^2} \le \sum_{j=1}^{J} |\alpha_j| \|R\psi_j\|_{L^2} \le \left(\sum_{j=1}^{J} |\alpha_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} \|R\psi_j\|_{L^2}^2\right)^{\frac{1}{2}} \le J^{1/2} \frac{\|R\|_{\mathcal{B}(H^{-\mathtt{N}},H^0)}}{(\lambda/2)^{a\mathtt{N}}} ,$$

where we used that the norm of ϕ_0 is smaller than the norm of ϕ and therefore is smaller than 1. From this the thesis follows. □

Lemma C.2. *Let M be a module, and let u be a function of the form*

$$u(x) = \sum_{\zeta \in M} \hat{u}_\zeta e^{\mathrm{i}\zeta\cdot x} ,$$

be such that

$$\|u\|_{H^{-\mathtt{N}}} \le K \tag{C.5}$$

Let $\beta \in M^{(c)}$ and consider $\tilde{\beta}$ defined as in (2.20)*, then one has*

$$\|e^{\mathrm{i}\tilde{\beta}\cdot x} u\|_{H^{-2\mathtt{N}}} \le \frac{K}{\langle (\tilde{\beta}+\kappa)_{M^\perp}\rangle^{\mathtt{N}}} . \tag{C.6}$$

Proof. One has

$$\|e^{\mathrm{i}\tilde{\beta}\cdot x} u\|_{H^{-2N}}^2 = \sum_{\zeta\in M} \langle \tilde{\beta}+\kappa+\zeta\rangle^{-2\mathtt{N}} |\hat{u}_\zeta|^2 . \tag{C.7}$$

We analyse, using (5.20) and (5.22), the term

$$\begin{aligned} \langle \tilde{\beta}+\kappa+\zeta\rangle^2 &= 1 + (\zeta+\tilde{\beta}+\kappa)_M^2 + (\zeta+\tilde{\beta}+\kappa)_{M^\perp}^2 \\ = 1 + (\zeta+\kappa')^2 + (\tilde{\beta}+\kappa)_{M^\perp}^2 &= \frac{1}{2} + (\zeta+\kappa')^2 + \frac{1}{2} + (\tilde{\beta}+\kappa)_{M^\perp}^2 \\ \ge 2\sqrt{\frac{1}{2} + (\zeta+\kappa')^2}\sqrt{\frac{1}{2} + (\tilde{\beta}+\kappa)_{M^\perp}^2} &\ge \langle \zeta+\kappa'\rangle \langle (\tilde{\beta}+\kappa)_{M^\perp}\rangle . \end{aligned}$$

Inserting in (C.7) one immediately gets the thesis. □

References

[AFKM15] Nalini Anantharaman, Clotilde Fermanian-Kammerer, and Fabricio Macià. Semiclassical completely integrable systems: long-time dynamics and observability via two-microlocal Wigner measures. *Amer. J. Math.*, 137(3):577–638, 2015.

[Bam17] Dario Bambusi. Reducibility of 1-d schrödinger equation with time quasiperiodic unbounded perturbations, ii. *Communications in Mathematical Physics*, 353(1):353–378, 2017.

[Bam18] Dario Bambusi. Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations. I. *Trans. Amer. Math. Soc.*, 370(3):1823–1865, 2018.

[BBHM18] Pietro Baldi, Massimiliano Berti, Emanuele Haus, and Riccardo Montalto. Time quasi-periodic gravity water waves in finite depth. *Invent. Math.*, 214(2):739–911, 2018.

[BBM14] Pietro Baldi, Massimiliano Berti, and Riccardo Montalto. Kam for quasi-linear and fully nonlinear forced perturbations of airy equation. *Mathematische Annalen*, 359(1-2):471–536, 2014.

[BGMR17] D. Bambusi, B. Grebert, A. Maspero, and D. Robert. Growth of Sobolev norms for abstract linear Schrödinger Equations. To appear in *Journal of the European Math. Society* (JEMS). Preprint *arXiv:1706.09708*, 2017.

[BGMR18] D. Bambusi, B. Grebert, A. Maspero, and D. Robert. Reducibility of the quantum Harmonic oscillator in d-dimensions with polynomial time dependent perturbation. *Analysis & PDEs*, 11(3):775–799, 2018.

[BL21] Dario Bambusi and Beatrice Langella. A C^∞ Nekhoroshev theorem. *Mathematics in Engineering*, 3 (2): 117, 2021.

[BLM19] Dario Bambusi, Beatrice Langella, and Riccardo Montalto. Reducibility of non-resonant transport equation on $\mathbb{T}^d$ with unbounded perturbations. *Ann. Henri Poincaré*, 20 (6):1893–1929, 2019.

[BLM20] Dario Bambusi, Beatrice Langella, and Riccardo Montalto. On the spectrum of the Schrödinger operator on $\mathbb{T}^d$: a normal form approach. *Communications in Partial Differential Equations*, 45 (4); 303320, 2020.

[BKP15] D. Bambusi, T. Kappeler, and T. Paul. From Toda to KdV. *Nonlinearity*, 28(7):2461–2496, 2015.

[BM16] M. Berti and R. Montalto. Quasi-periodic standing wave solutions of gravity-capillary water waves. *Memoirs of the American Mathematical Society,* Volume 263, Number 1273, 2020.

[BM19] M. Berti and A. Maspero. Long time dynamics of Schrödinger and wave equations on flat tori. *J. Differential Equations*, 267(2):1167–1200, 2019.

[Bou98] J. Bourgain. Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. *Ann. of Math. (2)*, 148(2):363–439, 1998.

[Bou04] Jean Bourgain. *Green's Function Estimates for Lattice Schrodinger Operators and Applications.(AM-158)*. Princeton University Press, 2004.

[EK10] L. Hakan Eliasson and Sergei B. Kuksin. KAM for the nonlinear Schrödinger equation. *Ann. of Math. (2)*, 172(1):371–435, 2010.

[ERT84] Gregory Eskin, James Ralston, and Eugene Trubowitz. On isospectral periodic potentials in $\mathbf{R}^n$. II. *Comm. Pure Appl. Math.*, 37(6):715–753, 1984.

[FGMP19] R. Feola, F. Giuliani, R. Montalto, and M. Procesi. Reducibility of first order linear operators on tori via Moser's theorem. *J. Funct. Anal.*, 276(3):932–970, 2019.

[FGN19] Roberto Feola, Benot Grbert, and Trung Nguyen. Reducibility of schrdinger equation on a zoll manifold with unbounded potential, 2019.

[FKT90] Joel Feldman, Horst Knörrer, and Eugene Trubowitz. The perturbatively stable spectrum of a periodic Schrödinger operator. *Invent. Math.*, 100(2):259–300, 1990.

[FKT91] Joel Feldman, Horst Knörrer, and Eugene Trubowitz. Perturbatively unstable eigenvalues of a periodic Schrödinger operator. *Comment. Math. Helv.*, 66(4):557–579, 1991.

[Fri90] Leonid Friedlander. On the spectrum of the periodic problem for the Schrödinger operator. *Comm. Partial Differential Equations*, 15(11):1631–1647, 1990.

[Gio03] Antonio Giorgilli. Notes on exponential stability of Hamiltonian systems. *Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi"*, 2003.

[Kar96] Yu. E. Karpeshina. Perturbation series for the Schrödinger operator with a periodic potential near planes of diffraction. *Comm. Anal. Geom.*, 4(3):339–413, 1996.

[Kar97] Yulia E. Karpeshina. *Perturbation theory for the Schrödinger operator with a periodic potential*, volume 1663 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[Mar86] V. Marchenko. *Sturm-Liouville operators and applications.* Birkhäuser, Basel, 1986.

[Mon19] Riccardo Montalto. A reducibility result for a class of linear wave equations on $\mathbb{T}^d$. *Int. Math. Res. Not. IMRN*, (6):1788–1862, 2019.

[Nek77] N. N. Nekhorošev. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. *Uspehi Mat. Nauk*, 32(6(198)):5–66, 287, 1977.

[Nek79] N. N. Nekhorošev. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II. *Trudy Sem. Petrovsk.*, (5):5–50, 1979.

[Par08] Leonid Parnovski. Bethe-Sommerfeld conjecture *Annales Henri Poincaré* 9 (2008), 457-508.

[PS10] Leonid Parnovski and Alexander V. Sobolev. Bethe-Sommerfeld conjecture for periodic operators with strong perturbations. *Invent. Math.*, 181(3):467–540, 2010.

[PS09] Leonid Parnovski and Roman Shterenberg. Asymptotic expansion of the integrated density of states of a two dimensional periodic Schrödinger operator. *Inventiones Math.* 176, 275-323, 2009.

[PS12] Leonid Parnovski and Roman Shterenberg. Complete asymptotic expansion of the integrated density of states of multidimensional almost-periodic Schrödinger operator. *Annals of Math.* 176 (2012), 1039-1096

[PX13] Michela Procesi and Xindong Xu. Quasi-Töplitz functions in KAM theorem. *SIAM J. Math. Anal.*, 45(4):2148–2181, 2013.

[Roy07] Nicolas Roy. A semi-classical K. A. M. theorem. *Comm. Partial Differential Equations*, 32(4-6):745–770, 2007.

[SVA] J. Saranen and G. Vainikko. Periodic integral and pseudodifferential equations with numerical approximation. *Springer Monographs in Mathematics, Springer-Verlag,* Berlin, 2002.

[Tay] M. E. Taylor. Pseudodifferential operators and nonlinear PDE. *Progress in Mathematics, Birkhäuser Boston*, vol. 100,, Inc., Boston, MA, 1991.

[Vel15] Oktay Veliev. *Multidimensional periodic Schrödinger operator*, volume 263 of *Springer Tracts in Modern Physics.* Springer, Cham, 2015. Perturbation theory and applications.

[Wan11] W.-M. Wang. Eigenfunction localization for the 2D periodic Schrödinger operator. *Int. Math. Res. Not. IMRN*, (8):1804–1838, 2011.

[Wei77] Alan Weinstein. Asymptotics of eigenvalue clusters for the Laplacian plus a potential. *Duke Math. J.*, 44(4):883–892, 1977.