

Some Neumann-Bessel series and the Laplacian on polygons

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Abstract: Several sums of Neumann series with Bessel and trigonometric functions are evaluated, as finite sums of trigonometric functions. They arise from a generalization of the Neumann expansion of the eigenstates of the Laplacian in regular polygons. A simple accurate approximation of $J_0(x)$ is found, on the interval $[0, 2]$.

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INTRODUCTION

The ground state of the Laplace equation in a regular polygon with Dirichlet boundary conditions at the n sides, has a natural expression as a Neumann series of Bessel and trigonometric functions,

$$\psi_n(r, \theta) = J_0(\lambda_n r) + 2 \sum_{k=1}^{\infty} h_{k,n} J_{kn}(\lambda_n r) \cos(kn\theta),$$

with coefficients $h_{k,n}$ to be found and eigenvalue $-\lambda_n^2$ that scales with the area. For the equilateral triangle and the square, the solutions are known as sums of few trigonometric functions of the coordinates $x = r \cos \theta$ and $y = r \sin \theta$. Such solutions have a corresponding Neumann expression [7]. For the square of area π :

$$\begin{aligned} J_0(r\sqrt{2\pi}) + 2 \sum_{k=1}^{\infty} J_{4k}(r\sqrt{2\pi}) \cos(4k\theta) \\ = \frac{1}{2} \cos(x\sqrt{2\pi}) + \frac{1}{2} \cos(y\sqrt{2\pi}) \end{aligned} \quad (1)$$

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The triangle requires some work to establish the equivalence:

$$\begin{aligned}
J_0(\lambda_3 r) + 2 \sum_{k=1}^{\infty} \frac{\cos(k\pi/2 - \pi/6)}{\cos(\pi/6)} J_{3k}(\lambda_3 r) \cos(3k\theta) \\
= \frac{2}{3\sqrt{3}} \sin\left(\frac{4\pi}{3R_3}x + \frac{2\pi}{3}\right) - \frac{2}{3\sqrt{3}} \left[\sin\left[\frac{2\pi}{3R_3}(x + y\sqrt{3}) - \frac{2\pi}{3}\right] \right. \\
\left. - \frac{2}{3\sqrt{3}} \sin\left[\frac{2\pi}{3R_3}(x - y\sqrt{3}) - \frac{2\pi}{3}\right] \right] \quad (2)
\end{aligned}$$

where, for area π , $\lambda_3^2 = 4\pi/\sqrt{3}$ and $R_3 = \frac{2}{3}\sqrt{\pi\sqrt{3}}$. In [7] I obtained a sum that generalizes the integrable cases $n = 3, 4$:

$$\begin{aligned}
f_n(x, y) &= J_0(r) + 2 \sum_{k=1}^{\infty} \frac{\cos[nk\frac{3\pi}{2} - \frac{\pi}{2n}]}{\cos(\frac{\pi}{2n})} J_{nk}(r) \cos(nk\theta) \\
&= \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{\cos[r \cos(\theta + \frac{2\pi}{n}\ell) + \frac{\pi}{2n}]}{\cos(\frac{\pi}{2n})} \\
&= \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{\cos[x \cos(\frac{2\pi}{n}\ell) - y \sin(\frac{2\pi}{n}\ell) + \frac{\pi}{2n}]}{\cos(\frac{\pi}{2n})} \quad (3)
\end{aligned}$$

For $n \rightarrow \infty$ the Riemann sum in the second line is $\int_0^{2\pi} \frac{dt}{2\pi} \cos(r \cos t) = J_0(r)$. For $n = 2$ it is $f_2(x, y) = \cos x$. For $n = 6$:

$$\begin{aligned}
f_6(x, y) &= J_0(r) + 2 \sum_{k=1}^{\infty} (-1)^k J_{6k}(r) \cos(6k\theta) \\
&= \frac{1}{3} \cos x + \frac{2}{3} \cos\left(\frac{1}{2}x\right) \cos\left(\frac{\sqrt{3}}{2}y\right) \quad (4)
\end{aligned}$$

The functions f_n are eigenfunctions of the Laplace operator with eigenvalue -1 but, for $n > 4$, they no longer vanish on the boundary of a n -polygon. I only remark that the level curves $f_n(x, y) = C$ are closed around the origin (where $f_n = 1$) up to a separatrix with n self-intersections, with values $C_5 = -0.334909$, $C_6 = -1/3$, $C_7 = 0.19633$, etc. The level lines and the separatrices for $n = 6, 7$ are shown in Fig.1.

The study of the Laplacian in polygons has a long history. The ground states beyond the square, $n > 4$, cannot be finite sums of trigonometric functions. They have been investigated analytically and numerically in $1/n$ expansion (see for example [7, 4, 3]).

In this paper I generalize the identity (3), and obtain a number of new formulas for Neumann series whose sums contain a finite number of terms. For certain values of the parameters, they are identities that are found in the tables by Gradshteyn and Ryzhik [2], Prudnikov, Brychkov and Marichev [9], a recent paper by Al-Jarrah, Dempsey and Glasser [1], and two old papers by Takizawa and Kobayasi [10, 5]. In the last ones, the Neumann series appear as correlation functions for the heat flow in coupled harmonic oscillators.

A side result is an accurate approximation of $J_0(x)$ on the interval $[0, 2]$,

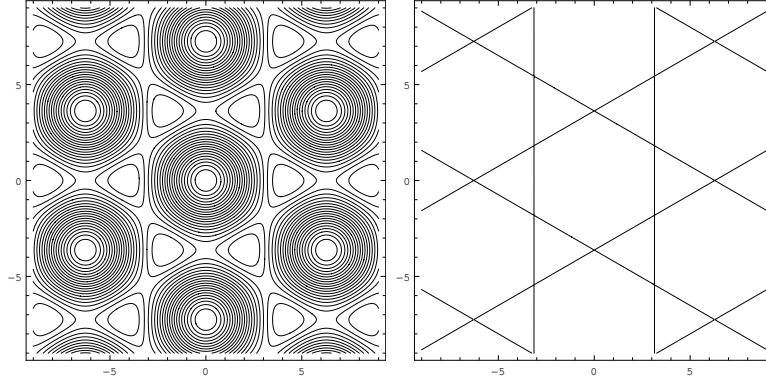


FIGURE 1. Left: the contour plot of the function f_6 in eq.(4). Right: the separatrix $f_6(x, y) = -1/3$ is a Kagomé lattice. It can be written as $0 = \cos(x/2)[\cos(x/2) + \cos(\sqrt{3}y/2)]$. Vertices are the double zeros.

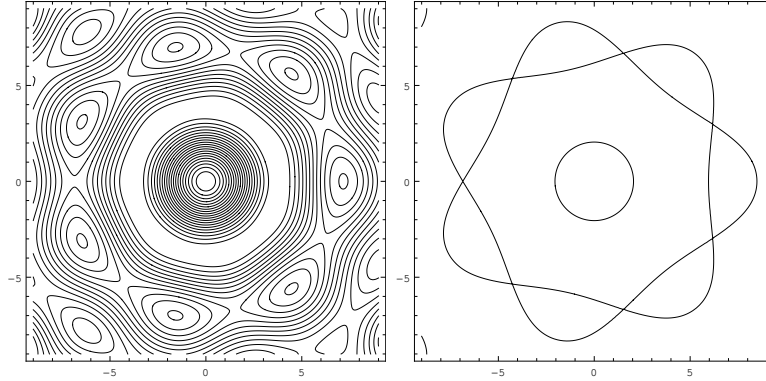


FIGURE 2. Left: the contour plot of f_7 in eq.(3). Right: the separatrix $f_7(x, y) = -1.9633...$

by three cosines (eq.18).

All figures and several checks were made with Wolfram *Mathematica*7.

THE SUMMATION FORMULA

The source equation of various sums in this paper is:

$$\sum_{k=-\infty}^{+\infty} J_{kn+p}(z) e^{ikny} = \frac{1}{n} \sum_{\ell=0}^{n-1} e^{iz \sin(y + \frac{2\pi}{n}\ell) - ip(y + \frac{2\pi}{n}\ell)} \quad (5)$$

$n = 1, 2, \dots$, $p = 0, 1, \dots, n - 1$, y is real, z is complex. For $y = 0$ and even n eq.(5) is eq.1 in [10]. Sums of this sort are tabulated only for $n = 1, 2$ in [9].

Proof. The result follows from the Fourier integral of a Bessel function of integer order. For $z \in \mathbb{C}$, the sum $\sum_{k=-\infty}^{\infty} e^{ikny} J_{kn+p}(z)$ is uniformly convergent in y by the bound $|J_{\pm m}(z)| \leq C|z/2|^m/m!$ (Nielsen, see §3.13 in [11]).

$$\begin{aligned} \sum_{k=-\infty}^{\infty} e^{ikny} J_{kn+p}(z) &= \sum_{k=-\infty}^{\infty} e^{ikny} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iz \sin \theta - i(kn+p)\theta} \\ &= \sum_{k=-\infty}^{\infty} e^{ikny} \sum_{j=0}^{n-1} \int_{\frac{2\pi}{n}j}^{\frac{2\pi}{n}(j+1)} \frac{d\theta}{2\pi} e^{iz \sin \theta - ip\theta} e^{-ikn\theta} \\ &= \sum_{j=0}^{n-1} \left[\sum_{k=-\infty}^{\infty} e^{ikny} \int_0^{\frac{2\pi}{n}} \frac{d\theta}{2\pi} e^{iz \sin(\theta + \frac{2\pi}{n}j) - ip(\theta + \frac{2\pi}{n}j)} e^{-ikn\theta} \right] \end{aligned}$$

The functions $\sqrt{n/2\pi} e^{ikny}$, $k \in \mathbb{Z}$, are a complete orthonormal basis in the Hilbert space $L^2(0, 2\pi/n)$. The infinite sum is the Fourier representation of $(1/n) \exp[iz \sin(y + \frac{2\pi}{n}j) - ip(y + \frac{2\pi}{n}j)]$. \square

In the following I give some examples.

1. The case $p = 0$ and $y = \frac{\pi}{2} + \alpha$ is an extension with angle α of the equations 19 and 20 in [1], where $\alpha = 0$. With $J_{-m}(z) = (-)^m J_m(z)$:

$$J_0(z) + 2 \sum_{k=1}^{\infty} e^{ikn\frac{\pi}{2}} J_{kn}(z) \cos(kn\alpha) = \frac{1}{n} \sum_{\ell=0}^{n-1} e^{iz \cos(\alpha + \frac{2\pi}{n}\ell)} \quad (6)$$

For $n = 1$, separation of even and odd parity terms in z gives the Jacobi expansions (eqs. 5.7.10.4 and 5 in [9]):

$$J_0(z) + 2 \sum_{k=1}^{\infty} (-)^k J_{2k}(z) \cos(2k\alpha) = \cos(z \cos \alpha) \quad (7)$$

$$\sum_{k=0}^{\infty} (-)^k J_{2k+1}(z) \cos[(2k+1)\alpha] = \frac{1}{2} \sin(z \cos \alpha) \quad (8)$$

If α is replaced by $\alpha + \pi/2$ they are eqs. 8.514.5 and 6 in [2] and 10.4, 10.5 in [6]:

$$J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos(2k\alpha) = \cos(z \sin \alpha) \quad (9)$$

$$\sum_{k=0}^{\infty} J_{2k+1}(z) \sin[(2k+1)\alpha] = \frac{1}{2} \sin(z \sin \alpha) \quad (10)$$

1.1. For n replaced by $2n$, eq.(6) is:

$$J_0(z) + 2 \sum_{k=1}^{\infty} (-)^{kn} J_{2kn}(z) \cos(2kn\alpha) = \frac{1}{n} \sum_{\ell=0}^{n-1} \cos[z \cos(\alpha + \frac{\pi}{n}\ell)] \quad (11)$$

where the finite sum was reduced by noting that terms ℓ and $n + \ell$ are the same.

- The value $y = \frac{\pi}{2}$ yields eq.(23) in [1].
- For $n = 1$ the derivative of (11) in $\alpha = \frac{\pi}{4}$ is:

$$2J_2(z) - 6J_6(z) + 10J_{10}(z) - 14J_{14}(z) + \dots = z^{\frac{\sqrt{2}}{4}} \sin(z^{\frac{\sqrt{2}}{2}}) \quad (12)$$

- For $n = 2$ eq.(11) becomes

$$J_0(z) + 2 \sum_{k=1}^{\infty} J_{4k}(z) \cos(4k\alpha) = \frac{1}{2} [\cos(z \sin \alpha) + \cos(z \cos \alpha)]. \quad (13)$$

The values $\alpha = 0, \frac{\pi}{4}$ give eqs.5.7.1.19 in [9]. The derivative is:

$$\sum_{k=1}^{\infty} k J_{4k}(z) \sin(4k\alpha) = \frac{z}{16} [\sin(z \sin \alpha) \cos \alpha - \sin(z \cos \alpha) \sin \alpha], \quad (14)$$

further, the expansion in small α gives:

$$\sum_{k=1}^{\infty} k^2 J_{4k}(z) = \frac{z}{64} (z - \sin z) \quad (15)$$

- Case $n = 3$ is eq.(4), $\alpha = 0$ gives eq. 5.7.1.21 in [9].

1.2. If n is replaced by $2n + 1$, eq.(6) is:

$$J_0(z) + 2 \sum_{k=1}^{\infty} e^{i(2n+1)k\frac{\pi}{2}} J_{(2n+1)k}(z) \cos[(2n+1)k\alpha] = \frac{1}{2n+1} \sum_{\ell=0}^{2n} e^{iz \cos(\alpha + \frac{2\pi}{2n+1}\ell)}$$

The even-parity and odd-parity parts in the exchange $z \rightarrow -z$ are:

$$\begin{aligned} J_0(z) + 2 \sum_{k=1}^{\infty} (-)^k J_{(4n+2)k}(z) \cos[(4n+2)k\alpha] \\ = \frac{1}{2n+1} \sum_{\ell=0}^{2n} \cos[z \cos(\alpha + \frac{2\pi}{2n+1}\ell)] \end{aligned} \quad (16)$$

$$\begin{aligned} 2 \sum_{k=0}^{\infty} (-)^{n+k} J_{(2n+1)(2k+1)}(z) \cos[(2n+1)(2k+1)\alpha] \\ = \frac{1}{2n+1} \sum_{\ell=0}^{2n} \sin[z \cos(\alpha + \frac{2\pi}{2n+1}\ell)] \end{aligned} \quad (17)$$

The first equation with $n = 1$ and $\alpha = \pi/12$, offers a simple approximation of $J_0(x)$ in $0 \leq x \leq 2$ with an error $|\epsilon| \leq 4 \times 10^{-9}$

$$J_0(x) = \frac{1}{3} [\cos(x \frac{1}{\sqrt{2}}) + \cos(x \frac{\sqrt{3}-1}{2\sqrt{2}}) + \cos(x \frac{\sqrt{3}+1}{2\sqrt{2}})] + \epsilon \quad (18)$$

At $j_{0,1} \approx 2.404$ (first zero of $J_0(x)$) the error is 3.4×10^{-8} .

Examples of the second equation, (17), are:

$$\sum_{k=0}^{\infty} (-)^k J_{6k+3}(z) \cos[(6k+3)\alpha] = -\frac{1}{6} \sum_{\ell=0}^2 \sin[z \cos(\alpha + \frac{2\pi}{3}\ell)] \quad (19)$$

$$\sum_{k=0}^{\infty} (-)^k J_{10k+5}(z) \cos[(10k+5)\alpha] = \frac{1}{10} \sum_{\ell=0}^4 \sin[z \cos(\alpha + \frac{2\pi}{5}\ell)] \quad (20)$$

The first one with $\alpha = \pi$ is eq.22 in [1].

Both sums are eigenfunctions of the Laplacian with eigenvalue $\lambda = -1$ (see Figs. 3,4). The finite sum in (20), with $z = r$, $x = r \cos \alpha$ and $y = r \sin \alpha$, is:

$$\begin{aligned} f(x, y) = \frac{1}{10} \sin x - \frac{1}{5} \sin(x \cos \frac{\pi}{5}) \cos(y \sin \frac{\pi}{5}) \\ + \frac{1}{5} \sin(x \cos \frac{2\pi}{5}) \cos(y \sin \frac{2\pi}{5}) \end{aligned} \quad (21)$$

1.3. Eq. (5) with $p = 0$ is multiplied by $\exp(i\beta)$, β real, and the real part is taken. The left hand side becomes:

$$\begin{aligned} J_0(x) \cos \beta + \sum_{k=1}^{\infty} J_{kn}(x) \operatorname{Re}[e^{i\beta}(e^{ikny} + e^{-ikn(y+\pi)})] \\ = J_0(x) \cos \beta + 2 \sum_{k=1}^{\infty} \cos(\beta - kn \frac{\pi}{2}) J_{kn}(x) \cos[kn(y + \frac{\pi}{2})] \end{aligned}$$

The identity (3) is obtained, when $\beta = \frac{\pi}{2n}$ and $y + \frac{\pi}{2} = \theta + \pi$.

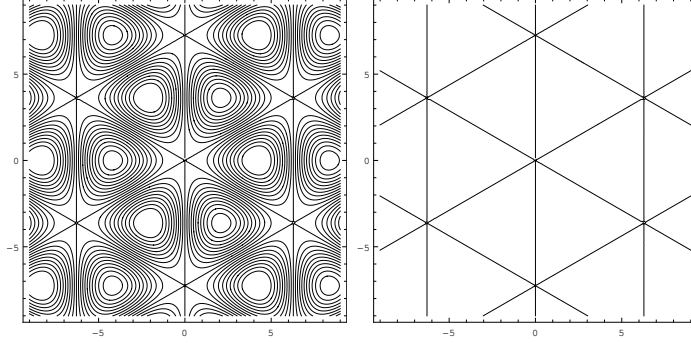


FIGURE 3. Contour plot of the sum (19). The function is the ground state of the Laplacian, vanishing on the boundary of equilateral triangles (no nodal lines) and is an excited state of the hexagon (the right figure is the contour for the value zero).

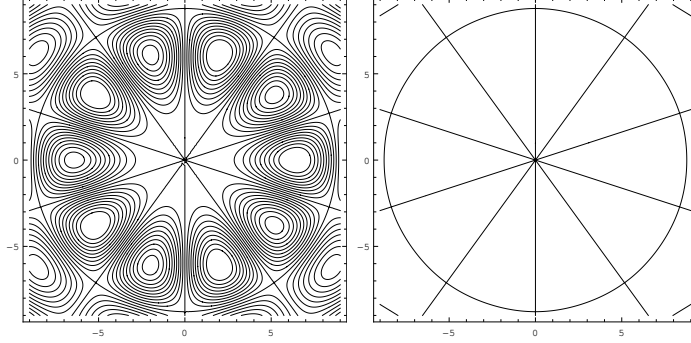


FIGURE 4. Left: contour plot of the sum (20), i.e the function f in (21). Right: the contour $f = 0$ is very close to the circumference of radius $j_{5,1}$ (the first zero of J_5), where $|f(x, y)| \leq 0.001$.

2. Parseval's identity is applied to (5):

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} J_{kn+p}^2(x) &= \frac{1}{n^2} \sum_{k\ell} e^{ip\frac{2\pi}{n}(k-\ell)} \int_0^{2\pi} \frac{dy}{2\pi} e^{ix \sin(y - \frac{\pi}{n}(k-\ell)) - ix \sin(y + \frac{\pi}{n}(k-\ell))} \\
 &= \frac{1}{n^2} \sum_{k\ell} e^{ip\frac{2\pi}{n}(k-\ell)} \int_0^{2\pi} \frac{dy}{2\pi} e^{-i2x \cos y \sin(\frac{\pi}{n}(k-\ell))} \\
 &= \frac{1}{n^2} \sum_{k\ell} e^{ip\frac{2\pi}{n}(k-\ell)} J_0[2x \sin(\frac{\pi}{n}(k-\ell))] \\
 &= \frac{1}{n} + \frac{2}{n^2} \sum_{k=1}^{n-1} k \cos(\frac{2\pi}{n}kp) J_0(2x \sin \frac{\pi k}{n})
 \end{aligned}$$

The last sum is unchanged if k is replaced by $n - k$:

$$\sum_{k \in \mathbb{Z}} J_{kn+p}^2(x) = \frac{1}{n} + \frac{1}{n} \sum_{k=1}^{n-1} \cos\left(\frac{2\pi}{n} kp\right) J_0\left(2x \sin \frac{\pi k}{n}\right) \quad (22)$$

The left-hand side is $J_p(x)^2 + \sum_{k=1}^{\infty} [J_{kn+p}^2(x) + J_{kn-p}^2(x)]$.

2.1. If $n \rightarrow 2n$ and $p = 0$, the sum (22) is amenable to eq.29 in [1]:

$$J_0^2(x) + 2 \sum_{k=1}^{\infty} J_{2kn}^2(x) = \frac{1}{2n} + \frac{1}{2n} J_0(2x) + \frac{1}{n} \sum_{k=1}^{n-1} J_0\left(2x \cos \frac{\pi k}{2n}\right) \quad (23)$$

- If $n \rightarrow 2n$ and $p = n$ in (22), with simple steps one obtains:

$$\sum_{k=0}^{\infty} J_{(2k+1)n}^2(x) = \frac{1}{4n} + \frac{(-1)^n}{4n} J_0(2x) + \frac{1}{2n} \sum_{\ell=1}^{n-1} (-1)^{\ell} J_0\left(2x \sin \frac{\pi \ell}{2n}\right) \quad (24)$$

Examples:

$$\sum_{k=0}^{\infty} J_{2+4k}^2(x) = \frac{1}{8} + \frac{1}{8} J_0(2x) - \frac{1}{4} J_0(x\sqrt{2}) \quad (25)$$

$$\sum_{k=0}^{\infty} J_{3+6k}^2(x) = \frac{1}{12} - \frac{1}{6} J_0(x) - \frac{1}{12} J_0(2x) + \frac{1}{6} J_0(x\sqrt{3}) \quad (26)$$

3. In eq.(5) the variable y is shifted to $y+2t$. The equation is multiplied by $e^{iz' \sin y - i q y}$ and integrated in y :

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} J_{p+kn}(z) J_{q-kn}(z') e^{i(kn+p)2t} \\ = \frac{1}{n} \sum_{\ell=0}^{n-1} e^{-ip \frac{2\pi}{n} \ell} \int_0^{2\pi} \frac{dy}{2\pi} e^{iz \sin(y+2t+\frac{2\pi}{n}\ell) + iz' \sin y - i(p+q)y} \end{aligned} \quad (27)$$

In the integral, the shift y to $y - t - \frac{\pi}{n}\ell$ changes the exponent to $i(z+z') \sin y \cos(t + \frac{\pi}{n}\ell) + i(z-z') \cos y \sin(t + \frac{\pi}{n}\ell) - i(p+q)(y - t - \frac{\pi}{n}\ell)$

3.1. With $z = z'$ we obtain eq.1 in [5]:

$$\sum_{k=-\infty}^{+\infty} J_{p+kn}(z) J_{q-kn}(z) e^{2iknt} = \frac{1}{n} \sum_{\ell=0}^{n-1} e^{-i(p-q)(t+\frac{\pi}{n}\ell)} J_{p+q}[2z \cos(t + \frac{\pi}{n}\ell)] \quad (28)$$

For $n = 1$, with a shift of the index k and renaming of parameter, it is eq.8.530 in [2].

For $n = 2$, $p = q$ and $t = \pi/4$ it is eq. 5.7.11.25 [9].

3.2. Eq.(27) with $p = q = 0$ and $t = 0$ is:

$$J_0(z)J_0(z') + 2 \sum_{k=1}^{\infty} (-)^{kn} J_{kn}(z)J_{kn}(z') = \frac{1}{n} \sum_{\ell=0}^{n-1} \int_0^{2\pi} \frac{dy}{2\pi} e^{iz \sin(y + \frac{2\pi}{n}\ell) + iz' \sin y} \quad (29)$$

For $n = 1, 2$ they are eqs. 5.7.11.1 and 5.7.11.18 in [9]; if also $z = z'$ they are eqs. 31, 32 in [1].

A new example is:

$$\sum_{k=1}^{\infty} J_{4k}(x)J_{4k}(y) = \frac{1}{8}[J_0(x+y) + J_0(x-y) - 4J_0(x)J_0(y)] + \frac{1}{4}J_0(\sqrt{x^2 + y^2}) \quad (30)$$

4. Multiplication of (5) by $\exp(-ay)$ ($a > 0$) with $p = 0$ and $n = 1$, and integration on \mathbb{R}^+ give:

$$\sum_{k=-\infty}^{\infty} J_{2k}(z) \frac{a}{a^2 + 4k^2} + 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \frac{2k+1}{a^2 + (2k+1)^2} = \int_0^{\infty} dy e^{iz \sin y - ay}$$

The last integral is done by expanding $\exp(iz \sin y)$ and using integrals eqs.3.895.1 and 3.895.4 [2]. The even and odd terms are:

$$\frac{1}{a^2} J_0(z) + 2 \sum_{k=1}^{\infty} \frac{J_{2k}(z)}{a^2 + 4k^2} = \sum_{k=0}^{\infty} \frac{(-)^k z^{2k}}{a^2(a^2 + 4) \dots (a^2 + 4k^2)} \quad (31)$$

$$\sum_{k=0}^{\infty} J_{2k+1}(z) \frac{2(2k+1)}{a^2 + (2k+1)^2} = \sum_{k=0}^{\infty} \frac{(-)^k z^{2k+1}}{(a^2 + 1)(a^2 + 9) \dots (a^2 + (2k+1)^2)} \quad (32)$$

For $a = i\nu$ the equations are expansions of the Lommel functions $s_{-1,\nu}(z)$ and $s_{0,\nu}(z)$ in series of Bessel functions (eq. 11.9.7 in [8]).

More and more identities can be obtained by derivation, or integration with functions. Here I limited myself to some examples.

Data availability. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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