



Discrete and Conservative Factorizations in $\mathbf{Fib}(B)$

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Abstract

We focus on the transfer of some known orthogonal factorization systems from \mathbf{Cat} to the 2-category $\mathbf{Fib}(B)$ of fibrations over a fixed base category B : the internal version of the *comprehensive factorization*, and the factorization systems given by (sequence of coidentifiers, discrete morphism) and (sequence of coinverters, conservative morphism) respectively. For the class of fibrewise opfibrations in $\mathbf{Fib}(B)$, the construction of the latter two simplify to a single coidentifier (respectively coinverter) followed by an internal discrete opfibration (resp. fibrewise opfibration in groupoids). We show how these results follow from their analogues in \mathbf{Cat} , providing suitable conditions on a 2-category \mathcal{C} , that allow the transfer of the construction of coinverters and coidentifiers from \mathcal{C} to $\mathbf{Fib}_{\mathcal{C}}(B)$.

Keywords Internal fibration · Factorization system · Coidentifier · Coinverter

Mathematics Subject Classification 18A32 · 18D05 · 18D30

1 Introduction

The crucial point of the work [2] was to recover Yoneda's Classification Theorem in [17, §3.2] as a result of the factorization of a *regular span* $S: X \rightarrow B \times A$ through an internal discrete opfibration, in the 2-category $\mathbf{Fib}(B)$ of fibrations over B . Yoneda's Theorem was the base to give a new interpretation of cohomology groups in the additive case. Theorem 3.2

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in [2], which extends the above factorization to any *fibrewise opfibration* (see Definition 4.7) with codomain a split fibration, allows us to enlarge this point of view to non-additive cases, such as cohomology of groups and of associative algebras.

The first aim of the present work is to show that the above mentioned result, obtained in [2] via an ad hoc construction, is actually an application of an orthogonal factorization system in $\text{Fib}(B)$, whose right class is given by internal discrete opfibrations. This is explained in Theorem 4.10, which generalizes to $\text{Fib}(B)$ the well known *comprehensive factorization system* of Cat [15]. The above theorem relies on the fact that, for the class of opfibrations, the comprehensive factorization system coincides with the orthogonal factorization system in Cat given by (sequence of coidentifiers, discrete functor), whose construction in this case simplifies to a single coidentifier (see Theorem 4.1 and Corollary 4.2). Actually, a similar phenomenon occurs in $\text{Fib}(B)$. In fact, $\text{Fib}(B)$ inherits from Cat the latter factorization system (see Theorem 4.5 (i)), which, restricted to fibrewise opfibrations, can be performed by a single coidentifier. This implies that for such class it coincides with the internal comprehensive factorization system (see Corollary 4.11).

As an obvious consequence of the results reported so far, we get a reflection of any morphism in $\text{Fib}(B)$ onto a discrete morphism. For some purposes, this may cause an undesired loss of information with respect to the initial data. So, one may look for a finer reflection, where fibres are turned into groupoids. This gives a richer structure which is at the base, for example, of the interpretation given in [1] of Schreier–Mac Lane Theorem on the classification of group extension and its further generalizations (see Proposition 2.7 in [1]).

The main goal of the present work is to show first in Theorem 4.5 (ii) that $\text{Fib}(B)$ inherits such a finer factorization from the one in Cat given by (sequence of coinverters, conservative functor) introduced in [7]; second, that for fibrewise opfibrations, it is performed by a single coinverter followed by a fibrewise opfibration whose fibres are groupoids (Theorem 4.12).

The achieved results rely on specific 2-categorical properties of Cat and of $\text{Fib}(B)$ which are studied in detail in Sect. 3. In particular, in Proposition 3.8 we detect a sufficient condition to transfer the construction of coidentifiers and coinverters from a 2-category \mathcal{C} to the 2-category $\text{Fib}_{\mathcal{C}}(B)$ of internal fibrations over a fixed object B . This happens when the 2-monad $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$, whose pseudo-algebras define internal fibrations (in the sense of Street [10]), preserves coidentifiers and coinverters of idteetes. In Propositions 3.13 and 3.16 we prove that the property stated above holds when $\mathcal{C} = \text{Cat}$ and $\mathcal{C} = \text{Fib}(B)$, for any B , thanks to the exponentiability of fibrations and opfibrations in Cat .

Throughout the paper, 2-limits and 2-colimits are to be understood in a strict sense.

2 Review of Internal Fibrations

Let \mathcal{C} be a finitely complete 2-category [11]. For a fixed object B in \mathcal{C} , we shall denote by \mathcal{C}/B the slice 2-category over B and by $\mathcal{C}//B$ the pseudo-slice 2-category over B .

We shall denote as follows the (strict) comma objects in \mathcal{C} of identities, along identities on the left and on the right respectively, and iso-comma along identities:

$$\begin{array}{ccccccc}
 B/B & \xrightarrow{d_1} & B & & B/f & \xrightarrow{d_1} & A \\
 d_0 \downarrow & \nearrow \mu_B & \downarrow 1 & & Rf \downarrow & \nearrow \varphi_f & \downarrow f \\
 B & \xrightarrow{1} & B & & B & \xrightarrow{1} & B \\
 & & & & & & \\
 f/B & \xrightarrow{L_f} & B & & f/\cong B & \xrightarrow{I_f} & B \\
 d_0 \downarrow & \nearrow \psi_f & \downarrow 1 & & w_f \downarrow & \nearrow \omega_f & \downarrow 1 \\
 A & \xrightarrow{f} & B & & A & \xrightarrow{f} & B
 \end{array} \quad (1)$$

Following [10], one can extend the assignment $f \mapsto Lf$ to 1-cells and 2-cells, yielding a 2-functor $L: \mathcal{C} // B \rightarrow \mathcal{C} // B$. Moreover there exist: a pseudo 2-natural transformation $u: 1_{\mathcal{C} // B} \rightarrow L$, a (strict) 2-natural transformation $m: L^2 \rightarrow L$ and a modification $\lambda: Lu \rightarrow uL$ such that (L, u, m, λ) gives rise to a KZ-doctrine in the sense of Definition 1.1 in [9]. In other words, this structure provides a lax-idempotent 2-monad.

Let us observe that, in fact, $L: \mathcal{C} // B \rightarrow \mathcal{C} // B$ factors through the inclusion of \mathcal{C}/B in $\mathcal{C} // B$, and the above 2-monad on $\mathcal{C} // B$ restricts to a *strict* 2-monad on \mathcal{C}/B , which is also part of a KZ-doctrine by the same λ . We will adopt the same notation for both monads as far as no confusion arises.

Like L , also the 2-functors R and I on $\mathcal{C} // B$, defined by the corresponding comma squares in (1), can be endowed with a structure of 2-monad, which is colax-idempotent in the case of R and pseudo-idempotent in the case of I . In both cases, these structures restrict to strict 2-monads (R, v, n, ρ) and (I, i, l, ι) on \mathcal{C}/B .

One of the most important features of KZ-doctrines is that the corresponding (pseudo-) algebra structures are unique up to isomorphism for each object and they are characterized as right (pseudo-)inverse left adjoint to the unit component of the monad. Applying this observation and its dual to the special cases of the 2-functors L , R and I described above, one can characterize (pseudo-)opfibrations (and dually fibrations) and isofibrations in \mathcal{C} .

Proposition 2.1 *For a morphism $f: A \rightarrow B$ in \mathcal{C} the following conditions are equivalent and define an internal fibration (respectively pseudo-fibration):*

1. (i) *For all X in \mathcal{C} , $\mathcal{C}(X, f): \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$ is a fibration (respectively pseudo-fibration) in \mathbf{Cat} ;*
 (ii) *for all $g: Y \rightarrow X$, the commutative square below is a morphism of fibrations (respectively pseudo-fibrations) in \mathbf{Cat} :*

$$\begin{array}{ccc} \mathcal{C}(X, A) & \xrightarrow{\mathcal{C}(g, A)} & \mathcal{C}(Y, A) \\ \mathcal{C}(X, f) \downarrow & & \downarrow \mathcal{C}(Y, f) \\ \mathcal{C}(X, B) & \xrightarrow{\mathcal{C}(g, B)} & \mathcal{C}(Y, B) \end{array}$$

2. *f admits a structure of pseudo-algebra for the 2-monad $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$ (respectively $R: \mathcal{C} // B \rightarrow \mathcal{C} // B$);*
3. *The morphism $v_f: f \rightarrow Rf$ admits a right adjoint in \mathcal{C}/B (respectively $\mathcal{C} // B$);*
4. *(Chevalley criterion) The morphism $f_1: A/A \rightarrow B/f$, determined by the equations*

$$\begin{cases} (Rf)f_1 = fd_0 \\ d_1f_1 = d_1 \\ \varphi_f f_1 = f\mu_A \end{cases}$$

admits a right adjoint in \mathcal{C} with counit an identity (respectively isomorphism).

In practice, given an internal fibration according to the above form 2 of Proposition 2.1, it is convenient to fix a corresponding pseudo-algebra structure once and for all (which in \mathbf{Cat} means to fix a cleavage). Accordingly, throughout the paper, $\text{Fib}_{\mathcal{C}}(B)$ will denote the 2-category whose objects are pseudo-algebras for the monad $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$, whose 1-cells are strict pseudo-algebra morphisms, and with the obvious 2-cells (we shall write just $\text{Fib}(B)$ for $\mathcal{C} = \mathbf{Cat}$).

Remark 2.2 The definition of internal fibration (resp. pseudo-fibration) in a representable 2-category appears in the form 2 of Proposition 2.1 in the works of Street [10,12]. The characterizations 1 and 3 in Proposition 2.1 are well-known and already present in the literature (see, for example, [16]). As for the Chevalley criterion, it was first proved by Gray [4] for fibrations in \mathbf{Cat} , while an internal version of it appears in [10, Proposition 9], asking for the unit to be an isomorphism. In fact, the latter characterizes pseudo-opfibrations (see (3.17) in [12]). This is the reason why we consider the characterization 4 also for internal (strict) fibrations.

Definition 2.3 An internal fibration is said to be *discrete* if the functor $\mathcal{C}(X, f)$ in 1.(i) of Proposition 2.1 is a discrete fibration.

The equivalent conditions of Proposition 2.1 may be easily adapted to define *internal opfibrations* (respectively pseudo-opfibrations), replacing the monad R with the monad L . For the reader's convenience, throughout the paper, most of the results are stated in terms of fibrations. Where not explicitly provided, the corresponding results for opfibrations can be obtained by duality.

Proposition 2.4 For a morphism $f: A \rightarrow B$ in \mathcal{C} the following conditions are equivalent and define an internal isofibration:

1. For all X in \mathcal{C} , $\mathcal{C}(X, f): \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$ is an isofibration in \mathbf{Cat} ;
2. f admits a structure of pseudo-algebra for the 2-monad $I: \mathcal{C}/B \rightarrow \mathcal{C}/B$;
3. the morphism $i_f: f \rightarrow If$ admits a right adjoint in \mathcal{C}/B .

3 Coinverters and Coidentifiers in $\mathbf{Fib}_{\mathcal{C}}(B)$

From now on, let \mathcal{C} be a finitely complete 2-category with coidentifiers and coinverters of reflexive 2-cells, whose definition we recall for the sake of completeness (the reader may refer to [8,13] for example).

Definition 3.1 The coidentifier (respectively coinverter) of a 2-cell α is a 1-cell q such that:

1. $q\alpha$ is an identity (resp. isomorphism);
2. for any other 1-cell f such that $f\alpha$ is an identity (resp. isomorphism), there exists a unique 1-cell t with $tq = f$;
3. for any 2-cell $\beta: f \rightarrow f'$ such that $f\alpha$ and $f'\alpha$ are identities (resp. isomorphisms), there exists a unique 2-cell γ with $\gamma q = \beta$;

In this paper we will consider in particular coidentifiers (coinverters) of idtees (inverttees). Given a 1-cell f , we denote by (K, κ) its idtee, where κ is the 2-universal 2-cell making $f\kappa$ an identity. We denote by (W, ω) the inverttee of f , where ω is the 2-universal 2-cell making $f\omega$ an isomorphism.

Later on, we will take advantage of the following results concerning isofibrations. Recall that a 2-cell is called *f-vertical* if its image under f is an identity.

Lemma 3.2 Let f be an isofibration and α a 2-cell such that $f\alpha$ is an isomorphism. Then α factorizes as $\alpha = \sigma \cdot \tau$, where τ is *f-vertical* and σ is an isomorphism.

Proof Since f is an isofibration, the isomorphism $f\alpha$ admits a cartesian lifting σ , which is an isomorphism, at the codomain of α . Then τ is the unique *f-vertical* factorization of α through σ . \square

Corollary 3.3 *Let f be an isofibration, then:*

- (i) *f is conservative if and only if its fibres are groupoids;*
- (ii) *the coinverter of the identee of f coincides with the coinverter of its invertree.*

Proof Let (W, ω) and $(K, \omega c)$ be the invertree and the identee, respectively, of f :

$$K \xrightarrow{c} W \begin{array}{c} \xrightarrow{w_0} \\ \Downarrow \omega \\ \xrightarrow{w_1} \end{array} A \xrightarrow{f} B.$$

Since $f\omega$ is an isomorphism by definition, as in Lemma 3.2, we can factorize ω as a composite $\omega = \sigma \cdot \tau$, where σ is a cartesian lifting of $f\omega$, and τ the unique f -vertical comparison 2-cell. τ being vertical, there is a unique $c': W \rightarrow K$ such that $\omega c c' = \tau$. So we have factorized ω as in the following diagram:

$$\begin{array}{ccccc} W & \xrightarrow{c'} & K & \xrightarrow{c} & W \\ & \searrow & \swarrow \sigma & & \downarrow w_1 \\ & & & & A \end{array} \quad \begin{array}{c} \xrightarrow{w_0} \\ \leftarrow \omega \\ \xrightarrow{w_1} \end{array}$$

It is now easy to see that f is conservative, i.e. its invertree is an isomorphism, if and only if its fibres are groupoids, i.e. its identee is an isomorphism.

As for the second statement, it suffices to observe that the coinverter of the identee ωc coinverts also $\omega = \sigma \cdot \omega c c'$. \square

Obviously the last result does not hold in general if f is not an isofibration, as it is witnessed by the non-constant functor from the arrow-category $\mathbf{2}$ to the groupoid \mathbf{I} with two objects and two non-trivial arrows.

Identees in \mathcal{C}/B are computed in \mathcal{C} , while this is not true for invertrees (and 2-limits in general). On the other hand, it is easy to check that the following holds for 2-colimits.

Lemma 3.4 *The forgetful 2-functor $\text{dom}: \mathcal{C}/B \rightarrow \mathcal{C}$ creates 2-colimits, and in particular coidentifiers and coinverters of identees.*

We are going to explore the behaviour of the monad R with respect to these limits and colimits. Analogous results can be proved for the monad L .

Remark 3.5 It is worth observing that the functor R can be described by means of the following construction:

$$\begin{array}{ccccc} B/f & \xrightarrow{d_1^* f} & B/B & \xrightarrow{d_0} & B \\ \downarrow d_1 & & \downarrow d_1 & \swarrow \mu_B & \parallel \\ A & \xrightarrow{f} & B & \xlongequal{\quad} & B \end{array} \quad \begin{array}{c} \xrightarrow{Rf} \\ \xrightarrow{d_0} \end{array}$$

That is, $R = (d_0)_! d_1^*$, i.e. the composite of the change-of-base 2-functor along d_1 with the composition 2-functor with d_0 , which is left adjoint to d_0^* .

Lemma 3.6 *The monad $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$ preserves identees.*

Proof By Remark 3.5, the thesis follows from the fact that d_1^* preserves limits, being a right adjoint, and $(d_0)_!$ preserves identees. \square

Lemma 3.7 *The identee of a morphism $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}_{\mathcal{C}}(B)$ can be computed as in \mathcal{C} .*

Proof Let

$$\begin{array}{ccccc}
 & & k_0 & & \\
 & & \searrow & & \\
 K & \xrightarrow{\quad} & A & \xrightarrow{p} & C \\
 & \Downarrow \kappa & \downarrow f & & \swarrow g \\
 & k_1 & B & & \\
 & \nearrow h & & &
 \end{array} \quad (2)$$

be an identee diagram in \mathcal{C}/B (which means that κ is also the identee of p in \mathcal{C}). Since p is a morphism in $\text{Fib}_{\mathcal{C}}(B)$ and R preserves identees by Lemma 3.6, it is straightforward to prove that the adjunctions $v_f \dashv r_f$ and $v_g \dashv r_g$ in the diagram

$$\begin{array}{ccccc}
 B/h & \xrightarrow{Rk_0} & B/f & \xrightarrow{Rp} & B/g \\
 \uparrow \lrcorner & \Downarrow R\kappa & \uparrow \lrcorner & & \uparrow \lrcorner \\
 & Rk_1 & & & \\
 v_h \lrcorner & & v_f \lrcorner & & v_g \lrcorner \\
 & \downarrow k_0 & \downarrow & & \downarrow \\
 K & \xrightarrow{\quad} & A & \xrightarrow{p} & C \\
 & \Downarrow \kappa & & & \\
 & k_1 & & &
 \end{array}$$

induce an adjunction $v_h \dashv r_h$ by the universal property of the identees. \square

There's no obvious reason why coinverters and coidentifiers should be preserved by the monad R , however this happens in some cases of interest which we will explore later on. So, for a given object B in \mathcal{C} , we shall consider the property

(\dagger) The monad $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$ preserves coinverters and coinverters of identees.

Proposition 3.8 *Let B be an object in \mathcal{C} satisfying (\dagger), $p: (A, f) \rightarrow (C, g)$ a morphism in $\text{Fib}_{\mathcal{C}}(B)$ and κ its identee in \mathcal{C} . Then we get a factorization*

$$\begin{array}{ccccc}
 & & p & & \\
 & & \searrow & & \\
 A & \xrightarrow{q} & Q & \xrightarrow{s} & C \\
 & \downarrow f & \downarrow gs & & \swarrow g \\
 & B & & &
 \end{array}$$

of p in $\text{Fib}_{\mathcal{C}}(B)$ in either of the following ways:

- (i) taking as q the coidentifier of κ in \mathcal{C} ; then $q: (A, f) \rightarrow (Q, gs)$ is the coidentifier of κ in $\text{Fib}_{\mathcal{C}}(B)$;
- (ii) taking as q the coinverter of κ in \mathcal{C} ; then $q: (A, f) \rightarrow (Q, gs)$ is the coinverter of κ in $\text{Fib}_{\mathcal{C}}(B)$.

Proof We shall only prove (ii), (i) is proved analogously.

Recall that κ is also the identee of p in \mathcal{C}/B and consider the corresponding identee diagram (2) in \mathcal{C}/B . Since p (and then f) coinverts κ , the morphisms s and g_s in the factorization above are uniquely determined by the universal property of q , and this explains why q is a coinverter of κ in \mathcal{C}/B .

Since f is a fibration, the unit component $v_f: (A, f) \rightarrow (B/f, Rf)$ admits a right adjoint r_f in \mathcal{C}/B . We call η_f and ϵ_f the corresponding unit and counit. Likewise, v_g has a right adjoint r_g , and $pr_f = r_g(Rp)$ since p is a morphism in $\text{Fib}_{\mathcal{C}}(B)$. Let us consider the following diagram:

$$\begin{array}{ccccccc}
 & & B/h & \xrightarrow{Rk_0} & B/f & \xrightarrow{Rp} & B/(gs) & \xrightarrow{Rs} & B/g \\
 & & \downarrow R\kappa & & \downarrow Rq & & \downarrow r_{gs} & & \downarrow r_g \\
 & & B/h & \xrightarrow{Rk_1} & B/f & \xrightarrow{Rq} & B/(gs) & \xrightarrow{Rs} & B/g \\
 & & \downarrow k_0 & & \downarrow v_f & & \downarrow v_{gs} & & \downarrow v_g \\
 K & \xrightarrow{v_h} & K & \xrightarrow{q} & A & \xrightarrow{q} & Q & \xrightarrow{s} & C \\
 & & \downarrow \kappa & & \downarrow p & & \downarrow p & & \downarrow p \\
 & & K & \xrightarrow{k_1} & A & \xrightarrow{q} & Q & \xrightarrow{s} & C
 \end{array}$$

Now, $pr_f(R\kappa) = r_g(Rp)(R\kappa) = 1$, hence $r_f(R\kappa)$ factors through κ and $qr_f(R\kappa)$ is an isomorphism. By the assumption (\dagger) , Rq is the coinverter of $R\kappa$, so there exists a unique $r_{gs}: B/(gs) \rightarrow Q$ such that $r_{gs}(Rq) = qr_f$. By the 2-dimensional universal property of the coinverters q and Rq , one can prove that a unit η_{gs} and a counit ϵ_{gs} are induced by η_f and ϵ_f respectively, making $v_{gs} \dashv r_{gs}$ an adjoint pair in \mathcal{C}/B , so that g_s is a fibration. As a consequence of this construction, q turns out to be a morphism of fibrations over B .

It remains to show that for each $c: (A, f) \rightarrow (Y, y)$ in $\text{Fib}_{\mathcal{C}}(B)$ such that $c\kappa$ is an isomorphism, the unique comparison morphism t in \mathcal{C}/B , induced by the coinverter q and such that $tq = c$, is actually a morphism in $\text{Fib}_{\mathcal{C}}(B)$. Let us denote by r_y the R -pseudo-algebra structure on y , i.e. the right adjoint to v_y , and observe that the diagram

$$\begin{array}{ccc}
 B/(gs) & \xrightarrow{Rt} & B/y \\
 r_{gs} \downarrow & & \downarrow r_y \\
 Q & \xrightarrow{t} & Y
 \end{array}$$

commutes since Rq is a coinverter, then epimorphic, and precomposition with Rq gives the commutative square presenting c as a morphism of R -pseudo-algebras. \square

Corollary 3.9 *Let B be an object in \mathcal{C} satisfying (\dagger) , $f: A \rightarrow B$ a fibration in \mathcal{C} , κ its identee in \mathcal{C} , and $q: A \rightarrow Q$ its coinverter (respectively coidentifier) in \mathcal{C} . Then the unique comparison morphism $s: Q \rightarrow B$, such that $sq = f$, is a fibration and q is the coinverter (respectively coidentifier) of κ in $\text{Fib}_{\mathcal{C}}(B)$.*

Proof Apply Proposition 3.8 to the morphism $f: (A, f) \rightarrow (B, 1_B)$ in $\text{Fib}_{\mathcal{C}}(B)$. \square

In the cases we are interested in, which will be studied in Sect. 3.1, the property (\dagger) relies upon the exponentiability of split opfibrations in \mathcal{C} . In this context, exponentiability is to be understood in a 2-categorical sense: a 1-cell f is exponentiable if the change-of-base 2-functor along f has a right adjoint.

Lemma 3.10 *If for an object B in \mathcal{C} , the comma projection d_1 in the diagram*

$$\begin{array}{ccc} B/B & \xrightarrow{d_0} & B \\ d_1 \downarrow & \swarrow \mu_B & \parallel \\ B & \xlongequal{\quad} & B \end{array}$$

is exponentiable, then the functor

$$R: \mathcal{C}/B \rightarrow \mathcal{C}/B$$

has a right adjoint, hence B satisfies the condition (\dagger) . In particular, this holds for any B when split opfibrations in \mathcal{C} are exponentiable.

Proof By Remark 3.5, $R = (d_0)_! d_1^*$. Hence R is left adjoint to $(d_1)_* d_0^*$, where $(d_1)_*$ denotes the right adjoint to d_1^* , which exists by assumption. \square

Proposition 3.11 *Under the assumptions of Lemma 3.10, the functor $\text{dom}: \text{Fib}_{\mathcal{C}}(B) \rightarrow \mathcal{C}$ creates 2-colimits.*

Proof One can repeat the same arguments of the proof of Proposition 3.8, using the fact that now R preserves any 2-colimit, and that $\text{dom}: \mathcal{C}/B \rightarrow \mathcal{C}$ creates 2-colimits. \square

Remark 3.12 If instead of (\dagger) we ask for

(\dagger') The monad $L: \mathcal{C}/B \rightarrow \mathcal{C}/B$ preserves coidentifiers and coinverters of idtees,

then the results of Proposition 3.8 hold with $\text{Fib}_{\mathcal{C}}(B)$ replaced by $\text{OpFib}_{\mathcal{C}}(B)$. Accordingly, if d_0 is exponentiable, and in particular when split fibrations are exponentiable in \mathcal{C} , then L admits a right adjoint, (\dagger') holds for B , and $\text{dom}: \text{OpFib}_{\mathcal{C}}(B) \rightarrow \mathcal{C}$ creates colimits.

3.1 Case Study: Cat and Fib(B)

It is well-known that fibrations in Cat are exponentiable [3] in the classical 1-categorical sense. As observed by Johnstone [6], this property holds also in the 2-categorical sense recalled above. As a consequence, by Lemma 3.10 and Remark 3.12, we have:

Proposition 3.13 *In the 2-category Cat , each object B satisfies the conditions (\dagger) and (\dagger') .*

We will see in Proposition 3.16 that one can extend the last property from Cat to $\text{Fib}(B)$ for each B , by means of the pseudo-functorial interpretation of fibrations in Cat .

Proposition 3.14 *Let B be a category. Then*

- (i) *2-colimits, in particular coidentifiers and coinverters of idtees, exist in $\text{Fib}(B)$;*
- (ii) *their construction can be performed fibrewise.*

Proof (i) The existence is guaranteed by Proposition 3.11, since Cat is 2-cocomplete. We shall prove (ii) just for coidentifiers, the general case can be treated likewise.

Let us consider a morphism $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}(B)$ and focus our attention on its restriction p_b to a single fibre over some b in B . We can consider the coidentifier $q_b: A_b \rightarrow Q_b$ of its idtee (K_b, κ_b) . Since f and g are fibrations, the assignments $b \mapsto A_b$ and $b \mapsto C_b$ are pseudo-functorial and the collection of the functors p_b gives rise to a

natural transformation of pseudo-functors from B^{op} to Cat . By the universal property of the coidentifiers q_b for each b , the assignment $b \mapsto Q_b$ is also pseudo-functorial and the q_b 's organize in a natural transformation. Let us briefly show how this can be proved.

In fact, the assignment $b \mapsto K_b$ is also pseudo-functorial and together with the collection of the κ_b 's, it determines the idtee (K, κ) of the cartesian functor p . Given an arrow $\beta: b' \rightarrow b$ in B , we always denote by β^* its associated change of base functor for any chosen fibration over B . Since $q_{b'}\beta^*\kappa_b = q_{b'}\kappa_{b'}\beta^* = 1$, by the universal property of the coidentifier q_b there is a unique functor $\beta^*: Q_b \rightarrow Q_{b'}$ such that $\beta^*q_b = q_{b'}\beta^*$:

$$\begin{array}{ccccc} K_b & \xrightarrow{\quad} & A_b & \xrightarrow{q_b} & Q_b \\ \beta^* \downarrow & \searrow \kappa_b & \downarrow \beta^* & & \downarrow \beta^* \\ K_{b'} & \xrightarrow{\quad} & A_{b'} & \xrightarrow{q_{b'}} & Q_{b'} \end{array}$$

Given a composable pair of arrows

$$b'' \xrightarrow{\beta'} b' \xrightarrow{\beta} b$$

in B , let $\phi_{\beta, \beta'}: (\beta')^*\beta^* \rightarrow (\beta\beta')^*$ be the corresponding coherence isomorphism induced by the fibration f . Since $q_{b''}\phi_{\beta, \beta'}$ is a 2-cell between $q_{b''}(\beta')^*\beta^* = (\beta')^*\beta^*q_b$ and $q_{b''}(\beta\beta')^* = (\beta\beta')^*q_b$, then by the universal property of the coidentifier q_b there exists a unique invertible 2-cell $\psi_{\beta, \beta'}$ such that $\psi_{\beta, \beta'}q_b = q_{b''}\phi_{\beta, \beta'}$.

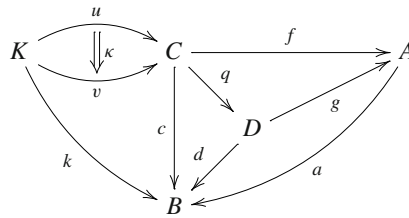
$$\begin{array}{ccccc} A_b & \xrightarrow{q_b} & Q_b & & \\ \downarrow \beta^* & & \downarrow \beta^* & & \\ \phi_{\beta, \beta'} \swarrow & & \searrow \psi_{\beta, \beta'} & & \\ A_{b'} & \xrightarrow{q_{b'}} & Q_{b'} & & \\ \downarrow (\beta')^* & & \downarrow (\beta')^* & & \\ A_{b''} & \xrightarrow{q_{b''}} & Q_{b''} & & \end{array}$$

Finally, the coherence conditions on the ψ 's making the assignment $b \mapsto Q_b$ into a pseudo-functor can be deduced once again by the universal property of the q_b 's. Since for a morphism $t: (A, f) \rightarrow (Y, y)$ in $\text{Fib}(B)$, $t\kappa = 1$ if and only if $t_b\kappa_b = 1$ for each b in B , q is actually the coidentifier of κ in $\text{Fib}(B)$. \square

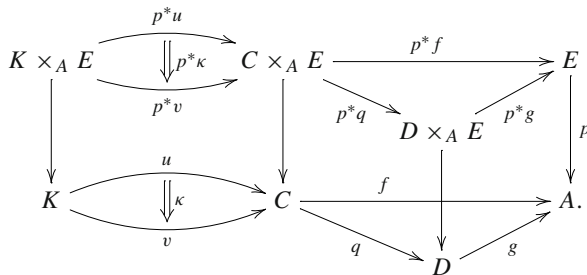
Lemma 3.15 *For each internal fibration (resp. opfibration) $p: (E, e) \rightarrow (A, a)$ in $\text{Fib}(B)$, the change of base 2-functor $p^*: \text{Fib}(B)/(A, a) \rightarrow \text{Fib}(B)/(E, e)$ preserves coidentifiers and coinverters of idtees.*

Proof We will prove the result concerning internal fibrations and coinverters, the variations involving opfibrations and coidentifiers are obtained analogously.

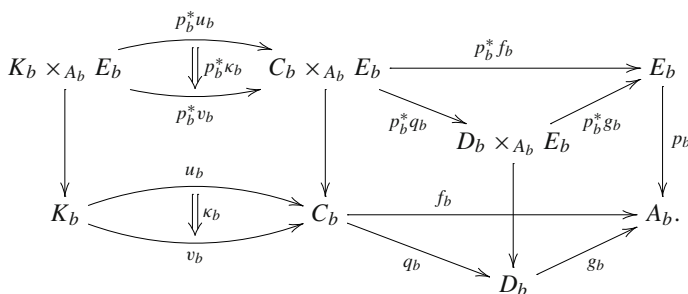
Let the arrow $q : ((C, c), f) \rightarrow ((D, d), g)$ in the diagram



be the coinverter in $\text{Fib}(B)/(A, a)$ of an identee κ , and consider its image under the change of base 2-functor p^* , i.e. the upper part of the next diagram (we omit all arrows over B , all pullbacks provide in fact fibrations over B):



We would like to show that p^*q is the coinverter of the identee $p^*\kappa$ in $\text{Fib}(B)/(E, e)$. To this end, we consider the restriction of the above diagram to the fibres over any object b in B . By limit commutation, the latter is the same as the corresponding change of base diagram in the fibres over b :



Now observe that, by Proposition 3.14 (ii), q_b is the coinverter of κ_b . Moreover, since p_b is a fibration in Cat by assumption, it is exponentiable, hence p_b^* is a left adjoint and $p_b^*q_b$ is the coinverter of $p_b^*\kappa_b$. Finally, again by Proposition 3.14, p^*q is the coinverter of $p^*\kappa$ in $\text{Fib}(B)$, and hence in $\text{Fib}(B)/(A, a)$. \square

Proposition 3.16 *In the 2-category $\text{Fib}(B)$, each object satisfies the conditions (\dagger) and (\dagger') .*

Proof Let $a: A \rightarrow B$ be a fibration of categories, then the projections d_0 and d_1 of the comma square in $\text{Fib}(B)$

$$\begin{array}{ccc} (A, a)/(A, a) & \xrightarrow{d_0} & (A, a) \\ d_1 \downarrow & \swarrow \mu_{(A,a)} & \parallel \\ (A, a) & \xlongequal{\quad} & (A, a) \end{array}$$

are an internal fibration and opfibration respectively (see Theorem 14 in [10]). As a consequence, by Lemma 3.15, the corresponding change of base 2-functors d_0^* and d_1^* preserve coidentifiers and coinverters of idtees. Now likewise in the proof of Lemma 3.10, the thesis follows from the fact that $R = (d_0)_! d_1^*$ and $(d_0)_!$ is a left adjoint (and similarly for L). \square

Proposition 3.17 *Let $p: (A, f) \rightarrow (C, g)$ be an internal fibration (resp. opfibration) in $\text{Fib}(B)$. Then the morphism s in the factorization of Proposition 3.8 (i) is an internal fibration (resp. opfibration) in $\text{Fib}(B)$.*

Proof By Proposition 3.16, we can apply Corollary 3.9 to the fibration p in $\text{Fib}(B)$. \square

4 Three Factorization Systems in $\text{Fib}(B)$

Let us recall that an (orthogonal) factorization system on a 1-category \mathcal{C} is given by a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms in \mathcal{C} such that:

- (i) \mathcal{E} and \mathcal{M} are closed under composition and contain isomorphisms;
- (ii) each morphism f in \mathcal{C} admits a factorization $f = m \cdot e$ with m in \mathcal{M} and e in \mathcal{E} ;
- (iii) for each commutative square

$$\begin{array}{ccc} & \xrightarrow{x} & \\ e \downarrow & \nearrow d & \downarrow m \\ & \xrightarrow{y} & \end{array}$$

with m in \mathcal{M} and e in \mathcal{E} , there exists a unique morphism d such that $md = y$ and $de = x$.

When \mathcal{C} is a 2-category such factorization system $(\mathcal{E}, \mathcal{M})$ becomes a *strict 2-categorical (or a Cat-enriched) factorization system* if the following additional property holds:

- (iv) for each diagram

$$\begin{array}{ccc} & \xrightarrow{x} & \\ e \downarrow & \begin{array}{c} \Downarrow \alpha \\ \nearrow x' \\ \searrow d' \\ \Downarrow \beta \end{array} & \downarrow m \\ & \xrightarrow{y} & \end{array}$$

where $m\alpha = \beta e$ and d and d' are determined by (x, y) and (x', y') respectively, there exists a unique δ such that $m\delta = \beta$ and $\delta e = \alpha$.

4.1 Three Factorization Systems in Cat

The *comprehensive factorization system* introduced in [15] and given by (initial functor, discrete opfibration) is a well known example of factorization system in \mathbf{Cat} , which is actually strict 2-categorical. We will call by the same name the dual factorization given by (final functor, discrete fibration).

Another example of factorization system in \mathbf{Cat} , which is 2-categorical as well, and has conservative functors as right class, was introduced in [7, Theorem C.0.31] relying on the notion of coinverter. Let us illustrate how this is obtained. For a functor f

$$\begin{array}{ccc} W & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \omega \\ \xrightarrow{\quad} \end{array} & A \xrightarrow{q} Q \\ & & \downarrow f \quad \swarrow s \\ & & B \end{array}$$

the factorization s of f through the coinverter q of the invertree ω of f is not conservative in general. One has to repeat this “invertree–coinverter” procedure possibly infinitely many times in order to get a conservative comparison, and an actual factorization system. A similar phenomenon occurs when taking the “identree–coidentifier” analogue of the previous procedure, which allows to factor any functor as a (possibly infinite) sequence of coidentifiers followed by a discrete functor, i.e. a functor whose fibres are discrete, yielding another (strict 2-categorical) factorization system in \mathbf{Cat} .

We are going to show that if we restrict ourselves to fibrations, the latter two transfinite procedures above reduce to a single step. We start with the second one and we show also that, for fibrations, it coincides with the comprehensive factorization system. This actually holds not only in \mathbf{Cat} , but in any 2-category $\mathbf{Cat}(\mathcal{E})$ of internal categories where the construction of the comprehensive factorization of any functor provided in [14] is still valid, as, for example, when \mathcal{E} is a finitely cocomplete locally cartesian closed category, like any topos \mathcal{E} .

Theorem 4.1 *Let $f: A \rightarrow B$ be a fibration in $\mathbf{Cat}(\mathcal{E})$ as above. The coidentifier of the identree of f factorizes f into a final functor followed by a discrete fibration, giving then the comprehensive factorization of f . Starting with f opfibration, the same procedure yields the dual factorization of f given by an initial functor followed by a discrete opfibration.*

Proof We consider just the case of fibrations. Following the approach of Section 3 in [14], we perform the comprehensive factorization of f by taking the free R -algebra Rf , which is a split fibration, and then reflecting it into a discrete fibration p :

By construction of the above reflection, d is the coidentifier of the identee κ_{Rf} of Rf . Considering the adjunction $v \dashv r$ provided by the fact that f is a fibration, we get $dv\kappa_f = d\kappa_{Rf}\bar{v} = 1$, where κ_f is the identee of f . Let now q be a functor such that $q\kappa_f = 1$, and consider the unit $\eta: 1 \rightarrow rv$ of the adjunction $v \dashv r$ in $\text{Cat}(\mathcal{E})/B$. Then $f\eta = 1$ and η is contained in κ_f , so that $q\eta = 1$ as well, and $qrv = q$. On the other hand, the counit $\epsilon: vr \rightarrow 1$ is such that $(Rf)\epsilon = 1$, so it is contained in κ_{Rf} and hence $d\epsilon = 1$ and $dvr = d$.

Now, $qr\kappa_{Rf} = q\kappa_f\bar{r} = 1$, so by the universal property of the coidentifier d there exists a unique t such that $td = qr$. Hence $q = qrv = tdv$, and t is unique with this last property. Indeed, if $t'dv = q$ for some t' , then $t'd = t'dvr = qr = td$ and hence $t' = t$ since d is epimorphic. This proves that dv is the coidentifier of κ_f , and then it is final [14]. \square

Corollary 4.2 *For a fibration (resp. opfibration) f , the factorization of f given by (sequence of coidentifiers, discrete functor) reduces to a single coidentifier and coincides with the comprehensive factorization.*

In the special case of Cat , Theorem 4.1 can be proved directly by means of the pseudo-functorial interpretation of fibrations. This indeed is what we are going to do in order to obtain the analogous result, where coidentifiers are replaced by coinverters and discrete fibrations are replaced by fibrations in groupoids (i.e. fibrations whose identee is an isomorphism).

Theorem 4.3 *Each fibration (respectively opfibration) $f: A \rightarrow B$ in Cat admits a factorization given by the coinverter of the identee of f followed by a fibration (resp. opfibration) in groupoids. This factorization of f coincides with the one given by (sequence of coinverters, conservative functor).*

Proof Let us denote by

$$\text{Cat} \begin{array}{c} \xrightarrow{\pi} \\ \perp \\ \xleftarrow{i} \end{array} \text{Gpd}$$

the reflection of categories in groupoids, where the left adjoint π can be obtained by taking as unit component, for each category A , the coinverter η_A of the 2-cell μ_A associated with the comma category A/A :

$$A/A \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \mu_A \\ \xrightarrow{\quad} \end{array} A \xrightarrow{\eta_A} \pi(A).$$

Consider now a fibration $f: A \rightarrow B$ and denote by $[f]: B^{\text{op}} \rightarrow \text{Cat}$ the corresponding pseudo-functor. The composite $\pi[f]: B^{\text{op}} \rightarrow \text{Gpd}$ gives rise to a fibration in groupoids

$\bar{f}: \bar{A} \rightarrow B$. On the other hand, $\eta[f]$ corresponds to a morphism $q: (A, f) \rightarrow (\bar{A}, \bar{f})$ in $\text{Fib}(B)$:

$$\begin{array}{ccc} A & \xrightarrow{q} & \bar{A} \\ & \searrow f & \swarrow \bar{f} \\ & B. & \end{array}$$

The component q_b of $\eta[f]$ at an object b of B is actually the coinverter η_{A_b} of μ_{A_b} :

$$A_b/A_b \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \mu_{A_b} \\ \xrightarrow{\quad} \end{array} A_b \xrightarrow{\eta_{A_b}} \pi(A_b).$$

It is not difficult to see that the pair $(A_b/A_b, \mu_{A_b})$ coincides with the restriction (K_b, κ_b) of the identee (K, κ) of f to the fibre over b . Hence, as explained in Proposition 3.14, q turns out to be the coinverter of κ in $\text{Fib}(B)$. Thanks to Corollary 3.3, \bar{f} is conservative and we get the desired factorization of f . \square

4.2 From Cat to Fib(B)

We are going to use now the results of the previous sections to produce analogous factorization systems in $\text{Fib}(B)$. First, we need a preliminary result.

Lemma 4.4 *For a morphism $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}(B)$, the coinverter q of the invertree of p in $\text{Fib}(B)$ is also the coinverter of the invertree of p in Cat .*

Proof The only non-trivial property to prove is that for an arrow α in A , $q\alpha$ is an isomorphism as soon as $p\alpha$ is an isomorphism. But if $p\alpha$ is an isomorphism, then α factors as $\alpha = \kappa \cdot \nu$ where κ is an f -cartesian lifting of $f\alpha$, hence an isomorphism, and ν is f -vertical with $p\nu$ isomorphism. So ν , and hence α , is inverted by q . \square

Theorem 4.5 *For each category B , $\text{Fib}(B)$ inherits from Cat two factorization systems given by*

- (i) *(sequence of coidentifiers, discrete cartesian functors);*
- (ii) *(sequence of coinverters, conservative cartesian functors).*

Proof Identees in $\text{Fib}(B)$ are computed in Cat by Lemma 3.7. Even if invertrees in $\text{Fib}(B)$ may differ from the corresponding invertrees computed in Cat , their coinverters coincide by Lemma 4.4. As a consequence, thanks to Proposition 3.11, such factorization systems are just performed in Cat . \square

Thanks to a result due to Bénabou, proving that a morphism in $\text{Fib}(B)$ is an internal fibration if and only if it is a fibration in Cat (see, for example, Theorem 4.16 in [5]), we easily get the following result.

Proposition 4.6 *$\text{Fib}(B)$ inherits from Cat the comprehensive factorization system having internal discrete fibrations as right class.*

Proof Let $p: (A, f) \rightarrow (C, g)$ be any morphism in $\text{Fib}(B)$. Take its factorization (q, s) in Cat , with q a final functor and s a discrete fibration. Then gs is a fibration and s is an internal fibration. It is easy to see that, since p is cartesian and s is discrete, q is cartesian as well. \square

Our next goal is to show that $\text{Fib}(B)$ admits also a comprehensive factorization system having internal discrete *opfibrations* as right class. We cannot repeat the above argument because internal opfibrations in $\text{Fib}(B)$ are not opfibrations in Cat . Let us recall from [2] the following definition.

Definition 4.7 (see [2, Definition 2.1]) We say that a morphism $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}(B)$ is a *fibrewise (discrete) opfibration* if, for every object b of B , the restriction $p_b: A_b \rightarrow C_b$ of p to the b -fibres is a (discrete) opfibration.

From Theorem 2.8 in [2] it follows that every internal opfibration in $\text{Fib}(B)$ is a fibrewise opfibration, while the latter is exactly a morphism in $\text{Fib}(B)$ which is an internal opfibration in Cat/B (see Propositions 2.5 and 2.7 in [2]). By Corollary 2.9 in [2], the two notions coincide in the discrete case. Recall also from [2] that Yoneda's *regular spans* and *two-sided fibrations* are instances, respectively, of fibrewise opfibrations and internal opfibrations in $\text{Fib}(B)$.

Proposition 4.8 Let $p: (A, f) \rightarrow (C, g)$ be a fibrewise (resp. internal) opfibration in $\text{Fib}(B)$. Then we get a factorization

$$\begin{array}{ccccc} & & p & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{q} & Q & \xrightarrow{s} & C \\ & \searrow f & \downarrow gs & \nearrow g & \\ & & B & & \end{array}$$

of p in $\text{Fib}(B)$ in either of the following ways:

- (i) taking as q the coidentifier of the identee of p ; then s is a discrete opfibration in $\text{Fib}(B)$;
- (ii) taking as q the coinverter of the identee of p ; then s is a fibrewise (resp. internal) opfibration in groupoids in $\text{Fib}(B)$.

Proof Let us consider a fibrewise opfibration $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}(B)$.

(i) Take the factorization (q, s) of p as in Proposition 3.8 (i), with q the coidentifier of the identee of p . If we restrict to a single fibre over some b in B , we get a factorization

$$\begin{array}{ccccc} A_b & \xrightarrow{q_b} & Q_b & \xrightarrow{s_b} & C_b, \\ & & \searrow p_b & & \end{array}$$

where, thanks to Proposition 3.14 (ii), q_b is the coidentifier of the identee of p_b . Since the latter is an opfibration, by Theorem 4.1 s_b is a discrete opfibration, hence s is an internal discrete opfibration in $\text{Fib}(B)$.

(ii) Take instead the factorization (q, s) of p as in Proposition 3.8 (ii). Likewise in (i), for each b in B , thanks to Theorem 4.3, we get a factorization (q_b, s_b) of p_b into a coinverter followed by an opfibration in groupoids.

If moreover p is an internal opfibration, then s is also an internal opfibration by Corollary 3.9 applied to $\text{Fib}(B)$, thanks to Proposition 3.16. \square

The result in Proposition 4.8 (i) (with g a split fibration) was obtained in Section 3.3 of [2], by providing an explicit construction of the discrete opfibration s together with an ad hoc definition of q , which was later on proved to be the coidentifier of the identee of p .

As a consequence of Proposition 4.8 (i), we get an internal comprehensive factorization system for $\text{Fib}(B)$. In the next results, by *initial* we mean a morphism which is left orthogonal

to internal discrete opfibrations. First we need the following lemma, whose dual version is in Proposition 3.5 in [14], for a particular case. For completeness, we formulate it here in a general finitely complete 2-category.

Lemma 4.9 *In a finitely complete 2-category, left adjoints are initial.*

Proof Consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ l \downarrow & & \downarrow m \\ C & \xrightarrow{g} & D \end{array}$$

where m is an internal discrete opfibration and l is left adjoint to some r , with unit η and counit ϵ , so that $\epsilon l \cdot l\eta = 1_l$ and $r\epsilon \cdot \eta r = 1_r$.

Let $\gamma: fr \rightarrow d$ denote the opcartesian m -lifting of $g\epsilon$ at fr , so that $md = g$. The 2-cell $d\epsilon$ being also an m -lifting of $g\epsilon$, it coincides with γ since m is a discrete opfibration. Then $m(d\epsilon l \cdot f\eta) = g\epsilon l \cdot m f\eta = g(\epsilon l \cdot l\eta) = 1_{gl}$.

$$\begin{array}{ccccc} A & \xrightarrow{l} & C & \xrightarrow{mfr} & D \\ & \searrow f\eta & \uparrow f\eta & \downarrow g\epsilon & \\ & & C & \xrightarrow{g} & D \\ & \searrow f & \xrightarrow{d\epsilon} d & \uparrow m & \\ & & B & & \\ & \swarrow \kappa & & & \\ K & & & & \end{array}$$

Hence $d\epsilon l \cdot f\eta$ factors through the identitee κ of m , which is discrete, so κ , and hence $d\epsilon l \cdot f\eta$, is an identity. Consequently, $dl = f$.

Suppose $d': C \rightarrow B$ is another morphism such that $md' = g$ and $d'l = f$. Then $d'\epsilon$ is another m -lifting of $g\epsilon$, so $d'\epsilon = d\epsilon$ and $d = d'$. \square

Theorem 4.10 *In $\text{Fib}(B)$ there exists a comprehensive factorization system given by (initial morphism, internal discrete opfibration).*

Proof Consider a morphism $p: (A, f) \rightarrow (C, g)$ in $\text{Fib}(B)$, and the monad $L: \text{Fib}(B)/(C, g) \rightarrow \text{Fib}(B)/(C, g)$. Lp is an internal opfibration, so we can factorize it in $\text{Fib}(B)$ as a coidentifier q followed by an internal discrete opfibration s , thanks to Proposition 4.8. Now consider the factorization

$$(A, f) \xrightarrow{u_p} p/(C, g) \xrightarrow{q} (Q, gs) \xrightarrow{s} (C, g)$$

p

of p , where the unit component u_p is initial by Lemma 4.9, since it is a left adjoint (see Corollary 6 in [10]). The result follows from Proposition 4.8 as it is easy to see that coidentifiers are initial. \square

Corollary 4.11 *For each fibrewise opfibration, the factorization of Theorem 4.10 coincides with the one given by (sequence of coidentifiers, discrete cartesian functor).*

As a consequence of Proposition 4.8 (ii), we get an extension of Theorem 4.3.

Theorem 4.12 *For every fibrewise opfibration $p: (A, f) \rightarrow (C, g)$ in $\mathbf{Fib}(B)$, the factorization of Proposition 4.8 (ii) coincides with the one given by (sequence of coinverters, conservative cartesian functor).*

Proof By Corollary 3.3 applied to p , which is an opfibration, and then an isofibration, in \mathbf{Cat}/B , p is conservative in \mathbf{Cat}/B , hence in $\mathbf{Fib}(B)$. Then the thesis follows from Theorem 4.5 (ii). \square

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