Inflation Targeting, Recursive Inattentiveness and Heterogeneous Beliefs

Anna Agliari[a,*] Domenico Massaro[b]

Nicolò Pecora[a] Alessandro Spelta[b]

March 2, 2016

[a] *Università Cattolica del Sacro Cuore, Piacenza,*
[b] *Università Cattolica del Sacro Cuore, Milano,*
[*] *corresponding author: anna.agliari@unicatt.it*

Abstract

In this paper we consider a scenario in which the monetary authority provides an explicit inflation target in order to anchor private sector expectations and align them with policy objectives. In this context, a biased perception of the target may arise due to imperfect information flows and idiosyncrasies in information processing lead to heterogenous beliefs about the target. We allow private sector expectations to be revised over time as new information becomes available and the direction of change is determined by the distance between agents' beliefs and actual realizations of macro variables. The recursive choice between alternative predictors is modeled as an optimization problem under rational inattention. Within this framework we investigate whether a simple interest rate rule can steer the economy toward the targeted equilibrium. Our findings suggest that standard policy advices, i.e., ensure determinacy under rational expectations, may not be sufficient to reach the target. Instead, a sound monetary policy should be fine-tuned to ensure that the signal sent by realizations of macro variables can correct biased agents' beliefs.

JEL codes: E52, D83, D84, C62.

Keywords: Inflation targeting, Monetary policy, Recursive inattentiveness, Heterogeneous expectations, New Keynesian model.

Acknowledgment: We would like to thank Mikhail Anufriev, Cars Hommes, an anonymous referee and participants of the 20th CEF International Conference at Oslo, June 22-24, 2014 for stimulating discussions and helpful comments. Financial support from the EU 7^{th} framework collaborative projects *"Complexity Research Initiative for Systemic InstabilitieS (CRISIS)"*, grant no. 288501 and from the Ministry of Education, Universities and Research of Italy (MIUR), program *SIR* (grant n. RBSI144KWH), is gratefully acknowledged. None of the above are responsible for errors in this paper.

1 Introduction

Modern monetary policy has emphasized that maintaining a stable monetary environment depends crucially on the ability of the policy regime to control inflation expectations. Woodford (2003) defines the activity of modern central banks (CB hereafter) as *management of expectations*. Therefore, in recent years policy makers developed communication strategies that aim explicitly to align expectations with their own policy objectives. The provision of an explicit numerical inflation target aimed at providing a focal point for private sector expectations is an example of such communication strategies. As argued by Svensson (2009) and Blinder, Ehrmann, Fratzscher, De Haan, and Jansen (2008), in an ideal world characterized by symmetric information between the CB and the rest of the economy and a fully informed private sector holding rational expectations, there is no specific role for CB communication. However, the importance that the debate on CB transparency and communication has assumed in recent years demonstrates that both theorists and policy makers are concerned with deviations from such ideal world.

In this paper we consider a scenario in which the CB announces the target in order to anchor private sector expectations but a biased perception of the target may arise due to information imperfections and transparency issues. In particular, due to idiosyncrasies in the process of understanding and processing information, heterogeneous beliefs about the true inflation target may arise. Heterogeneity in individual expectations has been abundantly documented using survey data on inflation expectations, see, e.g., Carroll (2003), Mankiw, Reis, and Wolfers (2003), Branch (2004), and Pfajfar and Santoro (2010) among others, as well as data on individual expectations collected in learning-to-forecast laboratory experiments, see, e.g., Adam (2007), Pfajfar and Zakelj (2010), Assenza, Heemeijer, Hommes, and Massaro (2011), and Hommes (2011) among others.

Although the private sector may have a biased view of the true target, we introduce discipline in the evolution of beliefs in order to minimize departures from models characterized by full information and rational expectations. In fact, we assume that private sector's beliefs about inflation are revised over time as new information becomes available and the direction of change is determined by the distance between agents beliefs and actual realizations. De Grauwe (2012) considers this willingness to learn via continuos evaluation of individual performance as *the most fundamental definition of rational behavior* (De Grauwe (2012), p. 7). Moreover, evidence for the evolution of heterogeneous forecasting strategies over time in reaction to past forecast errors has been provided by Frankel and Froot (1991), Bloomfield and Hales (2002), Branch (2004), Assenza, Heemeijer, Hommes, and Massaro (2011) and Hommes (2011), among others, using survey data as well as experimental data. Recent theoretical contributions analyzing inflation dynamics under endogenous selection of expectation rules include, among others, Brock and de Fontnouvelle (2000), Tuinstra and Wagener (2007), Brazier, Harrison, King, and Yates (2008), Branch and McGough (2010), De Grauwe (2011), Branch and Evans (2011), Anufriev, Assenza, Hommes, and Massaro (2013) and Hommes and Lustenhouwer (2016).[1]

Within this framework in which co-evolution of beliefs and realizations of aggregate variables emerges through the ongoing evaluation of such beliefs, we ask the following question: can a simple instrument rule implemented by the CB lead the economy to the targeted inflation? Intuitively, if the intended inflation target produces good forecasts, or in other words, if the monetary policy rule implemented by the CB keeps inflation close enough to the target, the probability that agents will rely on the true target will be high and dynamics will converge to the intended equilibrium. If, on the other hand, the true inflation target does not produce good forecasts, agents will adopt different predictors, causing the economy to move away from the targeted equilibrium. Our results suggest that, in the presence of imperfect information flows and recursive evaluation of beliefs, standard policy advices

[1]Another line of research relaxes the assumption of rational expectations and considers CB transparency and communications issues within the context of a New Keynesian framework with adaptive learning (see e.g., Orphanides and Williams (2005), Berardi and Duffy (2007), and Eusepi and Preston (2010) among others).

(i.e., obeying the Taylor principle) may not be sufficient to guarantee convergence to the target. Instead, the CB can ensure global stability of the target by fine-tuning monetary policy in order to correct agents' biased beliefs.

The paper is organized as follows. Section 2 presents the theoretical framework featuring recursive inattentiveness and heterogeneous biased beliefs in the presence of CB's inflation targeting. Section 3 derives policy results about inflation target stability. Section 4 contains concluding remarks.

2 The model

This section develops a New Keynesian (NK hereafter) environment extended to include possible biases in the perceived inflation targets, and describes the dynamics of such beliefs.

2.1 An NK economy with biased inflation target beliefs

We consider a NK DSGE model as in Woodford (2003) or Galí (2008). The demand side of the economy is composed by a continuum of households maximizing the expected present value of discounted utility subject to their budget constraint. On the supply side, a continuum of firms produces differentiated consumption goods under monopolistic competition and a staggered price setting mechanism as in Calvo (1983). We assume that firms are owned by households and maximize expected profits given the production function and the households' demand. The equations describing the demand and the supply side of the economy are given by

$$y_t = \bar{E}_t y_{t+1} - \sigma^{-1}(i_t - \bar{E}_t \pi_{t+1}) \quad (2.1)$$

$$\pi_t = k y_t + \beta \bar{E}_t \pi_{t+1}, \quad (2.2)$$

where y denotes the output gap, π refers to inflation, i is the interest rate set by the monetary authority and $\bar{E} = \int_i E_i$ denotes the average expectation across agents (indexed by i), which might have heterogeneous beliefs due to the presence of idiosyncrasies in information processing.[2]

The Central Bank (CB) in the model targets a level of inflation $\bar{\pi}$ via the following interest rate rule:

$$i_t = \bar{\pi} + \phi_\pi(\pi_t - \bar{\pi})\,. \tag{2.3}$$

In the remainder we will assume that the CB has a zero-inflation target, i.e., $\bar{\pi} = 0$, so that equation (2.3) reduces to

$$i_t = \phi_\pi \pi_t\,. \tag{2.4}$$

Although the CB announces the target in order to anchor private sector expectations, we consider a scenario in which a biased perception of the target may arise due to imperfect information flows. Such imperfections may be related to transparency issues or inaccurate information processing. The inaccuracy in the perception of the inflation target in the model generates a potential source of macroeconomic instability related to the lack of coordination among individuals, who then hold heterogeneous beliefs about the target. In particular, following Salle, Yildizoğlu, and Sénégas (2013) we assume that the inflation target perceived by agent i, $\bar{\pi}_i^p$, and the true inflation target are related via the relationship $\bar{\pi}_i^p = \bar{\pi} + \nu_i$, where ν_i represents a noise term. Agents in the model then use their perceived target to forecast future inflation. As for the expectations on the output gap, we assume

[2]We remark that this is the standard approach followed in the literature on monetary policy with diverse beliefs (see, e.g., Brazier, Harrison, King, and Yates (2008), De Grauwe (2011), and Arifovic, Bullard, and Kostyshyna (2013) among others). Micro-founded NK models consistent with heterogeneous expectations have been derived by Branch and McGough (2009), Kurz (2011) and Massaro (2013). Eqs. (2.1) and (2.2) correspond to the model developed by Branch and McGough (2009) or to the model derived in Kurz (2011). We note that in Kurz (2011) there are additional terms in the demand and supply equations, corresponding respectively to the deviation of the average of agents' forecasts of their individual future consumption from the average forecast of aggregate consumption ($\int_i E_{i,t}c_{i,t+1} - E_{i,t}c_{t+1}$) and a similar deviation of price forecasts ($\int_i E_{i,t}p_{i,t+1} - E_{i,t}p_{t+1}$). We treat these terms as i.i.d. disturbances (see Cornea, Hommes, and Massaro (2012) for an empirical assessment) and analyze the dynamics of the deterministic skeleton given by Eqs. (2.1) and (2.2).

that agents, given their perceived inflation target, form their beliefs about the output gap consistently with the structural equations of the canonical NK model. Consequently, given a certain belief $\bar{\pi}_i^p$ about inflation, the correspondent belief about the output gap is $(1-\beta)\bar{\pi}_i^p/k$.[3] This assumption allows us to have a minimal deviation from the standard rational expectations paradigm and it is in line with the interpretation of a rational inattention framework as an environment in which agents know the structural parameters but receive noisy information about the target implemented by the monetary authority.

2.2 Belief dynamics

As a result of imperfect information flows, agents in our model hold heterogeneous beliefs. We allow for individual expectations to change over time and we introduce discipline in the individual selection of the forecasting rule for inflation (and the implied rule for the output gap) by subjecting the choice of the forecasting heuristic to a fitness criterion.

In what follows we will consider a discrete support for the noise term ν_i linking true and perceived inflation target, implying a finite number of biased beliefs. A finite number of forecasting rules seems reasonable, as boundedly rational agents may exhibit digit preference and restrict their predictions, for example, to values in integer numbers or to half percentages.[4] We will relax this assumption in Section 3.2 and allow for the possibility of a continuum of biased beliefs.

We define the probability of choosing a certain predictor h from a set of H predictors conditional on the set of fitness measures $U=(U_1,...,U_H)$ as

$$P(h|U)=\frac{e^{\delta U_h}}{\sum_{h=1}^{H}e^{\delta U_h}}\ . \tag{2.5}$$

[3] Agliari, Pecora, and Spelta (2015) consider a scenario in which the choices of the inflation and the output gap predictors are unrelated.

[4] Digit preference has been observed both in survey measures of expectations and experimental data. See, e.g., Curtin (2005), Duffy and Lunn (2009), and Assenza, Heemeijer, Hommes, and Massaro (2011).

The multinomial logit expression described in Eq. (2.5) can be derived directly from a random utility model (see Manski and McFadden (1981) and Brock and Hommes (1997)) in which agents observe the performance of each rule h with some noise

$$\widetilde{U}_h = U_h + \epsilon_{h,i} \ , \tag{2.6}$$

where $\epsilon_{h,i}$ represent an idiosyncratic error term. Assuming that the noise term $\epsilon_{h,i}$ is drawn from a double exponential distribution, as the number of agents goes to infinity, the probability of agents choosing predictor h, is given by the multinomial logit formula in Eq. (2.5). The parameter δ is referred to as *intensity of choice* and it is inversely related to the variance of the noise term in Eq. (2.6). The intensity of choice reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy according to the fitness measure. The case $\delta = 0$ corresponds to the case of infinite variance in which differences in fitness can not be observed and all probabilities are constant and equal to $1/H$, where H is the total number of available predictors. The case $\delta = \infty$ corresponds to the case in which the deterministic part of the fitness can be perfectly observed and in every period all agents choose the best predictor.

Alternatively, Eq. (2.5) can be derived within a framework in which the choice between the predictor for future inflation, linked to the target announced by the CB, is modeled as an optimization problem under rational inattention. The notion of rational inattention as described by Sims (2003) implies that the true value of the options available to the agents can be investigated, but due to the agents' limited information processing capacity, it is too costly to know them with certainty. Therefore the performance of each option h, measured by U_h, is observed imprecisely, implying noise in the decision process and resulting in probabilistic choices of the agents. Appendix A, based on the results of Matějka and McKay (2015), shows that the probability of choosing predictor h under rational inattention has the same form of Eq. (2.5), with the intensity of choice being inversely related to the *unit cost of information* v. When $v = 0$ ($\delta = \infty$) agents always choose the best predictor, while as v rises the information cost on forecast attractiveness increases, and when $v = \infty$ ($\delta = 0$) agents only decide on the basis of their priors $1/H$.

Given that agents can switch between forecasts in each period, they solve the static predictor selection problem in each period and therefore, the probability, in period t, of choosing a prediction strategy h for inflation and the output gap is given by

$$n_{h,t} = P_t(h|U_{t-1}) = \frac{e^{\delta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\delta U_{h,t-1}}} \ . \tag{2.7}$$

In the light of the aforementioned empirical evidence for the evolution of forecasting strategies in reaction to past forecast errors in both survey and experimental data, we assume that the attractiveness of each predictor is negatively affected by past squared forecast errors. This fitness metric is also common in the theoretical literature (see, e.g., De Grauwe (2012), Branch and McGough (2010), and Anufriev, Assenza, Hommes, and Massaro (2013) among others). The attractiveness of predictor h at the beginning of period t is defined as:

$$U_{h,t-1} = -\sum_{x} \left(x_{t-1} - E_{h,t-2}x_{t-1}\right)^2 \ , \tag{2.8}$$

with $x \in \{\pi, y\}$. Since agents using the same predictor h will have identical expectations, the expectational terms in Eqs. (2.1) – (2.2) can be rewritten as $\bar{E} = \int_i E_i = \sum_{h=1}^{H} n_h E_h$. The full model under scrutiny is described by the following system of equations

$$
\begin{aligned}
y_t &= \sum_{h=1}^{H} n_{h,t} E_{h,t} y_{t+1} - \sigma^{-1} \left(i_t - \sum_{h=1}^{H} n_{h,t} E_{h,t} \pi_{t+1} \right) \\
\pi_t &= k y_t + \beta \sum_{h=1}^{H} n_{h,t} E_{h,t} \pi_{t+1} \\
i_t &= \phi_\pi \pi_t \\
n_{h,t} &= \frac{e^{\delta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\delta U_{h,t-1}}} \\
U_{h,t-1} &= -\sum_{x} \left(x_{t-1} - E_{h,t-2} x_{t-1} \right)^2 \ , \qquad (2.9)
\end{aligned}
$$

where $x \in \{\pi, y\}$ and the set of predictors $h = 1, ..., H$ is composed by pairs of beliefs, respectively for inflation and the output gap.

3 Inflation target stability and monetary policy

We now turn to the main research question: can a simple instrument rule, as in Eq. (2.3), implement the inflation rate targeted by the CB in the presence of biased perceptions and recursive evaluation of beliefs?

3.1 Few biased beliefs

We follow Anufriev, Assenza, Hommes, and Massaro (2013) and start with the simplest possible case in which there are only three types of beliefs. In particular, we will assume that agents may overestimate the target by an amount b_π, underestimate the target by an amount $-b_\pi$, or have correct beliefs about the target. We consider the assumption of a simple constant bias in agents forecasts as a parsimonious representation of a scenario in which agents have an incorrect belief about the target due to informational frictions. De Grauwe (2011) considers a similar environment in which the biases are time-varying and the divergence in beliefs is a function of the volatility of the endogenous variable being forecast, finding that the willingness to learn from past errors leads to waves of optimism and pessimism.

The simplifying assumption of constant beliefs will enable us to derive analytical results about global stability and build the intuition for possible dynamics in the case of many, possibly a continuum of, belief types considered in Section 3.2.[5] The set of predictors is then composed by the pairs[6]

$$\begin{aligned}
&\text{predictor 1:} \quad E_{1,t}\pi_{t+1} = 0, \qquad E_{1,t}y_{t+1} = 0 \\
&\text{predictor 2:} \quad E_{2,t}\pi_{t+1} = b_\pi, \qquad E_{2,t}y_{t+1} = (1-\beta)b_\pi/k \\
&\text{predictor 3:} \quad E_{3,t}\pi_{t+1} = -b_\pi, \quad E_{3,t}y_{t+1} = -(1-\beta)b_\pi/k \ .
\end{aligned}$$

Substituting the specified forecasting rules into system (2.9), we get

$$y_t = (1-\beta)k^{-1}b_\pi(n_{2,t} - n_{3,t}) - \sigma^{-1}i_t + \sigma^{-1}b_\pi(n_{2,t} - n_{3,t}) \tag{3.1a}$$

$$\pi_t = ky_t + \beta b_\pi(n_{2,t} - n_{3,t}) \tag{3.1b}$$

$$i_t = \phi_\pi \pi_t \tag{3.1c}$$

$$n_{h,t} = \frac{e^{\delta U_{h,t-1}}}{\sum_{h=1}^{3} e^{\delta U_{h,t-1}}} \tag{3.1d}$$

$$U_{h,t-1} = -\sum_x (x_{t-1} - E_{h,t-2}x_{t-1})^2 \ , \tag{3.1e}$$

where $h \in \{1, 2, 3\}$, $x \in \{x, y\}$.

Let us define $m_t = n_{2,t} - n_{3,t} = z(y_{t-1},\, \pi_{t-1})$, with $z(y_{t-1},\, \pi_{t-1})$ being described by Eqs. (3.1d) – (3.1e). Then, by substituting the policy rule (3.1c) into the aggregate demand equation (3.1a) and plugging the aggregate supply equation (3.1b) into the resulting expression, we obtain

$$y_t = b_\pi \, \Lambda \, m_t \ , \tag{3.2}$$

[5] We also remark that the type of dynamic behavior shown in De Grauwe (2011) can be found in our framework, e.g., in the presence of stable 2-cycles or in the presence of shocks buffeting the economy and causing dynamics to jump between different basins of attractions.

[6] In this example we assume “symmetric” beliefs, in the sense that positive and negative biases are exactly balanced around the targeted equilibrium. The main reason is that under this assumption the target is among the equilibria of the system and this allows us to address questions about its stability. However we remark that symmetry of beliefs is not essential for many qualitative features of the bifurcation scenario.

where $\Lambda \equiv \frac{(1-\beta)\sigma + k(1-\beta\phi_\pi)}{k(\sigma + k\phi_\pi)}$. Substituting then (3.2) into (3.1b) we get

$$\pi_t = b_\pi \, \Gamma \, m_t \, , \tag{3.3}$$

where $\Gamma \equiv \frac{k+\sigma}{\sigma + k\phi_\pi}$. In this way we can define the map T composed by (3.2) – (3.3) as:

$$T : \begin{cases} y_t = b_\pi \, \Lambda \, z \left(\pi_{t-1}, y_{t-1}\right) \\ \pi_t = b_\pi \, \Gamma \, z \left(\pi_{t-1}, y_{t-1}\right) \end{cases} \tag{3.4}$$

From the Jacobian matrix J of T, given by

$$J = b_\pi \begin{bmatrix} \Lambda z_y & \Lambda z_\pi \\ \Gamma z_y & \Gamma z_\pi \end{bmatrix}$$

it is straightforward to see that $\det J(y, \pi) = 0$. Thus, in any point of the phase space, one eigenvalue is equal to zero. From this consideration it follows that there ought to exist a one-dimensional invariant plane on which dynamics take place. We can indeed state the following

Proposition 3.1. *The straight line $y = \frac{\Lambda}{\Gamma}\pi$ is invariant.*

Proof. See Appendix B. □

The intuition behind Proposition 3.1 follows from the fact that the NK model under analysis is purely forward looking and from the assumption that agents use the structural equations of the model to form expectations about future output gap, given their beliefs about the inflation target. This allows to express output gap expectations as a function of the perceived inflation target. Therefore, the dynamics of (3.4) can be described by the restriction of the map T to the invariant line, that is the following 1D map[7]

$$m_t = f_\delta\left(m_{t-1}\right) = \frac{e^{-\delta\left[M - N m_{t-1}\right]} - e^{-\delta\left[M + N m_{t-1}\right]}}{1 + e^{-\delta\left[M - N m_{t-1}\right]} + e^{-\delta\left[M + N m_{t-1}\right]}}\ , \tag{3.5}$$

in terms of the variable m_t, which is easier to handle analytically. By characterising the dynamics for m_t we can pin down the dynamics of y_t and π_t via (3.2) – (3.3).

The map f_δ is monotonic,[8] bounded and symmetric with respect to the point $m = 0$, which implies that the map always owns the steady state $m^* = 0$ corresponding to the targeted equilibrium (see Appendix D). However, the steady state targeted by the CB, corresponding to $m^* = 0$ in terms of the dynamics described in Eq. (3.5), may not be globally or even locally stable. Dynamics may converge to other *non-fundamental* steady states denoted by $m^+ > 0$ and $m^- = -m^+ < 0$. In what follows we provide a complete analysis of the global dynamics of (3.5) and show how they depend on the parameters of interest, namely the intensity of choice δ and the monetary policy reaction coefficient ϕ_π.[9]

Let us define $\theta = [\beta,\, k,\, \sigma]$, which collects the structural parameters of the NK model, and introduce the positive constants $\phi_\pi^w = \phi_\pi^w(\theta)$, $\phi_\pi^m = \phi_\pi^m(\theta)$, $\phi_\pi^a = \phi_\pi^a(\theta)$, $\phi_\pi^o = \phi_\pi^o(\theta)$ defined in Appendix C, such that $\phi_\pi^w < \phi_\pi^m < \phi_\pi^a < \phi_\pi^o$. We will now identify different monetary policy regimes on the basis of the strength of the monetary policy reaction coefficient ϕ_π. When $\phi_\pi < \phi_\pi^w$ we define the monetary policy regime as *weak*; when $\phi_\pi^w < \phi_\pi < \phi_\pi^m$ the monetary policy regime is defined as *moderate*; when $\phi_\pi^m < \phi_\pi < \phi_\pi^a$ monetary policy is defined as *aggressive*; when $\phi_\pi^a < \phi_\pi < \phi_\pi^o$ we label the policy regime as *very aggressive*; finally, when $\phi_\pi > \phi_\pi^o$ we refer to the implemented policy as *overreacting*. Table 1 summarises the

[7] The expressions for M and N are derived in Appendix B.

[8] The monotonic intervals of f_δ are solely determined by the sign of N as shown in Appendix D.

[9] Consistently with the literature on dynamic predictor selection in NK models, we consider a linearized version of the standard model for the purpose of aggregation and study global dynamics of the nonlinear model resulting from the introduction of recursive evaluation of beliefs (see e.g., Branch and McGough (2010), De Grauwe (2012) and Anufriev, Assenza, Hommes, and Massaro (2013) among others). The analysis of non-fundamental equilibria in this framework is justified on the grounds that linearized coefficients are not sensitive to variations in the steady states.

monetary policy regimes.

Strength of ϕ_π	Monetary policy regime
$\phi_\pi < \phi_\pi^w$	*weak*
$\phi_\pi^w < \phi_\pi < \phi_\pi^m$	*moderate*
$\phi_\pi^m < \phi_\pi < \phi_\pi^a$	*aggressive*
$\phi_\pi^a < \phi_\pi < \phi_\pi^o$	*very aggressive*
$\phi_\pi > \phi_\pi^o$	*overreacting*

Table 1: Monetary policy regimes

Using the Clarida, Galí, and Gertler (2000) calibration, the threshold values for the different monetary policy regimes are: $\phi_\pi^w = 0.8661$, $\phi_\pi^m = 1.9710$, $\phi_\pi^a = 6.2877$ and $\phi_\pi^o = 14.8762$.[10] The corresponding dynamics are described respectively in Propositions 3.2 – 3.6.

Weak monetary policy

Proposition 3.2. *Let $\phi_\pi < \phi_\pi^w$ ("weak policy"). Then values $0 < \delta_1^* \leq \delta_2^*$ exist such that*

- *for $\delta < \delta_1^*$ the target steady state is unique and globally stable;*
- *for $\delta_1^* < \delta < \delta_2^*$ three steady states exist, the unstable target steady state m^*, and two other stable non-fundamental steady states, m^+ and m^-;*
- *for $\delta > \delta_2^*$ five steady states exist, three steady states are locally stable (m^*, m^+ and m^-) and two other steady states are unstable.*

Proof. See Appendix D. □

Figure 1 shows the map f_δ under a weak monetary policy regime ($\phi_\pi = 0.5$) for low, medium, and high values of the the parameter δ.[11] When δ is relatively

[10] Numerical values of monetary policy thresholds vary according to the preferred calibration. For example, using Woodford (1999) calibration leads to the following policy thresholds: $\phi_\pi^w = 0.9617$, $\phi_\pi^m = 1.2490$, $\phi_\pi^a = 1.7997$ and $\phi_\pi^o = 2.1562$. However, the specific values of the thresholds do not alter the qualitative results of the analysis.

[11] The bias parameter b_π is set to 0.25, corresponding to a bias of one percentage point in terms of annualised inflation. Different values of b_π only impact the values of δ at which the bifurcations occur, but they do not change the qualitative bifurcation scenario.

low, the target steady state, corresponding to $m^* = 0$, is unique and globally stable. The intuition for this result is the following. When the cost of information is prohibitively high, i.e., δ is low, agents decide mostly on the basis of their priors, meaning that they are more or less evenly distributed among the different predictors. Therefore, due to the symmetry of beliefs around the target, realized inflation and output will remain relatively close to the target equilibrium and dynamics will converge. As δ increases, two stable non-fundamental steady states are created, $m+ > 0$ and $m^- < 0$, while the target equilibrium $m^* = 0$ loses stability for intermediate values of δ, to become locally stable again for high values of δ, where two additional unstable steady states are created in a pitchfork bifurcation. The intuition for the existence of stable non-fundamental steady states for high values of δ is simple (cf. Proposition D.2 in Appendix D). Suppose that realizations of inflation and output gap are close to some biased beliefs. When the cost of information is low, i.e., intensity of choice is high, almost all agents will adopt the biased predictor, which is the best performing predictor in terms of forecast error. If the monetary policy reaction is weak, the signal sent by realizations of aggregate variables is not strong enough to "correct" agents' beliefs and dynamics may lock into non-fundamental equilibria.

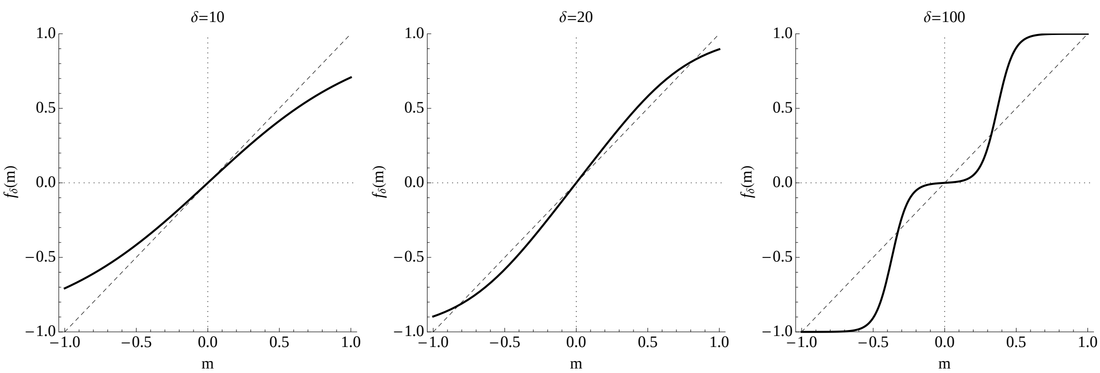

Figure 1: Map $f_\delta(m)$ for different values of δ in the *weak* monetary policy scenario. Parameter values are $\phi_\pi = 0.5$ and $b_\pi = 0.25$.

***Moderate* monetary policy**

Proposition 3.3. *Let $\phi_\pi^w < \phi_\pi < \phi_\pi^m$ ("moderate policy"). Then values $0 < \delta_1^* \leq \delta_2^*$ exist such that*

- *for $\delta < \delta_1^*$ the target steady state is unique and globally stable;*
- *for $\delta > \delta_2^*$ five steady states exist, three steady states (m^*, m^+ and m^-) are locally stable and two other steady states are unstable.*

Proof. See Appendix D. □

Dynamics under a moderate monetary policy are shown in Figure 2 for low, medium and high values of δ. The monetary policy reaction coefficient is set to $\phi_\pi = 1.5$, a value typically suggested in the literature (see, e.g., Taylor (1993)). As before, a decrease in the cost of information, i.e., higher δ, leads to the creation of stable non-fundamental steady states. The difference from the previous case is that the target equilibrium does not lose local stability. Therefore, an interest rate rule that reacts more than point to point to deviations of inflation from the target, leads to convergence to the fundamental equilibrium if the economy is sufficiently close to the target. However, the Taylor principle (i.e., $\phi_\pi > 1$) alone is not sufficient to ensure convergence to the target. In fact, even if the target equilibrium is determinate under rational expectations, the presence of misperception of the CB target coupled with recursive evaluation of forecasting heuristics may lead to convergence to non-fundamental equilibria. As before, the existence of multiple stable equilibria is due to the fact that monetary policy is not strong enough, even if the Taylor principle is satisfied, to correct wrong beliefs of agents about the target. We remark that in the presence of finite values of δ agents might not select in equilibrium the best performing predictor due to the presence of noise in the evaluation of the performances.[12] On the other hand, when $\delta \to \infty$ all agents select

[12]This result is driven by the general idea that agents cannot perfectly observe the performances of alternative predictors due to imperfect information flows, and qualitative insights are robust to the specific predictor selection mechanism.

the best performing predictor in each period. When all agents coordinate e.g., on a biased predictor, we have an almost self-fulfilling equilibrium in the sense that aggregate outcomes do not exactly coincide with agents predictions. This is due to the fact that we are now considering a scenario with a finite number of beliefs and agents are choosing the best forecasting model among those available to them. We will relax this assumption is Section 3.2 and consider an arbitrarily large number of predictors.

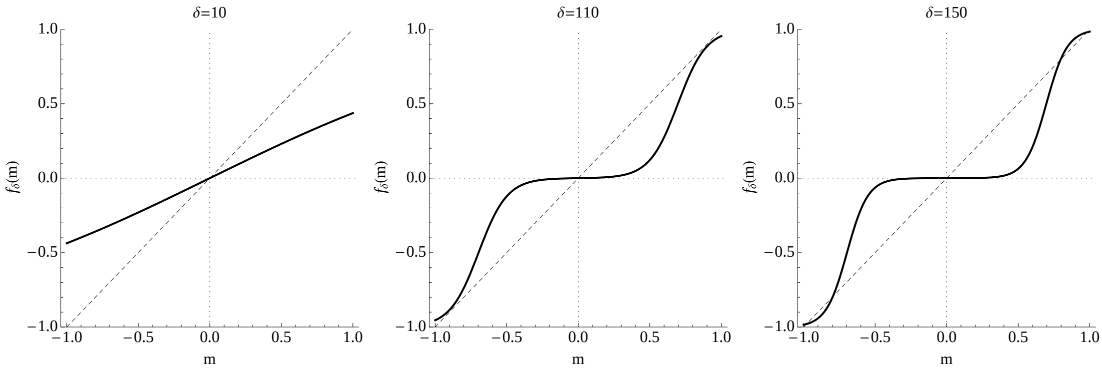

Figure 2: Map $f_\delta(m)$ for different values of δ in the *moderate* monetary policy scenario. Parameter values are $\phi_\pi = 1.5$ and $b_\pi = 0.25$.

Aggressive **monetary policy**

Proposition 3.4. *Let $\phi_\pi^m < \phi_\pi < \phi_\pi^a$ ("aggressive policy"). Then the target steady state is unique and globally stable for any δ.*

Proof. See Appendix D. □

When the nominal interest rate reacts aggressively to inflation, the CB avoids multiplicity of equilibria and the target equilibrium is globally stable. Adjustment dynamics differ according to whether the slope of the map is positive or negative, i.e., whether $\phi_\pi^m < \phi_\pi < \phi_\pi^*$ or $\phi_\pi^* < \phi_\pi < \phi_\pi^a$.[13] Figure 3 depicts the map (3.5) for different values of δ and $\phi_\pi = 2$ in the case of positive slope. The aggressive monetary policy regime of the CB reacts to deviations from the target in such a way that the fundamental equilibrium is closest to realizations of aggregate variables

[13]See Appendix C for the definition of ϕ_π^*.

and ongoing evaluations of forecasting rule lead more and more agents to believe in the true value of the target. Therefore, a properly designed monetary policy leads to uniqueness and global stability of the target steady state even in the presence of biased beliefs about the true target.

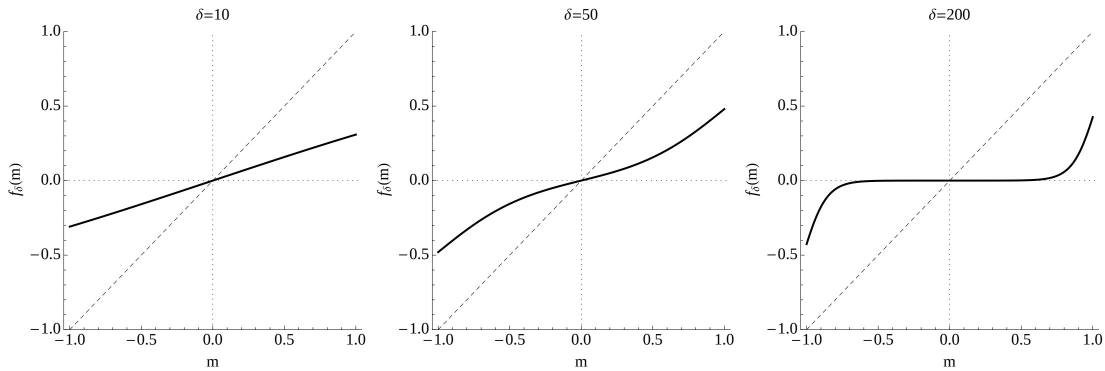

Figure 3: Map $f_\delta(m)$ for different values of δ in the *aggressive* monetary policy scenario. Parameter values are $\phi_\pi = 2$ and $b_\pi = 0.25$.

Very aggressive **monetary policy**

Proposition 3.5. *Let $\phi_\pi^a < \phi_\pi < \phi_\pi^o$ ("very aggressive policy"). Then a value $\delta_1^* > 0$ exists such that*

- *for $\delta < \delta_1^*$ the target steady state is unique and globally stable;*
- *for $\delta > \delta_1^*$ the locally stable target steady state and a stable 2-cycle coexist, separated by an unstable 2-cycle.*

Proof. See Appendix D. □

In this scenario macroeconomic variables follow an oscillatory path that can lead to convergence to the target equilibrium or to a stable 2-cycle. In fact, Figure 4 shows the creation of two stable non-fundamental steady state for the *second iterate* of map f_δ for high values of δ. The reason for this result is due to the strong negative effect of real interest rate on output, acting as a stabilizing force. Suppose that a positive cost-push shock hits the economy. Higher inflation causes the CB to raise the real interest rate which in turns lowers demand which reduces future inflation. However, if the reaction of the CB is too strong, the decrease in demand in the face of higher real rates will be high enough to push the economy out of the basin of attraction of the target steady state, and the system will lock in a stable 2-cycle. The reason is that when ϕ_π is relatively high, there will be a consistent decrease in output after, say a positive shock to inflation, and this will have a positive impact on the performance of the negative bias predictor causing more and more agents to adopt that predictor. If the intensity of choice δ is relatively low, agents do not respond fast to differences in predictors' performances and dynamics will slowly converge to the target equilibrium. However, when δ is higher, even small differences in predictors' performances may lead agents to switch massively among forecasting rules. These alternate waves of "optimism" and "pessimism" lead the system to a stable 2-cycle.

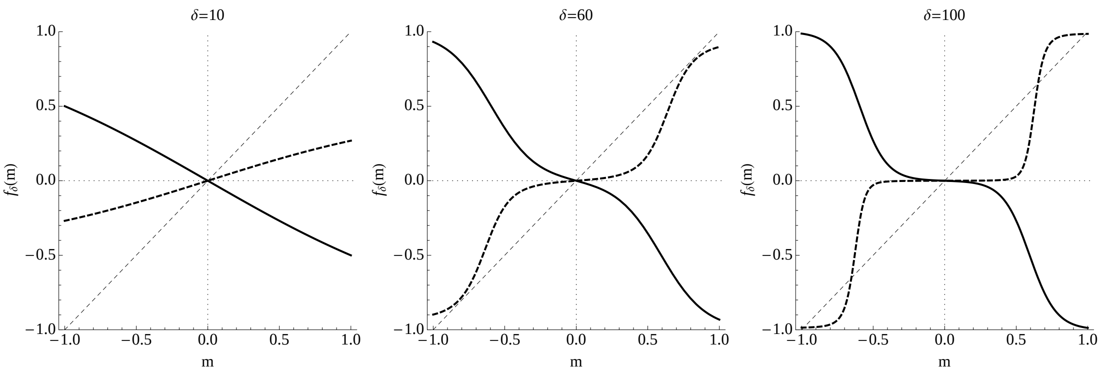

Figure 4: Map $f_\delta(m)$ (solid line) and second iterate $f^2_\delta(m)$ (thick dashed line) for different values of δ in the *very aggressive* monetary policy scenario. Parameter values are $\phi_\pi = 10$ and $b_\pi = 0.25$.

Overreacting monetary policy

Proposition 3.6. *Let $\phi_\pi > \phi_\pi^o$ ("overreacting policy"). Then values $0 < \delta_1^* < \delta_2^*$ exist such that*

- *for $\delta < \delta_1^*$ the target steady state is unique and globally stable;*
- *for $\delta_1^* < \delta < \delta_2^*$ the unstable target steady state and a stable 2-cycle coexist;*

- *for $\delta > \delta_2^*$ the locally stable target steady state and a stable 2-cycle coexist, separated by an unstable 2-cycle.*

Proof. See Appendix D. □

Dynamics under an overreacting monetary policy are described in Figure 5 for low, intermediate and high values of δ. As in the previous case, an increase in δ leads to the creation of a stable 2-cycle characterized by large shifts in agents beliefs. However, for intermediate values of δ the target equilibrium loses local stability.

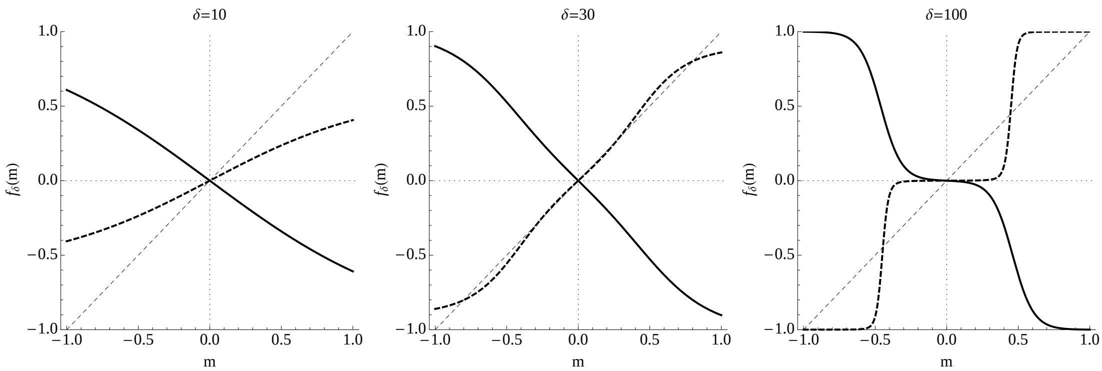

Figure 5: Map $f_\delta(m)$ (solid line) and second iterate $f_\delta^2(m)$ (thick dashed line) for different values of δ in the *overreacting* monetary policy scenario. Parameter values are $\phi_\pi = 15$ and $b_\pi = 0.25$.

The results of the analysis performed in this section show that in a scenario in which biased perceptions of the CB target arise due to imperfections in information processing, standard policy advices, such as the Taylor principle (i.e., $\phi_\pi > 1$), may not be sufficient to ensure convergence to the target. Rational inattention and recursive evaluation of beliefs as new information becomes available may result in co-evolution of aggregate variables and beliefs towards non-fundamental steady states or 2-cycles. Nevertheless, a properly designed monetary policy can ensure convergence to the target by impacting, via the interest rate, on realizations of macro variables in such a way to correct wrong agents' beliefs.

3.2 Many biased beliefs

In Section 3.1 we considered the simplest possible scenario in which information imperfections and individual idiosyncrasies gave rise to three different types of beliefs, corresponding to underestimation, overestimation and correct guess of the target. This example enabled us to derive analytical results and build the intuition for possible dynamics as a function of the key parameters δ, related to the cost of information, and ϕ_π, measuring the strength of monetary policy.

A similar analysis can be made for other examples with a larger number of heterogeneous beliefs resulting from imperfect information. Figure 6 shows the bifurcation diagram in the presence of five different beliefs, namely $\{-b_\pi, -b_\pi/2, 0, b_\pi/2, b_\pi\}$, with respect to the intensity of choice parameter δ.[14]

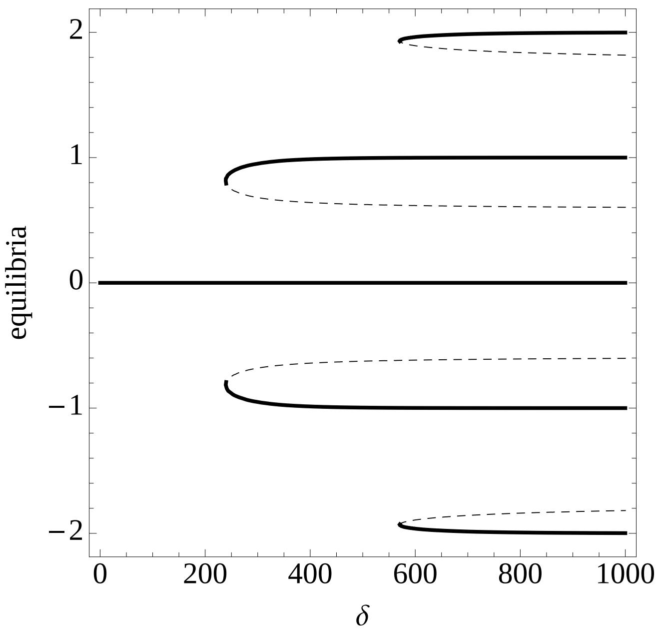

Figure 6: Bifurcation diagram for the system with five beliefs. Solid (dashed) lines indicate stable (unstable) equilibria. Parameter values are $\phi_\pi = 1.25$ and $b_\pi = 0.25$.

For high values of δ, additional steady states are created. The intuition is similar to the case of three beliefs. If the intensity of choice is high enough, more and more agents will adopt the belief yielding the most precise forecast, causing dynamics to

[14]The bifurcation diagram refers to system (3.1), where the variable m_t is now defined as $m_t = 2n_{2,t} - 2n_{3,t} + n_{4,t} - n_{5,t}$ and therefore bounded by -2 and +2.

lock into a self-fulfilling non-fundamental equilibrium.

In this section we want to revisit our main policy question, i.e, whether a simple interest rate rule can implement the inflation level targeted by the CB in the presence of biased perceptions and recursive evaluation of beliefs, when the number of beliefs H is arbitrarily large. In general, it is difficult to obtain analytical results for systems with many belief types. We will therefore resort to the *large type limit* concept (LTL henceforth) introduced in Brock, Hommes, and Wagener (2005) and used by Anufriev, Assenza, Hommes, and Massaro (2013) in a similar context. Given an arbitrary set of H inflation beliefs $b_h \in \mathbb{R}$, and correspondent output gap beliefs ab_h with $a \equiv (1-\beta)/k$, drawn from a common initial distribution with density $\psi(b)$, the average expectations terms in system (2.9) can be written as

$$\begin{aligned}
\bar{E}_t y_{t+1} &= \frac{\frac{1}{H}\sum_{h=1}^{H} ab_h \exp\left(-\delta\left((b_h-\pi_{t-1})^2+(ab_h-y_{t-1})^2\right)\right)}{\frac{1}{H}\sum_{h=1}^{H} \exp\left(-\delta\left((b_h-\pi_{t-1})^2+(ab_h-y_{t-1})^2\right)\right)} \\
\bar{E}_t \pi_{t+1} &= \frac{\frac{1}{H}\sum_{h=1}^{H} b_h \exp\left(-\delta\left((b_h-\pi_{t-1})^2+(ab_h-y_{t-1})^2\right)\right)}{\frac{1}{H}\sum_{h=1}^{H} \exp\left(-\delta\left((b_h-\pi_{t-1})^2+(ab_h-y_{t-1})^2\right)\right)} ,
\end{aligned}$$

where we divided both numerators and denominators by H. The LTL is obtained by replacing the sample mean with the population mean, yielding

$$\bar{E}_t y_{t+1} = \frac{\int ab \exp\left(-\delta((b-\pi_{t-1})^2+(ab-y_{t-1})^2)\right)\psi(b)db}{\int \exp\left(-\delta((b-\pi_{t-1})^2+(ab-y_{t-1})^2)\right)\psi(b)db} \tag{3.6a}$$

$$\bar{E}_t \pi_{t+1} = \frac{\int b \exp\left(-\delta((b-\pi_{t-1})^2+(ab-y_{t-1})^2)\right)\psi(b)db}{\int \exp\left(-\delta((b-\pi_{t-1})^2+(ab-y_{t-1})^2)\right)\psi(b)db} . \tag{3.6b}$$

As shown in Brock, Hommes, and Wagener (2005), when the number of beliefs H is sufficiently large the LTL dynamics well approximate the dynamics of the system with H beliefs. In particular, when H is large, with high probability the steady states and their local stability conditions coincide for both the LTL and the H-beliefs map. Hence, properties of the dynamical system with many types of beliefs can be studied using the LTL system.

For suitable distributions $\psi(b)$ of initial beliefs, Eqs. (3.6a) – (3.6b) can be computed explicitly. We follow Anufriev, Assenza, Hommes, and Massaro (2013) and consider a normal distribution $\psi(b) \sim N(0, s^2)$ centered around the CB inflation target. In this case a straightforward computation shows that Eqs. (3.6a) – (3.6b) reduce to

$$\bar{E}_t y_{t+1} = \frac{\delta 2 s^2}{1 + \delta 2 s^2 (1 + a^2)} a(\pi_{t-1} + a y_{t-1}) \tag{3.7a}$$

$$\bar{E}_t \pi_{t+1} = \frac{\delta 2 s^2}{1 + \delta 2 s^2 (1 + a^2)} (\pi_{t-1} + a y_{t-1}) \, . \tag{3.7b}$$

Using results (3.7a) – (3.7b), we can rewrite system (2.9) as

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \Lambda \cdot M_y & \Lambda \cdot M_\pi \\ \Gamma \cdot M_y & \Gamma \cdot M_\pi \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} \tag{3.8}$$

where Λ and Γ are defined as before (see Appendix B) and

$$\begin{aligned} M_y &= \frac{2k(1-\beta)\delta s^2}{k^2 + 2(k^2 + (\beta - 1)^2)\delta s^2} \\ M_\pi &= \frac{2k^2 \delta s^2}{k^2 + 2(k^2 + (\beta - 1)^2)\delta s^2} \, . \end{aligned}$$

Dynamics in the presence of a continuum of beliefs are described in the following proposition:

Proposition 3.7. *Consider the LTL dynamics described by linear system* (3.8)*.*

1. *Let $\phi_\pi < 1$. Then a value δ^* exists such that:*

    - *for $\delta < \delta^*$ the target steady state is unique and globally stable;*
    - *for $\delta > \delta^*$ the target steady state is unstable.*

2. Let $\phi_\pi > 1$. Under certain restrictions on structural parameters,[15] *a value ϕ_π^{**} exists such that:*

(2a) for $\phi_\pi < \phi_\pi^{**}$ the target steady state is unique and globally stable for any δ;

(2b) for $\phi_\pi > \phi_\pi^{**}$, a value δ^{**} exists such that:

- *for $\delta < \delta^{**}$ the target steady state is unique and globally stable;*
- *for $\delta > \delta^{**}$ the target steady state is unstable.*

Proof. See Appendix E. □

The general policy implications of agents' biased perception of the target and recursive evaluation of beliefs derived in the case of few biased perceptions carry over to the case of an arbitrarily large number of heterogeneous beliefs. In particular, when the cost of information is prohibitively high, the target steady state is globally stable. The intuition for this result is the same as in the case with few biased beliefs laid out in Section 3.1. When the coefficient $\phi_\pi < 1$, the monetary policy reaction to inflation is not strong enough to offset deviations of inflation from the target, leading to system instability. Once again, the Taylor principle, i.e., $\phi_\pi > 1$, is not a sufficient condition to guarantee convergence to the target. In fact, as in the case of few biased beliefs, monetary policy may overreact to deviations of inflation from the target, causing oscillatory dynamics moving away from the target.

4 Conclusions

This paper discusses the issue of inflation target implementability via simple instrument rules. In particular, we consider a scenario in which the CB announces the

[15] The set of parameter restrictions is described in Appendix E. Most commonly used calibrated values (see, e.g., Woodford (1999) and Clarida, Galí, and Gertler (2000) among others) satisfy these restrictions. If the restrictions are not satisfied, then, given $\phi_\pi > 1$, the target steady state is globally stable for any δ.

target to anchor private sector expectations, but biased perceptions of the target arise due to imperfect information.

Recursive evaluation of beliefs as new information becomes available leads to a dynamical system in which aggregate variables and private sector's expectations co-evolve over time. The specific form of beliefs dynamics can be derived alternatively from a random utility model in which agents observe the distance between predictions and realizations with some noise, or from an optimization problem under rational inattention, in which agents face an information processing capacity constraint. Both frameworks link the probability of agents' holding a certain belief to the performance of such belief in terms of forecasting error, via a multinomial logit model. Within this environment, we investigate whether the monetary authority can effectively manage private sector expectations via an interest rate rule, and lead the economy to the desired target.

Our results suggest that the CB's ability to implement the inflation target depends crucially upon the interplay between the strength of monetary policy reaction to inflation, and the key parameter regulating the evolution of beliefs over time. The latter is related to the noise with which agents observe predictors' performances within the random utility framework, or to the cost of information within the rational inattention environment. In a fully specified model the unit cost of information would be endogenous. For example, it could be that in periods of high-news-coverage of macroeconomic conditions related e.g., to high aggregate volatility, the unit cost of information is relatively lower due to the vast availability of information through the media;[16] or in general the level of technical change might determine the cost of collecting and processing information. In this paper we keep the parameter regulating the cost of information fixed over time for analytical simplicity and leave the case of a time-varying cost for future research.

At first we analyse the simplest possible scenario in which information imperfections give raise to only few biased beliefs. This allows us to derive analytical

[16]This would be consistent with the results of Carroll (2003) who finds that during high-news-coverage periods agents are better informed.

results about global stability and build the intuition for the resulting dynamics. We then consider a more general scenario in which an arbitrarily large number of biased perceptions may arise as a consequence of imperfect information flows.

We find that, when the cost of information is not prohibitively high, the monetary authority should react aggressively to deviations of inflation from the target. However, merely obeying to the Taylor principle, i.e., setting the interest rate to ensure determinacy, may not be sufficient to achieve the target. In fact, as argued by Branch and McGough (2010), in the presence of heterogeneous beliefs, determinacy under rational expectations may not be a robust criterion for policy advice. When monetary policy is passive (i.e., $\phi_\pi < 1$), the fundamental equilibrium is indeterminate under rational expectations, leading to the existence of sunspot equilibria which are closely connected to the existence of non-fundamental equilibria in our model. However, our results show that even when monetary policy is set to ensure determinacy under rational expectations (i.e., $\phi_\pi > 1$), multiple equilibria and excess volatility may arise under heterogeneous expectations, for example when the policy regime is moderate or overly aggressive as described in Section 3.1. Indeed, given the CGG calibration, a reaction coefficient $1 < \phi_\pi < \phi_\pi^m$ or $\phi_\pi > \phi_\pi^a > 1$ ensures determinacy under rational expectations, but not uniqueness and global stability under recursive inattentiveness and heterogeneous beliefs. The finding that determinacy under rational expectations may not be enough to insulate the economy from instability has also been showed in Benhabib, Schmitt-Grohé, and Uribe (2001), Benhabib and Eusepi (2005), Branch and McGough (2010) among others.

The CB could also act in order to increase the precision of information to the public in order to reduce the uncertainty about the target in the market. In particular, the CB could act in order to influence the cost of information, i.e., provide easier access to information about the target or in general about future policy actions. This recent shift in communication policies of CB, e.g., forward guidance, has been empirically documented in recent studies (see Campbell, Evans, Fisher, and

Justiniano (2012), Negro, Giannoni, and Patterson (2012) and Kool and Thornton (2012) among others). Nevertheless, our results show that reducing the cost of information is not enough, by itself, as a policy measure to guarantee global stability of the target. In fact, if monetary policy is not tight enough or if it overreacts to inflation fluctuations, lowering the cost of information may only guarantee the local stability of the target.[17] Instead, monetary policy should be fine-tuned in order to ensure that the signal sent by realizations of aggregate variables can correct wrong agents' beliefs. Indeed, our findings suggests that properly designed monetary policies lead to uniqueness and global stability of the target steady state and are robust to different levels of the cost of information.

An interesting extension of our analysis would be to include a fully rational predictor, perhaps available at some cost, among the set of beliefs. Notice in fact that, although some agents in our setup have correct beliefs about the target, their forecasts in each period do not coincide with perfect foresight, i.e., fully rational expectations, due to the presence of boundedly rational agents. The introduction of a fraction of boundedly rational agents in a setup with rational expectations may alter sensibly the determinacy properties of the model as shown for example by Branch and McGough (2009) and Massaro (2013) among others, who consider models in which a fixed fraction of agents has rational expectations, while another fixed fraction of agents has boundedly rational expectations. We leave the analysis of a richer model including (costly) fully rational expectations and recursive evaluation of beliefs for future research.

[17] See also Gaballo (2015) on the interaction between monetary policy and the social value of information.

Appendix

A Belief dynamics under Rational Inattention

Given the set of options available to agents, the performance of each option h, measured by U_h, is observed imprecisely, implying noise in the decision process and resulting in probabilistic choices of the agents. Agents do not know *a priori* which predictor will be more attractive in a certain period, so their prior probabilities of choosing each predictor, without processing any information, are symmetric and equal to $1/H$. Therefore we can apply the framework developed in Matějka and McKay (2015) to derive the probabilistic choice between predictors as the outcome of an optimization problem under rational inattention with discrete and symmetric options.[18] Under rational inattention, agents cannot fully observe the true values of $U = (U_1, ..., U_H)$, and they have some prior knowledge on the predictor attractiveness given by the joint pdf $g(U)$. Agents receive signals on the choices h and the cost of information is defined using an entropy-based measure. For a generic discrete random variable X, entropy is defined as

$$Q(X) = -\sum_k P(k) \log P(k) \,,$$

where $P(k)$ is the probability of state k. The cost function used in the rational inattention literature is given by the *mutual information* defined as the reduction in entropy of X due to signal Y about X, i.e., $Q(X) - Q(X|Y)$. In our setup the cost of information is therefore defined as

$$c(P, g) \equiv v \left(-\sum_h 1/H \log(1/H) + \int_U \sum_h P(h|U) \log(P(h|U)) g(U) dU \right) \tag{A.1}$$

where v is the *unit cost of information*, the first term is the prior uncertainty,[19] while the second term denotes the posterior uncertainty after observing the signal. Under rational

[18] See also Dräger (2015) for a recent application of the framework developed by Matějka and McKay (2015) to model the dynamic choice between a fully rational predictor and a sticky information predictor in changing macroeconomic conditions.

[19] Notice that we used the fact that we assumed uniform priors $1/H$.

inattention, the optimisation problem faced by the agents reads as follows:

$$\max_{P=\{P(h|U)\}_{h=1}^{H}} \sum_h \int_U U_h P(h|U) g(U) dU - c(P,g) \tag{A.2}$$

subject to

$$P(h|U) \geq 0 \tag{A.3}$$

$$\sum_h P(h|U) = 1, \tag{A.4}$$

where $c(P,g)$ is defined in Eq. (A.1). In other words, agents choose the conditional probabilities $\{P(h|U)\}_{h=1}^{H}$ in order to maximise the expected value of the predictor attractiveness U net of the cost of information processing. From the first order conditions of problem (A.2) – (A.4), Matějka and McKay (2015) show that the probability of choosing predictor h, given a set of values in U is given by

$$P(h|U) = \frac{e^{U_h/v}}{\sum_{h=1}^{H} e^{U_h/v}}.$$

Defining $v \equiv 1/\delta$ we obtain Eq. (2.5).

B Reduction to 1D map

Proof of Proposition 3.1. Points on the straight line $y = \frac{\Lambda}{\Gamma}\pi$ are given by the (parametric) representation:

$$T: \begin{cases} y = b_\pi \, \Lambda \, m \\ \pi = b_\pi \, \Gamma \, m \end{cases}$$

where $m \in \mathbb{R}$, $\Lambda \equiv \frac{(1-\beta)\sigma + k(1-\beta\phi_\pi)}{k(\sigma+k\phi_\pi)}$, and $\Gamma \equiv \frac{k+\sigma}{\sigma+k\phi_\pi}$.

Let $S = \{(b_\pi\Lambda m, b_\pi\Gamma m) \in \mathbb{R}^2 : m \in \mathbb{R}\}$ be the invariant set. The assertion of Proposition 3.1 follows by showing that $P' = T(P) \in S$ for any $P \in S$.

Recall the definition $m_t = z(y_{t-1}, \pi_{t-1})$. From $P = (b_\pi\Lambda m, b_\pi\Gamma m) \in S$ and applying map T, we get $P' = (b_\pi\Lambda z(b_\pi\Lambda m, b_\pi\Gamma m), b_\pi\Gamma z(b_\pi\Lambda m, b_\pi\Gamma m)) \in S$. Calling

$m' = z(b_\pi \Lambda m, b_\pi \Gamma m)$, the restriction of T on set S is the 1-D map

$$m' = f_\delta(m) = \frac{e^{-\delta[M-Nm]} - e^{-\delta[M+Nm]}}{1 + e^{-\delta[M-Nm]} + e^{-\delta[M+Nm]}}$$

where

$$M = \left(\frac{(1-\beta)\, b_\pi}{k}\right)^2 + b_\pi^2$$

and

$$N = 2b_\pi \left(\frac{(1-\beta)}{k} b_\pi \Lambda + b_\pi \Gamma\right).$$

The trajectories starting in S belong to it forever, while any point not belonging to S is mapped into S in one iteration. $\square$

C Monetary policy thresholds

Define the function $q(\phi_\pi) = \frac{M}{N}$. Given the theoretical restriction $0 < \beta < 1$, the function $q(\phi_\pi)$ has the following properties:

- $q(\phi_\pi)$ has an asymptote in $\phi_\pi = \phi_\pi^*$
- $0 < q(0) < 1$
- $q'(\phi_\pi) > 0$
- $\lim_{\phi_\pi \to +\infty} q(\phi_\pi) < 0$

where $\phi_\pi^* = \frac{\left(k+\sigma-k\beta-2\sigma\beta+k^2\sigma+\sigma\beta^2+k^3\right)}{k\beta(1-\beta)}$. Let us introduce two positive quantities ϕ_π^w and ϕ_π^o defined by the solution of equation $x^* - 1 = |q(\phi_\pi)|$ where $x^* \approx 1.46306$ is the solution of $2 + e^x - xe^x = 0$ (see Lemma 1 in Appendix C), and other two positive quantities ϕ_π^m and ϕ_π^a defined by the solution of the equation $1 = |q(\phi_\pi)|$. Given the properties of $q(\phi_\pi)$ we have that $\phi_\pi^w < \phi_\pi^m < \phi_\pi^* < \phi_\pi^a < \phi_\pi^o$. The function $q(\phi_\pi) = \frac{M}{N}$, together with the critical values of ϕ_π^w, ϕ_π^m, ϕ_π^a, ϕ_π^o, are shown in Figure 7 using the Clarida, Galí, and Gertler (2000) (CGG) calibration. Given the calibrated values for the structural parameters, the threshold values for the different monetary policy regimes are: $\phi_\pi^w = 0.8661$, $\phi_\pi^m = 1.9710$, $\phi_\pi^a = 6.2877$ and $\phi_\pi^o = 14.8762$.

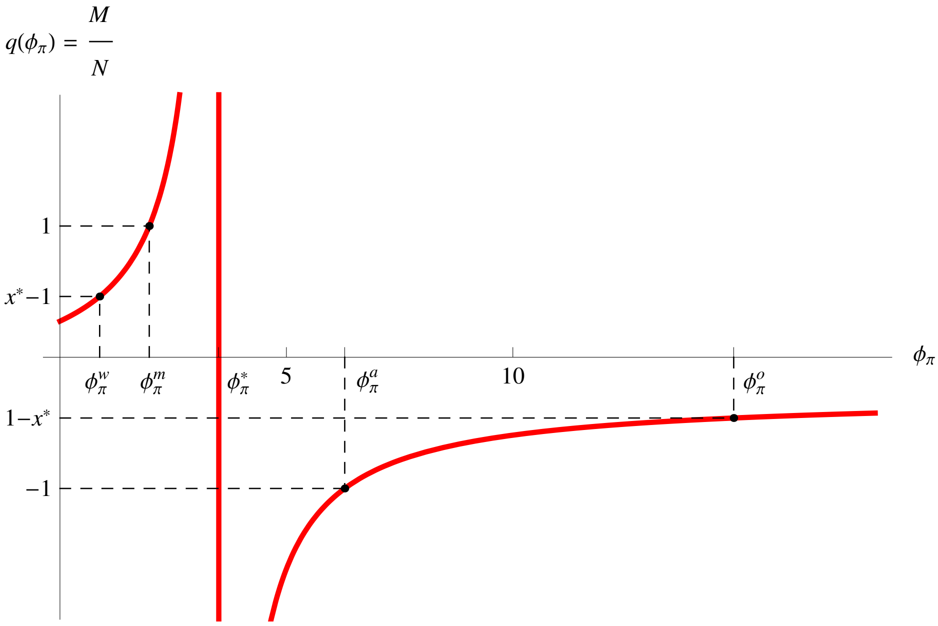

Figure 7: Function $q(\phi_\pi)$, CGG calibration.

D Dynamics of the model with few biased beliefs

In order to analyse the dynamics of the model with few biased beliefs we follow the strategy laid out in Anufriev, Assenza, Hommes, and Massaro (2013). The slope of the map in (3.5) is given by

$$f_\delta'(m) = \frac{\delta e^{\delta N m}\left(e^{\delta M} + 4e^{\delta N m} + e^{\delta(M+2Nm)}\right) N}{\left(1 + e^{\delta 2Nm} + e^{\delta(M+Nm)}\right)^2} . \tag{D.1}$$

Notice that the sign of the previous expression depends only from the parameter N, because $\delta \in [0, \infty)$ and M is always positive. The monotonic intervals of $f_\delta(m)$ are determined by the sign of N which, in turns, depends on the values of ϕ_π, since all the other structural parameters are set at the baseline calibration. Thus we calculate the ϕ_π values that satisfy $N = N(\phi_\pi) > 0$, i.e.

$$\frac{2}{k^2}\frac{b_\pi}{\sigma+k\phi_\pi}\left(k+\sigma-k\beta-2\sigma\beta+k^2\sigma+\sigma\beta^2+k^3-k\beta\phi_\pi+k\beta^2\phi_\pi\right)>0$$

Since all coefficients are positive, we find a threshold value for ϕ_π

$$\phi_\pi^*=\frac{\left(k+\sigma-k\beta-2\sigma\beta+k^2\sigma+\sigma\beta^2+k^3\right)}{k\beta\left(1-\beta\right)}$$

We can distinguish two cases, namely $\phi_\pi<\phi_\pi^*$, implying that $f_\delta(m)$ is increasing, and $\phi_\pi>\phi_\pi^*$ implying that $f_\delta(m)$ is decreasing. Using the CGG calibration, the threshold value such that $N=0$ is $\phi_\pi^*=3.5059$.

Moreover, the function $f_\delta(m)$ has the following properties:

- f maps the interval $[-1,1]$ into itself;
- f is bounded;
- f is odd because $f_\delta\left(-m\right)=-f_\delta\left(m\right)$.

The monotonicity of f ensures that, when $\phi_\pi<\phi_\pi^*$, from the minimum value $f(-1)$ it follows that $f(-1)\geq-1$ and from the maximum value $f(1)$ it follows that $f(1)\leq 1$. Since function $f_\delta\left(m\right)$ is bounded (either from below or above), no diverging trajectories are possible. Indeed m expresses the difference between fractions and it can assume only value in the interval $[-1,1]$.

The following lemmas are useful to prove the results described in Proposition 3.2.

Lemma 1. *Equation* $2+e^x-xe^x=0$ *has a unique solution* $x^*\in(1,2)$*. For* $x<x^*$ *we have* $2+e^x-xe^x>0$ *and for* $x>x^*$ *we have* $2+e^x-xe^x<0$.

Proof. Consider the function $g(x)=2+e^x-xe^x$. Notice that $\lim_{x\to-\infty}g(x)=2$, $\lim_{x\to\infty}g(x)=-\infty$, $g(0)=3$, and that derivative $g'(x)=-xe^x$. Hence, for $x\leq 0$ function g increases from 2 to 3 and has no zeros. For $x>0$ function g is strictly decreasing and has at most one zero. On the other hand, $g(1)=2>0$, while $g(2)=2-e^2<0$, because $e^x>1+x$ for $x=2$ becomes $e^2>3$. Applying the intermediate value theorem we obtain that there exists x^*, zero of function g, and that $x^*\in(1,2)$. $\square$

Lemma 2. *The function $f_\delta(m)$ defined on $(0, \infty)$ is concave for every $0 < \delta < \frac{\ln 4}{M}$*

Proof. The second derivative of f_δ is given by

$$f''_\delta(m) = -\frac{N^2\delta^2 e^{\delta Nm}(e^{2\delta Nm} - 1)}{(1 + e^{2\delta Nm} + e^{\delta(M+Nm)})^3}(e^{\delta M} + 8e^{\delta Nm} + e^{\delta(M+2Nm)} - e^{\delta(2M+Nm)})$$

The fraction in this expression is positive for $m > 0$. Hence the sign of the second derivative depends only on the term between brackets, which can be rewritten as

$$e^{\delta M}\left(1 + e^{2\delta Nm}\right) + e^{\delta Nm}\left(8 - e^{2\delta M}\right) \ .$$

When $m = 0$ this term becomes

$$2e^{\delta M} + 8 - e^{2\delta M} = \left(2 + e^{\delta M}\right)\left(4 - e^{\delta M}\right)$$

which is positive when $\left(4 - e^{\delta M}\right) > 0$ i.e. $0 < \delta < \frac{\log 4}{M}$. By continuity of the second derivative, $f''_\delta(m) < 0$ for small $m > 0$. With a further increase of m, the sign of the second derivative would change when the term between brackets is zero, i.e. when

$$\frac{e^{\delta M}}{8 - e^{2\delta M}} = -\frac{e^{\delta Nm}}{1 + e^{2\delta Nm}} \tag{D.2}$$

Fix $e^{\delta M} = x$. The left hand side can be re-written as $\frac{x}{8-x^2}$ and this function does not take values in the interval $[-0.5, 0)$. However the right hand side does take values only in this interval, as a function $\frac{-t}{1+t^2}$ where we set $t = e^{\delta Nm}$. It means that there is no m to satisfy equality (D.2) and $f''_\delta(m)$ does not change its sign. Thus we establish that $f''_\delta(m) < 0$ for $0 < \delta < \frac{\log 4}{M}$ and for any $m > 0$. This completes the proof.

□

The following result provides conditions for local stability of the inflation target of the CB for the case in which the map f_δ is increasing, i.e., when $\phi_\pi < \phi^*_\pi$.

Proposition D.1 (Local stability of the target for $\phi_\pi < \phi^*_\pi$)**.** *Consider the dynamics given by* (3.5)*. Let x^* denote the solution of the equation $2 + e^x - xe^x = 0$. The following cases are possible:*

1. *When $\phi_\pi < \phi_\pi^w$, two values $0 < \delta_1^* < \delta_2^*$ exist such that for $\delta \notin [\delta_1^*, \delta_2^*]$ the target steady state is locally stable, and for $\delta \in (\delta_1^*, \delta_2^*)$ the target steady state is unstable.*

2. *When $\phi_\pi > \phi_\pi^w$ the target steady state is locally stable for any $\delta \geq 0$.*

Proof. The derivative of map f_δ described in (D.1) computed in the target steady state is given by

$$f'_\delta(0) = \frac{2\delta N}{2 + e^{\delta M}} \,.$$

Since we are considering the case in which $N > 0$, the condition for local stability is given by $f'_\delta(0) < 1$, or, equivalently by $h(\delta) < \frac{1}{N}$, where function h is defined as

$$h(\delta) = \frac{2\delta}{2 + e^{\delta M}} \,. \tag{D.3}$$

Notice that $h(0) = 0$ and the derivative of the function in δ is given by

$$h' = \frac{2(2 + e^x - xe^x)}{(2 + e^x)^2} \,,$$

where we introduced the variable $x = \delta M$.

When $M > 0$, the variable x is positive and changes from 0 to ∞ together with δ. According to Lemma 1 we have then that the function h is initially increasing in δ and then decreasing. Function h takes its maximum value in the point where $x = x^*$, i.e., when $\delta = x^*/M$. The value of function h in this point is given by

$$h\left(\frac{x^*}{M}\right) = \frac{2x^*}{2 + e^{x^*}} \cdot \frac{1}{M} = \frac{2x^*}{2 + \frac{2}{x^*-1}} \cdot \frac{1}{M} = \frac{(x^* - 1)}{M} \,.$$

The maximum value of h is positive according to Lemma 1. If it is larger than $\frac{1}{N}$, i.e., if $x^* - 1 > \frac{M}{N}$, then the two solutions of equation $h(\delta) = \frac{1}{N}$ define an interval (δ_1, δ_2) where $h(\delta) > \frac{1}{N}$, and so the target steady state is unstable. In the opposite case, if the maximum value of h is smaller than $\frac{1}{N}$, then $h(\delta) < \frac{1}{N}$ for any δ and the target steady state is always locally stable. $\square$

Proposition D.2 (Steady states for $\delta = +\infty$)**.** *Consider the dynamics given by* (3.5) *for the special case of $\delta = +\infty$. Let us denote $m^* = 0$, $m^+ = 1$ and $m^- = -1$. When* *the slope of the map is positive,*[20] *i.e.,* $\phi_\pi < \phi_\pi^*$, *the following cases are possible:*

1. *When* $\phi_\pi < \phi_\pi^m$, *the system has three locally stable steady states,* m^*, m^+, *and* m^-. *The basin of attraction of the steady state is* $\left(-\frac{M}{N}, \frac{M}{N}\right)$.

2. *When* $\phi_\pi > \phi_\pi^m$ *there exists a unique, globally stable fundamental steady state.*

Proof. For $m_{t-1} > \frac{M}{N}$ we have that $M - Nm_{t-1} < 0$, therefore $f_\infty(m_{t-1}) = 1$. For $m_{t-1} \in \left(-\frac{M}{N}, \frac{M}{N}\right)$ we have that $M - Nm_{t-1} > 0$, therefore $f_\infty = 0$. Finally, for $m_{t-1} < -\frac{M}{N}$ we have that $M + Nm_{t-1} < 0$, therefore $f_\infty = -1$. The non-fundamental steady state m^+ exists if and only if the 45-degree line has an intersection with the upper horizontal parts of f_∞, i.e., when it intersects the line 1 at some $m > \frac{M}{N}$. The condition for this to happen is $\frac{M}{N} < 1$, i.e., $\phi_\pi < \phi_\pi^m$ (see Figure 7). $\square$

When the monetary policy is not aggressive enough, i.e., whenever $\phi_\pi < \phi_\pi^m$, we observe non-fundamental steady states for δ high enough, as suggested by the following

Lemma 3. *Suppose* $\phi_\pi < \phi_\pi^m$. *Then for* δ *high enough, the map described in* (3.5) *has two locally stable steady states,* $m^+ > 0$ *and* $m^- = -m^+ < 0$.

Proof. We prove the existence of m^+ (the existence of m^- follows from the symmetry of $f_\delta(m)$).
Let us fix $0 < \varepsilon < \frac{N-M}{M}$ and define $\gamma = \varepsilon\frac{M}{N} > 0$. Then consider the set $U = \{m : m > \frac{M}{N} + \gamma\}$. This set U is bounded from below and $\lim_{\delta\to+\infty} f_\delta(m) = 1$. For $0 < \varepsilon < \frac{N-M}{M}$, we have that $0 < \frac{M}{N}(1+\varepsilon) < 1$, hence $\forall m \in U$ and δ sufficiently large, we have that

$$f_\delta(m) > \frac{M}{N}(1+\varepsilon) = \frac{M}{N} + \gamma$$

Thus function $f_\delta(m)$, increasing and bounded from above, maps U into itself. Therefore there exists a locally stable steady state within the set U. $\square$

Using the results derived above we can now prove Proposition 3.2.

Proof of Proposition 3.2. The targeted equilibrium is locally stable for low values of δ, but loses and then gains local stability again through two subsequent pitchfork

[20]Dynamics for the case $\phi_\pi > \phi_\pi^*$ are analysed below in the proofs of Propositions 3.4 – 3.6

bifurcations. Together with the concavity of $f_\delta(m)$ proved in Lemma 2, it implies the global stability of the target steady state for small values of δ. Consider now the moment of the first pitchfork bifurcation at $\delta = \delta_1^*$: the target steady state loses stability and it might happen in two different ways. If function $f_\delta(m)$ is concave for $m > 0$, the bifurcation occurring at $\delta = \delta_1^*$ is supercritical and two stable non-fundamental stable steady states are created. But if function $f_\delta(m)$ is not concave (and in particular $f_\delta(m)$ is convex for small $m > 0$), then the bifurcation is subcritical and two new unstable steady states are created. The only way in which they can be created is via fold bifurcation.[21] As δ increases, the fundamental equilibrium regains its stability at $\delta = \delta_2^*$, when function is convex for small $m > 0$. Thus at $\delta = \delta_2^*$ a subcritical pitchfork bifurcation occurs and two new unstable steady states appears. But given that $f_\delta(m)$ is not decreasing and bounded, then there exist two other stable steady states. These five steady states are also observable for high δ values, as proved in Lemma 3.

We have checked the usual conditions for a pitchfork bifurcation to occur. Let $m = 0$ be a fixed point for the map $f_\delta(m) = F(m, \delta)$, $\delta_{1,2}^*$ the bifurcation values with $f'_{\delta_{1,2}^*}(0) = 1$ and

$$F_{mm}\left(0, \delta_{1,2}^*\right) = F_\delta\left(0, \delta_{1,2}^*\right) = 0$$

The non-degeneracy conditions $F_{m,\delta}\left(0, \delta_{1,2}^*\right) \neq 0$ and $F_{mmm}\left(0, \delta_{1,2}^*\right) \neq 0$ hold. Then there is a pitchfork bifurcation at $\left(0, \delta_{1,2}^*\right)$. Notice also that at $\delta = \delta_1^*$ we have $F_{mmm}(0, \delta_1^*) < 0$, therefore the pitchfork bifurcation is supercritical. On the other hand at $\delta = \delta_2^*$, $F_{mmm}(0, \delta_2^*) > 0$, hence the pitchfork bifurcation is subcritical. □

Proof of Proposition 3.3. According to Proposition D.1*(2)*, the target steady state is always locally stable when $\phi_\pi < \phi_\pi^m < \phi_\pi^*$. It is unique and, therefore, globally stable, when $f_\delta(m)$ is concave, i.e., for small δ values (see Lemma 2). On the other hand, when δ is sufficiently high, two other locally stable steady states exist, m^+ and m^- (see Lemma 3). These steady states could only be created via tangent bifurcation. Since we cannot rule out the possibility of a number of subsequent tangent bifurcations (where the

[21] Numerical analysis demonstrate that such scenario may happen for values of ϕ_π which are very close to ϕ_π^w. See Anufriev, Assenza, Hommes, and Massaro (2013) for details.

non-fundamental steady states are created and subsequently disappear), we denote as δ_1^* the instance of the first tangent bifurcation and as δ_2^* the instance of the last tangent bifurcation. However, in our numerical analysis we never encountered a case in which $\delta_1^* \neq \delta_2^*$. $\square$

Proof of Proposition 3.4 *($\phi_\pi^m < \phi_\pi < \phi_\pi^*$).* We will start by proving the global stability result for the case $\phi_\pi^m < \phi_\pi < \phi_\pi^*$, i.e., the map $f_\delta(m)$ is increasing. The proof for the case $\phi_\pi^* < \phi_\pi < \phi_\pi^a$ is provided below. When $\phi_\pi > \phi_\pi^m$, it follows from Proposition D.1*(2)* that the target steady state is locally stable. In order to prove that it is globally stable for any δ, we show that it is the unique steady state of the dynamics described by the map $f_\delta(m)$. Since $f_\delta(m)$ is an increasing function, uniqueness implies global stability. Map $f_\delta(m)$ can be re-written as

$$m_t = f_\delta(m_t) = \frac{e^{-\delta[M-Nm_{t-1}]} - e^{-\delta[M+Nm_{t-1}]}}{1 + e^{-\delta[M-Nm_{t-1}]} + e^{-\delta[M+Nm_{t-1}]}} = \frac{1 - e^{-2\delta Nm}}{1 + e^{\delta(M-Nm)} + e^{-2\delta Nm}}$$

Assume that $m > 0$. Since function $f_\delta(m)$ is bounded from above by the horizontal asymptote $f_\delta(m) \leq 1 \; \forall m$, no steady state can exist within the interval $[1, +\infty)$. Let us consider $m \in (0,1)$ and show that $f_\delta(m) \in (0, \frac{1}{2}]$. Since $f_\delta(m)$ is an increasing map, the following chain of inequalities holds

$$0 = f_\delta(0) \leq f_\delta(m) \leq f_\delta(1)$$

Furthermore the aggressive monetary policy scenario, i.e., $\phi_\pi \geq \phi_\pi^m$) implies that $\frac{M}{N(\phi_\pi)} \geq 1$ (see Figure 7), i.e. $M - N(\phi_\pi) \geq 0$. Then from $e^{-\delta(N-M)} \geq 1$, we can derive the following

$$f_\delta(m) \leq f_\delta(1) = \frac{1 - e^{-2\delta N}}{1 + e^{\delta(M-N)} + e^{-2\delta N}} \leq \frac{1}{2}\,.$$

The above expression implies that there are no fixed points for $m > \frac{1}{2}$.

Suppose now that $0 < m \leq \frac{1}{2}$. Applying the restriction $\phi_\pi \geq \phi_\pi^m$, i.e. $M - N(\phi_\pi) \geq 0$ we find that the condition on $m \in (0, \frac{1}{2}]$ implies that $e^{\delta(M/2)} < e^{\delta(M-Nm)}$. We obtain the following estimate of dynamics on the interval $(0, 1/2)$

$$f_\delta(m) = \frac{1 - e^{-2\delta Nm}}{1 + e^{\delta(M - Nm)} + e^{-2\delta Nm}} \leq \frac{M}{N} \frac{1 - e^{-2\delta Nm}}{1 + e^{\delta(M/2)} + e^{-2\delta Nm}}$$

Let the function on the right hand side be defined as $g(m)$. This function is increasing in m with first and second derivative respectively given by

$$g'(m) = 2\delta M \frac{e^{2\delta Nm}\left(2 + e^{\delta(M/2)}\right)}{\left(1 + e^{2\delta Nm}\left(1 + e^{\delta(M/2)}\right)\right)^2}$$

$$g''(m) = -4\delta^2 MN\left(2 + e^{\delta(M/2)}\right) \frac{-1 + e^{2\delta Nm}\left(1 + e^{\delta(M/2)}\right)}{\left(1 + e^{2\delta Nm}\left(1 + e^{\delta(M/2)}\right)\right)^3}$$

Note that $e^{2Mm\delta} > 1$ and $g''(m) < 0$ for $m > 0$ and $M > 0$. Note also that

$$g'(0) = 2\frac{M\delta}{e^{\delta(M/2)} + 2}$$

and, by fixing $M\delta = x$, we get

$$g'(0) = \frac{2x}{e^{x/2} + 2}$$

The first derivative of $l(x) = \frac{2x}{e^{x/2}+2}$ is $l'(x) = \frac{\left(2e^{x/2} - xe^{x/2} + 4\right)}{e^x + 4e^{x/2} + 4}$. Therefore a maximum point has to satisfy $2 - x + 4e^{-x/2} = 0$, which is $2x^*$, with x^* defined in Lemma 1. Hence we can state that $g'(0) < 1$.

Thanks to the concavity of g and $g'(0) < 1$ for $m > 0$, it follows that $g(m) < m \; \forall m > 0$. Given that $g(m)$ borders $f(m)$ then $f(m) < m \; \forall m \in (0, 1)$. Thus no positive fixed points are possible. Since function $f_\delta(m)$ is odd it also implies that no negative steady states are possible for $-1 < m < 0$. □

Proofs of Propositions 3.4 ($\phi_\pi^* < \phi_\pi < \phi_\pi^a$) – 3.6. As we have shown before, when $\phi_\pi > \phi_\pi^*$ the dynamics of m_t is described by a decreasing map. Thank to the characteristics of $f_\delta(m)$ it is possible to prove the dynamical properties of this case employing the same properties of the map when $\phi_\pi < \phi_\pi^*$.

Indeed it holds that

$$f_\delta\left(m, N, M\right) = f_\delta\left(-m, -N, M\right) \tag{D.4}$$

Since $f_\delta\left(m, N, M\right)$ is odd, it follows that the second iterate of $g_\delta\left(m\right)$, where $g_\delta\left(m\right) = f_\delta\left(-m\right) = -f_\delta\left(m\right)$, is equal to the second iterate of $f_\delta\left(m\right)$, i.e.

$$f_\delta^2\left(m\right) = g_\delta^2\left(m\right) \tag{D.5}$$

Furthermore we have proved that $f_\delta\left(m\right)$ is an increasing function. Then it has only fixed points and consequently $f_\delta^2\left(m\right)$ has the same fixed points and the same bifurcations of $f_\delta\left(m\right)$. Since $g_\delta\left(m\right)$ is decreasing, it has only one fixed points and possible (stable or unstable) 2-cycles.

Now, if δ^* is a bifurcation value for $f_\delta\left(m\right)$ given the corresponding $\left(\bar{N}, \bar{M}\right)$, then the same δ^* may be a bifurcation value for $g_\delta\left(m\right)$ given the parameter $\left(-\bar{N}, \bar{M}\right)$. Since we have proved that the target equilibrium has only pitchfork bifurcation for a given $\left(\bar{N}, \bar{M}\right)$, thus $\left(-\bar{N}, \bar{M}\right)$ corresponds to a flip bifurcation for $g_\delta\left(m\right)$. Furthermore if a tangent bifurcation occurs for $\left(\bar{N}, \bar{M}\right)$ for $f_\delta\left(m\right)$, then a tangent bifurcation will occur at $\left(-\bar{N}, \bar{M}\right)$ for $g_\delta^2\left(m\right)$ and this gives rise to two 2-cycles for $g_\delta\left(m\right)$.

Thanks to equations (D.4)-(D.5) and to the properties of the map $f_\delta(m)$, we can prove the results of Propositions 3.4 – 3.6 in the way previously adopted.
Moreover, concerning Proposition 3.6, we have checked the usual conditions for a period-doubling bifurcation to occur. Let $m = 0$ be a fixed point for the map $f_\delta\left(m\right) = F(m, \delta)$, $\delta_{1,2}^*$ the bifurcation values with $f'_{\delta_{1,2}^*}\left(0\right) = -1$ and

$$F_{mm}\left(0, \delta_{1,2}^*\right) = 0$$

Assume furthermore the non-degeneracy condition $F_\delta F_{mm} + 2F_{m\delta} \neq 0$ also holds. Then there is a flip bifurcation at $\left(0, \delta_{1,2}^*\right)$. Furthermore at $\delta = \delta_1^*$ we have that $-2F_{mmm} - 3\left(F_{mm}\right)^2 < 0$, therefore the flip bifurcation is supercritical. On the other hand at $\delta = \delta_2^*$, $-2F_{mmm} - 3\left(F_{mm}\right)^2 > 0$, hence the flip bifurcation is subcritical.

□

Fig. 8 summarizes the stability results that have been proved, showing the overlapping of the determinacy/stability regions in the (δ, ϕ_π)-space with the parameter restrictions we identified in Table 1. We notice that when the Taylor principle is not satisfied (i.e., $\phi_\pi < 1$), the fundamental steady state is indeterminate, leading to the existence of sunspot equilibria which are closely connected to the existence of non-fundamental equilibria in our model. Our results then show that even when monetary policy is set to ensure determinacy under rational expectations multiple equilibria and stable cycles may arise under heterogeneous expectations.

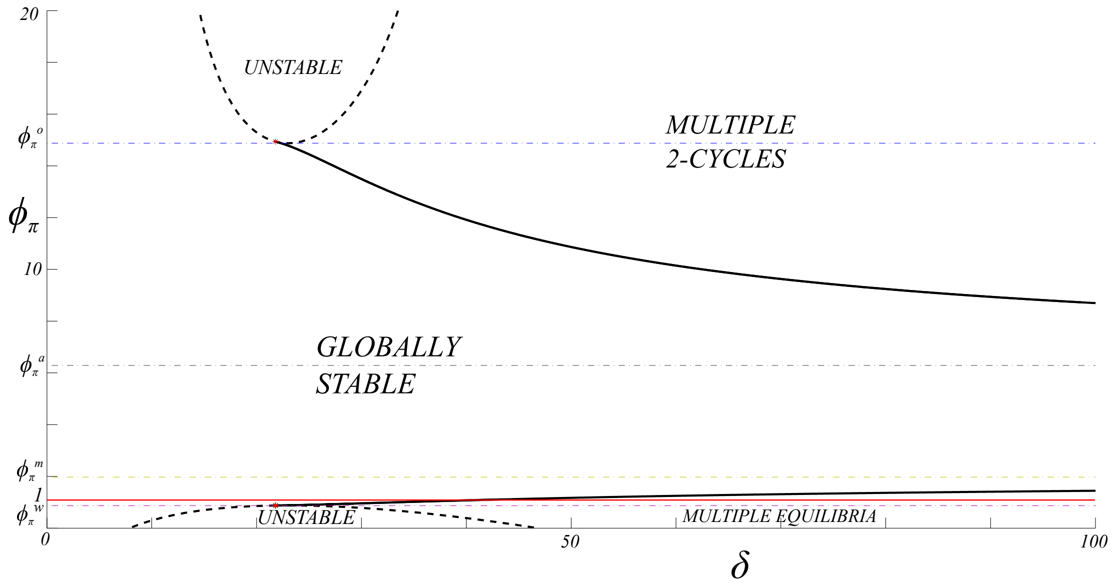

Figure 8: Bifurcation diagram of dynamics, in coordinates (δ, ϕ_π). Parameters are set at the CGG calibration with $b = 0.25$. The lower dashed curve gives all the parameters of the pitchfork bifurcation, the upper dashed curve gives all the parameters of the flip bifurcation while the solid line gives the parameters of tangent bifurcation. The red dots denote the points at which the curve of pitchfork (flip) bifurcation intersect with the curve of tangent bifurcation, while the red line $\phi_\pi = 1$ marks the boundary between the regions of determinacy vs. indeterminacy under rational expectations.

E LTL dynamics

Consider the LTL system described by (3.8). Straightforward computations show that the eigenvalues of the transition matrix are given by

$$\begin{aligned}\lambda_1 &= \frac{2s^2\delta(k^3 + k^2\sigma + (\beta-1)^2\sigma + k(\beta-1)(\beta\phi_\pi - 1))}{(k^2 + 2s^2(k^2 + (\beta-1)^2)\delta)(\sigma + k\phi_\pi)} \\ \lambda_2 &= 0\,.\end{aligned}$$

The stability analysis reduces to the study of the eigenvalue λ_1.

Proof of Propositions 3.7. When $\phi_\pi < 1$ we have that

$$\begin{aligned}\lambda_1(0) &= 0 \\ \lambda_1'(\delta) &= \frac{2k^2s^2(k^3 + k^2\sigma + (\beta-1)^2\sigma + k(\beta-1)(\beta\phi_\pi - 1))}{(k^2 + 2s^2(k^2 + (\beta-1)^2)\delta)^2(\sigma + k\phi_\pi)} > 0 \\ \lim_{\delta\to\infty} \lambda_1(\delta) &= \frac{k^3 + k^2\sigma + (\beta-1)^2\sigma + k(\beta-1)(\beta\phi_\pi - 1)}{(k^2 + (\beta-1)^2)(\sigma + k\phi_\pi)} > 1\,,\end{aligned}$$

implying that a value δ^* must exist such that $\lambda_1(\delta^*) = 1$.

When $\phi_\pi > 1$ we have that the following holds:

$$\lambda_1'(\delta) > (<)\, 0 \text{ if } \phi_\pi < (>)\, \phi_\pi^\bullet\,, \text{ with } \phi_\pi^\bullet = \frac{-k - k^3 + k\beta - \sigma - k^2\sigma + 2\beta\sigma - \beta^2\sigma}{-k\beta + k\beta^2}\,.$$

When $\phi_\pi < \phi_\pi^\bullet$ we have that

$$\begin{aligned}\lambda_1(0) &= 0 \\ \lambda_1'(\delta) &> 0 \\ \lim_{\delta\to\infty} \lambda_1(\delta) &< 1\,,\end{aligned}$$

which imply that target steady state is globally stable for any δ.

When $\phi_\pi > \phi_\pi^\bullet$ we can derive the following restrictions on structural parameters:

1. $0 < k < \frac{1}{2\sqrt{2}}$
2. $\frac{3}{4} - \frac{1}{4}\sqrt{1-8k^2} < \beta < \frac{3}{4} + \frac{1}{4}\sqrt{1-8k^2}$.

When restrictions 1 – 2 are satisfied, we can derive the threshold value

$$1 < \phi_\pi^\bullet < \phi_\pi^{**} = \frac{-k-k^3+k\beta-2\sigma-2k^2\sigma+4\beta\sigma-2\beta^2\sigma}{k+k^3-3k\beta+2k\beta^2}$$

such that, when $\phi_\pi < \phi_\pi^{**}$ we have that

$$\begin{aligned} \lambda_1(0) &= 0 \\ \lambda_1'(\delta) &< 0 \\ \lim_{\delta\to\infty} \lambda_1(\delta) &> -1 \, , \end{aligned}$$

implying that the target steady state is globally stable for any δ; while when $\phi_\pi > \phi_\pi^{**}$, we instead have that

$$\begin{aligned} \lambda_1(0) &= 0 \\ \lambda_1'(\delta) &< 0 \\ \lim_{\delta\to\infty} \lambda_1(\delta) &< -1 \, , \end{aligned}$$

implying that a value δ^{**} must exist such that $\lambda_1(\delta^{**}) = -1$.

When at least one between restrictions $1-2$ is not satisfied we have that

$$\begin{aligned} \lambda_1(0) &= 0 \\ \lambda_1'(\delta) &< 0 \\ \lim_{\delta\to\infty} \lambda_1(\delta) &> -1 \, , \end{aligned}$$

implying that the target steady state is globally stable for any δ. $\square$

References

Adam, K. (2007): "Experimental Evidence on the Persistence of Output and Inflation," *The Economic Journal*, 117(520), 603–636.

Agliari, A., N. Pecora, and A. Spelta (2015): "Coexistence of equilibria in a New Keynesian model with heterogeneous beliefs," *Chaos, Solitons and Fractals*, 79, 83 – 95.

Anufriev, M., T. Assenza, C. Hommes, and D. Massaro (2013): "Interest Rate Rules and Macroeconomic Stability under Heterogeneous Expectations," *Macroeconomic Dynamics*, 17, 1574–1604.

Arifovic, J., J. Bullard, and O. Kostyshyna (2013): "Social Learning and Monetary Policy Rules," *The Economic Journal*, 123, 38–76.

Assenza, T., P. Heemeijer, C. Hommes, and D. Massaro (2011): "Individual Expectations and Aggregate Macro Behavior," CeNDEF Working Paper 2011-1, University of Amsterdam.

Benhabib, J., and S. Eusepi (2005): "The design of monetary and fiscal policy: A global perspective," *Journal of Economic Theory*, 123(1), 40–73.

Benhabib, J., S. Schmitt-Grohé, and M. Uribe (2001): "The perils of Taylor rules," *Journal of Economic Theory*, 96(1), 40–69.

Berardi, M., and J. Duffy (2007): "The value of central bank transparency when agents are learning," *European Journal of Political Economy*, 23, 9–29.

Blinder, A., M. Ehrmann, M. Fratzscher, J. De Haan, and D. Jansen (2008): "Central Bank Communication and Monetary Policy: a Survey of Theory and Evidence," *Journal of Economic Literature*, 46(2), 910–945.

Bloomfield, R., and J. Hales (2002): "Predicting the Next Step of a Random

Walk: Experimental Evidence of Regime-Switching Beliefs," *Journal of Financial Economics*, 65, 397–414.

Branch, W. (2004): "The Theory of Rationally Heterogeneous Expectations: Evidence from Survey Data on Inflation Expectations," *The Economic Journal*, 114(497), 592–621.

Branch, W., and G. Evans (2011): "Monetary Policy and Heterogeneous Expectations," *Economic Theory*, 47, 365–393.

Branch, W., and B. McGough (2009): "Monetary Policy in a new Keynesian Model with Heterogeneous Expectations," *Journal of Economic Dynamics and Control*, 33, 1036–1051.

——— (2010): "Dynamic Predictors Selection in a New Keynesian Model with Heterogeneous Expectations," *Journal of Economic Dynamics and Control*, 34(8), 1492–1508.

Brazier, A., R. Harrison, M. King, and T. Yates (2008): "The Danger of Inflating Expectations of Macroeconomic Stability: Heuristic Switching in an Overlapping Generations Monetary Model," *International Journal of Central Banking*, 4(2), 219–254.

Brock, W., and P. de Fontnouvelle (2000): "Expectational diversity in monetary economies," *Journal of Economic Dynamics and Control*, 24(5-7), 725–759.

Brock, W., C. Hommes, and F. Wagener (2005): "Evolutionary Dynamics in Markets with Many Trader Types," *Journal of Mathematical Economics*, 41, 7–42.

Brock, W. A., and C. H. Hommes (1997): "A Rational Route to Randomness," *Econometrica*, 65(5), 1059–1095.

Calvo, G. (1983): "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12, 983–998.

Campbell, J. R., C. L. Evans, J. D. Fisher, and A. Justiniano (2012): "Macroeconomic Effects of Federal Reserve Forward Guidance," *Brookings Papers on Economic Activity*, 44(1), 1–80.

Carroll, C. (2003): "Macroeconomic expectations of households and professional forecasters," *Quarterly Journal of Economics*, 118, 269–298.

Clarida, R., J. Galí, and M. Gertler (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115(1), 147–180.

Cornea, A., C. Hommes, and D. Massaro (2012): "Behavioural Heterogeneity in U.S. Inflation Dynamics," CeNDEF Working Paper 2012-3, University of Amsterdam.

Curtin, R. (2005): "Inflation Expectations: Theoretical Models and Empirical Tests," mimeo, University of Michigan.

De Grauwe, P. (2011): "Animal Spirits and Monetary Policy," *Economic Theory*, 47, 423–457.

——— (2012): *Lectures on behavioral macroeconomics*. Princeton University Press, New Jersey.

Dräger, L. (2015): "Recursive Inattentiveness with Heterogeneous Expectations," *Macroeconomic Dynamics*, pp. 1–28, doi:10.1017/S1365100514000741.

Duffy, D., and P. D. Lunn (2009): "The Misperception of Inflation by Irish Consumers," *The Economic and Social Review*, 40, 139–163.

Eusepi, S., and B. Preston (2010): "Central bank communication and expectation stabilisation," *American Economic Journal: Macroeconomics*, 2, 235–271.

Frankel, J., and K. Froot (1991): "Chartists, Fundamentalists and the Demand for Dollars," in *Private Behaviour and Government Policy in Interdependent Economies*, ed. by A. Courakis, and M. Taylor. Oxford: Clarendon.

Gaballo, G. (2015): "Rational Inattention to News: The Perils of Forward Guidance," *American Economic Journal: Macroeconomics*, forthcoming.

Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton University Press.

Hommes, C., and J. Lustenhouwer (2016): "Managing heterogeneous and unanchored expectations: a monetary policy analysis," CeNDEF Working Paper 16-01, University of Amsterdam.

Hommes, C. H. (2011): "The Heterogeneous Expectations Hypothesis: Some Evidence from the Lab," *Journal of Economic Dynamics and Control*, 35, 1–24.

Kool, C. J. M., and D. L. Thornton (2012): "How effective is central bank forward guidance?," Working Papers 2012-063, Federal Reserve Bank of St. Louis.

Kurz, M. (2011): "A New Keynesian Model with Diverse Beliefs," Discussion paper, Stanford University.

Mankiw, N., R. Reis, and J. Wolfers (2003): "Disagreement about inflation expectations," in *NBER Macroeconomics Annual*, ed. by M. Gertler, and K. Rogoff.

Manski, C., and D. McFadden (1981): "Alternative Estimators and Sample Designs for Discrete Choice Analysis," in *Structural Analysis of Discrete Data*, ed. by C. Manski, and D. McFadden. MIT Press.

Massaro, D. (2013): "Heterogeneous Expectations in Monetary DSGE Models," *Journal of Economic Dynamics and Control*, 37(3), 680–692.

Matějka, F., and A. McKay (2015): “Rational Inattention to discrete choices: A new foundation for the multinomial logit model,” *American Economic Review*, 1(105), 272–298.

Negro, M. D., M. Giannoni, and C. Patterson (2012): “The forward guidance puzzle,” Staff Reports 574, Federal Reserve Bank of New York.

Orphanides, A., and J. Williams (2005): “Imperfect knowledge, inflation expectations and monetary policy,” in *Inflation Targeting*, ed. by B. Bernanke, and M. Woodford. University of Chicago Press.

Pfajfar, D., and E. Santoro (2010): “Heterogeneity, Learning and Information Stickiness in Inflation Expectations,” *Journal of Economic Behavior and Organization*, 75(3), 426–444.

Pfajfar, D., and B. Zakelj (2010): “Inflation Expectations and Monetary Policy Design: Evidence from the Laboratory,” CentER discussion paper 2011-091, University of Tilburg.

Salle, I., M. Yildizoğlu, and M. Sénégas (2013): “Inflation targeting in a learning economy: an ABM perspective,” *Economic Modelling*, 34, 114 – 128.

Sims, C. (2003): “Implications of Rational Inattention,” *Journal of Monetary Economics*, 50, 665–690.

Svensson, L. (2009): “Transparency under Flexible Inflation Targeting: Experiences and Challenges,” *Economic Review*, 1, 5–44.

Taylor, J. (1993): “Discretion versus Policy Rules in Practice,” *Carnegie-Rochester Conference Series on Public Policy*, 39(0), 195–214.

Tuinstra, J., and F. Wagener (2007): “On learning equilibria,” *Economic Theory*, 30(3), 493–513.

Woodford, M. (1999): "Optimal Monetary Policy Inertia," NBER Working Paper 7261.

Woodford, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.