# Asymptotics for a parabolic equation with critical exponential nonlinearity

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**Abstract**

We consider the Cauchy problem :

$$\begin{cases} \partial_t u = \Delta u - u + \lambda f(u) & \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases}$$

where $\lambda > 0$,

$$f(u) := 2\alpha_0 u e^{\alpha_0 u^2}, \quad \text{for some } \alpha_0 > 0,$$

with initial data $u_0 \in H^1(\mathbb{R}^2)$. The nonlinear term $f$ has a *critical* growth at infinity in the energy space $H^1(\mathbb{R}^2)$ in view of the Trudinger-Moser embedding. Our goal is to investigate from the initial data $u_0 \in H^1(\mathbb{R}^2)$ whether the solution blows up in finite time or the solution is global in time. For $0 < \lambda < \frac{1}{2\alpha_0}$, we prove that for initial data with energies below or equal to the ground state level, the dichotomy between finite time blow-up and global existence can be determined by means of a potential well argument.

## 1 Introduction and main results

**Model parabolic problem.** We consider the Cauchy problem for a two space dimensional parabolic equation with exponential-type nonlinearity, more precisely we focus the attention on the following model problem:

$$\begin{cases} \partial_t u = \Delta u - u + \lambda f(u) & \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\lambda > 0$,

$$f(u) := 2\alpha_0 u e^{\alpha_0 u^2}, \quad \text{for some } \alpha_0 > 0,$$

and we consider initial data in the *energy* space $H^1(\mathbb{R}^2)$, i.e.

$$u_0 \in H^1(\mathbb{R}^2).$$

In this framework, *energy* refers to the functional associated with the stationary problem:

$$I(v) := \frac{1}{2}\|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} F(v)\,dx,$$

where

$$\|v\|_{H^1} := \left( \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 \right)^{\frac{1}{2}}, \quad \text{and} \quad F(v) := \int_0^v f(\eta)\,d\eta = e^{\alpha_0 v^2} - 1.$$

The above functional is well defined in $H^1(\mathbb{R}^2)$, and the nonlinear term $f$ that we are considering has *critical* growth in the energy space in view of the Trudinger-Moser embedding [1, 32].

Concerning local existence and uniqueness for (1.1), Ibrahim, Jrad, Majdoub and Saanouni [14] proved that, for any $u_0 \in H^1(\mathbb{R}^2)$, the Cauchy problem (1.1) has a local in time solution

$$u \in \mathcal{C}\big( [0,T]; H^1(\mathbb{R}^2) \big)$$

for some finite time $T > 0$ (see Definition 2.1 and Remark 2.2), and the solution is *unique*. Then the smoothing effect of the heat kernel implies that $u$ is a *classical* solution; in fact, it belongs to the class

$$u \in L^\infty_{loc}\big( (0,T]; L^\infty(\mathbb{R}^2) \big) \cap \mathcal{C}^1\big( (0,T); L^2(\mathbb{R}^2) \big) \cap \mathcal{C}^{1,2}\big( (0,T) \times \mathbb{R}^2 \big),$$

see [20, Remark 4.1].

We define the maximal existence time $T_*$ of the solution $u$ as

$$T_* := \sup\Big\{ T > 0 \ : \ \text{the problem (1.1) admits a solution } u \in \mathcal{C}\big( [0,T]; H^1(\mathbb{R}^2) \big) \Big\} \in (0, +\infty].$$

If $T_* < +\infty$ then the $L^\infty$-norm of the solution blows up, i.e.

$$\text{if } T_* < +\infty \text{ then } \limsup_{t \to T_*} \|u(t)\|_{L^\infty} = +\infty,$$

see e.g. [5, Section 5.3]. In view of the definition of $T_*$, it is natural to try to understand whether $T_* < +\infty$ yields also the blow-up of the $H^1$-norm of the solution. This problem is related to the dependence of the local existence time of the solution to (1.1) from the *size* of the initial data $u_0 \in H^1(\mathbb{R}^2)$; this aspect will be emphasized in Section 2 in comparison with the *energy subcritical* problem. For the *energy subcritical* problem, the local existence time is uniform with respect to the $H^1$-norm, while for the *energy critical* Cauchy problem (1.1), we can find a *uniform* local existence time for *small* initial data only, and we *quantify* the smallness condition in Theorem 2.6.

As a consequence of Theorem 2.6, we deduce that if the $H^1$-norm of the solution $u$ to (1.1) is sufficiently small then $u$ is a *global* solution, see Corollary 2.8. Indeed, our aim is to find sufficient conditions in order to determine from the initial data $u_0 \in H^1(\mathbb{R}^2)$ whether the solution blows up in finite time (i.e. $T_* < +\infty$) or the solution is global in time (i.e. $T_* = +\infty$). The same problem for nonlinear hyperbolic and parabolic equations with polynomial nonlinearities has been widely studied via the potential well argument starting from the seminal papers by Sattinger [35], Tsutsumi [39], Ishii [21], and Payne and Sattinger [30]. Let us recall the central idea of this method in the parabolic case following the presentation given in [31].

**Polynomial case.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a smooth bounded domain, and let us consider

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0 & \text{in } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

with $1 < p \leq 2^* - 1$, and $2^* = \frac{2N}{N-2}$. For any initial data in the energy space $H^1_0(\Omega)$, there exists some finite time $T > 0$ and a local in time solution $u$ belonging to $\mathcal{C}\big( [0,T]; H^1_0(\Omega) \big)$ (this is a consequence of the $L^{p+1}$-existence result in [4] for any $1 < p \leq 2^* - 1$, and of the smoothing effect of the heat kernel). In this framework, the energy functional is given by

$$I_p(v) := \frac{1}{2}\|\nabla v\|^2_{L^2} - \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}}.$$

Let $v \in H_0^1(\Omega) \setminus \{0\}$, and let us analyze the energy of the function $\sigma v$ for any $\sigma \geq 0$. By an easy computation, one can show that

$$I_p(\sigma v) = \frac{\sigma^2}{2} \|\nabla v\|_{L^2}^2 - \frac{\sigma^{p+1}}{p+1} \|v\|_{L^{p+1}}^{p+1}$$

attains its unique maximum at a point $\bar{\sigma} = \bar{\sigma}(v) > 0$, and $\bar{v} := \bar{\sigma} v$ satisfies

$$\|\nabla \bar{v}\|_{L^2}^2 - \|\bar{v}\|_{L^{p+1}}^{p+1} = 0.$$

Therefore, the energy $I(\sigma v)$ has the structure of a potential well, and every ray $\sigma v$, for any $\sigma > 0$ and for $v \in H_0^1(\Omega) \setminus \{0\}$, has a *unique* intersection with the *Nehari manifold*

$$N = \{v \in H_0^1(\Omega) \setminus \{0\} \; : \; \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} = 0\}.$$

The depth of the well is given by the lowest pass over the ridge defined by all possible maps $\sigma \mapsto I_p(\sigma v)$ as $v$ ranges over $H_0^1(\Omega) \setminus \{0\}$, namely

$$d_p := \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \max_{\sigma \geq 0} I_p(\sigma v).$$

It is well known that $d_p$ can be characterized as

$$d_p = \inf_{v \in N} \; I_p(v), \qquad \text{and also} \qquad d_p = \frac{p-1}{2(p+1)} \Lambda^{2(p+1)/(p-1)},$$

where $\Lambda = \Lambda_{p+1}(\Omega)$ is the best constant in the Sobolev embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, i.e.

$$\Lambda = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla v\|_{L^2}}{\|v\|_{L^{p+1}}}.$$

If $1 < p < 2^* - 1$ then $d_p$ is the energy level of ground state solutions.

The potential well associated with the Cauchy problem (1.2) is the set (*stable* set)

$$W_p := \Big\{ v \in H_0^1(\Omega) : \; I_p(v) < d_p, \; \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} > 0 \Big\} \cup \{0\},$$

and the exterior of the potential well (*unstable* set) is

$$V_p := \Big\{ v \in H_0^1(\Omega) \; : \; I_p(v) < d_p, \; \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} < 0 \Big\}.$$

The sets $V_p$ and $W_p$ are both invariant under the flow associated with (1.2). Concerning the stable set, if $1 < p < 2^* - 1$, any solution which enters the stable set $W_p$ exists globally in time. This result is a direct consequence of the fact that, in the subcritical case, the time $T$ of local existence of the solution to (1.2) depends only on the size of the norm of the initial data in $H_0^1(\Omega)$, and for any $v \in W_p$ the Dirichlet norm $\|\nabla v\|_{L^2}$ is uniformly bounded (see [39]). Similar results have also been proven for $p = 2^* - 1$, where the situation is different because the local existence time of the solution to (1.2) depends on the specific initial data rather than its *size* (see [21], [19], [22], and [23], [38]). On the other side, if $1 < p \leq 2^* - 1$ then any solution which intersects the unstable set $V_p$ blows up in finite time (see [30] and [21]). Related studies can be found in [29, 28, 6, 18]. For the case $p = 2^* - 1$ and $\Omega = \mathbb{R}^N$, $N \geq 3$, we refer to [17], and to the recent result [8] in which the authors completely describe the dynamics near the ground state.

Related results on the asymptotic behavior of solutions for the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u - u + |u|^{p-1}u & \text{in } (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

when $N \geq 3$ and for the subcritical $1 < p < 2^* - 1$ case can be found in [7, 9, 11, 12] and references therein. Our forthcoming work [24] also contains results for the problem above with subcritical and critical nonlinearity.

When $N = 2$ any power nonlinearity is allowed, and the critical nonlinearity seems to be of exponential type as in (1.1). In the same spirit of the previous results, we show that for the Cauchy problem (1.1) if the *energy is below the ground state level* the dichotomy between blow-up and global existence is determined by means of a potential well argument.

**The stationary problem.** It is not difficult to show that the stationary problem associated with (1.1), i.e.

$$-\Delta v + v = \lambda f(v) \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

has no *non-trivial* $H^1$-solution if $\lambda \geq \frac{1}{2\alpha_0}$. Therefore, from now on, we will assume

$$0 < \lambda < \frac{1}{2\alpha_0}. \tag{1.4}$$

The existence of ground state solutions for (1.3) with $\lambda$ in the range (1.4) is proven in [33]. From [33], we also know that the mountain pass level

$$c := \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} I\big(\gamma(s)\big), \quad \Gamma := \Big\{ \gamma \in \mathcal{C}\big([0,1];\, H^1(\mathbb{R}^2)\big) \,:\, \gamma(0) = 0,\, I\big(\gamma(1)\big) < 0 \Big\}. \tag{1.5}$$

coincides with the ground state energy level, and ground state solutions can be characterized as minimizers of $I$ on a suitable constraint, i.e.

$$c = \inf\Big\{ I(v) \,:\, v \in H^1(\mathbb{R}^2) \setminus \{0\},\, \frac{1}{2}\|v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^2} F(v)\, dx = 0 \Big\}. \tag{1.6}$$

Moreover,

$$0 < c < \frac{2\pi}{\alpha_0}. \tag{1.7}$$

Another useful characterization of the mountain pass level $c$ can be obtained by means of the Nehari functional

$$J(v) := \langle dI(v), v \rangle = \|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} v f(v)\, dx. \tag{1.8}$$

Let

$$d := \inf\Big\{ I(v) \,:\, v \in H^1(\mathbb{R}^2) \setminus \{0\},\, J(v) = 0 \Big\},$$

then the existence of a mountain pass solution $\overline{v} \in H^1(\mathbb{R}^2) \setminus \{0\}$ to (1.3) implies $I(\overline{v}) = c$ and $dI(\overline{v}) \equiv 0$; therefore, $d \leq I(\overline{v}) = c$. The opposite inequality also holds, hence

$$c = d, \tag{1.9}$$

and this can be deduced from the geometry of $J$ and $I$ in the energy space. In particular, (1.9) is a consequence of the following property which gives also the potential well structure of the energy functional $I$.

**Proposition 1.1.** *Assume that $\lambda$ is as in (1.4). For any $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, there exists a unique $\overline{\sigma} = \overline{\sigma}(v) > 0$ such that*

$$J(\sigma v) \begin{cases} > 0 & \text{if } 0 < \sigma < \overline{\sigma}, \\ = 0 & \text{if } \sigma = \overline{\sigma}, \\ < 0 & \text{if } \sigma > \overline{\sigma}. \end{cases} \tag{1.10}$$

*Moreover,*

$$\lim_{\sigma \to +\infty} I(\sigma v) = -\infty, \tag{1.11}$$

*and $\overline{\sigma}$ is the unique maximum point of the map $\sigma \mapsto I(\sigma v)$ on $[0, +\infty)$.*

The proof of the above Proposition 1.1 follows by simple computations (see also Lemma 7.4 and Lemma 7.5 with $a = 1$ and $b = 0$). From Proposition 1.1, it is easy to deduce that $c \leq d$ by comparison with the auxiliary level

$$\tilde{c} = \inf_{v \in H^1(\mathbb{R}^2) \setminus \{0\}} \sup_{\sigma > 0} I(\sigma v),$$

see Proposition 7.1 with $a = 1$ and $b = 0$.

**Stable and unstable sets.** In view of Proposition 1.1, for any fixed $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, the function $\sigma \mapsto I(\sigma v)$ has the shape of a potential well. The idea of the potential well method is to trap the solution to (1.1) in the well to the left of $\overline{\sigma}(v)$ in order to guarantee global existence. To ensure that the solution to (1.1) is trapped, we have to find the lowest pass over the ridge defined by all possible $I(\sigma v)$ as $v$ ranges over $H^1(\mathbb{R}^2) \setminus \{0\}$. The height of the lower pass over the ridge is the mountain pass level $\tilde{c}$, and $\tilde{c} = d$.

Therefore, the potential well argument suggests to consider the splitting of the $d$-sublevel set of the energy $I$ determined by the Nehari functional $J$. More precisely, we consider the unstable set $V$ and the stable set $W$ defined respectively by

$$V := \Big\{ v \in H^1(\mathbb{R}^2) \; : \; I(v) < d, \; J(v) < 0 \Big\},$$

and

$$\begin{aligned} W := & \Big\{ v \in H^1(\mathbb{R}^2) \; : \; I(v) < d, \; J(v) \geq 0 \Big\} \\ = & \Big\{ v \in H^1(\mathbb{R}^2) \; : \; I(v) < d, \; J(v) > 0 \Big\} \cup \{0\}. \end{aligned}$$

**Theorem 1.2.** *Let $u \in \mathcal{C}\big( [0, T_*); H^1(\mathbb{R}^2) \big)$ be the maximal solution to (1.1) with $\lambda$ as in (1.4), and $u_0 \in H^1(\mathbb{R}^2)$.*

*(i) If $u(t_0) \in V$ for some $t_0 \in [0, T_*)$ then $T_* < +\infty$.*

*(ii) There exists $t_0 \in [0, T_*)$ such that $u(t_0) \in W$ if and only if*

$$T_* = +\infty, \qquad \text{and} \qquad \lim_{t \to +\infty} \|u(t)\|_{H^1} = 0. \tag{1.12}$$

The first part of the above Theorem 1.2 complements the blow-up result obtained in [14] for non-positive energies.

**Theorem 1.3** ([14, Theorem 2.1.3])**.** *Let $u \in \mathcal{C}\big([0,T_*);H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1.1) *with*

$$0 < \lambda \leq \frac{1}{2\alpha_0}, \tag{1.13}$$

*and $u_0 \in H^1(\mathbb{R}^2)$. If $I\big(u(t_0)\big) \leq 0$ and $u(t_0) \neq 0$ for some $t_0 \in [0,T_*)$ then $T_* < +\infty$.*

The above Theorem 1.3 is proved in [14] in the particular case $\alpha_0 = 1$ and $\lambda = \frac{1}{2}$, but the arguments of the proof in [14] can be adapted to cover the general case with $\alpha_0 > 0$ and $\lambda$ in the range (1.13), see Remark 4.4.

Up to our knowledge, Theorem 1.2 is a *new* application of the potential well argument to heat equations with *critical* exponential nonlinearities in the two space dimensional case. The same problem with *subcritical* exponential nonlinearities is studied in [10] and [34].

It is important to mention that similar results for dispersive equations are already available in the literature, for example: see [2] and [25] for the subcritical exponential case, and see [16] for the critical exponential case.

Differently from the dispersive framework, the energy associated with heat equations decreases along solutions, and this monotonicity property enables us to easily determine the dichotomy between blow-up and global existence also at the ground state energy level $d$.

**Theorem 1.4.** *Let $u \in \mathcal{C}\big([0,T_*);H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1.1) *with $\lambda$ as in* (1.4), *and $u_0 \in H^1(\mathbb{R}^2)$. Assume that $I\big(u(t_0)\big) = d$ for some $t_0 \in [0,T_*)$.*

- *(i) If $J\big(u(t_0)\big) < 0$ then $u(t) \in V$ for any $t \in (t_0,T_*)$.*
- *(ii) If $J\big(u(t_0)\big) > 0$ then $u(t) \in W$ for any $t \in (t_0,T_*)$.*
- *(iii) If $J\big(u(t_0)\big) = 0$ then $u(t_0)$ is a stationary ground state solution, and $u(t) = u(t_0)$ for any $t \in [t_0,+\infty)$.*

**Outline of the paper.** In Section 2, we discuss the dependence of the local existence time of the solution to (1.1) from the $H^1$-norm of the initial data, and we obtain a sufficient condition for global existence (see Corollary 2.8).

In Section 3, we collect some basic properties of the solution to (1.1) which will be crucial to prove the instability of the set $V$, and the stability of the set $W$.

Section 4 is devoted to the study of the unstable set $V$, and more precisely to the proof of Theorem 1.2(i). The proof is based on the classical concavity method due to Levine [26] (see Lemma 4.2) which applies to (1.1) due to the fact that the Nehari functional $J$ along solutions entering $V$ is – *not only* negative but – bounded away from zero by a negative constant (see Proposition 4.3).

Section 5 is devoted to the study of the stable set $W$, and more precisely to the proof of Theorem 1.2(ii). This second part of the statement of Theorem 1.2 is more accurate with respect to the first part concerning the instability in $V$, in fact Theorem 1.2(ii) gives a characterization of $W$ in terms of the necessary and sufficient condition (1.12).

The proof of the stability of the set $W$ is mainly based on Corollary 2.8. In order to show that solutions entering $W$ satisfy the assumptions of Corollary 2.8, it was important to realize that the following inclusion holds:

$$W \subseteq \Big\{v \in H^1(\mathbb{R}^2) \;:\; I(v) < d,\; P(v) \geq 0\Big\}, \tag{1.14}$$

where

$$P(v) := \frac{1}{2}\|v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^2} F(v)\,dx \tag{1.15}$$

is the so-called Pohozaev functional, i.e. the functional appearing in the characterization of the ground state energy level (1.6), as developed in [33]. The validity of the inclusion (1.14) is the idea underlying the argument of the proof of Proposition 5.2.

The positivity of the Nehari functional $J$ near the origin of $H^1(\mathbb{R}^2)$ (see Theorem 5.4) is crucial to show that (1.12) is a *sufficient* condition for the solution to (1.1) to enter $W$.

To show that the $H^1$-norm of solutions entering $W$ must decay to zero as time tends to infinity, we need a compactness result, see Proposition 5.6. In the proof of Proposition 5.6, we will consider the following auxiliary growth functions

$$\tilde{f}(u) := 2\alpha_0 u(e^{\alpha_0 u^2} - 1), \quad \text{and} \quad \tilde{F}(u) := e^{\alpha_0 u^2} - 1 - \alpha_0 u^2, \tag{1.16}$$

satisfying

$$f(u) = \tilde{f}(u) + 2\alpha_0 u, \quad \text{and} \quad F(u) = \tilde{F}(u) + \alpha_0 u^2.$$

Both $u\tilde{f}(u)$ and $\tilde{F}(u)$ are *critical* in the energy space with respect to the Trudinger-Moser inequality [1, 32], but these auxiliary growth functions are not affected by the lack of compactness at spatial infinity (i.e. the lack of compactness of the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$), in fact

$$\lim_{u\to 0} \frac{u\tilde{f}(u)}{u^2} = \lim_{u\to 0} \frac{\tilde{F}(u)}{u^2} = 0.$$

The description of the asymptotics at the ground state energy level given by Theorem 1.4 is developed in Section 6.

The validity of (1.14) may raise questions about the role of the splitting of the $d$-sublevel set of the energy $I$ determined by the sign of the Pohozaev functional $P$ with respect to the flow associated with the Cauchy problem (1.1). Indeed, the Pohozaev functional $P$ and the Nehari functional $J$ determine the *same* splitting below the ground state energy level $d$, as already observed in [16], see also [25]. In [16], a slightly different critical exponential nonlinearity is considered, and in Section 7, we show that the argument in [16] can be adapted to the energy functional associated with (1.1).

# 2 Uniform local existence time and blow-up alternative

Let $\Omega \subseteq \mathbb{R}^2$ be any smooth domain, and let us consider the more general Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + g(u) & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $u_0 \in H_0^1(\Omega)$, and $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfies

($g_1$) $g(0) = 0$, and

($g_2$) there exists $\alpha_0 > 0$ such that for any $\varepsilon > 0$ we have

$$|g(s_1) - g(s_2)| \le C_\varepsilon |s_1 - s_2| \big( e^{\alpha_0(1+\varepsilon)s_1^2} + e^{\alpha_0(1+\varepsilon)s_2^2} \big), \quad s_1,\ s_2 \in \mathbb{R},$$

for some positive constant $C_\varepsilon$.

Under the above assumptions on the nonlinear term $g$, the Cauchy problem (2.1) includes the model problem (1.1) as a particular case. Note that, by assuming condition $(g_2)$, we take into account nonlinear terms with *square* exponential growth at infinity, which are *critical* in the energy space. For any initial data in $H^1_0(\Omega)$, the argument introduced in [14] gives the local existence and uniqueness of the solution $u$ to (2.1) in the class of functions $\mathcal{C}\big([0,T]; H^1_0(\Omega)\big)$, for some $T > 0$.

**Definition 2.1.** Let $u_0 \in H^1_0(\Omega)$. We say that $u$ is a (mild) solution to (2.1) if $u \in \mathcal{C}\big([0,T]; H^1_0(\Omega)\big)$, and $u$ verifies the integral equation

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} g(u(s))ds.$$

**Remark 2.2.** As proved in [20, Proposition 4.1] and [13], $u$ is a (mild) solution to (2.1) if and only if $u$ satisfies

$$\partial_t u = \Delta u + g(u)$$

in the sense of distributions.

Combining the arguments of [14] with [20, Remark 4.1], we have the following result.

**Theorem 2.3** ([14], [20])**.** *Let $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$ satisfy $(g_1)$ and $(g_2)$, and assume $u_0 \in H^1_0(\Omega)$. There exist $T = T(u_0) > 0$ and a unique solution $u \in \mathcal{C}\big([0,T]; H^1_0(\Omega)\big)$ to* (2.1)*. Moreover,*

$$u \in L^\infty_{loc}\big((0,T]; L^\infty(\mathbb{R}^2)\big).$$

Let us introduce the maximal existence time of the solution $u$ to (2.1) as

$$T_* := \sup\Big\{ T > 0 \ : \ \text{the problem (2.1) admits a solution } u \in \mathcal{C}\big([0,T]; H^1_0(\Omega)\big) \Big\} \in (0, +\infty]. \quad (2.2)$$

Under the assumptions of Theorem 2.3, if the maximal existence time defined by (2.2) satisfies $T_* < +\infty$ then

$$\limsup_{t \to T_*} \|u(t)\|_{L^\infty} = +\infty,$$

see e.g. [5, Section 5.3].

In view of the definition of $T_*$, the following question arises:

$$\text{Does } T_* < +\infty \text{ imply } \limsup_{t \to T_*} \|u(t)\|_{H^1} = +\infty? \quad (2.3)$$

The above question remains *open*: the *critical* exponential nonlinearity that we consider could have an effect on the blow-up alternative (2.3), and in analogy with the *critical* polynomial case (see [36]), our guess is that there could exist initial data in $H^1_0(\Omega)$ for which $T_* < +\infty$ even if $u \in L^\infty\big([0,T_*), H^1_0(\Omega)\big)$.

As mentioned in Section 1, the above question about the blow-up alternative (2.3) is related to the dependence of the local existence time of the solution to (1.1) from the *size* of the initial data in $H^1_0(\Omega)$: if one could find a local existence time $T > 0$ which is *uniform* with respect to the $H^1$-norm of the initial data, i.e. $T = T(\|u_0\|_{H^1})$, then the blow-up alternative (2.3) would hold.

To explain this point of view, let us compare the energy *critical* problem with the *subcritical* and *supercritical* cases. To take into account nonlinear terms with *subcritical* or *supercritical* growth in the energy space, it is enough to replace $(g_2)$ respectively with:

$(g_2)_{sub}$ for any $\alpha_0 > 0$ there exists $C_{\alpha_0} > 0$ such that

$$|g(s_1) - g(s_2)| \le C_{\alpha_0} |s_1 - s_2| \big( e^{\alpha_0 s_1^2} + e^{\alpha_0 s_2^2} \big), \quad s_1,\ s_2 \in \mathbb{R};$$

$(g_2)_{sup}$ there exists $\gamma > 2$ and $\alpha_0 > 0$ such that

$$\liminf_{s\to+\infty} \frac{|g(s)|}{e^{\alpha_0 s^\gamma}} > 0.$$

The subcritical, critical or supercritical behavior of $g$ affects the local existence time of the solution to the Cauchy problem (2.1). In the *supercritical* case, we have a non-existence result for (2.1).

**Theorem 2.4.** *Let $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfy $(g_1)$ and $(g_2)_{sup}$, and assume that $g \geq 0$ on $\mathbb{R}$. There exists $u_0 \in H^1(\mathbb{R}^2)$, $u_0 \geq 0$, such that for any $T > 0$ the Cauchy problem* (2.1) *has* no *nonnegative solution in $\mathcal{C}\big([0, T), H^1(\mathbb{R}^2)\big) \cap L^\infty_{loc}\big((0, T), L^\infty(\mathbb{R}^2)\big)$.*

*Proof.* Let $\gamma > 2$ be as in $(g_2)_{sup}$, and define

$$u_0(x) := \begin{cases} \left(\log \frac{1}{|x|}\right)^{\frac{1}{\gamma}} \log\left(\log \frac{1}{|x|}\right) & |x| \leq \frac{1}{e}, \\ 0 & |x| > \frac{1}{e}. \end{cases}$$

Then $u_0 \in H^1_0(B_{1/e}(0))$, and arguing as in [20, Section 3] it is not difficult to deduce non-existence. □

In the *subcritical* case, the solution to (2.1) exists up to some finite time which depends only on the *size* of the initial data in $H^1_0(\Omega)$.

**Theorem 2.5.** *Let $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfy $(g_1)$ and $(g_2)_{sub}$, and let $M > 0$. There exists $T = T(M) > 0$ such that, for any $u_0 \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} \leq M$, the Cauchy problem* (2.1) *has a unique solution $u \in \mathcal{C}\big(\,[0, T]; H^1_0(\Omega)\big)$.*

We omit the proof of the above Theorem 2.5, since it can be obtained by means of a standard fixed point argument by exploiting the integral representation formula and the smoothing effect of the heat kernel (see also the proof of the following Theorem 2.6). In the *subcritical* case, it is clear that the blow-up alternative holds:

$$\text{if } T_* < +\infty \text{ then } \limsup_{t\to T_*} \|u(t)\|_{H^1} = +\infty. \tag{2.4}$$

Indeed, if not we could extend the solution to (2.1) beyond the time $T_* < +\infty$, using Theorem 2.5, and reach a contradiction.

In the *critical* case, from [14], we *cannot* deduce that the local existence time $T > 0$ is bounded away from zero by a positive constant depending only on the $H^1$-norm of the initial data, and we expect that the smallness of the local existence time $T$ depends on the specific initial data and *not* only on its *size*. Nevertheless, if we consider *small* initial data, we can find a *uniform* local existence time for the solution to (2.1).

**Theorem 2.6.** *Let $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfy $(g_1)$ and $(g_2)$. Let $0 < m < \frac{4\pi}{\alpha_0}$, and $M > 0$. There exists $T = T(m, M) > 0$ such that, for any $u_0 \in H^1(\mathbb{R}^2)$ with*

$$\|\nabla u_0\|^2_{L^2} \leq m \quad \textit{and} \quad \|u_0\|^2_{L^2} \leq M, \tag{2.5}$$

*the Cauchy problem* (1.1) *has a unique solution $u \in \mathcal{C}\big(\,[0, T]; H^1_0(\Omega)\big)$.*

The *smallness* condition (2.5) with $0 < m < \frac{4\pi}{\alpha_0}$ comes from the following scale invariant form of the Trudinger-Moser inequality in $H^1(\mathbb{R}^2)$.

**Theorem 2.7** ([1])**.** *If $\alpha \in (0, 4\pi)$ then there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^2} (e^{\alpha v^2} - 1)\, dx \le C_\alpha \|v\|_{L^2}^2, \quad \text{for any } v \in H^1(\mathbb{R}^2) \text{ with } \|\nabla v\|_{L^2} \le 1, \tag{2.6}$$

*and the above inequality* fails *if $\alpha \ge 4\pi$.*

*Proof of Theorem 2.6.* In order to prove the existence of a unique solution $u \in \mathcal{C}\big([0,T]; H_0^1(\Omega)\big)$, let us first write the equation in (1.1) in the equivalent integral formulation (see [20, Proposition 4.1] and [13] for a justification of this equivalence)

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} g(u(s)) ds. \tag{2.7}$$

Since $0 < m < \frac{4\pi}{\alpha_0}$ there exists $\varepsilon \in (0,1)$ such that $m = \frac{4\pi}{\alpha_0}(1-\varepsilon)$. Let us consider the set

$$X = X(m,M) = \left\{ u \in L^\infty\big((0,T), H^1(\mathbb{R}^2)\big) : \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 \le \frac{4\pi}{\alpha_0}\left(1 - \frac{\varepsilon}{2}\right);\ \sup_{t\in[0,T]} \|u(t)\|_{L^2}^2 \le 2M \right\}.$$

This set endowed with the distance

$$d(u,w) = \sup_{t\in[0,T]} \|\nabla u(t) - \nabla w(t)\|_{L^2} + \sup_{t\in[0,T]} \|u(t) - w(t)\|_{L^2}$$

is a complete metric space. We show that if $T > 0$ is small enough the map

$$(\Phi u)(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} g(u(s)) ds$$

is a contraction from $X$ into itself. We remark that $u_0$ satisfies

$$\|\nabla u_0\|_{L^2} \le \sqrt{m} = \sqrt{\frac{4\pi}{\alpha_0}(1-\varepsilon)} < \sqrt{\frac{4\pi}{\alpha_0}(1-\frac{\varepsilon}{2})}, \qquad \|u_0\|_{L^2} \le \sqrt{M}.$$

Let us first prove that $\Phi$ maps $X$ into itself. Indeed, thanks to property $(g_2)$ for any $u \in X(m,M)$ and for any $t \in [0,T]$ we obtain

$$\begin{aligned}
\|(\Phi u)(t)\|_{L^2} &\le \|e^{t\Delta} u_0\|_{L^2} + \int_0^t \|e^{(t-s)\Delta} g(u(s))\|_{L^2} ds \\
&\le \|u_0]\|_{L^2} + C_\varepsilon \int_0^t \left\| e^{(t-s)\Delta} \left[ |u(s)| \left( e^{\alpha_0(1+\varepsilon)u^2(s)} + 1 \right) \right] \right\|_{L^2} ds \\
&\le \sqrt{M} + 2C_\varepsilon \int_0^t \|e^{(t-s)\Delta} |u(s)|\|_{L^2} ds + C_\varepsilon \int_0^t \left\| e^{(t-s)\Delta} \left[ |u(s)| \left( e^{\alpha_0(1+\varepsilon)u^2(s)} - 1 \right) \right] \right\|_{L^2} ds \\
&\le \sqrt{M} + 2C_\varepsilon t \sqrt{2M} + C_{\varepsilon,r} \int_0^t \frac{1}{(t-s)^{\frac{1}{r}-\frac{1}{2}}} \left\| |u(s)| \left( e^{\alpha_0(1+\varepsilon)u^2(s)} - 1 \right) \right\|_{L^r} ds,
\end{aligned}$$

where $1 < r < 2$ will be chosen later. If we could prove that there exists a constant $C = C(m,M)$ such that for any $s \in [0,T]$ we have

$$\left\| u(s) \left( e^{\alpha_0(1+\varepsilon)u^2(s)} - 1 \right) \right\|_{L^r}^r \le C \tag{2.8}$$

then we would obtain

$$\begin{aligned}\|(\Phi u)(t)\|_{L^2} &\le \sqrt{M} + 2C_\varepsilon t\sqrt{2M} + \tilde{C}_{\varepsilon,r}\int_0^t \frac{1}{(t-s)^{\frac{1}{r}-\frac{1}{2}}}ds \\ &= \sqrt{M} + 2C_\varepsilon t\sqrt{2M} + \tilde{C}_{\varepsilon,r} t^{3/2-1/r}.\end{aligned}$$

Therefore if $T$ is sufficiently small depending only on $m, M$ then

$$\|(\Phi u)(t)\|_{L^2} \le \sqrt{2M}.$$

The estimate (2.8) can be obtained via the scale invariant Trudinger-Moser inequality (2.6). Indeed, for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and provided that $rp \ge 2$ we have

$$\begin{aligned}\|u(s)\left(e^{\alpha_0(1+\varepsilon)u^2(s)} - 1\right)\|_{L^r}^r &\le \int_{\mathbb{R}^2} |u(s)|^r \left(e^{\alpha_0 r(1+\varepsilon)u^2(s)} - 1\right)dx \\ &\le \left(\int_{\mathbb{R}^2} |u(s)|^{rp}dx\right)^{1/p}\left(\int_{\mathbb{R}^2}\left(e^{\alpha_0 rq(1+\varepsilon)u^2(s)} - 1\right)dx\right)^{1/q} \\ &\le C\|u(s)\|_{H^1}^r \left(\int_{\mathbb{R}^2}\left(e^{\alpha_0 rq(1+\varepsilon)u^2(s)} - 1\right)dx\right)^{1/q}.\end{aligned}$$

Now choosing $q = 1 + \varepsilon^2$, $r = 1 + \varepsilon^4$, and since $\alpha_0 = \frac{4\pi(1-\varepsilon)}{m}$, we can estimate

$$\begin{aligned}\left(\int_{\mathbb{R}^2}\left(e^{\alpha_0 rq(1+\varepsilon)u^2(s)} - 1\right)dx\right)^{1/q} &= \left(\int_{\mathbb{R}^2}\left(e^{4\pi(1-\varepsilon^8)\left(\frac{u(s)}{\sqrt{m}}\right)^2} - 1\right)dx\right)^{1/q} \\ &\le C\left(\frac{\|u(s)\|_{L^2}^2}{m}\right)^{1/q} \le C\left(\frac{2M}{m}\right)^{1/q}.\end{aligned}$$

We remark that with this choice of $q$ we obtain $p = \frac{1+\varepsilon^2}{\varepsilon^2}$ and $rp \ge 2$. Therefore, we obtain

$$\left\|u(s)\left(e^{\alpha_0(1+\varepsilon)u^2(s)} - 1\right)\right\|_{L^r}^r \le C.$$

Next for any $u \in X$ and for any $t \in [0, T]$, thanks to (2.8), we obtain

$$\begin{aligned}\|\nabla(\Phi u)(t)\|_{L^2} &\le \|e^{t\Delta}\nabla u_0\|_{L^2} + \int_0^t \|\nabla e^{(t-s)\Delta} g(u(s))\|_{L^2} ds \\ &\le \|\nabla u_0]\|_{L^2} + C_\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{\alpha_0(1+\varepsilon)u^2(s)} + 1\right)\right]\right\|_{L^2} ds \\ &\le \sqrt{m} + 2C_\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\|\sqrt{t-s}\nabla e^{(t-s)\Delta}|u(s)|\|_{L^2} ds \\ &+ C_\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{\alpha_0(1+\varepsilon)u^2(s)} - 1\right)\right]\right\|_{L^2} ds \\ &\le \sqrt{m} + 2C_\varepsilon\sqrt{t}\sqrt{2M} + C_{\varepsilon,r}\int_0^t \frac{1}{(t-s)^{\frac{1}{r}}}\left\||u(s)|\left(e^{\alpha_0(1+\varepsilon)u^2(s)} - 1\right)\right\|_{L^r} ds \\ &\le \sqrt{m} + 2C_\varepsilon\sqrt{t}\sqrt{2M} + \tilde{C}_{\varepsilon,r}\int_0^t \frac{1}{(t-s)^{\frac{1}{r}}}ds \\ &\le \sqrt{m} + 2C_\varepsilon\sqrt{t}\sqrt{2M} + \tilde{C}_{\varepsilon,r} t^{1-\frac{1}{r}},\end{aligned}$$

with the same $1 < r < 2$ chosen above. Therefore if $T$ is sufficiently small depending only on $m, M$ then

$$\sup_{t\in[0,T]} \|\nabla(\Phi u)(t)\|_{L^2} \leq \sqrt{\frac{4\pi}{\alpha_0}\Big(1 - \frac{\varepsilon}{2}\Big)}.$$

In a similar way it is possible to prove that for $T = T(m, M)$ small enough the map $\Phi$ is a contraction on $X$. Finally by using the standard regularizing properties of the heat kernel it is possible to prove that the fixed point $u \in X$ of $\Phi$ satisfies $u \in \mathcal{C}\big([0,T], H^1(\mathbb{R}^2)\big)$. □

From Theorem 2.6, we deduce a sufficient condition for global existence.

**Corollary 2.8.** *Let $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfy $(g_1)$ and $(g_2)$, and let $u_0 \in H^1_0(\Omega)$. If the maximal solution $u \in \mathcal{C}\big(\,[0,T_*); H^1(\mathbb{R}^2)\big)$ to (2.1) satisfies*

$$\sup_{t\in[t_0,T_*)} \|\nabla u(t)\|^2_{L^2} < \frac{4\pi}{\alpha_0}, \quad \textit{and} \quad \sup_{t\in[t_0,T_*)} \|u(t)\|^2_{L^2} < +\infty, \quad \textit{for some } \bar{t}_0 \in [0,T_*),$$

*then $u$ is global in time, i.e. $T_* = +\infty$.*

*Proof.* As in the subcritical case, if we assume $T_* < +\infty$ then we can apply Theorem 2.6 to extend the solution $u$ beyond the maximal existence time $T_*$, and reach a contradiction. □

# 3 Basic properties of the solution to the model problem (1.1)

Let $u_0 \in H^1(\mathbb{R}^2)$. Let $u \in \mathcal{C}\big([0,T]; H^1(\mathbb{R}^2)\big)$ be the local in time solution to (1.1) found in [14], where $T = T(u_0) > 0$ is the local time of existence. We already pointed out that $u \in L^\infty_{loc}\big((0,T]; L^\infty(\mathbb{R}^2)\big)$, see [20, Remark 4.1]. Moreover, by using the integral formulation of the equation and the growth property of the nonlinearity it is possible to prove (see also [5, Chapter 5]) that

$$\Delta u \in \mathcal{C}\big(\,(0,T],\ L^2(\mathbb{R}^2)\,\big), \quad \text{and} \quad u \in \mathcal{C}^1\big(\,(0,T],\ L^2(\mathbb{R}^2)\,\big).$$

Then by standard arguments, $u$ is a *classical* solution for (1.1), i.e.

$$u \in \mathcal{C}^{1,2}\big(\,(0,T) \times \mathbb{R}^2\,\big).$$

**Proposition 3.1.** *For any $t \in (0,T)$, we have*

$$\|\partial_t u(t)\|^2_{L^2} = -\frac{d}{dt} I\big(u(t)\big), \tag{3.1}$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = -\|u(t)\|^2_{H^1} + \lambda \int_{\mathbb{R}^2} u(t) f\big(u(t)\big)\, dx, \tag{3.2}$$

*and*

$$|\langle dI\big(u(t)\big), \varphi\rangle| \leq \|\partial_t u(t)\|_{L^2}\|\varphi\|_{L^2}, \quad \textit{for any } \varphi \in H^1(\mathbb{R}^2). \tag{3.3}$$

*Proof.* The monotonicity of the energy (3.1) follows by multiplying the equation in (1.1) by $\partial_t u$, integrating over $\mathbb{R}^2$, and applying density arguments as in [5, Lemma 5.4.5].

Since $u \in \mathcal{C}\big(\,[0,T]; H^1(\mathbb{R}^2)\,\big) \cap \mathcal{C}^1\big(\,(0,T]; L^2(\mathbb{R}^2)\,\big)$, and $\Delta u \in \mathcal{C}\big(\,(0,T],\ L^2(\mathbb{R}^2)\,\big)$, (3.2) follows by multiplying the equation in (1.1) by $u$ and integrating over $\mathbb{R}^2$. Finally, to deduce (3.3) we multiply the equation in (1.1) by $\varphi \in H^1(\mathbb{R}^2)$, and we integrate over $\mathbb{R}^2$, obtaining

$$\int_{\mathbb{R}^2} \partial_t u(t)\varphi\, dx = -\int_{\mathbb{R}^2} \big(\nabla u(t) \cdot \nabla\varphi + u(t)\varphi\big)\, dx + \lambda \int_{\mathbb{R}^2} f\big(u(t)\big)\varphi\, dx = -\langle dI\big(u(t)\big), \varphi\rangle.$$

□

We complete this section with the following continuity result that can be proven arguing as in [14, Proposition 3.6].

**Lemma 3.2.** *If $T > 0$ and $u \in \mathcal{C}\big( [0,T], H^1(\mathbb{R}^2) \big)$ then*

$$F(u) \in \mathcal{C}\big( [0,T], L^1(\mathbb{R}^2) \big), \quad \textit{and} \quad uf(u) \in \mathcal{C}\big( [0,T], L^1(\mathbb{R}^2) \big).$$

*Hence, $J(u) \in \mathcal{C}\big( [0,T], \mathbb{R} \big)$.*

# 4 Blow-up in $V$

If $v \in V$, and $\overline{\sigma} = \overline{\sigma}(v) > 0$ is given by Proposition 1.1, then $\overline{\sigma} \in (0,1)$, and hence

$$2d \le 2I(\overline{\sigma} v) \le \|\overline{\sigma} v\|_{H^1}^2 < \|v\|_{H^1}^2. \tag{4.1}$$

To prove the invariance of the set $V$ under the flow associated with (1.1), it is crucial to recall that, from (1.7) and (1.9), we know that

$$d > 0. \tag{4.2}$$

**Lemma 4.1.** *Let $u \in \mathcal{C}\big( [0,T_*); H^1(\mathbb{R}^2) \big)$ be the maximal solution to* (1.1) *with $\lambda$ as in* (1.4)*, and $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in V$ for some $t_0 \in [0,T_*)$ then $u(t) \in V$ for any $t \in [t_0, T_*)$.*

*Proof.* In view of the monotonicity of the energy (3.1), and since $J(u) \in \mathcal{C}\big([0,T_*); \mathbb{R}\big)$, it is enough to prove that $J\big(u(t)\big) \neq 0$ for any $t \in (t_0, T_*)$. If $J\big(u(t)\big) = 0$ for some $t \in (t_0, T_*)$ then there exists $t_1 \in (t_0, T_*)$ such that

$$J\big(u(t)\big) < 0 \text{ for any } t \in [t_0, t_1), \text{ and } J\big(u(t_1)\big) = 0.$$

Therefore, $u(t) \in V$ for any $t \in [t_0, t_1)$, and

- either $u(t_1) \neq 0$. Hence $d \le I\big(u(t_1)\big)$, which is *not* possible due to the monotonicity of the energy (3.1);
- or $u(t_1) = 0$ which yields
$$\lim_{t \to t_1^-} \|u(t)\|_{H^1} = \|u(t_1)\|_{H^1} = 0,$$
  and this contradicts (4.1) and (4.2).

□

In order to prove that solutions entering $V$ blow up in finite time, we will apply the following blow-up Lemma containing the classical idea of the concavity method due to Levine [26].

**Lemma 4.2** ([26])**.** *There exists no non-negative and increasing function $y \in \mathcal{C}^2(\bar{t}, +\infty)$, with $\bar{t} \in \mathbb{R}$, such that, for some $\beta > 0$,*

$$y(t)y''(t) \ge (\beta + 1)[y'(t)]^2 \ \textit{on} \ (\bar{t}, +\infty),$$

*and*

$$\lim_{t \to +\infty} y(t) = +\infty. \tag{4.3}$$

*Proof.* For the sake of completeness, we briefly sketch the proof. By contradiction, we assume that such a function $y$ exists. In view of (4.3), $h(t) := y^{-\beta}(t)$ is well defined on the half-line $(t', +\infty)$, for some $t' \geq \bar{t}$ sufficiently large. Moreover,

$$\lim_{t \to +\infty} h(t) = 0. \tag{4.4}$$

For any $t > t'$, we can compute

$$h'(t) = -\beta [y(t)]^{-\beta-1} y'(t) < 0,$$

and

$$h''(t) = \beta [y(t)]^{-\beta-2} \Big( (\beta+1)[y'(t)]^2 - y(t) y''(t) \Big) \leq 0.$$

Therefore, $h$ is concave and decreasing on $(t', +\infty)$, and this contradicts (4.4). □

The concavity method works in our setting due to the fact the Nehari functional along solutions entering $V$ is bounded away from zero by a strictly negative constant.

**Proposition 4.3.** *Let $u \in \mathcal{C}\big([0, T_*); H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1.1) *with $\lambda$ as in* (1.4)*, and $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in V$ for some $t_0 \in [0, T_*)$ then there exists $\varepsilon > 0$ such that $J\big(u(t)\big) < -\varepsilon$ for any $t \in [t_0, T_*)$.*

*Proof.* Let

$$d' := \inf \Big\{ \, H(v) \; : \; v \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) \leq 0 \, \Big\},$$

where

$$H(v) := I(v) - \frac{1}{2} J(v) = \lambda \int_{\mathbb{R}^2} \left( \frac{1}{2} v f(v) - F(v) \right) dx. \tag{4.5}$$

Then $d = d'$. In fact, clearly $d' \leq d$, and in order to deduce that $d \leq d'$, it is enough to show that

$$d \leq H(v) \quad \text{for any } v \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ with } J(v) < 0.$$

Let $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, and let $\overline{\sigma} = \overline{\sigma}(v) > 0$ be as in Proposition 1.1. If $J(v) < 0$, then $\overline{\sigma} v \in (0, 1)$, and we can estimate

$$\begin{aligned}
d \leq I(\overline{\sigma} v) &= \lambda \int_{\mathbb{R}^2} \left( \frac{1}{2} \overline{\sigma} v f(\overline{\sigma} v) - F(\overline{\sigma} v) \right) dx = \lambda \sum_{k=2}^{+\infty} \frac{\alpha_0^k}{k!} (k-1) \|\overline{\sigma} v\|_{L^{2k}}^{2k} \\
&= \lambda \sum_{k=2}^{+\infty} \frac{\alpha_0^k}{k!} (k-1) \overline{\sigma}^{\,2k} \|v\|_{L^{2k}}^{2k} \leq \lambda \sum_{k=2}^{+\infty} \frac{\alpha_0^k}{k!} (k-1) \|v\|_{L^{2k}}^{2k} \\
&= \lambda \int_{\mathbb{R}^2} \left( \frac{1}{2} v f(v) - F(v) \right) dx = H(v).
\end{aligned}$$

With the above characterization of $d$, it is easy to show that for any $\varepsilon > 0$

$$d_\varepsilon := \inf \Big\{ \, I(v) \; : \; v \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) = -\varepsilon \, \Big\} \geq d - \frac{\varepsilon}{2}. \tag{4.6}$$

In fact, by direct computations

$$\begin{aligned}
d_\varepsilon &= \inf \Big\{ \, I(v) + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \; : \; v \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) = -\varepsilon \, \Big\} \\
&= \inf \Big\{ \, I(v) - \frac{1}{2} J(v) - \frac{\varepsilon}{2} \; : \; u \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) = -\varepsilon \, \Big\} \\
&= \inf \Big\{ \, I(v) - \frac{1}{2} J(v) \; : \; v \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) = -\varepsilon \, \Big\} - \frac{\varepsilon}{2} \\
&= \inf \Big\{ \, H(v) \; : \; v \in H^1(\mathbb{R}^2) \setminus \{0\}, \; J(v) = -\varepsilon \, \Big\} - \frac{\varepsilon}{2} \geq d - \frac{\varepsilon}{2}.
\end{aligned}$$

Next, we assume that the maximal solution $u$ to (1.1) satisfies $u(t_0) \in V$ for some $t_0 \in [0, T_*)$. Then there exists $\varepsilon > 0$ such that

$$\min\{d - I(u(t_0)),\ -J(u(t_0))\} > \varepsilon.$$

In view of (4.6) and the monotonicity of the energy (3.1), we get

$$d_\varepsilon \geq d - \frac{\varepsilon}{2} > I(u(t_0)) \geq I(u(t)), \quad \text{for any } t \in [t_0, T_*). \tag{4.7}$$

Assume that $J(u(t_1)) = -\varepsilon$ for some $t_1 \in (t_0, T_*)$. Then $d_\varepsilon \leq I(u(t_1))$, which contradicts (4.7).

Summarizing, we have $J(u(t_0)) < -\varepsilon$, and $J(u(t)) \neq -\varepsilon$ for any $t \in [t_0, T_*)$. Therefore, the proof is complete in view of the continuity of $J$ along the solution, see Lemma 3.2. □

*Proof of Theorem 1.2.(i).* We argue by contradiction assuming that the solution $u$ is *global*, i.e. $T_* = +\infty$, and we apply the blow-up Lemma 4.2 to the non-negative and increasing $\mathcal{C}^2$-function defined by

$$y(t) := \frac{1}{2}\int_{t_0}^{t} \|u(s)\|_{L^2}^2\, ds, \quad t \in [t_0, +\infty). \tag{4.8}$$

In view of (3.2), we have

$$y''(t) = \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -J(u(t)) > \varepsilon, \quad t \in (t_0, +\infty), \tag{4.9}$$

where $\varepsilon > 0$ is given by Proposition 4.3. From (4.9), we deduce that

$$\lim_{t\to+\infty} y'(t) = \lim_{t\to+\infty} y(t) = +\infty. \tag{4.10}$$

Let $\tilde{f}$ and $\tilde{F}$ be as in (1.16). Since there exists $\theta > 2$ such that

$$\theta \tilde{F}(s) \leq s\tilde{f}(s), \quad \text{for any } s \in \mathbb{R},$$

we can estimate

$$\begin{aligned} y''(t) = -J(u(t)) &= -\Big(\|\nabla u(t)\|_{L^2}^2 + (1 - 2\alpha_0\lambda)\|u(t)\|_{L^2}^2\Big) + \lambda \int_{\mathbb{R}^2} u(t)\tilde{f}(u(t))\, dx \\ &\geq -\Big(\|\nabla u(t)\|_{L^2}^2 + (1 - 2\alpha_0\lambda)\|u(t)\|_{L^2}^2\Big) + \lambda\theta \int_{\mathbb{R}^2} \tilde{F}(u(t))\, dx \\ &\geq -\theta I(u(t)) + \left(\frac{\theta}{2} - 1\right)\Big(\|\nabla u(t)\|_{L^2}^2 + (1 - 2\alpha_0\lambda)\|u(t)\|_{L^2}^2\Big) \\ &\geq -\theta I(u(t)) + Cy'(t), \end{aligned} \tag{4.11}$$

where $C = C(\theta, \alpha_0, \lambda) := (\theta - 2)(1 - 2\lambda\alpha_0) > 0$.

Using (3.2), we get

$$-I(u(t)) = \int_{t_0}^{t} \|\partial_s u(s)\|_{L^2}^2\, ds - I(u(t_0)), \tag{4.12}$$

and hence

$$\begin{aligned} y(t)y''(t) &\geq \frac{\theta}{2}\left(\int_{t_0}^{t} \|u(s)\|_{L^2}^2\, ds\right)\left(\int_{t_0}^{t} \|\partial_s u(s)\|_{L^2}^2\, ds\right) + y(t)\Big(Cy'(t) - \theta I(u(t_0))\Big) \\ &\geq \frac{\theta}{2}\left(\int_{t_0}^{t}\left(\int_{\mathbb{R}^2} u(s)\partial_s u(s)\, dx\right) ds\right)^2 + y(t)\Big(Cy'(t) - \theta I(u(t_0))\Big) \\ &\geq \frac{\theta}{2}\left(\int_{t_0}^{t} \frac{1}{2}\frac{d}{ds}\|u(s)\|_{L^2}^2\, ds\right)^2 + y(t)\Big(Cy'(t) - \theta I(u(t_0))\Big) \\ &= \frac{\theta}{2}\Big(y'(t) - y'(t_0)\Big)^2 + y(t)\Big(Cy'(t) - \theta I(u(t_0))\Big). \end{aligned} \tag{4.13}$$

In view of (4.10), for any $\beta \in (0,1)$ there exists $t_\beta > 1$ such that

$$y(t)y''(t) \geq \frac{\theta}{2}\beta[y'(t)]^2, \quad \text{for any } t \geq t_\beta. \tag{4.14}$$

If we choose $\beta > 0$ such that $\frac{2}{\theta} < \beta < 1$ then we are in the framework of the blow-up Lemma 4.2, and we reach a contradiction. □

**Remark 4.4.** The proof of Theorem 1.3 in [14] is given in the particular case $\alpha_0 = 1$ and $\lambda = \frac{1}{2}$, but it can be adapted to cover the general case

$$0 < \lambda \leq \frac{1}{2\alpha_0}, \quad \text{for some } \alpha_0 > 0. \tag{4.15}$$

In fact, as showed in [14], it is easier to apply the concavity method of Levine if the energy of the solution becomes negative. For completeness, we briefly show how to modify the previous arguments to prove Theorem 1.3 in the general case (4.15).

First, assume that $u_0 \in H^1(\mathbb{R}^2)$, $I\big(u(t_0)\big) = 0$ and $u(t_0) \neq 0$ for some $t_0 \in [0, T_*)$, then there exists $t_1 \in (t_0, T_*)$ such that $I\big(u(t_1)\big) < 0$. If not then the monotonicity of the energy (3.1) yields $I\big(u(t)\big) = 0$ for any $t \in [t_0, T_*)$, and $u(t) = u(t_0)$ a.e. in $\mathbb{R}^2$, for any $t \in [t_0, T_*)$. Therefore $u(t_0)$ solves the stationary problem (1.3), in particular $J\big(u(t_0)\big) = 0$. Since

$$I(v) = \frac{1}{2}J(v) + \lambda \int_{\mathbb{R}^2} \Big(\frac{1}{2}vf(v) - F(v)\Big)\,dx \geq \frac{1}{2}\Big(J(v) + \alpha_0^2\lambda\|v\|_{L^4}^4\Big), \quad \text{for any } v \in H^1(\mathbb{R}^2), \tag{4.16}$$

we deduce that $u(t_0) = 0$, which is *not* possible.

Next, assume that $u_0 \in H^1(\mathbb{R}^2)$, and $I\big(u(t_0)\big) < 0$ for some $t_0 \in [0, T_*)$. Following the proof of Theorem 1.2.(i), we argue by contradiction assuming that $T_* = +\infty$, and we consider the function $y$ defined in (4.8). Combining (4.9) with (4.16) and the monotonicity of the energy (3.1), we get

$$y''(t) = -J\big(u(t)\big) \geq -2I\big(u(t)\big) \geq -2I\big(u(t_0)\big) > 0, \quad t \in (t_0, +\infty),$$

therefore it is enough to obtain (4.14) to reach a contradiction. From (4.11), and recalling that $\theta > 2$ and $1 - 2\alpha_0\lambda \geq 0$, we get

$$y''(t) \geq -\theta I\big(u(t)\big).$$

Moreover, since $I\big(u(t_0)\big) < 0$, (4.12) yields

$$-I\big(u(t)\big) > \int_{t_0}^{t} \|\partial_s u(s)\|_{L^2}^2\,ds,$$

and hence, arguing as in (4.13), we conclude that

$$y(t)y''(t) \geq \frac{\theta}{2}\Big(y'(t) - y'(t_0)\Big)^2,$$

which gives (4.14).

## 5 Global existence and destiny of the orbits in $W$

The uniqueness of the solution to (1.1) plays a role in the proof of the invariance of the set $W$ under the flow associated with (1.1).

**Lemma 5.1.** *Let $u \in \mathcal{C}\big( [0,T_*); H^1(\mathbb{R}^2) \big)$ be the maximal solution to* (1.1) *with $\lambda$ as in* (1.4)*, and $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in W$ for some $t_0 \in [0,T_*)$ then $u(t) \in W$ for any $t \in [t_0, T_*)$.*

*Proof.* We argue by contradiction assuming that $u(t_1) \in V$ for some $t_1 \in (t_0, T_*)$. Since $J(u) \in \mathcal{C}\big( [0,T_*);\ \mathbb{R} \big)$, we have $J\big(u(t_2)\big) = 0$ for some $t_2 \in [t_0, t_1)$. Therefore, either $d \le I\big(u(t_2)\big)$ or $u(t_2) = 0$.

The monotonicity of the energy (3.1) yields $I\big(u(t_2)\big) \le I\big(u(t_0)\big) < d$ , and hence $u(t_2) = 0$. Therefore, by uniqueness, $u(t) = 0$ for any $t \in [t_2, T_*)$, which contradicts $u(t_1) \in V$. □

In order to prove that solutions entering $W$ are global in time, the idea is to apply Theorem 2.6 in view of the following property of $W$ in the energy space.

**Proposition 5.2.** *If $\lambda$ is as in* (1.4) *then, for any $v \in W$, we have $\|\nabla v\|_{L^2}^2 < 2d$.*

*Proof.* If $v \in W$ then in particular

$$\frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^2} F(v)\, dx < d,$$

and to complete the proof it is enough to show that

$$P(v) := \frac{1}{2}\|v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^2} F(v)\, dx \ge 0.$$

Note that the auxiliary functional $P$ is strictly related to the definition of the set $W$; in fact, we already pointed out in (1.6) and (1.9) that

$$d = \inf\big\{ \ I(v)\ :\ v \in H^1(\mathbb{R}^2) \setminus \{0\},\ P(v) = 0 \ \big\}. \tag{5.1}$$

It is not difficult to obtain the analogue of (1.10) for the functional $P$, and show that for any $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ there exists a unique $\tilde\sigma = \tilde\sigma(v) > 0$ such that

$$P(\sigma v) \begin{cases} > 0 & \text{if } 0 < \sigma < \tilde\sigma, \\ = 0 & \text{if } \sigma = \tilde\sigma, \\ < 0 & \text{if } \sigma > \tilde\sigma. \end{cases} \tag{5.2}$$

In fact, $P(\sigma v) = 0$ if and only if

$$\|v\|_{L^2}^2 = \frac{2\lambda}{\sigma^2} \int_{\mathbb{R}^2} F(\sigma v)\, dx,$$

and the function

$$h(\sigma) := \frac{2\lambda}{\sigma^2} \int_{\mathbb{R}^2} F(\sigma v)\, dx = 2\lambda \sum_{k=1}^{+\infty} \frac{\alpha_0^k}{k!} \sigma^{2(k-1)} \|v\|_{L^{2k}}^{2k}$$

satisfies

$$\lim_{\sigma \to 0^+} h(\sigma) = 2\alpha_0 \lambda \|v\|_{L^2}^2, \qquad \lim_{\sigma \to +\infty} h(\sigma) = +\infty, \quad \text{and} \quad h' > 0 \text{ on } (0, +\infty).$$

Using (5.2), we prove that if $v \in W \setminus \{0\}$ then $P(v) \ge 0$. If this was not true then $P(v) < 0$, and $\tilde\sigma = \tilde\sigma(v) \in (0,1)$. Hence, the characterization of the level $d$ given by (5.1) yields

$$d \le I(\tilde\sigma v). \tag{5.3}$$

The point is that (5.3) *cannot* happen. In fact, since $v \in W \setminus \{0\}$ then Proposition 1.1 implies

$$\frac{d}{d\sigma} I(\sigma v) = \frac{1}{\sigma} J(\sigma v) > 0, \quad \text{for any } \sigma \in (0,1],$$

and in particular

$$I(\tilde{\sigma} v) < I(v) < d.$$

□

The set $W$ is stable, and more precisely

**Theorem 5.3.** *Let $u \in \mathcal{C}\big( [0,T_*); H^1(\mathbb{R}^2) \big)$ be the maximal solution to (1.1) with $\lambda$ as in (1.4), and $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in W$ for some $t_0 \in [0,T_*)$ then $T_* = +\infty$.*

*Proof.* Without loss of generality, we may assume that $u(t) \neq 0$ for any $t \in [t_0, T_*)$. From Lemma 5.1, we see that $u(t) \in W$ for any $t \in [t_0, T_*)$. On the one hand, (3.2) yields for any $t \in (t_0, T_*)$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -J\big(u(t)\big) < 0, \tag{5.4}$$

and

$$\sup_{t\in[t_0,T_*)} \|u(t)\|_{L^2} < +\infty.$$

On the other hand, from Proposition 5.2, we get

$$\sup_{t\in[t_0,T_*)} \|\nabla u(t)\|_{L^2}^2 \leq 2d, \tag{5.5}$$

and it is crucial to recall that, from (1.7) and (1.9), we know that

$$2d < \frac{4\pi}{\alpha_0}. \tag{5.6}$$

Therefore, we are under the assumptions of Corollary 2.8 which guarantees that $T_* = +\infty$. □

**Theorem 5.4.** *Assume that $\lambda$ is as in (1.4). There exists $m = m(\alpha_0, \lambda) > 0$ such that*

$$J(v) > 0 \quad \textit{for any } v \in H^1(\mathbb{R}^2) \setminus \{0\} \textit{ with } \|\nabla v\|_{L^2} \leq m, \tag{5.7}$$

*and $W$ contains a neighborhood of the origin in $H^1(\mathbb{R}^2)$. Therefore, if the maximal solution $u \in \mathcal{C}\big( [0,T_*); H^1(\mathbb{R}^2) \big)$ to (1.1) with $u_0 \in H^1(\mathbb{R}^2)$ is* global *(i.e. $T_* = +\infty$) and*

$$\lim_{t\to+\infty} \|u(t)\|_{H^1} = 0$$

*then there exists $t_0 \in [0,+\infty)$ such that $u(t) \in W$ for any $t \in [t_0, +\infty)$.*

*Proof.* The relevant part of the proof is to show that (5.7) holds. In fact, it is clear that (5.7) implies that $W$ contains a neighborhood of the origin in $H^1(\mathbb{R}^2)$: if we set

$$S_\delta := \big\{ v \in H^1(\mathbb{R}^2) \ : \ \|v\|_{H^1} < \delta \big\}, \quad \text{with} \quad 0 < \delta < \min\big\{\sqrt{2d}, m(\alpha_0,\lambda)\big\},$$

then we have for any $v \in S_\delta$

$$I(v) \leq \frac{1}{2}\|v\|_{H^1}^2 < \frac{1}{2}\delta^2 < d,$$

and (5.7) yields $J(v) > 0$ provided $v \neq 0$. Therefore $S_\delta \subseteq W$.

In the second part of the statement of Theorem 5.4, since the maximal solution $u$ is global and

$$\lim_{t\to+\infty} \|u(t)\|_{H^1} = 0,$$

we have $u(t) \in S_\delta$ for any $t > 0$ sufficiently large, and we get the desired conclusion.

In order to prove (5.7), we will follow the argument developed in [16, Lemma 2.1], and we begin by recalling the Gagliardo-Nirenberg inequality:

$$\|v\|_{L^q}^q \leq C_q \|\nabla v\|_{L^2}^{q-2} \|v\|_{L^2}^2, \quad \text{for any } v \in H^1(\mathbb{R}^2), \ \text{ with } q \geq 2. \tag{5.8}$$

For any $v \in H^1(\mathbb{R}^2)$, we can estimate

$$\begin{aligned}
\lambda \int_{\mathbb{R}^2} v f(v)\, dx &= 2\alpha_0 \lambda \left[ \|v\|_{L^2}^2 + \int_{\mathbb{R}^2} v^2 (e^{\alpha_0 v^2} - 1)\, dx \right] \\
&\leq 2\alpha_0\lambda \left[ \|v\|_{L^2}^2 + \left( \int_{\mathbb{R}^2} |v|^{2q} \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^2} (e^{\alpha_0 q' v^2} - 1)\, dx \right)^{\frac{1}{q'}} \right] \\
&\leq 2\alpha_0\lambda \|v\|_{L^2}^2 + C \|\nabla v\|_{L^2}^{2-\frac{2}{q}} \|v\|_{L^2}^{\frac{2}{q}} \left( \int_{\mathbb{R}^2} (e^{\alpha_0 q' v^2} - 1)\, dx \right)^{\frac{1}{q'}},
\end{aligned}$$

where we used Hölder's inequality with $q$, $q' > 1$ satisfying $\frac{1}{q} + \frac{1}{q'} = 1$, and the Gagliardo-Nirenberg inequality (5.8).

If $0 < m < \sqrt{\frac{2\pi}{\alpha_0 q'}}$ then we can apply the scale invariant Trudinger-Moser inequality (2.6) to any $v \in H^1(\mathbb{R}^2)$ with $\|\nabla v\|_{L^2} \leq m$, and get

$$\int_{\mathbb{R}^2} (e^{\alpha_0 q' v^2} - 1)\, dx = \int_{\mathbb{R}^2} \left(e^{2\pi \left(\sqrt{\frac{\alpha_0 q'}{2\pi}} v\right)^2} - 1\right) dx \leq C \|v\|_{L^2}^2,$$

where the constant $C = C(\alpha_0, q) > 0$ is *independent* of $m$.

Summarizing, for any $q > 1$, there exists a constant $C = C(\alpha_0, \lambda, q) > 0$ such that if $v \in H^1(\mathbb{R}^2)$ satisfies $\|\nabla v\|_{L^2} \leq m$, for some $0 < m < \sqrt{\frac{2\pi}{\alpha_0 q'}}$, then we have

$$\begin{aligned}
\lambda \int_{\mathbb{R}^2} v f(v)\, dx &\leq 2\alpha_0\lambda \|v\|_{L^2}^2 + C \|\nabla v\|_{L^2}^{\frac{2}{q'}} \|v\|_{L^2}^2 \\
&\leq 2\alpha_0\lambda \|v\|_{L^2}^2 + C m^{\frac{2}{q'}} \|v\|_{L^2}^2,
\end{aligned}$$

and hence

$$\begin{aligned}
J(v) &\geq \|\nabla v\|_{L^2}^2 + (1 - 2\alpha_0\lambda) \|v\|_{L^2}^2 - C m^{\frac{2}{q'}} \|v\|_{L^2}^2 \\
&\geq \left[ (1 - 2\alpha_0\lambda) - C m^{\frac{2}{q'}} \right] \|v\|_{L^2}^2.
\end{aligned}$$

Since $1 - 2\alpha_0\lambda > 0$, and the constant $C > 0$ is independent of $m$, if we choose $m$ sufficiently small then we reach the desired conclusion. □

Actually, all the solutions entering $W$ (which are *global*, i.e. $T_* = +\infty$, in view of Theorem 5.3) have the same destiny in the following sense.

**Theorem 5.5.** *Let $u \in \mathcal{C}\big([0,T_*); H^1(\mathbb{R}^2)\big)$ be the maximal solution to (1.1) with $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in W$ for some $t_0 \in [0,T_*)$ then*

$$\lim_{t\to+\infty} \|u(t)\|_{H^1} = 0.$$

In order to prove the above Theorem 5.5, we will use the following convergence result.

**Proposition 5.6.** *Let $\{v_n\}_n \subset W$ be such that*

$$M := \sup_n \|v_n\|_{L^2} < +\infty. \tag{5.9}$$

*If*

$$\lim_{n\to+\infty} J(v_n) = 0, \tag{5.10}$$

*and*

$$\lim_{n\to+\infty} I(v_n) =: I_\infty \in (-\infty, d), \tag{5.11}$$

*then*

$$\lim_{n\to+\infty} \|v_n\|_{H^1} = 0, \qquad \textit{and} \qquad I_\infty = 0. \tag{5.12}$$

*Proof.* Let $H$ be the functional defined by (4.5), i.e.

$$H(v) := I(v) - \frac{1}{2}J(v) = \lambda \int_{\mathbb{R}^2} \left(\frac{1}{2}vf(v) - F(v)\right) dx, \quad \text{for any } v \in H^1(\mathbb{R}^2),$$

and recall that in the proof of Proposition 4.3, we emphasized that the level $d$ can be characterized as

$$d = \inf\Big\{ H(v) \ : \ v \in H^1(\mathbb{R}^2) \setminus \{0\},\ J(v) \le 0 \Big\}. \tag{5.13}$$

Let $\tilde{f}$ be as in (1.16). If we show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} v_n \tilde{f}(v_n)\, dx = 0, \quad \text{and} \quad \lim_{n\to+\infty} H(v_n) = 0, \tag{5.14}$$

then the proof is complete. Indeed, we can rewrite

$$J(v_n) = \|\nabla v_n\|_{L^2}^2 + (1 - 2\lambda\alpha_0)\|v_n\|_{L^2}^2 - \lambda \int_{\mathbb{R}^2} v_n \tilde{f}(v_n)\, dx,$$

or equivalently

$$\|\nabla v_n\|_{L^2}^2 + (1 - 2\lambda\alpha_0)\|v_n\|_{L^2}^2 = J(v_n) + \lambda \int_{\mathbb{R}^2} v_n \tilde{f}(v_n)\, dx,$$

and hence combining (5.10) with (5.14), we deduce that

$$\lim_{n\to+\infty} \|\nabla v_n\|_{L^2}^2 + (1 - 2\lambda\alpha_0)\|v_n\|_{L^2}^2 = 0.$$

Since $1 - 2\alpha_0\lambda > 0$, this is enough to obtain the first part of (5.12). The second part of (5.12) (i.e. $I_\infty = 0$) is a direct consequence of (5.10), (5.14), and the following identity

$$I(v_n) = H(v_n) + \frac{1}{2}J(v_n).$$

The rest of the proof is dedicated to showing (5.14), and we begin by summarizing some properties of the sequence $\{v_n\}_n$ which will be useful to obtain (5.14). Since $\{v_n\}_n \subset W$, we have

$$J(v_n) > 0, \quad \text{for any } n \geq 1, \tag{5.15}$$

and as a consequence of Proposition 5.2, we also have

$$m := \sup_n \|\nabla v_n\|_2^2 \leq 2d. \tag{5.16}$$

In particular, recalling (5.6), we know that

$$m < \frac{4\pi}{\alpha_0}, \tag{5.17}$$

and this strict inequality will be crucial in the proof of the convergence result expressed by (5.14).

In order to prove (5.14), we consider the Schwarz symmetrized sequence $\{v_n^*\}_n \subset H^1_{\text{rad}}(\mathbb{R}^2)$, i.e. the sequence of the non-negative spherically symmetric and decreasing rearrangements of $\{v_n\}_n$ (see e.g. [27, Chapter 3]). In view of the properties of Schwarz symmetrization, we have

$$\int_{\mathbb{R}^2} v_n \tilde{f}(v_n)\, dx = \int_{\mathbb{R}^2} v_n^* \tilde{f}(v_n^*)\, dx, \quad \text{and} \quad H(v_n) = H(v_n^*),$$

and to obtain (5.14), it is enough to show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} v_n^* \tilde{f}(v_n^*)\, dx = 0, \quad \text{and} \quad \lim_{n\to+\infty} H(v_n^*) = 0. \tag{5.18}$$

Using again the properties of Schwarz symmetrization, together with (5.16) and (5.9), we get

$$\sup_n \|\nabla v_n^*\|_{L^2}^2 \leq m, \quad \text{and} \quad \sup_n \|v_n^*\|_{L^2} = M < +\infty.$$

In particular, up to subsequences, $v_n^* \rightharpoonup w$ in $H^1(\mathbb{R}^2)$, and $v_n^* \to w$ a.e. in $\mathbb{R}^2$. We divide the proof of (5.18) into two steps: first,

1. we show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} v_n^* \tilde{f}(v_n^*)\, dx = \int_{\mathbb{R}^2} w \tilde{f}(w)\, dx, \quad \text{and} \quad \lim_{n\to+\infty} H(v_n^*) = H(w), \tag{5.19}$$

   and finally,

2. we deduce that $w = 0$.

*Step 1.* The proof of (5.19) is a direct consequence of the compactness result [15, Theorem 1.5(C)] related to the new scale invariant Trudinger-Moser inequality with the exact growth condition. For the sake of completeness, we show that (5.19) can be deduced as well by the classical compactness lemma of Strauss [37, Compactness Lemma 2] (see also [3, Theorem A.I]). From (5.17), we have the existence of $\varepsilon \in (0,1)$ such that

$$m = \frac{4\pi}{\alpha_0}(1-\varepsilon), \tag{5.20}$$

and according to the notations used in [3, Theorem A.I], we introduce the auxiliary growth function

$$Q(s) := e^{\alpha_0(1+\varepsilon)s^2} - 1.$$

If $P : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

(i) $\displaystyle\lim_{|s|\to+\infty}\frac{P(s)}{Q(s)}=0, \quad \text{and} \quad \lim_{s\to 0}\frac{P(s)}{Q(s)}=0,$

(ii) $\displaystyle\sup_n \int_{\mathbb{R}^2} Q(v_n^*)\,dx<+\infty,$

(iii) $P(v_n^*)\to P(w)$ a.e. in $\mathbb{R}^2$, and

(iv) $v_n^*(x)\to 0$ as $|x|\to+\infty$ uniformly with respect to $n$,

then the compactness lemma of Strauss guarantees that

$$P(v_n^*)\to P(w) \quad \text{in } L^1(\mathbb{R}^2). \tag{5.21}$$

To see that (ii) holds, we renormalize each $v_n^*$ by setting

$$w_n:=\frac{v_n^*}{\sqrt{m}},$$

so that (5.16) yields $\|\nabla w_n\|_{L^2}\le 1$. In view of the scale invariant Trudinger-Moser inequality (2.6), we can estimate

$$\begin{aligned}\int_{\mathbb{R}^2} Q(v_n^*)\,dx &= \int_{\mathbb{R}^2}\big(e^{\alpha_0(1+\varepsilon)[v_n^*]^2}-1\big)\,dx=\int_{\mathbb{R}^2}\big(e^{\alpha_0(1+\varepsilon)mw_n^2}-1\big)\,dx=\int_{\mathbb{R}^2}\big(e^{4\pi(1-\varepsilon^2)w_n^2}-1\big)\,dx\\ &\le C_\varepsilon\|w_n\|_{L^2}^2=C_\varepsilon\frac{\|v_n^*\|_{L^2}^2}{m}\le C_\varepsilon\frac{M^2}{m},\end{aligned}$$

where we also used (5.20) and (5.9).

The a.e.-convergence of the sequence $\{v_n^*\}_n$ and the continuity of $P$ yield (iii). Moreover, the radial symmetry and the boundedness of $\{v_n^*\}_n$ in $H^1(\mathbb{R}^2)$ give the uniform decay at infinity expressed by (iv).

Finally, if we set

$$\text{either} \quad P(s):=s\tilde f(s) \quad \text{or} \quad P(s):=\frac12 sf(s)-F(s)$$

then also the assumption (i) of the compactness lemma of Strauss is satisfied, and hence (5.21) holds. This completes the proof of (5.19).

*Step 2.* First, we show that $J(w)\le 0$. Since $1-2\alpha_0\lambda>0$, using the weak convergence $v_n^*\rightharpoonup w$ in $H^1(\mathbb{R}^2)$, (5.19), and (5.10), we can estimate

$$\begin{aligned}J(w)&=\|\nabla w\|_{L^2}^2+(1-2\alpha_0\lambda)\|w\|_{L^2}^2-\lambda\int_{\mathbb{R}^2} w\tilde f(w)\,dx\\ &\le\liminf_{n\to+\infty}\Big(\|\nabla v_n^*\|_{L^2}^2+(1-2\alpha_0\lambda)\|v_n^*\|_{L^2}^2-\lambda\int_{\mathbb{R}^2} v_n^*\tilde f(v_n^*)\,dx\Big)\\ &=\liminf_{n\to+\infty} J(v_n^*)\le\liminf_{n\to+\infty} J(v_n)=0.\end{aligned}$$

Next, we argue by contradiction assuming that $w\neq 0$. On the one hand, since $w\neq 0$ and $J(w)\le 0$, (5.13) yields

$$d\le H(w).$$

On the other hand, from (5.19) and (5.15), we deduce that

$$H(w)=\lim_{n\to+\infty} H(v_n^*)=\lim_{n\to+\infty} H(v_n)=\lim_{n\to+\infty}\Big(I(v_n)-\frac12 J(v_n)\Big)\le\lim_{n\to+\infty} I(v_n)=I_\infty.$$

Since by assumption (5.11), we have $I_\infty < d$ then

$$H(w) < d,$$

and we reach a contradiction. $\square$

*Proof of Theorem 5.5.* As in the proof of Theorem 5.3, we have the monotonicity of the $L^2$-norm of the solution (5.4) which ensures both

$$\sup_{t\in[t_0,+\infty)} \|u(t)\|_{L^2} < +\infty \tag{5.22}$$

and

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} \quad \text{exists,} \tag{5.23}$$

and we also have (5.5) and (5.6), i.e.

$$\sup_{t\in[t_0,+\infty)} \|\nabla u(t)\|_{L^2}^2 \leq 2d < \frac{4\pi}{\alpha_0}. \tag{5.24}$$

Moreover, the monotonicity of the energy (3.1) and [14, Theorem 2.1.3] (see also Theorem 1.3) guarantee that

$$\lim_{t\to+\infty} I\big(u(t)\big) = I_\infty \in [0, d). \tag{5.25}$$

Next, we find a sequence $t_n \in [t_0, +\infty)$ satisfying

$$\lim_{n\to+\infty} t_n = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} J\big(u(t_n)\big) = 0. \tag{5.26}$$

We point out that (5.25), and in particular the fact that $I_\infty \geq 0$, implies

$$\limsup_{t\to+\infty} \frac{d}{dt} I\big(u(t)\big) = 0. \tag{5.27}$$

Combining (5.27) with the identity (3.1), we deduce the existence of a sequence $t_n \in [t_0, +\infty)$ such that

$$\lim_{n\to+\infty} t_n = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \|\partial_t u(t_n)\|_{L^2} = 0.$$

The sequence $\{u(t_n)\}_n$ is a Palais-Smale sequence for the energy functional at the level $I_\infty$. More precisely, there exists $\{\varepsilon_n\}_n \subset \mathbb{R}^+$ satisfying

$$|\langle dI\big(u(t_n)\big), \varphi\rangle| \leq \varepsilon_n \|\varphi\|_{L^2} \quad \text{for any } \varphi \in H^1(\mathbb{R}^2), \quad \text{and} \lim_{n\to+\infty} \varepsilon_n = 0. \tag{5.28}$$

In fact, recalling (3.3), it is enough to choose $\varepsilon_n := \|\partial_t u(t_n)\|_{L^2}$. Since $u(t_n) \in W$ for any $n \geq 1$, we have $J(u_n) > 0$, and as a particular case of (5.28), we deduce that

$$0 < J\big(u(t_n)\big) = \langle dI\big(u(t_n)\big), u(t_n)\rangle \leq \varepsilon_n \|u(t_n)\|_{L^2}.$$

Therefore, with the help of (5.22), we get

$$\lim_{n\to+\infty} J\big(u(t_n)\big) = 0.$$

We are now in position to prove that $\|u(t)\|_{H^1} \to 0$ as $t \to \infty$. Recalling (5.22) and (5.25), since $u(t_n) \in W$ satisfies (5.26) then we can apply Proposition 5.6 to conclude that

$$\lim_{n\to+\infty} \|u(t_n)\|_{H^1} = 0, \quad \text{and} \quad I_\infty = 0. \tag{5.29}$$

In particular,

$$\lim_{n\to+\infty} \|u(t_n)\|_{L^2} = 0,$$

and hence, in view of (5.23), we have

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0.$$

If we set

$$w(t) := \frac{u(t)}{\sqrt{2d}}, \quad \text{for any } t \in [t_0, +\infty),$$

then (5.24) yields $\|\nabla w(t)\|_{L^2} \leq 1$. Applying the scale invariant Trudinger-Moser inequality (2.6), we can estimate

$$\int_{\mathbb{R}^2} F\big(u(t)\big)\,dx = \int_{\mathbb{R}^2} \big(e^{2d\alpha_0[w(t)]^2} - 1\big)\,dx \leq C_d\|w(t)\|_{L^2}^2 = C_d\frac{\|u(t)\|_{L^2}^2}{2d}, \quad \text{for any } t \in [t_0, +\infty).$$

Note that, it is possible to apply (2.6) in view of the fact that $2d\alpha_0 < 4\pi$, see (5.24). Therefore,

$$\lim_{t\to+\infty} \int_{\mathbb{R}^2} F\big(u(t)\big)\,dx = 0,$$

and

$$\lim_{t\to+\infty} \|u(t)\|_{H^1}^2 = \lim_{t\to+\infty} \left(2I\big(u(t)\big) + 2\lambda \int_{\mathbb{R}^2} F\big(u(t)\big)\,dx\right) = 2I_\infty.$$

But from (5.29), we know that $I_\infty = 0$, and hence the proof of the Theorem is complete.

□

**Remark 5.7.** The validity of the Palais-Smale compactness condition for the energy functional $I$ in the region $(-\infty, \frac{2\pi}{\alpha_0})$ is still an open question. This problem will be addressed in a forthcoming paper.

# 6 Asymptotics at the ground state level

In this section we prove Theorem 1.4 concerning the asymptotic behavior of the solution for initial data with the same energy as the ground state solution. The key property is expressed by the following Lemma.

**Lemma 6.1.** *Assume that there exists $t_0 \in [0, T_*)$ such that*

$$I(u(t_0)) = d \quad \text{and} \quad J(u(t_0)) \neq 0.$$

*Then $I(u(t)) < d$ for any $t \in (t_0, T_*)$.*

*Proof.* By contradiction let us assume that there exists $t_1 \in (t_0, T_*)$ such that $I(u(t_1)) = d$. Then by the monotonicity of the energy $I(u(t)) = d$, for any $t \in [t_0, t_1]$. Moreover,

$$\|\partial_t u(t)\|_{L^2}^2 = -\frac{d}{dt} I(u(t)) = 0,$$

for any $t \in (t_0, t_1)$, and hence $u(t) = u(t_0)$ a.e. in $\mathbb{R}^2$ for any $t \in [t_0, t_1]$. Therefore, $u(t_0)$ is a stationary solution and $dI(u(t_0)) \equiv 0$. In particular, $J(u(t_0)) = \langle dI(u(t_0)), u(t_0)\rangle = 0$ and we reach a contradiction. □

*Proof of Theorem 1.4.* The results in (i) and (ii) follow directly from Lemma 6.1 and the arguments in Lemma 4.1 and in Lemma 5.1. Let us now prove (iii). First, we remark that if $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ is such that $I(v) = d$ and $J(v) = 0$ then $v$ is a critical point for the energy functional. Indeed thanks to the property $J(v) = 0$, we deduce that the map $\sigma \to I(\sigma v)$, for $\sigma > 0$, attains its unique maximum at $\sigma = 1$. In particular ,

$$I(\sigma v) < I(v) = d, \quad \forall\, \sigma \in (0,1) \cup (1, +\infty). \tag{6.1}$$

From (5.2) we deduce the existence of a unique $\tilde{\sigma} = \tilde{\sigma}(v) > 0$ such that $P(\tilde{\sigma} v) = 0$, where $P$ is the Pohozaev functional defined in Section 5. Therefore if we use the characterization (1.6) of the ground state energy level $d$ in terms of the Pohozaev functional, we get

$$d \leq I(\tilde{\sigma} v).$$

In view of (6.1), the above inequality holds if and only if $\tilde{\sigma} = 1$.

By using again the characterization (1.6) of the level $d$, we deduce that $v$ is a minimizer, therefore there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi \, dx = \theta \left( \int_{\mathbb{R}^2} v \varphi \, dx - \lambda \int_{\mathbb{R}^2} f(v) \varphi \, dx \right), \quad \text{for any } \varphi \in H^1(\mathbb{R}^2).$$

In particular

$$\theta = \frac{\|\nabla v\|_{L^2}^2}{\int_{\mathbb{R}^2} v^2 \, dx - \lambda \int_{\mathbb{R}^2} f(v) v \, dx} = \frac{\|\nabla v\|_{L^2}^2}{J(v) - \|\nabla v\|_{L^2}^2} = -1.$$

Hence for any $\varphi \in H^1(\mathbb{R}^2)$ we have

$$\langle dI(v), \varphi \rangle = \int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} v \varphi \, dx - \lambda \int_{\mathbb{R}^2} f(v) \varphi \, dx = 0.$$

Therefore if there exists $t_0 \in [0, T_*)$ such that

$$I(u(t_0)) = d, \quad \textit{and} \quad J(u(t_0)) = 0,$$

then $u(t_0)$ is a stationary solution (more precisely a ground state) and by uniqueness, $u$ is global and it coincides with the ground state $u(t_0)$ a.e. in $\mathbb{R}^2$ after the time $t_0$. □

# 7 Indistinguishable splittings

In this section, we adapt the arguments of [16] to show that the Nehari functional $J$ defined by (1.8) and the Pohozaev functional $P$ defined by (1.15) determine the *same* splitting below the ground state energy level $d$. In particular, we will follow the arguments of the variational part of the paper

[16], and we point out that the energy critical nonlinearity that we are considering does not fulfill the hypothesis required in [16].

Given two parameters $(a, b) \in \mathbb{R}^2$ satisfying

$$a \geq 0, \quad \text{and} \quad b \geq 0, \quad \text{with} \quad (a, b) \neq (0, 0), \tag{7.1}$$

we define the functional

$$J_{a,b}(v) := a\|\nabla v\|_{L^2}^2 + (a+b)\|v\|_{L^2}^2 - a\int_{\mathbb{R}^2} \lambda v f(v)\,dx - 2b\int_{\mathbb{R}^2} \lambda F(v)\,dx.$$

The above functional is relevant for the two-parameter rescaling function

$$v_{a,b}^\sigma(x) := \sigma^a v\big(\sigma^{-b}x\big), \quad \sigma > 0, \quad x \in \mathbb{R}^2, \tag{7.2}$$

in fact

$$\frac{d}{d\sigma} I\big(v_{a,b}^\sigma\big) = \frac{1}{\sigma} J_{a,b}\big(v_{a,b}^\sigma\big), \quad \sigma > 0. \tag{7.3}$$

Note that, if $a = 1$ and $b = 0$ then

$$J_{1,0}(v) := \|v\|_{H^1}^2 - \lambda\int_{\mathbb{R}^2} v f(v)\,dx$$

is the Nehari functional $J$ defined by (1.8). If $a = 0$ and $b = 1$ then

$$J_{0,1}(v) := \|v\|_{L^2}^2 - 2\lambda\int_{\mathbb{R}^2} F(v)\,dx$$

is related to the Pohozaev functional $P$ defined by (1.15), and more precisely $J_{0,1} = 2P$. Therefore, $J_{a,b}$ interpolates between the Pohozaev and Nehari functionals in the following sense

$$J_{a,b}(v) = aJ(v) + 2bP(v). \tag{7.4}$$

If we consider the constrained minimization problem

$$d_{a,b} := \inf\Big\{I(v) \ : \ v \in H^1(\mathbb{R}^2) \setminus \{0\},\ J_{a,b}(v) = 0\Big\}$$

then [16, Lemma 2.6] suggests that $d_{a,b}$ is positive and independent of $a$ and $b$.

**Proposition 7.1.** *Assume that $\lambda$ is as in* (1.4)*, and $(a, b)$ as in* (7.1)*. Then the level $d_{a,b}$ is independent of $a$ and $b$, and more precisely $d_{a,b} = c$, where $c$ is the mountain pass level* (1.5)*. Hence,*

$$0 < d_{a,b} < \frac{2\pi}{\alpha_0}. \tag{7.5}$$

In view of (1.7), it is clear that (7.5) is a direct consequence of the first part of the statement of Proposition 7.1.

Next, we consider the sets

$$W_{a,b} := \Big\{\, v \in H^1(\mathbb{R}^2) \ : \ I(v) < d_{a,b},\ J_{a,b}(v) \geq 0 \,\Big\},$$

and

$$V_{a,b} := \Big\{\, v \in H^1(\mathbb{R}^2) \ : \ I(v) < d_{a,b},\ J_{a,b}(v) < 0 \,\Big\}.$$

Adapting the arguments of the proof of [16, Lemma 2.9] to our framework, we show that the functionals $J_{a,b}$ define the same splitting below the ground state energy level $d_{a,b} = d$ independently of $a$ and $b$ in the range (7.1).

**Proposition 7.2.** *Assume that $\lambda$ is as in (1.4), and $(a,b)$ as in (7.1). The sets $W_{a,b}$ and $V_{a,b}$ are independent of $a$ and $b$.*

The proof of Proposition 7.1 and Proposition 7.2 can be found in Section 7.2 and Section 7.3 respectively. More precisely, the remaining part of this section is organized as follows. In Section 7.1, we study the geometry of the functionals $J_{a,b}$ and $I$ along the rescaling (7.2). The results of Section 7.1 will enable us to prove Proposition 7.1 (in Section 7.2) and Proposition 7.2 (in Section 7.3).

**Remark 7.3.** We mention that in view of the results in [16] and [25], we expect that Proposition 7.1 and Proposition 7.2 hold also when the parameters $(a,b)$ are in the range

$$a \geq 0, \quad \text{and} \quad a+b \geq 0, \quad \text{with} \quad (a,b) \neq (0,0).$$

However, for simplicity, we restrict the attention to the range (7.1).

## 7.1 Geometry of the functionals along the rescaling

The energy functional has a unique global maximum point on the rescaling $\sigma \in (0,+\infty) \mapsto v^{\sigma}_{a,b}$.

**Lemma 7.4.** *Assume that $\lambda$ is as in (1.4), and $(a,b)$ as in (7.1) with $a>0$. For any $v \in H^1(\mathbb{R}^2)\setminus\{0\}$ there exists $\overline{\sigma} = \overline{\sigma}(v) > 0$ such that*

$$J_{a,b}(v^{\sigma}_{a,b}) \begin{cases} > 0 & \text{if } 0 < \sigma < \overline{\sigma}, \\ = 0 & \text{if } \sigma = \overline{\sigma}, \\ < 0 & \text{if } \sigma > \overline{\sigma}. \end{cases}$$

*Moreover, the map $\sigma \in (0,+\infty) \mapsto I\big(v^{\sigma}_{a,b}\big)$ is monotone strictly increasing on $(0,\overline{\sigma})$, stricly decreasing on $(\overline{\sigma},+\infty)$, and attains its unique maximum at $\overline{\sigma}$.*

*Proof.* Note that it is enough to prove the first part of the statement, then the second part is a direct consequence of (7.3).

We have

$$J_{a,b}\big(v^{\sigma}_{a,b}\big) = a\sigma^{2a}\|\nabla v\|^2_{L^2} + (a+b)\sigma^{2(a+b)}\|v\|^2_{L^2} - \lambda\sigma^{2b}\int_{\mathbb{R}^2}\Big[\, a\sigma^a v f\big(\sigma^a v\big) + 2bF\big(\sigma^a v\big)\,\Big]\, dx.$$

First, we can rewrite

$$asf(s) + 2bF(s) = 2\sum_{k=1}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)s^{2k}, \qquad s \in \mathbb{R},$$

and

$$\begin{aligned}\int_{\mathbb{R}^2}\Big[\, a\sigma^a v f\big(\sigma^a v\big) + 2bF\big(\sigma^a v\big)\,\Big]\, dx &= 2\sum_{k=1}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\big\|\sigma^a v\big\|^{2k}_{L^{2k}} \\ &= 2\sigma^{2a}\sum_{k=1}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}}.\end{aligned}$$

Therefore

$$\begin{aligned}J_{a,b}\big(v^\sigma_{a,b}\big) =& a\sigma^{2a}\|\nabla v\|^2_{L^2} + (a+b)\sigma^{2(a+b)}\|v\|^2_{L^2} - 2\lambda\sigma^{2(a+b)}\sum_{k=1}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}}\\ =& a\sigma^{2a}\|\nabla v\|^2_{L^2} + (a+b)\sigma^{2(a+b)}(1-2\lambda\alpha_0)\|v\|^2_{L^2}\\ &- 2\lambda\sigma^{2(a+b)}\sum_{k=2}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}}.\end{aligned}$$

Next, we distinguish two cases according to $b>0$ or $b=0$. First, we assume $b>0$, and we rewrite

$$J_{a,b}\big(v^\sigma_{a,b}\big) = \sigma^{2(a+b)}h(\sigma),$$

where

$$h(\sigma) := a\sigma^{-2b}\|\nabla v\|^2_{L^2} + (a+b)(1-2\alpha_0\lambda)\|v\|^2_{L^2} - 2\lambda\sum_{k=2}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}}.$$

In particular, the map $\sigma\in(0,+\infty)\mapsto J_{a,b}\big(v^\sigma_{a,b}\big)$ has the same sign as $h$. Since $b>0$, and by assumption we also have $a>0$, it is clear that

$$\lim_{\sigma\to0^+}h(\sigma)=+\infty,\quad\text{and}\quad\lim_{\sigma\to+\infty}h(\sigma)=-\infty,$$

and

$$h'(\sigma) = -2ab\sigma^{-2b-1}\|\nabla v\|^2_{L^2} - 2\lambda\sum_{k=2}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)2(k-1)a\sigma^{2(k-1)a-1}\|v\|^{2k}_{L^{2k}} < 0.$$

Therefore, there exists $\overline{\sigma}=\overline{\sigma}(v)>0$ such that

$$h(\sigma)\begin{cases} >0 & \text{if } 0<\sigma<\overline{\sigma},\\ =0 & \text{if } \sigma=\overline{\sigma},\\ <0 & \text{if } \sigma>\overline{\sigma}.\end{cases}$$

To complete the proof, it remains to consider the case $b=0$. In this case, we rewrite

$$J_{a,b}\big(v^\sigma_{a,b}\big) = \sigma^{2a}\tilde{h}(\sigma),$$

where

$$\tilde{h}(\sigma) := a\Big[\|\nabla v\|^2_{L^2} + (1-2\alpha_0\lambda)\|v\|^2_{L^2}\Big] - 2\lambda\sum_{k=2}^{+\infty}\frac{\alpha_0^k}{k!}(ka+b)\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}}.$$

Since $a>0$, we deduce that

$$\tilde{h}(0)>0,\quad\lim_{\sigma\to+\infty}\tilde{h}(\sigma)=-\infty,\quad\text{and}\quad\tilde{h}'(\sigma)<0,$$

and the proof is complete. □

**Lemma 7.5.** *Assume that $\lambda$ is as in (1.4), and $(a,b)$ as in (7.1) with $a>0$. For any $v\in H^1(\mathbb{R}^2)\setminus\{0\}$, we have*

$$\lim_{\sigma\to+\infty}I\big(v^\sigma_{a,b}\big) = -\infty.$$

*Proof.* We can rewrite

$$
\begin{aligned}
I\big(v^\sigma_{a,b}\big) &= \frac{1}{2}\Big(\sigma^{2a}\|\nabla v\|^2_{L^2} + \sigma^{2(a+b)}\|v\|^2_{L^2}\Big) - \lambda\sigma^{2b}\int_{\mathbb{R}^2} F\big(\sigma^a v\big)\,dx \\
&= \frac{1}{2}\Big(\sigma^{2a}\|\nabla v\|^2_{L^2} + \sigma^{2(a+b)}\|v\|^2_{L^2}\Big) - \lambda\sigma^{2(a+b)}\sum_{k=1}^{+\infty}\frac{\alpha_0^k}{k!}\sigma^{2(k-1)a}\|v\|^{2k}_{L^{2k}} \\
&\le \frac{1}{2}\Big(\sigma^{2a}\|\nabla v\|^2_{L^2} + \sigma^{2(a+b)}\|v\|^2_{L^2}\Big) - \lambda\sigma^{2(a+b)}\frac{\alpha_0^2}{2}\sigma^{2a}\|v\|^4_{L^4},
\end{aligned}
$$

and the right hand side goes to $-\infty$ as $\sigma \to +\infty$, provided $a > 0$. □

## 7.2 The level $d_{a,b}$ is independent of $(a, b)$ in the range (7.1)

This section is devoted to the proof of Proposition 7.1. As already mentioned in Section 1, from the existence result in [33], we know that the level $d_{0,1}$ coincides with the mountain pass level $c$ associated with the energy functional $I$, i.e.

$$d_{0,1} = c, \tag{7.6}$$

see (1.6). Moreover, [33, Theorem 4, Propositions 1 and 2] gives the existence of $\overline{v} \in H^1(\mathbb{R}^2) \setminus \{0\}$ satisfying

$$I(\overline{v}) = c, \quad P(\overline{v}) = 0, \quad \text{and} \quad J(\overline{v}) = 0. \tag{7.7}$$

If $a = 0$ and $b > 0$ then $J_{0,b}(v) = 2bP(v)$ for any $v \in H^1(\mathbb{R}^2)$, and hence

$$d_{0,b} = c, \quad \text{for any } b > 0.$$

To complete the proof of Proposition 7.1, we will show that

$$d_{a,b} = c \quad \text{for any } a > 0 \text{ and } b \ge 0, \tag{7.8}$$

and from now on, we assume that $a > 0$ and $b \ge 0$.

Combining (7.4) with (7.7), we get

$$d_{a,b} \le I(\overline{v}) = c \tag{7.9}$$

Next, we introduce the auxiliary level

$$c_{a,b} := \inf_{v\in H^1(\mathbb{R}^2)\setminus\{0\}} \max_{\sigma>0} I\big(v^\sigma_{a,b}\big).$$

The proof of (7.8) is complete if we show that

$$c \le c_{a,b}, \quad \text{and} \quad c_{a,b} = d_{a,b}.$$

*Step 1: $c \le c_{a,b}$.* Let $v \in H^1(\mathbb{R}^2) \setminus \{0\}$. From Lemma 7.5, we deduce the existence of $\tilde{\sigma} = \tilde{\sigma}(v) > 0$ such that

$$I\big(v^{\tilde{\sigma}}_{a,b}\big) < 0.$$

We define $\gamma \in \mathcal{C}\big([0,1], H^1(\mathbb{R}^2)\big)$ as

$$\gamma(s) := \begin{cases} v^{s\tilde{\sigma}}_{a,b} & \text{if } 0 < s \le 1, \\ 0 & \text{if } s = 0, \end{cases}$$

so that

$$\gamma \in \Gamma := \Big\{ \gamma \in \mathcal{C}\big([0,1];\ H^1(\mathbb{R}^2)\big) \ :\ \gamma(0) = 0,\ I\big(\gamma(1)\big) < 0 \Big\}.$$

Note that the continuity of $\gamma$ at $s = 0$ is a consequence of the conditions on $(a, b)$. In fact, since $a > 0$ and $b \geq 0$ then we have

$$\lim_{\sigma \to 0^+} \big\| v^{\sigma}_{a,b} \big\|^2_{H^1} = \lim_{\sigma \to 0^+} \big( \sigma^{2a} \|\nabla v\|^2_{L^2} + \sigma^{2(a+b)} \|v\|^2_{L^2} \big) = 0. \tag{7.10}$$

By construction

$$c \leq \max_{s \in [0,1]} I\big(\gamma(s)\big) \leq \max_{\sigma > 0} I\big(v^{\sigma}_{a,b}\big),$$

and hence

$$c \leq c_{a,b}.$$

*Step 2:* $d_{a,b} \leq c_{a,b}$. For any $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, applying Lemma 7.4, we have the existence of $\overline{\sigma} = \overline{\sigma}(v) > 0$ such that $J_{a,b}\big(v^{\overline{\sigma}}_{a,b}\big) = 0$, and

$$d_{a,b} \leq I\big(v^{\overline{\sigma}}_{a,b}\big) = \max_{\sigma > 0} I\big(v^{\sigma}_{a,b}\big).$$

Therefore,

$$d_{a,b} \leq c_{a,b}.$$

*Step 3:* $d_{a,b} \geq c_{a,b}$. Let $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ be such that $J_{a,b}(v) = 0$. From Lemma 7.4, we deduce that $\overline{\sigma} = \overline{\sigma}(v) = 1$ and

$$c_{a,b} \leq \max_{\sigma > 0} I\big(v^{\sigma}_{a,b}\big) = I\big(v^{1}_{a,b}\big) = I(v),$$

and hence

$$c_{a,b} \leq d_{a,b}.$$

### 7.3 The sets $W_{a,b}$ and $V_{a,b}$ are independent of $(a, b)$ in the range (7.1)

This section is devoted to the proof of Proposition 7.2. First, following [16, Lemma 2.1], we show that the functional $J_{a,b}$ is positive near the origin of $H^1(\mathbb{R}^2)$.

**Lemma 7.6.** *Assume that $\lambda$ is as in* (1.4)*, and $(a, b)$ as in* (7.1)*. There exists $m = m(\alpha_0, \lambda, a+b) > 0$ such that*

$$J_{a,b}(v) > 0 \quad \text{for any } v \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ with } \|\nabla v\|_{L^2} \leq m, \tag{7.11}$$

*and $W_{a,b}$ contains a neighborhood of the origin in $H^1(\mathbb{R}^2)$.*

*Proof.* As in Theorem 5.4, the relevant part of the proof is to show (7.11). Since $2F(s) \leq sf(s)$ for any $s \in \mathbb{R}$, and $(a, b)$ is in the range (7.1), we can estimate the nonlinear part of the functional $J_{a,b}$ as

$$a \int_{\mathbb{R}^2} \lambda v f(v)\, dx + 2b \int_{\mathbb{R}^2} \lambda F(v)\, dx \leq \lambda(a+b) \int_{\mathbb{R}^2} v f(v)\, dx, \quad \text{for any } v \in H^1(\mathbb{R}^2).$$

Therefore arguing as in the proof of Theorem 5.4, for any $q > 1$, we get the existence of a constant $C = C(\alpha_0, \lambda, q, a, b) > 0$ such that if $v \in H^1(\mathbb{R}^2)$ satisfies $\|\nabla v\|_{L^2} \leq m$, for some $0 < m < \sqrt{\frac{2\pi}{\alpha_0 q'}}$, then

$$\begin{aligned} a \int_{\mathbb{R}^2} \lambda v f(v)\, dx + 2b \int_{\mathbb{R}^2} \lambda F(v)\, dx &\leq 2\alpha_0 \lambda (a+b) \|v\|^2_{L^2} + C \|\nabla v\|^{\frac{2}{q'}}_{L^2} \|v\|^2_{L^2} \\ &\leq 2\alpha_0 \lambda (a+b) \|v\|^2_{L^2} + C m^{\frac{2}{q'}} \|v\|^2_{L^2}, \end{aligned}$$

and hence

$$J_{a,b}(v) \geq \left[(a+b)(1-2\alpha_0\lambda) - Cm^{\frac{2}{q'}}\right]\|v\|_{L^2}^2.$$

Since $(a+b)(1-2\alpha_0\lambda) > 0$, and the constant $C > 0$ is independent of $m$, if we choose $m$ sufficiently small then we reach the desired conclusion. □

We point out that

$$W_{a,b} = \left\{ v \in H^1(\mathbb{R}^2) \ : \ I(v) < d_{a,b}, \ J_{a,b}(v) > 0 \right\} \cup \{0\}. \tag{7.12}$$

In fact, on one hand, clearly

$$\left\{ v \in H^1(\mathbb{R}^2) \ : \ I(v) < d_{a,b}, \ J_{a,b}(v) > 0 \right\} \cup \{0\} \subseteq W_{a,b}.$$

On the other hand, if $v \in W_{a,b} \setminus \{0\}$ then $J_{a,b}(v) > 0$. If not then $J_{a,b}(v) = 0$, hence $d_{a,b} \leq I(v)$, and we easily reach a contradiction.

Next, following [16, Lemma 2.9], we show that the set $W_{a,b}$ is path connected.

**Lemma 7.7.** *Assume that $\lambda$ is as in* (1.4)*, and $(a,b)$ as in* (7.1) *with $a > 0$. The set $W_{a,b} \setminus \{0\}$ is contracted to $\{0\}$ by the rescaling $\sigma \in (0,1] \mapsto v_{a,b}^\sigma$.*

*Proof.* Let $v \in W_{a,b} \setminus \{0\}$. Arguing as in (7.10), we have that $\|v_{a,b}^\sigma\|_{H^1} \to 0$ as $\sigma \to 0^+$, and hence it is enough to show that

$$v_{a,b}^\sigma \in W_{a,b}, \quad \text{for any } \sigma \in (0,1).$$

Since $v \in W_{a,b} \setminus \{0\}$, we have $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, $I(v) < d_{a,b}$, and $J_{a,b}(v) > 0$. Applying Lemma 7.4, we deduce that $\overline{\sigma} = \overline{\sigma}(v) > 1$, and

$$J_{a,b}\big(v_{a,b}^\sigma\big) > 0, \quad \text{for any } \sigma \in (0,1).$$

Moreover, the map $\sigma \in (0,1] \mapsto I\big(v_{a,b}^\sigma\big)$ is monotone strictly increasing, and in particular

$$I\big(v_{a,b}^\sigma\big) < I\big(v_{a,b}^1\big) = I(v) < d_{a,b}.$$

□

To complete the proof of Proposition 7.2, we follow closely [16, Lemma 2.9].

*Proof of Proposition 7.2.* In view of Proposition 7.1, the union of the disjoint sets $W_{a,b}$ and $V_{a,b}$ is independent of $a$ and $b$. Therefore, it is enough to show that $W_{a,b}$ is independent of $a$ and $b$.

By definition, the set $V_{a,b}$ is open in $H^1(\mathbb{R}^2)$. Also, $W_{a,b}$ is open in $H^1(\mathbb{R}^2)$: in fact, we have (7.12), and Lemma 7.6 guarantees that $W_{a,b}$ contains a neighborhood of the origin in $H^1(\mathbb{R}^2)$.

Let $a' > 0$ and $b' \geq 0$ then the set $W_{a',b'}$ is connected, see Lemma 7.7. For any $(a,b)$ in the range (7.1), we have $W_{a',b'} = (W_{a',b'} \cap W_{a,b}) \cup (W_{a',b'} \cap V_{a,b})$, and hence

$$W_{a',b'} = W_{a',b'} \cap W_{a,b} \subseteq W_{a,b}.$$

In particular, the set $W_{a,b}$ is independent of $(a,b)$ if $a > 0$ and $b \geq 0$. Therefore, we set

$$W := W_{a,b} \quad \text{for any } (a,b) \text{ in the range (7.1) with } a > 0.$$

If $(a,b)$ is in the range (7.1) with $a = 0$, i.e. $a = 0$ and $b > 0$, then there exists a sequence $\{a_n\}_n$ of positive real numbers converging to $a = 0$. More precisely, $\{a_n\}_n \subset \mathbb{R}^+$ and

$$\lim_{n\to+\infty} a_n = 0.$$

We know that $d_{0,b} = d_{a_n,b}$, and since $a_n > 0$, $W_{a_n,b} = W$. Clearly, for any fixed $v \in H^1(\mathbb{R}^2)$, we have

$$\lim_{n\to+\infty} J_{a_n,b}(v) = J_{0,b}(v),$$

and hence

$$W_{0,b} = \bigcup_n W_{a_n,b} = W.$$

□

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