

QUANTITATIVE STABILITY FOR HYPERSURFACES WITH ALMOST CONSTANT CURVATURE IN SPACE FORMS

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ABSTRACT. The Alexandrov Soap Bubble Theorem asserts that the distance spheres are the only embedded closed connected hypersurfaces in space forms having constant mean curvature. The theorem can be extended to more general functions of the principal curvatures $f(k_1, \dots, k_{n-1})$ satisfying suitable conditions.

In this paper we give sharp quantitative estimates of proximity to a single sphere for Alexandrov Soap Bubble Theorem in space forms when the curvature operator f is close to a constant. Under an assumption that prevents bubbling, the proximity to a single sphere is optimally quantified in terms of the oscillation of the curvature function f . Our approach provides a unified picture of quantitative studies of the method of moving planes in space forms.

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1. INTRODUCTION

In the celebrated papers [4, 5], Alexandrov proved the so-called soap bubble theorem:

Alexandrov theorem: *round spheres are the only C^2 -regular, connected, closed hypersurfaces embedded in the Euclidean space having constant mean curvature.*

As Alexandrov observed in [3, 5], the result can be extended to space forms \mathbb{M}_+^n , i.e. to the hyperbolic space \mathbb{H}^n and the hemisphere \mathbb{S}_+^n (but not in the whole sphere). Here and in the rest of the paper, we denote the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n and the sphere \mathbb{S}^n by

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the same symbol \mathbb{M}^n , while when we write \mathbb{M}_+^n we mean that we are considering \mathbb{R}^n , \mathbb{H}^n and the hemisphere \mathbb{S}_+^n .

Alexandrov's theorem has been widely studied and extended in several directions. It is well-known that the theorem is in general false for non-embedded submanifolds (see e.g. [27] and [46] for classical counterexamples in higher dimension and in \mathbb{R}^3 , respectively). However, for immersed hypersurfaces, one can add some condition in order to guarantee that S is a sphere: in particular Hopf proved in [25] that every constant mean curvature C^2 -regular sphere *immersed* in the 3-dimensional Euclidean space is necessary a round sphere (see also [1, 37, 38] for a very recent generalization of Hopf's theorem to simply-connected homogeneous 3-manifolds), and Barbosa and DoCarmo [6] proved that every compact, orientable and stable hypersurface *immersed* in \mathbb{R}^n is a round sphere (see also [7] for generalizations of this result). Moreover there exists *non-closed* constant mean curvature hypersurfaces embedded in \mathbb{R}^3 which are not diffeomorphic to a sphere, like for instance the unduloids (see [17] and [28] for the generalization to higher dimensions).

In [31] and [44] it is showed that Alexandrov's theorem generalizes to a large class of curvature operators (see also [5, 12, 22, 29, 33, 42, 43, 45, 47]). More precisely, let S be a C^2 -regular, connected, closed hypersurface embedded in \mathbb{M}_+^n . Then S is always the boundary of a relatively compact connected open set $\Omega \subset \mathbb{M}_+^n$ and we orient S by using the normal vector field to S inward with respect to Ω . Let $\{\kappa_1, \dots, \kappa_{n-1}\}$ be the principal curvatures of S ordered increasingly. We denote by H_S one of the following functions:

- i) the mean curvature $H := \frac{1}{n-1} \sum_i \kappa_i$;
- ii) $f(\kappa_1, \dots, \kappa_{n-1})$, where

$$f: \{x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : x_1 \leq x_2 \leq \dots \leq x_{n-1}\} \rightarrow \mathbb{R},$$

is a C^2 -function such that

$$f(x) > 0, \text{ if } x_i > 0 \text{ for every } i = 1, \dots, n-1$$

and f is concave on the component Γ of $\{x \in \mathbb{R}^{n-1} : f(x) > 0\}$ containing $\{x \in \mathbb{R}^{n-1} : x_i > 0\}$.

For instance, if H_r denotes the r -higher order curvature of S defined as the elementary symmetric polynomial of degree r in the principal curvatures of S , then $H_r^{1/r}$ satisfies ii). By using this notation, Alexandrov's theorem can be stated as follows:

Alexandrov theorem II: *distance spheres are the only C^2 -regular, connected, closed hypersurfaces embedded in \mathbb{M}_+^n such that H_S is constant.*

The main result in this paper is the following theorem which gives sharp stability estimates of proximity to a single sphere for Alexandrov theorem II.

Theorem 1.1. *Let S be a C^2 -regular, connected, closed hypersurface embedded in \mathbb{M}_+^n satisfying a uniform touching ball condition of radius ρ . There exist constants $\varepsilon, C > 0$ such that if*

$$(1) \quad \text{osc}(H_S) \leq \varepsilon,$$

then there are two concentric balls B_r^d and B_R^d of \mathbb{M}_+^n such that

$$(2) \quad S \subset \overline{B_R^d} \setminus B_r^d,$$

and

$$(3) \quad R - r \leq C \text{osc}(H_S).$$

The constants ε and C depend only on n and upper bounds on ρ^{-1} and on the area of S .

Here and in the following $\text{osc}(\mathbf{H}_S)$ will be the oscillation of the function \mathbf{H}_S , i.e.

$$\text{osc}(\mathbf{H}_S) := \max_{p \in S} \mathbf{H}_S(p) - \min_{p \in S} \mathbf{H}_S(p).$$

The *uniform touching ball condition* of radius ρ in theorem 1.1 means that at any point of S there exist two balls of radius ρ both tangent to S one from inside and one from outside. Since the constant ρ is fixed, a bubbling phenomenon can not appear (see for instance [14]). As can be shown by a calculation for ellipsoids, the estimate in (3) is optimal and it is new in the general setting of theorem 1.1. In the Euclidean space, quantitative studies for the mean curvature were available in literature only for convex domains and they were not optimal (see the discussion in [15]); hence (3) remarkably improves these results.

Theorem 1.1 follows the research line initiated in [15] and pursued in [16]. The aim of the present paper is twofold: on the one hand we generalize the results in [15] and [16] to the hemisphere and on the other hand we extend our analysis to a larger class of curvature operators (in [15] and [16] we only focused on the mean curvature). To achieve this goal, we provide a uniform quantitative approach of the method of the moving planes in space forms. Together with the wide generality of the assumptions of this theorem, the unified treatment of the problem in space forms is a major point of this manuscript.

As a consequence of theorem 1.1 we have the following

Corollary 1.2. *Let $\rho_0, A_0 > 0$ and $n \in \mathbb{N}$ be fixed. There exists $\varepsilon > 0$, depending on n, ρ_0 and A_0 , such that if S is a connected closed C^2 hypersurface embedded in \mathbb{M}_+^n having area bounded by A_0 , satisfying a touching ball condition of radius $\rho \geq \rho_0$, and such that*

$$(4) \quad \text{osc}(\mathbf{H}_S) \leq \varepsilon,$$

then S is C^1 -close to a sphere and there exists a C^2 -regular map $\Psi : \partial B_r^d \rightarrow \mathbb{R}$ such that

$$F(p) = \exp_x(\Psi(p)N_p)$$

defines a C^2 -diffeomorphism from B_r^d to S and

$$(5) \quad \|\Psi\|_{C^1(\partial B_r)} \leq C \text{osc}(\mathbf{H}_S)^{1/2},$$

where N is a normal vector field to ∂B_r^d .

As already mentioned, we tackle the problem by doing a quantitative study of the method of the moving planes, i.e. we study the original proof of Alexandrov from a quantitative point of view. There are other possible approaches for proving the symmetry result which are based on integral and geometric identities. The interested reader can refer to [39, 41, 43, 42], and to [10] for a recent generalization (see also [18] for minimal assumptions on the regularity of S). The approach in [41] has been exploited in [14] to tackle the problem of bubbling (see also [19] and [32]). Symmetry and quantitative studies of proximity to a single sphere by using an integral approach can be found in [9, 11, 13, 20, 23, 24, 34, 35, 36, 40].

The paper is organized as follows. In section 2 we review the method of the moving planes in space forms and set up some notation. In section 3 we give technical local quantitative estimates in space forms. In section 4 we find estimates on curvatures of projected surfaces in conformally Euclidean spaces. In section 5 we prove the approximate symmetry in one direction. In section 6 we show how to prove global approximate symmetry result by using the approximate symmetry in any direction. In section 7 we complete the proof of the main theorems.

We will use the following models of space forms:

- \mathbb{H}^n is the half-space $\{x \in \mathbb{R}^n : x_n > 0\}$ with the Riemannian metric

$$(6) \quad g_x = \frac{1}{x_n} \langle \cdot, \cdot \rangle, \quad \text{for every } x \in \mathbb{R}^n$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean product on \mathbb{R}^n ;

- \mathbb{S}^n is the n -dimensional unitary sphere $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$ with the round metric g induced by the Euclidean metric in \mathbb{R}^{n+1} . Here we recall that if we consider the stereographic projection $\mathbb{S}^n \setminus \{\text{one point}\} \rightarrow \mathbb{R}^n$, then g is projected to the Riemann metric

$$(7) \quad g_x = \frac{4}{(1 + |x|^2)^2} \langle \cdot, \cdot \rangle, \quad \text{for every } x \in \mathbb{R}^n.$$

In order to simplify the reading of the paper, there is a list of symbols at the end of the manuscript.

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2. THE METHOD OF THE MOVING PLANES

In this preliminary section we recall the method of the moving planes in \mathbb{M}_+^n (see [3, 4, 5]).

We begin by recalling the definition of *center of mass* in the context of Riemannian geometry (see e.g. [30] for more details).

Let (M, g) be an oriented complete Riemannian manifold and let Ω be a bounded domain (i.e. a bounded connected open set). Let $P_\Omega: M \rightarrow \mathbb{R}$ be the function

$$P_\Omega(x) = \frac{1}{2|\Omega|_g} \int_\Omega d(x, a)^2 da,$$

where $|\Omega|_g$ is the volume of Ω with respect to g . Then the gradient of P_Ω takes the following expression

$$(8) \quad \nabla P_\Omega(x) = -\frac{1}{|\Omega|_g} \int_\Omega \exp_x^{-1}(a) da.$$

In some cases P_Ω is a convex map and it attains the minimum at only one point \mathcal{O} , which is usually called the *center of mass* of Ω . For instance this occurs in the following cases:

- all the sectional curvatures of M are nonpositive;
- Ω is contained in a geodesic ball of radius $r < \frac{1}{2} \min \left\{ \text{inj } M, \frac{\pi}{2\sqrt{K}} \right\}$, where K is an upper bound on the sectional curvatures of M ;
- $M = \mathbb{S}^n$ and Ω is contained in \mathbb{S}_+^n .

We further recall that a Riemannian manifold (M, g) is a symmetric space if for every $p \in M$ there exists an isometry $f: M \rightarrow M$ such that $f(p) = p$ and $f_{*|p} = -\text{Id}$. In the following statement, by a *hyperplane* in a symmetric space we mean a complete totally geodesic hypersurface.

Lemma 2.1. *Let (M, g) be a symmetric space, Ω a bounded domain in M and $x \in M$ be such that $\nabla P_\Omega(x) = 0$. Assume that for every hyperplane π in M not containing x there exists a hyperplane π_1 passing through x and such that $\pi \cap \pi_1 \cap \Omega = \emptyset$. Then every hyperplane of symmetry for Ω contains x .*

Proof. Assume by contradiction that there exists a hyperplane π of symmetry for Ω not containing x . Let π_1 be a hyperplane passing through x and disjoint from π inside Ω , i.e. $\pi \cap \pi_1 \cap \Omega = \emptyset$. Since π_1 and π are disjoint, they subdivide Ω in three disjoint subsets Ω_1 , Ω_2 and Ω_3 , with $|\Omega_i|_g > 0$, $i = 1, 2, 3$. Since Ω is symmetric about π , we have that

$$|\Omega_1|_g + |\Omega_2|_g = |\Omega_3|_g.$$

Moreover formula (8) implies

$$(9) \quad \int_{\Omega_1} \exp_x^{-1}(a) da = - \int_{\Omega_2 \cup \Omega_3} \exp_x^{-1}(a) da.$$

Let $f: M \rightarrow M$ be the isometry such that $f(x) = x$ and $f|_{*x} = -\text{Id}$. Then

$$\exp_x^{-1} f(a) = -\exp_x^{-1}(a)$$

and

$$- \int_{\Omega_2 \cup \Omega_3} \exp_x^{-1}(a) da = \int_{f(\Omega_2 \cup \Omega_3)} \exp_x^{-1}(a) da.$$

Therefore (9) implies

$$|\Omega_1|_g = |\Omega_2|_g + |\Omega_3|_g.$$

which gives a contradiction since $|\Omega_2|_g > 0$. \square

Lemma 2.1 can be in particular applied in space forms \mathbb{M}_+^n , where we have the uniqueness of the center of mass.

Proposition 2.2 (Characterization of the distance balls in \mathbb{M}_+^n). *Let $S = \partial\Omega$ be a C^2 -regular, connected, closed hypersurface embedded in \mathbb{M}_+^n , where Ω is a relatively compact domain. Assume that for every geodesic path $\gamma: \mathbb{R} \rightarrow \mathbb{M}^n$ there exists a hyperplane π orthogonal to γ such that S is symmetric about π . Then S is a distance sphere about \mathcal{O} .*

Proof. In view of lemma 2.1 any hyperplane of symmetry of S contains the point \mathcal{O} . Therefore S is invariant by reflections about every hyperplane passing through \mathcal{O} . Since every orientation preserving isometry of \mathbb{M}^n fixing \mathcal{O} can be obtained as the composition of reflections about hyperplanes containing \mathcal{O} , S is invariant by rotations and then it is a distance sphere. \square

We describe the method of the moving planes in \mathbb{M}_+^n and we introduce some notation. The method consists in moving hyperplanes along a geodesic path orthogonal to a fixed direction and it is similar in \mathbb{R}^n , \mathbb{H}^n and \mathbb{S}_+^n . The method can be described in terms of a point \mathfrak{o} that we fix. Since \mathbb{M}_+^n is a homogeneous space, the construction does not depend on the choice of the point we fix. In particular we choose \mathfrak{o} to be: the origin in \mathbb{R}^n , e_n in \mathbb{H}^n and the north pole in \mathbb{S}_+^n . For every direction $v \in T_{\mathfrak{o}}\mathbb{M}_+^n$, we consider the geodesic path $\gamma_v: I \rightarrow \mathbb{M}_+^n$ satisfying $\gamma_v(0) = \mathfrak{o}$ and $\dot{\gamma}_v(0) = v$. The domain of γ_v is $I = \mathbb{R}$ in \mathbb{R}^n and \mathbb{H}^n and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ in \mathbb{S}_+^n . For any $s \in I$ we denote by $\pi_{v,s}$ the hyperplane passing through $\gamma_v(s)$ and orthogonal to $\dot{\gamma}_v(s)$, and we define

$$S_{v,s} = \{p \in S : p \in \pi_{v,t} \text{ for some } t > s\}.$$

We denote by $S_{v,s}^\pi$ be the reflection of $S_{v,s}$ about $\pi_{v,s}$. Note that

- $S_{v,s} = \{p \in S : p \cdot v > s\}$, if $\mathbb{M}_+^n = \mathbb{R}^n$;
- $S_{v,s} = \{p \in S : p \cdot \dot{\gamma}_v(s) > 0\}$, if $\mathbb{M}_+^n = \mathbb{S}_+^n$.

In the hyperbolic case, giving an explicit description of $S_{v,s}$ is more complicated, but it can be simplified by assuming $v = e_1$. This assumption is not restrictive since we can always rotate every direction v in e_1 by using an isometry of \mathbb{H}^n . In this case we have

- $S_{e_1,s} = \{p \in S : p \cdot e_1 > s\}$, if $\mathbb{M}_+^n = \mathbb{H}^n$.

Let

$$m_v = \inf\{s \in I : S_{v,t}^\pi \subset \Omega \text{ for every } t > s\}.$$

The hyperplane $\pi_v := \pi_{v,m_v}$ is called the *critical hyperplane*. By construction S_{v,m_v}^π is contained in $\bar{\Omega}$ and S and S_{v,m_v}^π are tangent at some point p_0 which can be either interior to S_{v,m_v}^π , or on $\partial S_{v,m_v}^\pi$ (and in this last case $p_0 \in \pi_v$).

Now for the sake of completeness we recall the generalized version of Alexandrov's theorem that we study in this paper and its proof in \mathbb{M}_+^n (see [5, 31, 39, 41, 42, 43]).

Theorem 2.3. *The only closed C^2 -regular connected hypersurfaces embedded in \mathbb{M}_+^n and such that H_S is constant are the distance spheres.*

Proof. The proof consists in showing that for every unitary vector $v \in T_0\mathbb{M}_+^n$, S is symmetric about π_v . This is obtained by showing that $S \cap S_{v,m_v}^\pi$ is open and closed in S_{v,m_v}^π . Note that $S \cap S_{v,m_v}^\pi$ is not empty since S_{v,m_v}^π is tangent to S at some point and $S \cap S_{v,m_v}^\pi$ is closed in S_{v,m_v}^π . The only nontrivial step is that $S \cap S_{v,m_v}^\pi$ is open, which is obtained by using maximum principles for solutions to elliptic equations.

Let $p_0 \in S \cap S_{v,m_v}^\pi$. By construction we have that

$$T_{p_0}S = T_{p_0}S^\pi,$$

where S^π is the reflection of S about π_v . From the implicit function theorem, S and S^π are locally the Euclidean graph of C^2 -regular functions u and \hat{u} , respectively, defined in a ball B_r of radius r centered at the origin O in $T_{p_0}S$. The functions u and \hat{u} satisfy the elliptic equation $Lu(x) = H_S(x, u(x))$ for $x \in B_r$; here the operator L is the mean curvature operator or, more generally, a fully nonlinear operator. The ellipticity of L is standard in the case of the mean curvature operator and it follows from [31] for the other cases considered. The proof is standard and hence it is omitted.

Since H_S is constant, the difference $u - \hat{u}$ satisfies an elliptic equation of the form $\mathcal{L}(u - \hat{u}) = 0$ with $u(O) - \hat{u}(O) = 0$.

If p_0 is interior to S_{v,m_v}^π then we can choose r sufficiently small such that $u - \hat{u} \geq 0$ in B_r and by the strong maximum principle we obtain that $u - \hat{u} \equiv 0$ in B_r .

If p_0 is on the boundary of S_{v,m_v}^π , then by construction $u - \hat{u} \geq 0$ in a half ball B_r^+ , $u(O) - \hat{u}(O) = 0$ and $\nabla u(O) = \nabla \hat{u}(O) = 0$. By applying Hopf's boundary point lemma at the point O we obtain that $u - \hat{u} \equiv 0$ in B_r^+ .

Hence, we have proved that $S \cap S_{v,m_v}^\pi$ is open, and the conclusion follows. \square

3. LOCAL QUANTITATIVE ESTIMATES IN \mathbb{M}^n

In this section we prove some preliminary estimates which we will use in the proof of the main theorem. We have the following preliminary lemma about the local equivalences of distances.

Lemma 3.1. *• Let d be the distance induced by the hyperbolic metric (6) in \mathbb{H}^n and let q be such that $d(q, e_n) < R$; then*

$$(10) \quad c|q - e_n| \leq d(q, e_n) \leq C|q - e_n|,$$

for some positive constants c and C depending only on R .

• Let d be the distance induced by the round metric (7) in \mathbb{R}^n and let p, q in \mathbb{R}^n be such that $|p|, |q| \leq R$. Then

$$(11) \quad \frac{2}{1+R^2}|p - q| \leq d(p, q) \leq \pi|p - q|.$$

Let us consider now a C^2 -regular, connected, closed hypersurface $S = \partial\Omega$ embedded in \mathbb{M}_+^n , where Ω is a relatively compact domain and denote by N the inward normal vector field (inward with respect to Ω). For $p \in S$ we denote by $\varphi_p: \mathbb{M}_+^n \rightarrow \mathbb{R}^n$ the following function whose definition depends on the geometry of \mathbb{M}^n :

- if \mathbb{M}^n is \mathbb{R}^n , $\varphi_p \in \text{SO}(n) \ltimes \mathbb{R}^n$ and it is such that $\varphi_p(p) = 0$ and $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$;
- if \mathbb{M}^n is \mathbb{H}^n , φ_p is an orientation preserving isometry of \mathbb{H}^n such that $\varphi_p(p) = e_n$, $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$;
- if \mathbb{M}^n is \mathbb{S}^n , φ_p is the stereographic projection from the antipodal point to p restricted to \mathbb{S}_+^n composed with a rotation of \mathbb{R}^n in order to have $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$.

Note that in all the three cases we have that $\varphi_p(S)$ is a hypersurface embedded in \mathbb{R}^n and

$$\varphi_{p*|_p}(T_p S) = \{x_n = 0\}.$$

For $r > 0$, we denote by $\mathcal{U}_r(p)$ the open neighborhood of p in S such that $\varphi_p(\mathcal{U}_r(p))$ is the (Euclidean) graph of a C^2 -function $u: B_r \rightarrow \mathbb{R}$ defined in the ball of radius r of \mathbb{R}^{n-1} centered at the origin. Even if we don't have a canonical choice of φ_p , the subsets $\mathcal{U}_r(p)$ do not depend on the choice of φ_p . Moreover, the implicit function theorem implies that any $p \in S$ has a neighborhood $\mathcal{U}_r(p)$ for r sufficiently small. In order to establish the quantitative estimates we need in the proof of the main theorem, we have to show that r can be uniformly bounded from below with a bound depending only on ρ . The following lemma also introduces the quantity ρ_1 which will be largely used in the paper.

Lemma 3.2. *Let S be a C^2 -regular closed hypersurface embedded in \mathbb{M}_+^n and satisfying a touching ball condition of radius ρ and let ρ_1 be defined in the following way:*

- $\rho_1 = \rho$, if $\mathbb{M}^n = \mathbb{R}^n$;
- $\rho_1 = (1 - e^{-\rho} \sinh \rho) e^{-\rho} \sinh \rho$, if $\mathbb{M}^n = \mathbb{H}^n$;
- $\rho_1 = \frac{\rho}{\pi}$, if $\mathbb{M}^n = \mathbb{S}^n$.

Then

- (i) *any point $p \in S$ admits a neighborhood $\mathcal{U}_{\rho_1}(p)$ and $\varphi_p(\mathcal{U}_{\rho_1}(p))$ is the graph of a C^2 -function $u: B_{\rho_1} \rightarrow \mathbb{R}$ satisfying*

$$(12) \quad |u(x) - u(O)| \leq \rho_1 - \sqrt{\rho_1^2 - |x|^2}, \quad |\nabla u(x)| \leq \frac{|x|}{\sqrt{\rho_1^2 - |x|^2}};$$

- (ii) *there exists a universal constant C such that for any $0 < \alpha < \frac{1}{2} \min(1, \rho_1^{-1})$ and q in $\mathcal{U}_{\alpha\rho_1}(p)$ we have*

$$(13) \quad d_S(p, q) \leq \alpha C \rho_1,$$

where d_S is the geodesic distance on S .

Proof. We refer to [15, Lemma 2.1] for the Euclidean case and we show how to deduce the statement from the Euclidean case (the hyperbolic case has been already done in [16]).

- (i) It is enough to observe that $\varphi_p(S)$ satisfies an Euclidean touching ball condition of radius ρ_1 (this motivates the choice of ρ_1). This can be easily deduced by using lemma 3.1. Hence the first item of the statement follows from the Euclidean case.
- (ii) Let $q \in \mathcal{U}_{\rho_1}(p)$. Then $\varphi_p(q) = (x, v(x))$ for some $|x| < \rho_1$. Let $\gamma: [0, 1] \rightarrow \varphi_p(S)$ be the curve joining $\varphi_p(p)$ to $\varphi_p(q)$ defined as $\gamma(t) = (tx, v(tx))$. Then

$$\dot{\gamma}(t) = (x, \nabla v(tx) \cdot x)$$

and the Cauchy-Schwarz inequality implies

$$|\dot{\gamma}(t)| \leq |x| \sqrt{1 + |\nabla v(tx)|^2}.$$

Then (12) yields

$$|\dot{\gamma}(t)| \leq \frac{\rho_1 |x|}{\sqrt{\rho_1^2 - t^2 |x|^2}} \leq \frac{|x|}{\sqrt{1 - \alpha^2}} \leq \frac{2}{\sqrt{3}} |x|,$$

for $0 \leq |x| \leq \alpha \rho_1$. Since

$$d_S(p, q) \leq l(\gamma)$$

and

$$\begin{aligned} l(\gamma) &= \int_0^1 \frac{|\dot{\gamma}|}{v(tx)} dt, \text{ in the hyperbolic case,} \\ l(\gamma) &= \int_0^1 \frac{2}{1 + |\gamma|^2} |\dot{\gamma}| dt, \text{ in the spherical case,} \end{aligned}$$

we obtain that

$$d_S(p, q) \leq C|x|$$

for a universal constant C and (13) follows. □

The following result follows from Lemma 3.2

Corollary 3.3. *Let S be a compact C^2 -regular embedded hypersurface in \mathbb{M}_+^n satisfying a touching ball condition of radius ρ . Let $q \in \mathcal{U}_{\alpha\rho_1}(p)$, with $0 < \alpha < \frac{1}{2} \min(1, \rho_1^{-1})$. Then*

$$d_S(p, q) \leq C d(p, q),$$

where C depends only on ρ . In particular for every $p \in S$ the geodesic ball $\mathcal{B}_r(p)$ centered at p and with radius $r < \frac{1}{2} \min(1, \rho_1^{-1})$ satisfies

$$(14) \quad \text{Area}(\mathcal{B}_r(p)) \geq cr^{n-1},$$

and c is a constant depending only on n .

3.1. Quantitative stability of the parallel transport. We study quantitative estimates involving the parallel transport. We adopt the following notation (see also [16]):

given $p, q \in \mathbb{M}_+^n$ with $p \neq q$, we denote by $\tau_p^q: T_p \mathbb{M}^n \rightarrow T_q \mathbb{M}^n$ the parallel transport along the unique geodesic path σ connecting p to q . We further assume that τ_p^p is the identity of $T_p \mathbb{M}^n$.

Remark 3.4. If \mathbb{M}^n is \mathbb{R}^n , then τ_p^q is the identity for every p, q .

In the case $\mathbb{M}^n = \mathbb{H}^n$, if p and q belongs to the same vertical line, then we have $\tau_p^q(v) = \frac{p_n}{q_n} v$.

In the case \mathbb{S}_+^n , we have $\tau_p^q = (P_q \circ R_\alpha)|_{T_p \mathbb{S}^n}$, where R_α acts as the rotation of the angle $\alpha = d(p, q)$ in the plane π containing σ and as the identity in the orthogonal complement of π and P_q is the orthogonal projection onto $T_q \mathbb{S}^n$.

Here we introduce the following notation:

- given $p \in \mathbb{M}^n$ and $v \in T_p \mathbb{M}^n$, $|v|_p := g_p(v, v)^{1/2}$;
- if $S = \partial\Omega$ is a compact C^2 -regular embedded hypersurface in \mathbb{M}_+^n , where Ω is a relatively compact domain in \mathbb{M}_+^n , N is the inward unitary normal vector field on S .

The first result we prove is the following

Proposition 3.5. *Let S be a compact C^2 -regular embedded hypersurface in \mathbb{M}_+^n satisfying a touching ball condition of radius ρ . There exists $\delta_0 = \delta_0(\rho)$ such that if $p, q \in S$ with $d_S(p, q) \leq \delta_0$ then*

$$(15) \quad g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2 d_S(p, q)^2} \quad \text{and} \quad |N_p - \tau_q^p(N_q)|_p \leq C d_S(p, q),$$

where C is a constant depending only on ρ .

Proof. We have already proved the assertion in the Euclidean and in the hyperbolic space in [15] and [16], respectively, and here we focus on the spherical case. It is convenient to regard S as a hypersurface of \mathbb{R}^n equipped with the spherical metric (7). We may further assume that p is the origin O of \mathbb{R}^n and q belongs to a straight line passing through O .

Let $\delta_0 = \min(\rho_1, \frac{1}{C})$, where C will be specified later. If $d(p, q) \leq \rho_1 = \frac{\rho}{\pi}$, then we can apply the Euclidean estimates in [15, Lemma 2.1] and obtain

$$\nu_p \cdot \nu_q \geq \sqrt{1 - \frac{|p - q|^2}{\rho_1^2}},$$

where ν denotes the Euclidean inward normal vector field on S . Since

$$d(p, q) \leq \pi |p - q|,$$

we have

$$(16) \quad \nu_p \cdot \nu_q \geq \sqrt{1 - C^2 d_S(p, q)^2},$$

where C depends only on ρ . Moreover the inward g -unitary normal vector field N to S satisfies

$$N_p = \frac{1}{2} \nu_p, \quad \frac{2}{1 + |q|^2} N_q = \nu_q = 2\tau_q^p(N_q),$$

and (16) implies

$$g_p(N_p, \tau_q^p(N_q)) \geq \frac{1}{2} \sqrt{1 - C^2 d_S(p, q)^2},$$

which is the first inequality in (15). The second inequality in (15) follows by a direct computation and the claim follows. \square

We recall the following technical lemma proved in [16]. This is a useful lemma since it is the starting point from which we deduce the C^1 -closedness of two hypersurfaces in terms of the hyperbolic metric starting from an Euclidean information.

Lemma 3.6. *Let Σ and $\hat{\Sigma}$ be two compact embedded hypersurfaces in \mathbb{H}^n satisfying both a touching ball condition of radius ρ . Assume that $e_n \in \Sigma$ and $T_{e_n}\Sigma = \{x_n = 0\}$ and that there exist two local parametrizations $u, \hat{u} : B_r \rightarrow \mathbb{R}$ of Σ and $\hat{\Sigma}$, respectively, with $0 < r \leq \rho_1$ and such that $u - \hat{u} \geq 0$. Let $p_1 = (x_1, u(x_1))$ and $\hat{p}_1^* = (x_1, \hat{u}(x_1))$, with $x_1 \in \partial B_{r/4}$, and denote by γ the geodesic path starting from p_1 and tangent to $-\nu_{p_1}$ at p_1 . Assume that*

$$(17) \quad d(p_1, \hat{p}_1^*) + |\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq \theta,$$

for some $\theta \in [0, 1/2]$, where ν is the Euclidean unitary normal vector field to Σ . There exists \bar{r} depending only on ρ such that if $r \leq \bar{r}$ we have that $\gamma \cap \hat{\Sigma} \neq \emptyset$ and, if we denote by \hat{p}_1 the first intersection point between γ and $\hat{\Sigma}$, then

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta,$$

where C is a constant depending only on n and ρ , and provided that $C\theta < 1/2$.

The last result of this section is the “spherical counterpart” of lemma 3.6.

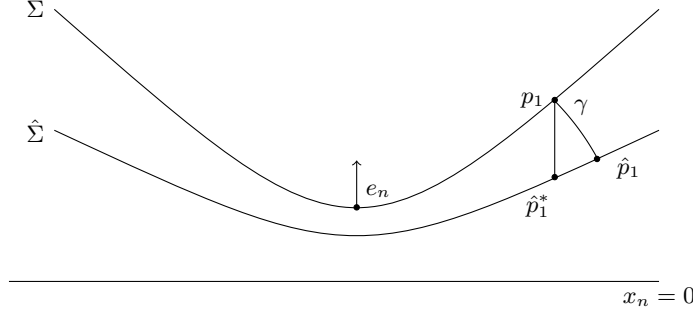


FIGURE 1. The statement of lemma 3.6

Lemma 3.7. *Let Σ and $\hat{\Sigma}$ be two compact embedded hypersurfaces in \mathbb{R}^n with the round metric (7) satisfying a touching ball condition of radius ρ . Assume $O \in \Sigma$ and $T_O \Sigma = \{x_n = 0\}$ and that there exist two local parametrizations $u, \hat{u} : B_r \rightarrow \mathbb{R}$ of Σ and $\hat{\Sigma}$, respectively, with $0 < r \leq \rho_1$ and such that $u - \hat{u} \geq 0$. Let $p_1 = (x_1, u(x_1))$ and $\hat{p}_1^* = (x_1, \hat{u}(x_1))$, with $x_1 \in \partial B_{r/4}$, and denote by γ the geodesic path starting from p_1 and tangent to $-\nu_{p_1}$ at p_1 . Assume that*

$$(18) \quad d(p_1, \hat{p}_1^*) + |\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq \theta,$$

for some $\theta \in [0, 1/2]$, where ν is the Euclidean unitary normal vector field to Σ . There exists \bar{r} depending only on ρ such that if $r \leq \bar{r}$ we have that $\gamma \cap \hat{\Sigma} \neq \emptyset$ and, if we denote by \hat{p}_1 the first intersection point between γ and $\hat{\Sigma}$, then

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta,$$

where C is a constant depending only on n and ρ , and provided that $C\theta < 1/2$.

Proof. We first notice that, by choosing r small enough in terms of ρ , from lemma 3.2 we have that $|\nu_{p_1} - e_n| \leq 1/4$. We observe that the geodesic γ is almost flat, i.e., viewed as an Euclidean circle its radius R satisfies

$$(19) \quad R = O\left(\frac{1}{|x_1|^2}\right) \text{ as } |x_1| \rightarrow 0.$$

Indeed, up to apply a rotation, we may assume that both p_1 and ν_{p_1} belong to the plane π_1 spanned by $\{e_1, e_2\}$. In this way, the geodesic path γ belongs to the plane π_1 and we can work in a “bidimensional way”. We can write $p_1 = (x_1, y_1)$ and $\nu_{p_1} = (\nu_1, \nu_2)$ and we compute the geodesic γ passing through p_1 and tangent to ν_{p_1} . We solve

$$\begin{cases} (x_1 + a)^2 + (y_1 + b)^2 = 1 + a^2 + b^2 \\ (x_1 + a, y_1 + b) \cdot (\nu_1, \nu_2) = 0 \end{cases}$$

and we find

$$a = \frac{(1 - |p_1|^2)\nu_2 + 2y_1(p_1 \cdot \nu)}{2(x_1\nu_2 - y_1\nu_1)}, \quad b = \frac{-(1 - |p_1|^2)\nu_1 - 2x_1(p_1 \cdot \nu)}{2(x_1\nu_2 - y_1\nu_1)}.$$

If $|x_1| \rightarrow 0$, according to lemma 3.2, we have that $y_1 = O(x_1^2)$, $\nu_1 = O(x_1)$ and $\nu_2 = 1 + o(1)$; so we get that $a \sim \frac{1}{2x_1}$, b is bounded and (19) follows.

Let B^+ and B^- be the exterior and interior touching balls of $\hat{\Sigma}$ at $\hat{p}_0 = (0, \hat{u}(0))$, respectively. A standard geometrical argument shows that it is possible to choose \bar{r} small enough in terms of ρ such that γ intersects B^+ and B^- at points which are distant from the origin less than \bar{r} . This implies the existence of the point \hat{p}_1 in the assertion for any $r \leq \bar{r}$.

Now we estimate the distance between p_1 and \hat{p}_1 as follows. Let q be the unique point having distance 2ε from p_1 and lying on the geodesic path containing p_1 and \hat{p}_1^* . Let T be the geodesic right-angle triangle having vertices p_1 and q and hypotenuse contained in the geodesic passing through p_1 and \hat{p}_1 (see figure 2). Since the angle α at the vertex p_1 is such that $|\sin \alpha| \leq 1/4$, then from the cosine rule for spherical triangles we have that

$$(20) \quad d(p_1, \hat{p}_1) \leq C\theta.$$

Moreover, the triangle inequality gives that

$$(21) \quad d(\hat{p}_1^*, \hat{p}_1) \leq C\theta$$

for some constant C , and from (15) we obtain that

$$(22) \quad |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq |N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} + |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}.$$

Since p_1 and \hat{p}_1^* are on the same vertical line (18) implies

$$(23) \quad |N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} = |\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq C\theta.$$

As next step we show that

$$(24) \quad |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta.$$

We obtain (24) by showing that if $p, q \in \mathbb{R}^n$ belong to the Euclidean ball centered at the origin and having radius s , then

$$(25) \quad 2|\tau_p^O(v) - \tau_q^O(w)| \leq \frac{(1+s)^2}{4} (d(p, O)^2 + d(q, O)^2) + |v - \tau_q^p(w)|_p + \frac{9}{2}d(p, q).$$

for every $v, w \in \mathbb{R}^n$, $|v|_q = |w|_{\hat{q}} = 1$. We have

$$\tau_p^O(v) = \frac{1}{1+|p|^2}v, \quad \tau_q^O(w) = \frac{1}{1+|q|^2}w.$$

and using Cauchy-Schwarz inequality and taking into account lemma 3.1 we have

$$||q|^2 - |p|^2| = |(q-p) \cdot (q+p)| \leq |q-p||q+p| \leq s(1+s^2)d(p, q).$$

Since

$$|w| = \frac{1+|\hat{q}|^2}{2}$$

we have

$$2 \left| \frac{1}{1+|q|^2} \tau_p^O(w) - \frac{1}{1+|q|^2} \tau_q^O(w) \right| = 2 \left| \frac{|q|^2 - |p|^2}{(1+|p|^2)(1+|q|^2)} \right| \leq 2s(1+s^2)d(p, q).$$

Now using

$$2|\tau_p^O(v)| = 1, \quad 2|\tau_q^O(w)| = 1;$$

and

$$|v| = \frac{1+|p|^2}{2}, \quad |w| = \frac{1+|q|^2}{2},$$

we compute

$$\begin{aligned}
2|\tau_p^O(v) - \tau_q^O(w)| &\leq 2 \left| \tau_p^O(v) - \frac{1}{1+|p|^2} \tau_p^O(v) \right| + 2 \left| \frac{1}{1+|p|^2} \tau_p^O(v) - \tau_q^O(w) \right| \\
&\leq 2 \left| 1 - \frac{1}{1+|p|^2} \right| |\tau_p^O(v)| + 2 \left| \frac{1}{1+|p|^2} \tau_q^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| \\
&\quad + 2 \left| \frac{1}{1+|q|^2} \tau_p^O(w) - \frac{1}{1+|q|^2} \tau_q^O(w) \right| + 2 \left| 1 - \frac{1}{1+|q|^2} \right| |\tau_q^O(w)| \\
&\leq \left| 1 - \frac{1}{1+|p|^2} \right| + 2 \left| \frac{1}{1+|p|^2} \tau_q^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| + 4d(p, q) + \left| 1 - \frac{1}{1+|q|^2} \right| \\
&\leq |p|^2 + 2 \left| \frac{1}{1+|p|^2} \tau_q^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| + 4d(p, q) + |q|^2 \\
&\leq \frac{(1+s)^2}{4} (d(p, O)^2 + d(q, O)^2) + 2 \left| \frac{1}{1+|p|^2} \tau_p^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| + 4d(p, q).
\end{aligned}$$

Now we show that

$$2 \left| \frac{1}{1+|p|^2} \tau_p^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| \leq |v - \tau_q^p(w)|_p + \frac{1}{2} d(p, q).$$

Let σ be the geodesic path connecting p with q . Then σ is contained in a circle of \mathbb{R}^n and denotes by C its center and by α the angle between $p - C$ and $q - C$. Then

$$\frac{1+|p|^2}{1+|q|^2} w = R_\alpha \tau_q^p w, \text{ for every } w \in \mathbb{R}^n,$$

where R_α is the rotation (clockwise or anti-clockwise) about α in the plane containing C and is the identity in the complement. Therefore we have

$$\begin{aligned}
2 \left| \frac{1}{1+|p|^2} \tau_p^O(v) - \frac{1}{1+|q|^2} \tau_p^O(w) \right| &= \left| \frac{1}{1+|p|^2} v - \frac{1}{1+|q|^2} w \right|_p \leq \left| v - \frac{1+|p|^2}{1+|q|^2} w \right|_p = |v - R_\alpha \tau_q^p w|_p \\
&\leq |v - \tau_q^p w|_p + |\tau_q^p w - R_\alpha \tau_q^p w|_p
\end{aligned}$$

and, consequently, we deduce,

$$|\tau_q^p w - R_\alpha \tau_q^p w|_p \leq |\alpha| \leq \frac{1}{2} d(p, q)$$

which implies (25).

Therefore, by applying (25) to $|\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}$, we have

$$\begin{aligned}
|\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} &\leq \frac{(1+C\theta)^2}{4} (d(p_1, \hat{p}_1^*)^2 + d(p_1, \hat{p}_1)^2) + |N_{\hat{p}_1^*} - \tau_{\hat{p}_1}^{\hat{p}_1^*}(N_{\hat{p}_1^*})|_{\hat{p}_1^*} + \frac{9}{2} d(\hat{p}_1, \hat{p}_1^*) \\
&\leq C\theta,
\end{aligned}$$

where the last inequality follows from (18),(20),(21) and (15). This last inequality, (22) and (23) imply that

$$|N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta,$$

and therefore from (20) we conclude. \square

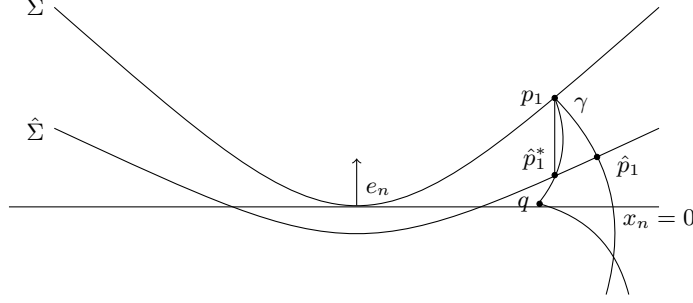


FIGURE 2. The statement and the proof of lemma 3.7

4. CURVATURES OF PROJECTED SURFACES IN CONFORMALLY EUCLIDEAN SPACES

In this section we consider a connected open set Ω in \mathbb{R}^n equipped with a metric $g = h^2 \langle \cdot, \cdot \rangle$ conformal to the Euclidean metric. We further assume the existence of an Euclidean hyperplane π of \mathbb{R}^n such that $\Omega \cap \pi$ is a totally geodesic hypersurface in Ω . This setting includes the Euclidean space, the hyperbolic space and \mathbb{R}^n with the round metric (7). For instance in the half-space model of the hyperbolic space we can take as π any vertical Euclidean hyperplane; in the spherical case we can consider Euclidean hyperplanes passing through the origin.

For our purposes, we consider a hypersurface U of class C^2 embedded in Ω which intersects π transversally. The implicit function theorem implies that $U' = U \cap \pi$ is a C^2 -submanifold of π . Furthermore if ν_q is an Euclidean unit normal vector field to U and w is a unit normal vector to π , we have that

$$\nu'_q = \frac{(-1)^n * (*(\nu_q \wedge w) \wedge w)}{|*(*(\nu_q \wedge w) \wedge w)|}, \quad q \in U',$$

is an Euclidean unitary normal vector field to U' in π , where $*$ is the Euclidean Hodge star operator in \mathbb{R}^n . In particular U' is orientable in π . Let

$$N_q = \frac{1}{h(q)} \nu_q, \quad N'_q = \frac{1}{h(q)} \nu'_q$$

be the normal vectors with respect to the metric g and

$$\omega_p = \frac{1}{h(p)} w, \quad p \in \Omega.$$

Proposition 4.1. *Let κ_j , $j = 1, \dots, n-1$ be the principal curvatures of U with respect to the metric g and to the orientation induced by N . Then the principal curvatures κ'_i of U' (viewed as submanifold of π) with respect to the orientation induced by N' satisfy*

$$(26) \quad \frac{1}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \kappa_1(q) \leq \kappa'_i(q) \leq \frac{1}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \kappa_{n-1}(q),$$

for every $q \in U'$. Moreover the principal curvatures $\tilde{\kappa}'_i$ of U' , seen as a hypersurface of U , satisfy

$$(27) \quad |\tilde{\kappa}'_i(q)| \leq \frac{|g_q(\omega_q, N_q)|}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\},$$

for every $q \in U'$.

Proof. Let $v \in T_q U'$ satisfy $|v|_q = 1$ and

$$\kappa_q(v) = g_q(\nabla_v \tilde{N}, v),$$

where \tilde{N} denotes an extension of N in Ω and ∇ is the Levi-Civita connection of g . For $p \in U'$, N_q is orthogonal to $T_q U'$ and consequently it lies on the plane spanned by w and N'_q and hence

$$N = a w + b N',$$

where a is a function on U' and

$$b = g(N, N').$$

If \tilde{a} , \tilde{b} and \tilde{N}' are extensions of a , b and N' in Ω ,

$$\tilde{N} = \tilde{a} w + \tilde{b} \tilde{N}'$$

defines an extension of N and a direct computation yields

$$\kappa_p(v) = a(p) g_p(\nabla_v w, v) + b(p) g_p(\nabla_v \tilde{N}', v) = b(p) g_p(\nabla_v \tilde{N}', v),$$

where we used that $\pi \cap \Omega$ is totally geodesic. Therefore

$$\frac{1}{g_q(N_q, N'_q)} \kappa_q(v) = g_q(\nabla_v \tilde{N}', v)$$

and consequently

$$\frac{1}{g_q(N_q, N'_q)} \kappa_1(q) \leq \kappa'_i(q) \leq \frac{1}{g_q(N_q, N'_q)} \kappa_{n-1}(q)$$

for every $q \in U'$ and $i = 1, \dots, n-2$.

Now we show

$$(28) \quad g_q(N_q, N'_q) = \sqrt{1 - g_q(\omega_q, N_q)^2},$$

which implies (26). We have

$$(29) \quad \nu_q \cdot \nu'_q = \frac{(-1)^n * (\nu_q \wedge w) \wedge w \cdot \nu_q}{| * (\nu_q \wedge w) \wedge w |} = \frac{(-1)^n * (\nu_q \wedge w) \wedge w \cdot * \nu_q}{| * (\nu_q \wedge w) \wedge w |}.$$

Let $\{\nu_q, e_1, \dots, e_{n-1}\}$ be a positive-oriented orthonormal basis of \mathbb{R}^n such that

- $\{e_1, \dots, e_{n-1}\}$ is a positive-oriented Euclidean-orthonormal basis of $T_q U$;
- $\{e_2, \dots, e_{n-1}\}$ is a basis of $T_q U'$.

In this way $w \in \langle \nu_q, e_1 \rangle$,

$$*(\nu_q \wedge w) = (w \cdot e_1) e_2 \wedge \dots \wedge e_{n-1}, \quad * \nu_q = e_1 \wedge \dots \wedge e_{n-1},$$

and

$$| * (\nu_q \wedge w) \wedge w | = | * (\nu_q \wedge w) \wedge w | = w \cdot e_1$$

Hence, from (29), we get

$$\begin{aligned} \nu_q \cdot \nu'_q &= \frac{(-1)^n}{w \cdot e_1} * (\nu_q \wedge w) \wedge w \cdot * \nu_q \\ &= \frac{(-1)^n}{w \cdot e_1} (w \cdot e_1) e_2 \wedge \dots \wedge e_{n-1} \wedge w \cdot e_1 \wedge \dots \wedge e_{n-1} \\ &= (-1)^n (w \cdot e_1) e_2 \wedge \dots \wedge e_{n-1} \wedge e_1 \cdot e_1 \wedge \dots \wedge e_{n-1} \\ &= (w \cdot e_1) e_1 \wedge \dots \wedge e_{n-1} \cdot e_1 \wedge \dots \wedge e_{n-1} \\ &= w \cdot e_1. \end{aligned}$$

Since $|w| = 1$, we have $w \cdot e_1 = \sqrt{1 - (w \cdot \nu_q)^2}$ and so

$$(30) \quad \nu_q \cdot \nu'_q = \sqrt{1 - (w \cdot \nu_q)^2}.$$

Since

$$\nu_q \cdot \nu'_q = g_q(N_q, N'_q), \quad \text{and} \quad w \cdot \nu_q = g_q(\omega_q, N_q),$$

(28) follows.

Now we prove (27). In this case we regard U' as a submanifold of U . Let $q \in U'$, $v \in T_q U'$ such that $|v|_q = 1$ and let $\alpha: (-\delta, \delta) \rightarrow S$ be a curve satisfying $\alpha(0) = q$, $\dot{\alpha}(0) = v$, $|\dot{\alpha}|_\alpha = 1$. Let \tilde{N}' be a unitary normal vector field of U' in U near q . We may complete v with an orthonormal basis $\{v, v_2, \dots, v_{n-2}\}$ of $T_q U'$ such that

$$\tilde{N}'_q = *_q(N_q \wedge v \wedge v_2 \wedge \dots \wedge v_{n-2}),$$

where $*_q$ is the Hodge star operator at q in Ω with respect to g and to the standard orientation. Let

$$\check{\kappa}'_q(v) = g_q(*_q(\tilde{N}'_q \wedge v \wedge v_2 \wedge \dots \wedge v_{n-2}), D_t \dot{\alpha}|_{t=0}),$$

where D_t is the covariant derivative in (Ω, g) . Since $D_t \dot{\alpha}|_{t=0} \in \pi$, we have

$$\check{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(*_q(\omega_q \wedge v \wedge v_2 \wedge \dots \wedge v_{n-2}), D_t \dot{\alpha}|_{t=0}).$$

Now, $*_q(\omega_q \wedge v \wedge v_2 \wedge \dots \wedge v_{n-2})$ is a normal vector to $T_q U'$ in π and so

$$\check{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(\nabla_v \tilde{N}, v),$$

where \tilde{N} is an arbitrary extension of N in a neighborhood of q . From (26) we obtain

$$|\check{\kappa}'_q(v)| \leq \frac{|g_q(N_q, \omega_q)|}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\},$$

as required. \square

Remark 4.2. It may be convenient to explain the meaning of (30) when $n = 3$. In this case $*(v \wedge w)$ is the vector product $v \times w$ so

$$\nu'_q = -\frac{(\nu_q \times w) \times w}{|(\nu_q \times w) \times w|} \quad \text{and} \quad |(\nu_q \times w) \times w| = \sqrt{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}.$$

So

$$\begin{aligned} \nu_q \cdot \nu'_q &= -\frac{1}{\sqrt{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}} ((\nu_q \times w) \times w) \cdot \nu_q \\ &= -\frac{1}{\sqrt{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}} (\nu_q \times w) \cdot (w \times \nu_q) \\ &= \frac{|\nu_q \times w|^2}{\sqrt{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}} \\ &= \frac{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}{\sqrt{|\nu_q|^2 |w|^2 - (\nu_q \cdot w)^2}} \\ &= \sqrt{1 - (\nu_q \cdot w)^2}. \end{aligned}$$

Now we focus in a different setting. Let $\bar{\Omega}$ be the projection of Ω onto $\{x_n = 0\}$ and let $\pi \subseteq \Omega$ be the graph of a C^2 function $F: A \rightarrow \mathbb{R}$, where $A \subseteq \bar{\Omega}$ is an open subset.

Proposition 4.3. Let U' be a C^2 regular oriented hypersurface of π and let U'' be the orthogonal projection of U' onto $\{x_n = 0\}$. Then the principal curvatures of U'' satisfy

(31)

$$|\kappa''_i(\bar{q})| \leq \frac{h(q)}{\sqrt{1 + |\nabla F(\bar{q})|^2}} \left((\nu'_q \cdot e_n)^2 + \frac{1}{1 + |\nabla F(\bar{q})|^2} \right)^{-3/2} \left(\max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} + 4 \frac{|\nabla h(q)|}{h(q)^2} \right),$$

for every $i = 1, \dots, n-2$, where $\{\kappa_i''\}$ are the principal curvatures of U'' with respect to the Euclidean metric, $q \equiv (\bar{q}, q_n) \in U'$ and $\nu'_q = h(q)N'_q$.

Proof. If X is a local positive oriented parametrization of U' , then $\bar{X} = X - (X \cdot e_n)e_n$ is a local parametrization of U'' , and we can orient U'' with

$$(32) \quad \nu'' \circ \bar{X} := \text{vers}(*(\bar{X}_1 \wedge \bar{X}_2 \wedge \dots \wedge \bar{X}_{n-2} \wedge e_n)),$$

where \bar{X}_k is the k^{th} derivative of \bar{X} with respect to the coordinates of its domain and $*$ is the Hodge star operator in \mathbb{R}^n with respect to the the Euclidean metric and the standard orientation.

Now we prove inequalities (31). Fix a point $q = (\bar{q}, q_n) \in U'$ and $\bar{v} \in T_{\bar{q}}U'$ be nonzero. Let $\beta: (-\delta, \delta) \rightarrow U''$ be an arbitrary regular curve contained in U'' such that

$$\beta(0) = \bar{q}, \quad \dot{\beta}(0) = \bar{v}.$$

Then

$$\kappa_{\bar{q}}''(\bar{v}) = \frac{1}{|\bar{v}|^2} \nu''_{\bar{q}} \cdot \ddot{\beta}(0)$$

is the normal curvature of U'' at (\bar{q}, \bar{v}) , viewed as hypersurface of $\{x_n = 0\}$ with the Euclidean metric. We can write

$$\kappa_{\bar{q}}''(\bar{v}) = \frac{1}{|\bar{v}|^2} \nu''_{\bar{q}} \cdot \ddot{\alpha}(0)$$

where $\alpha = (\beta, \alpha_n)$ whose projection onto U' is β . From

$$\bar{X}_k = X_k - (X_k \cdot e_n)e_n,$$

and the definition of ν'' (32) we have

$$\kappa_{\bar{q}}''(\bar{v}) = \frac{*(X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0)}{|\dot{\beta}(0)|^2 |X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge e_n|}.$$

We may assume that $\{X_1(q), \dots, X_{n-2}(q)\}$ is an orthonormal basis of $T_q U'$ with respect to the Euclidean metric. Let

$$\mathbf{N}_q = \frac{(-\nabla F(\bar{q}), 1)}{\sqrt{1 + |\nabla F(\bar{q})|^2}}$$

be the Euclidean normal vector to π at q and let

$$a = \sqrt{1 + |\nabla F(\bar{q})|^2}.$$

Therefore $\{X_1(q), \dots, X_{n-2}(q), \nu'_q, \mathbf{N}_q\}$ is an Euclidean orthonormal basis of \mathbb{R}^n and we can split \mathbb{R}^n in

$$(33) \quad \mathbb{R}^n = T_q U' \oplus \langle \nu'_q \rangle \oplus \langle \mathbf{N}_q \rangle,$$

and e_n splits accordingly into

$$e_n = e'_n + e''_n + e'''_n.$$

Therefore

$$*(X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) = *(X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge e'''_n) \cdot \ddot{\alpha}(0),$$

i.e.

$$*(X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) = \frac{1}{a} * (X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge \mathbf{N}_q) \cdot \ddot{\alpha}(0).$$

Since

$$\nu'_q = *(X_1(q) \wedge \dots \wedge X_{n-2}(q) \wedge \mathbf{N}_q)$$

we obtain

$$\kappa_q''(\bar{v}) = \frac{1}{a|\dot{\beta}(0)|^2} \frac{\nu_q' \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|}.$$

We may assume that α is parametrized by arc length with respect to the metric g , i.e.

$$|\dot{\alpha}|^2 = h(\alpha)^{-2}$$

and so

$$|\dot{\beta}|^2 = h(\alpha)^{-2} - \dot{\alpha}_n^2,$$

which implies

$$(34) \quad \kappa_q''(\bar{v}) = \frac{1}{a(h(q)^{-2} - v_n^2)} \frac{\nu_q' \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|}.$$

Since

$$X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n = X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n'' + X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n'''$$

and

$$\begin{aligned} X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n'' &= (\nu_q' \cdot e_n) X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \nu_q', \\ X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n''' &= \frac{1}{a} X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \mathbf{N}_q, \end{aligned}$$

we obtain

$$|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n| = \left((\nu_q' \cdot e_n)^2 + \frac{1}{a^2} \right)^{1/2}.$$

On the other hand

$$\kappa_q'(v) = g_q(N_q', D_t \dot{\alpha}|_{t=0})$$

where D_t is the covariant derivative in π . It is well-known that the Christoffel symbols of g are given by

$$\Gamma_{ij}^k = \delta_i^k \partial_j f + \delta_j^k \partial_i f - \delta_i^j \partial_k f,$$

where $f = \log h$. We have

$$\begin{aligned} D_t \dot{\alpha} &= \ddot{\alpha} + \sum_{i,j,k=1}^n \Gamma_{ij}^k(\alpha) \dot{\alpha}_i \dot{\alpha}_j e_k = \ddot{\alpha} + \sum_{i,j,k=1}^n (\delta_i^k \partial_j f(\alpha) + \delta_j^k \partial_i f(\alpha) - \delta_i^j \partial_k f(\alpha)) \dot{\alpha}_i \dot{\alpha}_j e_k \\ &= \ddot{\alpha} + \sum_{i,k=1}^n (2\partial_i f(\alpha) \dot{\alpha}_i \dot{\alpha}_k - \dot{\alpha}_i^2 \partial_k f(\alpha)) e_k + \sum_{k=1}^n \partial_k f(\alpha) \dot{\alpha}_k^2 e_k \end{aligned}$$

and

$$D_t \dot{\alpha}|_{t=0} = \ddot{\alpha}(0) + \sum_{i,k=1}^n (2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)) e_k + \sum_{k=1}^n \partial_k f(q) v_k^2 e_k.$$

Therefore

$$\begin{aligned}
\kappa'_q(v) &= g_q \left(N'_q, \ddot{\alpha}(0) + \sum_{i,k=1}^n (2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)) e_k + \sum_{k=1}^n \partial_k f(q) v_k^2 e_k \right) \\
&= h(q) \nu'_q \cdot \ddot{\alpha}(0) + h(q) \sum_{i,k=1}^n (2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)) e_k \cdot \nu'_q + h(q) \sum_{k=1}^n \partial_k f(q) v_k^2 \nu'_q \cdot e_k \\
&= h(q) \nu'_q \cdot \ddot{\alpha}(0) + \sum_{i,k=1}^n (2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)) e_k \cdot \nu'_q + \sum_{k=1}^n \partial_k h(q) v_k^2 \nu'_q \cdot e_k,
\end{aligned}$$

and we get

$$\nu'_q \cdot \ddot{\alpha}(0) = \frac{\kappa'_q(v)}{h(q)} - \frac{1}{h(q)} \sum_{i,k=1}^n (2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)) e_k \cdot \nu'_q - \frac{1}{h(q)} \sum_{k=1}^n \partial_k h(q) v_k^2 \nu'_q \cdot e_k.$$

From (34) we deduce

$$\begin{aligned}
\kappa''_{\bar{q}}(\bar{v}) &= \frac{1}{a(h(q)^{-2} - v_n^2)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{-1/2} \\
&\quad \left(\frac{\kappa'_q(v)}{h(q)} - \frac{1}{h(q)} \sum_{i,k=1}^n (2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)) e_k \cdot \nu'_q - \frac{1}{h(q)} \sum_{k=1}^n \partial_k h(q) v_k^2 \nu'_q \cdot e_k \right),
\end{aligned}$$

for every $v \in T_q U'$ such that $g_q(v, v) = 1$. Therefore

$$\begin{aligned}
\kappa''_1(\bar{q}) &= \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{-1/2} \inf_{v \in \mathbb{S}_q^{n-2}} A_q(v), \\
\kappa''_{n-2}(\bar{q}) &= \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} A_q(v),
\end{aligned}$$

where

$$A_q(v) = \frac{1}{h(q)^{-2} - v_n^2} \left(\kappa'_q(v) - \sum_{i,k=1}^n (2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)) e_k \cdot \nu'_q - \sum_{k=1}^n \partial_k h(q) v_k^2 \nu'_q \cdot e_k \right)$$

and $\mathbb{S}_q^{n-2} = \{v \in T_q U' : |v|_q = 1\}$. Since $|v|_q^2 = 1$, then $|v|^2 = h(q)^{-2}$ and we can rewrite $A_q(v)$ as

$$\begin{aligned}
A_q(v) &= \frac{1}{h(q)^{-2} - v_n^2} \left(\kappa'_q(v) - \nu'_q \cdot \left(2(\nabla h(q) \cdot v) v - h(q)^{-2} \nabla h(q) + \sum_{k=1}^n \partial_k h(q) v_k^2 e_k \right) \right) \\
&= \frac{1}{h(q)^{-2} - v_n^2} \left(\kappa'_q(v) - \nu'_q \cdot \left(2(\nabla h(q) \cdot v) v - h(q)^{-2} \nabla h(q) + \sum_{k=1}^n \partial_k h(q) v_k^2 e_k \right) \right).
\end{aligned}$$

Since $|\kappa''_i(\bar{q})| \leq \max\{|\kappa''_1(\bar{q})|, |\kappa''_{n-2}(\bar{q})|\}$, $i = 1, \dots, n-2$, we obtain

$$(35) \quad |\kappa''_i(\bar{q})| \leq \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} |A_q(v)|.$$

We have

$$\begin{aligned}
|A_q(v)| &= \frac{1}{h(q)^{-2} - v_n^2} \left| \kappa'_q(v) + \nu'_q \cdot \left(2(\nabla h(q) \cdot v) v - \frac{1}{h(q)^2} \nabla h(q) + \sum_{k=1}^n \partial_k h(q) v_k^2 e_k \right) \right| \\
&\leq \frac{1}{h(q)^{-2} - v_n^2} \left(|\kappa'_q(v)| + \left| 2(\nabla h(q) \cdot v) v - \frac{1}{h(q)^2} \nabla h(q) + \sum_{k=1}^n \partial_k h(q) v_k^2 e_k \right| \right) \\
&\leq \frac{1}{h(q)^{-2} - v_n^2} \left(|\kappa'_q(v)| + 2|(\nabla h(q) \cdot v) v| + \frac{1}{h(q)^2} |\nabla h(q)| + \left| \sum_{k=1}^n \partial_k h(q) v_k^2 e_k \right| \right) \\
&\leq \frac{1}{h(q)^{-2} - v_n^2} \left(|\kappa'_q(v)| + \frac{2}{h(q)^2} |\nabla h(q)| + \frac{1}{h(q)^2} |\nabla h(q)| + \frac{1}{h(q)^2} |\nabla h(q)| \right)
\end{aligned}$$

i.e.,

$$|A_q(v)| \leq \frac{1}{h(q)^{-2} - v_n^2} \left(|\kappa'_q(v)| + \frac{4}{h(q)^2} |\nabla h(q)| \right).$$

Since $\mathbb{R}^n = T_q U' \oplus \langle \mathbf{N}_q \rangle \oplus \langle \nu'_q \rangle$, we can write

$$e_n = e_n'' + \frac{1}{a} + \nu_q \cdot e_n \nu'_q,$$

where \tilde{e}_n is the orthogonal projection of e_n onto $T_q U'$. Therefore

$$1 - |e_n'|^2 = \frac{1}{a^2} + (\nu'_q \cdot e_n)^2.$$

Since $h(q)v$ lies in $T_q U'$ and it has unitary Euclidean norm, we have

$$|e_n'|^2 \geq h(q)^2 (e_n \cdot v)^2 = h(q)^2 v_n^2$$

and so

$$1 - h(q)^2 v_n^2 \geq \frac{1}{a^2} + (\nu'_q \cdot e_n)^2,$$

i.e.

$$h(q)^{-2} - v_n^2 \geq \left(\frac{1}{a^2} + (\nu'_q \cdot e_n)^2 \right) h(q)^{-2}.$$

Hence

$$|A_q(v)| \leq (|\kappa'_q(v)| + 4h(q)^{-2} |\nabla h(q)|) \left(\frac{1}{a^2} + (\nu'_q \cdot e_n)^2 \right)^{-1} h(q)^2,$$

which yields

$$(36) \quad |\kappa_i''(\bar{q})| \leq \frac{h(q)}{a} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{-3/2} \sup_{v \in \mathbb{S}_q^{n-2}} (|\kappa'_q(v)| + 4h(q)^{-2} |\nabla h(q)|),$$

which implies (31). □

Now we use (31) in space forms.

In the *Euclidean space* we have $\Omega = \mathbb{R}^n$ and $h(q) = 1$ and (31) reduces to

$$|\kappa_i''(\bar{q})| \leq \frac{1}{\sqrt{1 + |\nabla F(\bar{q})|^2}} \left((\nu'_q \cdot e_n)^2 + \frac{1}{1 + |\nabla F(\bar{q})|^2} \right)^{-3/2} \max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\},$$

which was already found in [15, Proposition 2.8] when π is a hyperplane, $\omega_1 = \frac{(\nabla F(\bar{q}), -1)}{\sqrt{1 + |\nabla F(\bar{q})|^2}}$ and $\omega_2 = e_n$.

In the *Hyperbolic space* we have $\Omega = \{q_n > 0\}$ and $h(q) = \frac{1}{q_n}$ and (31) reduces to

$$|\kappa_i''(\bar{q})| \leq \frac{1}{q_n \sqrt{1 + |\nabla F(\bar{q})|^2}} \left((\nu'_q \cdot e_n)^2 + \frac{1}{1 + |\nabla F(\bar{q})|^2} \right)^{-3/2} (\max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} + 4) ,$$

(see also [16, Proposition 4.3]).

Now we focus \mathbb{R}^n equipped with the spherical metric. In this case $h(q) = \frac{2}{1+|q|^2}$ and (31) gives

$$|\kappa_i''(\bar{q})| \leq \frac{2}{(1 + |q|^2) \sqrt{1 + |\nabla F(\bar{q})|^2}} \left((\nu'_q \cdot e_n)^2 + \frac{1}{1 + |\nabla F(\bar{q})|^2} \right)^{-3/2} (\max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} + 4|q|) .$$

In particular if π is the hemisphere of some hyperplane which does not contain the origin, then we have

$$(37) \quad |\kappa_i''(\bar{q})| \leq \frac{h(q)|(q - O_\pi)_n|}{R} \left((\nu'_q \cdot e_n)^2 + \frac{(q - O_\pi)_n^2}{R^2} \right)^{-3/2} (\max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} + 4|q|) ,$$

for every $i = 1, \dots, n-2$, where O_π and R are the center and the radius of π , respectively.

5. APPROXIMATE SYMMETRY IN ONE DIRECTION

We consider the following set-up: let $S = \partial\Omega$ be a C^2 -regular connected closed hypersurface embedded in \mathbb{M}_+^n , where Ω is a bounded domain. Assume that S satisfies a uniform touching ball condition of radius $\rho > 0$. We fix a direction v in $T_o\mathbb{M}^n$ and we apply the method of the moving planes as described in section 2. Let $\pi = \pi_{v, m_v}$ be the critical hyperplane and in order to simplify the notation we set

$$\begin{aligned} S_+ &= \{p \in S : p \in \pi_{v, t} \text{ for some } t > m_v\} , \\ S_- &= \{p \in S : p \in \pi_{v, t} \text{ for some } t < m_v\} . \end{aligned}$$

From the method of the moving planes we have that the reflection S_+^π of S_+ with respect to π is contained in Ω and it is tangent to S_- at a point p_0 (internally or at the boundary). Let Σ and $\hat{\Sigma}$ be the connected components of S_+^π and S_- containing p_0 , respectively.

The main result in this section is the following

Theorem 5.1. *There exists $\varepsilon > 0$ such that if*

$$\text{osc}(\mathbf{H}_S) \leq \varepsilon ,$$

then for any $p \in \Sigma$ there exists $\hat{p} \in \hat{\Sigma}$ such that

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \leq C \text{osc}(\mathbf{H}_S) .$$

Here, the constants ε and C depend only on n , ρ and the area of S . In particular ε and C do not depend on the direction v .

Moreover, $\partial\Omega$ is contained in a neighbourhood of radius $C \text{osc}(\mathbf{H}_S)$ of $\Sigma \cup \Sigma^\pi$ (Σ^π is the reflection of Σ about π), i.e.

$$d(p, \Sigma \cup \Sigma^\pi) \leq C \text{osc}(\mathbf{H}_S) ,$$

for every $p \in \partial\Omega$.

Before giving the proof of theorem 5.1, we provide two preliminary results about the geometry of Σ . For $t > 0$ we set

$$\Sigma_t = \{p \in \Sigma : d_\Sigma(p, \partial\Sigma) > t\} .$$

The following lemmas quantitatively show that Σ_t is connected for t small enough.

Here we use the results in section 4 and we consider the unitary normal vector field ω to π directed as the geodesic γ in \mathbb{M}_+^n satisfying $\dot{\gamma}(0) = v$.

Lemma 5.2. *Assume*

$$(38) \quad g_p(N_p, \omega_p) \leq \mu$$

for every p on the boundary of Σ , for some $\mu \leq 1/2$, and let $t_0 = \rho\sqrt{1-\mu^2}$. Then Σ_t is connected for any $0 < t < t_0$.

Proof. We can work in \mathbb{R}^n for every space form considered, and we may assume that π is an Euclidean hyperplane of \mathbb{R}^n (in the spherical case we can consider the projection from a point antipodal to a point inside π).

Let Σ' be the subset of π obtained by projecting Σ onto π (for any point $p \in \Sigma$ we define the projection of p onto π as the point on π which realizes the distance d of p from π). Σ' is an open set of π with $\partial\Sigma' = \partial\Sigma$. Proposition 4.1 gives

$$|\kappa'_i(p)| \leq \frac{1}{\sqrt{1 - (g_p(N_p, \omega_p))^2}} \max\{|\kappa_1(p)|, |\kappa_{n-1}(p)|\},$$

for any $p \in \partial\Sigma$ and $i = 1, \dots, n-1$, where κ'_i are the principal curvatures of $\partial\Sigma$ viewed as a hypersurface of π . Since S satisfies a touching ball condition of radius ρ , we have

$$\max\{|\kappa_1(p)|, |\kappa_{n-1}(p)|\} \leq \frac{1}{\rho}$$

and, consequently,

$$(39) \quad |\kappa'_i(p)| \leq \frac{1}{\rho\sqrt{1 - (g_p(N_p, \omega_p))^2}},$$

for $i = 1, \dots, n-1$. From (38) and (39) we have that $\partial\Sigma'$ satisfies a touching ball condition of radius

$$\rho' \geq \rho\sqrt{1 - (g_p(N_p, \omega_p))^2} \geq t_0.$$

Therefore if $s < t_0$,

$$\mathcal{C}_s = \{z \in \pi : d(z, \partial\Sigma) < s\}$$

is a collar neighborhood of $\partial\Sigma$ in Σ' of radius s . Since π is a critical hyperplane in the method of moving planes, if p belongs to the maximal cap S_+ then any point on the geodesic path connecting p to its projection onto π is contained in the closure of Ω . It follows that the preimage of \mathcal{C}_s via the projection contains a collar neighborhood of $\partial\Sigma$ of radius s in Σ . This implies that Σ can be retracted in Σ_t for any $t \leq s$ which completes the proof. \square

Lemma 5.3. *There exists $\bar{\delta} > 0$ depending only on ρ with the following property. Assume that there exists a connected component Γ_δ of Σ_δ , for some $0 < \delta \leq \bar{\delta}$, such that one of the following two assumptions is fulfilled:*

- i) $0 \leq g_q(N_q, \omega_q) \leq \frac{1}{8}$ for any $q \in \partial\Gamma_\delta$,
- ii) for any $q \in \partial\Gamma_\delta$ there exists $\hat{q} \in \hat{\Sigma}$ such that

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \leq \delta.$$

Then

$$(40) \quad 0 \leq g_q(N_q, \omega_q) \leq \frac{1}{4}$$

for any $q \in \partial\Sigma$ and Σ_δ is connected.

Proof. Case i). The crucial observation is that we can choose $\bar{\delta}$ small enough such that $\bar{\delta} \leq \delta_0$, where δ_0 is the bound appearing in proposition 3.5, and the set $\Sigma \setminus \Gamma_\delta$ is enclosed by π and the set obtained as the union of all the exterior and interior touching balls to the reflection of S about π , S^π . This implies that for any $p \in \Sigma \setminus \Gamma_\delta$ there exists $q \in \partial\Gamma_\delta$ such that $d_\Sigma(p, q) \leq \delta$ and we can apply the estimates in proposition 3.5. Indeed from (15) we have that

$$|N_p - \tau_q^p(N_q)|_p \leq C\delta, \text{ and } g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2\delta^2},$$

where $C = C(\rho)$. Therefore

$$g_p(N_p, \omega_p) = g_p(N_p - \tau_q^p(N_q), \omega_p) + g_p(\tau_q^p(N_q) \cdot \omega_p) \leq C\delta + g_p(\tau_q^p(N_q), \omega_p)$$

and by using

$$g_p(\tau_q^p(N_q), \omega_p) = g_q(N_q, \tau_p^q(\omega_q))$$

we obtain

$$g_p(N_p, \omega_p) \leq C\delta + g_q(N_q, \omega_q) + g_q(N_q, \tau_p^q(\omega_q) - \omega_q) \leq C\delta + g_q(N_q, \omega_q) + |\tau_p^q(\omega_q) - \omega_q|_q.$$

Since

$$|\tau_p^q(\omega_q) - \omega_q|_q = 0,$$

we deduce

$$g_p(N_p, \omega_p) \leq C\delta + g_q(N_q, \omega_q).$$

This last bound holds for every $p \in \partial\Sigma$ and by choosing δ small enough in terms of ρ we obtain (40), as required.

Case ii): Γ_δ satisfies ii). Let $q \in \partial\Gamma_\delta$. By construction of the method of moving planes, $g_q(N_q, \omega_q) \geq 0$. We denote by q^π the reflection of q about π and we have

$$d(q^\pi, \hat{q}) \leq d(q^\pi, q) + d(q, \hat{q}) \leq 3\delta.$$

Up to consider a smaller δ in terms of ρ , from corollary 3.3 we find $C = C(\rho)$ such that $d_S(q^\pi, \hat{q}) \leq C\delta$ and $q^\pi \in \mathcal{U}_{\rho_1}(\hat{q})$. Hence we can apply (15) and obtain

$$g_{\hat{q}}(N_{\hat{q}} \cdot \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})) \geq \sqrt{1 - C^2\delta^2} \quad \text{and} \quad |N_{\hat{q}} - \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})|_{\hat{q}} \leq C\delta.$$

Since N_{q^π} and q^π are the reflection of N_q and q about π , respectively, we have that

$$g_q(N_q, \omega_q) = -g(\tau_{q^\pi}^q(N_{q^\pi}), \omega_q),$$

and hence

$$2g_q(N_q, \omega_q) = g_q(N_q - \tau_{q^\pi}^q(N_{q^\pi}), \omega_q) = g_q(N_q - \tau_{\hat{q}}^q(N_{\hat{q}}), \omega_q) + g_q(\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi}), \omega_q).$$

This implies that

$$0 \leq 2g_q(N_q, \omega_q) \leq |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q + |\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi})|_q.$$

Next we observe that

$$|\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi})|_q = |N_{\hat{q}} - \tau_{\hat{q}}^{\hat{q}}\tau_{q^\pi}^{\hat{q}}(N_{q^\pi})|_{\hat{q}} \leq c(\delta)|N_{\hat{q}} - \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})|_{\hat{q}}$$

where $c(\delta) \rightarrow 0$ when $\delta \rightarrow 0$. Hence for a suitable choice of $\bar{\delta}$ we get

$$(41) \quad 0 \leq 2g_q(N_q, \omega_q) \leq \frac{1}{8},$$

and the claim follows from case i). □

Now we can focus on the proof of the first part of theorem 5.1, and show that there exist constants ε and C , depending only on n , ρ and $|S|_g$, such that if

$$\text{osc}(\mathbf{H}_S) \leq \varepsilon,$$

then for any p in Σ there exists \hat{p} in $\hat{\Sigma}$ satisfying

$$(42) \quad d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \leq C \text{osc}(\mathbf{H}_S).$$

In the proof of theorem 5.1 we are going to choose a number $\delta > 0$ sufficiently small in terms of ρ , n and $|S|_g$. A first requirement on δ is that the assumptions of lemmas 5.2 and 5.3 are satisfied. Other restrictions on the value of δ will be done in the development of the proof. We subdivide the proof of the first part of the statement in four cases depending on whether the distances of p_0 and p from $\partial\Sigma$ are greater or less than δ .

5.0.1. Case 1. $d_\Sigma(p_0, \partial\Sigma) > \delta$ and $d_\Sigma(p, \partial\Sigma) \geq \delta$. In this first case we assume that p_0 and p are interior points of Σ , which are far from $\partial\Sigma$ more than δ . We first assume that p_0 and p are in the same connected component of Σ_δ ; then, lemma 5.3 will be used in order to show that Σ_δ is in fact connected.

Let $r_0 > 0$ be such that $\mathcal{U}_{r_0}(p_i) \subset \Sigma$ for every $p_i \in \Sigma_\delta$. The value of r_0 follows from lemma 3.2 by letting

$$(43) \quad r_0 = \min(\bar{r}, \alpha\rho_1),$$

where \bar{r} is given by lemmas 3.6 and 3.7, $\alpha \in (0, \frac{1}{2} \min(1, \rho_1^{-1}))$ is such that $\alpha C\rho_1 \leq \frac{\delta}{4}$, and C is the constant appearing in (13).

Lemma 5.4. *Let $\varepsilon_0 \in [0, 1/2]$, p_0 and p be in a connected component of Σ_δ and $r_i = (1 - \varepsilon_0^2)^i r_0$. There exist an integer $J \leq J_\delta$, where*

$$(44) \quad J_\delta := \max\left(4, \frac{2^{n-1}|S|_g}{\delta^{n-1}}\right),$$

and a sequence of points $\{p_1, \dots, p_J\}$ in $\Sigma_{\delta/2}$ such that

$$\begin{aligned} p_0, p &\in \bigcup_{i=0}^J \overline{\mathcal{U}_{r_i/4}}(p_i), \\ \mathcal{U}_{r_0}(p_i) &\subseteq \Sigma, \quad i = 0, \dots, J, \\ p_{i+1} &\in \overline{\mathcal{U}_{r_i/4}}(p_i), \quad i = 0, \dots, J-1. \end{aligned}$$

Proof. In view of corollary 3.3, for every z in Σ and $r \leq \rho_0$, the geodesic ball $\mathcal{B}_r(z)$ in Σ satisfies

$$\text{Area}(\mathcal{B}_r(z)) \geq cr^{n-1},$$

where c is a constant depending only on n . A general result for Riemannian manifolds with boundary (see e.g. [16, Proposition A.1]) implies that there exists a piecewise geodesic path parametrized by arc length $\gamma: [0, L] \rightarrow \Sigma_{\delta/2}$ connecting p_0 to p and of length L bounded by δJ_δ , where J_δ is given by (44).

We define $p_i = \gamma(r_i/4)$, for $i = 1, \dots, J-1$ and $p_J = p$. Our choice of r_0 guarantees that $\mathcal{U}_{r_0}(p_i) \subset \Sigma$, for every $i = 0, \dots, J$, and the other required properties are satisfied by construction. \square

Since p and p_0 are in a connected component of Σ_δ , there exists a sequence of points p_1, \dots, p_J in the connected component of $\Sigma_{\delta/2}$ containing p_0 , with $J \in \mathbb{N}$ and $p_J = p$, and a chain of subsets $\{\mathcal{U}_{r_0}(p_i)\}_{i=0, \dots, J}$ of Σ as in lemma 5.4. We notice that Σ and $\hat{\Sigma}$ are tangent at p_0 and that in

particular the two normal vectors to Σ and $\hat{\Sigma}$ at p_0 coincide. Now we apply the map φ_{p_0} (see section 3). Then $\varphi_{p_0}(\Sigma)$ and $\varphi_{p_0}(\hat{\Sigma})$ can be locally parametrized near $\varphi_{p_0}(p_0)$ as graphs of two functions $u_0, \hat{u}_0: B_{r_0} \subset \{x_n = 0\} \rightarrow \mathbb{R}$. Lemma 3.2 implies that $|\nabla u_0|, |\nabla \hat{u}_0| \leq M$ in B_{r_0} , where M is some constant which depends only on r_0 , i.e. only on ρ . Hence the difference $u_0 - \hat{u}_0$ solves a second-order linear uniformly elliptic equation of the form

$$\mathcal{L}(u_0 - \hat{u}_0)(x) = \mathbf{H}_{\Gamma(u_0)}(x, u_0(x)) - \mathbf{H}_{\Gamma(\hat{u}_0)}(x, \hat{u}_0(x))$$

with ellipticity constants uniformly bounded by a constant depending only on n and ρ . Since $u_0(0) = \hat{u}_0(0)$ and $u_0 \geq \hat{u}_0$, Harnack's inequality (see Theorems 8.17 and 8.18 in [21]) yields

$$\sup_{B_{r_0/2}} (u_0 - \hat{u}_0) \leq C \operatorname{osc}(\mathbf{H}_S),$$

and from interior regularity estimates (see e.g. [21, Theorem 8.32]) we obtain

$$(45) \quad \|u_0 - \hat{u}_0\|_{C^1(B_{r_0/4})} \leq C \operatorname{osc}(\mathbf{H}_S),$$

where C depends only on ρ and n . Now we use lemmas 3.6 and 3.7. Since $p_1 \in \partial \mathcal{U}_{r_0/4}(p_0)$, we can write $\varphi_{p_0}(p_1) = (x_1, u_0(x_1))$, with $x_1 \in \partial B_{r_0/4}$. Let $\hat{p}_1^* \in \hat{\Sigma}$ be such that

$$\varphi_{p_0}(\hat{p}_1^*) = (x_1, \hat{u}_0(x_1))$$

and let \hat{p}_1 be the first intersection point between $\hat{\Sigma}$ and the geodesic path γ starting from p_1 and tangent to $-N_{p_1}$ at p_1 . From (45) we have

$$(46) \quad d(\varphi_{p_0}(p_1), \varphi_{p_0}(\hat{p}_1^*)) + |\nu_{\varphi_{p_0}(p_1)} - \nu_{\varphi_{p_0}(\hat{p}_1^*)}| \leq C \operatorname{osc}(\mathbf{H}_S)$$

which implies that the assumptions in lemmas 3.6 and 3.7 are fulfilled, and we obtain

$$(47) \quad d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C \operatorname{osc}(\mathbf{H}_S),$$

where C depends only on n and ρ .

Now we apply φ_{p_1} . By definition of φ_{p_1} , we have $\varphi_{p_1}(\hat{p}_1) = te_n$ for some $t \in \mathbb{R}$ (t depends on the geometry of the ambient space). A standard computation yields

$$|\nu_{\varphi_{p_1}(p_1)} - \nu_{\varphi_{p_1}(\hat{p}_1)}| = |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}$$

which in view of (47) implies

$$|\nu_{p_1} - \nu_{\hat{p}_1}| \leq C \operatorname{osc}(\mathbf{H}_S),$$

where C is a constant that depends only on ρ and n . Since $\operatorname{osc}(\mathbf{H}_S) \leq \varepsilon$ then $|\nu_{p_1} - \nu_{\hat{p}_1}| < C\varepsilon$ and by choosing ε such that $C\varepsilon < 1$, we can apply [15, Lemma 3.4] and obtain that Σ and $\hat{\Sigma}$ are locally graphs of two functions

$$u_1, \hat{u}_1: B_{r_1} \rightarrow \mathbb{R}^+,$$

such that $u_1(0) = \varphi_{p_1}(p_1)$ and $\hat{u}_1(0) = \varphi_{p_1}(\hat{p}_1)$ and where

$$r_1 = (1 - C^2\varepsilon^2)r < \rho_1.$$

Now, we can iterate the argument we did before. Indeed, since

$$0 \leq \inf_{B_{r_1/2}} (u_1 - \hat{u}_1) \leq u_1(0) - \hat{u}_1(0) \leq C \operatorname{osc}(\mathbf{H}_S),$$

by applying Harnack's inequality we obtain that

$$\sup_{B_{r_1/2}} (u_1 - \hat{u}_1) \leq C \operatorname{osc}(\mathbf{H}_S)$$

and from interior regularity estimates we find

$$(48) \quad \|u_1 - \hat{u}_1\|_{C^1(B_{r_1/4})} \leq C \operatorname{osc}(\mathbf{H}_S),$$

where C depends only on ρ and n . Hence, (48) is the analogue of (45), and we can iterate the argument. The iteration goes on until we arrive at $p_N = p$ and obtain a point $\hat{p}_N \in \hat{\Sigma}$ such that

$$d(p, \hat{p}_N) + |N_p - \tau_{\hat{p}_N}^p(N_{\hat{p}_N})|_p \leq C \operatorname{osc}(\mathbf{H}_S).$$

In view of lemma 5.3 we have that Σ_δ is connected and the claim follows.

5.0.2. *Case 2:* $d_\Sigma(p_0, \partial\Sigma) \geq \delta$ and $d_\Sigma(p, \partial\Sigma) < \delta$. We extend the estimates found in case 1 to a point p which is far less than δ from the boundary of Σ . Let $q \in \Sigma$ and $p_{\min} \in \partial\Sigma$ be such that

$$d_\Sigma(q, \partial\Sigma) = \delta, \quad d_\Sigma(p, q) + d_\Sigma(p, \partial\Sigma) = \delta, \quad \text{and} \quad d_\Sigma(p, p_{\min}) = d_\Sigma(p, \partial\Sigma).$$

From case 1 we have that there exists \hat{q} in $\hat{\Sigma}$ such that

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \leq C \operatorname{osc}(\mathbf{H}_S).$$

Lemma 5.3 (case (ii)) yields that

$$(49) \quad 0 \leq g_z(N_z, \omega_z) \leq \frac{1}{4},$$

for any $z \in \partial\Sigma$ and Σ_δ is connected.

Let $q^\pi \in S$ be the reflection of q about π and fix $r \leq \rho_1$ in order to define $\mathcal{U}_r(q^\pi)$. We denote by $U_r(q)$ the reflection of $\mathcal{U}_r(q^\pi) \cap S$ about π and $U' = \mathcal{U}_r(q^\pi) \cap \pi$. Proposition 4.1 implies that U' is a hypersurface of π with an induced orientation and its principal curvatures κ'_i satisfy the following bounds

$$\frac{1}{\sqrt{1 - g_z(N_z, \omega_z)^2}} \kappa_1(z) \leq \kappa'_i(z) \leq \frac{1}{\sqrt{1 - g_z(N_z, \omega_z)^2}} \kappa_{n-1}(z),$$

for every $z \in U'$ and $i = 1, \dots, n-1$. From (49) and since $|\kappa_i(z)| \leq \rho^{-1}$ for any $z \in S$ (this follows from the touching ball condition), we have

$$(50) \quad |\kappa'_i(z)| \leq \frac{2}{\rho},$$

for any $z \in U'$. Let U'' be the Euclidean orthogonal projection of $\varphi_q(U')$ onto $\{x_n = 0\}$. In order to apply Carleson estimates in [8, Theorem 1.3], we need to prove the following

Lemma 5.5. *Let $\{\kappa''_1, \dots, \kappa''_{n-2}\}$ be the Euclidean principal curvature of U'' viewed as a hypersurface of \mathbb{R}^{n-1} . Then*

$$(51) \quad \|\kappa''_i\|_\infty \leq C, \quad i = 1, \dots, n-2,$$

for some constant $C = C(\rho)$.

Proof. We refer to [15] and [16] for the proof of the assertion in the Euclidean and the Hyperbolic spaces, respectively, and we focus here on the spherical case. Here we use the same notation as in section 4. In particular we recall that for $z \in \varphi_q(U')$, \bar{z} is the projection of z onto U'' . Proposition 4.3 yields

$$|\kappa''_i(\bar{z})| \leq \frac{4|(z - O_\pi)_n|}{R(1 + |z|^2)^2} \left((\nu'_z \cdot e_n)^2 + \frac{(z - O_\pi)_n^2}{R^2} \right)^{-3/2} (\max\{|\kappa'_1(z)|, |\kappa'_{n-1}(z)|\} + 4|z|)$$

for every $z \in U'$, where O_π is the center of $\varphi_q(\pi)$ in \mathbb{R}^n .

We notice that the radius of $\varphi_q(\pi)$ is given by

$$R = \frac{1}{2a} + \frac{a}{2},$$

where a is the Euclidean distance between $\varphi_q(\pi)$ and the origin of \mathbb{R}^n . This follows from the proof of lemma 3.7: indeed, up to apply a rotation, we may assume that the point on π having minimal Euclidean distance from the origin is $(a, 0, \dots, 0)$, with $a > 0$, and that the normal to π in $(a, 0, \dots, 0)$ is e_1 . From a straightforward calculation we obtain the value of R .

Since $d(q, \pi) < \delta$ we have $a < \delta$ which implies

$$(52) \quad R > \frac{1}{2\delta}.$$

Since $U' \subset \mathcal{U}_r(q)$ and $\varphi_q(q) = O$, our choice of r implies that for any $z \in \varphi_q(U')$ we have

$$|z| \leq \rho_1 = \frac{\rho}{\pi}$$

and

$$|z - O_\pi| \leq |z - q| + |q - O_\pi| \leq \delta + R \leq 2R,$$

and then (52) implies

$$(53) \quad |\kappa_i''(\bar{z})| \leq 8(\nu'_z \cdot e_n)^{-3} (\max\{|\kappa_1'(z)|, |\kappa_{n-1}'(z)|\} + 4\rho_1).$$

Now we give a lower bound of $\nu'_z \cdot e_n$. We can write

$$\nu'_z \cdot e_n = \nu'_z \cdot (e_n - \nu_z) + \nu'_z \cdot \nu_z = \nu'_z \cdot (e_n - \nu_z) + g_z(N'_z, N_z).$$

We firstly observe that

$$(54) \quad g_z(N'_z, N_z) \geq \frac{1}{2}.$$

Indeed formula (30) yields

$$g_z(N'_z, N_z) = \sqrt{1 - g_z(\omega_z, N_z)^2},$$

and (40) implies (54) (here we can change configuration by considering the stereographic projection from the antipodal point to $\varphi_q^{-1}(z)$ in order to regard π as a vertical hyperplane of \mathbb{R}^n). Moreover, from lemma 2.1 in [15] we can choose r small enough in terms of ρ_1 in order to obtain $|e_n - \nu_z| \leq 1/4$. Hence

$$\nu'_z \cdot e_n = \nu'_z \cdot (e_n - \nu_z) + \nu'_z \cdot \nu_z \geq \frac{1}{2} - |\nu'_z \cdot (e_n - \nu_z)| \geq \frac{1}{4}$$

and

$$|\kappa_i''(\bar{z})| \leq C,$$

for some constant $C = C(\rho)$, as required. \square

We denote by x and E_r the projections of $\varphi_q(p_{min})$ and $\varphi_q(U_r(q))$ onto $\{x_n = 0\}$, respectively. The Euclidean distance of x from ∂E_r is less than $C\delta$ where C depends only on ρ and, up to chose a smaller δ in terms of ρ , the projection of p_{min} stays close to $U'' \subset \partial E_r$ and we can apply theorem 1.3 in [8], corollary 8.36 in [21] and Harnack's inequality (see e.g. [21, Corollary 8.36]) to obtain

$$(55) \quad \sup_{B_{2\delta C}(x) \cap E_r} (u - \hat{u}) \leq C(u - \hat{u})(z) + \text{osc}(\mathbf{H}_S)$$

with $z = x + 4C\delta\nu''_x$, where ν''_x is the interior normal to U'' at x . Thanks to (51) and by choosing δ small enough in terms of ρ , from (55) and Harnack's inequality we obtain

$$(56) \quad 0 \leq \|u - \hat{u}\|_{C^1(B_{C\delta}(x) \cap E_r)} \leq C((u(0) - \hat{u}(0)) + \text{osc}(\mathbf{H}_S)).$$

Since $d_\Sigma(q, \partial\Sigma) = \delta$, from Case 1 we know that

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \leq C \text{osc}(\mathbf{H}_S),$$

and from (56) we obtain that

$$(57) \quad 0 \leq \|u - \hat{u}\|_{C^1(B_{C\delta}(x) \cap E_r)} \leq C \operatorname{osc}(\mathbf{H}_S).$$

From lemma 3.7 we deduce

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \leq C \operatorname{osc}(\mathbf{H}_S),$$

as required.

5.0.3. *Case 3:* $0 < d_\Sigma(p_0, \partial\Sigma) < \delta$. We first prove the following preliminary lemma which implies via lemma 5.2 that Σ is connected.

Lemma 5.6. *By choosing δ small enough in terms of ρ , the following inequality holds*

$$(58) \quad 0 \leq g_{p_0}(N_{p_0}, \omega_{p_0}) \leq \frac{1}{4}.$$

Proof. We assume the statement in the Euclidean case (see [15, Section 4.1.3]) and we show how to deduce the claim in the hyperbolic and in the spherical case. We first consider the Hyperbolic case. Up to apply an isometry we can assume that $p_0 = e_n$ and $\pi = \{x_1 = 0\}$. Our assumptions on S imply that its diameter is bounded in terms of ρ and $|S|_g$ (see e.g. [16, Proposition A.2]). Therefore S is contained in an Euclidean ball about the origin and of radius depending only on ρ and $|S|_g$. Up to choose δ small enough in terms of ρ , we have that [15, Section 4.1.3] implies

$$0 \leq \nu_{p_0} \cdot e_1 \leq \frac{1}{4}.$$

Since $\nu_{p_0} \cdot e_1 = g_{p_0}(N_{p_0}, \omega_{p_0})$, the claim follows. In the spherical case the proof is analogue once the setting is modified as follows: we work in (\mathbb{R}^n, g) , where g is the round metric (7), assuming that $p_0 = O$ and π is an Euclidean hyperplane. \square

Then we prove the existence of a point $q \in \Sigma$ such that

$$(59) \quad \begin{cases} d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \leq C \operatorname{osc}(\mathbf{H}_S) \\ d_\Sigma(q, \partial\Sigma) \geq \delta \end{cases}$$

and we apply cases 1 and 2 to conclude.

In the same fashion as in case 2, we can locally write $\varphi_{p_0}(\Sigma)$ and $\varphi_{p_0}(\hat{\Sigma})$ as graphs of function $u, \hat{u}: E_r \rightarrow \mathbb{R}$ near $\varphi_{p_0}(p_0)$, respectively. Without loss of generality we can assume $r < 1$ (indeed r must be chosen small enough in terms of δ). Let $U'' \subset \partial E_r$ be the projection of $\varphi_{p_0}(U_r(p_0) \cap \pi)$ onto $\{x_n = 0\}$. Analogously to case 2, the Euclidean principal curvatures of U'' are bounded by a constant \mathcal{K} depending only on ρ . Then we can argue as in [15, Section 4.1.3] and find a point $y \in E_r$ of the form

$$y = \bar{x} + 2c_*\delta\nu_{\bar{x}}'',$$

where c_* is 1 in the Euclidean and in the spherical case, while it is the constant c appearing in (10) in the hyperbolic case and $\bar{x} \in U''$ is such that

$$|\bar{x}| = \min_{x \in U''} |x|,$$

(see figure 3). By choosing δ sufficiently small in terms of ρ we have

$$d(w, \hat{w}^*) + |\nu_w - \nu_{\hat{w}^*}| \leq C \operatorname{osc}(\mathbf{H}_S),$$

where $w = (y, u(y))$, $\hat{w}^* = (y, \hat{u}(y))$. Lemmas 3.6 and 3.7 yield

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \leq C \operatorname{osc}(\mathbf{H}_S),$$

where $q = \varphi_{p_0}^{-1}(w)$, $\hat{q}^* = \varphi_{p_0}^{-1}(\hat{w}^*)$ and \hat{q} is the first intersection point between $\hat{\Sigma}$ and the geodesic path starting from q and tangent to $-N_q$ at q .

Let z be a point on $\partial U_r(p_0)$ realizing $d(q, \partial U_r(p_0))$. By construction and from lemma 3.1 we have

$$d_\Sigma(q, \partial\Sigma) \geq d(q, z) \geq c_* |\varphi_{p_0}(q) - \varphi_{p_0}(z)| \geq 2\delta.$$

Since $d_\Sigma(q, \partial\Sigma) \geq \delta$, q satisfies (59) and the claim follows.

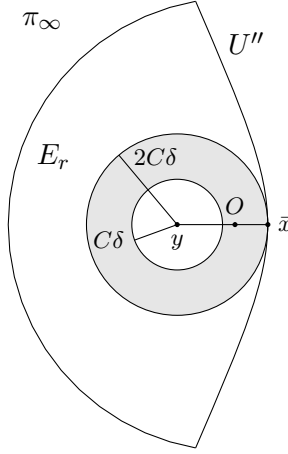


FIGURE 3. Case 3 in the proof of theorem 5.1.

5.0.4. *Case 4:* $p_0 \in \partial\Sigma$. This is the limit configuration of case 3 when $d_\Sigma(p_0, \partial\Sigma) \rightarrow 0$. Indeed, here E_r is a half-ball in \mathbb{R}^n and the argument used in case 3 can be easily adapted. This completes the proof of the first part of theorem 5.1.

5.0.5. *Last step:* $d(x, \Sigma \cup \Sigma^\pi) \leq C \operatorname{osc}(\mathbf{H}_S)$ for every $x \in \partial\Omega$. Assume by contradiction that

$$d(x, \Sigma \cup \Sigma^\pi) > C \operatorname{osc}(\mathbf{H}_S)$$

for some x in $\partial\Omega$. Since Ω is connected, it is possible to find $y \in \Omega$, such that

$$y \in \Omega_- \quad \text{and} \quad C \operatorname{osc}(\mathbf{H}_S) < d(y, \Sigma) \leq 2C \operatorname{osc}(\mathbf{H}_S),$$

where

$$\begin{aligned} \Omega_+ &= \{p \in \Omega : p \in \pi_{v,t} \text{ for some } t > m_v\}, \\ \Omega_- &= \{p \in \Omega : p \in \pi_{v,t} \text{ for some } t < m_v\}. \end{aligned}$$

Let p be a projection of y over $\Sigma \cup \Sigma^\pi$. If $p \in \Omega_+$, then y belongs to the exterior touching ball of S at p , which gives a contradiction. The same contradiction is obtained when $p \in \pi$ since, in that case $g_p(N_p, \omega_p) \leq 1/4$. If $p \in \Omega_-$, we can find a point $\hat{p} \in S$ such that \hat{p} and p lies on the geodesic γ starting from p and orthogonal to Σ^π and such that

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})| \leq C \operatorname{osc}(\mathbf{H}_S).$$

By the smallness of $\operatorname{osc}(\mathbf{H}_S)$ we obtain that y belongs to the exterior touching ball of S at p , which is a contradiction. \square

6. GLOBAL APPROXIMATE SYMMETRY

From the previous section we have that if a C^2 -regular closed hypersurface $S = \partial\Omega$ embedded in \mathbb{M}_+^n satisfies the assumptions of theorem 5.1 then it is almost symmetric with respect to any direction, with the almost symmetry quantified by the deficit $\text{osc}(H_S)$. In this section we show how this result leads to the almost radial symmetry of S . Such procedure is not peculiar of the kind of deficit considered, but it can be applied whenever one has the approximate symmetry in any direction with respect to some deficit. More precisely we consider the following

Definition 6.1. Let Υ be the space of open sets Ω in \mathbb{M}_+^n whose boundary is a C^2 -regular connected closed embedded hypersurface, with the topology induced by the Hausdorff distance. A *deficit function* is any continuous function $\text{def} : \Upsilon \rightarrow [0, +\infty)$ such that $\text{def}(\Omega) = 0$ if and only if Ω is a ball.

From now on we fix a deficit function def .

Definition 6.2. We say that a bounded open set Ω satisfies the *approximate symmetry property (ASP)* if there exists a constant $\mathcal{K} > 0$ satisfying the following condition: for every direction v there exists a connected component Σ of the maximal cap in the direction v such that

$$d(p, \Sigma \cup \Sigma^{\pi v}) \leq \mathcal{K} \text{def}(\Omega),$$

for every $p \in \partial\Omega$.

The main theorem in this section is the following.

Theorem 6.3. *Let $S = \partial\Omega$ be a C^2 -regular closed hypersurface embedded in \mathbb{M}_+^n , with Ω satisfying (ASP) and*

$$(60) \quad \text{def}(\Omega) \leq \frac{|\Omega|_g}{4\mathcal{K}}.$$

There exist \mathcal{O} in \mathbb{M}_+^n and two balls B_r^d and B_R^d centered at \mathcal{O} of radius r and R , respectively, with $r \leq R$, such that

$$B_r^d(\mathcal{O}) \subseteq \Omega \subseteq B_R^d(\mathcal{O})$$

and

$$(61) \quad R - r \leq C \text{def}(\Omega),$$

where C depends on $n, \rho, |S|_g$ and \mathcal{K} .

The following lemma is needed in order to prove theorem 6.3.

Lemma 6.4. *Let $S = \partial\Omega$ be a C^2 -regular closed hypersurface embedded in \mathbb{M}_+^n , with Ω satisfying (ASP) and (60). Then there exists \mathcal{O} in \mathbb{M}_+^n such that*

$$d(\mathcal{O}, \pi_v) \leq C \text{def}(\Omega),$$

for every direction v in $T_{\mathcal{O}}\mathbb{M}^n$, where C depends on $n, \rho, |S|_g$ and \mathcal{K} .

Proof. We fix an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space at the “origin” \mathbf{o} and we consider the corresponding critical hyperplanes π_{e_i} . We define an approximate center of symmetry \mathcal{O} as follows:

$$\mathcal{O} := \bigcap_{i=1}^n \pi_{e_i}.$$

We notice that in the Euclidean case \mathcal{O} is well-defined. Although in the hyperbolic space n orthogonal hyperplanes do not always intersect, we can work as in [16, Lemma 6.1] and show

that (60) implies the existence of \mathcal{O} . In \mathbb{S}_+^n the existence of \mathcal{O} is always guaranteed. Indeed every π_{e_i} is given by the intersection of a plane Π_{e_i} of \mathbb{R}^{n+1} with \mathbb{S}_+^n and the intersection of all the Π_{e_i} 's is a straight line r which, by construction, can not lie in the plane $\{x_{n+1} = 0\}$; hence $\mathcal{O} = r \cap \mathbb{S}_+^n \neq \emptyset$.

Let \mathcal{R} be the reflection about \mathcal{O} . Note that

$$\mathcal{R}(p) = \pi_{e_1} \circ \cdots \circ \pi_{e_n}(p),$$

where we identify π_{e_i} with the reflection about the corresponding hyperplane.

Let v be a fixed direction and let Ω_v be the corresponding maximal cap. Since Ω satisfies (ASP), we have

$$(62) \quad |\Omega_v|_g \geq \frac{|\Omega|_g}{2} - C \text{def}(\Omega),$$

for some constant C depending only on $n, \rho, |S|_g$ and \mathcal{K} . Moreover we have

$$(63) \quad |\Omega \Delta \Omega^\pi|_g = 2(|\Omega|_g - 2|\Omega_v|_g) \leq 4C \text{def}(\Omega),$$

where $\Omega \Delta \Omega^\pi$ denotes the symmetric difference between Ω and Ω^π . Next we work as in lemma 4.1 in [13]. Here we only sketch the argument referring to [13] for details (see also [15]).

Without loss of generalities, we may assume $\mathcal{O} \in \pi_{v, m_v - \mu}$, for some $\mu > 0$ and for $k \in \mathbb{N}$ we define

$$\mu_k = \left| \{p \in \Omega \cap \pi_{v, s} : m_v + (k-1)\mu < s < m_v + k\mu\} \right|_g.$$

By construction μ_k is decreasing and, in particular,

$$\mu_k \leq \mu_0 := \left| \{\Omega \cap \pi_{v, s} : m_v - \mu < s < m_v\} \right|_g.$$

Moreover, μ_0 is bounded by $C \text{def}(\Omega)$. Indeed, formula (62) yields

$$|\Omega \Delta \mathcal{R}(\Omega)|_g \leq C \text{def}(\Omega),$$

and then we obtain

$$|\Omega \cap \mathcal{R}(\Omega_v)|_g \geq |\Omega_v|_g - |\Omega \Delta \mathcal{R}(\Omega)|_g \geq \frac{|\Omega|_g}{2} - C \text{def}(\Omega).$$

Since

$$\mathcal{R}(\Omega_v) \subset \bigcup_{s < 0} \pi_{v, m_v - s},$$

we obtain that

$$\mu_0 := \left| \{\Omega \cap \pi_{v, s} : m_v - \mu < s < m_v\} \right|_g \leq C \text{def}(\Omega).$$

Therefore

$$(64) \quad \mu_k \leq C \text{def}(\Omega)$$

for every k in \mathbb{N} . From (62) we get

$$\frac{|\Omega|_g}{2} - C \text{def}(\Omega) \leq |\Omega_v|_g \leq \sum_{k=0}^{k_0} \mu_k \leq k_0 \mu_0 \leq \frac{\text{diam}(\Omega)}{m_v} C \text{def}(\Omega)$$

where k_0 is the integer part of $\frac{\text{diam}(\Omega)}{m_v}$. From Proposition A.1 in [16] we have

$$m_v \leq C \text{def}(\Omega),$$

where C depends only on $n, \rho, |S|_g$ and \mathcal{K} , as required. \square

Proof of theorem 6.3. Let \mathcal{O} be as in lemma 6.4 and define

$$r = \sup\{s > 0 : B_s^d(\mathcal{O}) \subset \Omega\} \quad \text{and} \quad R = \inf\{s > 0 : B_s^d(\mathcal{O}) \supset \Omega\},$$

so that $B_r^d(\mathcal{O}) \subseteq \Omega \subseteq B_R^d(\mathcal{O})$.

Let $p, q \in S$ be such that $d(p, \mathcal{O}) = r$ and $d(q, \mathcal{O}) = R$. We can assume that $p \neq q$ (otherwise $r = R$ and S is a round sphere). Let $v \in T_{\mathcal{O}}\mathbb{M}^n$ be the direction

$$v := \frac{1}{d(p, q)} \tau_p^{\circ}(\exp_p^{-1}(q))$$

and π_v the critical hyperplane in the v -direction. We denote by γ the geodesic path passing through p and q and let s_p and s_q in \mathbb{R} be such that

$$\gamma(s_p) = p \text{ and } \gamma(s_q) = q.$$

Let $z \in \pi_v$ be such that $d(z, \mathcal{O}) = d(\mathcal{O}, \pi_v)$. We have

$$p \in \pi_{v, s_p}, \quad q \in \pi_{v, s_q}, \quad s_q = s_p + t;$$

see section 2 for the definition of π_{v, s_p} and π_{v, s_q} . We first show that $d(q, z) \leq d(p, z)$. Assume by contradiction that $d(q, z) > d(p, z)$. Since q and p belong to a geodesic orthogonal to the hyperplanes $\pi_{v, s}$ and $s_p < s_q$, then $s_q > m_v$. Since $\pi_v = \pi_{v, m_v}$ corresponds to the critical position of the method of the moving planes in the direction v , we have that $\gamma(s) \in \Omega$ for any $s \in (m_v, s_q)$. Since $s_p < s_q$ we have that $|s_p - m_v| \geq |s_q - m_v|$ and since γ is orthogonal to π_v we obtain $d(q, z) \leq d(p, z)$, which gives a contradiction. Since $d(q, z) \leq d(p, z)$ we have

$$r \geq R - d(\mathcal{O}, z) = R - d(\mathcal{O}, \pi_v)$$

and lemma 6.4 implies (61). \square

7. PROOF OF THE MAIN RESULTS

We have all the ingredients to prove theorem 1.1 and corollary 1.2.

Proof of theorem 1.1. Let $S = \partial\Omega$ be a C^2 -regular, connected, closed hypersurface embedded in \mathbb{M}_+^n satisfying a uniform touching ball condition of radius ρ , where Ω is a relatively compact domain. Theorem 5.1 implies that there exist ϵ and C positive such that if

$$\text{osc}(\mathbf{H}_S) \leq \epsilon,$$

then

$$d(p, \Sigma \cup \Sigma^{\pi_v}) \leq C \text{osc}(\mathbf{H}_S),$$

for every $p \in \partial\Omega$. \square

Proof of corollary 1.2. The proof consists in one more application of the method of the moving planes and it is in the spirit of [13, Theorems 1.2 and 1.5]. Let $B_r^d(\mathcal{O})$ and $B_R^d(\mathcal{O})$ be as in theorem 6.3 and let $0 < t < r - C \text{def}(\Omega)$. We aim at proving that for any $p \in S$, there exist two cones with vertex at p and of fixed aperture, one contained in Ω and one contained in the complementary of Ω . The first cone $C^-(p)$, is obtained by considering all the geodesic path connecting p to the boundary of $B_t^d(\mathcal{O})$ tangentially. The second cone $C^+(p)$ is the reflection of $C^-(p)$ with respect to p . We show that $C^-(p)$ is contained in Ω and an analogous argument shows that $C^+(p)$ is contained in the complementary of Ω . We assume, by contradiction, that $p \notin B_r^d(\mathcal{O})$ (otherwise the claim is trivial) and that there exists a point $q \in C^-(p) \cap \partial B_t^d(\mathcal{O})$

such that the geodesic path γ connecting q to p is not contained in Ω . We apply the method of the moving planes in the direction v defined by

$$v := \frac{1}{d(p, q)} \tau_q^\circ(\exp_q^{-1}(p)).$$

Since γ is not contained in Ω , the method of the moving planes stops before reaching q and one can prove that

$$d(\mathcal{O}, \pi_\omega) \geq r - t.$$

Since $0 < t < r - C\text{def}(\Omega)$, from lemma 6.4, we obtain

$$C\text{def}(\Omega) < r - t \leq d(\mathcal{O}, \pi_\omega) \leq C\text{def}(\Omega),$$

which gives a contradiction. The argument above shows also that for any $p \in S$ the geodesic path connecting p to \mathcal{O} is contained in Ω . This implies that there exists a C^2 -regular map $\Psi : \partial B_r^d(\mathcal{O}) \rightarrow \mathbb{R}$ such that

$$F(p) = \exp_x(\Psi(p)N_p),$$

defines a C^2 -diffeomorphism from $B_r^d(\mathcal{O})$ to S . By choosing $t = r - \sqrt{C\text{def}(\Omega)}$ we have that for any $p \in S$ there exists a uniform cone of opening $\pi - \sqrt{C\text{def}(\Omega)}$ with vertex at p and axis on the geodesic connecting p to \mathcal{O} . This implies that Ψ is locally Lipschitz and the bound (5) on $\|\Psi\|_{C^1}$ follows (see also [13, Theorem 1.2]). \square

Remark 7.1. We observe that if $H_S = H$ is the mean curvature of $\partial\Omega$, then (5) can be improved and we can obtain the optimal linear bound $\|\Psi\|_{C^{1,\alpha}} \leq C \text{osc}(H)$ by using elliptic regularity. Indeed, let $\phi : U \rightarrow \partial B_r^d(\mathcal{O})$ be a local parametrization of $\partial B_r^d(\mathcal{O})$, where U is an open set of \mathbb{R}^{n-1} . From the proof of corollary 1.2, $F \circ \phi$ gives a local parametrization of S . A standard computation yields that

$$L(\Psi \circ \phi) = H(F \circ \phi) - H_{B_r}$$

where H_{B_r} is the mean curvature of ∂B_r and L is an elliptic operator which, thanks to the bounds on Ψ above, can be seen as a second order linear operator acting on $\Psi \circ \phi$. Then [21, Theorem 8.32] implies the bound on the $C^{1,\alpha}$ -norm of Ψ , as required.

LIST OF SYMBOLS

In this last section we collect some symbols we used in the paper.

\mathbb{M}^n denotes one of the following manifolds: the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n , the hemisphere \mathbb{S}^n .

\mathbb{M}_+^n denotes one of the following manifolds: the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n , the hemisphere \mathbb{S}_+^n .

S denotes a C^2 -regular, connected, closed hypersurface embedded in \mathbb{M}_+^n , $\Omega \subset \mathbb{M}_+^n$ denotes a relatively compact connected open set such that $\partial\Omega = S$.

$\kappa_1, \dots, \kappa_{n-1}$ denote the principal curvatures of S ordered increasingly.

H denotes the mean curvature of S .

H_S denotes a more general function of the principal curvatures of S (see the definition in the introduction).

$\text{osc}(H_S)$ denotes the oscillation of H_S on S .

\exp_x denotes the exponential map of a generic Riemannian manifold (M, g) at $x \in M$.

inj denotes the injectivity radius.

\mathcal{O} denotes either the center of mass (in section 2) or the approximate center of mass (in all other sections).

O denotes either the origin of $T_p S$ (in section 2) or the origin in \mathbb{R}^{n-1} (in all other sections).

\mathbf{o} denotes the origin in \mathbb{R}^n , e_n in \mathbb{H}^n and the north pole in \mathbb{S}_+^n .

$T_p M$ denotes the tangent space to a generic Riemannian manifold (M, g) at $p \in M$ and $T_p S$ denotes the tangent hyperplane to S at p .

d denotes the geodesic distance in \mathbb{M}^n induced by the Riemannian metric and d_S denotes the distance in S .

B_r denotes either a Euclidean ball of radius r centred at the origin O of $T_p S$ (in section 2) or a Euclidean ball of radius r centred at the origin O of \mathbb{R}^{n-1} (in all other sections).

$B_r^d(p)$ denotes the ball, with respect to d , of radius r in \mathbb{M}^n centred at $p \in \mathbb{M}^n$.

$\mathcal{B}_r(p)$ denotes the geodesic ball of S of radius r centred at $p \in S$.

$\varphi_p: \mathbb{M}_+^n \rightarrow \mathbb{R}^n$, for $p \in S$, denotes the following function whose definition depends on the geometry of \mathbb{M}^n :

- if \mathbb{M}^n is \mathbb{R}^n , $\varphi_p \in \mathrm{SO}(n) \ltimes \mathbb{R}^n$ and it is such that $\varphi_p(p) = 0$ and $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$;
- if \mathbb{M}^n is \mathbb{H}^n , φ_p is an orientation preserving isometry of \mathbb{H}^n such that $\varphi_p(p) = e_n$, $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$;
- if \mathbb{M}^n is \mathbb{S}^n , φ_p is the stereographic projection from the antipodal point to p restricted to \mathbb{S}_+^n composed with a rotation of \mathbb{R}^n in order to have $\varphi_{p*|_p}(T_p S) = \{x_n = 0\}$.

$\mathcal{U}_r(p)$ denotes the open neighborhood of p in S such that $\varphi_p(\mathcal{U}_r(p))$ is the (Euclidean) graph of a C^2 -function $u: B_r \rightarrow \mathbb{R}$ defined in the ball of radius r of \mathbb{R}^{n-1} centered at the origin.

ρ denotes the radius of the touching ball condition of S and ρ_1 denotes the following quantity:

- $\rho_1 = \rho$, if $\mathbb{M}^n = \mathbb{R}^n$;
- $\rho_1 = (1 - e^{-\rho} \sinh \rho) e^{-\rho} \sinh \rho$, if $\mathbb{M}^n = \mathbb{H}^n$;
- $\rho_1 = \frac{\rho}{\pi}$, if $\mathbb{M}^n = \mathbb{S}^n$.

τ_p^q denotes the parallel transport along the unique geodesic path connecting p to q .

$|\cdot|_p := g_p(\cdot, \cdot)^{1/2}$ where $p \in \mathbb{M}^n$.

$|\cdot|$ denotes the Euclidean norm.

$|\cdot|_g$ denotes either the volume with respect to the Riemannian metric g or the area with respect to the Riemannian metric g .

N denotes the inward unitary normal vector field on S .

ν denotes the Euclidean inward unitary normal vector field on S .

π , S_+ , S_- and p_0 see section 5.

S_+^π denotes the reflection of S_+ with respect to π .

Σ and $\hat{\Sigma}$ denote the connected components of S_+^π and S_- containing p_0 , respectively.

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