DE GRUYTER Forum Math. 2020; 32(6): 1487–1513

Research Article

Marco Abbadini*

Operations that preserve integrability, and truncated Riesz spaces

https://doi.org/10.1515/forum-2018-0244
Received October 9, 2018; revised June 17, 2020

Abstract: For any real number $p \in [1, +\infty)$, we characterise the operations $\mathbb{R}^I \to \mathbb{R}$ that preserve p-integrability, i.e., the operations under which, for every measure μ, the set $\mathcal{L}^p(\mu)$ is closed. We investigate the infinitary variety of algebras whose operations are exactly such functions. It turns out that this variety coincides with the category of Dedekind σ-complete truncated Riesz spaces, where truncation is meant in the sense of R. N. Ball. We also prove that $\mathbb{R}$ generates this variety. From this, we exhibit a concrete model of the free Dedekind σ-complete truncated Riesz spaces. Analogous results are obtained for operations that preserve p-integrability over finite measure spaces: the corresponding variety is shown to coincide with the much studied category of Dedekind σ-complete Riesz spaces with weak unit, $\mathbb{R}$ is proved to generate this variety, and a concrete model of the free Dedekind σ-complete Riesz spaces with weak unit is exhibited.

Keywords: Integrable functions, L^p, Riesz space, vector lattice, σ-completeness, weak unit, infinitary variety, equational classes, axiomatisation, free algebra, generation

MSC 2010: Primary 06F20; secondary 03C05, 08A65, 08B20, 46E30

Communicated by: Manfred Droste

1 Introduction

1.1 Operations that preserve integrability

In this work we investigate the operations which are somehow implicit in the theory of integration by addressing the following question: which operations preserve integrability, in the sense that they return integrable functions when applied to integrable functions?

Let us clarify the question by recalling some definitions.

For $(\Omega, \mathcal{F}, \mu)$ a measure space (with the range of μ in $[0, +\infty]$) and $p \in [1, +\infty)$, we adopt the notation $\mathcal{L}^p(\mu) := \{f : \Omega \to \mathbb{R} \mid f \text{ is } \mathcal{F}\text{-measurable and } \int_\Omega |f|^p \, d\mu < \infty\}$. It is well known that, for $f, g \in \mathcal{L}^p(\mu)$, we have $f + g \in \mathcal{L}^p(\mu)$, that is, $\mathcal{L}^p(\mu)$ is closed under the pointwise addition induced by addition of real numbers $+ : \mathbb{R}^2 \to \mathbb{R}$. More generally, consider a set I and a function $\tau : \mathbb{R}^I \to \mathbb{R}$, which we shall call an *operation of arity* $|I|$. We say $\mathcal{L}^p(\mu)$ *is closed under* τ if τ returns functions in $\mathcal{L}^p(\mu)$ when applied to functions in $\mathcal{L}^p(\mu)$, that is, for every $(f_i)_{i \in I} \subseteq \mathcal{L}^p(\mu)$, the function $\tau((f_i)_{i \in I}) : \Omega \to \mathbb{R}$ given by $x \in \Omega \mapsto \tau((f_i(x))_{i \in I})$ belongs to $\mathcal{L}^p(\mu)$. If $\mathcal{L}^p(\mu)$ is closed under τ, we also say that τ *preserves* p*-integrability over* $(\Omega, \mathcal{F}, \mu)$. Finally, we say that τ *preserves* p*-integrability* if τ preserves p-integrability over every measure space.

In Part I of this paper we characterise those operations that preserve integrability. Indeed, the first question we address is the following.

***Corresponding author: Marco Abbadini,** Dipartimento di Matematica Federigo Enriques, Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy, e-mail: marco.abbadini@unimi.it

Question 1.1. Under which operations $\mathbb{R}^I \to \mathbb{R}$ are $\mathcal{L}^p$ spaces closed? Equivalently, which operations preserve p-integrability?

Examples of such operations are the constant 0, the addition +, the binary supremum $\vee$ and infimum $\wedge$, and, for $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda(\cdot)$ by λ. A further example is the operation of countably infinite arity $\curlyvee$ defined as

$$\curlyvee(y, x_0, x_1, \dots) := \sup_{n\in\omega}\{x_n \wedge y\}.$$

Yet another example is the unary operation

$$\begin{aligned} \overline{\cdot}\colon \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto \overline{x} := x \wedge 1, \end{aligned}$$

called *truncation*. Here, although the constant function 1 belongs to $\mathcal{L}^p(\mu)$ if, and only if, μ is finite, it is always the case that $f \in \mathcal{L}^p(\mu)$ implies $\overline{f} \in \mathcal{L}^p(\mu)$.

It turns out that, for any given p, the operations that preserve p-integrability are essentially just 0, +, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\overline{\cdot}$, in the sense that every operation that preserves p-integrability may be obtained from these by composition. This we prove in Theorem 2.3.

We also have an explicit characterisation of the operations that preserve p-integrability. Denoting with $\mathbb{R}^+$ the set $\{\lambda \in \mathbb{R} \mid \lambda \geqslant 0\}$, for $n \in \omega$ and $\tau\colon \mathbb{R}^n \to \mathbb{R}$, we will prove that τ preserves p-integrability precisely when τ is Borel measurable and there exist $\lambda_0, \dots, \lambda_{n-1} \in \mathbb{R}^+$ such that, for every $x \in \mathbb{R}^n$, we have

$$|\tau(x)| \leqslant \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

Theorem 2.1 tackles the general case of arbitrary arity, settling Question 1.1.

In Part I we also address a variation of Question 1.1 where we restrict attention to finite measures. Recall that a measure μ on a measurable space $(\Omega, \mathcal{F})$ is *finite* if $\mu(\Omega) < \infty$. The question becomes:

Question 1.2. Under which operations $\mathbb{R}^I \to \mathbb{R}$ are $\mathcal{L}^p$ spaces of finite measure closed? Equivalently, which operations preserve p-integrability over finite measure spaces?

As mentioned, the function constantly equal to 1 belongs to $\mathcal{L}^p(\mu)$ for every finite measure μ. We prove in Theorem 2.4 that, for any given p, the operations that preserve p-integrability over finite measure spaces are essentially just 0, +, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and 1, in the same sense as in the above.

Theorem 2.2 provides an explicit characterisation of the operations that preserve p-integrability over finite measure spaces. In particular, for $n \in \omega$ and $\tau\colon \mathbb{R}^n \to \mathbb{R}$, τ preserves p-integrability over finite measure spaces precisely when τ is Borel measurable and there exist $\lambda_0, \dots, \lambda_{n-1}, k \in \mathbb{R}^+$ such that, for every $x \in \mathbb{R}^n$, we have

$$|\tau(x)| \leqslant k + \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

1.2 Truncated Riesz spaces and weak units

In Part II of this paper we investigate the equational laws satisfied by the operations that preserve p-integrability. (As it is shown by Theorems 2.1 and 2.2, the fact that an operation preserves p-integrability – over arbitrary and finite measure spaces, respectively – does not depend on the choice of p. Hence, we say that the operation *preserves integrability*.) We therefore work in the setting of varieties of algebras [4]. In this paper, under the term *variety* we include also infinitary varieties, i.e., varieties admitting primitive operations of infinite arity. For background please see [16].

We assume familiarity with the basic theory of Riesz spaces, also known as vector lattices. All needed background can be found, for example, in the standard reference [12]. As usual, for a Riesz space G, we set $G^+ := \{x \in G \mid x \geqslant 0\}$.

A *truncated Riesz space* is a Riesz space G endowed with a function $\overline{\cdot}\colon G^+ \to G^+$, called *truncation*, which has the following properties for all $f, g \in G^+$.

(B1) $f \wedge \overline{g} \leqslant \overline{f} \leqslant f$.

(B2) If $\overline{f} = 0$, then $f = 0$.

(B3) If $nf = \overline{nf}$ for every $n \in \omega$, then $f = 0$.

The notion of truncation is due to R. N. Ball [2], who introduced it in the context of lattice-ordered groups. Please see Section 8 for further details.

Let us say that a partially ordered set B is *Dedekind σ-complete* if every nonempty countable subset $A \subseteq B$ that admits an upper bound admits a supremum. Theorem 10.2 proves that the category of Dedekind σ-complete truncated Riesz spaces is a variety generated by $\mathbb{R}$. This variety can be presented as having operations of finite arity only, together with the single operation $\curlyvee$ of countably infinite arity. Moreover, we prove that the variety is finitely axiomatisable by equations over the theory of Riesz spaces. One consequence (Corollary 10.4) is that the free Dedekind σ-complete truncated Riesz space over a set I (exists, and) is

$$\mathrm{F}_{\mathrm{t}}(I) := \{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ preserves integrability}\}.$$

We prove results analogous to the foregoing for operations that preserve integrability over finite measure spaces. An element 1 of a Riesz space G is a *weak* (*order*) *unit* if $1 \geqslant 0$ and, for all $f \in G$, $f \wedge 1 = 0$ implies $f = 0$. Theorem 12.2 shows that the category of Dedekind σ-complete Riesz spaces with weak unit is a variety generated by $\mathbb{R}$, again with primitive operations of countable arity. It, too, is finitely axiomatisable by equations over the theory of Riesz spaces. By Corollary 12.4, the free Dedekind σ-complete Riesz space with weak unit over a set I (exists, and) is

$$\mathrm{F}_{\mathrm{u}}(I) := \{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ preserves integrability over finite measure spaces}\}.$$

The varietal presentation of Dedekind σ-complete Riesz spaces with weak unit was already obtained in [1]. Here we add the representation theorem for free algebras, and we establish the relationship between Dedekind σ-complete Riesz spaces with weak unit and operations that preserve integrability. The proofs in the present paper are independent of [1]. On the other hand, the results in this paper do depend on a version of the Loomis–Sikorski Theorem for Riesz spaces, namely Theorem 9.3 below. A proof can be found in [7], and can also be recovered from the combination of [5] and [6]. The theorem and its variants have a long history: for a fuller bibliographic account please see [5].

1.3 Outline

In Part I we characterise the operations that preserve integrability, and we provide a simple set of operations that generate them. Specifically, we characterise the operations that preserve measurability, integrability, and integrability over finite measure spaces, respectively in Sections 3, 4, and 5. In Section 6 we show that the operations 0, $+$, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\overline{\cdot}$ generate the operations that preserve integrability, and that 0, $+$, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and 1 generate the operations that preserve integrability over finite measure spaces.

In Part II we prove that the categories of Dedekind σ-complete truncated Riesz spaces and Dedekind σ-complete Riesz spaces with weak unit are varieties generated by $\mathbb{R}$. In more detail, in Section 7 we define the operation $\curlyvee$, in Section 8 we define truncated lattice-ordered abelian groups, in Section 9 we prove a version of the Loomis–Sikorski Theorem for truncated ℓ-groups, in Section 10 we show the category of Dedekind σ-complete truncated Riesz spaces to be generated by $\mathbb{R}$, in Section 11 we prove a version of the Loomis–Sikorski Theorem for ℓ-groups with weak unit, in Section 12 we show the category of Dedekind σ-complete Riesz spaces with weak unit to be generated by $\mathbb{R}$.

Finally, as an additional result, in the Appendix we provide an explicit characterisation of the operations that preserve ∞-integrability.

Notation. We let ω denote the set $\{0, 1, 2, \dots\}$.

Part I: **Operations that preserve integrability**

2 Main results of Part I

In this section we state the main results of Part I, together with the needed definitions. The first two main results (Theorems 2.1 and 2.2) are a characterisation of the operations that preserve p-integrability over arbitrary and finite measure spaces, respectively. The other two main results (Theorems 2.3 and 2.4) provide a set of generators for these operations. To state the theorems, we introduce a little piece of terminology.

For a set I, and $i \in I$, we denote by $\pi_i \colon \mathbb{R}^I \to \mathbb{R}$ the projection onto the i-th coordinate. The *cylinder σ-algebra on* $\mathbb{R}^I$ (notation: $\mathrm{Cyl}(\mathbb{R}^I)$) is the smallest σ-algebra which makes each projection function $\pi_i \colon \mathbb{R}^I \to \mathbb{R}$ measurable. If $|I| \leqslant |\omega|$, then the cylinder σ-algebra on $\mathbb{R}^I$ coincides with the Borel σ-algebra (see [10, Lemma 1.2]).

Theorem 2.1. *Let I be a set, $\tau \colon \mathbb{R}^I \to \mathbb{R}$ and $p \in [1, +\infty)$. The following conditions are equivalent.*
(1) *τ preserves p-integrability.*
(2) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have*

$$|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|.$$

Theorem 2.2. *Let I be a set, $\tau \colon \mathbb{R}^I \to \mathbb{R}$ and $p \in [1, +\infty)$. The following conditions are equivalent.*
(1) *τ preserves p-integrability over every finite measure space.*
(2) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have*

$$|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j |v_j|.$$

Theorems 2.1 and 2.2 show that the fact that an operation preserves p-integrability – over arbitrary and finite measure spaces, respectively – does not depend on the choice of p. Hence, once Theorems 2.1 and 2.2 will be settled, we will simply say that the operation *preserves integrability*.

The other two main results of Part I (Theorems 2.3 and 2.4 below) provide a set of generators for the operations that preserve integrability over arbitrary and finite measure spaces, respectively. To state the theorems, we start by defining, for any set $\mathcal{C}$ of operations $\tau \colon \mathbb{R}^{J_\tau} \to \mathbb{R}$, what we mean by *operations generated by* $\mathcal{C}$. Given two sets Ω and I, a subset $S \subseteq \mathbb{R}^\Omega$, and a function $\tau \colon \mathbb{R}^I \to \mathbb{R}$, we say that S is *closed under* τ if, for every family $(f_i)_{i\in I}$ of elements of S, we have that $\tau((f_i)_{i\in I})$ (which is the function from Ω to $\mathbb{R}$ which maps x to $\tau((f_i(x))_{i\in I})$) belongs to S. Consider a set $\mathcal{C}$ of functions $\tau \colon \mathbb{R}^{J_\tau} \to \mathbb{R}$, where the set J_τ depends on τ. We say that a function $f \colon \mathbb{R}^I \to \mathbb{R}$ is *generated by* $\mathcal{C}$ if f belongs to the smallest subset of $\mathbb{R}^{\mathbb{R}^I}$ which contains, for each $i \in I$, the projection function $\pi_i \colon \mathbb{R}^I \to \mathbb{R}$, and which is closed under each element of $\mathcal{C}$.

Theorem 2.3. *For every set I, the operations $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability are exactly those generated by the operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, and $\dot{-}$.*

Theorem 2.4. *For every set I, the operations $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability over every finite measure space are exactly those generated by the operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, and 1.*

The rest of Part I is devoted to a proof of Theorems 2.1–2.4.

3 Operations that preserve measurability

In this section we study measurability, which is a necessary condition for integrability. In particular, we characterise the operations that preserve measurability (Theorem 3.3). This result will be of use in the following sections as preservation of measurability is necessary to preservation of integrability (Lemma 4.2). Let us start by defining precisely what we mean by “to preserve measurability”.

Definition 3.1. Let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. For $(\Omega, \mathcal{F})$ a measurable space, we say that the function τ *preserves measurability over* $(\Omega, \mathcal{F})$ if, for every family $(f_i)_{i\in I}$ of $\mathcal{F}$-measurable functions from Ω to $\mathbb{R}$, the function $\tau((f_i)_{i\in I})\colon \Omega \to \mathbb{R}$ is also $\mathcal{F}$-measurable. We say that τ *preserves measurability* if τ preserves measurability over every measurable space.

When we regard $\mathbb{R}$ as a measurable space, we always do so with respect to the Borel σ-algebra, denoted by $\mathcal{B}_{\mathbb{R}}$.

Lemma 3.2. *Let $(\Omega, \mathcal{F})$ be a measurable space, I a set and $f\colon \Omega \to \mathbb{R}^I$ a function. Then f is $\mathcal{F}$-$\mathrm{Cyl}(\mathbb{R}^I)$-measurable if, and only if, for every $i \in I$ the function $\pi_i \circ f\colon \Omega \to \mathbb{R}$ is $\mathcal{F}$-$\mathcal{B}_{\mathbb{R}}$-measurable.*

Proof. See [17, Theorem 3.1.29 (ii)]. □

Now we can obtain a characterisation of the operations that preserve measurability.

Theorem 3.3. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. The following are equivalent.*
(1) *τ preserves measurability.*
(2) *τ preserves measurability over $(\mathbb{R}^I, \mathrm{Cyl}(\mathbb{R}^I))$.*
(3) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable.*

Proof. (1) ⇒ (2) Trivial.

(2) ⇒ (3) For every $i \in I$, $\pi_i\colon \mathbb{R}^I \to \mathbb{R}$ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable. Since τ preserves measurability, $\tau((\pi_i)_{i\in I})$ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable. Since $(\pi_i)_{i\in I}\colon \mathbb{R}^I \to \mathbb{R}^I$ is the identity, $\tau((\pi_i)_{i\in I}) = \tau \circ (\pi_i)_{i\in I} = \tau$ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable.

(3) ⇒ (1) Let us consider a measurable space $(\Omega, \mathcal{F})$ and a family $(f_i)_{i\in I}$ of measurable functions $f_i\colon \Omega \to \mathbb{R}$. Consider the function $(f_i)_{i\in I}\colon \Omega \to \mathbb{R}^I$, $x \mapsto (f_i(x))_{i\in I}$. We have $\pi_i \circ (f_i)_{i\in I} = f_i$, therefore $\pi_i \circ (f_i)_{i\in I}$ is measurable for every $i \in I$. Thus, by Lemma 3.2, $(f_i)_{i\in I}$ is measurable. Thus $\tau((f_i)_{i\in I}) = \tau \circ (f_i)_{i\in I}$ is measurable, because it is a composition of measurable functions. □

3.1 The operations that preserve measurability depend on countably many coordinates

A fact that will be of use in the following sections is that the operations that preserve measurability depend on countably many coordinates. This we show in Corollary 3.6 below. Let us start by recalling what is meant with “to depend on countably many coordinates”.

Definition 3.4. Given a set I.
(1) Let $S \subseteq \mathbb{R}^I$. For $J \subseteq I$, we say that S *depends only on J* if, given any $x, y \in \mathbb{R}^I$ such that $x_j = y_j$ for all $j \in J$, we have $x \in S \Leftrightarrow y \in S$. We say that S *depends on countably many coordinates* if there exists a countable subset $J \subseteq I$ such that S depends only on J.
(2) Let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. For $J \subseteq I$, we say that τ *depends only on J* if, given any $x, y \in \mathbb{R}^I$ such that $x_j = y_j$ for all $j \in J$, we have $\tau(x) = \tau(y)$. We say that τ *depends on countably many coordinates* if there exists a countable subset $J \subseteq I$ such that τ depends only on J.

We believe that the following proposition is folklore, but we were not able to locate an appropriate reference.

Proposition 3.5. *If $\tau\colon \mathbb{R}^I \to \mathbb{R}$ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable, then τ depends on countably many coordinates.*

Proof. First, every element of $\mathrm{Cyl}(\mathbb{R}^I)$ depends on countably many coordinates: indeed, the set of elements of $\mathrm{Cyl}(\mathbb{R}^I)$ which depend on countably many coordinates is a σ-subalgebra of $\mathrm{Cyl}(\mathbb{R}^I)$ which makes the projection functions measurable (see also [9, 254M(c)]). Second, let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be $\mathrm{Cyl}(\mathbb{R}^I)$-measurable. The idea that we will use is that τ is determined by the family $(\tau^{-1}((a, +\infty)))_{a\in\mathbb{Q}}$. For every $a \in \mathbb{Q}$, there exists a countable subset $J \subseteq I$ such that the measurable set $\tau^{-1}((a, +\infty))$ depends only on J_a. Then $J := \bigcup_{a\in\mathbb{Q}} J_a$ has the property that, for each $b \in \mathbb{Q}$, $\tau^{-1}((b, +\infty))$ depends only on J. We claim that τ depends only on J. Let $x, y \in \mathbb{R}^I$

be such that $x_j = y_j$ for every $j \in J$. We shall prove $\tau(x) = \tau(y)$. Suppose $\tau(x) \neq \tau(y)$. Without loss of generality, $\tau(x) < \tau(y)$. Let $a \in \mathbb{Q}$ be such that $\tau(x) < a < \tau(y)$. Then $x \notin \tau^{-1}((a, +\infty))$ and $y \in \tau^{-1}((a, +\infty))$. This implies that it is not true that $\tau^{-1}((a, +\infty))$ depends only on J. □

Corollary 3.6. *Let I be a set and $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. If τ preserves measurability, then τ depends on countably many coordinates.*

Proof. If τ preserves measurability, then τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable by Theorem 3.3. By Proposition 3.5, the function τ depends on countably many coordinates. □

3.2 The case of uncountable Polish spaces

The remaining results in this section are not used in the proofs of our main results.

One may think that, for an operation $\tau\colon \mathbb{R}^I \to \mathbb{R}$, the condition "$\tau$ preserve measurability over *every* measurable space" is too strong because we may not be interested in all measurable spaces. However, Proposition 3.7 shows that this condition is equivalent to "τ preserve measurability over $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$" (if τ has countable arity).

Proposition 3.7. *For a set I such that $|I| \leqslant |\omega|$ and a function $\tau\colon \mathbb{R}^I \to \mathbb{R}$, τ preserves measurability if, and only if, τ preserves measurability over $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.*

Proof. If $I = \emptyset$, then τ is a constant function. Hence τ preserves measurability over every measurable space. Let us consider the case $I \neq \emptyset$. By Theorem 3.3, τ preserves measurability if, and only if, τ preserves measurability over $(\mathbb{R}, \mathrm{Cyl}(\mathbb{R}^I))$. Since $\mathbb{R}^I$ and $\mathbb{R}$ are uncountable Polish spaces with Borel σ-algebras $\mathrm{Cyl}(\mathbb{R}^I)$ and $\mathcal{B}_{\mathbb{R}}$, respectively, $(\mathbb{R}^I, \mathrm{Cyl}(\mathbb{R}^I))$ and $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ are isomorphic measurable spaces (see [17, Theorem 3.3.13]). (Recall that an isomorphism of measurable spaces $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ is a bijective measurable function $f\colon \Omega \to \Omega'$ such that its inverse is measurable.) □

Remark 3.8. In Proposition 3.7 above, one may replace the measurable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ by any of its isomorphic copies. In particular, one may replace it with the measurable space given by any uncountable Polish space endowed with its Borel σ-algebra (see [17, Chapter 3]).

4 Operations that preserve integrability

The goal of this section is to prove Theorem 2.1, i.e., to characterise the operations that preserve p-integrability.

Remark 4.1. Let $(\Omega, \mathcal{F})$ be a measurable space, and let μ_0 be the null-measure on $(\Omega, \mathcal{F})$: for each $A \in \mathcal{F}$, $\mu_0(A) = 0$. Then $\mathcal{L}^p(\mu_0)$ is the set of $\mathcal{F}$-measurable functions from Ω to $\mathbb{R}$. Hence, preservation of p-integrability over $(\Omega, \mathcal{F}, \mu_0)$ is equivalent to preservation of measurability over $(\Omega, \mathcal{F})$.

An immediate consequence of Remark 4.1 is the following lemma.

Lemma 4.2. *Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$ and $p \in [1, +\infty)$. If τ preserves p-integrability, then τ preserves measurability.*

Lemma 4.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f, g\colon \Omega \to \mathbb{R}$ be functions, and let $\lambda \in \mathbb{R}$. Then the following properties hold.*
(1) *If $f \in \mathcal{L}^p(\mu)$, then $|f| \in \mathcal{L}^p(\mu)$.*
(2) *If $f \in \mathcal{L}^p(\mu)$, then $\lambda f \in \mathcal{L}^p(\mu)$.*
(3) *If $f, g \in \mathcal{L}^p(\mu)$, then $f + g \in \mathcal{L}^p(\mu)$.*
(4) *If $g \in \mathcal{L}^p(\mu)$, $|f| \leqslant |g|$ and f is $\mathcal{F}$-measurable, then $f \in \mathcal{L}^p(\mu)$.*

Proof. Statement (1) is immediate by definition of $\mathcal{L}^p(\mu)$, (2) follows from linearity of the integration operator, (4) follows from the monotonicity of the integration operator, while (3) follows from the Minkowski inequality (see [14, Theorem 3.5]):

$$\left(\int_\Omega |f+g|^p \, d\mu\right)^{\frac{1}{p}} \leqslant \left(\int_\Omega (|f|+|g|)^p \, d\mu\right)^{\frac{1}{p}} \overset{\text{Mink.}}{\leqslant} \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p \, d\mu\right)^{\frac{1}{p}}. \qquad \square$$

The next lemma settles the easiest direction of the characterisation of operations that preserve p-integrability, i.e., the implication (2) ⇒ (1) in Theorem 2.1.

Lemma 4.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, I a set, $\tau \colon \mathbb{R}^I \to \mathbb{R}$ an operation that preserves measurability over $(\Omega, \mathcal{F})$ and $p \in [1, +\infty)$. If there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|$, then τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.*

Proof. Let $(f_i)_{i\in I}$ be a family in $\mathcal{L}^p(\mu)$; since τ preserves measurability over $(\Omega, \mathcal{F})$, it follows that $\tau((f_i)_{i\in I})$ is $\mathcal{F}$-measurable. For each $x \in \Omega$, $|\tau((f_i(x))_{i\in I})| \leqslant \sum_{j\in J} \lambda_j |f_j(x)|$. Thus $|\tau((f_i)_{i\in I})| \leqslant \sum_{j\in J} \lambda_j |f_j|$. Hence, by Lemma 4.3, $\tau((f_i)_{i\in I}) \in \mathcal{L}^p(\mu)$. □

This shows that the condition $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|$ is sufficient for preservation of p-integrability. We are left to prove the converse direction: when τ does not satisfy this condition, there exists a measure space over which τ does not preserve p-integrability. As we shall see, at least when the arity of τ is countable, this space can always be taken to be $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathrm{Leb})$ where Leb is the restriction to $\mathcal{B}_\mathbb{R}$ of the Lebesgue measure, and this happens because $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathrm{Leb})$ is what we call a partitionable measure space.

Definition 4.5. A measure space $(\Omega, \mathcal{F}, \mu)$ is called *partitionable* if, for every sequence $(a_n)_{n\in\omega}$ of elements of $\mathbb{R}^+$, there exists a sequence $(A_n)_{n\in\omega}$ of disjoint elements of $\mathcal{F}$ such that $\mu(A_n) = a_n$.

Remark 4.6. The measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathrm{Leb})$ is partitionable.

The role of partitionable measure spaces is clarified by the following result.

Lemma 4.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $p \in [1, +\infty)$, let I be a set and let $\tau \colon \mathbb{R}^I \to \mathbb{R}$ be a function. Suppose $|I| \leqslant |\omega|$ and suppose $(\Omega, \mathcal{F}, \mu)$ is partitionable. If τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$, then there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have*

$$|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|.$$

Proof. We give the proof for $I = \omega$. The case $|I| < |\omega|$ relies on an analogous argument.

We suppose, contrapositively, that, for every finite subset of indices $J \subseteq I$ and every J-tuple $(\lambda_j)_{j\in J}$ of nonnegative real numbers, there exists $v \in \mathbb{R}^I$ such that $|\tau(v)| > \sum_{j\in J} \lambda_j |v_j|$; we shall prove that τ does not preserve p-integrability. For each $n \in \omega$, we let v^n be an element of $\mathbb{R}^I$ such that $|\tau(v^n)| > \sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v_j^n|$. Set $C := \Omega \setminus \bigcup_{n\in\omega} A_n$. For each $i \in \omega$, we set

$$f_i \colon \Omega \to \mathbb{R}, \qquad x \mapsto \begin{cases} v_i^n & \text{if } x \in A_n, \\ 0 & \text{if } x \in C. \end{cases}$$

Let $(A_n)_{n\in\omega}$ be a sequence of disjoint elements of $\mathcal{F}$ such that $\mu(A_n) = \frac{1}{|\tau(v^n)|^p}$; one such sequence exists because $(\Omega, \mathcal{F}, \mu)$ is partitionable. Then

$$\begin{aligned} \int_\Omega |\tau((f_i)_{i\in\omega})|^p \, d\mu &= \int_C |\tau((f_i)_{i\in\omega})|^p \, d\mu + \sum_{n\in\omega} \int_{A_n} |\tau((f_i)_{i\in\omega})|^p \, d\mu \\ &\geqslant \sum_{n\in\omega} |\tau((v_i^n)_{i\in\omega})|^p \mu(A_n) \\ &= \sum_{n\in\omega} |\tau(v^n)|^p \frac{1}{|\tau(v^n)|^p} = \sum_{n\in\omega} 1 = \infty. \end{aligned} \tag{4.1}$$

The following chain of inequalities holds:

$$\begin{aligned}\int_{\Omega} |f_i|^p \, d\mu &= \sum_{n\in\omega} |v_i^n|^p \mu(A_n) = \sum_{n\in\omega} |v_i^n|^p \frac{1}{|\tau(v^n)|^p} \\ &\leqslant M + \sum_{n>i, v_i^n\neq 0} |v_i^n|^p \frac{1}{|\tau(v^n)|^p} \quad (\text{for some } M \in \mathbb{R}^+) \\ &\leqslant M + \sum_{n>i, v_i^n\neq 0} |v_i^n|^p \frac{1}{(\sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v_j^n|)^p} \leqslant M + \sum_{n>i, v_i^n\neq 0} |v_i^n|^p \frac{1}{(2^{\frac{n}{p}} |v_i^n|)^p} \\ &\leqslant M + \sum_{n>i, v_i^n\neq 0} \frac{1}{2^n} < \infty. \end{aligned} \tag{4.2}$$

The first inequality holds for some $M \in \mathbb{R}^+$ because with the condition $n > i$ we ignore finitely many terms of the series, while with the condition $v_i^n \neq 0$ we ignore some null terms. The third inequality holds because $n > i \Rightarrow i \in \{0, \dots, n-1\}$.

From equations (4.1) and (4.2) we conclude that τ does not preserve p-integrability. □

Lemma 4.8. *If I is a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$ a function, $p \in [1, +\infty)$ and $(\Omega, \mathcal{F}, \mu)$ a partitionable measure space, then the following conditions are equivalent.*

(1) *τ preserves p-integrability.*

(2) *τ preserves measurability, and τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.*

(3) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|$.*

Proof. (1) ⇒ (2) If τ preserves p-integrability, then, by Lemma 4.2, τ preserves measurability. Trivially, τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.

(2) ⇒ (3) If τ preserves measurability, then, by Theorem 3.3, τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable. By Proposition 3.5, τ depends on countably many coordinates, hence Lemma 4.7 applies and the proof of the implication is complete.

(3) ⇒ (1) By Theorem 3.3, τ preserves measurability. By Lemma 4.4, the thesis is proved. □

Proof of Theorem 2.1. There exist partitionable measure spaces, see, e.g., Remark 4.6. Theorem 2.1 is the equivalence (1) ⇔ (3) in Lemma 4.8. □

4.1 Examples

Example 4.9. Let $n \in \omega$ and $\tau\colon \mathbb{R}^n \to \mathbb{R}$. Then τ preserves p-integrability if, and only if, τ is Borel measurable and there exist $\lambda_0, \dots, \lambda_{n-1} \in \mathbb{R}^+$ such that, for every $x \in \mathbb{R}^n$, we have

$$|\tau(x)| \leqslant \sum_{j=0}^{n-1} \lambda_i |x_i|.$$

Example 4.10. A function $\tau\colon \mathbb{R}^\omega \to \mathbb{R}$ preserves p-integrability if, and only if, τ is Borel measurable and there exist a finite subset of indices $J \subseteq \omega$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have

$$|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j |v_j|.$$

4.2 The case of $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathrm{Leb})$ and the discrete case

The remaining results in this section are not used in the proofs of our main results.

One may think that, for an operation $\tau\colon \mathbb{R}^I \to \mathbb{R}$, the condition "$\tau$ preserve p-integrability over *every* measure space" is too strong because we may not be interested in all measure spaces. However, Proposition 4.11 shows that this condition is equivalent to "τ preserve p-integrability over $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \text{Leb})$" (if τ has countable arity), and Proposition 4.13 provides an analogous result for a discrete measure space.

Proposition 4.11. *Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$, with $|I| \leqslant |\omega|$, and $p \in [1, +\infty)$. Then τ preserves p-integrability if, and only if, τ preserves p-integrability over $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \text{Leb})$.*

Proof. Trivially, if τ preserves p-integrability, then τ preserves p-integrability over $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \text{Leb})$. For the converse, by Proposition 3.7, if τ preserves p-integrability over $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \text{Leb})$ then τ preserves measurability. By Remark 4.6, $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \text{Leb})$ is partitionable. An application of (2) $\Rightarrow$ (1) in Lemma 4.8 concludes the proof. □

We next provide an analogue of Proposition 4.11 for a discrete measure space. We denote by $\mathcal{P}(X)$ the power set of a set X.

Lemma 4.12. *There exists a measure μ on $(\omega, \mathcal{P}(\omega))$ such that $(\omega, \mathcal{P}(\omega), \mu)$ is partitionable.*

Proof. We define a measure μ on $(\omega \times \mathbb{Z}, \mathcal{P}(\omega \times \mathbb{Z}))$, by setting $\mu(\{(n, z)\}) = 2^z$. For every $n \in \omega$, there exists $K_n \subseteq \mathbb{Z}$ such that $a_n = \sum_{z \in K_n} 2^z$. Set $A_n := \{(n, z) \mid z \in K_n\}$. Then $\mu(A_n) = \sum_{z \in K_n} \mu(\{(n, z)\}) = \sum_{z \in K_n} 2^z = a_n$. Moreover, for any pair of distinct $n, m \in \omega$, the sets A_n and A_m are disjoint. The set $\omega \times \mathbb{Z}$ is countably infinite, hence $(\omega \times \mathbb{Z}, \mathcal{P}(\omega \times \mathbb{Z}))$ and $(\omega, \mathcal{P}(\omega))$ are isomorphic measurable spaces, which concludes the proof. □

Proposition 4.13. *There exists a measure μ on $(\omega, \mathcal{P}(\omega))$ such that, for every set I, every function $\tau\colon \mathbb{R}^I \to \mathbb{R}$ and every $p \in [1, +\infty)$, τ preserves p-integrability if, and only if, τ preserves measurability and τ preserves p-integrability over $(\omega, \mathcal{P}(\omega), \mu)$.*

Proof. By Lemma 4.12, there exists a measure μ on $(\omega, \mathcal{P}(\omega))$ such that $(\omega, \mathcal{P}(\omega), \mu)$ is partitionable. The thesis follows from (1) $\Leftrightarrow$ (2) in Lemma 4.8. □

5 Operations that preserve integrability over finite measure spaces

The goal of this section is to prove Theorem 2.2, i.e., to characterise the operations that preserve p-integrability over finite measure spaces. We follow the same strategy of Section 4, with the appropriate adjustments.

Lemma 5.1. *Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$ and $p \in [1, +\infty)$. If τ preserves p-integrability over every finite measure space, then τ preserves measurability.*

Proof. By Remark 4.1. □

Lemma 5.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, I a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$ an operation that preserves measurability over $(\Omega, \mathcal{F})$ and $p \in [1, +\infty)$. If there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j \in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant k + \sum_{j \in J} \lambda_j |v_j|$, then τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.*

Proof. Let $(f_i)_{i \in I}$ be a family in $\mathcal{L}^p(\mu)$; since τ preserves measurability over $(\Omega, \mathcal{F})$, we have that $\tau((f_i)_{i \in I})$ is $\mathcal{F}$-measurable. For each $x \in \Omega$, $|\tau((f_i(x))_{i \in I})| \leqslant k + \sum_{j \in J} \lambda_j |f_j(x)|$. Thus $|\tau((f_i)_{i \in I})| \leqslant k + \sum_{j \in J} \lambda_j |f_j|$. Note that the function $k\colon \Omega \to \mathbb{R}$, $x \mapsto k$ belongs to $\mathcal{L}^p(\mu)$, because μ is finite. Hence, by Lemma 4.3, $\tau((f_i)_{i \in I}) \in \mathcal{L}^p(\mu)$. □

It is not difficult to see that no finite measure space is partitionable: thus we replace the concept of partitionability with a slightly different one.

Definition 5.3. A measure space $(\Omega, \mathcal{F}, \mu)$ is called *conditionally partitionable* if there exists a sequence $(b_n)_{n \in \omega}$ of strictly positive real numbers such that, for every sequence $(a_n)_{n \in \omega}$ of elements of $\mathbb{R}^+$ satisfying $a_n \leqslant b_n$ for every $n \in \omega$, there exists a sequence $(A_n)_{n \in \omega}$ of disjoint elements of $\mathcal{F}$ such that $\mu(A_n) = a_n$.

Remark 5.4. The measure space $([0, 1], \mathcal{B}_{[0,1]}, \text{Leb})$, where Leb is the Lebesgue measure, is conditionally partitionable (take $b_n = \frac{1}{2^{n+1}}$).

Lemma 5.5. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $p \in [1, +\infty)$, let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. Suppose that $|I| \leqslant |\omega|$ and that $(\Omega, \mathcal{F}, \mu)$ is conditionally partitionable. If τ preserves p-integrability* *over $(\Omega, \mathcal{F}, \mu)$, then there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have*

$$|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j |v_j|.$$

Proof. We give the proof for $I = \omega$. The case $|I| < |\omega|$ relies on an analogous argument.

We suppose, contrapositively, that, for every finite subset of indices $J \subseteq I$, every J-tuple $(\lambda_j)_{j\in J}$ of nonnegative real numbers and every $k \in \mathbb{R}^+$, there exists $v \in \mathbb{R}^I$ such that $|\tau(v)| > k + \sum_{j\in J} \lambda_j |v_j|$; we shall prove that τ does not preserve p-integrability. Since $(\Omega, \mathcal{F}, \mu)$ is conditionally partitionable, there exists a sequence $(b_n)_{n\in\omega}$ of strictly positive real numbers such that, for every sequence $(a_n)_{n\in\omega}$ of elements of $\mathbb{R}^+$ satisfying $a_n \leqslant b_n$ for every $n \in \omega$, there exists a sequence $(A_n)_{n\in\omega}$ of disjoint elements of $\mathcal{F}$ such that $\mu(A_n) = a_n$.

For each $n \in \omega$, we let v^n be an element of $\mathbb{R}^I$ such that

$$|\tau(v^n)| > \left(\frac{1}{b_n}\right)^{\frac{1}{p}} + \sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v^n_j|.$$

Then we have

$$\frac{1}{|\tau(v^n)|^p} < \frac{1}{((\frac{1}{b_n})^{\frac{1}{p}} + \sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v^n_j|)^p} \leqslant \frac{1}{((\frac{1}{b_n})^{\frac{1}{p}})^p} = b_n.$$

Therefore, there exists a sequence $(A_n)_{n\in\omega}$ of disjoint elements of $\mathcal{F}$ such that $\mu(A_n) = \frac{1}{|\tau(v^n)|^p}$. Since

$$|\tau(v^n)| > \left(\frac{1}{b_n}\right)^{\frac{1}{p}} + \sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v^n_j| > \sum_{j=0}^{n-1} 2^{\frac{n}{p}} |v^n_j|,$$

the remaining part of the proof runs as for Lemma 4.8. □

Lemma 5.6. *Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$ a function, $p \in [1, +\infty)$ and $(\Omega, \mathcal{F}, \mu)$ a conditionally partitionable finite measure space. The following conditions are equivalent.*
(1) *τ preserves p-integrability over every finite measure space.*
(2) *τ preserves measurability, and τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.*
(3) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j |v_j|$.*

Proof. (1) ⇒ (2) If τ preserves p-integrability over every finite measure space, then, by Lemma 5.1, τ preserves measurability. Trivially, τ preserves p-integrability over $(\Omega, \mathcal{F}, \mu)$.

(2) ⇒ (3) If τ preserves measurability, then, by Theorem 3.3, τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable. By Proposition 3.5, τ depends on countably many coordinates, hence Lemma 5.5 applies and the proof of the implication is complete.

(3) ⇒ (1) By Theorem 3.3, τ preserves measurability. By Lemma 5.2, the thesis is proved. □

Proof of Theorem 2.2. There exist conditionally partitionable finite measure spaces, see, e.g., Remark 5.4. Theorem 2.2 is the equivalence (1) ⇔ (3) in Lemma 5.6. □

5.1 Examples

Example 5.7. Let $n \in \omega$ and $\tau\colon \mathbb{R}^n \to \mathbb{R}$. Then τ preserves p-integrability over every finite measure space if, and only if, τ is Borel measurable and there exist $\lambda_0, \dots, \lambda_{n-1}, k \in \mathbb{R}^+$ such that, for every $x \in \mathbb{R}^n$, we have

$$|\tau(x)| \leqslant k + \sum_{j=0}^{n-1} \lambda_j |x_j|.$$

Example 5.8. A function $\tau\colon \mathbb{R}^\omega \to \mathbb{R}$ preserves p-integrability over every finite measure space if, and only if, τ is Borel measurable and there exist a finite subset of indices $J \subseteq \omega$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ and k such that, for every $v \in \mathbb{R}^I$, we have

$$|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j |v_j|.$$

5.2 The case of $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$ and the discrete case

The remaining results in this section are not used in the proofs of our main results.

One may think that, for an operation $\tau\colon \mathbb{R}^I \to \mathbb{R}$, the condition "$\tau$ preserve p-integrability over *every* finite measure space" is too strong because we may not be interested in all finite measure spaces. However, Proposition 5.9 shows that this condition is equivalent to "τ preserve p-integrability over $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$" (at least when τ has countable arity), and Proposition 5.11 provides an analogous result for a discrete finite measure space.

Proposition 5.9. *Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$, with $|I| \leqslant |\omega|$, and $p \in [1,+\infty)$. Then τ preserves p-integrability over every finite measure space if, and only if, τ preserves p-integrability over* $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$.

Proof. Trivially, if τ preserves p-integrability, then τ preserves p-integrability over $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$. For the converse, by Proposition 3.7 and Remark 3.8, if τ preserves p-integrability over $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$, then τ preserves measurability. By Remark 5.4, $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$ is conditionally partitionable. An application of (2) ⇒ (1) in Lemma 5.6 concludes the proof. □

Similarly to the case of arbitrary measure, we next provide an analogue of Proposition 5.9 for a discrete finite measure space.

Lemma 5.10. *There exists a probability measure μ on $(\omega, \mathcal{P}(\omega))$ such that the measure space $(\omega, \mathcal{P}(\omega), \mu)$ is conditionally partitionable.*

Proof. Let $X := \{(n,m) \in \omega \times \omega \mid m \geqslant n\}$. We let ν be the unique measure on $(X, \mathcal{P}(X))$ such that, for every $(n,m) \in X$, we have $\nu(\{(n,m)\}) = \frac{1}{2^m}$. Then,

$$\sum_{(n,m)\in X} \nu(\{(n,m)\}) = \sum_{n\in\omega} \sum_{m\in\omega,\, m\geqslant n} \nu(\{(n,m)\}) = \sum_{n\in\omega} \sum_{m\in\omega,\, m\geqslant n} \frac{1}{2^m}$$
$$= \sum_{n\in\omega} \frac{2}{2^n} = 4.$$

Hence, ν is a finite measure.

We prove that $(X, \mathcal{P}(X), \nu)$ is conditionally partitionable. For $n \in \omega$, let $b_n := \frac{1}{2^{n-1}}$. Further, let $(a_n)_{n\in\omega}$ be a sequence of elements of $\mathbb{R}^+$ satisfying $a_n \leqslant b_n$ for every $n \in \omega$. For every $n \in \mathbb{N}$, since $0 \leqslant a_n \leqslant \frac{1}{2^{n-1}}$, there exists a subset K_n of $\{k \in \omega \mid k \geqslant n\}$ such that $a_n = \sum_{k\in K_n} \frac{1}{2^k}$. Set $A_n := \{(n,m) \mid m \in K_n\}$. Note that $A_n \subseteq X$. Then $\mu(A_n) = \sum_{m\in K_n} \mu(\{(n,m)\}) = \sum_{m\in K_n} \frac{1}{2^m} = a_n$. Moreover, for any pair of distinct $n, m \in \omega$, the sets A_n and A_m are disjoint. This proves that $(X, \mathcal{P}(X), \nu)$ is conditionally partitionable.

Define the measure $\frac{\nu}{4}$ on $(X, \mathcal{P}(X))$ by setting $\frac{\nu}{4}(A) = \frac{\nu(A)}{4}$. Using the fact that $(X, \mathcal{P}(X), \nu)$ is a conditionally partitionable measure space, it is not difficult to see that $(X, \mathcal{P}(X), \frac{\nu}{4})$ is a conditionally partitionable measure space, too. We have $\frac{\nu}{4}(X) = \frac{\nu(X)}{4} = \frac{4}{4}$; thus $\frac{\nu}{4}$ is a probability measure.

The set X is countably infinite, hence $(X, \mathcal{P}(X))$ and $(\omega, \mathcal{P}(\omega))$ are isomorphic measurable spaces, which concludes the proof. □

Proposition 5.11. *There exists a probability measure μ on $(\omega, \mathcal{P}(\omega))$ such that, for every set I, every function $\tau\colon \mathbb{R}^I \to \mathbb{R}$ and every $p \in [1,+\infty)$, τ preserves p-integrability over every finite measure space if, and only if, τ preserves measurability and τ preserves p-integrability over $(\omega, \mathcal{P}(\omega), \mu)$.*

Proof. By Lemma 5.10, there exists a probability measure μ on $(\omega, \mathcal{P}(\omega))$ such that $(\omega, \mathcal{P}(\omega), \mu)$ is conditionally partitionable. The thesis follows from (1) ⇔ (2) in Lemma 5.6. □

6 Generation

The goal of this section is to prove Theorems 2.3 and 2.4, which exhibit a generating set for the class of operations that preserve integrability over arbitrary and finite measure spaces, respectively.

As it is shown by Theorems 2.1 and 2.2, the fact that an operation preserves p-integrability – over arbitrary and finite measure spaces, respectively – does not depend on the choice of p. Hence, we say that the operation *preserves integrability*.

Recall from the introduction the operation

$$\curlyvee(y, x_0, x_1, \dots) := \sup_{n\in\omega}\{x_n \wedge y\}.$$

We adopt the notation

$$\overset{y}{\underset{n\in\omega}{\curlyvee}} x_n := \curlyvee(y, x_0, x_1, \dots).$$

From the operations 0, +, $\vee$ and $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$) we generate the operations

$$\begin{aligned} f \wedge g &:= -((-f) \vee (-g)),\\ f^+ &:= f \vee 0,\\ f^- &:= -(f \wedge 0),\\ |f| &:= f^+ - f^-. \end{aligned}$$

Additionally, using $\curlyvee$, we generate

$$\overset{g}{\underset{n\in\omega}{\curlywedge}} f_n := \inf_{n\in\omega}\{f_n \vee g\} = -\overset{-g}{\underset{n\in\omega}{\curlyvee}} -f_n.$$

Let Ω be a set and let $S \subseteq \mathbb{R}^\Omega$. We let $\sigma(S)$ denote the smallest σ-algebra $\mathcal{F}$ of subsets of Ω such that every $s \in S$ is $\mathcal{F}$-measurable.

Lemma 6.1. *Let Ω be a set and let $S \subseteq \mathbb{R}^\Omega$. Then $\sigma(S)$ is the σ-algebra of subsets of Ω generated by the set $\{g^{-1}((\lambda, +\infty)) \mid g \in S, \lambda \in \mathbb{R}\}$.*

Proof. See [8, Proposition 2.3]. □

Lemma 6.2. *Let Ω be a set, let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, let K be an element of the σ-algebra of subsets of Ω generated by $\mathcal{A}$, and let $K \subseteq Y \subseteq \Omega$. Then K belongs to any σ-algebra $\mathcal{G}$ of subsets of Y such that $A \cap Y \in \mathcal{G}$ for each $A \in \mathcal{A}$.*

Proof. Let $\Sigma := \{S \subseteq \Omega \mid S \cap Y \in \mathcal{G}\}$. A straightforward verification shows that Σ is a σ-algebra of subsets of Ω. Moreover, $\mathcal{A} \subseteq \Sigma$. Therefore, by definition of $\mathcal{F}$, $\mathcal{F} \subseteq \Sigma$. Hence, $K \in \Sigma$, which means $K = K \cap Y \in \mathcal{G}$. □

Given $S \subseteq \mathbb{R}^\Omega$, we denote by $\langle S\rangle$ the closure of S under 0, +, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\overline{\cdot}$. Given $A \subseteq \Omega$, we write $\mathbb{1}_A$ for the characteristic function of A in Ω.

Lemma 6.3. *Let Ω be a set, let $S \subseteq \mathbb{R}^\Omega$, let $K \in \sigma(S)$ and let $K \subseteq Y \subseteq \Omega$ be such that $\mathbb{1}_Y \in \langle S\rangle$. Then $\mathbb{1}_K \in \langle S\rangle$.*

Proof. Set $\mathcal{G} := \{C \subseteq Y \mid \mathbb{1}_C \in \langle S\rangle\}$. Note that $\mathcal{G}$ is a σ-algebra of subsets of Y. Indeed, $\mathbb{1}_Y \in \langle S\rangle$, and, for $C_0, C_1 \subseteq Y$, we have $\mathbb{1}_{C_0\cap C_1} = \mathbb{1}_{C_0} \wedge \mathbb{1}_{C_1}$ and $\mathbb{1}_{Y\setminus C_0} = \mathbb{1}_Y - \mathbb{1}_{C_0}$. Further, let $(C_n)_{n\in\omega}$ be a family with $C_n \subseteq Y$. The characteristic function of $\bigcup_{n\in\omega} C_n$ is $\curlyvee^{\mathbb{1}_Y}_{n\in\omega} \mathbb{1}_{C_n}$.

By Lemma 6.1, the σ-algebra $\sigma(S)$ is generated by $\mathcal{A} := \{g^{-1}((\lambda, +\infty)) \mid g \in S, \lambda \in \mathbb{R}\}$. Let $A \in \mathcal{A}$, and write $A = g^{-1}((\lambda, +\infty))$ for some $g \in S$ and some $\lambda \in \mathbb{R}^+$. We have

$$\mathbb{1}_{A\cap Y} := \overset{\mathbb{1}_Y}{\underset{n\in\omega}{\curlyvee}} n(g - \lambda\mathbb{1}_Y)^+. \tag{6.1}$$

Indeed, for $x \in A \cap Y$, we have $g(x) > \lambda$ and $\mathbb{1}_Y(x) = 1$, hence

$$\overset{\mathbb{1}_Y(x)}{\underset{n\in\omega}{\curlyvee}} n(g(x) - \lambda\mathbb{1}_Y(x))^+ = \overset{1}{\underset{n\in\omega}{\curlyvee}} n(\underbrace{g(x) - \lambda}_{>0})^+ = 1.$$

For $x \in \Omega \setminus Y$, we have $\mathbb{1}_Y(x) = 0$, and therefore

$$\overset{\mathbb{1}_Y(x)}{\underset{n\in\omega}{\curlyvee}} n(g(x) - \lambda\mathbb{1}_Y(x))^+ = \overset{0}{\underset{n\in\omega}{\curlyvee}} n(g(x))^+ = 0.$$

For $x \in Y \setminus A$, we have $g(x) \leqslant \lambda$ and $\mathbb{1}_Y(x) = 1$, hence

$$\curlyvee_{n\in\omega}^{\mathbb{1}_Y(x)} n(g(x) - \lambda\mathbb{1}_Y(x))^+ = \curlyvee_{n\in\omega}^{1} n(g(x) - \lambda)^+ = \curlyvee_{n\in\omega}^{1} 0 = 0.$$

Given equation (6.1), we have $\mathbb{1}_{A\cap Y} \in \langle S\rangle$, which means $A \cap Y \in \mathcal{G}$. By Lemma 6.2, $K \in \mathcal{G}$. □

The truncation operation $\overline{\cdot}$ comes into play in the following lemma.

Lemma 6.4. *Let $\lambda \in \mathbb{R}^+ \setminus \{0\}$. The operations*

$$\mathbb{1}_{\cdot>\lambda} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x > \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbb{1}_{\cdot\geqslant\lambda} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x \geqslant \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

are generated by the operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee, \overline{\cdot}$.

Proof. Computation shows $\mathbb{1}_{f>1} = \curlyvee_{n\in\omega}^{\overline{f}} n(f - \overline{f})$. Moreover, $\mathbb{1}_{f>\lambda} = \mathbb{1}_{\frac{1}{\lambda}f>1}$. Finally, let $0 < q_0 < q_1 < \cdots$ be a sequence of elements of $\mathbb{R}$ such that $q_n \to \lambda$. Then $\mathbb{1}_{f\geqslant\lambda} = \curlywedge_{n\in\omega}^{0} \mathbb{1}_{f>q_n}$. □

Lemma 6.5. *Let $S \subseteq \mathbb{R}^\Omega$, let $g \in \langle S\rangle$, $A \in \sigma(S)$, $\lambda \in \mathbb{R}^+$ be such that $\lambda\mathbb{1}_A \leqslant g$. Then $\lambda\mathbb{1}_A \in \langle S\rangle$.*

Proof. We have $0 \in \langle S\rangle$, hence the thesis is immediate for $\lambda = 0$. Suppose $\lambda > 0$. Then $A \subseteq \{x \in \Omega \mid g(x) \geqslant \lambda\}$. By Lemma 6.4, $\mathbb{1}_{\{x\in\Omega \mid g(x)\geqslant\lambda\}} = \mathbb{1}_{g\geqslant\lambda} \in \langle S\rangle$. By Lemma 6.3, $\mathbb{1}_A \in \langle S\rangle$, hence $\lambda\mathbb{1}_A \in \langle S\rangle$. □

Lemma 6.6. *Let $S \subseteq \mathbb{R}^\Omega$, let $g \in \langle S\rangle$ and let $f \in \mathbb{R}^\Omega$ be $\sigma(S)$-measurable and such that $|f| \leqslant g$. Then $f \in \langle S\rangle$.*

Proof. First, we prove the statement for $f \geqslant 0$. Given that f is positive and $\sigma(S)$-measurable, f is the supremum in $\mathbb{R}^\Omega$ of a positive increasing sequence $(s_n)_{n\in\omega}$ of $\sigma(S)$-measurable simple functions (see [14, Theorem 1.17]). By Lemma 6.5, $s_n \in \langle S\rangle$ for every $n \in \omega$. Hence

$$f = \sup_{n\in\omega} s_n = \sup_{n\in\omega} s_n \wedge g = \curlyvee_{n\in\omega}^{g} s_n \in \langle S\rangle.$$

For f not necessarily positive, the previous part of the proof shows that f^+ and f^- belong to $\langle S\rangle$. Then $f = f^+ - f^- \in \langle S\rangle$. □

Lemma 6.7. *Let $(\Omega, \mathcal{F})$ be a measurable space, and, for each $n \in \omega$, let $f_n \colon \Omega \to \mathbb{R}$ be a measurable function. If, for every $x \in \Omega$, $\sup_{n\in\omega} f_n(x) \in \mathbb{R}$, then $\sup f_n \colon \Omega \to \mathbb{R}$ is measurable. Analogously, if, for every $x \in \Omega$, $\inf_{n\in\omega} f_n(x) \in \mathbb{R}$, then the function $\inf_{n\in\omega} f_n \colon \Omega \to \mathbb{R}$ is measurable.*

Proof. By [14, Theorem 1.14]. □

Lemma 6.8. *The operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\overline{\cdot}$ preserve integrability.*

Proof. The operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$) and $\overline{\cdot}$ preserve integrability. Moreover,

$$\curlyvee_{n\in\omega}^{g} f_n = \sup_{n\in\omega}\{f_n \wedge g\}$$

and therefore, by Lemma 6.7, $\curlyvee$ preserves measurability. The constant function 0 is always integrable, therefore 0 preserves integrability. By (3) in Lemma 4.3, + preserves integrability. The operation $|\cdot|$ is immediately seen to preserve integrability. Since, for every f, g functions, $|f \vee g| \leqslant |f| + |g|$, then $\vee$ preserves integrability by (4) in Lemma 4.3. We have $\curlyvee_{n\in\omega}^{g} f_n = \sup_{n\in\omega}\{f_n \wedge g\}$, and therefore $f_0 \wedge g \leqslant \curlyvee_{n\in\omega}^{g} f_n \leqslant g$. Hence, $|\curlyvee_{n\in\omega}^{g} f_n| \leqslant |g| + |f_0|$. Thus, $\curlyvee$ preserves integrability. Finally, $|\overline{f}| \leqslant |f|$, and therefore $\overline{\cdot}$ preserve integrability, by (4) in Lemma 4.3. □

Proof of Theorem 2.3. The operations $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\overline{\cdot}$ preserve integrability by Lemma 6.8. Moreover, by definition, the class of integrability-preserving operations is closed under every integrability-preserving operation and contains the projection functions. Therefore, every operation generated by 0, +, ∨, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and $\bar{\cdot}$ preserves integrability.

To prove the converse, we use Theorem 2.1. Let J be a finite subset of I, and let $(\lambda_j)_{j\in J}$ be a J-tuple of nonnegative real numbers. Then $\sum_{j\in J} \lambda_j|\pi_j| \in \langle\{\pi_i \mid i \in I\}\rangle$. Let τ be $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and such that for every $v \in \mathbb{R}^I$ we have $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j|v_j|$, i.e., $|\tau| \leqslant \sum_{j\in J} \lambda_j|\pi_j|$. Note that $\mathrm{Cyl}(\mathbb{R}^I) = \sigma(\{\pi_i \mid i \in I\})$, by definition. Then $\tau \in \langle\{\pi_i \mid i \in I\}\rangle$, by Lemma 6.6. Therefore, τ is generated by 0, +, ∨, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, $\bar{\cdot}$. □

It is worth recalling that, in the proof of Theorem 2.3, the role of the truncation operation $\bar{\cdot}$ lies in Lemma 6.4.

Proof of Theorem 2.4. Note that the operations 0, +, ∨, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and 1 preserve integrability over finite measure spaces. Moreover, by definition, the class of the operations that preserve integrability over finite measure spaces is closed under every integrability-preserving operation and contains the projection functions. Therefore, every operation generated by 0, +, ∨, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and 1 preserves integrability over every finite measure space.

To prove the converse, we use Theorem 2.2. Note that the truncation is generated by ∨, $-1(\cdot)$ (i.e., scalar multiplication by -1), and 1; indeed, $\bar{f} = f \wedge 1 = -((-f) \vee (-1))$. Let J be a finite subset of I, let $(\lambda_j)_{j\in J}$ be a J-tuple of nonnegative real numbers, and let $k \in \mathbb{R}^+$. Then $k + \sum_{j\in J} \lambda_j|\pi_j| \in \langle\{\pi_i \mid i \in I\} \cup \{k\}\rangle$. Let τ be $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and such that for every $v \in \mathbb{R}^I$ we have $|\tau(v)| \leqslant k + \sum_{j\in J} \lambda_j|v_j|$, i.e., $|\tau| \leqslant k + \sum_{j\in J} \lambda_j|\pi_j|$. Note that $\mathrm{Cyl}(\mathbb{R}^I) = \sigma(\{\pi_i \mid i \in I\}) = \sigma(\{\pi_i \mid i \in I\} \cup \{1\})$, by definition. Then we have $\tau \in \langle\{\pi_i \mid i \in I\} \cup \{1\}\rangle$, by Lemma 6.6. Therefore, τ is generated by 0, +, ∨, $\lambda-$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, 1. □

Part II: Truncated Riesz spaces and weak units

7 The operation $\curlyvee$

We now investigate the operation $\curlyvee$, defined on $\mathbb{R}$ in Section 6, for more general lattices. Given a Dedekind σ-complete (not necessarily bounded) lattice B we write $\curlyvee$ for the operation on B of countably infinite arity defined as

$$\curlyvee(g, f_0, f_1, \dots) := \sup_{n\in\omega}\{f_n \wedge g\}$$

We adopt the notation

$$\curlyvee_{n\in\omega}^{g} f_n := \curlyvee(g, f_0, f_1, \dots).$$

Proposition 7.1. *If B is a Dedekind σ-complete lattice, then the following properties hold for every $g, h \in B$ and all $(f_n)_{n\in\omega} \subseteq B$.*
(TS1) $\curlyvee_{n\in\omega}^{g} f_n = \curlyvee_{n\in\omega}^{g} (f_n \wedge g)$.
(TS2) $\curlyvee_{n\in\omega}^{g} f_n = (f_0 \wedge g) \vee (\curlyvee_{n\in\omega\setminus\{0\}}^{g} f_n)$.
(TS3) $\curlyvee_{n\in\omega}^{g} (f_n \wedge h) \leqslant h$.

Proof. Straightforward verification. □

Conversely, we have the following.

Proposition 7.2. *If B is a lattice endowed with an operation $\curlyvee$ of countably infinite arity which satisfies* (TS1), (TS2) *and* (TS3), *then B is Dedekind σ-complete and $\curlyvee_{n\in\omega}^{g} f_n = \sup_{n\in\omega}\{f_n \wedge g\}$.*

Proof. By induction on $k \in \omega$, (TS2) entails

$$\curlyvee_{n\in\omega}^{g} f_n = (f_0 \wedge g) \vee \dots \vee (f_k \wedge g) \vee \left(\curlyvee_{n\geqslant k+1}^{g} f_n \right).$$

Thus $f_k \wedge g \leqslant (f_0 \wedge g) \vee \cdots \vee (f_k \wedge g) \vee (\curlyvee^{g}_{n \geqslant k+1} f_n) = \curlyvee^{g}_{n\in\omega} f_n$. Thus, $\curlyvee^{g}_{n\in\omega} f_n$ is an upper bound of $(f_k \wedge g)_{k\in\omega}$. Suppose now that $f_n \wedge g \leqslant h$ for every $n \in \omega$. Then

$$\curlyvee^{g}_{n\in\omega} f_n \overset{\text{(TS1)}}{=} \curlyvee^{g}_{n\in\omega} (f_n \wedge g) \overset{f_n\wedge g\leqslant h}{=} \curlyvee^{g}_{n\in\omega} (f_n \wedge g \wedge h) \overset{\text{(TS3)}}{\leqslant} h.$$

This shows $\curlyvee^{g}_{n\in\omega} f_n = \sup_{n\in\omega}\{f_n \wedge g\}$. To prove that B is Dedekind σ-complete, let $(f_n)_{n\in\omega} \subseteq B$ and $g \in B$ be such that $f_n \leqslant g$ for all $n \in \omega$. Then

$$\curlyvee^{g}_{n\in\omega} f_n = \sup_{n\in\omega}\{f_n \wedge g\} \overset{f_n\leqslant g}{=} \sup_{n\in\omega} f_n.$$ □

A map between two partially ordered sets is *σ-continuous* if it preserves all existing countable suprema.

Proposition 7.3. *Let $\varphi\colon B \to C$ be a lattice morphism between two Dedekind σ-complete lattices. Then φ is σ-continuous if, and only if, φ preserves $\curlyvee$.*

Proof. First, suppose φ preserves $\curlyvee$. Let $(f_n)_{n\in\omega} \subseteq B$ and $f = \sup_{n\in\omega} f_n$. Then

$$\begin{aligned}
\varphi\Big(\sup_{n\in\omega} f_n\Big) &= \varphi\Big(\sup_{n\in\omega}\{f_n \wedge f\}\Big) && \text{(because } f_n \leqslant f)\\
&= \varphi\Big(\curlyvee^{f}_{n\in\omega} f_n\Big)\\
&= \curlyvee^{\varphi(f)}_{n\in\omega} \varphi(f_n) && \text{(because } \varphi \text{ preserves } \curlyvee)\\
&= \sup_{n\in\omega}\{\varphi(f_n) \wedge \varphi(f)\}\\
&= \sup_{n\in\omega} \varphi(f_n \wedge f) && \text{(because } \varphi \text{ preserves } \wedge)\\
&= \sup_{n\in\omega} \varphi(f_n) && \text{(because } f_n \leqslant f).
\end{aligned}$$

Therefore, φ is σ-continuous.

For the converse implication, suppose that φ is σ-continuous. Let $(f_n)_{n\in\omega} \subseteq B$ and $g \in B$. Then

$$\begin{aligned}
\varphi\Big(\curlyvee^{g}_{n\in\omega} f_n\Big) &= \varphi\Big(\sup_{n\in\omega}\{f_n \wedge g\}\Big)\\
&= \sup_{n\in\omega} \varphi(f_n \wedge g) && \text{(because } \varphi \text{ preserves count. sups)}\\
&= \sup_{n\in\omega}\{\varphi(f_n) \wedge \varphi(g)\} && \text{(because } \varphi \text{ preserves } \wedge)\\
&= \curlyvee^{\varphi(g)}_{n\in\omega} \varphi(f_n).
\end{aligned}$$

Hence, φ preserves $\curlyvee$. □

Remark 7.4. Propositions 7.1, 7.2 and 7.3 show that, whenever $\mathcal{V}$ is a variety with a lattice reduct, then its subcategory of Dedekind σ-complete objects, with σ-continuous morphisms, is a variety which has, as primitive operations, the operations of $\mathcal{V}$ together with $\curlyvee$, and, as axioms, the axioms of $\mathcal{V}$ together with (TS1), (TS2) and (TS3).

8 Truncated ℓ-groups

We assume familiarity with the basic theory of ℓ-groups. All needed background can be found, for example, in the standard reference [3]. In [2], R. N. Ball defines a truncated ℓ-group as an abelian divisible ℓ-group that is

endowed with a function $\overline{\cdot}\colon G^+ \to G^+$, called *truncation*, which has the following properties for all $f, g \in G^+$.
(B1) $f \wedge \overline{g} \leqslant \overline{f} \leqslant f$.
(B2) If $\overline{f} = 0$, then $f = 0$.
(B3) If $nf = \overline{nf}$ for every $n \in \omega$, then $f = 0$.
In this paper, we do not assume divisibility. The truncation $\overline{\cdot}$ may be extended to an operation on G, by setting $\overline{f} = \overline{f^+} - f^-$. Here, as is standard, we set $f^+ := f \vee 0$, and $f^- := -(f \wedge 0)$. Then Ball's definition may be reformulated as follows.

Definition 8.1. A *truncated ℓ-group* is an abelian ℓ-group that is endowed with a unary operation $\overline{\cdot}\colon G \to G$, called *truncation*, which has the following properties.
(T1) For all $f \in G$, we have $\overline{f} = \overline{f^+} - f^-$.
(T2) For all $f \in G^+$, we have $\overline{f} \in G^+$.
(T3) For all $f, g \in G^+$, we have $f \wedge \overline{g} \leqslant \overline{f} \leqslant f$.
(T4) For all $f \in G^+$, if $\overline{f} = 0$, then $f = 0$.
(T5) For all $f \in G^+$, if $nf = \overline{nf}$ for every $n \in \omega$, then $f = 0$.

Axiom (T2) ensures that $\overline{\cdot}$ may be restricted to an operation on G^+. Axiom (T1) gives the one-to-one correspondence with Ball's definition. Axioms (T3), (T4), (T5) correspond, respectively, to Axioms (B1), (B2), (B3). An ℓ-homomorphism φ between truncated ℓ-groups preserves $\overline{\cdot}$ if, and only if, φ preserves $\overline{\cdot}$ over positive elements; indeed, if φ preserves $\overline{\cdot}$ over positive elements, then, for $f \in G$,

$$\varphi(\overline{f}) = \varphi(\overline{f^+} - f^-) = \varphi(\overline{f^+}) - \varphi(f^-) = \overline{\varphi(f^+)} - \varphi(f^-) = \overline{\varphi(f)^+} - \varphi(f)^- = \overline{\varphi(f)}.$$

This ensures that the equivalence with Ball's definition also holds for morphisms.

Note that (T1), (T2) and (T3) are (essentially) equational axioms. This is evident for (T1); (T2) can be written as $\forall f\ \overline{f^+} \wedge 0 = 0$; (T3) is the conjunction of the two equations $\forall f, g\ f^+ \wedge \overline{g^+} \vee \overline{f^+} = \overline{f^+}$ and $\forall f\ \overline{f^+} \vee f^+ = f^+$. The axioms (T4) and (T5) cannot be expressed in such equational terms. However, as we shall see, this becomes possible when we add the hypothesis of Dedekind σ-completeness.

It is well known that a Dedekind σ-complete ℓ-group is archimedean and thus abelian. Let G be a Dedekind σ-complete ℓ-group, endowed with a unary operation $\overline{\cdot}$. We denote by (T4') and (T5') the following properties, which may or may not hold in G.
(T4') For all $f \in G^+$, we have $f = \curlyvee^{f}_{n\in\omega} n\overline{f}$.
(T5') For all $f \in G^+$, we have $f = \curlyvee^{f}_{n\in\omega} (nf - \overline{nf})$.
Note that (T4') and (T5'), are (essentially) equational axioms: indeed, (T4') is equivalent to $\forall f\ f^+ = \curlyvee^{f^+}_{n\in\omega} n\overline{f^+}$, and (T5') is equivalent to $\forall f\ f^+ = \curlyvee^{f^+}_{n\in\omega} (nf^+ - \overline{nf^+})$.

Our aim in this section, met in Propositions 8.2, 8.5 and 8.8, is to show that, for a Dedekind σ-complete ℓ-group endowed with a unary operation $\overline{\cdot}$ which satisfies (T1), (T2) and (T3), the axioms (T4) and (T5) may be equivalently replaced by the equational axioms (T4') and (T5'). This will show the axioms of Dedekind σ-complete truncated ℓ-groups to be equational.

Proposition 8.2. *Let G be an abelian ℓ-group endowed with a unary operation $\overline{\cdot}$. Then* (T4') *implies* (T4), *and* (T5') *implies* (T5).

Proof. Suppose (T4'). Let $f \in G^+$ be such that $\overline{f} = 0$. By (T4'),

$$f = \curlyvee^{f}_{n\in\omega} n\overline{f} = \curlyvee^{f}_{n\in\omega} 0 = 0.$$

Hence, (T4) holds. Suppose (T5'). Let $f \in G^+$ be such that $nf = \overline{nf}$ for every $n \in \omega$. By (T5'),

$$f = \curlyvee^{f}_{n\in\omega} (nf - \overline{nf}) = \curlyvee^{f}_{n\in\omega} 0 = 0.$$

Hence (T5) holds. □

We shall use the following standard distributivity result.

Lemma 8.3. *Let G be an ℓ-group, I a set and $(x_i)_{i\in I} \subseteq G$. If $\sup_{i\in I} x_i$ exists, then, for every $a \in G$, $\sup_{i\in I}\{a \wedge x_i\}$ exists and*

$$a \wedge \Big(\sup_{i\in I} x_i\Big) = \sup_{i\in I}\{a \wedge x_i\}.$$

Proof. See [3, Proposition 6.1.2]. □

Lemma 8.4. *Let G be a Dedekind σ-complete ℓ-group, let $g \in G$, $h \in G^+$ and $(f_n)_{n\in\omega} \subseteq G$. Then*

$$\mathop{\curlyvee}_{n\in\omega}^{g} (f_n + h) = \Bigg(\Bigg(\mathop{\curlyvee}_{n\in\omega}^{g} f_n\Bigg) + h\Bigg) \wedge g.$$

Proof. We have

$$\begin{aligned}
\mathop{\curlyvee}_{n\in\omega}^{g} (f_n + h) &= \sup_{n\in\omega}\{(f_n + h) \wedge g\} \\
&= \sup_{n\in\omega}\{(f_n + h) \wedge (g + h) \wedge g\} \quad \text{(because } h \geqslant 0) \\
&= \sup_{n\in\omega}\{(f_n + h) \wedge (g + h)\} \wedge g \quad \text{(by Lemma 8.3)} \\
&= \sup_{n\in\omega}\{(f_n \wedge g) + h\} \wedge g \\
&= \Big(\sup_{n\in\omega}\{f_n \wedge g\} + h\Big) \wedge g \\
&= \Bigg(\Bigg(\mathop{\curlyvee}_{n\in\omega}^{g} f_n\Bigg) + h\Bigg) \wedge g.
\end{aligned}$$ □

Proposition 8.5. *Let G be a Dedekind σ-complete ℓ-group endowed with a unary operation $\overline{\cdot}$ such that* (T2), (T3) *and* (T4) *hold. Then* (T4') *holds, i.e., for all $f \in G^+$,*

$$f = \mathop{\curlyvee}_{n\in\omega}^{f} n\overline{f}.$$

Proof. By (T2), $\overline{f} \in G^+$. Therefore $0\overline{f} \leqslant 1\overline{f} \leqslant 2\overline{f} \leqslant 3\overline{f} \leqslant \cdots$. Hence,

$$\mathop{\curlyvee}_{n\in\omega}^{f} n\overline{f} = \mathop{\curlyvee}_{n\in\omega\setminus\{0\}}^{f} n\overline{f} = \mathop{\curlyvee}_{n\in\omega}^{f} (n+1)\overline{f} = \mathop{\curlyvee}_{n\in\omega}^{f} (n\overline{f} + \overline{f}) = \Bigg(\Bigg(\mathop{\curlyvee}_{n\in\omega}^{f} n\overline{f}\Bigg) + \overline{f}\Bigg) \wedge f \quad \text{(by Lemma 8.4)}.$$

Therefore, setting $b \coloneqq \curlyvee^{f}_{n\in\omega} n\overline{f}$, we have

$$0 = ((b + \overline{f}) \wedge f) - b = \overline{f} \wedge (f - b) = \overline{f - b},$$

where the last equality holds because, by (T3), we have $\overline{f} \wedge (f - b) \leqslant \overline{f - b}$ and, for the opposite inequality, we have $\overline{f - b} \leqslant f - b$ and $\overline{f - b} = \overline{f - b \wedge f} \leqslant \overline{f}$.

By (T4), since $\overline{f - b} = 0$, we have $f - b = 0$, i.e., $f = \curlyvee^{f}_{n\in\omega} n\overline{f}$. □

Lemma 8.6. *Let G be a Dedekind σ-complete ℓ-group endowed with a unary operation $\overline{\cdot}$ such that* (T2) *and* (T3) *holds. Let $a, b \in G^+$. Then*

$$\overline{a + b} \leqslant \overline{a} + \overline{b}.$$

Proof. By (T3), $\overline{a+b} \leqslant a+b$. By (T2), $\overline{a+b} \geqslant 0$, thus $b \wedge \overline{(a+b)} \geqslant 0$, and therefore $\overline{a+b} \leqslant \overline{a+b} + (b \wedge \overline{(a+b)})$. Hence,

$$\begin{aligned}
\overline{a + b} &\leqslant [(a + b) \wedge (a + \overline{(a + b)})] \wedge [\overline{(a + b)} + (b \wedge \overline{(a + b)})] \\
&= [a + (b \wedge \overline{(a + b)})] \wedge [\overline{(a + b)} + (b \wedge \overline{(a + b)})] \\
&= (a \wedge \overline{(a + b)}) + (b \wedge \overline{(a + b)}) \\
&\leqslant \overline{a} + \overline{b} \qquad \text{(by (T3))}.
\end{aligned}$$ □

Lemma 8.7. *Let G be an abelian ℓ-group endowed with a unary operation $\overline{\cdot}$ such that* (T3) *holds. Then, for all $a, b \in G^+$, if $a \leqslant b$, then $a - \overline{a} \leqslant b - \overline{b}$.*

Proof. Since $a \leqslant b$, we have $\overline{b} - b \leqslant \overline{b} - a$. By (T3), $\overline{b} - b \leqslant 0$. Hence,

$$\begin{aligned} \overline{b} - b &\leqslant (\overline{b} - a) \wedge 0 \\ &= (\overline{b} \wedge a) - a \quad &&\text{(because + distributes over } \wedge) \\ &\leqslant \overline{a} - a &&\text{(by (T3))} \end{aligned}$$

as desired. □

Proposition 8.8. *Let G be a Dedekind σ-complete ℓ-group endowed with a unary operation $\overline{\cdot}$ such that* (T2), (T3) *and* (T5) *hold. Then* (T5') *holds, i.e., for all $f \in G^+$,*

$$f = \curlyvee_{n\in\omega}^{f} (nf - \overline{nf}).$$

Proof. Let $k \in \omega$. By (T3) we have $0 \leqslant kf - \overline{kf}$. We have

$$\begin{aligned} \curlyvee_{n\in\omega}^{f} (nf - \overline{nf}) &\geqslant \curlyvee_{n\in\omega\setminus\{0,\dots,k-1\}}^{f} (nf - \overline{nf}) \\ &= \curlyvee_{n\in\omega}^{f} ((n+k)f - \overline{(n+k)f}) \\ &\geqslant \curlyvee_{n\in\omega}^{f} (nf - \overline{nf} + kf - \overline{kf}) &&\text{(by Lemma 8.6)} \\ &= \left(\left(\curlyvee_{n\in\omega}^{f} (nf - \overline{nf})\right) + kf - \overline{kf}\right) \wedge f &&\text{(by Lemma 8.4).} \end{aligned}$$

The opposite inequality is immediate. Therefore, setting $b := \curlyvee_{n\in\omega}^{f}(nf - \overline{nf})$, we have $b = (b + kf - \overline{kf}) \wedge f$, which implies

$$0 = ((b + kf - \overline{kf}) \wedge f) - b = (kf - \overline{kf}) \wedge (f - b).$$

We set $a := f - b$. We have $0 \leqslant a \leqslant f$, because $0 \leqslant b \leqslant f$. By (T3) and Lemma 8.7, $0 \leqslant ka - \overline{ka} \leqslant kf - \overline{kf}$. Therefore, $0 = (ka - \overline{ka}) \wedge a$. It is elementary that, in any abelian group, $x \wedge y = 0$ implies $(nx) \wedge y = 0$ for each $n \in \omega$. Therefore,

$$0 = (ka - \overline{ka}) \wedge ka \overset{\text{(T2)}}{=} (ka - \overline{ka}).$$

Hence, $ka = \overline{ka}$. Since k is arbitrary, by (T5) we infer $a = 0$, i.e., $f - \curlyvee_{n\in\omega}^{f}(nf - \overline{nf}) = 0$. □

To sum up, Propositions 8.2, 8.5 and 8.8 show that, for Dedekind σ-complete ℓ-groups endowed with a unary operation $\overline{\cdot}$, Axioms (T1)-(T5) are equivalent to Axioms (T1)-(T3) together with Axioms (T4') and (T5').

We denote by $\sigma\ell\mathbb{G}_t$ the category whose objects are Dedekind σ-complete truncated ℓ-groups, and whose morphisms are σ-continuous ℓ-homomorphisms that preserve $\overline{\cdot}$. Since Axioms (T1), (T2), (T3), (T4') and (T5') are equational, $\sigma\ell\mathbb{G}_t$ is a variety, whose operations are the operations of ℓ-groups, together with $\overline{\cdot}$ and $\curlyvee$, and whose axioms are the axioms of ℓ-groups, together with the following ones.

(TS1) $\curlyvee_{n\in\omega}^{g} f_n = \curlyvee_{n\in\omega}^{g} (f_n \wedge g)$.
(TS2) $\curlyvee_{n\in\omega}^{g} f_n = (f_0 \wedge g) \vee (\curlyvee_{n\in\omega\setminus\{0\}}^{g} f_n)$.
(TS3) $\curlyvee_{n\in\omega}^{g} (f_n \wedge h) \leqslant h$.
(T1) For all $f \in G$, we have $\overline{f} = \overline{f^+} - f^-$.
(T2) For all $f \in G^+$, we have $\overline{f} \in G^+$.
(T3) For all $f, g \in G^+$, we have $f \wedge \overline{g} \leqslant \overline{f} \leqslant f$.
(T4') For all $f \in G^+$, we have $f = \curlyvee_{n\in\omega}^{f} \overline{nf}$.
(T5') For all $f \in G^+$, we have $f = \curlyvee_{n\in\omega}^{f} (nf - \overline{nf})$.

9 The Loomis–Sikorski Theorem for truncated ℓ-groups

Definition 9.1. Given a set X, a *σ-ideal of subsets of X* is a set $\mathcal{I}$ of subsets of X such that the following conditions hold.
(1) $\emptyset \in \mathcal{I}$.
(2) $B \in \mathcal{I}, A \subseteq B \Rightarrow A \in \mathcal{I}$.
(3) $(A_n)_{n\in\omega} \subseteq \mathcal{I} \Rightarrow \bigcup_{n\in\omega} A_n \in \mathcal{I}$.

If $\mathcal{I}$ is a σ-ideal of subsets of X, we say that a property P *holds for $\mathcal{I}$-almost every $x \in X$* if $\{x \in X \mid P$ does not hold for $x\} \in \mathcal{I}$. A σ-ideal $\mathcal{I}$ of subsets of X induces on $\mathbb{R}^X$ an equivalence relation $\sim$, defined by $f \sim g$ if, and only if, $f(x) = g(x)$ for $\mathcal{I}$-almost every $x \in X$. We write $\frac{\mathbb{R}^X}{\mathcal{I}}$ for the quotient $\frac{\mathbb{R}^X}{\sim}$. Every operation τ of countable arity on $\mathbb{R}$ induces an operation $\tilde{\tau}$ on $\frac{\mathbb{R}^X}{\mathcal{I}}$, by setting $\tilde{\tau}(([f_i]_{\mathcal{I}})_{i\in I}) := [g]_{\mathcal{I}}$, where $g(x) = \tau((f_i(x))_{i\in I})$. The assumption that $\mathcal{I}$ is closed under countable unions guarantees that this definition is well posed. Therefore, by Remark 7.4, $\frac{\mathbb{R}^X}{\mathcal{I}}$ is a Dedekind σ-complete truncated ℓ-group.

The aim of this section is to prove the following theorem.

Theorem 9.2 (Loomis–Sikorski Theorem for truncated ℓ-groups). *Let G be a Dedekind σ-complete truncated ℓ-group. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ such that, for every $f \in G$, $\iota(\overline{f}) = \iota(f) \wedge [1]_{\mathcal{I}}$.*

We will give a proof that is rather self-contained, with the main exception of the use of Theorem 9.3 below. Anyway, we believe that a shorter (but not self-contained) way to prove Theorem 9.2 above (even in the less restrictive hypothesis that G is an archimedean truncated ℓ-group) may be the following. First, show that the divisible hull G^d of G admits a truncation that extends the truncation of G. Then embed G^d in $\frac{\mathbb{R}^X}{\mathcal{I}}$ via [2, Theorem 5.3.6 (1)]. Finally, using arguments similar to those in [13, Theorem 6.2], show that this embedding preserves all countable suprema.

Theorem 9.3 (Loomis–Sikorski Theorem for Riesz spaces). *Let G be a Dedekind σ-complete Riesz space. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X and an injective σ-continuous Riesz morphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$.*

For a proof of Theorem 9.3 see [7], or [5] and [6].

Corollary 9.4 (Loomis–Sikorski Theorem for ℓ-groups). *Let G be a Dedekind σ-complete ℓ-group. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$.*

Proof. There exist a Dedekind σ-complete Riesz space H and an injective ℓ-morphism $\varphi\colon G \hookrightarrow H$ that preserves every existing supremum; see [11]. Applying Theorem 9.3 to the Dedekind σ-complete Riesz space H, we obtain an injective σ-continuous Riesz morphism $\varphi'\colon H \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$. The composition $\iota = \varphi' \circ \varphi\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ is an injective σ-continuous ℓ-morphism, since both φ and φ' are injective σ-continuous ℓ-morphisms. □

Our strategy to prove Theorem 9.2 is the following. Lemma 9.12 will prove Theorem 9.2 for countably generated algebras. This will imply that $\mathbb{R}$ generates the variety of Dedekind σ-complete truncated ℓ-groups, and from this fact Theorem 9.2 is derived.

Lemma 9.5. *Let G be a Dedekind σ-complete truncated ℓ-group generated by a subset $S \subseteq G$. Then, for every $g \in G$, there exist $s_0, \dots, s_{n-1} \in S$ such that $|g| \leqslant |s_0| + \cdots + |s_{n-1}|$.*

Proof. Let $T := \{h \in G \mid$ there exist $s_0, \dots, s_{n-1} \in G$: $|h| \leqslant |s_0| + \cdots + |s_{n-1}|\}$. It is clear that $S \subseteq T$ and standard that T is a convex ℓ-subgroup of G. Moreover, for every $g \in G$, and every $(f_n)_{n\in\omega} \subseteq G$, the following hold.
(1) $\curlyvee^{g}_{n\in\omega} f_n = \sup_{n\in\omega}\{f_n \wedge g\}$, and therefore $f_0 \wedge g \leqslant \curlyvee^{g}_{n\in\omega} f_n \leqslant g$. Hence,

$$\Big|\underset{n\in\omega}{\overset{g}{\curlyvee}} f_n\Big| = \Big(\underset{n\in\omega}{\overset{g}{\curlyvee}} f_n\Big) \vee \Big(-\underset{n\in\omega}{\overset{g}{\curlyvee}} f_n\Big) \leqslant g \vee [-(f_0 \wedge g)] \leqslant g \vee [(-f_0) \vee (-g)] \leqslant |g| \vee |f_0|.$$

(2) $|\overline{g}| = |\overline{g^+ - g^-}| \leqslant |\overline{g^+}| + |\overline{g^-}| \overset{\text{(T2)}}{=} \overline{g^+} + \overline{g^-} \overset{\text{(T3)}}{\leqslant} g^+ + g^- = |g|$.
Since T is a convex ℓ-subgroup of G, (1) and (2) imply that T is closed under $\curlyvee$ and $\overline{\cdot}$. □

Lemma 9.6. *Let X be a set, and $\mathcal{I}$ a σ-ideal of subsets of X. Let $(g_n)_{n\in\omega}$ be a sequence of functions from X to $\mathbb{R}$. Suppose that, for $\mathcal{I}$-almost every $x \in X$, $\sup_{n\in\omega} g_n(x) \in \mathbb{R}$. Then the set $\{[g_n]_{\mathcal{I}} \mid n \in \omega\}$ admits a supremum in $\frac{\mathbb{R}^X}{\mathcal{I}}$.*

Proof. Let $A \in \mathcal{I}$ be such that, for every $x \in X \setminus A$, $\sup_{n\in\omega} g_n(x) \in \mathbb{R}$. Let $v\colon X \to \mathbb{R}$ be any function such that, for every $x \in X \setminus A$, $v(x) = \sup_{n\in\omega} g_n(x)$. Then $[v]_{\mathcal{I}}$ is the supremum of $\{[g_n]_{\mathcal{I}} \mid n \in \omega\}$ in $\frac{\mathbb{R}^X}{\mathcal{I}}$. □

Lemma 9.7. *Let G be a Dedekind σ-complete truncated ℓ-group, let $f \in G^+$ and let $(f_i)_{i\in\omega} \subseteq G^+$. Then*

$$f = \bigvee_{i\in\omega}^{f}\left(if - \bigvee_{k\in\omega}^{if} \overline{f_k}\right).$$

Proof. Trivially, $f \leqslant \bigvee_{i\in\omega}^{f}(if - \bigvee_{k\in\omega}^{if} \overline{f_k})$. We prove the opposite inequality. By (T3), for every $k \in \omega$, we have $\overline{f_k} \wedge (if) \leqslant \overline{if}$, and therefore we have $\bigvee_{k\in\omega}^{if} \overline{f_k} = \sup_{i\in\omega}\{\overline{f_k} \wedge (if)\} \leqslant \overline{if}$. Hence, $if - \bigvee_{k\in\omega}^{if} \overline{f_k} \geqslant if - \overline{if}$. Therefore, we have

$$\bigvee_{i\in\omega}^{f}\left(if - \bigvee_{k\in\omega}^{if} \overline{f_k}\right) \geqslant \bigvee_{i\in\omega}^{f}(if - \overline{if}) \overset{\text{(T5')}}{=} f. \qquad \square$$

Lemma 9.8. *Let G be an abelian ℓ-group, let $a \in G$ and let $u \in G^+$. Then $(a^+ \wedge u) - a^- = a \wedge u$.*

Proof. We have $(a^+ \wedge u) - a^- = (a^+ - a^-) \wedge (u - a^-) = a \wedge (u + (a \wedge 0)) = a \wedge (u + a) \wedge u = a \wedge u$. □

Lemma 9.9. *Let G be a countably generated Dedekind σ-complete truncated ℓ-group. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X, an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ and an element $u \in \frac{\mathbb{R}^X}{\mathcal{I}}$ such that, for every $f \in G$, $\iota(\overline{f}) = \iota(f) \wedge u$.*

Proof. By Corollary 9.4, there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$.

Let S be a countable generating set of G and let $F := \{|s_0| + \cdots + |s_{n-1}| \mid s_0, \ldots, s_{n-1} \in S\}$. Let us enumerate F as $F = \{f_0, f_1, f_2, \ldots\}$. We shall prove that the set $\{\iota(\overline{f_n}) \mid n \in \omega\}$, admits a supremum $u \in \frac{\mathbb{R}^X}{\mathcal{I}}$ that satisfies the statement of the lemma.

By Lemma 9.7, for each $n \in \omega$, we have

$$\overline{f_n} = \bigvee_{i\in\omega}^{\overline{f_n}}\left(i\overline{f_n} - \bigvee_{k\in\omega}^{i\overline{f_n}} \overline{f_k}\right).$$

Since ι is a σ-continuous ℓ-homomorphism, using Proposition 7.3, we have the following.
(1) For each $n \in \omega$, $\iota(\overline{f_n}) = \bigvee_{i\in\omega}^{\iota(\overline{f_n})}(i\iota(\overline{f_n}) - \bigvee_{k\in\omega}^{i\iota(\overline{f_n})} \iota(\overline{f_k}))$.
For every $n \in \omega$, let $g_n \in \mathbb{R}^X$ be such that $[g_n]_{\mathcal{I}} = \iota(\overline{f_n})$. Then, by (1), for $\mathcal{I}$-almost every $x \in X$, the following conditions hold.
(1') For each $n \in \omega$, $g_n(x) = \bigvee_{i\in\omega}^{g_n(x)}(ig_n(x) - \bigvee_{k\in\omega}^{ig_n(x)} g_k(x))$.
Let x be such that (1') hold. Suppose by way of contradiction that $\sup_{n\in\omega} g_n(x) = \infty$. Then there exists $n \in \omega$ such that $g_n(x) > 0$. Therefore, we have

$$g_n(x) = \bigvee_{i\in\omega}^{g_n(x)}\left(ig_n(x) - \bigvee_{k\in\omega}^{ig_n(x)} g_k(x)\right) > 0,$$

which implies that there exists $i \in \omega$ such that $ig_n(x) - \bigvee_{k\in\omega}^{ig_n(x)} g_k(x) > 0$. Thus, $\bigvee_{k\in\omega}^{ig_n(x)} g_k(x) < ig_n(x)$. But $\sup_{n\in\omega} g_n(x) = \infty$ implies $\bigvee_{k\in\omega}^{ig_n(x)} g_k(x) = ig_n(x)$, a contradiction. Therefore, $\sup_{n\in\omega} g_n(x) \in \mathbb{R}$ holds for each $x \in X$ satisfying (1′), and thus for $\mathcal{I}$-almost every $x \in X$. By Lemma 9.6, the set $\{[g_n]_{\mathcal{I}} \mid n \in \omega\} = \{\iota(\overline{f_n}) \mid n \in \omega\}$ admits a supremum u.

Let $f \in G^+$. Then

$$\begin{aligned}
\iota(f) \wedge u &= \iota(f) \wedge \sup_{n\in\omega} \iota(\overline{f_n}) \\
&= \sup_{n\in\omega}\{\iota(f) \wedge \iota(\overline{f_n})\} \quad \text{(by Lemma 8.3)} \\
&= \sup_{n\in\omega}\{\iota(f \wedge \overline{f_n})\} \\
&\leqslant \iota(\overline{f}) \qquad\qquad\qquad \text{(by (T3))}.
\end{aligned}$$

For the opposite inequality, by Lemma 9.5 there exists $m \in \omega$ such that $\overline{f} \leqslant f_m$. Then $\overline{f} = \overline{f} \wedge f_m \overset{(T3)}{\leqslant} \overline{f_m}$. Therefore $\iota(\overline{f}) \leqslant \iota(\overline{f_m}) \leqslant u$, and moreover $\iota(\overline{f}) \leqslant \iota(f)$ by (T3). Thus, $\iota(\overline{f}) \leqslant \iota(f) \wedge u$. For an arbitrary $f \in G$, $\overline{f} = \overline{f^+} - f^-$ by (T1), hence $\iota(\overline{f}) = \iota(\overline{f^+}) - \iota(f^-) = (\iota(f^+) \wedge u) - \iota(f^-) \overset{\text{Lem. 9.8}}{=} \iota(f) \wedge u$. □

Let G be a Dedekind σ-complete ℓ-group, let $H \subseteq G$, and let $u \in G$. We say that u is a *weak unit for H* if $u \geqslant 0$ and, for every $h \in H$,

$$|h| = \curlyvee_{n \in \omega}^{|h|} n(|h| \wedge u).$$

Remark 9.10. We will see in Lemma 11.2 that a weak unit for G in the sense above is the same as a weak unit of G in the usual sense.

Lemma 9.11. *Let Y be a set, $\mathcal{J}$ a σ-ideal of subsets of Y, $H \subseteq \frac{\mathbb{R}^Y}{\mathcal{J}}$ an ℓ-subgroup, and $u \in \frac{\mathbb{R}^Y}{\mathcal{J}}$ a weak unit for H. Then there exists a set X, a σ-ideal $\mathfrak{J}$ of subsets of X, and a σ-continuous ℓ-homomorphism $\psi\colon \frac{\mathbb{R}^Y}{\mathcal{J}} \to \frac{\mathbb{R}^X}{\mathfrak{J}}$ such that the restriction of ψ to H is injective and $\psi(u) = [1]_{\mathfrak{J}}$.*

Proof. Let $v \in \mathbb{R}^Y$ be such that $[v]_{\mathcal{J}} = u$. Since $u \geqslant 0$, we may choose $v \geqslant 0$. Let $X := \{y \in Y \mid v(y) > 0\}$. Let $\mathfrak{J} := \{J \cap X \mid J \in \mathcal{J}\} = \{J \in \mathcal{J} \mid J \subseteq X\}$. Let $(\,\cdot\,)_{|X}\colon \mathbb{R}^Y \to \mathbb{R}^X$ be the restriction map that sends $f \in \mathbb{R}^Y$ to $f_{|X} \in \mathbb{R}^X$, where $f_{|X}(x) = f(x)$ for each $x \in X$. Write $[\,\cdot\,]_{\mathcal{J}}\colon \mathbb{R}^Y \twoheadrightarrow \frac{\mathbb{R}^Y}{\mathcal{J}}$ for the natural quotient map, and similarly for $[\,\cdot\,]_{\mathfrak{J}}\colon \mathbb{R}^X \twoheadrightarrow \frac{\mathbb{R}^X}{\mathfrak{J}}$. Since $\ker[\,\cdot\,]_{\mathcal{J}} \subseteq \ker([\,\cdot\,]_{\mathfrak{J}} \circ (\,\cdot\,)_{|X})$, by the universal property of the quotient there exists a unique σ-continuous ℓ-homomorphism $\rho\colon \frac{\mathbb{R}^Y}{\mathcal{J}} \to \frac{\mathbb{R}^X}{\mathfrak{J}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{R}^Y & \xrightarrow{(\,\cdot\,)_{|X}} & \mathbb{R}^X \\
{\scriptstyle [\,\cdot\,]_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle [\,\cdot\,]_{\mathfrak{J}}} \\
\frac{\mathbb{R}^Y}{\mathcal{J}} & \xrightarrow{\rho} & \frac{\mathbb{R}^X}{\mathfrak{J}}.
\end{array}$$

We claim that the restriction of ρ to H is injective. Indeed, let $h \in H^+$ be such that $\rho(h) = 0$. Let $g \in \mathbb{R}^Y$ be such that $[g]_{\mathcal{J}} = h$. Since $h \geqslant 0$, we may choose $g \geqslant 0$. We have that $[g_{|X}]_{\mathfrak{J}} = 0$. Therefore, for $\mathfrak{J}$-almost every $x \in X$, $g(x) = 0$. Therefore, for $\mathcal{J}$-almost every $y \in Y$, $g(y) = 0$ or $y \in Y \setminus X$, i.e., $g(y) = 0$ or $v(y) = 0$. Since $h = \curlyvee_{n\in\omega}^{h} n(h \wedge u)$, we have $g(y) = \curlyvee_{n\in\omega}^{g(y)} n(g(y) \wedge v(y))$ for $\mathcal{J}$-almost every $y \in Y$. Therefore, for $\mathcal{J}$-almost every $y \in Y$, if $v(y) = 0$, then $g(y) = \curlyvee_{n\in\omega}^{g(y)} n(g(y) \wedge 0) = \curlyvee_{n\in\omega}^{g(y)} 0 = 0$, i.e., $g(y) = 0$. Hence, for $\mathcal{J}$-almost every $y \in Y$, $g(y) = 0$. Thus, $h = 0$.

For every $\lambda \in \mathbb{R}^+ \setminus \{0\}$, the function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \mathbb{R}$ which maps x to λx is an isomorphism of Dedekind σ-complete ℓ-groups. Indeed, its inverse is the map $\frac{1}{\lambda}(\,\cdot\,)$. Then, the map $m\colon \mathbb{R}^X \to \mathbb{R}^X$ which maps f to the function $m(f)$ defined by $(m(f))(x) = \frac{1}{v(x)} f(x)$ is an isomorphism of Dedekind σ-complete ℓ-groups; indeed, its inverse is $m^{-1}\colon \mathbb{R}^X \to \mathbb{R}^X$ defined by $(m^{-1}(g))(x) = v(x)g(x)$. For every $f, g \in \mathbb{R}^X$, $[f]_{\mathfrak{J}} = [g]_{\mathfrak{J}}$ if, and only if, $[m(f)]_{\mathfrak{J}} = [m(g)]_{\mathfrak{J}}$. Hence, $\ker[\,\cdot\,]_{\mathcal{J}} = \ker([\,\cdot\,]_{\mathcal{J}} \circ m)$. Therefore, there exists an isomorphism $\eta\colon \frac{\mathbb{R}^X}{\mathfrak{J}} \xrightarrow{\sim} \frac{\mathbb{R}^X}{\mathfrak{J}}$ of Dedekind σ-complete ℓ-groups which makes the following diagram commute:

$$\begin{array}{ccc}
\mathbb{R}^X & \underset{\sim}{\xrightarrow{m}} & \mathbb{R}^X \\
{\scriptstyle [\,\cdot\,]_{\mathfrak{J}}}\downarrow & & \downarrow{\scriptstyle [\,\cdot\,]_{\mathfrak{J}}} \\
\frac{\mathbb{R}^X}{\mathfrak{J}} & \underset{\sim}{\xrightarrow{\eta}} & \frac{\mathbb{R}^X}{\mathfrak{J}}.
\end{array}$$

We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{R}^Y & \xrightarrow{(\,\cdot\,)_{|X}} & \mathbb{R}^X & \underset{\sim}{\xrightarrow{m}} & \mathbb{R}^X \\
{\scriptstyle [\,\cdot\,]_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle [\,\cdot\,]_{\mathfrak{J}}} & & \downarrow{\scriptstyle [\,\cdot\,]_{\mathfrak{J}}} \\
\frac{\mathbb{R}^Y}{\mathcal{J}} & \xrightarrow{\rho} & \frac{\mathbb{R}^X}{\mathfrak{J}} & \underset{\sim}{\xrightarrow{\eta}} & \frac{\mathbb{R}^X}{\mathfrak{J}}.
\end{array}$$

We set $\psi := \eta \circ \rho$. Note that $m(v_{|X}) \in \mathbb{R}^X$ is the function constantly equal to 1: indeed, $m(v_{|X})(x) = \frac{1}{v(x)} v_X(x) = 1$. Thus, $\psi(u) = \eta(\rho(u)) = \eta(\rho([v]_{\mathcal{J}})) = [m(v_X)]_{\mathfrak{I}} = [1]_{\mathfrak{I}}$. Since the restriction of ρ to H is injective, and η is bijective, the restriction of ψ to H is injective. □

Lemma 9.12. *Let G be a countably generated Dedekind σ-complete truncated ℓ-group. Then there exist a set X, a σ-ideal $\mathfrak{I}$ of subsets of X and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathfrak{I}}$ such that, for every $f \in G$, $\iota(\overline{f}) = \iota(f) \wedge [1]_{\mathfrak{I}}$.*

Proof. By Lemma 9.9, there exist a set Y, a σ-ideal $\mathcal{J}$ of subsets of Y, an injective σ-continuous ℓ-homomorphism $\varphi\colon G \hookrightarrow \frac{\mathbb{R}^Y}{\mathcal{J}}$ and an element $u \in \frac{\mathbb{R}^Y}{\mathcal{J}}$ such that, for every $f \in G$,

$$\varphi(\overline{f}) = \varphi(f) \wedge u.$$

First, $0 \leqslant \varphi(\overline{0}) = 0 \wedge u$, hence $u \geqslant 0$. Since, for all $f \in G$, $|f| = \curlyvee_{n\in\omega}^{|f|} n\overline{|f|}$ by (T4'), we have

$$|\varphi(f)| = \bigcurlyvee_{n\in\omega}^{|\varphi(f)|} n(|\varphi(f)| \wedge u).$$

Therefore, setting H equal to the image of G, u is a weak unit for H. By Lemma 9.11, there exist a set X, a σ-ideal $\mathfrak{I}$ of subsets of X, and a σ-continuous ℓ-homomorphism $\psi\colon \frac{\mathbb{R}^Y}{\mathcal{J}} \to \frac{\mathbb{R}^X}{\mathfrak{I}}$ such that the restriction of ψ to H is injective and $\psi(u) = [1]_{\mathfrak{I}}$. The function $\iota := \psi \circ \varphi$ has the required properties. □

Theorem 9.13. *The variety $\sigma\ell\mathbb{G}_t$ of Dedekind σ-complete truncated ℓ-groups is generated by $\mathbb{R}$.*

Proof. Let G be a Dedekind σ-complete truncated ℓ-group. Suppose that an equation $\tau = \rho$ (in the language of Dedekind σ-complete truncated ℓ-groups) does not hold in G. Since τ and ρ have countably many arguments, the equation $\tau = \rho$ does not hold in a countably generated Dedekind σ-complete truncated ℓ-group G'. By Lemma 9.12, $\tau = \rho$ does not hold in $\mathbb{R}$. The statement follows by the HSP Theorem for (infinitary) varieties (see [16, Theorem (9.1)]). □

Proof of Theorem 9.2. Since the variety of Dedekind σ-complete truncated ℓ-groups is generated by $\mathbb{R}$, there exists a set X, a $\sigma\ell\mathbb{G}_t$-subalgebra $H \subseteq \mathbb{R}^X$, and a surjective morphism $\psi\colon H \to G$ of Dedekind σ-complete truncated ℓ-groups. Let

$$\mathfrak{I} := \{A \subseteq X \mid \text{there exists } (f_n)_{n\in\omega} \subseteq \ker\psi \text{ such that for all } a \in A \text{ there exists } n \in \omega \text{ such that } f_n(a) \neq 0\}.$$

Note that $\mathfrak{I}$ is a σ-ideal of subsets of X. Therefore we have the projection map $\mathbb{R}^X \to \frac{\mathbb{R}^X}{\mathfrak{I}}$ which is a morphism of Dedekind σ-complete truncated ℓ-groups. If $f \in \ker\psi$, then $f(x) = 0$ for $\mathfrak{I}$-almost every $x \in X$. In other words, if $f \in \ker\psi$, then $[f]_{\mathfrak{I}} = 0$. For the universal property of quotients, there exists a morphism $\iota\colon G \to \frac{R^X}{\mathfrak{I}}$ of Dedekind σ-complete truncated ℓ-groups such that the following diagram commutes:

$$\begin{array}{ccc} H & \overset{\iota}{\hookrightarrow} & \mathbb{R}^X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle[\cdot]_{\mathfrak{I}}} \\ G & \overset{\iota}{\dashrightarrow} & \frac{\mathbb{R}^X}{\mathfrak{I}}. \end{array}$$

Let $f \in H$ be such that $\iota(\psi(f)) = [f]_{\mathfrak{I}} = 0$. Then there exists a set $A \in \mathfrak{I}$ such that $f(x) = 0$ for every $x \in X \setminus A$. Since $A \in \mathfrak{I}$, there exists a sequence $(f_n)_{n\in\omega}$ of elements of $\ker\psi$ such that, for every $a \in A$, there exists $n \in \omega$ such that $f_n(a) \neq 0$. Let us show

$$|f| = \bigcurlyvee_{n,k\in\omega}^{|f|} k|f_n|. \tag{9.1}$$

Equation (9.1) holds if, and only if, for every $a \in X$, $|f(a)| = \curlyvee_{n,k\in\omega}^{|f(a)|} k|f_n(a)|$. If $a \notin A$, then both sides equal 0. If $a \in A$, then there exists $m \in \omega$ such that $f_m(a) \neq 0$, and therefore

$$\bigcurlyvee_{n,k\in\omega}^{|f(a)|} k|f_n(a)| \geqslant \bigcurlyvee_{k\in\omega}^{|f(a)|} k|f_m(a)| = |f(a)|.$$

Since the opposite inequality is trivial, (9.1) is shown. By (9.1),

$$|\psi(f)| = \mathop{\curlyvee}_{n,k\in\omega}^{|\psi(f)|} k|\psi(f_n)| \overset{f_n\in\ker\psi}{=} \mathop{\curlyvee}_{n,k\in\omega}^{|\psi(f)|} 0 = 0.$$

Therefore $\psi(f) = 0$, and thus $f \in \ker\psi$. This implies that ι is injective. □

10 $\mathbb{R}$ generates Dedekind σ-complete truncated Riesz spaces

Theorem 10.1 (Loomis–Sikorski Theorem for truncated Riesz spaces). *Let G be a Dedekind σ-complete truncated Riesz space. Then there exist a set X, a σ-ideal $\mathcal{J}$ of subsets of X, and an injective σ-continuous Riesz morphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{J}}$ such that, for every $f \in G$, $\iota(\bar{f}) = \iota(f) \wedge [1]_{\mathcal{J}}$.*

Proof. By Theorem 9.2, there exist a set X, a σ-ideal $\mathcal{J}$ of subsets of X, and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{J}}$ such that, for every $f \in G$, $\iota(\bar{f}) = \iota(f) \wedge [1]_{\mathcal{J}}$. Since $\frac{\mathbb{R}^X}{\mathcal{J}}$ is Dedekind σ-complete, it is archimedean; by [15, Corollary 11.53], ι is a Riesz morphism. □

We denote by $\sigma\mathbb{RS}_t$ the variety of Dedekind σ-complete truncated Riesz spaces, whose primitive operations are 0, $+$, $\vee$, $\lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, and $\overline{\cdot}$, and whose axioms are the axioms of Riesz spaces, together with (TS1), (TS2), (TS3), (T1), (T2), (T3), (T4') and (T5').

We can now obtain the first main result of Part II, as a consequence of Theorem 10.1.

Theorem 10.2. *The variety $\sigma\mathbb{RS}_t$ of Dedekind σ-complete truncated Riesz spaces is generated by $\mathbb{R}$.*

Proof. Let G be a Dedekind σ-complete truncated Riesz space. By Theorem 10.1, there exist a set X, a σ-ideal $\mathcal{J}$ of subsets of X, and an injective σ-continuous Riesz morphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{J}}$ such that, for every $f \in G$, $\iota(\bar{f}) = \iota(f) \wedge [1]_{\mathcal{J}}$. Regarding $\frac{\mathbb{R}^X}{\mathcal{J}}$ as an object of $\sigma\mathbb{RS}_t$ with the structure induced from $\mathbb{R}$, we conclude that G is a subalgebra of a quotient of a power of $\mathbb{R}$. □

Remark 10.3. From [1, Theorem 7.4], it follows that $\mathbb{R}$ actually generates $\sigma\mathbb{RS}_t$ as a quasi-variety, where quasi-equations are allowed to have countably many premises only.

Corollary 10.4. *For any set I,*

$$\begin{aligned}\mathrm{F_t}(I) &:= \left\{ f\colon \mathbb{R}^I \to \mathbb{R} \;\middle|\; f \text{ is } \mathrm{Cyl}(\mathbb{R}^I)\text{-measurable and there exist } J \subseteq I \text{ finite and } (\lambda_j)_{j\in J} \subseteq \mathbb{R}^+\colon |f| \leqslant \sum_{j\in J} \lambda_j|\pi_j| \right\} \\ &= \{ f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ preserves integrability} \}\end{aligned}$$

is the Dedekind σ-complete truncated Riesz space freely generated by the projections $\pi_i\colon \mathbb{R}^I \to \mathbb{R}$ ($i \in I$).

Proof. By Theorem 10.2, the variety $\sigma\mathbb{RS}_t$ of Dedekind σ-complete truncated Riesz spaces is generated by $\mathbb{R}$. Therefore, by a standard result in general algebra, the smallest $\sigma\mathbb{RS}_t$-subalgebra S of $\mathbb{R}^{\mathbb{R}^I}$ that contains the set of projection functions $\{\pi_i\colon \mathbb{R}^I \to \mathbb{R} \mid i \in I\}$ is freely generated by the projection functions. The set S is the smallest subset of $\mathbb{R}^{\mathbb{R}^I}$ that contains, for each $i \in I$, the projection function $\pi_i\colon \mathbb{R}^I \to \mathbb{R}$, and which is closed under every primitive operation of $\sigma\mathbb{RS}_t$. By Theorem 2.4, S consists precisely of all operations $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability. An application of Theorem 2.1 completes the proof. □

Write $\pi\colon I \to \mathrm{F_t}(I)$ for the function $\pi(i) = \pi_i$. Corollary 10.4 asserts the following. For any set I, for every Dedekind σ-complete truncated Riesz space G, for every function $f\colon I \to G$, there exists a unique σ-continuous truncation-preserving Riesz morphism $\varphi\colon \mathrm{F_t}(I) \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{\ \pi\ } & \mathrm{F_t}(I) \\ & \searrow^{\forall f} & \Big\downarrow{\scriptstyle \exists!\varphi} \\ & & G. \end{array}$$

11 The Loomis–Sikorski Theorem for ℓ-groups with weak unit

An element 1 of an abelian ℓ-group G is a *weak unit* if $1 \geqslant 0$ and, for every $f \in G$, $f \wedge 1 = 0$ implies $f = 0$.

Remark 11.1. Let G be an archimedean abelian ℓ-group, and let 1 be a weak unit. Then $f \mapsto f \wedge 1$ is a truncation. Indeed, the following show that (T1)–(T5) hold.
(1) $f \wedge u = (f^+ \wedge u) - f^-$ by Lemma 9.8.
(2) For all $f \in G^+$, $f \wedge 1 \in G^+$.
(3) For all $f, g \in G^+$, $f \wedge (g \wedge 1) = (f \wedge 1) \wedge g \leqslant f \wedge 1 \leqslant f$.
(4) For all $f \in G^+$, if $f \wedge 1 = 0$, then $f = 0$.
(5) For all $f \in G^+$, if $nf = (nf) \wedge 1$ for every $n \in \omega$, then $nf \leqslant 1$ for every $n \in \omega$. Since G is archimedean, $f = 0$.

Lemma 11.2. *Let G be a Dedekind σ-complete ℓ-group G, and let $1 \in G$. Then 1 is a weak unit if, and only if, the following conditions hold.*
(W1) $1 \geqslant 0$.
(W2) *For all $f \in G^+$, $f = \curlyvee^{f}_{n\in\omega} n(f \wedge 1)$.*

Proof. Since G is Dedekind σ-complete, G is archimedean. If 1 is a weak unit, then $1 \geqslant 0$ and, by Remark 11.1 and Proposition 8.5, for all $f \in G^+$, $f = \curlyvee^{f}_{n\in\omega} n(f \wedge 1)$. Conversely, suppose that (W1) and (W2) hold. If $f \wedge 1 = 0$, then $f = \curlyvee^{f}_{n\in\omega} n(f \wedge 1) = \curlyvee^{f}_{n\in\omega} 0 = 0$, and so 1 is a weak unit. □

Note that, in the language of Dedekind σ-complete ℓ-groups, axioms (W1) and (W2) are equational. Indeed, (W1) corresponds to $1 \wedge 0 = 0$, and (W2) corresponds to $\forall f\ f^+ = \curlyvee^{f^+}_{n\in\omega} n(f^+ \wedge 1)$.

Theorem 11.3 (Loomis–Sikorski Theorem for ℓ-groups with weak unit). *Suppose G is a Dedekind σ-complete ℓ-group with weak unit 1. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X, and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ such that $\iota(1) = [1]_{\mathcal{I}}$.*

Proof. By Remark 11.1, G is a Dedekind σ-complete truncated ℓ-group, with the truncation given by $f \mapsto f \wedge 1$. Then, by Theorem 9.2, there exist a set Y, a σ-ideal $\mathcal{J}$ of subsets of Y and an injective σ-continuous ℓ-homomorphism $\varphi\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{J}}$ such that, for every $f \in G$, $\varphi(f \wedge 1) = \varphi(f) \wedge [1]_{\mathcal{J}}$. The element $\varphi(1)$ is a weak unit for the image of G under φ. Therefore, by Lemma 9.11, there exists a set X, a σ-ideal $\mathcal{I}$ of subsets of X, and a σ-continuous ℓ-homomorphism $\psi\colon \frac{\mathbb{R}^Y}{\mathcal{J}} \to \frac{\mathbb{R}^X}{\mathcal{I}}$ such that the restriction of ψ to H is injective and $\psi(\varphi(1)) = [1]_{\mathcal{I}}$. The function $\iota := \psi \circ \varphi$ has the desired properties. □

Corollary 11.4. *The variety of Dedekind σ-complete ℓ-groups with weak unit is generated by* $\mathbb{R}$.

Proof. Let G be a Dedekind σ-complete ℓ-group with weak unit. By Theorem 11.3, G is a subalgebra of a quotient of a power of $\mathbb{R}$. □

12 $\mathbb{R}$ generates Dedekind σ-complete Riesz spaces with weak unit

Theorem 12.1 (Loomis–Sikorski Theorem for Riesz spaces with weak unit). *Let G be a Dedekind σ-complete Riesz space with weak unit. Then there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X, and an injective σ-continuous Riesz morphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ such that $\iota(1) = [1]_{\mathcal{I}}$.*

Proof. By Theorem 10.1, there exist a set X, a σ-ideal $\mathcal{I}$ of subsets of X and an injective σ-continuous ℓ-homomorphism $\iota\colon G \hookrightarrow \frac{\mathbb{R}^X}{\mathcal{I}}$ such that, for every $f \in G$, $\iota(1) = [1]_{\mathcal{I}}$. Since $\frac{\mathbb{R}^X}{\mathcal{I}}$ is Dedekind σ-complete, and thus archimedean, by [15, Corollary 11.53], ι is a Riesz morphism. □

We denote by $\sigma\mathbb{RS}_u$ the variety of Dedekind σ-complete Riesz spaces with weak unit, whose primitive operations are $0, +, \vee, \lambda(\cdot)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$, and 1, and whose axioms are the axioms of Riesz spaces, together with (TS1), (TS2), (TS3), (W1), (W2).

As the second main result of Part II, we now deduce a theorem that was already obtained in [1].

Theorem 12.2. *The variety* $\sigma\mathbb{RS}_u$ *of Dedekind* σ*-complete Riesz spaces with weak unit is generated by* $\mathbb{R}$.

Proof. Let G be a Dedekind σ-complete truncated Riesz space. By Theorem 12.1, G is a subalgebra of a quotient of a power of $\mathbb{R}$. □

Remark 12.3. It has been shown in [1] that $\mathbb{R}$ actually generates $\sigma\mathbb{RS}_u$ as a quasi-variety, in the sense of Remark 10.3.

Corollary 12.4. *For any set* I,

$$\begin{aligned}\mathrm{F_u}(I) &:= \Big\{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ is } \mathrm{Cyl}(\mathbb{R}^I)\text{-measurable and there exist } J \subseteq I \text{ finite, } (\lambda_j)_{j\in J} \subseteq \mathbb{R}^+,\ k \in \mathbb{R}^+ \\ &\qquad \text{such that } |f| \leqslant k + \sum_{j\in J}\lambda_j|\pi_j|\Big\} \\ &= \{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ preserves integrability over finite measure spaces}\}\end{aligned}$$

is the Dedekind σ*-complete Riesz space with weak unit freely generated by the elements* $\{\pi_i\}_{i\in I}$*, where, for* $i \in I$, $\pi_i\colon \mathbb{R}^I \to \mathbb{R}$ *is the projection on the* i*-th coordinate.*

The proof is analogous to the proof of Corollary 10.4, and $\mathrm{F_u}(I)$ is characterised by a universal property analogous to the one that characterises $\mathrm{F_t}(I)$.

A Operations that preserve ∞-integrability

In Section 4 it has been shown that, for any $p \in [1, +\infty)$, a function $\tau\colon \mathbb{R}^I \to \mathbb{R}$ preserves p-integrability if, and only if, τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and nonnegative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant \sum_{j\in J}\lambda_j|v_j|$. Does the same hold for $p = \infty$? The answer is no. Indeed, the function $(\,\cdot\,)^2\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is an example of operation which preserves ∞-integrability but not p-integrability, for every $p \in [1, +\infty)$. In Theorem A.5, we will answer the following question.

Question A.1. Which operations $\mathbb{R}^I \to \mathbb{R}$ preserve ∞-integrability?

We will see that an operation $\mathbb{R}^I \to \mathbb{R}$ preserve ∞-integrability if, and only if, roughly speaking, it is measurable and it maps coordinatewise-bounded subsets of $\mathbb{R}^I$ onto bounded subsets of $\mathbb{R}$. To make this precise, we introduce some definitions.

Given a measure space $(\Omega, \mathcal{F}, \mu)$, we define $\mathcal{L}^\infty(\mu)$ as the set of $\mathcal{F}$-measurable functions from Ω to $\mathbb{R}$ that are bounded outside of a measurable set of null μ-measure.

Definition A.2. Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$. We say that τ *preserves* ∞*-integrability* if for every measure space $(\Omega, \mathcal{F}, \mu)$ and every family $(f_i)_{i\in I} \subseteq \mathcal{L}^\infty(\mu)$ we have $\tau((f_i)_{i\in I}) \in \mathcal{L}^\infty(\mu)$.

We can now state the answer to Question A.1 precisely. Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. Then τ preserves ∞-integrability if, and only if, τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and, for every $(M_i)_{i\in I} \subseteq \mathbb{R}^+$, the restriction of τ to $\prod_{i\in I}[-M_i, M_i]$ is bounded. This will follow from Theorem A.5.

A.1 Operations that preserve boundedness

As a preliminary step, in Theorem A.4, we characterise the operations which preserve boundedness.

Definition A.3. Let I be a set, $\tau\colon \mathbb{R}^I \to \mathbb{R}$. We say that τ *preserves boundedness* if for every set Ω and every family $(f_i)_{i\in I}$ of bounded functions $f_i\colon \Omega \to \mathbb{R}$, we have that $\tau((f_i)_{i\in I})\colon \Omega \to \mathbb{R}$ is also bounded.

Theorem A.4. *Let* I *be a set and* $\tau\colon \mathbb{R}^I \to \mathbb{R}$*. The following conditions are equivalent.*
(1) τ *preserves boundedness.*
(2) *For every* $(M_i)_{i\in I} \subseteq \mathbb{R}^+$*, the restriction of* τ *to* $\prod_{i\in I}[-M_i, M_i]$ *is bounded.*

Proof. We prove (1) ⇒ (2). Fix $(M_i)_{i\in I} \subseteq \mathbb{R}^+$. Take $\Omega := \prod_{i\in I}[-M_i, M_i]$ and, for every $i \in I$, let f_i be the restriction of the projection function $\pi_i\colon \mathbb{R}^I \to \mathbb{R}$ to Ω. Since f_i maps Ω onto $[-M_i, M_i]$, f_i is bounded. Thus $\tau((f_i)_{i\in I})$ is bounded, i.e., there exists $\tilde{M}$ such that for every $x \in \Omega$ we have $\tau((f_i(x))_{i\in I}) \in [-\tilde{M}, \tilde{M}]$. Let $x \in \Omega$. Then $\tau(x) = \tau((\pi_i(x))_{i\in I}) = \tau((f_i(x))_{i\in I}) \in [-\tilde{M}, \tilde{M}]$. Thus (2) holds.

We now prove (2) ⇒ (1). Let Ω be a set, and let $(f_i)_{i\in I}$ be a family of bounded functions from Ω to $\mathbb{R}$. For each $i \in I$, let $M_i \in \mathbb{R}^+$ be such that the image of f_i is contained in $[-M_i, M_i]$. Let $\tilde{M}$ be such that τ maps $\prod_{i\in I}[-M_i, M_i]$ onto a subset of $[-\tilde{M}, \tilde{M}]$. Then, for each $x \in \Omega$, $\tau((f_i)_{i\in I})(x) = \tau((f_i(x))_{i\in I}) \in [-\tilde{M}, \tilde{M}]$. □

A.2 Operations that preserve ∞-integrability

The following is the main theorem of this section.

Theorem A.5. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. The following conditions are equivalent.*
(1) *τ preserves ∞-integrability.*
(2) *τ preserves measurability and boundedness.*
(3) *τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and, for every $(M_i)_{i\in I} \subseteq \mathbb{R}^+$, the restriction of τ to $\prod_{i\in I}[-M_i, M_i]$ is bounded.*

In order to prove Theorem A.5, we need some lemmas.

Lemma A.6. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. If τ preserves ∞-integrability, then τ preserves measurability.*

Proof. Every measurable space $(\Omega, \mathcal{F})$ may be endowed with the null measure μ_0: for each $A \in \mathcal{F}$, $\mu_0(A) = 0$. Then $\mathcal{L}^\infty(\mu_0)$ is the set of $\mathcal{F}$-measurable functions from Ω to $\mathbb{R}$. Hence, preservation of ∞-integrability over $(\Omega, \mathcal{F}, \mu_0)$ is equivalent to preservation of measurability over $(\Omega, \mathcal{F})$. □

Lemma A.7. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. If τ preserves ∞-integrability, then τ preserves boundedness.*

Proof. Let us suppose that τ does not preserve boundedness. By Theorem A.4, there exists $(M_i)_{i\in I} \subseteq \mathbb{R}^+$ such that the restriction of τ to $\prod_{i\in I}[-M_i, M_i]$ is not bounded. Fix one such family $(M_i)_{i\in I}$; let $\Omega := \prod_{i\in I}[-M_i, M_i]$. Let $(\omega_n)_{n\in\omega}$ be a sequence in Ω such that $|\tau(\omega_0)| < |\tau(\omega_1)| < \cdots$ and $|\tau(\omega_n)| \to \infty$ as $n \to \infty$. Consider on $(\Omega, \mathcal{P}(\Omega))$ the discrete measure μ such that $\mu(\{\omega_n\}) = \frac{1}{2^n}$ for every $n \in \omega$ and $\mu(\Omega \setminus \{\omega_0, \omega_1, \dots\}) = 0$. Then $(\Omega, \mathcal{P}(\Omega), \mu)$ is a finite measure space. For $i \in I$, the restriction $(\pi_i)_{|\Omega}$ of π_i to Ω is bounded, since its image is $[-M_i, M_i]$. Moreover, $(\pi_i)_{|\Omega}$ is $\mathcal{P}(\Omega)$-measurable. Therefore, $(\pi_i)_{|\Omega} \in \mathcal{L}^\infty(\mu)$. We have $\tau_{|\Omega} \notin \mathcal{L}^\infty(\mu)$; indeed, let A be a subset of Ω of null μ-measure. Then $\omega_n \notin A$ for every $n \in \omega$. Therefore $\tau_{|\Omega}$ is not bounded outside of A. □

Lemma A.8. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. If τ preserves measurability and boundedness, then τ preserves ∞-integrability.*

Proof. By Corollary 3.6, τ depends on a countable subset $J \subseteq I$. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and consider a family $(f_i)_{i\in I} \subseteq \mathcal{L}^\infty(\mu)$. For each $j \in J$, let A_j be a measurable subset of Ω such that $\mu(A_j) = 0$ and f_j is bounded outside of A_j. Set $A := \bigcup_{j\in J} A_j$. Then $\mu(A) = 0$. For each $i \in I$, define $\tilde{f}_i$ as f_i if $i \in J$, otherwise let $\tilde{f}_i$ be the function constantly equal to 0. Since τ depends only on J, we have $\tau((f_i)_{i\in I}) = \tau((\tilde{f}_i)_{i\in I})$. For every $i \in I$, the restriction $(\tilde{f}_i)_{|\Omega\setminus A}$ is bounded. We have that $\tau((f_i)_{i\in I})_{|\Omega\setminus A} = \tau((\tilde{f}_i)_{i\in I})_{|\Omega\setminus A} = \tau(((\tilde{f}_i)_{|\Omega\setminus A})_{i\in I})$ is bounded since τ preserves boundedness and, for every $i \in I$, $(\tilde{f}_i)_{|\Omega\setminus A}$ is bounded. Thus $\tau((f_i)_{i\in I})$ is bounded outside of a set of null measure. Moreover, $\tau((f_i)_{i\in I})$ is measurable because τ preserve measurability. Therefore $\tau((f_i)_{i\in I}) \in \mathcal{L}^\infty(\mu)$. □

Proof of Theorem A.5. By Lemmas A.6 and A.7, we have (1) ⇒ (2). Lemma A.8, we have (2) ⇒ (1). By Theorems 3.3 and A.4, we have (2) ⇔ (3). □

Corollary A.9. *Let I be a set and let $\tau\colon \mathbb{R}^I \to \mathbb{R}$ be a function. If τ preserves p-integrability for some $p \in [1, +\infty)$, then τ preserves ∞-integrability.*

Proof. By Theorem 2.1, τ is $\mathrm{Cyl}(\mathbb{R}^I)$-measurable and there exist a finite subset of indices $J \subseteq I$ and non-negative real numbers $(\lambda_j)_{j\in J}$ such that, for every $v \in \mathbb{R}^I$, we have $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j|$. Let $(M_i)_{i\in I} \subseteq \mathbb{R}^+$. Let $v \in \prod_{i\in I}[-M_i, M_i]$. Then $|\tau(v)| \leqslant \sum_{j\in J} \lambda_j |v_j| \leqslant \sum_{j\in J} \lambda_j M_j$. Thus, the restriction of τ to $\prod_{i\in I}[-M_i, M_i]$ is bounded. Therefore, by Theorem A.5, τ preserves ∞-integrability. □

Remark A.10. The converse of Corollary A.9, as mentioned at the beginning of this section, is not true, as shown by the function $(\cdot)^2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$.

Acknowledgment: The author is deeply grateful to his Ph.D. advisor Professor Vincenzo Marra for the many helpful discussions. Moreover, the author gratefully acknowledges the anonymous referee for his or her valuable suggestions.

References

[1] M. Abbadini, Dedekind σ-complete ℓ-groups and Riesz spaces as varieties, *Positivity* (2019), DOI 10.1007/s11117-019-00718-9.
[2] R. N. Ball, Truncated abelian lattice-ordered groups I: The pointed (Yosida) representation, *Topology Appl.* **162** (2014), 43–65.
[3] A. Bigard, K. Keimel and S. Wolfenstein, *Groupes et anneaux réticulés*, Lecture Notes in Math. 608, Springer, Berlin, 1977.
[4] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Grad. Texts in Math. 78, Springer, New York, 1981.
[5] G. Buskes, B. de Pagter and A. van Rooij, The Loomis–Sikorski theorem revisited, *Algebra Universalis* **58** (2008), no. 4, 413–426.
[6] G. Buskes and A. van Rooij, Small Riesz spaces, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), no. 3, 523–536.
[7] G. Buskes and A. Van Rooij, Representation of Riesz spaces without the Axiom of Choice, *Nepali Math. Sci. Rep.* **16** (1997), no. 1–2, 19–22.
[8] G. B. Folland, *Real Analysis*, 2nd ed., Pure Appl. Math. (New York), John Wiley & Sons, New York, 1999.
[9] D. H. Fremlin, *Measure Theory. Vol. 2*, Torres Fremlin, Colchester, 2003.
[10] O. Kallenberg, *Foundations of Modern Probability*, Probab. Appl. (N. Y.), Springer, New York, 1997.
[11] M. A. Lepellere and A. Valente, Embedding of Archimedean *l*-groups in Riesz spaces, *Atti Semin. Mat. Fis. Univ. Modena* **46** (1998), no. 1, 249–254.
[12] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces. Vol. I*, North-Holland, Amsterdam, 1971.
[13] D. Mundici, Tensor products and the Loomis–Sikorski theorem for MV-algebras, *Adv. in Appl. Math.* **22** (1999), no. 2, 227–248.
[14] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.
[15] E. Schechter, *Handbook of Analysis and its Foundations*, Academic Press, San Diego, 1999.
[16] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Instytut Matematyczny Polskiej Akademi Nauk, Wrosław, 1959.
[17] S. M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer, New York, 1998.