On the global stability of smooth solutions of the Navier-Stokes equations

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Abstract

In [24] and previous papers by the same authors, a general smooth setting was proposed for the incompressible Navier-Stokes (NS) Cauchy problem on a torus of any dimension $d \geqslant 2$, and the *a posteriori* analysis of its approximate solutions. In this note, using the same setting I propose an elementary proof of the following statement: global existence and time decay of the NS solutions are stable properties with respect to perturbations of the initial datum. Fully explicit estimates are derived, using Sobolev norms of arbitrarily high order. An application is proposed, in which the initial data are generalized Beltrami flows. A comparison with the related literature is performed.

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1 Introduction

Let us consider the Cauchy problem for the (homogeneous, incompressible) Navier-Stokes (NS) equations on a torus $\mathbf{T}^d$; this reads

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathcal{P}(u, u) \ , \qquad u(x, 0) = u_0(x) \ . \tag{1.1}$$

Here: any dimension $d \geqslant 2$ is admitted; $u = u(x, t)$ is the divergence free velocity field, depending on $x \in \mathbf{T}^d$ and on time t; $\nu > 0$ is the kinematic viscosity; Δ is the Laplacian of $\mathbf{T}^d$; $\mathcal{P}$ is the bilinear map sending any two sufficiently regular vector fields $v, w : \mathbf{T}^d \to \mathbf{R}^d$ into

$$\mathcal{P}(v, w) := -\mathfrak{L}((v \bullet \nabla) w) \ . \tag{1.2}$$

In the above $(v \bullet \nabla) w : \mathbf{T}^d \to \mathbf{R}^d$ is the vector field of components $((v \bullet \nabla) w)_r = \sum_{s=1}^d v_s \partial_s w_r$, and $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields.

In [24], a smooth functional setting was proposed for the Cauchy problem (1.1); this relies on the Fréchet space of C^∞ vector fields on $\mathbf{T}^d$ with vanishing divergence and mean, which is represented as an intersection of H^p type Sobolev spaces of arbitrarily high order p (i.e., as a Sobolev space of infinite order); here and in the sequel $\| \ \|_p$ stands for the H^p norm, for any real p. Working within this framework, a general strategy was presented in [24] to infer estimates on the solution u of (1.1) from the *a posteriori* analysis of any approximate solution $u_\mathtt{a}$; such estimates concern the time interval of existence of u and the distance $\|u(t) - u_\mathtt{a}(t)\|_p$ of arbitrarily high order p. ([1]) This setting also applies to the inviscid limit $\nu = 0$ (giving Euler's equations) and to the case where an external forcing is present; however, such extensions will not be considered in the present note.

A similar approach to the approximate solutions of (1.1) in Sobolev spaces of finite order was presented in [20]; both [20] and [24] are greatly indebted to the seminal works [4] [27] on the *a posteriori* analysis of NS approximants. In typical applications of the approach of [20] [24], the approximant $u_\mathtt{a}$ is provided by the Galerkin method (see the same works), or by a truncated power expansion in the "Reynolds number" $1/\nu$ [23] [16] (see also [15], where the limit $\nu = 0$ is considered and $u_\mathtt{a}$ is a truncated power expansion in t). ([2])

The aim of this note is to discuss the global, time decaying NS solutions and their stability with the methods of [24]; the necessary tools from that work are reviewed

[1] Of course, for each time t, $u(t)$ means the map $x \in \mathbf{T}^d \mapsto u(x, t)$; $u_\mathtt{a}(t)$ and analogous symbols appearing in the sequel must be understood similarly.

[2] In addition let me mention papers [18] [19], presenting earlier variants of the same ideas on NS approximants, and the recent work [25] which extends the methods of [24] to (homogeneous, incompressible) magnetohydrodynamics.

in Sections 2 and 3. Given a smooth, global NS solution v, there are several notions of decay for v: for example, one can ask that

$$\|v(t)\|_n \to 0 \qquad \text{for } t \to +\infty \tag{1.3}$$

for some n. In Section 4 of the note I point out that condition (1.3) for some $n > d/2+1$, and other reasonable decay conditions, are in fact equivalent to exponential decay in all Sobolev norms:

$$\|v(t)\|_p \leqslant \text{const.} \times e^{-\nu t} \qquad \text{for all } p \in \mathbf{R} \text{ and } t \geqslant 0 \tag{1.4}$$

(where the constants depend on v and p). In Section 5, I derive the stability of the above properties with respect to small perturbations of the initial data. More precisely, I show what follows: *if the NS solution v with an initial datum v_0 is global and decaying (in any sense equivalent to (1.4)), the same happens for the NS solution u with any datum u_0 such that $\|u_0 - v_0\|_n$ is sufficiently small for some $n > d/2+1$.* Again for $\|u_0 - v_0\|_n$ small, I derive estimates on $u - v$ having essentially the form

$$\|u(t) - v(t)\|_p \leqslant \text{const.} \times \|u_0 - v_0\|_p \, e^{-\nu t} \qquad \text{for all } p \geqslant n \text{ and } t \geqslant 0\,. \tag{1.5}$$

The precise formulation of the above statements can be found in Theorem 5.1; to prove these results, I apply the machinery of [24] viewing v as an approximate solution for the NS Cauchy problem with datum u_0.

In Section 6, that concludes this note, I consider as an example the case when the initial datum v_0 is a generalized Beltrami flow [29] [31]. A datum of this kind can be arbitrarily large and is known to produce a global, decaying NS solution v with an elementary expression; the results of Section 5 are applied to this case.

The statement that the global nature of the NS solutions is preserved by small perturbations of the initial datum, under suitable decay assumptions for the unperturbed solution, is not at all new in the literature. The first result of this kind is [26]; the subject has been investigated until present time, see e.g. the recent works [3] [6]. ([3]) Most papers in this area discuss global stability in terms of some reference Banach space of velocity fields, which is an H^1 type space in [26], the so-called $\mathcal{X}^{-1}$ space in [3] ([4]), and, essentially, an H^r type space of fixed, integer order $r \geqslant 1$ in [6]; this makes some difference with respect to the present note, that refers systematically to a Fréchet space of velocity fields with infinitely many H^p norms.

[3] Other authors have discussed the preservation (under small changes of the initial datum) of the global nature of some special, nondecaying NS solutions in presence of external forces. In particular, global stability results have been obtained for the small amplitude, almost periodic solutions of the NS equations with time quasi-periodic external forcing: see [14] and references therein.

[4] this is made of the velocity fields $u : \mathbf{R}^3 \to \mathbf{R}^3$ with Fourier transform $\mathcal{F}u : k \mapsto (\mathcal{F}u)(k)$ such that $+\infty > \|u\|_{\mathcal{X}^{-1}} := \int_{\mathbf{R}^3} dk\, |k|^{-1} |(\mathcal{F}u)(k)|$.

In addition, most of the cited works refer to the case of space dimension $d = 3$ (admittedly, the most interesting case), while the present results hold for d arbitrary. Let me repeat that the space domain in this note is the torus $\mathbf{T}^d$; for a comparison with the literature, I will mention for example that [26] refers to a domain $\Omega \subset \mathbf{R}^3$, [3] to the whole space $\mathbf{R}^3$ and [6] to $\mathbf{T}^3$. ([5])

Another point I would mention is that, with notable exceptions like [3], most papers on this subject present estimates relying on unspecified constants. On the contrary, all estimates in this note are fully explicit: this holds, in particular, for the bounds on $\|u_0 - v_0\|_n$ yielding the stability result of Section 5 and for the constant in Eq. (1.5). This is made possible by the quantitative analysis performed in [17] [21] [22] for the constants in some fundamental inequalities on the NS bilinear map (1.2).

Concerning the form of the stability result, the earlier work to which the present note is closer is [6]; for a more detailed comparison, see Remark 4.4 (iv) (on the functional settings and decay conditions) and Remark 5.2 (ii) (about Eq. (1.5) and its analogue in [6]).

Apart from technical differences with respect to the previous literature, a feature of this note is that the present stability result arises almost automatically from the main theorem of [24] on NS approximants (to be used as indicated after Eq. (1.5)). It might be of some interest that the same result of [24] also has the applications mentioned before, rather different from the one presented here.

2 Preliminaries

Some function spaces. Throughout this note we work on a torus

$$\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d \qquad (d = 2, 3, ...) \tag{2.1}$$

and refer to the functional setting of [24] (and [20]), reproduced in this section to make the note self-contained.

Let us consider the space $D'(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{D}'$ of $\mathbf{R}^d$-valued distributions on $\mathbf{T}^d$; each $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$ with coefficients $v_k = \overline{v_{-k}} \in \mathbf{C}^d$, where $e_k(x) := (2\pi)^{-d/2} e^{ik \bullet x}$. The mean value $\langle v \rangle$ is, by definition, the action of v on the constant test function $(2\pi)^{-d}$, and $\langle v \rangle = (2\pi)^{-d/2} v_0$. The Laplacian of $v \in \mathbb{D}'$ has Fourier coefficients $(\Delta v)_k = -|k|^2 v_k$; if $\langle v \rangle = 0$ and $p \in \mathbf{R}$,

[5] One of the referees suggested a comment upon the extension of the present results to space domains different from $\mathbf{T}^d$. Indeed $\mathbf{T}^d$ has several nice features, namely, the strict positivity of the spectrum of $-\Delta$ on NS velocity fields (under the condition of mean zero) and the inequalities for the NS bilinear map (1.2) reviewed in Section 2; all of them are essential for the setting of [24] and for its present application. It is not obvious that the totality of these features would be preserved passing to other space domains.

we define $(-\Delta)^{p/2} v$ to be the element of $\mathbb{D}'$ with mean zero and Fourier coefficients $((-\Delta)^{p/2} v)_k = |k|^p v_k$ for $k \in \mathbf{Z}^d \setminus \{0\}$. We consider as well the space $L^2(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^2$, with the standard inner product $\langle v | w \rangle_{L^2} := \int_{\mathbf{T}^d} \overline{v} {\scriptstyle\bullet} w \, dx$. For any $p \in \mathbf{R}$, let us introduce the Sobolev space

$$\mathbb{H}^p_{\Sigma 0} := \{ v \in \mathbb{D}' \mid \ \operatorname{div} v = 0, \ \langle v \rangle = 0, \ (-\Delta)^{p/2} v \in \mathbb{L}^2 \ \} \tag{2.2}$$

$$= \{ v \in \mathbb{D}' \mid \ k {\scriptstyle\bullet} v_k = 0 \ \forall k \in \mathbf{Z}^d, \ \ v_0 = 0, \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2p} |v_k|^2 < +\infty \}$$

(the subscripts Σ and 0 indicate the vanishing of the divergence and of the mean); this carries the inner product and the norm

$$\langle v | w \rangle_p := \langle (-\Delta)^{p/2} v | (-\Delta)^{p/2} w \rangle_{L^2} = \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2p} \overline{v_k} {\scriptstyle\bullet} w_k \,, \tag{2.3}$$

$$\| v \|_p := \sqrt{\langle v | v \rangle_p} = \| (-\Delta)^{p/2} v \|_{L^2} \ .$$

For real $p \geqslant \ell$ one has $\mathbb{H}^p_{\Sigma 0} \subset \mathbb{H}^\ell_{\Sigma 0}$ and $\| \ \|_p \geqslant \| \ \|_\ell$. The vector space

$$\mathbb{H}^\infty_{\Sigma 0} := \bigcap_{p \in \mathbf{R}} \mathbb{H}^p_{\Sigma 0} \tag{2.4}$$

can be equipped with the topology induced by the family of all Sobolev norms $\| \ \|_p$ $(p \in \mathbf{R})$, which coincides with that induced by the countable subfamily $\| \ \|_p$ $(p \in \mathbf{N})$; so, we have a Fréchet space. Due to standard Sobolev imbeddings,

$$\mathbb{H}^\infty_{\Sigma 0} = \{ \ v \in C^\infty(\mathbf{T}^d, \mathbf{R}^d) \mid \ \operatorname{div} v = 0, \ \langle v \rangle = 0 \ \} \tag{2.5}$$

and the above mentioned topology on $\mathbb{H}^\infty_{\Sigma 0}$ coincides with that induced by the family of norms $\| \ \|_{C^r}$ $(r \in \mathbf{N})$, where $\| \ \|_{C^r}$ is the sup norm for the derivatives of all orders $\leqslant r$ (for $r \in \mathbf{N}$ and $p \in \mathbf{R}$ one has $\| \ \|_{C^r} \leqslant \text{const.} \| \ \|_p$ if $p > r + d/2$, and $\| \ \|_p \leqslant \text{const.} \| \ \|_{C^r}$ if $p \leqslant r$).

The fundamental bilinear map for NS equations. In the Introduction I have already mentioned the bilinear map sending two (sufficiently regular) divergence free, mean zero vector fields $v, w : \mathbf{T}^d \to \mathbf{R}^d$ into $\mathcal{P}(v, w) := -\mathfrak{L}((v {\scriptstyle\bullet} \nabla) w)$ (see Eq. (1.2) and the subsequent comments; a precise definition of the Leray projection $\mathfrak{L}$ is given, e.g., in [20]). Let $p, n \in \mathbf{R}$; it is known that $p > d/2$, $v \in \mathbb{H}^p_{\Sigma 0}$, $w \in \mathbb{H}^{p+1}_{\Sigma 0}$ $\Rightarrow \mathcal{P}(v, w) \in \mathbb{H}^p_{\Sigma 0}$ and that, for p, n as below, there are constants K_p, G_p, K_{pn}, G_{pn} $\in (0, +\infty)$ such that the following holds:

$$\|\mathcal{P}(v,w)\|_p \leqslant K_p \|v\|_p \|w\|_{p+1} \qquad \text{if } p > d/2,\ v \in \mathbb{H}^p_{\Sigma 0},\ w \in \mathbb{H}^{p+1}_{\Sigma 0}\ , \tag{2.6}$$

$$|\langle \mathcal{P}(v,w)|w\rangle_p| \leqslant G_p \|v\|_p \|w\|_p^2 \qquad \text{if } p > d/2+1,\ v \in \mathbb{H}^p_{\Sigma 0},\ w \in \mathbb{H}^{p+1}_{\Sigma 0}\ , \tag{2.7}$$

$$\|\mathcal{P}(v,w)\|_p \leqslant \frac{1}{2} K_{pn} (\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}) \tag{2.8}$$
$$\text{if } p \geqslant n > d/2,\ v \in \mathbb{H}^p_{\Sigma 0},\ w \in \mathbb{H}^{p+1}_{\Sigma 0}\ ,$$

$$|\langle \mathcal{P}(v,w)|w\rangle_p| \leqslant \frac{1}{2} G_{pn} (\|v\|_p \|w\|_n + \|v\|_n \|w\|_p) \|w\|_p \tag{2.9}$$
$$\text{if } p \geqslant n > d/2+1,\ v \in \mathbb{H}^p_{\Sigma 0},\ w \in \mathbb{H}^{p+1}_{\Sigma 0}\ .$$

Note that (2.8) with $n = p$ gives (2.6), with $K_p := K_{pp}$; similarly, (2.9) with $n = p$ gives (2.7) with $G_p := G_{pp}$. Eq. (2.6) (with the implication in the text before it) indicates that $\mathcal{P}$ maps continuously $\mathbb{H}^p_{\Sigma 0} \times \mathbb{H}^{p+1}_{\Sigma 0}$ to $\mathbb{H}^p_{\Sigma 0}$ for all $p > d/2$, and $\mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}$ to $\mathbb{H}^\infty_{\Sigma 0}$.

Eq. (2.6) and its generalization (2.8) are closely related to the basic norm inequalities about multiplication in Sobolev spaces. Eq. (2.7) was discovered in [9] for integer p, and extended in [5] to noninteger cases; inequalities very similar to (2.9) were proposed in [2] [30] [28]. Fully quantitative, upper and lower bounds for the sharp constants K_p, G_p, K_{pn}, G_{pn} in Eqs. (2.6-2.9) were proposed in [21] [22] [17]. From here to the end of the paper $K_p, ..., G_{pn}$ are (possibly nonsharp) constants fulfilling the above inequalities.

The NS Cauchy problem in a smooth framework. From here to the end of the paper we fix a viscosity

$$\nu \in (0, +\infty)\ . \tag{2.10}$$

The *(homogeneous, incompressible) NS Cauchy problem* with initial datum $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$ is the following:

$$\text{Find } u \in C^\infty([0,T), \mathbb{H}^\infty_{\Sigma 0}) \text{ such that } \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u,u)\ , \quad u(0) = u_0\ , \tag{2.11}$$

with $T = T_u \in (0, +\infty]$; obviously enough, a solution of (2.11) with $T = +\infty$ is said to be *global*. It is known that (2.11) has a unique maximal (i.e., not extendable) solution u, whose domain $[0, T)$ depends in principle on the datum u_0 [2] [8] [9] [10] [11] [12] [30]. ([6])

[6] Most of these classical works refer to Sobolev spaces of finite order, but there are standard arguments for passing to the infinite order case (i.e., to the setting based on $\mathbb{H}^\infty_{\Sigma 0}$); these arguments are reviewed, e.g., in [24].

3 Approximate solutions of the NS Cauchy problem. Control inequalities

Let $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$; hereafter I report a definition from [24], and the main result proved in the same work.

3.1 ***Definition.*** An *approximate solution* of the NS Cauchy problem (2.11) is any map $u_\mathtt{a} \in C^1([0, T_\mathtt{a}), \mathbb{H}^\infty_{\Sigma 0})$, with $T_\mathtt{a} \in (0, +\infty]$. Given such a function, (i) and (ii) are stipulated.
i) The differential error *and the* datum error *of $u_\mathtt{a}$ are, respectively:*

$$e(u_\mathtt{a}) := \frac{du_\mathtt{a}}{dt} - \nu \Delta u_\mathtt{a} - \mathcal{P}(u_\mathtt{a}, u_\mathtt{a}) \ \in C([0, T_\mathtt{a}), \mathbb{H}^\infty_{\Sigma 0}) \ ; \quad u_0 - u_\mathtt{a}(0) \in \mathbb{H}^\infty_{\Sigma 0} \ . \tag{3.1}$$

ii) Let $p \in \mathbf{R}$. A differential error estimator*, a* datum error estimator *and a* growth estimator *of order p for $u_\mathtt{a}$ are, respectively, a function $\epsilon_p \in C([0, T_\mathtt{a}), [0, +\infty))$, a number $\delta_p \in [0, +\infty)$ and a function $\mathcal{D}_p \in C([0, T_\mathtt{a}), [0, +\infty))$ such that*

$$\|e(u_\mathtt{a})(t)\|_p \leqslant \epsilon_p(t) \quad \textit{for } t \in [0, T_\mathtt{a}) \ , \quad \|u_0 - u_\mathtt{a}(0)\|_p \leqslant \delta_p \ , \tag{3.2}$$

$$\|u_\mathtt{a}(t)\|_p \leqslant \mathcal{D}_p(t) \quad \textit{for } t \in [0, T_\mathtt{a}) \ .$$

In particular the function $\epsilon_p(t) := \|e(u_\mathtt{a})(t)\|_p$, the number $\delta_p := \|u_0 - u_\mathtt{a}(0)\|_p$ and the function $\mathcal{D}_p(t) := \|u_\mathtt{a}(t)\|_p$ will be called the tautological *estimators of order p for the differential error, the datum error and the growth of $u_\mathtt{a}$.*

3.2 ***Proposition.*** *Let $u_\mathtt{a} \in C^1([0, T_\mathtt{a}), \mathbb{H}^\infty_{\Sigma 0})$ be an approximate solution of the NS Cauchy problem (2.11). Assume that, for some $n \in (d/2+1, +\infty)$, $u_\mathtt{a}$ has differential error, datum error and growth estimators of order n or $n+1$, indicated with $\epsilon_n, \delta_n, \mathcal{D}_n$ and $\mathcal{D}_{n+1}$, and that there is a function $\mathcal{R}_n \in C([0, T_\mathtt{c}), \mathbf{R})$, with $T_\mathtt{c} \in (0, T_\mathtt{a}]$, fulfilling the following* control inequalities*:*

$$\frac{d^+\mathcal{R}_n}{dt} \geqslant -\nu\mathcal{R}_n + (G_n\mathcal{D}_n + K_n\mathcal{D}_{n+1})\mathcal{R}_n + G_n\mathcal{R}_n^2 + \epsilon_n \ \textit{everywhere on } [0, T_\mathtt{c}), \quad \mathcal{R}_n(0) \geqslant \delta_n \tag{3.3}$$

(K_n, G_n as in Eqs. (2.6) (2.7), with p replaced by n; in the above we use the right, upper Dini derivative $(d^+\mathcal{R}_n(t)/dt)(t) := \limsup_{h \to 0^+} (\mathcal{R}_n(t+h) - \mathcal{R}_n(t))/h$). Consider the maximal solution $u \in C^\infty([0, T), \mathbb{H}^\infty_{\Sigma 0})$ of problem (2.11); then (i)(ii) hold.
i) u and its existence time T are such that

$$T \geqslant T_\mathtt{c} \ , \qquad \|u(t) - u_\mathtt{a}(t)\|_n \leqslant \mathcal{R}_n(t) \qquad \textit{for } t \in [0, T_\mathtt{c}) \ . \tag{3.4}$$

In particular, if $\mathcal{R}_n$ is global ($T_\mathtt{c} = +\infty$), then u is global as well ($T = +\infty$).

ii) Consider any $p \in (n, +\infty)$, and let ϵ_p, δ_p, $\mathcal{D}_p, \mathcal{D}_{p+1}$ be differential error, datum error and growth estimators of order p or $p+1$ for $u_{\mathtt{a}}$. Let $\mathcal{R}_p \in C([0, T_{\mathtt{c}}), \mathbf{R})$ be a function fulfilling the linear, order p control inequalities

$$\frac{d^+\mathcal{R}_p}{dt} \geqslant -\nu\mathcal{R}_p + (G_p\mathcal{D}_p + K_p\mathcal{D}_{p+1} + G_{pn}\mathcal{R}_n)\mathcal{R}_p + \epsilon_p \text{ everywhere on } [0, T_{\mathtt{c}}) , \quad \mathcal{R}_p(0) \geqslant \delta_p \tag{3.5}$$

(K_p, G_p, G_{pn} as in Eqs. (2.6) (2.7) (2.9); again, d^+/dt stands for the right, upper Dini derivative). Then

$$\|u(t) - u_{\mathtt{a}}(t)\|_p \leqslant \mathcal{R}_p(t) \qquad \text{for } t \in [0, T_{\mathtt{c}}) . \tag{3.6}$$

The relations (3.5) are both fulfilled as equalities by a unique function $\mathcal{R}_p \in C^1([0, T_{\mathtt{c}}), \mathbf{R})$, which is given explicitly by

$$\mathcal{R}_p(t) = e^{-\nu t + \mathcal{A}_p(t)} \Big(\delta_p + \int_0^t ds\, e^{\nu s - \mathcal{A}_p(s)} \epsilon_p(s)\Big) , \tag{3.7}$$

$$\mathcal{A}_p(t) := \int_0^t ds \,\big(G_p\mathcal{D}_p(s) + K_p\mathcal{D}_{p+1}(s) + G_{pn}\mathcal{R}_n(s)\big) . \qquad \square$$

I have already mentioned in the Introduction a number of applications of Proposition 3.2 in which $u_{\mathtt{a}}$ is, e.g., a Galerkin approximant or a truncated power expansion in $1/\nu$. From the viewpoint of the present note, Proposition 3.2 is the basic tool yielding, by elementary manipulations, a global stability result for the NS Cauchy problem: see Section 5.

4 Global, decaying NS solutions

4.1 ***Lemma.*** *Let $w_0 \in \mathbb{H}^{\infty}_{\Sigma 0}$ be such that $\|w_0\|_n < \nu/G_n$ for some $n \in (d/2 + 1, +\infty)$. Then, the (maximal) solution w of the NS Cauchy problem (2.11) with initial datum w_0 is global and, for each $p \in \mathbf{R}$, there is a constant $C_p \in [0, +\infty)$ such that $\|w(t)\|_p \leqslant C_p e^{-\nu t}$ for all $t \in [0, +\infty)$.*

Proof. This follows immediately from Proposition 5.1 of [24]. ([7]) $\square$

Using the previous Lemma, we can easily prove that many natural decay conditions for a global NS solution v are in fact equivalent, and are indeed equivalent to the requirement that $v(t_0)$ be sufficiently small at just one time t_0.

[7]The cited Proposition from [24] gives slightly more refined estimates on the norms of $w(t)$ (see Eq. (5.4) therein), which could be used to infer explicit expressions for the constants C_p in terms of the norms of w_0. These explicit expression are not relevant for our present purposes.

4.2 *Proposition.* *Let $v \in C^\infty([0,+\infty), \mathbb{H}^\infty_{\Sigma 0})$ be a global NS solution: $dv/dt = \nu\Delta v + \mathcal{P}(v,v)$. The following statements are equivalent:*
a) For some $n \in (d/2+1,+\infty)$ and $t_0 \in [0,+\infty)$, it is $\|v(t_0)\|_n < \nu/G_n$.
b) For some $n \in (d/2+1,+\infty)$, it is $\|v(t)\|_n \to 0$ for $t \to +\infty$.
c) For all $p \in \mathbf{R}$, it is $\|v(t)\|_p \to 0$ for $t \to +\infty$ (i.e., $v(t) \to 0$ in the Fréchet space $\mathbb{H}^\infty_{\Sigma 0}$).
d) For some $n \in (d/2+1,+\infty)$ and $\gamma \in (0,+\infty)$, it is $\int_0^{+\infty} dt\, \|v(t)\|_n^\gamma < +\infty$.
e) For all $p \in \mathbf{R}$ and $\gamma \in (0,+\infty)$, it is $\int_0^{+\infty} dt\, \|v(t)\|_p^\gamma < +\infty$.
f) For some $n \in (d/2+1,+\infty)$, there is constant $C_n \in [0,+\infty)$ such that $\|v(t)\|_n \leqslant C_n\, e^{-\nu t}$ for all $t \in [0,+\infty)$.
g) For each $p \in \mathbf{R}$, there is constant $C_p \in [0,+\infty)$ such that $\|v(t)\|_p \leqslant C_p\, e^{-\nu t}$ for all $t \in [0,+\infty)$.

Proof. It suffices to show that (a) ⇒ (g) and that, for (x) = (b),(c),(d),(e),(f), it is (g) ⇒ (x) ⇒ (a). Here are the proofs.
(a) ⇒ (g). Let n, t_0 be as in (a), that we assume to hold. Setting $w_0 := v(t_0)$ we see that $v(t) = w(t - t_0)$ for $t \in [t_0, +\infty)$, where w is the (maximal) solution of the NS Cauchy problem with datum w_0, which has the features predicted by Lemma 4.1. Thus, v fulfills (g).
(g) ⇒ (b),(c),...,(f). Obvious
(b) ⇒ (a); (c) ⇒ (a); (f) ⇒ (a). Obvious.
(d) ⇒ (a). Let n, γ be as in (d), that we assume to hold; hereafter we show that, for each $\eta > 0$, there is $t_0 \in [0,+\infty)$ such that $\|v(t_0)\|_n < \eta$. Indeed, if not so, we would have $\|v(t)\|_n \geqslant \eta$ for all $t \in [0,+\infty)$ and this would imply $\int_0^{+\infty} dt\, \|v(t)\|_n^\gamma \geqslant \eta^\gamma \int_0^{+\infty} dt = +\infty$, against (d). Now, with $\eta = \nu/G_n$ we get the thesis (a).
(e) ⇒ (a). In fact, it is evident that (e) ⇒ (d) and we know that (d) ⇒ (a). □

4.3 *Definition.* *i) A* global, decaying NS solution *is a global solution v with the equivalent properties (a)-(g) of Proposition 4.2.*
ii) We say that $v_0 \in \mathbb{H}^\infty_{\Sigma 0}$ gives rise to a global, decaying solution for the NS Cauchy problem *if such features are possessed by the maximal solution v of problem (2.11) with initial datum v_0.*
iii) The subset of $\mathbb{H}^\infty_{\Sigma 0}$ formed by the initial data v_0 as in (ii) will be indicated with $\mathbb{E}^\infty_{\Sigma 0\,\nu} \equiv \mathbb{E}^\infty_{\Sigma 0}$. □

4.4 Remarks. i) Lemma 4.1 indicates that, for any $n \in (d/2+1,+\infty)$, $\mathbb{E}^\infty_{\Sigma 0}$ contains the ball $\{v_0 \in \mathbb{H}^\infty_{\Sigma 0} \mid \|v_0\|_n < \nu/G_n\}$.

ii) One can give examples of arbitrarily large initial data $v_0 \in \mathbb{E}^\infty_{\Sigma 0}$; among them are the generalized Beltrami flows of Section 6.

iii) Consider a global NS solution v. Using the equivalence between the family of norms $\| \ \|_p$ ($p \in \mathbf{R}$) and the family of norms $\| \ \|_{C^r}$ ($r \in \mathbf{N}$) on $\mathbb{H}^\infty_{\Sigma 0}$ (see after

Eq. (2.5)), it is possible to construct further equivalents of conditions (a)-(g) in Proposition 4.2, e.g. the following ones:

d') For some $r \in \mathbf{N}$ with $r > d/2+1$ and some $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|v(t)\|^{\gamma}_{C^r} < +\infty$.

e') For all $r \in \mathbf{N}$ and $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|v(t)\|^{\gamma}_{C^r} < +\infty$.

In fact: (e) $\Leftrightarrow$ (e') due to the inequalities $\| \; \|_{C^r} \leqslant$ const. $\| \; \|_p$ for $p > r + d/2$ and $\| \; \|_p \leqslant$ const. $\| \; \|_{C^r}$ for $p \leqslant r$; (e') $\Rightarrow$ (d') (obvious);
(d') $\Rightarrow$ (d) since $\| \; \|_n \leqslant$ const. $\| \; \|_{C^r}$ for $n = r$. Thus (d') and (e') are both equivalent to (d) (e), and hence to all items in Proposition 4.2.

iv) Paper [6], already mentioned in the Introduction, considers for $d = 3$ the global NS solutions v with (divergence free, mean zero) data v_0, such that $\int_0^{+\infty} dt\, \|v(t)\|^2_{W^{r,\infty}} < +\infty$ for some $r \in \mathbf{N} \setminus \{0\}$; if v_0 has a minimal regularity (say, $v_0 \in \mathbb{H}^1_{\Sigma 0}$), for all $t > 0$ $v(t)$ is C^∞ [13] (and thus in $\mathbb{H}^\infty_{\Sigma 0}$), whence $\|v(t)\|_{W^{r,\infty}} = \|v(t)\|_{C^r}$. With the stronger assumption $v_0 \in \mathbb{H}^\infty_{\Sigma 0}$, a comparison can be made with the present note keeping in mind the previous Remark (iii); it turns out that the condition $\int_0^{+\infty} dt\, \|v(t)\|^2_{C^r} < +\infty$ is implied by those in Proposition 4.2 if $r = 1, 2$, and is equivalent to them if $r \geqslant 3$.

5 A global stability result for the NS Cauchy problem

5.1 *Theorem.* *Let $v_0 \in \mathbb{E}^\infty_{\Sigma 0}$ (see Definition 4.3). Denote with $v \in C^\infty([0, +\infty), \mathbb{H}^\infty_{\Sigma 0})$ the global, decaying NS solution with initial datum v_0, and set*

$$J_p := \int_0^{+\infty} dt\, \|v(t)\|_p < +\infty \quad \textit{for } p \in \mathbf{R} \; ; \tag{5.1}$$

in addition, choose any $n \in (d/2 + 1, +\infty)$ and define

$$\rho_n := \frac{\nu}{G_n} e^{-G_n J_n - K_n J_{n+1}} \; . \tag{5.2}$$

Then

$$u_0 \in \mathbb{H}^\infty_{\Sigma 0} \; , \; \|u_0 - v_0\|_n < \rho_n \quad \Longrightarrow \quad u_0 \in \mathbb{E}^\infty_{\Sigma 0} \; . \tag{5.3}$$

If $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$, $\|u_0 - v_0\|_n < \rho_n$ and u is the global, decaying NS solution with datum u_0, for all $t \in [0, +\infty)$ and $p \in (n, +\infty)$ we have

$$\|u(t) - v(t)\|_n \leqslant \frac{e^{G_n J_n + K_n J_{n+1}}}{1 - \delta_n/\rho_n}\, \delta_n\, e^{-\nu t}, \quad \delta_n := \|u_0 - v_0\|_n \, ; \tag{5.4}$$

$$\|u(t) - v(t)\|_p \leqslant e^{G_p J_p + K_p J_{p+1} + \dfrac{G_{pn}\, \delta_n/\rho_n}{G_n(1 - \delta_n/\rho_n)}}\, \delta_p\, e^{-\nu t}, \quad \delta_p := \|u_0 - v_0\|_p \, .$$

Under the stronger assumptions $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$ and $\delta_n \equiv \|u_0 - v_0\|_n \leqslant \rho_n/2$, the bounds (5.4) imply these simpler bounds, with linear dependence on both variables δ_n and δ_p: for $t \in [0,+\infty)$ and $p \in (n,+\infty)$,

$$\|u(t)-v(t)\|_n \leqslant 2\, e^{G_n J_n + K_n J_{n+1}}\, \delta_n\, e^{-\nu t} \ , \tag{5.5}$$

$$\|u(t)-v(t)\|_p \leqslant e^{G_p J_p + K_p J_{p+1} + \frac{G_{pn}}{G_n}}\, \delta_p\, e^{-\nu t} \ .$$

Proof. Let us consider a datum $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$, for the moment arbitrary. We are interested in the NS Cauchy problem (2.11) with datum u_0, and in its (maximal) exact solution u. We apply to this Cauchy problem Proposition 3.2 on approximate solutions, with

$$u_{\mathsf{a}} := v \ . \tag{5.6}$$

Since v solves exactly the NS equations with datum v_0, the differential error of v is zero and the datum error (with respect to (2.11)) is $u_0 - v_0$; we will use the tautological error and growth estimators associated to v according to Definition 3.1, which are

$$\epsilon_p(t) := 0 \ , \quad \delta_p := \|u_0 - v_0\|_p \ , \quad \mathcal{D}_p(t) := \|v(t)\|_p \quad \text{for } p \in \mathbf{R},\ t \in [0,+\infty). \tag{5.7}$$

In the sequel we will also refer to the primitive functions

$$\mathfrak{J}_p(t) := \int_0^t ds\, \|v(s)\|_p \ \leqslant J_p \qquad \text{for } p \in \mathbf{R},\ t \in [0,+\infty) \tag{5.8}$$

(the last inequality comes from comparison with (5.1)).
Let us choose $n \in (d/2+1,+\infty)$. We use the estimators (5.7) with $p = n$ or $n+1$ and try to fulfill the control inequalities (3.3) as equalities for an unknown C^1 function; this yields the Cauchy problem

$$\frac{d\mathcal{R}_n}{dt} = -\nu \mathcal{R}_n + (G_n\|v\|_n + K_n\|v\|_{n+1})\mathcal{R}_n + G_n \mathcal{R}_n^2 \ , \qquad \mathcal{R}_n(0) = \delta_n \tag{5.9}$$

for an unknown function $\mathcal{R}_n \in C^1([0,T_{\mathsf{c}}),\mathbf{R})$. The (maximal) solution of (5.9) is as follows:

$$\mathcal{R}_n(t) := \delta_n\, \frac{e^{-\nu t + G_n \mathfrak{J}_n(t) + K_n \mathfrak{J}_{n+1}(t)}}{1 - G_n \delta_n \mathcal{L}_n(t)} \quad \text{for } t \in [0,T_{\mathsf{c}}) \ , \tag{5.10}$$

$$\mathcal{L}_n(t) := \int_0^t ds\, e^{-\nu s + G_n \mathfrak{J}_n(s) + K_n \mathfrak{J}_{n+1}(s)} \quad \text{for } t \in [0,+\infty) \tag{5.11}$$

$$T_{\mathsf{c}} := \begin{cases} +\infty, \ \text{if } G_n\delta_n\mathcal{L}_n(t) \neq 1 \text{ for all } t \in (0,+\infty); \\ \text{the unique } t \in (0,+\infty) \text{ s.t. } G_n\delta_n\mathcal{L}_n(t) = 1, \\ \text{if this exists} \end{cases}$$

(note that $\mathcal{L}_n$ is strictly increasing on $[0,+\infty)$).

According to (5.8) $\mathcal{J}_n(s) \leqslant J_n$, $\mathcal{J}_{n+1}(s) \leqslant J_{n+1}$ for all $s \geqslant 0$, so that

$$\mathcal{L}_n(t) \leqslant e^{G_n J_n + K_n J_{n+1}} \int_0^t ds\, e^{-\nu s} = e^{G_n J_n + K_n J_{n+1}} \frac{1 - e^{-\nu t}}{\nu} \tag{5.12}$$
$$\leqslant \frac{1}{\nu} e^{G_n J_n + K_n J_{n+1}} = \frac{1}{G_n \rho_n} \quad \text{for } t \in [0, +\infty)$$

(as for the last equality, recall Eq. (5.2)). From now on we assume, as in (5.3),

$$\delta_n \equiv \|u_0 - v_0\|_n < \rho_n \ . \tag{5.13}$$

Then, due to (5.12),

$$G_n \delta_n \mathcal{L}_n(t) \leqslant \frac{\delta_n}{\rho_n} < 1 \text{ for all } t \in [0, +\infty) \tag{5.14}$$

so that $T_c = +\infty$, i.e., the solution $\mathcal{R}_n$ in Eqs. (5.10) (5.11) is globally defined. Due to Proposition 3.2, this implies that the solution u of the NS Cauchy problem (2.11) with datum u_0 is global as well, and that

$$\|u(t) - v(t)\|_n \leqslant \mathcal{R}_n(t) \quad \text{for } t \in [0, +\infty) \ ; \tag{5.15}$$

$$\|u(t) - v(t)\|_p \leqslant \mathcal{R}_p(t) \quad \text{for } p \in (n, +\infty),\ t \in [0, +\infty), \tag{5.16}$$
$$\mathcal{R}_p(t) := \delta_p\, e^{-\nu t + \mathcal{A}_p(t)} \ , \quad \mathcal{A}_p(t) := G_p \mathcal{J}_p(t) + K_p \mathcal{J}_{p+1}(t) + G_{pn} \int_0^t ds\, \mathcal{R}_n(s).$$

(To derive Eq. (5.16) one uses Eqs. (3.6) and (3.7), recalling the form (5.7) of the error and growth estimators and Eq. (5.8)). Now, let us return to the expression (5.10) for $\mathcal{R}_n$ in which we insert the inequalities $\mathcal{J}_n(t) \leqslant J_n$, $\mathcal{J}_{n+1}(t) \leqslant J_{n+1}$ (recall again (5.8)) and the inequality (5.14) for $\mathcal{L}_n$; this gives

$$\mathcal{R}_n(t) \leqslant \frac{e^{G_n J_n + K_n J_{n+1}}}{1 - \delta_n/\rho_n}\, \delta_n\, e^{-\nu t} = \frac{\nu\, \delta_n/\rho_n}{G_n(1 - \delta_n/\rho_n)}\, e^{-\nu t} \quad \text{for } t \in [0, +\infty) \ ; \tag{5.17}$$
$$\int_0^{+\infty} dt\, \mathcal{R}_n(t) \leqslant \frac{\delta_n/\rho_n}{G_n(1 - \delta_n/\rho_n)}$$

(the above equality follows from (5.2); the bound on the integral is a consequence of the bound on $\mathcal{R}_n$). Now, $\|u(t)\|_n \leqslant \|v(t)\|_n + \|u(t) - v(t)\|_n \leqslant \|v(t)\|_n + \mathcal{R}_n(t)$; from here we infer, using the definition (5.1) of J_n and the inequality in (5.17) for the integral of $\mathcal{R}_n$,

$$\int_0^{+\infty} dt\, \|u(t)\|_n \leqslant J_n + \frac{\delta_n/\rho_n}{G_n(1 - \delta_n/\rho_n)} < +\infty \ . \tag{5.18}$$

Thus the solution u of the NS Cauchy problem with datum u_0, besides being global is decaying: this means that $u_0 \in \mathbb{E}^\infty_{\Sigma 0}$, so statement (5.3) is proved.

To go on, let us insert the bound (5.17) for $\mathcal{R}_n$ in Eq. (5.15); this yields the bound on $\|u(t) - v(t)\|_n$ in (5.4).

Now, let $p \in (n, +\infty)$ and let us consider the definition of $\mathcal{A}_p(t)$ in Eq. (5.16); inserting therein the inequalities $\mathcal{J}_p(t) \leqslant J_p$, $\mathcal{J}_{p+1}(t) \leqslant J_{p+1}$ (see once more (5.8)), writing $\int_0^t ds\, \mathcal{R}_n(s) \leqslant \int_0^{+\infty} ds\, \mathcal{R}_n(s)$ and using for the last integral the bound (5.17), we obtain

$$\mathcal{A}_p(t) \leqslant G_p J_p + K_p J_{p+1} + \frac{G_{pn}\, \delta_n/\rho_n}{G_n(1 - \delta_n/\rho_n)} \quad \text{for } t \in [0, +\infty) \; . \tag{5.19}$$

Eqs. (5.19) and (5.16) yield the bound on $\|u(t) - v(t)\|_p$ in (5.4). To conclude, let us make the stronger assumption $\delta_n \leqslant \rho_n/2$; then the bounds (5.4) yield the simpler bounds (5.5), noting that $1/(1 - \delta_n/\rho_n) \leqslant 2$ and $(\delta_n/\rho_n)/(1 - \delta_n/\rho_n) \leqslant 1$. □

5.2 Remarks. i) Eq. (5.3) indicates that $\mathbb{E}^\infty_{\Sigma 0}$ is an *open* subset of $\mathbb{H}^\infty_{\Sigma 0}$ in the Fréchet topology.
ii) For a comparison between the present Theorem 5.1 and [6], let us recall that the cited work considers for $d = 3$ a global NS solution v with (divergence free, mean zero) initial datum v_0, fulfilling the decay condition discussed in Remark 4.4 (iv). According to Theorem 3.1 and Corollary 3.2 of [6], for each datum u_0 with $\|u_0 - v_0\|_r$ sufficiently small for some $r \in \mathbf{N} \setminus \{0\}$, the corresponding NS solution u is global as well, and $u - v$ fulfills bounds having essentially the form $\|u(t) - v(t)\|_m \leqslant \text{const.} \times \|u_0 - v_0\|_m\, e^{-\sigma \nu t}$ for each $\sigma \in (0, 1)$, $m \in \{0, 1, ..., r\}$, and $t \geqslant 0$, with unspecified constants depending (among others) on σ (the norms $\| \; \|_m$ are for each m as in the present note). The approach of [6] does not seem to encompass the $\sigma \to 1$ limit, that would give estimates more similar to those in the present Eq. (5.5).

6 Generalized Beltrami flows as initial data

Let us define a *generalized Beltrami flow on* $\mathbf{T}^d$ to be a vector field v_0 such that

$$v_0 \in \mathbb{H}^\infty_{\Sigma 0} \; , \quad \Delta v_0 = -\kappa^2\, v_0 \;\; (\kappa \in (0, +\infty)), \quad \mathcal{P}(v_0, v_0) = 0 \; . \tag{6.1}$$

For $d = 3$, the above notion (or its analog on $\mathbf{R}^3$) is considered in [29] [31]. Assuming again $d = 3$, for (6.1) to hold it suffices that $v_0 \in \mathbb{H}^\infty_{\Sigma 0}$ and $\operatorname{rot} v_0 = \pm \kappa v_0$ with

$\kappa \in (0,+\infty)$ ([8]); this is the case usually referred to as a Beltrami (or Beltrami-Trkal) flow [1] [6] [7].
From now on $d \in \{2,3,...\}$, v_0 is a generalized Beltrami flow, and κ is as in (6.1). Writing the condition $\Delta v_0 = -\kappa^2 v_0$ in terms of the Fourier components of v_0 one checks that $v_0 = 0$ (trivial case), or that $\kappa = |k| \geqslant 1$ for some $k \in \mathbf{Z}^d \setminus \{0\}$ and $v_{0k} = 0$ for $k \in \mathbf{Z}^d$, $|k| \neq \kappa$; these facts imply

$$(-\Delta)^{p/2} v_0 = \kappa^p v_0 \ , \quad \|v_0\|_p = \kappa^p \|v_0\|_{L^2} \quad \text{for } p \in \mathbf{R} . \tag{6.2}$$

To go on let us observe that, due to (6.1), the NS Cauchy problem (2.11) with initial datum v_0 has the global, decaying solution

$$v(t) = e^{-\kappa^2 \nu t} v_0 \qquad \text{for } t \in [0,+\infty) \ ; \tag{6.3}$$

thus, $v_0 \in \mathbb{E}^\infty_{\Sigma 0}$. Theorem 5.1 can be applied to any generalized Beltrami flow v_0; Eq.s (5.1) (6.2) (6.3) and (5.2) (for some $n > d/2+1$) give

$$J_p = \int_0^{+\infty} dt \, \|v(t)\|_p = \frac{\|v_0\|_p}{\kappa^2 \nu} = \frac{\kappa^{p-2}}{\nu} \|v_0\|_{L^2} \quad \text{for } p \in \mathbf{R} \, , \tag{6.4}$$

$$G_p J_p + K_p J_{p+1} = \frac{(G_p + K_p \kappa)\kappa^{p-2}}{\nu} \ \|v_0\|_{L^2} \quad \text{for } p \in \mathbf{R} \, , \tag{6.5}$$

$$\rho_n = \frac{\nu}{G_n} e^{-\dfrac{(G_n + K_n \kappa)\kappa^{n-2}}{\nu} \|v_0\|_{L^2}} \, .$$

Any datum $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$ with $\|u_0 - v_0\| < \rho_n$ produces a global, decaying solution u, and we have for $u(t) - v(t)$ the bounds (5.4) (or (5.5), if $\|u_0 - v_0\| \leqslant \rho_n/2$). Let me exhibit a generalized Beltrami flow v_0 for any d (which is not Beltrami if $d = 3$ and $A \neq 0$ below); this is

$$v_0(x) := \frac{\sqrt{2}}{(2\pi)^{d/2}} \, A \sin(k \bullet x) \quad \text{for } x \in \mathbf{T}^d \quad (A \in \mathbf{R}^d, k \in \mathbf{Z}^d \setminus \{0\}, A \bullet k = 0) \tag{6.6}$$

($\operatorname{div} v_0 = 0$ due to $A \bullet k = 0$; $\mathcal{P}(v_0, v_0) = -\mathfrak{L}((v_0 \bullet \nabla) v_0)$ vanishes because $((v_0 \bullet \nabla) v_0)(x) = (2\pi)^{-d} (A \bullet k) A \sin(2k \bullet x) = 0$). In the present case

$$\kappa = |k| \ , \quad \|v_0\|_{L^2} = |A| \ , \quad \|v_0\|_p = |k|^p |A| \quad \text{for all } p \in \mathbf{R} \ . \tag{6.7}$$

So, for each real p, $\|v_0\|_p$ can be arbitrarily large.

[8] in fact we have $\Delta v_0 = -\operatorname{rot} \operatorname{rot} v_0 + \nabla(\operatorname{div} v_0) = -\operatorname{rot} \operatorname{rot} v_0 = -\kappa^2 v_0$ and $(v_0 \bullet \nabla) v_0 = (\operatorname{rot} v_0) \wedge v_0 + \nabla(|v_0|^2/2) = \nabla(|v_0|^2/2)$, which implies $\mathcal{P}(v_0, v_0) = -\mathfrak{L}((v_0 \bullet \nabla) v_0) = 0$ because $\mathfrak{L}$ annihilates gradients.

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