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STOCHASTIC EQUATIONS WITH FRACTIONAL NOISE: CONTINUITY IN LAW AND APPLICATIONS

Mat06 / Probabilità e Statistica Matematica

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Preface

In this work, we will study some problems emerging from the theory and applications of stochastic equations driven by a fractional Brownian motion (fBm). A fBm $\{B_t^H,\ t \in \mathbb{R}\}$, defined for any $H \in (0,1)$, is a stochastic process which has been introduced first by Mandelbrot and Van Ness in 1968 (see [MaVa68]), with the aim of generalizing the standard Brownian motion (sBm) to a family of processes depending on a parameter H, which still enjoy most of the properties of the sBm.

The need of introducing such a process came from the world of applications. The hydrologist Harold Hurst, while studying the distribution of the intensity of the floods of the Nile river, discovered a rather unexpected fact: the range of the distribution of the floods was not compatible with the assumption of independence of increments which is intrinsic in a sBm setting. It was the first time that such a phenomenon was observed, and for this reason the parameter $H \in (0,1)$ of a fBm B^H is named *Hurst parameter* (see [Hur51]).

A fBm B^H, for all $H \in (0,1)$, is a Gaussian process, meaning that its finite dimensional distributions are Gaussian random vectors. It is characterized by having zero mean, i.e. $\mathrm{E}[B_t^H] = 0$, for all $t \in \mathbb{R}$, and covariance structure given by

$$\mathrm{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \text{ for all } s,t \in \mathbb{R}.$$

From the expression of the covariance, it is immediate to notice that for $H = \frac{1}{2}$ the fBm $B^{\frac{1}{2}}$ reduces to a sBm: thus, it is truly a generalization.

Our main focus, and the most original part of this thesis, is the study of stochastic partial differential equations (SPDEs), driven by a noise W^H which is partly a fBm. The theory of SPDEs has had a very huge interest in the last decades, due to both independent mathematical interest and possible applications in modelling. There are two main research lines, that approached this kind of equations in two different ways: the setting of infinite dimensional Hilbert space-valued SDEs (see [DPZa]) and the random field setting (see [Wal86]). We will focus on the latter approach, following also more recent works like [Dal99, BJQ15, HHLNT17].

Our main result is the continuity in law of the solutions of such equations with respect to the Hurst parameter H. We recall that the continuity in law with respect to fractionality indices has been studied in other related contexts ([JoVi07, JoVi07, JoVi10, JoVi10, WuXi09, AiSg17]). We point out that showing the continuity property with respect to H of the solution, for both time (SDE) and time-space (SPDE) stochastic differential equations driven by fractional noises, is a very interesting problem not only from a theoretical point of view, but also in the modelling applications. Indeed, if one uses such a model in applications it is very important that the estimates of H are stable and the above described continuity property may help in this direction. We will see one of these applications in Chapter 4.

We give a bit more of context of our setting: let $W^H = \{W^H(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ be a Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with zero mean and with covariance structure given, for $t, s \in [0,T]$ and $x, y \in \mathbb{R}$, by:

$$\mathrm{E}\left[W^H(t,x)W^H(s,y)\right] = \frac{1}{2}(t \wedge s)\Big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\Big).$$

This process is thus a sBm in the *time variable* $t \in [0,T]$ and a fBm of parameter $H \in (0,1)$ in the *space variable* $x \in \mathbb{R}$. The SPDEs that we consider are

$$L_i u^H(t,x) = b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{1}$$

where L_i, for $i \in \{w, h\}$, is either the wave or the heat differential operator, i.e.

$$L_h u^H(t,x) = \frac{\partial u^H}{\partial t}(t,x) - \frac{\partial^2 u^H}{\partial x^2}(t,x); \qquad L_w u^H(t,x) = \frac{\partial^2 u^H}{\partial t^2}(t,x) - \frac{1}{2}\frac{\partial^2 u^H}{\partial x^2}(t,x).$$

The noise $\dot{W}^H$ is the formal derivative of the process W^H we defined above. Due to the lack of regularity of the noise, to give meaning to (1) one needs to introduce a weaker concept of solution, denoted in literature as *mild solution*. It is an integral formulation of (1), that we describe precisely in Chapter 2.

In this setting, our result consists in showing that $u^{H_n} \xrightarrow{d} u^{H_0}$, whenever $\{H_n,\, n \in \mathbb{N}\}$ is a sequence of Hurst indices such that $H_n \to H_0$ as $n \to \infty$. This convergence, that we denote with $\xrightarrow{d}$, is the convergence in law (also denoted *weak convergence*), in the path space $\mathcal{C}([0,T] \times \mathbb{R})$. This problem has been already studied in [Bez16], in the case of the heat equation and for $H_0 \in [\frac{1}{2}, 1)$, with a general σ and with the space variable $x \in \mathbb{R}^d$. We will extend this result to the case of the wave equation and to a general H_0 (which varies in dependence of the various cases we consider), restricting for the sake of simplicity to the one-dimensional case $d = 1$.

We investigate this problem in two different settings: in [GJQ20] we studied the semilinear additive case, which is the case when b is a Lipschitz function and $\sigma \equiv 1$, and in [GJQ19] we studied the linear multiplicative case, which is the case when $b \equiv 0$ and $\sigma(u) = u$. Depending on the setting we are considering, we will also need to impose different initial conditions for the problem. We also remark that in the semilinear additive case we are able to prove the result for any limiting value $H_0 \in (0,1)$, while in the linear multiplicative case we have to restrict to $H_0 \in (\frac{1}{4}, 1)$, due to well-posedness problems that arise in (1).

We give a brief outline of the work that we need to do in order to show the result, in both cases. For the semilinear additive case, we first focus on the linear version of the equations ($b = 0$), for which the existence and uniqueness of the solution, together with the existence of a continuous modification, for any $H \in (0,1)$, is well-known, for example, from [Bal12], for $H \leq \frac{1}{2}$ and [Dal99] for $H > \frac{1}{2}$. In this case, since the solutions u^H are still Gaussian processes, the convergence in law of u^{H_n} to u^{H_0} reduces to analyse the convergence of a family of centred Gaussian processes. In order to prove this, we first check the tightness of the corresponding family of probability laws induced by $\{u^{H_n},\, k \in \mathbb{N}\}$ on $\mathcal{C}([0,T] \times \mathbb{R})$. Tightness is a measure-theoretic property that implies relative compactness, and thus the existence of a limit measure Y for a subsequence $\{u^{H_{n_k}}, k \in \mathbb{N}\}$. We are only left to identify the limit Y as u^{H_0}. This is quite straightforward, thanks to the Gaussianity of the solutions. We point out that in this case the proof is the same for both the wave and the heat equation.

In the case when b is a general Lipschitz function, we first show that both equations in (1) admit a unique solution for any $H \in (0,1)$. As far as we know, it is a relatively novel result for the case $H < \frac{1}{2}$ (the case $H > \frac{1}{2}$ is in [DaQu11]).

The analysis of the weak convergence in the semilinear case does not admit a unified proof for wave and heat equations. More precisely, for the wave equation, the convergence in law of u^{H_n} to u^{H_0}, whenever $H_n \to H_0$, follows from a deterministic pathwise argument: we prove that, for almost all $\omega \in \Omega$, the solution can be seen as the image of the solution in the linear additive case through a certain continuous functional $F : \mathcal{C}([0,T] \times \mathbb{R}) \to \mathcal{C}([0,T] \times \mathbb{R})$. In the case of the heat equation, the previous argument can only be applied partially, since the associated deterministic equation which has to be solved in order to define the above-mentioned functional seems to be ill-posed for an unbounded general coefficient b. We overcome this difficulty by first assuming that b is a bounded function and then by using a truncation argument. It is also worthy to point out that, in the analysis of the wave equation and the heat equation with bounded b, we have established ad hoc versions of Grönwall lemma which have been crucial to complete the corresponding proofs.

We consider then the linear multiplicative case $\sigma(u) = u$, which is also named in the literature as Hyperbolic Anderson Model (HAM) and Parabolic Anderson Model (PAM), respectively (see [BJQ15, HHNT15, BJQ17] and references therein). There are several well-posedness results for these equations. For the case $H < \frac{1}{2}$ some results may be found in [BJQ15, HNZ17], while the case $H \geq \frac{1}{2}$ falls in the general framework of Walsh and Dalang [Wal86, Dal99, DaQu11].

In these cases, the fact that $H < \frac{1}{2}$ entails important technical difficulties in order to define stochastic integrals with respect to the noise W^H. Moreover, as proved in [BJQ15], the above equations admit a unique solution if and only if $H > \frac{1}{4}$.

In this setting, we show again that the weak convergence $u^{H_n} \xrightarrow{d} u^{H_0}$ holds. The proof, as in the semilinear additive case, consists in establishing that the family of measures induced by $\{u^{H_n}, n \in \mathbb{N}\}$ is tight on $\mathcal{C}([0,T] \times \mathbb{R})$, followed by the identification of the limit law as u^{H_0}. The proof of the tightness is quite involved from the computational point of view, since we need to extend several results appearing in [BJQ15, BJQ16, BJQ17].

Regarding the limit identification, one of the key elements of the proof is to show a representation formula for the integrals with respect to W^H of a deterministic function f, i.e. to show that for any $H \in (0,1)$ the following equality holds

$$\int_0^T \int_{\mathbb{R}} f(t,x) dW^H(t,x) = \int_0^T \int_{\mathbb{R}} (T^H f)(t,x) d\widetilde{W}(t,x),$$

where $\{T^H, H \in (0,1)\}$ is a family of transformations of f and $\widetilde{W}$ is a complex Gaussian measure which is independent from H. This result turns out to be crucial in the identification of the limit in the linear multiplicative case (see Section 2.4.5). For this purpose, we also use some results of Malliavin Calculus, a stochastic calculus of variation theory which is useful to study many results in the theory of SPDEs.

A possible future extension of the work described above is to study these results within the framework of regularity structures. A first attempt towards this direction is done here in Chapter 3 by considering the continuity problem for a stochastic differential equation (SDE) driven by a fBm B^H, in the rough paths theory setting ([Gub04, FrHa, FrVi, FrVi11]). We consider the SDE

$$dY_t^H = \mu(Y_t^H)dt + \sigma(Y_t^H)dB_t^H. \tag{2}$$

We clarify and slightly extend a continuity result appearing in Chapter 15 of [FrVi], giving also a brief introduction to rough paths theory.

The result is the following: under the assumption $H \in (\frac{1}{3}, \frac{1}{2})$, consider the solution Y^H of (2), which defines a probability distribution on the space $\mathcal{C}^{1/3}([0,T])$ of $\frac{1}{3}$-Hölder continuous functions. It is possible to show that, whenever $H \to H_0 \in (\frac{1}{3}, \frac{1}{2})$, it holds $Y^H \xrightarrow{d} Y^{H_0}$, where $\xrightarrow{d}$ denotes the convergence in distribution on $\mathcal{C}^{1/3}([0,T])$.

The proof of this fact relies on a fundamental tool in rough paths theory: given an equation like (2), it is possible to define a solution operator which maps the noise B^H into the solution Y^H. In classical stochastic settings, like for example in Itô's theory, this map is discontinuous, and it can be shown that it is impossible to make it continuous (see [FrHa]). The main idea of rough paths theory is to enrich the noise B^H, by postulating a further component (the integral $\mathbb{B}^H$ of B^H with respect to itself) and by considering the noise as the couple $(B^H, \mathbb{B}^H)$. In this enriched setting, the solution map $(B^H, \mathbb{B}^H) \to Y^H$ can be made continuous (see [FrVi, FrHa]).

This crucial observation has an immediate effect on our continuity problem: it is sufficient to show that $(B^H, \mathbb{B}^H) \xrightarrow{d} (B^{H_0}, \mathbb{B}^{H_0})$ and exploit the continuity of the solution map to notice that $Y^H \xrightarrow{d} Y^{H_0}$. In the present work, we prove the continuity result $(B^H, \mathbb{B}^H) \xrightarrow{d} (B^{H_0}, \mathbb{B}^{H_0})$ by exploiting the usual scheme, that is to establish the tightness and then to identify the limit. Our contribution consists in the fact that we proved the tightness in the specific case of the fBm considering a slightly weaker assumption than the one in [FrVi]

The last part of the work is devoted to the development of a model driven, among other factors, by a fBm B^H. This model is then used to forecast the future prices in the Italian wholesale electricity market (available at [Prices]). We propose a stochastic differential equation of the type (2), coupled with a deterministic seasonal term and a jump component which will be modelled through a Hawkes process.

Hawkes processes are a generalization, studied first by Hawkes in [Haw71(1), Haw71(2)], of the classical Poisson point processes. In a Hawkes process, the intensity function, which is the function that models the frequency of the random jump times, is assumed to be self-exciting, instead of being constant. This means that every time a jump occurs, the instantaneous probability that a subsequent jump occurs is higher than in "normal periods". This is an interesting effect for our study case, since the italian electricity market shows the presence of several jumps, some of which appear to be clustered over short time periods.

Regarding the practical implementation of the model, we first study, following [JTWW13, NTW13, Wer14], the problems of parameter estimation and dataset filtering. These are crucial steps in the pre-processing phase of the model. After that, we finally evaluate the performance of the model. To do this, we use the model to produce forecasts of future electricity prices, at different forecasting horizons (from 1 to 30 days in the future). These forecasts are given in the form of *interval forecasts* (studied in [Wer14] and [NoWe18]) instead of the more classical point forecasts. This choice aims at evaluating more in detail the quality of the forecasts in the distributional sense, instead of giving a single prediction value. These kind of forecasts are then evaluated by using adequate metrics, like the Winkler score and the Pinball loss function.

Contents

1 | An introduction to the fractional Brownian motion

In this Chapter we introduce the central object of this thesis: the fractional Brownian motion (fBm). We first discuss its definition and basic properties. Then, we introduce a pair of integral representations for the fBm (Proposition 1.10 and Proposition 1.14), which are known from the existing literature. Finally, we present the theory of Wiener integration with respect to the fBm, that is, the theory of integration of deterministic functions with respect to the fBm. In this framework, we prove an almost sure representation formula (Proposition 1.18) for the Wiener integral with respect to a fBm, based on one of the integral representations presented before. This result slightly extends the results in Section 3 of [PiTa00].

1.1 Definition and basic properties

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, i.e. a probability space for which, whenever $B \subset \mathcal{F}$ is such that $\mathbb{P}(B) = 0$, then any $A \subset B$ satisfies $A \in \mathcal{F}$.

Definition 1.1. Let $H \in (0,1)$. We define the *fractional Brownian motion* (fBm) of Hurst parameter H as the Gaussian process $\{B_t^H, t \in \mathbb{R}\}$ characterized by

i) $B_0^H = 0$

ii) $\mathrm{E}[B_t^H] = 0$ for any $t \in \mathbb{R}$.

iii) $\mathrm{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) =: K^H(s,t)$, for all $s, t \in \mathbb{R}$.

The fact that the fBm exists is a consequence of the fact that the covariance function K^H is non-negative definite, for all $H \in (0,1)$, meaning that for every $(s_1, \dots, s_n), (x_1, \dots, x_n) \in \mathbb{R}^n$ it holds

$$\sum_{j=1}^{n} \sum_{k=1}^{n} K^H(s_j, s_k) x_j x_k \geq 0.$$

This has been proven in [SaTa], Lemma 2.10.8.

Remark 1.2. It is immediate to notice that $H = \frac{1}{2}$ yields $\mathrm{E}[B_t^H B_s^H] = \min(t,s)$, which is the covariance of a standard Brownian motion (sBm). This shows that the fBm is a generalization of the sBm.

Remark 1.3. One can define the fBm also for $H = 1$ as $B_t^1 = t\,B$, for all $t \in \mathbb{R}$, where B is a centred Gaussian distributed random variable with variance equal to 1.

Remark 1.4. We clarify why the fBm is characterized by its mean and covariance function. It is well-known that any $\mathbb{R}^n$-valued Gaussian random vector $(X_1, \dots, X_n)$ is characterized by its mean $(\mu_1, \dots, \mu_n)$ and covariance matrix $(\sigma_{i,j})_{i,j=1,\dots,n}$. This can be shown easily through characteristic functions.

We report now some interesting properties of the fBm. Not all of these facts will be important for our purposes, but they allow us to get a clearer insight on the nature of such a process.

Definition 1.5. Let $X = \{X_t,\, t \in \mathbb{R}\}$ and $Y = \{Y_t,\, t \in \mathbb{R}\}$ be two stochastic processes. We say that X is equal in distribution to Y, and we denote it with $X \overset{d}{=} Y$, if the two processes have the same finite dimensional distributions. This means that, for every $n \in \mathbb{N}$ and for every $(t_1, \ldots, t_n) \in \mathbb{R}^n$, the random vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ induce the same probability distribution on $\mathbb{R}^n$.

Definition 1.6. Let $\alpha > 0$. Let $X = \{X_t, t \in \mathbb{R}\}$ be a stochastic process. Define for any $b \in \mathbb{R}$ the process $\tilde{X}_b = \{\tilde{X}_{b,t} := X_{bt},\, t \in \mathbb{R}\}$. We say that X is α-self similar if, for every $b \in \mathbb{R}$, it holds $\tilde{X}_b \overset{d}{=} |b|^\alpha X$.

Proposition 1.7 (Basic properties of the fBm)**.** *Let $H \in (0,1)$. Then, the following hold:*

1) Let $\alpha < H$. The fBm B^H has a continuous modification, which can be chosen such that the trajectories are α-Hölder continuous.

2) The fBm B^H is H-self similar.

3) The fBm has stationary increments, i.e. for any $h > 0$ it holds $B^H_{t+h} - B^H_h \overset{d}{=} B^H_t$.

4) Let $t_1 < t_2 < t_3 < t_4$. One has that

$$\mathrm{E}\Big[(B^H_{t_4} - B^H_{t_3})(B^H_{t_2} - B^H_{t_1})\Big] \begin{cases} = 0 & \text{if } H = \frac{1}{2}, \\ > 0 & \text{if } H > \frac{1}{2}, \\ < 0 & \text{if } H < \frac{1}{2}. \end{cases}$$

This means that the fBm has negatively correlated increments when $H < \frac{1}{2}$ and positively correlated increments when $H > \frac{1}{2}$.

Proof. To prove *1)*, we notice that condition *iii)* in Definition 1.1 implies that $\mathrm{E}[(B^H_t - B^H_s)^2] = |t-s|^{2H}$. This, by Kolmogorov continuity theorem (Theorem A.2), implies that the fBm of parameter $H \in (0,1)$ has a continuous modification. Still by Kolmogorov continuity theorem, this modification can be chosen such that the trajectories are Hölder continuous for any $\alpha < H$.

For *2)*, we refer to page 7 of [Mis]. To prove *3)*, it is sufficient to observe that $\mathrm{E}[B_{t+h} - B^H_h] = \mathrm{E}[B^H_t] = 0$ and that, for every $t, s \in \mathbb{R}$, it holds

$$\mathrm{E}\Big[(B^H_{t+h} - B^H_h)(B^H_{s+h} - B^H_h)\Big] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big) = \mathrm{E}[B^H_t B^H_s],$$

which concludes the proof thanks to Gaussianity.

To prove *4)*, it suffices to notice that for $t_1 < t_2 < t_3 < t_4$, and for $H \in (0,1) \setminus \{\frac{1}{2}\}$ it holds

$$\begin{aligned}\mathrm{E}\Big[(B^H_{t_4} - B^H_{t_3})(B^H_{t_2} - B^H_{t_1})\Big] &= \frac{1}{2}\Big(|t_4 - t_1|^{2H} + |t_3 - t_2|^{2H} - |t_4 - t_2|^{2H} - |t_3 - t_1|^{2H}\Big) \\ &= (2H-1)H \int_{t_1}^{t_2} \int_{t_3}^{t_4} (s-r)^{2H-2}\, ds\, dr.\end{aligned}$$

The integral appearing is always positive (the integrand is positive), so the sign only depends on $2H - 1$, being positive for $\frac{1}{2} < H < 1$ and negative for $0 < H < \frac{1}{2}$.

□

Remark 1.8. Property *4)* in Propostion 1.7 is the key reason for the use of fBm in stochastic modelling. Indeed, the usual modelling with a sBm assumes that the noise of the input data is uncorrelated, whenever it is considered on disjoint time intervals. This hypothesis is often proved to be too restrictive, and sometimes even false. A possible solution to this is to consider fBm as the driving noise in modelling. We will analyse in Chapter 4 the advantages and disadvantages of this approach in detail.

There is another nice characterization of a fBm which involves the self-similarity and the stationarity of the increments:

Proposition 1.9. *Let $H \in (0,1]$ and let $\{X_t, t \in \mathbb{R}\}$ be a stochastic process with $\mathrm{E}[X_1^2] = 1$ and $X_0 = 0$. The following two statements are equivalent:*

1) X is a Gaussian process which is H-self similar and with stationary increments

2) X is a fBm with Hurst parameter H.

Proof. The case $H = 1$ is immediate, via the definition given in Remark 1.3. Let $H \in (0,1)$: we only have to prove that *1) ⇒ 2)*, since the converse has been proven in points *2)* and *3)* of Proposition 1.7.

We have to prove that X has mean 0 and covariance given by

$$\mathrm{E}[X_s X_t] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}).$$

We compute, using stationarity of the increments and self-similarity:

$$\begin{aligned}
\mathrm{E}[X_s X_t] =& \frac{1}{2}\mathrm{E}\Big[(X_s)^2 + (X_t)^2 - (X_t - X_s)^2\Big] \\
=& \frac{1}{2}\mathrm{E}\Big[(X_s)^2 + (X_t)^2 - (X_{t-s} - X_0)^2\Big] \\
=& \frac{1}{2}\Big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\Big)\mathrm{E}[X_1^2] \\
=& \frac{1}{2}\Big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\Big),
\end{aligned} \tag{1.1}$$

which concludes the proof. □

It is also worth pointing out that fBm is not a semimartingale, when $H \neq \frac{1}{2}$. Semimartigales are a large class of processes that are relevant in the theory of stochastic integration. Indeed, they are the largest class of processes for which it is possible to define Itô integral and Stratonovich integral (see e.g. [ReYo]) so that the integral satisfies a minimal continuity property.

1.2 Representations in law of the fBm

A useful property of the fBm is the fact that it has several integral representations, some of which are relatively simple and quite useful for applications, as we will see in Chapter 2. We refer to [SaTa], Section 7.2, for a more complete outlook on the topic.

First of all, we consider the so-called *moving average representation.*:

Proposition 1.10 ([SaTa], page 320). *Let $H \in (0,1)$. Let $\{W_t, t \in \mathbb{R}\}$ be a sBm. Consider, for all $t \in \mathbb{R}$,*

$$\tilde{B}_t^H := \frac{1}{C_1(H)} \int_{\mathbb{R}} \Big(((t-x)_+)^{H-\frac{1}{2}} - ((-x)_+)^{H-\frac{1}{2}} \Big) dW_x, \tag{1.2}$$

where the constant $C_1(H)$ is given by

$$C_1(H) := \left(\int_0^\infty \left((1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}} \right)^2 dx + \frac{1}{2H} \right)^{\frac{1}{2}}.$$

Then we have that $\tilde{B}^H \stackrel{d}{=} B^H$.

Remark 1.11. For a $\mathbb{R}$-valued function f, the notation f_+ denotes the function

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0. \end{cases}$$

Remark 1.12. When $H = \frac{1}{2}$, one has $C(\frac{1}{2}) = 1$ and we could read (1.2) as $\int_0^t W_x$ for $t > 0$ and $-\int_t^0 W_x$ for $t < 0$.

Proof (Proposition 1.10). We denote with $f_{H,t}$ the integrand of (1.2). First, we have to check that $f_{H,t} \in L^2(\mathbb{R})$ for every $t \in \mathbb{R}$, $H \in (0,1)$. When $t = 0$, $f_{H,0} \equiv 0$ trivially, and there is nothing to check. Suppose that $t \neq 0$: it is immediate to notice that for $x \to 0$ we have $f_{H,t}(x) \sim x^{H-\frac{1}{2}}$, which is square-integrable, and for $x \to t$ we have that $f_{H,t}(x) \sim (t-x)^{H-\frac{1}{2}}$, which is square-integrable too. Moreover, we have that whenever $x > t$ then $f_{H,t}(x) = 0$, so we are only left to show that $f_{H,t}$ is integrable for $x \to -\infty$. But we have that $(t-x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}} \sim C(-x)^{H-\frac{3}{2}}$ (one can easily check it with Taylor expansions). Thus $f_{H,t} \in L^2(\mathbb{R})$.

Now we denote with X_t the right-hand side in (1.2), and we show that X_t has the covariance structure of Definition 1.1. We compute the covariance $\mathrm{E}[X_s X_t]$ in the same way as (1.1). We need then to compute only $\mathrm{E}[X_t^2], \mathrm{E}[X_s^2]$ and $\mathrm{E}[(X_t - X_s)^2]$. Thanks to classical Itô isometry we have, supposing, without loss of generality, that $t > 0$:

$$\begin{aligned}
\mathrm{E}[X_t^2] =& \frac{1}{C_1(H)^2} \int_{\mathbb{R}} \left((t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right)^2 dx \\
=& \frac{t^{2H}}{C_1(H)^2} \int_{\mathbb{R}} \left((1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right)^2 dx \\
=& \frac{t^{2H}}{C_1(H)^2} \Big[\int_{-\infty}^0 \left((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right)^2 dx + \int_0^1 (1-u)^{2H-1} du \Big] \\
=& \frac{t^{2H}}{C_1(H)^2} \Big[\int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 dx + \frac{1}{2H} \Big] = t^{2H},
\end{aligned}$$

where we used the change of variables $u = \frac{x}{t}$ and the definition of $C_1(H)$. This can be done similarly for $t < 0$. For the term $\mathrm{E}[(X_t - X_s)^2]$, it suffices to notice that thanks to Itô isometry one has

$$\begin{aligned}
\mathrm{E}[(X_s - X_t)^2] =& \frac{1}{C_1(H)^2} \int_{\mathbb{R}} \left(((s-x)_+)^{H-\frac{1}{2}} - ((t-x)_+)^{H-\frac{1}{2}} \right)^2 dx \\
=& \frac{1}{C_1(H)^2} \int_{\mathbb{R}} \left(((s-t-y)_+)^{H-\frac{1}{2}} - ((-y)_+)^{H-\frac{1}{2}} \right)^2 dy = |t-s|^{2H},
\end{aligned}$$

where we used the change of variables $y = x - t$. This concludes the proof, since X is Gaussian, being the integral of a deterministic function with respect to a sBm. □

The next integral representation that we introduce is the so-called *spectral representation* [PiTa00], or *harmonizable representation* [SaTa]. This representation will be further generalized in Chapter 2 and it will be a key tool for our results on stochastic partial differential equations with multiplicative noise.

In order to state and prove it, first we briefly introduce the notion of complex random measure. Our aim is to be able to define a complex Gaussian measure $\tilde{M}$ which will play the role of the integrator in (1.2).

Definition 1.13. We define the complex Gaussian measure $\tilde{M}$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$, as a measure given by $\tilde{M} = \tilde{M}^1 + i\tilde{M}^2$, where $\tilde{M}^1, \tilde{M}^2$ are independent (real) centred Gaussian measures such that for every Borel set $A \subset \mathbb{R}$ it holds

i) $\tilde{M}^1(A) = \tilde{M}^1(-A)$ and $\tilde{M}^2(A) = -\tilde{M}^2(-A)$.

ii) $\mathrm{E}\Big[|\tilde{M}^1(A)|^2\Big] = \mathrm{E}\Big[|\tilde{M}^2(A)|^2\Big] = |A|/2$, where we denote as $|A|$ the Lebesgue measure of A.

Given Definition 1.13, it is immediate to notice that it holds $\mathrm{E}\Big[|\tilde{M}(A)|^2\Big] = |A|$, for every Borel set $A \subset \mathbb{R}$. It is possible to define an integral with respect to this measure, similarly to Itô integral. We refer to [SaTa], Section 7.2.2 for more details. For our purposes, it is sufficient to say that, given a complex-valued (deterministic) function $f \in L^2(\mathbb{R})$, the integral

$$\int_{\mathbb{R}} f(x) d\tilde{M}_x,$$

where we denoted $d\tilde{M}_x := \tilde{M}(dx)$, satisfies the Itô isometry in the sense that for $f, g \in L^2(\mathbb{R})$ (possibly complex-valued)

$$\begin{aligned} &1) \quad \mathrm{E}\Big[\Big(\int_{\mathbb{R}} f(x) d\tilde{M}_x\Big)\overline{\Big(\int_{\mathbb{R}} g(x) d\tilde{M}_x\Big)}\Big] = \int_{\mathbb{R}} f(x)\overline{g(x)} dx, \\ &2) \quad \mathrm{E}\Big[\Big|\int_{\mathbb{R}} f(x) d\tilde{M}_x\Big|^2\Big] = \int_{\mathbb{R}} |f(x)|^2 dx. \end{aligned} \tag{1.3}$$

We are now ready to state the spectral representation result.

Proposition 1.14 ([SaTa], Proposition 7.2.8)**.** *Let $\tilde{M}$ be a complex Gaussian measure as defined in Definition 1.13. Let $H \in (0,1)$. Then, the fBm B^H of parameter H has the following integral representation, for all $t \in \mathbb{R}$. Let $\tilde{B}^H$ be the process defined by*

$$\tilde{B}^H_t = C_2(H) \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} |x|^{\frac{1}{2} - H} d\tilde{M}_x, \tag{1.4}$$

where

$$C_2(H) = \Big(\frac{H\Gamma(2H)\sin(\pi H)}{\pi}\Big)^{\frac{1}{2}}.$$

Then, we have that $\tilde{B}^H \overset{d}{=} B^H$.

Proof. We report the proof for the sake of completeness: again, we denote in (1.4) the integrand as $f_{H,t}$ and the whole integral as X_t.

It is easy to check that for every $t \in \mathbb{R}$ and $H \in (0,1)$ it holds $f_{H,t} \in L^2(\mathbb{R})$. Indeed, $|f_{H,t}(x)| \sim |x|^{-H-\frac{1}{2}}$ as $|x| \to \infty$. Moreover, the function $\Big|\frac{e^{itx}-1}{ix}\Big|$ is bounded for $|x| \to 0$, and thus the only possible singularity comes from $|x|^{\frac{1}{2}-H}$, which is square integrable anyway. This means that $f_{H,t} \in L^2(\mathbb{R})$.

We show now that X is a Gaussian, H-self similar process with stationary increments: thanks to the characterization property Proposition 1.9 this is sufficient to show that X is a fBm of parameter H. First of all, the gaussianity of X comes from the fact that it is defined as the integral of a deterministic function (also called *Wiener integral*) with respect to a Gaussian process.

We are left to check that X is H-self similar and its increments are stationary. We first check that $X_{at} \overset{d}{=} a^H X_t$. To do this, it is sufficient to check that they have the same mean and covariance structure. Being a Wiener integral with respect to a Gaussian process, X_{at} has zero mean for all $t \in \mathbb{R}$, and we are left to show that

$$\mathrm{E}[X_{at}X_{as}] = a^{2H}\mathrm{E}[X_tX_s].$$

We compute explicitly, thanks to Itô isometry,

$$\begin{aligned}
\mathrm{E}[X_{at}X_{as}] =& \int_{\mathbb{R}} \Big(\frac{e^{iatx}-1}{ix}\Big)\overline{\Big(\frac{e^{iasx}-1}{ix}\Big)}|x|^{1-2H}dx \\
=& \int_{\mathbb{R}} (e^{iatx}-1)(e^{-iasx}-1)|x|^{-1-2H}dx \\
=& \int_{\mathbb{R}} (e^{ity}-1)(e^{-isy}-1)|y|^{-1-2H}a^{1+2H}\frac{dy}{a} \\
=& a^{2H}\int_{\mathbb{R}} (e^{ity}-1)(e^{-isy}-1)|y|^{-1-2H}dy \\
=& a^{2H}\mathrm{E}[X_tX_s].
\end{aligned}$$

We check that the increments are stationary. Fix $h > 0$. We exploit gaussianity again, plus the fact that for every t we have $\mathrm{E}[X_{t+h} - X_h] = 0 = \mathrm{E}[X_t]$. We are left again to check that the covariances are the same: notice first that

$$X_{t+h} - X_h = \int_{\mathbb{R}} \frac{e^{ihx}(e^{itx}-1)}{ix}|x|^{\frac{1}{2}-H}d\tilde{M}_x.$$

We have then

$$\begin{aligned}
\mathrm{E}\Big[(X_{t+h}-X_h)(X_{s+h}-X_h)\Big] =& \int_{\mathbb{R}} \Big(\frac{e^{ihx}(e^{itx}-1)}{ix}\Big)\Big(\overline{\frac{e^{ihx}(e^{isx}-1)}{ix}}\Big)|x|^{1-2H}dx \\
=& \int_{\mathbb{R}} \Big(\frac{e^{itx}-1}{ix}\Big)\Big(\overline{\frac{e^{itx}-1}{ix}}\Big)|x|^{1-2H}dx \\
=& \mathrm{E}[X_tX_s].
\end{aligned}$$

We are only left to prove now that $\mathrm{E}\Big[|X_1|^2\Big] = 1$. This is only a matter of computing the integral

$$\int_{\mathbb{R}} |f_{H,1}(x)|^2dx.$$

We refer to the proof of Proposition 7.2.8 in [SaTa] for the computations.

□

Remark 1.15. The integrand $f_{H,t}$ in Proposition 1.7 can be seen as $f_{H,t}(x) = \mathcal{F}1_{[0,t]}(x)|x|^{\frac{1}{2}-H}$, where we denote with $\mathcal{F}g$ the Fourier transform of a function $g \in L^1(\mathbb{R})$, defined as

$$\mathcal{F}g(\xi) := \int_{\mathbb{R}} e^{-i\xi x}g(x)dx. \tag{1.5}$$

This fact will be crucial when using the representation (1.4) to obtain a spectral representation also for Wiener integrals, as we will see in Proposition 1.18 and, in the multidimensional case, in Chapter 2.

We observe that the representation result Proposition 1.14 can be used as an alternative definition of the fBm. Let $\tilde{B}^H$ be the process defined, for every $t \in \mathbb{R}$, by

$$\tilde{B}^H_t := C_2(H)\int_{\mathbb{R}} \mathcal{F}1_{[0,t]}(x)|x|^{\frac{1}{2}-H}d\tilde{M}_x,$$

where $\tilde{M}$ is the complex Gaussian measure defined in Definition 1.13. Then, we showed in Proposition 1.14 that $\tilde{B}^H$ has the same distribution as B^H.

This is useful because we can define the fBm of Hurst parameter $H \in (0,1)$ via this relation; hence, we can define the whole family $\{B^H, H \in (0,1)\}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.16 (fBm, alternative definition)**.** Let $\tilde{M}$ be a complex Gaussian measure as defined in Definition 1.13. For every $H \in (0,1)$, we define the fBm process B^H on the space $(\Omega, \mathcal{F}, \mathbb{P})$, defined for every $t \in \mathbb{R}$ as

$$B_t^H = C_2(H) \int_{\mathbb{R}} \mathcal{F}1_{[0,t]}(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x.$$

From now on, we consider this last one as the standing definition of fBm. Thus, there exists a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all of the fBm's B^H are defined, for every $H \in (0,1)$. This fact will have further consequences, as we will see in Proposition 1.18.

1.3 Wiener integral with respect to the fBm

When developing a theory of stochastic calculus with respect to a process X, the first step we take is to define the integral of a (possibly large) class of deterministic functions with respect to X. This kind of integrals are also called in the literature *Wiener integrals*. When $X = B^{1/2}$, i.e. in the sBm case, we have that the natural class of such integrands is $L^2(\mathbb{R})$. In the case of the fBm with $H \neq \frac{1}{2}$ it is more difficult to identify the space of integrands. Indeed, when $H > \frac{1}{2}$, the "natural" space contains not only functions, but also pure distributions. For a detailed overview of the topic, we refer to [PiTa00] and [Jol06].

Let $H \in (0,1)$, and let f be a deterministic elementary function, i.e. a function given, for some $c \in \mathbb{R}$ and for some $t_1, t_2 \in \mathbb{R}$, with $t_1 \leq t_2$ by $f(x) = c\,1_{[t_1,t_2]}(t)$. Imagine that we wish to define the integral

$$I_H(f) := \int_{\mathbb{R}} f(t) dB_t^H = \int_{\mathbb{R}} c 1_{[t_1,t_2]}(t) dB_t^H.$$

A natural choice would be to define, similarly to the construction of the Riemann-Stieltjes integral, $I_H(f) := c(B_{t_2}^H - B_{t_1}^H)$. Let now $\mathcal{E}$ be the vector space of simple functions, i.e. finite linear combinations of elementary functions. The map I_H is clearly linear from $\mathcal{E}$ to $L^0(\Omega)$, which is defined as the space of measurable random variables.

Let us suppose for a moment that $H = \frac{1}{2}$; the linear map $f \mapsto I_{1/2}(f)$ defined on $\mathcal{E}$ takes values in $L^2(\Omega)$ and defines an isometry between $\mathcal{E}$ and $L^2(\Omega)$, if we endow $\mathcal{E}$ with the $L^2(\mathbb{R})$ norm, which we denote by $||\cdot||_{L^2(\mathbb{R})}$. We check it briefly: suppose without loss of generality that $f(t) = \sum_{j \leq N} c_j 1_{[t_{j-1},t_j]}(t)$, where $-\infty < t_0 < t_1 < \cdots < t_N < \infty$. We can do this for two reasons: we can always split a finite number of superimposed intervals in a (larger) finite number of disjoint intervals, and we can add intervals with $c_j = 0$, if needed. We have that

$$I_{1/2}(f) = \sum_{j=1}^{N} c_j (B_{t_j}^{1/2} - B_{t_{j-1}}^{1/2}),$$

which implies

$$\mathrm{E}\Big[|I_{1/2}(f)|^2\Big] = \sum_{j=1}^{N} c_j^2\, \mathrm{E}\Big[(B_{t_j}^{1/2} - B_{t_{j-1}}^{1/2})^2\Big] = \sum_{j=1}^{N} c_j^2 |t_j - t_{j-1}| = ||f||_{L^2(\mathbb{R})}^2,$$

which is the desired isometry. By linearity and isometric extension, we can thus define $I_{1/2}$ for every function in $L^2(\mathbb{R})$. This is true because the set $\mathcal{E}$ of simple functions is dense in $L^2(\mathbb{R})$.

Actually, one can prove more: the isometry between $L^2(\mathbb{R})$ and $L^2(\Omega)$ is an isometry between Hilbert spaces, since it preserves also the inner products:

$$\mathrm{E}\Big[I_{1/2}(f)I_{1/2}(g)\Big] = \int_{\mathbb{R}} f(x)g(x)\,dx.$$

In the pure fBm case $H \neq \frac{1}{2}$, we wish to find a suitable space of integrands that replaces $L^2(\mathbb{R})$. In [PiTa00], various choices of space have been proposed. Here, we will only define and use a particular choice, called in [PiTa00] the *spectral domain* of B^H, whose construction is inspired by the spectral representation (1.4).

First, we notice the following fact: let $f \in \mathcal{E}$ be a simple function of the form $f = \sum_{j\leq N} f_j = \sum_{j\leq N} c_j 1_{[s_j,t_j]}$, with $s_j \leq t_j$ for every $j = 1, \ldots, n$. By definition of the integral of elementary functions, we have that

$$\int_{\mathbb{R}} f_j(t)dB_t^H = c_j(B_{t_j}^H - B_{s_j}^H).$$

Thanks to Definition 1.16, we have that

$$\begin{aligned}
c_j(B_{t_j}^H - B_{s_j}^H) =& c_j C_2(H) \int_{\mathbb{R}} (\mathcal{F}1_{[0,t_j]}(x) - \mathcal{F}1_{[0,s_j]}(x))|x|^{\frac{1}{2}-H} d\tilde{M}_x \\
=& c_j C_2(H) \int_{\mathbb{R}} \mathcal{F}1_{[s_j,t_j]}(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x \\
=& C_2(H) \int_{\mathbb{R}} \mathcal{F}f_j(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x.
\end{aligned}$$

Thanks to the linearity of the integral and of the Fourier transform, we can write that

$$\begin{aligned}
I_H(f) =& \int_{\mathbb{R}} f(t)dB_t^H \\
=& \sum_{j=1}^N \int_{\mathbb{R}} f_j(t)dB_t^H \\
=& \sum_{j=1}^N C_2(H) \int_{\mathbb{R}} \mathcal{F}f_j(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x \\
=& C_2(H) \int_{\mathbb{R}} \mathcal{F}\Big(\sum_{j=1}^N f_j\Big)(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x \\
=& C_2(H) \int_{\mathbb{R}} \mathcal{F}f(x)|x|^{\frac{1}{2}-H} d\tilde{M}_x.
\end{aligned}$$

These considerations lead us to, thanks to classical Itô isometry,

$$\mathrm{E}\Big[|I_H(f)|^2\Big] = C_2(H)^2 \int_{\mathbb{R}} |\mathcal{F}f(x)|^2|x|^{1-2H}dx, \tag{1.6}$$

for any $f \in \mathcal{E}$. More generally, for $f, g \in \mathcal{E}$, we have

$$\mathrm{E}\Big[I_H(f)I_H(g)\Big] = C_2(H)^2 \int_{\mathbb{R}} \mathcal{F}f(x)\overline{\mathcal{F}g(x)}|x|^{1-2H}dx.$$

This property leads us to the definition of a natural space of integrands:

$$\Lambda_H := \Big\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\mathcal{F}f(x)|^2|x|^{1-2H}dx < \infty \Big\}.$$

We endow the vector space Λ_H with an inner product defined in the natural way as

$$\langle f, g\rangle_H := C_2(H)^2 \int_{\mathbb{R}} \mathcal{F}f(x)\overline{\mathcal{F}g(x)}|x|^{1-2H}dx,$$

and with the induced norm $||f||_H := \langle f, f\rangle_H^{1/2}$. We already showed (when we proved that $f_{H,t} \in L^2(\mathbb{R})$) that $\mathcal{E} \subset \Lambda_H$, but we are interested in knowing whether $\mathcal{E}$ is dense in Λ_H or not.

Proposition 1.17. *Let $H \in (0,1)$. The space Λ_H is a inner product space endowed with the inner product $\langle \cdot, \cdot\rangle_H$, but it is not complete. Moreover, the space $\mathcal{E}$ of simple functions is dense in Λ_H with the norm $|| \cdot ||_H$. As a consequence, we can define, for every $f \in \Lambda_H$, the integral*

$$I_H(f) = \int_{\mathbb{R}} f(t)dB_t^H,$$

via a standard isometric extension. This integral satisfies the Itô isometry in the sense that for every $f, g \in \Lambda_H$

$$\mathrm{E}\Big[I_H(f)I_H(g)\Big] = \langle f, g\rangle_H = \int_{\mathbb{R}} \mathcal{F}f(x)\overline{\mathcal{F}g(x)}|x|^{\frac{1}{2}-H}dx. \tag{1.7}$$

Proof. See Section 5.1 of [PiTa00]. □

With this definition in mind, and since the family $\{B^H, H \in (0,1)\}$ is defined on a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we are able to prove the following result, which will be further generalized in Chapter 2 in a 2-dimensional case, and for the n-th order iterated integral.

Proposition 1.18. *Let $\tilde{M}$ be the complex Gaussian measure defined in Definition 1.13. Then, for every $f \in \Lambda_H$, it holds that, $\mathbb{P}$-almost surely:*

$$\int_{\mathbb{R}} f(t)dB_t^H = C_2(H)\int_{\mathbb{R}} \mathcal{F}f(x)|x|^{\frac{1}{2}-H}d\tilde{M}_x. \tag{1.8}$$

Proof. Recall that the noises $\{B^H,\ H \in (0,1)\}$ are now defined on the same probability space (Definition 1.16). We already proved the relation (1.8) for $f \in \mathcal{E}$, and it holds $\mathbb{P}$-almost surely. Let now $f \in \Lambda_H$. By Proposition 1.17, there exists a sequence of simple functions $\{f_n, n \in \mathbb{N}\} \subset \mathcal{E}$ which converges to f in the $|| \cdot ||_H$ norm. We have that (1.8) holds for $f = f_n$, for every $n \in \mathbb{N}$, and in addition we have that, by the Itô isometry

$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}\Big[\Big(\int_{\mathbb{R}}(f(t) - f_n(t))dB_t^H\Big)^2\Big] &= \lim_{n\to\infty} \mathrm{E}\Big[\Big(C_2(H)\int_{\mathbb{R}}(\mathcal{F}f(x) - \mathcal{F}f_n(x))|x|^{\frac{1}{2}-H}d\tilde{M}_x\Big)^2\Big] \\ &= \lim_{n\to\infty} C_2(H)^2 \int_{\mathbb{R}} |\mathcal{F}(f - f_n)(x)|^2|x|^{\frac{1}{2}-H}dx = 0,\end{aligned}$$

since $||f - f_n||_H \to 0$ by hypothesis, as $n \to \infty$.

□

In the proof of Proposition 1.18, we could integrate $f(t) - f_n(t)$ with respect to B^H and $C_2(H)(\mathcal{F}(f - f_n)(x))|x|^{\frac{1}{2}-H}$ with respect to $\tilde{M}$ essentially for the same reason: the fact that $f - f_n \in \Lambda_H$ implies that $C_2(H)(\mathcal{F}(f - f_n)(x))|x|^{\frac{1}{2}-H} \in L^2(\mathbb{R})$ by the very definition of Λ_H.

Remark 1.19. Proposition 1.18 may seem redundant, but anyway it turns out to be an important tool in estimates for the following reason: it allows us to compare Wiener integrals relative to different noises (in the family $\{B^H, H \in (0,1)\}$) in the strong $L^p(\Omega)$ sense, allowing us to obtain stronger estimates.

2 | SPDEs with fractional noise: continuity in law

In this chapter, we discuss the main result of this thesis, that is the continuity with respect to the Hurst parameter H of some classes of wave and heat SPDEs driven by a noise W^H, which behaves. in the space variable. like a fractional Brownian motion of Hurst parameter $H \in (0,1)$ and, in the time variable, like a standard Brownian motion (sBm),

Before introducing the main results, we give an introduction to the theory of stochastic partial differential equations (SPDEs), in the framework of the random field approach, first developed by Walsh in [Wal86]. We will put a special focus on the stochastic heat and wave equations, driven by the noise W^H. We consider both the case of additive and multiplicative noise.

In Section 2.1, we discuss briefly and rather informally the basic concepts and intuitions behind the study of SPDEs. Then, from Section 2.2 onwards we will give a rigorous treatment of the topic. We will privilege a particular class of equations, driven by a Gaussian noise W^H which behaves in time like a sBm and in space like a fBm of Hurst parameter $H \in (0,1)$. We recall and extend the solution theory for equations of this type in Section 2.3. For the solutions u^H of this class of equations, in Section 2.4 we will study the weak continuity problem with respect to the parameter H.

2.1 SPDEs in the random field approach

Informally speaking, by SPDE one can mean any type of PDE which is influenced by some type of randomness. We will focus here on SPDEs which are defined for $(t,x) \in [0,T] \times \mathbb{R}$ and are of the form

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{X}(t,x). \tag{2.1}$$

Here, we denote by L a second-order differential operator, $b, \sigma : \mathbb{R} \to \mathbb{R}$ are two functions with suitable regularity conditions and $\dot{X}$ denotes a noise forcing term, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this setting, the term $\dot{X}$ will be the only source of randomness for the equation. The value $T > 0$ that we fixed represents the time horizon of our problem. We will consider two choices of the operator L, the wave operator $\frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ and the heat operator $\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$. Depending on the operator L, we will consider also different initial conditions: we will impose $u_0(x) = u(0,x)$ and $v_0(x) = \frac{\partial}{\partial t}u(0,x)$ for the wave operator and only $u_0(x) = u(0,x)$ for the heat operator. We will discuss later the regularity that we have to impose on u_0, v_0 from case to case.

Example 2.1. We consider an illustrative example for equations of the form (2.1). Set $L = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ (the heat operator), $b \equiv 0$, $\sigma \equiv 1$ and $\dot{X} = \dot{W}$, where $\dot{W} = \dot{W}(t,x)$ is a 2-dimensional Gaussian white noise. Then equation (2.1), that we consider with initial condition u_0 regular enough, reads

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x).$$

We see that this is just a version of the usual heat equation, to which we add a stochastic forcing term $\dot{W}$. The white noise $\dot{W}$ is characterized by having zero mean and covariance $\mathrm{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y)$, where we denoted by δ the Dirac delta distribution at 0.

Consider now the classical intuition behind the heat equation: for a fixed $t_0 \in [0,T]$ the function $x \mapsto u(t_0,x)$ describes the distribution of heat in a one-dimensional medium at time t_0. As the time t progresses, the form of the equation causes a "diffusion" of heat, which tends to spread out evenly in the x variable, like the physical intuition suggests. What do we change by adding a white noise term in the equation? The intuitive answer is that, in the stochastic version of the equation, the diffusion process still takes places, but it is perturbed by the term $\dot{W}$, which adds a random noisy perturbation in the system.

Another important problem in SPDE theory is the regularity problem. Many natural choices of noise term $\dot{X}$ are highly irregular. Consider Example 2.1; in this case the noise term $\dot{W}$ is space-time white noise: this means that it is a random distribution which is the distributional derivative of a Brownian sheet (a realization $\dot{W}(\cdot,\cdot)(\omega)$ of white noise is a collection, indexed by $(t,x) \in [0,T] \times \mathbb{R}$, of the outcomes of a family of independent Gaussian random variables with zero mean and unitary variance). We expect this lack of regularity of the noise to propagate to the solution u, even we expect also some influence from the smoothing effect of the heat operator.

This heuristic reasoning translates to the need, for most of the noises $\dot{X}$, of a weaker notion of solution, compared to the classical one suggested by (2.1). This leads to the so-called *mild formulation* of an SPDE, which we discuss in full details in Section 2.1. Anyway, we will still write formally our SPDEs in the form (2.1) from time to time. When we do this, we refer to (2.1) as the *formal* version of our SPDEs.

Once we have a proper notion of solution, we can ask ourselves some classical questions, starting from the basic existence and uniqueness problem for (2.1). We will discuss about classical and more recent developments in this topic in Section 2.4. Apart from existence and uniqueness, there are various other questions that one can ask about the solutions of an SPDE. We give some examples: a first interesting property that is studied is the a.s. path regularity of the solution. Given a solution u of an SPDE of the type (2.1), the idea is to check whether for almost every $\omega \in \Omega$ the paths $u(\cdot,\cdot)(\omega)$ have some regularity in $[0,T] \times \mathbb{R}$ (i.e. continuity, Hölder continuity, see [SaSa00], [SaSa02]). We will see some examples of this problems in Section 2.4. Another interesting property is the intermittency property, which has been studied e.g. in [DaMu09], [HHNT15], [BJQ17]. Informally speaking, the solution u of an SPDE is said to be *intermittent* if it presents large and quickly disappearing spikes.

Our main result will concern another type of problem: consider an equation of the form (2.1), but driven by a noise $\dot{W}^H$ which behaves in time like white noise and in space like fractional noise of Hurst parameter $H \in (0,1)$. Then, if the solution exists for every $H \in (a,b)$, with $0 \le a \le b \le 1$, we can consider the family of solutions $\{u^H,\, H \in (a,b)\}$. A natural question is: is this family continuous with respect to the parameter H? We will answer this question in the framework of weak convergence (also termed convergence in distribution) on the space $\mathcal{C}([0,T] \times \mathbb{R})$ of continuous functions, endowed with the norm of uniform convergence on compact sets. We give a first informal statement of our result

Consider equation (2.1) driven by the noise $\dot{W}^H$, defined for $(t,x) \in [0,T] \times \mathbb{R}$ as

$$Lu^H(t,x) = b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x).$$

We will prove that, under each of the three sets of hypotheses A, B1, C, defined in Section 2.4, it holds the following: let $\{H_n,\, n \in \mathbb{N}\}$ be a sequence of Hurst indexes such that $H_n \to H_0$, where both the $\{H_n,\, n \in \mathbb{N}\}$ and H_0 are admissible values for the set of hypotheses that we are considering.

Then it holds that $u^{H_n} \xrightarrow{d} u^{H_0}$, in the sense of weak convergence in the space $\mathcal{C}([0,T]\times\mathbb{R})$, endowed with the metric of uniform convergence in compact sets.

This result, apart from being natural and interesting from the mathematical point of view, is a first step towards using a model like (2.1) driven by W^H in applications. Indeed, if one uses such a model, usually has to estimate the parameter H of the model from previously available data. This, even if done through exacts methods, can never give an exact result, due to the finiteness of observable data. One has then to check that the error made in the estimation of the parameter H propagates reasonably on the solution u^H. The continuity in distribution of u^H with respect to H is a first step in this direction. Even if the result is not quantitative, it gives a first positive answer to this sensitivity problem.

In Chapter 4 we will see an example of application of such kind of fractional noises to an electricity market. In that case the driving noise will be a simple fBm B^H instead of the 2-dimensional noise W^H that we are considering here.

We give meaning now to the main concepts of SPDE theory. We start from the above mentioned concept of mild solution. Before being able to define it, we need several tools.

The concept of fundamental solution is very important. Given a differential operator L, if one can find a fundamental solution G for L, then the solution of the non-homogeneous problem $Lu=f$ can be found by convolving G with f. We do not enter into the details of this, but we will see the utility of fundamental solutions also in the stochastic setting soon. We report the two cases of fundamental solutions of our interest:

1) When we are in the wave equation case $L=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$, the fundamental solution $G=G_t(x)$ is given by
$$G_t(x)=\frac{1}{2}1_{\{|x|\le t\}}. \tag{2.2}$$

2) When we are in the heat equation case $L=\frac{\partial}{\partial t}-\frac{1}{2}\frac{\partial^2}{\partial x^2}$ the fundamental solution $G=G_t(x)$ is given by
$$G_t(x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}. \tag{2.3}$$

The constant $\frac{1}{2}$ appearing in the Heat equation operator is just there by convention. The equation is completely equivalent to the form $\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}$.

We also have to define properly the noise $\dot{X}$; in Section 2.2 we will interpret $\dot{X}$ as the formal derivative of a stochastic process $X=X(t,x)$ defined of a probability space $(\Omega,\mathcal{F},\mathbb{P})$, endowed with a suitable filtration $\{\mathcal{F}_t,\, t\in[0,T]\}$.

Remark 2.2. We give some intuition about what we mean as *formal derivative.* Our general objective will be to define an integral with respect to the process X. Assume $X:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a smooth deterministic function. We already saw in Chapter 2 (in the 1-dimensional case) that one can define the Riemann-Stieltjes integral with respect to X as a limit of Riemann sums. If X is, as we are assuming, at least differentiable, the Riemann-Stieltjes integral can be also defined as
$$\int_0^T\int_{\mathbb{R}} f(t,x)dX(t,x) := \int_0^T\int_{\mathbb{R}} f(t,x)\dot{X}(t,x)dtdx.$$
This definition is consistent with the limit of Riemann sums. This explains why we denote the noise as $\dot{X}$, while in the following we will consider its formal integral X. The idea is that we will "integrate" the expression (2.1) in order to obtain an integral formulation of it. This reasoning can be made precise in the framework of random distributions, but this goes beyond the scope of this work and we will skip it.

Regarding the construction of the filtration $\{\mathcal{F}_t\}$, we define it as the natural filtration associated to the noise, conveniently completed. See later for a precise definition. We are ready to define:

Definition 2.3. We say that a stochastic process $\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ is a *mild solution* for (2.1) if it is adapted with respect to $\mathcal{F}_t$, jointly measurable and it satisfies for every $(t,x)\in[0,T]\times\mathbb{R}$

$$u(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}}b(u(s,y))G_{t-s}(x-y)dy\,ds+\int_0^t\int_{\mathbb{R}}\sigma(u(s,y))G_{t-s}(x-y)X(ds,dy), \tag{2.4}$$

where we denoted as I_0 the solution of the deterministic equation $Lu=0$ associated with (2.1), with the same initial conditions.

We recall that a process $Y=Y(t,x)$ is said to be jointly measurable if the map $(t,x,\omega)\mapsto Y(t,x)(\omega)$ is measurable with respect to the σ-algebra $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$ (the codomain $\mathbb{R}$ is obviously endowed with the σ-algebra $\mathcal{B}(\mathbb{R})$), where we denote by $\mathcal{B}(A)$ the Borel sigma-algebra on $A\subseteq\mathbb{R}$

For the moment, we did not say anything about the initial conditions of the problem. These will in general depend on the form of the operator L. We give two examples: in the case of the heat equation, we only have to impose $u(0,x)=u_0(x)$. In the case of the wave equation, we need to impose $u(0,x)=u_0(x)$ and $\frac{\partial u}{\partial t}(0,x)=v_0(x)$. We will discuss more precisely the issue of initial conditions later.

In the right-hand side of (2.4), the rightmost addend is an integral of a stochastic process with respect to our noise $\dot{X}$. Being able to define a proper notion of integral with respect to $\dot{X}$ is a key tool in the theory of SPDEs. The possible definitions of integral strongly depend on the form of the noise $\dot{X}$. For our purposes, we will only work with the assumption that $\dot{X}$ is a Gaussian noise and, more specifically, that $\dot{X}$ behaves like a sBm in the time variable t and behaves like a fBm in the space variable x.

2.2 Spatially homogeneous Gaussian noise

We will define here the concept of spatially homogeneous Gaussian noise. We will see that all the noises of our interest W^H, for all $H\in(0,1)$, will be interpreted as spatially homogeneous Gaussian noises. The family of spatially homogeneous Gaussian noises is useful because it is the family that we will use to construct a proper integration theory for both deterministic functions and stochastic processes. With this integration theory, we are able to give an adequate meaning to equations of the form (2.1) in the mild sense of Definition 2.3.

2.2.1 Definition and examples

We define now the concepts that we need in order to give meaning to Definition 2.3. As a general framework, we lie into the theory of martingale measure stochastic integrals, introduced by Walsh in [Wal86]. The theory developed by Walsh has undergone a series of more recent developments and generalizations (see e.g. [Dal99], [NuQu07], [DaQu11]), which have allowed to study a larger class of equations.

Here, we will only consider the fruitful notion of spatially homogeneous Gaussian noise, and define the integral with respect to it. In particular, we will later on restrict to a special class of noises W^H, which behave in time like a sBm and in space like a fBm of Hurst parameter $H\in(0,1)$. The theory of stochastic integration with respect to such noises lies partially into the general theory of spatially homogeneous Gaussian noises exposed in [DaQu11], except from the case $H \in (0, \frac{1}{2})$, which has been handled separately in [BJQ15] and [HHLNT17] (in the former case using some ideas from [BGP12]), due to the more irregular nature of the noise W^H. We highlight that, even if it is possible to define a stochastic integral with respect to W^H when $H \in (0, \frac{1}{2})$, it is not possible to show existence and uniqueness of a solution for (2.4) whenever $H < \frac{1}{4}$. This limitation is somehow consistent with the existing boundaries for this kind of problems involving a fBm. See [BJQ17] for a significant non-existence result for the wave equation driven by W^H, when $H < \frac{1}{4}$.

Before restricting to the case of the noises W^H, we briefly introduce the basic theory of spatially homogeneous Gaussian noises. For a function φ, we denote with $\mathcal{F}\varphi(t,\cdot)(\xi)$ its Fourier transform with respect to the space variable $x \in \mathbb{R}$.

Definition 2.4. Let $X = \{X(\varphi),\, \varphi \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R})\}$ be a Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed on the space $\mathcal{C}_0^\infty([0,\infty) \times \mathbb{R})$ of smooth functions with compact support. We say that X is a *spatially homogeneous Gaussian noise* if it has zero mean, i.e. $\mathrm{E}\Big[X(\varphi)\Big] = 0$ for all $\varphi \in \mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})$, and satisfies, for every $\varphi, \psi \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R})$,

$$\mathrm{E}\Big[X(\varphi)X(\psi)\Big] = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\mu(d\xi)dt =: \langle \varphi, \psi\rangle_X, \tag{2.5}$$

for some temperate non-negative measure μ on $\mathbb{R}$, which we will call the *spectral measure* of X and has to be symmetric (in the sense that $\mu(-A) = \mu(A)$, for any $A \in \mathcal{B}(\mathbb{R})$).

Remark 2.5. With this definition, the map $\varphi \mapsto X(\varphi)$ is linear. Being X Gaussian, it suffices to check that

$$\mathrm{E}\Big[\Big(X(a\varphi + b\psi) - aX(\varphi) - bX(\psi)\Big)^2\Big] = 0,$$

which is immediate to check by applying (2.5) (see [Nua], Definition 1.1.1).

The integrability conditions on μ that are needed in order to be able to define the stochastic integral with respect to X allow the space of integrands with respect to X to be rich enough to define a good integration theory.

The most general approach to the spatially homogeneous Gaussian noise consists in defining the covariance structure starting from a tempered distribution Φ. Tempered distributions over $\mathbb{R}$ are elements of $\mathcal{S}'(\mathbb{R})$, the dual space of the Schwartz space $\mathcal{S}(\mathbb{R})$. The space $\mathcal{S}(\mathbb{R})$ is defined as the space of functions $g : \mathbb{R} \to \mathbb{R}$ that are $\mathcal{C}^\infty$ and such that g and its derivatives, multiplied to a polynomial of any order, decay to 0 for $x \to \infty$. Its dual space is the space of linear continuous functionals $\Phi : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$. We will not define the metric that is used on $\mathcal{S}(\mathbb{R})$, since it goes beyond the scope of this thesis. We refer to [Sch] for a more detailed study of this topic.

Given a non-negative-definite tempered distribution Φ, which means that, for every $g \in \mathcal{S}(\mathbb{R})$,

$$\Phi(g * \tilde{g}) \geq 0,$$

where $\tilde{g}(x) = g(-x)$, it is possible to define a spatially homogeneous Gaussian noise $\{X = X(\varphi),\, \varphi \in \mathcal{C}([0,\infty) \times \mathbb{R})\}$ again as a process with zero mean and covariance (see [BJQ15] for example)

$$\mathrm{E}\Big[X(\varphi)X(\psi)\Big] := \int_0^\infty \Phi\Big(\varphi(t,\cdot) * \tilde{\psi}(t,\cdot)\Big)dt. \tag{2.6}$$

This definition can be specialized to the case in which Φ defines a measure Λ on $\mathbb{R}$. See for example [DaQu11], [NuQu07]. Let Λ be a non-negative, and non-negative definite measure on $\mathbb{R}$. This means that $\Lambda(A) \geq 0$ for every $A \in \mathcal{B}(\mathbb{R})$, and that for every g integrable with respect to Λ,

$$\int_{\mathbb{R}} (g * \tilde{g})(x)\Lambda(dx) \geq 0.$$

We can define a spatially homogeneous Gaussian noise $\{X = X(\varphi),\, \varphi \in \mathcal{C}([0,\infty)\times\mathbb{R})\}$ as a stochastic process which has zero mean and covariance given, for every $\varphi, \psi \in \mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})$, by

$$\mathrm{E}\Big[X(\varphi)X(\psi)\Big] := \int_0^\infty dt \int_{\mathbb{R}} (\varphi(t,\cdot) * \tilde{\psi}(t,\cdot))(x)\Lambda(dx). \tag{2.7}$$

A more specific case of covariance structure is when Φ defines a function f. This case was first studied in [DaSa80] (see also [Dal99]): consider a function $f : \mathbb{R} \to [0,\infty)$ which is continuous on $\mathbb{R}\setminus\{0\}$ and symmetric, in the sense that $f(-x) = f(x)$. One could define a spatially homogeneous Gaussian noise $\{X = X(\varphi),\, \varphi \in \mathcal{C}([0,\infty)\times\mathbb{R})\}$ as a process with zero mean and covariance structure given, for every $\varphi, \psi \in \mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})$, by

$$\mathrm{E}\Big[X(\varphi)X(\psi)\Big] := \int_0^\infty dt \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t,x)\psi(t,y)f(x-y)dydx. \tag{2.8}$$

The interesting fact about these definitions is that all of them coincide, whenever it is possible to define all of them: indeed, if one chooses f as in (2.8) then f induces a measure $\Lambda(A) := \int_A f(x)dx$ and a tempered distribution $\Phi(\varphi) := \int_{\mathbb{R}} f(x)\varphi(x)dx$. And if one has a measure Λ, this obviously defines a tempered distribution $\Phi(\varphi) := \int_{\mathbb{R}} \varphi(x)\Lambda(dx)$. We have then, in this case,

$$\begin{aligned}\int_0^\infty dt \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t,x)\psi(t,y)f(x-y)dydx &= \int_0^\infty dt \int_{\mathbb{R}} (\varphi(t,\cdot) * \tilde{\psi}(t,\cdot))(x)\Lambda(dx)\\ &= \int_0^\infty \Phi(\varphi(t,\cdot) * \tilde{\psi}(t,\cdot))dt,\end{aligned}$$

where we interpret $\tilde{\psi}(t,x) = \psi(t,-x)$.

Another interesting fact about this direct approach to spatially homogeneous Gaussian noises is the fact, consequence of Bochner's Theorem, that Φ is a non-negative tempered distribution if and only if it is the Fourier transform in $\mathcal{S}'(\mathbb{R})$ of a symmetric non-negative tempered measure μ. We recall that a non-negative measure is tempered if and only if it holds

$$\int_{\mathbb{R}} \frac{1}{(1+|\xi|^2)^m}\,\mu(d\xi) < \infty,$$

for some $m \geq 1$ (see [Dal99], [DaQu11] [BJQ15]).

We recall that in our case we will consider a specific type of noise W^H, which is a one-parameter family of noises indexed by $H \in (0,1)$. We give a formal definition.

Definition 2.6. Consider the family of measures $\{\mu_H,\, H \in (0,1)\}$ associated with $\{W^H,\, H \in (0,1)\}$, defined by

$$\mu_H = c_H|\xi|^{1-2H}d\xi, \qquad c_H := \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{2.9}$$

This family satisfies (2.24), for every $H \in (0,1)$. For every $H \in (0,1)$, we define W^H as in Definition 2.4, where $\mu = \mu_H$.

The Fourier transform of μ_H in $\mathcal{S}'(\mathbb{R})$, defined as the distribution Φ such that for every $\varphi \in \mathcal{S}(\mathbb{R})$ it holds

$$\int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\mu_H(d\xi) = \Phi(\varphi).$$

This Fourier transform of μ_H is equal to the integrable function $f_H(x) = H(2H-1)|x|^{2H-2}$ when $H > \frac{1}{2}$, and it is equal to the genuine distribution

$$\Phi_H(\varphi) = H(2H-1)\int_{\mathbb{R}} (\varphi(x) - \varphi(0))|x|^{2H-2}dx$$

when $H < \frac{1}{2}$. Due to the different behaviour, the problem has to be handled with different techniques in the two cases. The case $H > \frac{1}{2}$ belongs to the general theory studied, for example, in [Dal99], while the case $H < \frac{1}{2}$ is more involved and was handled in [BJQ15] and [HHLNT17]

As we already suggested in the right-hand side of (2.5), the covariance structure of a spatially homogeneous Gaussian noise defines an inner product on the space $\mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})$, which is not complete under this inner product. We define

$$\mathcal{H}_X := \overline{\mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})}^{\langle\cdot,\cdot\rangle_{\mathcal{H}_X}},$$

the completion of $\mathcal{C}_0^\infty([0,\infty)\times\mathbb{R})$ with respect to the inner product $\langle\cdot,\cdot\rangle_X$. This makes $\mathcal{H}_X$ a Hilbert space. An interesting fact is that the space $\mathcal{H}_X$ is not necessarily a space of functions, but it can also contain some proper distributions. This is the case for the space $\mathcal{H}_H$ induced by μ_H whenever $H > \frac{1}{2}$, whereas in the case $H < \frac{1}{2}$ the space $\mathcal{H}_H$ is a space of functions. We refer to [Jol10] and [PiTa00] for an analysis of these problems.

Clearly, since $\mathcal{H}_X$ is a Hilbert space, its inner product naturally defines a norm, given for $\varphi \in \mathcal{H}_X$ by

$$||\varphi||_{\mathcal{H}_X} := \Big(\langle\varphi,\varphi\rangle_{\mathcal{H}_X}\Big)^{\frac{1}{2}}.$$

2.2.2 Wiener integral

We define now the stochastic integral of a deterministic function with respect to X. As always, we put a special focus on the case $X = W^H$, which is the one of interest for us. The definition of Wiener integral is quite straightforward: the construction of a complete space allows us to extend the definition of spatially homogeneous Gaussian noise to allow elements of the type $X(\varphi)$, with $\varphi \in \mathcal{H}_X$. This can be done by isometric extension of a linear operator with values in $L^2(\Omega)$.

Definition 2.7. Let $X = \{X(\varphi),\, \varphi \in \mathcal{H}_X\}$ be a spatially homogeneous Gaussian noise on $[0,T]\times\mathbb{R}$. For any $\varphi \in \mathcal{H}_X$, we say that $X(\varphi)$ is the *Wiener integral* of φ with respect to X, and we denote it as

$$\int_0^\infty \int_{\mathbb{R}} \varphi(t,x) X(dt,dx) := X(\varphi). \tag{2.10}$$

Remark 2.8. Thanks to the linearity of X, the definition satisfies the basic requirement of an integral, i.e. the linearity with respect to the integrand functions. Clearly this is not sufficient to justify the fact that we call $X(\varphi)$ an integral. We will go into details now.

The definition of Wiener integral may look artificial, but it has a natural interpretation. Suppose that for $s < t$ and $x < y$ the function $\varphi = 1_{(s,t]\times(x,y]} \in \mathcal{H}_X$; then, in order to have a good definition of integral, one hopes to have

$$\int_0^\infty \int_{\mathbb{R}} 1_{(s,t]\times(x,y]}(r,z) X(dr,dz) = X(t,y) - X(t,x) - X(s,y) + X(s,x)$$

for some 2-dimensional random field $X = \{X(t,x),\, (t,x) \in [0,T]\times\mathbb{R}\}$ which is defined consistently with $X = X(\varphi)$. The idea is to check that $1_{(0,t]\times(0,x]} \in \mathcal{H}_X$, for every $(t,x) \in [0,T]\times\mathbb{R}$ and define the random field $X(t,x) = X(1_{(0,t]\times(0,x]})$. We use the same notation X for the spatially homogeneous Gaussian noise and for the random field, since there is no risk of confusion.

Let $H \in (0,1)$. When $X = W^H$, it holds that $1_{(s,t]\times(x,y]} \in \mathcal{H}_X$ (see [BJQ15], Remark 2.1), and thus it is always possible to define the random field $\{W^H(t,x),\, t \in [0,T],\, x \in \mathbb{R}\}$ in the way we just described.

The definition of the random field $W^H(t,x)$ makes clear why we say that W^H behaves like a sBm in the time variable t and like a fBm of Hurst parameter $H \in (0,1)$ in the space variable x. Indeed, we have that

$$\begin{aligned}
\mathrm{E}[W^H(t,x)W^H(s,y)] =& c_H \int_0^\infty \int_{\mathbb{R}} 1_{[0,t]}(r)1_{[0,s]}(r)\mathcal{F}1_{[0,x]}(\xi)\overline{\mathcal{F}1_{[0,y]}(\xi)}|\xi|^{1-2H}d\xi dr \\
=& c_H \int_0^\infty \int_{\mathbb{R}} 1_{[0,(t\wedge s)]}(r)\mathcal{F}1_{[0,x]}(\xi)\overline{\mathcal{F}1_{[0,y]}(\xi)}|\xi|^{1-2H}d\xi dr \\
=& \Big(\int_0^{t\wedge s} dr\Big) c_H \int_{\mathbb{R}} \mathcal{F}1_{[0,x]}(\xi)\overline{\mathcal{F}1_{[0,y]}(\xi)}|\xi|^{1-2H}d\xi \\
=& \frac{1}{2}(t\wedge s)\Big(|x|^{2H}+|y|^{2H}-|x-y|^{2H}\Big)
\end{aligned}$$

The last step is a consequence of Proposition 1.14.

Remark 2.9. To define a theory of stochastic integration with respect to a noise X, it is also possible to start directly from a random field $X = X(t,x)$. In [BGP12], the authors construct, in such a framework, the space of deterministic integrands starting from elementary functions of the type $1_{(s,t]\times(x,y]}$, and define the space $\mathcal{H}_X$ starting from them.

The covariance structure of the spatially homogeneous Gaussian noise becomes an Itô's type isometry, now that we interpret $X(\varphi)$ as a Wiener integral. Indeed, we have, for any $\varphi \in \mathcal{H}_X$

$$\mathrm{E}[X(\varphi)^2] = \int_0^\infty \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2\mu(d\xi).$$

When we consider $X = W^H$ (and thus $\mu = \mu_H$), this reads, for every $\varphi \in \mathcal{H}_H$

$$\mathrm{E}[X(\varphi)^2] = c_H \int_0^\infty \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2|\xi|^{1-2H}d\xi.$$

The definition of Wiener integral can be extended to the n-dimensional case with the iterated Wiener integral. We refer to [Nua] for a good reference on the topic. The idea is to integrate functions that belong to the Hilbert space $\mathcal{H}_X^{\otimes n}$, for any $n > 1$. Consider an orthonormal basis of $\mathcal{H}_X$ with elements $\{e_1, e_2, \dots\}$. Take an elementary element of $\mathcal{H}_X^{\otimes n}$ of the form

$$\varphi = c_{i_1,\dots,i_n} e_{i_1} \hat{\otimes} \dots \hat{\otimes} e_{i_n},$$

where $\hat{\otimes}$ denotes the symmetrized tensor product, defined as

$$a_1 \hat{\otimes} a_2 \hat{\otimes} \cdots \hat{\otimes} a_n := \frac{1}{n!} \sum_{\pi\in\mathcal{P}(n)} a_{\pi(1)} \otimes a_{\pi(2)} \otimes \cdots \otimes a_{\pi(n)},$$

where $\mathcal{P}(n)$ is the set of permutations of order n. The set of linear combinations of elementary elements is dense in $\mathcal{H}_X^{\otimes n}$. An element of this form can be more conveniently (for our purposes) written as

$$\varphi = c_{j_1,\dots,j_m} e_{j_1}^{\otimes k_1} \hat{\otimes} \dots \hat{\otimes} e_{j_m}^{\otimes k_m}, \tag{2.11}$$

where all the $j_1, \dots j_m$ are different and $k_1 + \cdots + k_m = n$. We define for such an element the n-th order multiple Wiener integral as

$$I_n^X(\varphi) = c_{j_1,\dots,j_m} P_{k_1}(X(e_{j_1})) \cdots P_{k_m}(X(e_{j_m})), \tag{2.12}$$

where we denote by P_k the normalized k-th Hermite polynomial, defined as

$$P_k(x) = \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k}\Big(e^{-\frac{x^2}{2}}\Big),$$

where $k \geq 0$. The multiple integral is then extended by linearity to linear combinations of elementary functions of the type

$$\varphi = \sum_{\text{finite}} c_{i_1,\ldots,i_n} e_{i_1} \hat{\otimes} \ldots \hat{\otimes} e_{i_n}.$$

On this class of functions one has the natural norm induced by the norm $||\cdot||_{\mathcal{H}_X}$:

$$||\varphi||_{\mathcal{H}_X^{\otimes n}} := \sum_{\text{finite}} |c_{i_1,\ldots,i_n}|^2,$$

and it holds

$$\mathrm{E}\Big[|I_n^X(\varphi)|^2\Big] = n!\, ||\tilde{\varphi}||^2_{\mathcal{H}_X^{\otimes n}},$$

where

$$\tilde{\varphi}(t_1, x_1, \ldots, t_n, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}(n)} \varphi(t_{\sigma(1)}, x_{\sigma(1)}, \ldots, t_{\sigma(n)}, x_{\sigma(n)})$$

denoted the symmetrized version of φ, as defined in [Nua]. Thus the integral can be extended isometrically as an $L^2(\Omega)$-limit to the completion of the space of elementary functions which is exactly the whole space $\mathcal{H}_X^{\otimes n}$. For a general $\varphi \in \mathcal{H}_X^{\otimes n}$, we have $I_n^X(\varphi) := I_n^X(\tilde{\varphi})$.

Moreover, we have that, for any $\varphi \in \mathcal{H}_X^{\otimes n}$,

$$\mathrm{E}\Big[|I_n^X(\varphi)|^2\Big] = \mathrm{E}\Big[|I_n^X(\tilde{\varphi})|^2\Big] = n!\, ||\tilde{\varphi}||^2_{\mathcal{H}_X^{\otimes n}}.$$

For a general element φ of $\mathcal{H}_X^{\otimes n}$, the norm $||\varphi||_{\mathcal{H}_X^{\otimes n}}$ is given by

$$||\varphi||_{\mathcal{H}_X^{\otimes n}} = \int_{[0,\infty)^n} \int_{\mathbb{R}^n} |\mathcal{F}\varphi(t_1, \cdot, t_2, \cdot, \ldots, t_n, \cdot)(\xi_1, \ldots, \xi_n)|^2 \mu(d\xi_1) \cdots \mu(d\xi_n) dt_1 \cdots dt_n. \quad (2.13)$$

Here, we still denoted with $\mathcal{F}$ the Fourier transform on a function defined on $\mathbb{R}^n$. It is given, for $\xi \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$, by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x) dx$$

where we denoted by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^n$.

2.2.3 Spectral representation of W^H

For the moment we defined a general spatially homogeneous Gaussian noise X. In this section we prove a set of results which are relative to the special case $X = W^H$, that we already defined. In particular, we will give a useful representation of the iterated Wiener integral with respect to the noise W^H. This representation is the 2-dimensional version of the one of Proposition 1.18, which is a consequence of the representation result for the fBm Proposition 1.14. We will follow the same steps: first, we define an integral representation for W^H, and from this we will derive a representation result for the Wiener integral with respect to W^H.

As we introduced in the previous section, $W^H = \{W^H(\varphi),\, \varphi \in \mathcal{H}_H\}$ is a spatially homogeneous Gaussian noise defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and characterized by the covariance structure

$$\mathrm{E}\Big[W^H(\varphi) W^H(\psi)\Big] := c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi) \overline{\mathcal{F}\psi(t,\cdot)(\xi)} |\xi|^{1-2H} d\xi dt,$$

where c_H is the constant defined in (2.9), which we recall to be

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}.$$

Let us denote in this case the space $\mathcal{H}_{W^H}$ as $\mathcal{H}_H$ We also denote the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_H}$ as $\langle \cdot, \cdot \rangle_H$ and the norm $|| \cdot ||_{\mathcal{H}_H}$ as $|| \cdot ||_H$. We define, as usual, the random field $W^H(t,x) := W^H(1_{(0,t]\times(0,x]})$. From now on, with an abuse of notation, we will denote with W^H both the spatially homogeneous Gaussian noise and the random field.

As we saw in the one dimensional case, representations like the one of Proposition 1.14 allow us to define our family of processes on the same probability space. In the 2-dimensional case, we will define our family of random fields $\{W^H,\, H \in (0,1)\}$ using a single complex Gaussian process $\tilde{W}$ defined of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the following:

Definition 2.10. Let $\tilde{W}$ be the $\mathbb{C}$-valued random measure on $\mathcal{B}([0,\infty)\times\mathbb{R})$ defined as $\tilde{W} = \tilde{W}_1 + i\tilde{W}_2$, where

i) For all $A, B \in \mathcal{B}([0,\infty)\times\mathbb{R})$, we have that

$$\mathrm{E}\Big[\tilde{W}_j(A)\tilde{W}_j(B)\Big] = \frac{|A\cap B|}{2},$$

for any $j = 1,2$, where $|A \cap B|$ denotes the Lebesgue measure of $A \cap B$.

The form of $\tilde{W}$ implies that for all $A \in \mathcal{B}([0,\infty)\times\mathbb{R})$ one has $\mathrm{E}\Big[\tilde{W}(A)\Big] = |A|$. The measure $\tilde{W}$ is thus essentially a white noise. Moreover, it is closely related to the measure defined in Definition 1.13. Indeed, if one defines for every $t \in [0,\infty)$ the process $\tilde{W}_t(A) := \tilde{W}([0,t]\times A)$, it is immediate to notice that, for t fixed, the measure $A \mapsto \tilde{W}_t(A)$ is a modification of the measure defined in Definition 1.13, whose variance is only multiplied by the constant t.

Proposition 2.11. *Let $\tilde{W}$ be the random measure introduced in Definition 2.10 and consider:*

$$\tilde{W}^H(t,x) := C(H)^{\frac{1}{2}} \int_0^t \int_{\mathbb{R}} 1_{[0,t]}(s)\mathcal{F}(1_{[0,x]})(y)|y|^{1/2-H}\tilde{W}(ds,dy). \tag{2.14}$$

Then, $\tilde{W}^H$ is a Gaussian process which has the same distribution as the random field W^H.

Proof. Clearly, being $\tilde{W}^H$ the integral of a deterministic function with respect to a Gaussian process, it is a Gaussian process. The fact that it has the same covariance structure as W^H comes from a straightforward application of the standard Itô isometry. □

We prove now a representation result for the multiple Wiener integral (2.12) with respect to W^H. Before proving it, we need the following corollary of [BGP12], Theorem 4.3:

Proposition 2.12 (of [BGP12], Theorem 4.3)**.** *Every element in the Banach space $\mathcal{H}_H$ can be approximated by elementary functions of the form*

$$\varphi(t,x) = \sum_{\textit{finite}} c1_{(r,s]\times(y,z]}(t,x).$$

Proof. The result is a direct consequence of [BGP12], Theorem 4.3. Indeed, in that case the authors showed that every element of the space of predictable processes $X = X(t,x)(\omega)$ whose $||\cdot||_H$ norm is in $L^2(\Omega)$ can be approximated by simple processes of the form $1_G(\omega)1_{(s,t]}1_{(x,y]}$, for G measurable. To prove our result, it is sufficient to observe that if we choose a deterministic element φ in their proof, also its approximating sequence φ_n is deterministic. Moreover, the norm they define on the space of processes Λ_X, defined as

$$\Lambda_X := \Big\{[0,\infty)\times\Omega \to \mathcal{S}(\mathbb{R}) : \varphi \text{ is predictable, } \mathcal{F}\varphi_t(\omega) \text{ is a function}$$
$$\text{for all } (\omega,t), \text{ and } \mathrm{E}\Big[\int_0^\infty \int_{\mathbb{R}} |\mathcal{F}\varphi_t(x)|^2 dt \mu_X(dx)\Big] < \infty\Big\},$$

is equal to the norm $||\cdot||_H$ on $\mathcal{H}_H$, whenever we compute it for a deterministic element φ. Indeed, for a $\varphi \in \mathcal{H}_H$, we have

$$||\varphi||_{\Lambda_X} := \mathrm{E}\Big[c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}f(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\Big] = c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}f(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt =: ||\varphi||_H.$$

This two facts conclude the proof.

□

We have then that we can define the whole family $\{W^H,\ H \in (0,1)\}$ on the same probability space. This will allow us to compare directly solutions of SPDEs relative to different values of H. From now onwards, we will denote $W^H = \tilde{W}^H$, without risk of confusion.

Remark 2.13. If we denote, for every $H \in (0,1)$, with $\mathcal{F}_t^H$ the filtration generated by $\{W^H(s,x),\ s \in [0,t],\ x \in \mathbb{R}\}$, we clearly have from (2.14) that $\mathcal{F}_t^H \subset \mathcal{F}_t$, where $\mathcal{F}_t$ is the filtration generated by $\{\tilde{W}(s,A),\ s \in [0,t],\ A \in \mathcal{B}(\mathbb{R})\}$. From now on we will denote with $\mathcal{F}_t$ the filtrations defined here.

Theorem 2.14. *Let $f \in \mathcal{H}_H^{\otimes n}$. Denote with $I_n^H(f)$ the n-th multiple integral with respect to the noise W^H. Let $\hat{f}$ be the function defined by*

$$\begin{aligned}&\hat{f}(t_1,x_1,t_2,x_2,\ldots,t_n,x_n)\\ &\quad = (c_H)^{\frac{n}{2}} \mathcal{F}(f(t_1,\cdot,t_2,\cdot,\ldots,t_n,\cdot))(x_1,\ldots,x_n)|x_1|^{1/2-H}\cdots|x_n|^{1/2-H}.\end{aligned}$$

Then it holds that

$$I_n^H(f) = \tilde{I}_n(\hat{f}), \tag{2.15}$$

where the integral $\tilde{I}_n$ is the n-th order Wiener integral with respect to a complex Brownian motion $\tilde{W}$. The constant c_H is the one given in (2.70)

Proof. We first check that the result is true for the first-order Wiener integral I_1^H. Given $\varphi \in \mathcal{H}_H$, let

$$\tilde{\varphi}(t,x) := (c_H)^{\frac{1}{2}} \mathcal{F}(\varphi(t,\cdot))(x)|x|^{1/2-H}$$

We prove that, for $\varphi \in \mathcal{H}_H$, we have

$$I_1^H(\varphi) = \tilde{I}_1(\tilde{\varphi}),$$

which means

$$\int_0^T \int_{\mathbb{R}} \varphi(t,x) W^H(dt,dx) = (c_H)^{\frac{1}{2}} \int_0^T \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(x)|x|^{1/2-H} \tilde{W}(dt,dx).$$

This is true for step functions of the form $\varphi_{el} = 1_{(r,s]\times(y,z]}$. Indeed, for these functions it holds, thanks to Proposition 2.11 and thanks to the linearity of the integral and of the Fourier transform,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \varphi_{el}(t,x) W^H(dt,dx) =& W^H(s,z) - W^H(r,z) - W^H(s,y) + W^H(r,y)\\ =& (c_H)^{\frac{1}{2}} \int_0^T \int_{\mathbb{R}} 1_{(r,s]}(t) \mathcal{F}1_{(y,z]}(\xi)|\xi|^{1/2-H} \tilde{W}(dt,d\xi).\end{aligned}$$

By Corollary 2.12, we have that the linear combinations of step functions are dense in $\mathcal{H}_H$, which implies that the first-order identification is true for all $\varphi \in \mathcal{H}_H$. Indeed, if for any $\varphi \in \mathcal{H}_H$ we see both integrals

$$\int_0^T \int_{\mathbb{R}} \varphi(t,x) W^H(dt,dx) \quad \text{and} \quad (c_H)^{\frac{1}{2}} \int_0^T \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(x)|x|^{1/2-H} \tilde{W}(dt,dx)$$

as a limit of integrals of step functions in $L^2(\Omega)$, we see that the norm in which we are taking the limit (after using Itô isometry) is the same in both cases.

We extend now this correspondence also to the n-th order integral. By definition, the n-th dimensional Wiener integral with respect to W^H of an elementary function $f \in \mathcal{H}_H^{\otimes n}$ written in the form (2.11) is given by

$$\begin{aligned}
I_n^H(f) =& c_{j_1,\dots,j_m} P_{k_1}(W^H(e_{j_1})) \cdots P_{k_m}(W^H(e_{j_m})) \\
=& c_{j_1,\dots,j_m} P_{k_1}\Big[\tilde{I}_1\Big((c_H)^{\frac{1}{2}}\mathcal{F}e_{j_1}(t_{j_1},\cdot)(\xi_{j_1})|\xi_{j_1}|^{1/2-H}\Big)\Big] \\
&\qquad \cdots \times P_{k_m}\Big[\tilde{I}_1\Big((c_H)^{\frac{1}{2}}\mathcal{F}e_{j_m}(t_{j_m},\cdot)(\xi_{j_m})|\xi_{j_m}|^{1/2-H}\Big)\Big] \\
=& \tilde{I}_n\Big[c_{j_1,\dots,j_m}\Big((c_H)^{\frac{1}{2}}\mathcal{F}e_{j_1}(t_{j_1},\cdot)(\xi_{j_1})|\xi_{j_1}|^{1/2-H}\Big)^{\otimes k_1} \otimes \cdots \\
&\qquad \cdots \otimes \Big((c_H)^{\frac{1}{2}}\mathcal{F}e_{j_m}(t_{j_m},\cdot)(\xi_{j_m})|\xi_{j_m}|^{1/2-H}\Big)^{\otimes k_m}\Big] \\
=& \tilde{I}_n\Big[(c_H)^{\frac{n}{2}}\mathcal{F}f(t_1,\dots,t_m,\bullet)(\xi_1,\dots,\xi_n)|\xi_1|^{1/2-H}\cdots|\xi_n|^{1/2-H}\Big].
\end{aligned}$$

This proves (2.15) for elementary functions. The extension to a general function $f \in \mathcal{H}_H^{\otimes n}$ is straightforward. It holds by definition that

$$||f||_{\mathcal{H}_H^{\otimes n}} = ||\hat{f}||_{(L^2(\mathbb{R}_+\times\mathbb{R}))^{\otimes n}}.$$

Then, if f_k is a sequence of simple functions converging to f in the norm of $\mathcal{H}_H^{\otimes n}$ it holds immediately that $\hat{f}_k \to \hat{f}$ in $L^2(\mathbb{R}_+ \times \mathbb{R})$, and by the uniqueness of the limit in $L^2(\Omega)$, we have that

$$I_n^H(f) = \tilde{I}_n(\hat{f}),$$

which is our thesis.

□

2.2.4 Skorohod stochastic integral

Here we introduce briefly the theory of Skorohod integration with respect to the noise W^H. This techniques work for a more general class of noises, but since we are going to use only briefly this framework in the following, we think that it is better to restrict directly to the noise W^H of our interest. Here, we will introduce the Skorohod integral using techniques inspired by Malliavin Calculus. For further details about Malliavin Calculus, or any of the topics mentioned in this subsection, we refer to [Nua]. We are going to define now only the minimal set of objects and concepts which permit us to work in this setting in the following.

The idea of Malliavin Calculus is to define a theory of calculus for random variables, defining for example objects like the derivative of a random variable. Malliavin Calculus also allows to define a notion of stochastic integral with respect to W^H. This notion of integral will be related to the one defined later in Subsection 2.2.5 by the forthcoming Theorem 2.27.

Let $\mathcal{G}$ be the σ-algebra generated by $\{W^H(\varphi),\ \varphi \in \mathcal{H}_H\}$. By Theorem 1.1.1 of [Nua], we have that $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ can be represented as

$$F = \mathrm{E}[F] + \sum_{n\geq 1} F_n. \tag{2.16}$$

Here $F_n \in \mathcal{H}_{H,n}$, where $\mathcal{H}_{H,n}$ is the n-th Wiener chaos space associated to W^H. The n-th Wiener chaos space is a space of random variables which has the following property: every $F_n \in \mathcal{H}_{H,n}$

can be represented as $I_n^H(f_n)$, where $f_n \in \mathcal{H}_H^{\otimes n}$. This means that for every $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ we can rewrite (2.16) as

$$F = \mathrm{E}[F] + \sum_{n \geq 1} I_n^H(f_n), \tag{2.17}$$

for some sequence $\{f_n,\, n \geq 1\}$ of functions such that $f_n \in \mathcal{H}_H^{\otimes n}$. These Wiener spaces are orthogonal between them, in the sense that if $\varphi \in \mathcal{H}_{H,n}$ and $\psi \in \mathcal{H}_{H,m}$, with $m \neq n$, then

$$\mathrm{E}\Big[I_n^H(\varphi) I_m^H(\psi)\Big] = 0.$$

Let now $\mathcal{S}$ be the class of random variables F that can be written as

$$F = f(W^H(\varphi_1), \dots, W^H(\varphi_n)), \tag{2.18}$$

where $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and the $\varphi_j \in \mathcal{H}_H$, for every $j = 1, \dots, n$. For any $F \in \mathcal{S}$, we define the *Malliavin derivative* of F as the $\mathcal{H}_H$-valued random variable DF given by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W^H(\varphi_1), \dots, W^H(\varphi_n))\varphi_j. \tag{2.19}$$

If we endow $\mathcal{S}$ with the norm $||F||_{\mathbb{D}^{1,2}} := \mathrm{E}[|F|^2]^{\frac{1}{2}} + \mathrm{E}\Big[||DF||^2_{\mathcal{H}_H}\Big]^{\frac{1}{2}}$, it turns out that the operator D can be extended to the completion of $\mathcal{S}$ with respect to $||\cdot||_{\mathbb{D}^{1,2}}$, which we will denote by $\mathbb{D}^{1,2}$. We define now the *Divergence operator* δ, which is the adjoint of D. The divergence operator is defined on its domain $\mathrm{Dom}(\delta)$, which is the space of $\mathcal{H}_H$-valued random variables such that $u \in L^2(\Omega; \mathcal{H}_H)$ and

$$\Big|\mathrm{E}\Big[\langle DF, u\rangle_H\Big]\Big| \leq c_u \mathrm{E}[|F|^2]^{\frac{1}{2}}, \qquad \text{for all } F \in \mathbb{D}^{1,2},$$

where the constant $c = c_u$ depends on u. Being the adjoint of D the divergence operator δ is defined for any $u \in \mathrm{Dom}(\delta)$ by the duality relation, holding for every $F \in \mathbb{D}^{1,2}$:

$$\mathrm{E}\Big[\langle DF, u\rangle_H\Big] = \mathrm{E}[F\delta(u)].$$

From the duality relation and the fact that for any constant $F = c$ it holds $DF = 0$, one can deduce that $\mathrm{E}[\delta(u)] = 0$, for every $u \in \mathrm{Dom}(\delta)$. The divergence operator δ has a fruitful interpretation as a stochastic integral. We will denote, for any $u \in \mathrm{Dom}(\delta)$, the *Skorohod integral* of u as

$$\int_0^\infty \int_{\mathbb{R}} u(t,x) W^H(\delta t, \delta x) := \delta(u).$$

This notion of integral, as it is evident, does not rely on any adaptability condition for the integrand process u. Thus, it is suitable also for studying SPDEs in which it is not possible to define a natural filtration with respect to which we solve the problem. Anyway, it is also compatible with Itô's type definition of integral for adapted processes, as we will see in Subsection 2.2.6.

2.2.5 Itô stochastic integral

We introduce now the theory of stochastic integration with respect to the spatially homogeneous Gaussian noise $X = W^H$. This time we wish to integrate stochastic processes, and not functions like in the Wiener integral. There are various approaches in the literature (see for example [Dal99], [Wal86], [BJQ15], [HHLNT17]); the idea that they have in common is to define the integral $\int_0^\infty \int_{\mathbb{R}} g(t,x) X(dt, dx)$ for a sufficiently large class of processes and to make it satisfy a Burkholder's type estimate. With this two things, it is possible to define a proper solution theory for equations like (2.4), for example using fixed point arguments in some suitable Banach space to show that a solution to (2.4) exists. We outline the standard construction in the general case of a spatially homogeneous Gaussian noise X, to give an idea of the general construction. We will then specialize to the case of the noise W^H.

We defined W^H as a spatially homogeneous Gaussian noise on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we saw that it is possible to define a random field, which we still denote as W^H, by defining $W^H(t,x) := W^H(1_{[0,t]\times[0,x]})$. This can be done because, for any $H \in (0,1)$, for any $0 \le s \le t \le T$, and for any $x \le y$, the function $1_{(s,t]\times[x,y]} \in \mathcal{H}_H$. We define the natural filtration on $\mathcal{F}$ given by $\mathcal{F}_t := \sigma(\{W^H(s,x),\ s \in [0,t],\ x \in \mathbb{R}\}) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the sets $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$.

Definition 2.15. We say that g is an *elementary process* if it is a finite linear combination of processes of the form

$$g(t,x,\omega) := Y(\omega)1_{(r,s]\times(y,z]}(t,x), \tag{2.20}$$

for some $0 \le r \le s \le T$, for some $x \le y$ and for some $\mathcal{F}_r$-measurable random variable Y. We denote the space of such processes as $\mathcal{E}$.

For a process g of the form (2.20), we define its stochastic integral with respect to W^H as the process $(g \cdot W^H)_t$ given by

$$\begin{aligned}(g \cdot W^H)_t =& \int_0^t \int_{\mathbb{R}} g(\tau,x)W^H(d\tau,dx)\\ :=& Y\Big(W^H(t\wedge s,z) - W^H(t\wedge s,y) - W^H(t\wedge r,z) + W^H(t\wedge r,y)\Big).\end{aligned} \tag{2.21}$$

If we consider (2.21) for $t > s$, we can get rid of the minimums appearing and we obtain the rectangular increment of W^H over the rectangle $(r,s]\times(y,z]$, thus making the integral notion a natural one. The definition can be extended to elements of $\mathcal{E}$ by linearity.

Remark 2.16. We built the definition of elementary process (2.20) on functions of the type $1_{(r,s]\times(y,z]}$. This is not the only possible choice. In the classical references [Wal86] and [Dal99], the authors consider elementary functions of the form

$$\tilde{g}(t,x,\omega) = Y(\omega)1_{(r,s]\times A}(t,x),$$

where $A \in \mathcal{B}(\mathbb{R})$ has bounded Lebesgue measure. This framework leads naturally to the concept of Walsh's martingale measure; consider a spatially homogeneous Gaussian noise X. Then we can define a martingale measure as $M(t,A) := X(1_{(0,t]\times A})$, and it gives rise to an integral of the type

$$\int_0^t \int_{\mathbb{R}} \tilde{g}(\tau,x)M(d\tau,dx) := Y\Big(M(t\wedge s,A) - M(t\wedge r,A)\Big)$$

These two different approaches are equivalent if both of them are well-defined. Note that the latter approach fails when we consider the driving noise W^H for $H \in (0,\frac{1}{2})$. Indeed, in [BJQ15], Appendix C, the authors showed that in that case there exists a bounded Borel set A such that $1_{(r,s]\times A} \notin \mathcal{H}_H$, in the sense that

$$\int_0^\infty \int_{\mathbb{R}} |\mathcal{F}(1_{(r,s]\times A})(t,\cdot)(\xi)|^2|\xi|^{1-2H}d\xi dt = \infty.$$

This is the main reason for which we defined an elementary process of the form (2.20).

Given the definition of integral (2.21), the next step is to extend it to a larger class of integrand processes. The classical idea is to define an isometry between a suitable space of integrands and a suitable space of integrals, which allows to extend the integral to a general class of integrands (the completion of $\mathcal{E}$ with respect to the norm we are using on it) by isometric extension of the integral operator. We refer to [Dal99] for a general, yet not exhaustive, case.

Following [Dal99] and [BJQ15] we endow $\mathcal{E}$ with the norm

$$||g||_0 := \mathrm{E}\Big[c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}g(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\Big], \tag{2.22}$$

and we define $\mathcal{P}_0$ as the completion of $\mathcal{E}$ with respect to the norm $||\cdot||_0$. It turns out that the integral (2.21), with $X = W^H$, is an isometry from the space $\mathcal{E}$ to the space of continuous real-valued martingales $\{M = M(t),\, t \in [0,T]\}$, adapted with respect to $\mathcal{F}_t$, and endowed with the norm $||M|| := \mathrm{E}[M_T^2]^{1/2}$ (see [BJQ15], page 9). Thus, the map can be extended isometrically to the whole space $\mathcal{P}_0$. We have then that the integral of a process g in $\mathcal{P}_0$ is defined, given a sequence $g_n \to g$ in $\mathcal{P}_0$, as

$$(g \cdot W^H)_t := \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} g_n(s,x) W^H(ds,dx),$$

where the limit is meant in the space of continuous real-valued martingales M, with the norm defined above.

We have skipped the details of this construction, and specifically the proof that there is this isometry between the space of elementary integrands and the space of martingales. This has been proven in [BGP12] in a general case, and involves the construction of a complex-valued Walsh martingale measure. We omit the details here, since they would complicate the exposition and they have few relevance in the following.

An interesting question is to determine which kind of processes belong to $\mathcal{P}_0$. This is a crucial question in order to determine which kind of equations driven by W^H we will be able to solve. We will present now two criteria of integrability, one for $H \geq \frac{1}{2}$ and the other one for $H < \frac{1}{2}$.

Theorem 2.17 ([DaQu11], Proposition 2.9)**.** *Suppose $H \in [\frac{1}{2}, 1)$. Let $\Gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that, for all $t \in (0,T]$, $\Gamma(t)$ is a non-negative function with rapid decrease and such that its Fourier transform $\mathcal{F}\Gamma(t,\cdot)$ (computed in $\mathcal{S}'(\mathbb{R})$) satisfies*

$$c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}\Gamma(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt < \infty.$$

Moreover, we assume that $\Gamma(t,dx)dt$ defines a non-negative measure such that

$$\sup_{t\in[0,T]} \Gamma(t,\mathbb{R}) < \infty.$$

Let $Z : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a predictable stochastic process such that it holds

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|Z(t,x)|^2\Big] < \infty.$$

Then, the stochastic process $\{S = S(t,x) := Z(t,x)\Gamma(t,x),\, (t,x) \in [0,T]\times\mathbb{R}\}$ belongs to $\mathcal{P}_0$ and, if Z satisfies, for some $p \geq 2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|Z(t,x)|^p\Big] < \infty,$$

then we have the following Burkholder-Davis-Gundy inequality

$$\mathrm{E}\Big[|(S\cdot W^H)_t|^p\Big] \leq z_p (\nu_t)^{\frac{p}{2}-1} \int_0^t ds \Big(\sup_{x\in\mathbb{R}} \mathrm{E}\Big[|Z(s,x)|^p\Big]\Big) \int_{\mathbb{R}} c_H |\mathcal{F}\Gamma(s,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi, \tag{2.23}$$

where the constant z_p is the constant appearing in the classical Burkholder-Davis-Gundy inequality for continuous martingales, and ν_t is given by

$$\nu_t := c_H \int_0^t ds \int_{\mathbb{R}} |\mathcal{F}\Gamma(s,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi.$$

Remark 2.18. Theorem 2.17 holds also for a general spectral measure μ, provided that the measure satisfies the so-called *Dalang condition*

$$\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\, \mu(d\xi) < \infty. \tag{2.24}$$

This is implicitly satisfied for our measure μ_H, for every $H \in [\frac{1}{2}, 1)$. Indeed, the measure μ_H satisfies (2.24) for every $H \in (0,1)$.

Theorem 2.19 ([BJQ15], Theorem 2.9)**.** *Suppose $H \in (0, \frac{1}{2})$. Let $S : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a predictable function. Assume that for every (ω, t), $S(\omega, t, \cdot)$ is a tempered function, whose Fourier transform $\mathcal{F}S(\omega, t, \cdot)$ (computed in $\mathcal{S}'(\mathbb{R})$) is a locally integrable function such that*

$$I(T) := \mathrm{E}\Big[c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\Big] < \infty.$$

Then, $S \in \mathcal{P}_0$ and it holds the isometry property $\mathrm{E}\Big[|(S \cdot W^H)_T|^2\Big] = I(T)$. Moreover, we have the Burkholder-Davis-Gundy inequality

$$\mathrm{E}\Big[|(S \cdot W^H)_T|^p\Big] \le z_p (c_H)^{\frac{p}{2}} \mathrm{E}\Big[\int_0^T \int_{\mathbb{R}} |\mathcal{F}S(t,\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi dt\Big]^{\frac{p}{2}}, \tag{2.25}$$

where the constant z_p is the constant appearing in the classical Burkholder-Davis-Gundy inequality for continuous martingales.

Remark 2.20. The two Theorems 2.17 and 2.19 have a similar thesis, i.e. the integrability for a certain class of predictable processes and a Burkholder-Davis-Gundy inequality for these processes. But there is an important difference: Theorem 2.17, which is valid when $H \ge \frac{1}{2}$, gives a more flexible type of estimate; indeed, the Fourier transform in the Burkholder inequality is computed only for the deterministic part Γ of the integrand process S. This quantity can be often computed explicitly. This is not the case in Theorem 2.19, which works under the hypothesis $H < \frac{1}{2}$. Indeed, in this case the Fourier transform has to be computed for the whole integrand S. We will see later how this makes our calculations more involved whenever $H < \frac{1}{2}$.

Theorem 2.19 can be rewritten in an equivalent non-spectral form thanks to the following:

Proposition 2.21 ([BJQ15], Proposition 2.8)**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a tempered function whose Fourier transform in $\mathcal{S}'(\mathbb{R})$ is a locally integrable function. Then, for any $H \in (0, \frac{1}{2})$*

$$c_H \int_{\mathbb{R}} |\mathcal{F}g(\xi)|^2 |\xi|^{1-2H} d\xi = C_H \int_{\mathbb{R}^2} |g(x) - g(y)|^2 |x-y|^{2H-2} dy dx, \tag{2.26}$$

whenever one of the two integrals is finite. The constant C_H is given by $C_H = H(1-2H)/2$.

Remark 2.22. We rewrite the quantities appearing in Theorem 2.19 in an equivalent form as

$$I(T) := \mathrm{E}\Big[C_H \int_0^T \int_{\mathbb{R}^2} |S(t,x) - S(t,y)|^2 |x-y|^{2H-2} dy dx dt\Big] < \infty$$

and

$$\mathrm{E}\Big[|(S \cdot W^H)_T|^p\Big] \le z_p (C_H)^{\frac{p}{2}} \mathrm{E}\Big[\int_0^T \int_{\mathbb{R}} |S(t,x) - S(t,y)|^2 |x-y|^{2H-2} dy dx dt\Big]^{\frac{p}{2}}.$$

This is consistent with the original version of these estimates which appears in Theorem 2.9 of [BJQ15].

2.2.6 Relation between Itô integral and Skorohod integral

As we already mentioned, the integral $(S \cdot W^H)_t$ defined in Subsection 2.2.5 can be related to the integral $\delta(u)$ defined in the previous Subsection 2.2.4 by the following Theorem 2.27, which is an extension of Theorem 4.2 of [BJQ17] to the case $H \geq \frac{1}{2}$. To do this, we need some preliminaries.

First, let $A \in \mathcal{B}([0,\infty))$. We define for every element $f \in \mathcal{H}_H^{\otimes n}$ the element $f 1_A^{\otimes n} \in \mathcal{H}_H^{\otimes n}$ in the following way: if f is a function, we define it naturally as the function $f 1_A^{\otimes n}$. If f is a general element of $\mathcal{H}_H^{\otimes n}$, we take any sequence $\{f_k,\, k \in \mathbb{N}\}$ of functions in $\mathcal{H}_H^{\otimes n}$ such that $f_k \to f$ in $\mathcal{H}_H^{\otimes n}$ as $k \to \infty$. We define then

$$f 1_A^{\otimes n} := \lim_{k \to \infty} f_k 1_A^{\otimes n}.$$

This limit exists; indeed, we have that f_k is Cauchy in $\mathcal{H}_H^{\otimes n}$, and then

$$||f_k 1_A^{\otimes n} - f_\ell 1_A^{\otimes n}||_{\mathcal{H}_H^{\otimes n}} \leq ||f_k - f_\ell||_{\mathcal{H}_H^{\otimes n}}$$

implies that $f_k 1_A^{\otimes n}$ is also Cauchy in $\mathcal{H}_H^{\otimes n}$. The limit clearly does not depend on the chosen approximating sequence. We recall now some results from [BJQ17] that we will use in the proof.

Lemma 2.23 (Lemma A.1, [BJQ17])**.** *Let $F \in L^2(\Omega)$ with Wiener Chaos expansion given by $F = \sum_n I_n^H(f_n)$, where the $f_n \in \mathcal{H}_H^{\otimes n}$ are symmetric, and let $A \in \mathcal{B}([0,\infty))$. Then it holds*

$$\mathrm{E}[F|\mathcal{F}_A] = \sum_{n \geq 0} I_n^H(f_n 1_A^{\otimes n}).$$

Proof. The proof is exactly like the one in [BJQ17]. We only observe that if $h \in \mathcal{H}_H^{\otimes n}$ is symmetric, it can be written as the limit of a sequence of symmetric functions, which in turn can be written as the limit of linear combinations of functions of the type $f^{\otimes n}$, where $f \in \mathcal{H}_H$ and $||f||_{\mathcal{H}_H} = 1$. □

Lemma 2.24 (Proposition 1.3.3, [Nua])**.** *Let $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}(\delta)$ such that $Fu \in L^2(\Omega; \mathcal{H}_H)$. Then, $Fu \in \mathrm{Dom}(\delta)$ and it holds*

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H.$$

Lemma 2.25 (Proposition 1.3.6, [Nua])**.** *Let $u \in L^2(\Omega; \mathcal{H}_H)$ and $\{u_n,\, n \geq 1\} \subset \mathrm{Dom}(\delta)$ such that $\mathrm{E}\Big[||u_n - u||^2_{\mathcal{H}_H}\Big] \to 0$ as $n \to \infty$. Suppose that there exists a random variable $G \in L^2(\Omega)$ such that, for all $F \in \mathcal{S}$,*

$$\mathrm{E}\Big[\delta(u_n)F\Big] \to \mathrm{E}[GF].$$

Then $u \in \mathrm{Dom}(\delta)$ and $\delta(u) = G$.

We now define the contraction $\otimes_1$: let $h \in \mathcal{H}_H^{\otimes n}$. We define, for an element of the canonical basis of $\mathcal{H}_H^{\otimes n}$,

$$(e_1 \otimes \cdots \otimes e_n) \otimes_1 h := (e_1 \otimes \cdots \otimes e_{n-1}) \langle e_n, h \rangle_H,$$

and we extend it to a generic $f \in \mathcal{H}_H^{\otimes n}$ by linearity and density. We have the following lemma:

Lemma 2.26 (Theorem 4.3.8, [Stu04])**.** *Let $F = \sum_n I_n^H(f_n)$, with $f_n \in \mathcal{H}_H^{\otimes n}$ symmetric. Then $F \in \mathbb{D}^{1,2}$ if and only if*

$$\sum_{n \geq 1} n\, n! ||f_n||^2_{\mathcal{H}_H^{\otimes n}} < \infty,$$

and in this case, for every $h \in \mathcal{H}_H$ it holds

$$\langle DF, h \rangle_H = \sum_{n \geq 1} n I_{n-1}^H(f_n \otimes_1 h).$$

Theorem 2.27. *Let $H \in [\frac{1}{4}, 1)$ and let $u = \{u(t,x),\ (t,x) \in [0,\infty) \times \mathbb{R}\}$ be a stochastic process such that, restricted to $t \in [0,T]$, belongs to $\mathcal{P}_0$. Then for any $t \in [0,\infty)$, $u1_{[0,t]} \in \mathrm{Dom}(\delta)$ and its Skorohod integral coincides with the Itô integral, that is*

$$\int_0^\infty \int_{\mathbb{R}} u(s,x)1_{[0,t]}(s)W^H(\delta s, \delta x) = \int_0^t \int_{\mathbb{R}} u(s,x)W^H(ds,dx)$$

Proof. The proof is an adaptation of the one of Theorem 4.2 of [BJQ17]. The only difference is that here the general element of $\mathcal{H}_H$ is not a function, since the space $\mathcal{H}_H$ also contains genuine distributions for $H \geq \frac{1}{2}$. Referring to the proof given in the appendix of [BJQ17], we only have to adapt the proof of Case 1, since the general case can be carried out identically.

Let $g = g(s,x,\omega) = Y(\omega)1_{[a,b]}(s)1_{[u,v]}(x)$, where we assume Y to be bounded, $\mathcal{F}_a = \mathcal{F}_{[0,a]}$-measurable, and $Y \in \mathbb{D}^{1,2}$. We have to check that $g1_{[0,t]} \in \mathrm{Dom}(\delta)$ and that it holds $\delta(g1_{[0,t]}) = (g \cdot W^H)_t$, which is our thesis for the type of functions g we are considering. Notice first that $g1_{[0,t]} = Y1_{[a\wedge t, b\wedge t]\times[u,v]}$. Since $Y \in \mathbb{D}^{1,2}$ and $1_{[a\wedge t,b\wedge t]\times[u,v]} \in \mathrm{Dom}(\delta)$, we can apply Lemma 2.24 to conclude that $g1_{[0,t]} \in \mathrm{Dom}(\delta)$ and

$$\delta(g1_{[0,t]}) = Y\delta(1_{[a\wedge t,b\wedge t]\times[u,v]}) - \langle DY, 1_{[a\wedge t,b\wedge t]\times[u,v]}\rangle_H,$$

if the right-hand side belongs to $L^2(\Omega)$. We have that $Y\delta(1_{[a\wedge t,b\wedge t]\times[u,v]}) \in L^2(\Omega)$, and if we show that $\langle DY, 1_{[a\wedge t,b\wedge t]\times[u,v]}\rangle_H = 0$ we have finished.

We prove it: let us denote $h := 1_{[a\wedge t,b\wedge t]\times[u,v]}$ to simplify the notation. Since Y is $\mathcal{F}_a$-measurable, we have

$$Y = \mathrm{E}[Y|\mathcal{F}_a] = \sum_{n\geq 0} I_n^H(g_n 1_{[0,a]}^{\otimes n}),$$

for some symmetric $g_n \in \mathcal{H}_H^{\otimes n}$, and then, thanks to Lemma 2.26 we have that

$$\langle DY, h\rangle_H = \sum_{n\geq 0} I_n^H(g_n 1_{[0,a]}^{\otimes n} \otimes_1 h).$$

But now we notice that $g1_{[0,a]}^{\otimes n} \otimes_1 h = 0$, for all $g \in \mathcal{H}_H^{\otimes n}$. Indeed, we show it for $g = e^{\otimes n}$, where $e \in \mathcal{H}_H$ is a function. We have that

$$e^{\otimes n}1_{[0,a]}^{\otimes n} \otimes_1 h = e^{\otimes(n-1)}1_{[0,a]}^{\otimes(n-1)}\langle e1_{[0,a]}, h\rangle_H,$$

but we have that

$$\langle e1_{[0,a]}, h\rangle_H = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}e(s,\cdot)(\xi)1_{[0,a]}(s)\overline{\mathcal{F}1_{[u,v]}(\xi)}1_{[a\wedge t,b\wedge t]}(s)d\xi ds = 0.$$

This can be extended to a generic element in $\mathcal{H}_H^{\otimes n}$ by linearity and density.

The proof is then extended to the general element $u1_{[0,t]} \in \mathcal{P}_0$, following exactly the same steps of Theorem 4.2 of [BJQ17].

□

Remark 2.28. We want to remark that the hypothesis that $u \in \mathcal{P}_0$ implies (almost by definition) that u is adapted, so that it makes sense to compare the two integrals with this restriction. We can say that, when we restrict the Skorohod integral to the class of adapted processes, it coincides with the Itô integral.

2.3 Existence and uniqueness of solution

Consider the equation (2.1) driven by W^H:

$$Lu^H(t,x) = b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x), \tag{2.27}$$

where the operator L is either the wave operator $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ or the heat operator $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, giving rise to

$$\frac{\partial^2 u^H}{\partial t^2}(t,x) = \frac{\partial^2 u^H}{\partial x^2}(t,x) + b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x), \tag{SWE}$$

and

$$\frac{\partial u^H}{\partial t}(t,x) = \frac{\partial^2 u^H}{\partial x^2}(t,x) + b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x), \tag{SHE}$$

which will be the two equations that we will study. By a solution to (2.27) we mean a mild solution, as defined in Definition 2.3. This means that we ask a solution u^H to (2.27) to be an adapted process with respect to the filtration $\{\mathcal{F}_t, t \in [0,T]\}$ defined in Remark 2.13, jointly measurable and such that for every $(t,x) \in [0,T] \times \mathbb{R}$ it holds

$$u^H(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}} b(u^H(s,y))G_{t-s}(x-y)dy\,ds + \int_0^t \int_{\mathbb{R}} \sigma(u^H(s,y))G_{t-s}(x-y)W^H(ds,dy), \tag{2.28}$$

where G is the fundamental solution associated to L and I_0 is the deterministic solution of the PDE given by $Lu = 0$, with the same initial conditions that we impose on our stochastic problem. The explicit form of I_0 is given by:

$$I_0(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}(u_0(x+t) - u_0(x-t))$$

for the wave equation (where have to impose $u_0(x) := u^H(0,x)$ and $v_0(x) := \frac{\partial}{\partial t}u^H(0,x)$), and by:

$$I_0(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy$$

for the heat equation, where we only have to impose $u_0(x) := u^H(0,x)$. From now on, we will refer to (2.28) as the general form of our equation, but the reader should keep in mind that, unless explicitly stated, we are considering both cases of (SWE) and (SHE) at the same time, with the natural changes on I_0 and G.

We will work under 3 different sets of hypotheses, which are relative to the form of the functions b, σ, and to the initial conditions u_0 and v_0.

Hypothesis A: [Linear additive case]

We assume that $b \equiv 0$ and $\sigma \equiv 1$. Regarding the initial conditions, we assume

(a) *Wave equation*: u_0 is continuous and $v_0 \in L^1_{loc}(\mathbb{R})$.

(b) *Heat equation*: u_0 is continuous and bounded.

Hypothesis B: [Semilinear additive case]

We assume that $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, and $\sigma \equiv 1$. For the initial conditions, we assume

(a) *Wave equation*: u_0 and v_0 are H-Hölder continuous and bounded.

(b) *Heat equation*: u_0 is H-Hölder continuous and bounded.

Hypothesis C: [Linear multiplicative case]

We assume that $b \equiv 0$, and $\sigma(u) = u$. The initial conditions are

(a) *Wave equation*: $u_0 \equiv \eta$ and $v_0 \equiv 0$, for some $\eta \in \mathbb{R}$.

(b) *Heat equation*: $u_0 \equiv \eta$, for some $\eta \in \mathbb{R}$.

Let us postpone, for the moment, the discussion about the initial conditions. The need for different type of initial conditions comes, indeed, from technical reasons, and it does not have a great interest for us, apart from the fact that we will be trying to use the most general hypotheses that we can. The specific hypotheses will be discussed in Sections 2.4.3, 2.4.4 and 2.4.5. When proving our main result, we will also have to introduce a slightly modified version of Hypothesis B, which we define later as Hypothesis B1 (see Section 2.4.4).

Remark 2.29. We explain the motivation of the names: the term *linear* comes from the fact that $b \equiv 0$, and thus the SPDE, if we remove the noise term W^H, is a PDE of the form $Lu = 0$. We termed *semilinear* the case in which $b \neq 0$, consistently with the PDE literature. The term *additive* in Hypothesis A and B comes from the fact that the noise $W^H(t,x)$ enters into the equation without any dependence on the current values of the solution $u^(t,x)$. On the other hand, we define as *multiplicative* the case in which the noise term depends linearly on the solution u^H. In Hypothesis C, the term *linear* is also relative to the fact that $\sigma(u) = u$ is a linear function.

Each of these sets of hypotheses gives rise to different difficulties. We will now discuss each of these cases, stating an explicit result on existence and uniqueness of a solution for (2.28) in the specific case.

2.3.1 Linear additive case

In the Linear additive case (Hypothesis A), equation (2.27) reads

$$Lu^H(t,x) = \dot{W}^H(t,x),$$

and the mild fomulation (2.28) reads

$$u^H(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy). \tag{2.29}$$

We observe two things: first, the mild formulation in this case is an explicit formulation, since the solution u^H only appears on the left-hand side. Thus, it is sufficient to see that the right-hand side is well defined to conclude that a solution exists and is unique. Secondly, the integral appearing in the right-hand side is a Wiener integral, since G is a deterministic function. Thus, there is no need to define stochastic integrals with respect to W^H to give a solution theory for this equation under Hypothesis A. This will allow us to have existence and uniqueness for every $H \in (0,1)$. Conditions (a) and (b) in Hypothesis A easily imply that there exists a unique and continuous solution $I_0 : [0,T] \times \mathbb{R} \to \mathbb{R}$ of the deterministic associated problem $Lu^H = 0$. Summarizing, it holds the following basic result

Theorem 2.30. *Let $H \in (0,1)$ and suppose we are under Hypothesis A. For every $T > 0$ there exists a unique solution $u^H = \{u^H(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ of equation* (2.29) *Moreover, the random field u^H admits a modification with continuous sample paths.*

Proof. To show the existence and uniqueness of a solution, it is sufficient to observe that the Wiener integral appearing in (2.29) is well-defined and that the solution of the deterministic equation I_0 exists and is unique under our current hypothesis. The first fact is an immediate consequence of Lemma 2.46, which is stated later, while the second relies on the results in Section 4 of [DaQu11].

We are only left to prove that the solution u^H has a modification with continuous paths. Indeed, since I_0 is deterministic and continuous, we check that the stochastic convolution $\tilde{u}^H(t,x) := u^H(t,x) - I_0(t,x)$ admits a continuous modification. This is a direct consequence of Step 1 in the proof of Theorem 2.45 in Section 2.4.3. More precisely, for any $p \geq 2$, there exists a constant C (depending only on p) such that, for all $t, t' \in [0,T]$ and $x, x' \in \mathbb{R}$, it holds

$$\mathrm{E}\Big[|\tilde{u}^H(t,x) - \tilde{u}^H(t',x')|^p\Big] \leq C\left\{|t-t'|^{\alpha p} + |x-x'|^{pH}\right\},$$

where $\alpha = H$ for the wave equation and $\alpha = \frac{H}{2}$ for the heat equation. An application of Kolmogorov's continuity criterion concludes the proof. □

Remark 2.31. In the case of the heat equation, the assumptions of Theorem 2.30 indeed imply that, for all $p \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] < \infty.$$

For the wave equation, this property can be obtained by slightly strengthening the hypotheses of u_0 and v_0, e.g. assuming that they are bounded functions (see [DaQu11], Lemma 4.2).

Remark 2.32. The proof of Theorem 2.30 actually implies that the stochastic convolution in equation (2.29) has a modification which is (locally) β_1-Hölder continuous in time for any $\beta_1 \in (0,\alpha)$, with α as before, and (locally) β_2-Hölder continuous in space for any $\beta_2 \in (0,H)$.

2.3.2 Semilinear additive case

In the semilinear additive case (Hypothesis B), the formal equation (2.27) reads

$$Lu^H(t,x) = b(u^H(t,x)) + \dot{W}^H(t,x),$$

and the mild formulation (2.28) is given by

$$u^H(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b(u^H(s,y)) dy\, ds + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy). \quad (2.30)$$

In this case, the integral equation is an implicit equation, but still there is no genuine stochastic integral appearing. This allows us to prove existence and uniqueness the equation for every $H \in (0,1)$. Our result about existence and uniqueness for (2.30) is a special case of Theorem 13 of [Dal99]. In that reference, the author considers only the case $H \geq \frac{1}{2}$, with a null initial condition, while we use it with the general initial conditions given in Hypothesis B. For the sake of completeness, we give a complete proof of the result in our setting.

Theorem 2.33. *Let $H \in (0,1)$, and assume we are in the setting of Hypothesis B. Let $p \geq 2$ and $T > 0$. Then, equation (2.30) has a unique solution u^H in the space of $L^2(\Omega)$-continuous and adapted stochastic processes satisfying*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] < \infty.$$

Proof. We follow similar arguments as those used in [Dal99]. We split the proof in four parts.

Step 1: We define the following Picard iteration scheme. For $n = 0$, we set

$$u_0^H(t,x) := I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy), \quad (2.31)$$

and for $n \geq 1$ we define

$$u_n^H(t,x) := u_0^H(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b(u_{n-1}^H(s,y)) dy\, ds. \tag{2.32}$$

Clearly, the process u_0^H is adapted and, by step 1 in Section 2.4.4, it is $L^2(\Omega)$-continuous. Then, u_0^H admits a jointly measurable modification (cf. [BQS18], Proposition B.1), which will be denoted in the same way.

Owing to Lemma 2.35, we obtain that, for every $n \geq 0$, the Picard iteration u_n^H is $L^2(\Omega)$-continuous, and thus has a jointly measurable modification. Moreover, by Lemma 2.36 below, u_n^H is uniformly bounded in $L^p(\Omega)$, i.e.

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(t,x)|^p\Big] < \infty.$$

The above two facts imply that u_n^H is well-defined, for all $n \geq 0$. On the other hand, it is clear that any Picard iteration defines an adapted process.

Step 2: We prove that the Picard iteration scheme converges in the space of $L^2(\Omega)$-continuous, adapted and $L^p(\Omega)$-uniformly bounded processes, which is a complete normed space when endowed with the norm

$$||u^H||_p = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \Big(\mathrm{E}\Big[|u^H(t,x)|^p\Big]\Big)^{1/p}.$$

Indeed, it can be seen as the closed subset formed by adapted process of the space

$$L^\infty([0,T]\times\mathbb{R}; L^p(\Omega)),$$

which is a Banach space for any $p \geq 2$.

Then, it is sufficient to show that the sequence of Picard iterations is Cauchy with respect to $||\cdot||_p$ to infer the existence of a limit.

We use that b is Lipschitz and Minkowski inequality for integrals to obtain

$$\begin{aligned}
&\Big(\mathrm{E}\Big[|u_{n+1}^H(t,x) - u_n^H(t,x)|^p\Big]\Big)^{1/p} \\
&= \Big(\mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)[b(u_n^H(s,y)) - b(u_{n-1}^H(s,y))] dy\, ds\Big|^p\Big]\Big)^{1/p} \\
&\leq C\Big(\mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)|u_n^H(s,y) - u_{n-1}^H(s,y)| dy\, ds\Big|^p\Big]\Big)^{1/p} \\
&\leq C\int_0^t \int_{\mathbb{R}} \Big(\mathrm{E}\Big[G_{t-s}(x-y)^p|u_n^H(s,y) - u_{n-1}^H(s,y)|^p\Big]\Big)^{1/p} dy\, ds \\
&\leq C\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}} \Big(\mathrm{E}\Big[|u_n^H(s',y) - u_{n-1}^H(s',y)|^p\Big]\Big)^{1/p} dy\, ds \\
&= C\int_0^t \sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}} \Big(\mathrm{E}\Big[|u_n^H(s',y) - u_{n-1}^H(s',y)|^p\Big]\Big)^{1/p} ds.
\end{aligned}$$

This inequality implies that

$$\sup_{\substack{x\in\mathbb{R},\\ s\in[0,t]}} \Big(\mathrm{E}\Big[|u_{n+1}^H(s,x) - u_n^H(s,x)|^p\Big]\Big)^{1/p} \leq C\int_0^t \sup_{\substack{y\in\mathbb{R},\\ s'\in[0,s]}} \Big(\mathrm{E}\Big[|u_n^H(s',y) - u_{n-1}^H(s',y)|^p\Big]\Big)^{1/p} ds$$

If we define

$$f_n(t) := \sup_{\substack{x\in\mathbb{R},\\ s\in[0,t]}} \Big(\mathrm{E}\Big[|u^H_{n+1}(s,x) - u^H_n(s,x)|^p\Big]\Big)^{1/p},$$

we have that

$$f_n(t) \le C\int_0^t f_{n-1}(s)ds.$$

Thanks to Lemma 2.36, we have that f_0 is a bounded function on $[0,T]$, and thus integrable. Then, by Grönwall lemma, we can conclude that $\{u^H_n\}_{n\ge0}$ defines a Cauchy sequence in the underlying space, and therefore it converges to a limit u^H, namely

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H_n(t,x) - u^H(t,x)|^p\Big] = 0.$$

Since any u^H_n is $L^2(\Omega)$-continuous and adapted, u^H has the same properties. In particular, $L^2(\Omega)$-continuity implies the existence of a joint-measurable version of u^H.

Step 3: We check that the process u^H is a solution of (2.30). To do this, we take $n\to\infty$ with respect to the uniform $L^p(\Omega)$-norm in the expression

$$u^H_{n+1}(t,x) = u^H_0(t,x) + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u^H_n(s,y))dy\,ds.$$

The left-hand side, by its definition, converges to u^H, while for the non-constant (with respect to n) part of the right-hand side, we argue as follows:

$$\begin{aligned}
&\Big(\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)(b(u^H_n(s,y)) - b(u^H(s,y)))dy\,ds\Big|^p\Big]\Big)^{1/p}\\
&\le C\Big(\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)|u^H_n(s,y) - u^H(s,y)|dy\,ds\Big|^p\Big]\Big)^{1/p}\\
&\le C\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\Big(\mathrm{E}\Big[|u^H_n(s,y) - u^H(s,y)|^p\Big]\Big)^{1/p}dy\,ds\\
&\le C\int_0^t \sup_{(s,y)\in[0,T]\times\mathbb{R}} \Big(\mathrm{E}\Big[|u^H_n(s,y) - u^H(s,y)|^p\Big]\Big)^{1/p}ds\\
&\le C\sup_{(s,y)\in[0,T]\times\mathbb{R}} \Big(\mathrm{E}\Big[|u^H_n(s,y) - u^H(s,y)|^p\Big]\Big)^{1/p}.
\end{aligned}$$

We note that the latter term converges to zero as $n\to\infty$. Thus, we have that u^H satisfies (2.30).

Step 4: Uniqueness can be checked by using analogous arguments as those used in the previous steps. □

We have the following property of the sample paths of the solution u^H.

Theorem 2.34. *Let $p\ge2$. Assume that Hypothesis B is fulfilled. Let u^H be the solution of (2.30). Then, for any $t,t'\in[0,T]$ and $x,x'\in\mathbb{R}$ such that $|t'-t|\le1$ and $|x'-x|\le1$, the following inequalities hold true:*

$$\sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u^H(t',x) - u^H(t,x)|^p\Big] \le C_p|t'-t|^{\gamma p} \tag{2.33}$$

and

$$\sup_{t\in[0,T]} \mathrm{E}\Big[|u^H(t,x') - u^H(t,x)|^p\Big] \le C_p|x'-x|^{Hp}, \tag{2.34}$$

where $\gamma = H$ for the wave equation and $\gamma = \frac{H}{2}$ for the heat equation. Hence, the process u^H has a modification whose trajectories are almost surely γ'-Hölder continuous in time, for all $\gamma' < \gamma$, and H'-Hölder continuous in space for all $H' < H$.

Proof. The bounds (2.33) and (2.34) are an easy corollary of the stronger results obtained in Step 1 of the proof of Theorem 2.45 under Hypothesis B1, given in Section 2.4.4. Indeed, we proved the same kind of estimates, but uniformly with respect to the Hurst index H, when restricted on a compact set $H \in [a, b] \subset (0, 1)$. Here, we need to obtain (2.33) and (2.34) only for a fixed $H \in (0, 1)$. $\square$

We conclude this part by stating and proving the two lemmas that we used in step 1 of the proof of Theorem 2.33 above.

Lemma 2.35. *For each $n \geq 0$, the process u_n^H defined by (2.31) and (2.32) satisfies the following. There exists a constant $C = C(n, H)$ such that, for any $t \in [0, T]$ and $h \in \mathbb{R}$ with $t+h \leq T$, it holds*

$$\sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_n^H(t+h, x) - u_n^H(t, x)|^2\Big] \leq \begin{cases} Ch^{\min(2H,1)}, & \textit{wave equation,} \\ Ch^H, & \textit{heat equation.} \end{cases} \tag{2.35}$$

and, for any $x \in \mathbb{R}$ and $h \in \mathbb{R}$ with $|h| < 1$,

$$\sup_{t\in[0,T]} \mathrm{E}\Big[|u_n^H(t, x+h) - u_n^H(t, x)|^2\Big] \leq Ch^{2H}. \tag{2.36}$$

In particular, the process u_n^H is $L^2(\Omega)$-continuous.

Proof. We proceed by induction. In the case $n = 0$, first we study the time increments. We focus on the right continuity. The computations for the left continuity are analogous. We have

$$\mathrm{E}\Big[|u_0^H(t+h, x) - u_0^H(t, x)|^2\Big] \leq 2(A_1 + A_2),$$

where

$$\begin{aligned} A_1 &= |I_0(t+h, x) - I_0(t, x)|^2 \\ A_2 &= \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} [G_{t+h-s}(x-y) - G_{t-s}(x-y)] W^H(ds, dy) \\ &\qquad\qquad + \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) W^H(ds, dy)\Big|^2\Big]. \end{aligned}$$

In Theorem 3.7 of [BJQ15], it is shown that

$$A_1 \leq \begin{cases} Ch^{2H} & \text{for the wave equation,} \\ Ch^H & \text{for the heat equation.} \end{cases}$$

Concerning the term A_2, we have

$$A_2 \leq 2(A_{2,1} + A_{2,2}),$$

where

$$\begin{aligned} A_{2,1} &= \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} [G_{t+h-s}(x-y) - G_{t-s}(x-y)] W^H(ds, dy)\Big|^2\Big], \\ A_{2,2} &= \mathrm{E}\Big[\Big|\int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) W^H(ds, dy)\Big|^2\Big]. \end{aligned}$$

These terms have been studied in the proof of Theorem 2.45 in the linear additive case (see Section 2.4.3) concretely $A_{2,1}$ corresponds to term J_1 in that theorem and term $A_{2,2}$ corresponds to I_1. So,

$$A_{2,1} \leq \begin{cases} Ch^{1+2H}, & \text{for the wave equation,} \\ Ch^{\frac{1}{2}+H}, & \text{for the heat equation,} \end{cases}$$

and

$$A_{2,2} \leq \begin{cases} Ch^{1+2H}, & \text{for the wave equation,} \\ Ch^{\frac{1}{2}+H}, & \text{for the heat equation.} \end{cases}$$

Putting together the above estimates, we obtain the validity of (2.35) for $n = 0$.

Regarding the space increments, we have, for any $h \in \mathbb{R}$ with $|h| < 1$,

$$\mathrm{E}\Big[|u_0^H(t, x+h) - u_0^H(t, x)|^2\Big] \leq 2(B_1 + B_2),$$

where

$$\begin{aligned} B_1 &= |I_0(t, x+h) - I_0(t, x)|^2, \\ B_2 &= \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h-y) - G_{t-s}(x-y)] W^H(ds, dy)\Big|^2\Big]. \end{aligned}$$

As before, by [BJQ15], Theorem 3.7, we have

$$B_1 \leq Ch^{2H}$$

for both heat and wave equations. The term B_2 corresponds to the term J_2 in the proof of Theorem 2.45 in the linear additive case (again, we refer to Section 2.4.3). Hence

$$B_2 \leq C|h|^{1+2H}.$$

So, we have proved (2.36) for $n = 0$.

We suppose now by induction hypothesis that u_n^H satisfies (2.35) and (2.36). Let us compute the time increments of u_{n+1}^H, for $0 < h << 1$:

$$\mathrm{E}\Big[|u_{n+1}^H(t+h, x) - u_{n+1}^H(t, x)|^2\Big] \leq 3(D_1 + D_2 + D_3),$$

where

$$\begin{aligned} D_1 &= \mathrm{E}\Big[|u_0^H(t+h, x) - u_0^H(t, x)|^2\Big], \\ D_2 &= \mathrm{E}\Big[\Big(\int_0^t \int_{\mathbb{R}} G_s(y)|b(u_n^H(t+h-s, x-y)) - b(u_n^H(t-s, y)|\, dy\, ds\Big)^2\Big], \\ D_3 &= \mathrm{E}\Big[\Big(\int_t^{t+h} \int_{\mathbb{R}} G_s(y)|b(u_n(t+h-s, x-y))|\, dy\, ds\Big)^2\Big]. \end{aligned}$$

We already showed that D_1 is bounded as the right hand side of (2.35), so we only need to handle D_2 and D_3. As in Lemma 19 of [Dal99], first we compute D_2. Namely, using that b is Lipschitz and applying Cauchy-Schwarz inequality and Fubini theorem, we have

$$\begin{aligned} D_2 &\leq C\Big(\int_0^t \int_{\mathbb{R}} G_s(y) dy\, ds\Big) \mathrm{E}\Big[\int_0^t \int_{\mathbb{R}} G_s(y)|u_n^H(t+h-s, x-y) - u_n^H(t-s, x-y)|^2\, dy\, ds\Big] \\ &\leq C\, \mathrm{E}\Big[\int_0^t \int_{\mathbb{R}} G_s(y)|u_n^H(t+h-s, x-y) - u_n^H(t-s, x-y)|^2\, dy\, ds\Big] \\ &= C \int_0^t \int_{\mathbb{R}} G_s(y) \mathrm{E}\Big[|u_n^H(t+h-s, x-y) - u_n^H(t-s, x-y)|^2\Big]\, dy\, ds \\ &\leq \begin{cases} Ch^{2H}, & \text{wave equation,} \\ Ch^{H}, & \text{heat equation.} \end{cases} \end{aligned}$$

Notice that in the last inequality we used the induction hypothesis.

Regarding D_3, we have

$$D_3 \leq C \int_t^{t+h} \int_{\mathbb{R}} \Big(1 + \mathrm{E}\Big[|u_n^H(t+h-s, x-y)|^2\Big]\Big) G_s(y) dy\, ds.$$

The uniform boundedness in $L^2(\Omega)$ of u_n^H (by Lemma 2.36) gives that

$$D_3 \leq C \int_t^{t+h} \int_{\mathbb{R}} G_s(y) dy\, ds \leq Ch,$$

for both wave and heat equations. Thus, taking into account the above estimates for J_1, J_2 and J_3, we obtain that u_{n+1}^H satisfies (2.35).

We are left to deal with the spatial increments of u_{n+1}^H. Indeed, we have

$$\mathrm{E}\Big[|u_{n+1}^H(t, x+h) - u_{n+1}^H(t,x)|^2\Big] \leq 2(K_1 + K_2),$$

where

$$K_1 = \mathrm{E}\Big[|u_0^H(t, x+h) - u_0^H(t,x)|^2\Big],$$
$$K_2 = \mathrm{E}\Big[\Big(\int_0^t \int_{\mathbb{R}} |b(u_n^H(t-s, x+h-y)) - b(u_n^H(t-s, x-y))| G_s(y) dy\, ds\Big)^2\Big].$$

The term K_1 has already been studied, and K_2 can be treated as the term J_2, obtaining that $K_2 \leq C|h|^{2H}$. So we can infer that (2.36) is fulfilled for u_{n+1}^H. □

Lemma 2.36. *Let $p \geq 2$ and $[a,b] \subset (0,1)$. Let u_n^H, $n \geq 0$, be the Picard iteration scheme defined in* (2.31) *and* (2.32)*. Then,*

$$\sup_{n\geq 0} \sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times R} \mathrm{E}\Big[|u_n^H(t,x)|^p\Big] < \infty.$$

Proof. First, we have

$$\mathrm{E}\Big[|u_0^H(t,x)|^p\Big] \leq C_p\Big(|I_0(t,x)|^p + \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds, dy)\Big|^p\Big]\Big).$$

By [DaQu11], Lemma 4.2, we have that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0(t,x)| < \infty,$$

and this is uniform in H, since we are considering the same initial conditions for every H. Regarding the stochastic term, arguing as in (2.45) and applying Lemma 2.46, we get

$$\mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) W^H(ds,dy)\Big|^p\Big] = z_p c_H^{p/2} \Big[\int_0^t \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi\, ds\Big]^{p/2}$$
$$\leq \begin{cases} C_p\left(t^{1+2H}\right)^{p/2}, & \text{wave equation,} \\ C_p\left(t^{H}\right)^{p/2}, & \text{heat equation.} \end{cases}$$

The last inequality comes from an estimate essentially identical to the one already computed in (2.46). All above constants which are dependent on H can be uniformly bounded, provided that H is in the compact interval $[a,b] \subset (0,1)$. The above considerations yield

$$\sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_0^H(t,x)|^p\Big] < \infty.$$

Next, owing to (2.32) we can infer that

$$\mathrm{E}\Big[|u_{n+1}^H(t,x)|^p\Big] \le C\Big(1+\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u_n^H(s,y))dy\,ds\Big|^p\Big]\Big).$$

If we apply Hölder inequality, we obtain

$$\begin{aligned}
&\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u_n^H(s,y))dy\,ds\Big|^p\Big] \\
&\quad\le C\mathrm{E}\Big[\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\Big(1+|u_n^H(s,y))|^p\Big)dy\,ds\Big] \\
&\quad= C_1 + C_2\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|u_n^H(s,y))|^p\Big]dy\,ds \qquad (2.37)\\
&\quad\le C_1 + C_2\int_0^t\int_{\mathbb{R}} \sup_{H\in[a,b]}\sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(s',y))|^p\Big]G_{t-s}(x-y)dy\,ds \\
&\quad\le C_1 + C_2\int_0^t \sup_{H\in[a,b]}\sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(s',y))|^p\Big]ds.
\end{aligned}$$

The constants appearing in the previous calculations are clearly independent of H. Then, we have

$$\begin{aligned}
&\sup_{H\in[a,b]}\sup_{(t',y)\in[0,t]\times\mathbb{R}} \mathrm{E}\Big[|u_{n+1}^H(t',y)|^p\Big] \\
&\qquad\qquad\le C_1 + C_2\int_0^t \sup_{H\in[a,b]}\sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(s',y))|^p\Big]ds.
\end{aligned}$$

We conclude the proof by applying Grönwall lemma. □

2.3.3 Linear multiplicative case

We move now to the linear multiplicative case (Hypothesis C). In this case, the hypothesis that we impose implies that $I_0(t,x)=\eta$, for all $(t,x)\in[0,T]\times\mathbb{R}$. The formal equation (2.27) has the form

$$Lu^H(t,x) = u^H(t,x)\dot{W}^H(t,x),$$

and the mild formulation (2.28) reads

$$u^H(t,x) = \eta + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u^H(s,y)W^H(ds,dy). \qquad (2.38)$$

In this case, the integral appearing in (2.38) is a genuine stochastic integral, and for the moment we will consider the Itô's type integral, i.e. the one defined in Section 2.2.5. The solution theory for equations of this form has been developed in [Dal99] and [DaQu11] for the case $H\in(\frac{1}{2},1)$ and in [BJQ15] and [HHLNT17] for $H\in(\frac{1}{4},\frac{1}{2})$. We will see later in Theorem 2.42 how this notion of solution relates to the Skorohod notion of solution, when the integral in (2.38) is interpreted in the sense of Skorohod, as defined in Section 2.2.4.

The fact that the integral in (2.38) is really a stochastic integral does not allow us to solve the equation when $H<\frac{1}{4}$. We report here the relevant existence and uniqueness results:

Theorem 2.37 ([DaQu11], Theorem 4.3)**.** *Let $H\in[\frac{1}{2},1)$. Consider the setting of Hypothesis C. Then, there exists a unique mild solution u^H to equation* (2.38)*. Moreover, the solution u^H is $L^2(\Omega)$-continuous and satisfies, for every $p\ge 1$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] < \infty.$$

Remark 2.38. In [DaQu11], the result is stated for a general second-order differential operator L and for a general noise X. The case of the heat and wave operators fall under their hypotheses, and so does the noise W^H, provided that $H \geq \frac{1}{2}$. This result, indeed, holds also for the more general form of equation

$$Lu^H(t,x) = b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x),$$

whenever b, σ are Lipschitz continuous functions. We will not consider this more general setting, as we prefer to study the same equation for all admissible values of H.

Theorem 2.39 ([BJQ15], Theorem 1.1)**.** *Let $H \in (\frac{1}{4}, \frac{1}{2})$. Under Hypothesis C, there exists a unique mild solution u^H of* (2.38)*. Moreover, the solution u^H is $L^2(\Omega)$-continuous and satisfies, for every $p \geq 2$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] < \infty \tag{2.39}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^T \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\mathrm{E}\Big[|u^H(s,y) - u^H(s,z)|^p\Big]^{\frac{2}{p}}}{|y-z|^{2-2H}} dydzds < \infty. \tag{2.40}$$

We see that in the case $H \in (\frac{1}{4}, \frac{1}{2})$ the solution u^H satisfies, in addition to (2.39) the further constraint (2.40). This comes from the fact that, when $H \in (\frac{1}{4}, \frac{1}{2})$, we have to look for a solution of (2.38) in the space of $L^2(\Omega)$-continuous, adapted and jointly measurable processes endowed with a Sobolev's type norm which includes a term of the form (2.40).

For the moment, we have not said anything about the path continuity of the solutions. As in the linear additive and the semilinear additive cases, we have that the paths of u^H are almost surely H'-Hölder continuous in space, for every $H' < H$ for both wave and heat equation. In the time variable, we have that the solution u^H is almost surely $\frac{H'}{2}$-Hölder continuous, for every $H' < H$ in the heat equation case, and H'-Hölder continuous, for every $H' < H$, in the wave equation case. This has been proved in [BJQ16] in the case $H \in (\frac{1}{4}, \frac{1}{2})$. For the case $H \in [\frac{1}{2}, 1)$, we refer to [SaSa00, SaSa02] and to [Wal86]. Moreover, like in the semilinear additive case this result can be shown as an immediate consequence of the stronger results Proposition 2.61 and Propostion 2.68 proven in Section 2.4.5. We state a general result here for the sake of completeness.

Theorem 2.40. *Let $H \in (\frac{1}{4}, 1)$. Then, the solution u^H to* (2.38) *satisfies the same conditions of Theorem 2.34, i.e.*

$$\sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u^H(t',x) - u^H(t,x)|^p\Big] \leq C_p|t'-t|^{\gamma p}$$

and

$$\sup_{t\in[0,T]} \mathrm{E}\Big[|u^H(t,x') - u^H(t,x)|^p\Big] \leq C_p|x'-x|^{Hp},$$

where $\gamma = H$ for the wave equation and $\gamma = \frac{H}{2}$ for the heat equation. Thus, the process u^H has a modification whose trajectories are almost surely γ'-Hölder continuous in time, for all $\gamma' < \gamma$, and H'-Hölder continuous in space for all $H' < H$.

Proof. As in Theorem 2.34, the estimates we need are a consequence of the stronger result obtained in Proposition 2.61 and Proposition 2.68 in Section 2.4.5. Indeed, in that case we proved the same estimates, but uniformly with respect to H. □

We give now an equivalent result for the mild equation (2.38), where the stochastic integral is interpreted in the Skorohod sense. We have that the equation in this case reads:

$$u^H(t,x) = \eta + \int_0^\infty \int_{\mathbb{R}} 1_{[0,t]}(s) G_{t-s}(x-y) u^H(s,y) W^H(\delta s, \delta y). \tag{2.41}$$

We say in this case that u is a *Skorohod mild solution* of (2.41) if it is a process adapted with respect to $\{\mathcal{F}_t, t \in [0,\infty)\}$ which satisfies for almost every $(t,x) \in [0,T] \times \mathbb{R}$ equation (2.41). We have the following

Theorem 2.41 ([BJQ17], [HHLNT17])**.** *Let $H \in (\frac{1}{4}, 1)$ and $T > 0$. Equation (2.41) has a unique adapted solution.*

Proof. This result has been already proved in [BJQ17], Theorem 4.3 for $H \in (\frac{1}{4}, \frac{1}{2})$ in the case of wave equation. In page 49 of [HHLNT17], the authors notice that this is true also in the case of heat equation, always under the constraint $H \in (\frac{1}{4}, \frac{1}{2})$. We can extend it to the case $H \in [\frac{1}{2}, 1)$ thanks to Theorem 2.27. Indeed, Theorem 2.27 shows that Itô integral and Skorohod Integral coincide, and this implies immediately that the existence and uniqueness of a mild solution in the Itô sense implies the existence and uniqueness of a solution in the Skorohod sense. □

We are now ready to prove the extension of Theorem 4.3 of [BJQ17] that is of our interest.

Theorem 2.42. *Let $H \in (\frac{1}{4}, 1)$ and let u^H be the mild solution to (2.38) in the Itô sense. Let $\tilde{u}^H$ be the mild solution to (2.41) in the Skorohod sense. Then, the two solutions coincide. Moreover, the Picard iterations defined for $m \geq 0$ by*

$$\begin{aligned} u_0^H(t,x) :=& I_0(t,x) \\ u_{m+1}^H(t,x) :=& I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u_m^H(s,y) W^H(ds,dy) \end{aligned} \tag{2.42}$$

satisfy (up to a predictable modification)

$$u_m^H(t,x) = \sum_{j=0}^m I_j^H(g_j(\cdot,t,x)),$$

where I_j^H is the j-th multiple Wiener integral with respect to W^H and g_j is given by

$$g_n(t_1,x_1,t_2,x_2,\ldots,t_n,x_n) := G_{t-t_n}(x-x_n)\cdots G_{t_2-t_1}(x_2-x_1) \times \eta 1_{\{0<t_1<\cdots<t_n<t\}}. \tag{2.43}$$

Proof. The result has been already proven in Theorem 4.3 of [BJQ17] for the wave equation in the case $H < \frac{1}{2}$. We are extending it to the case of the heat equation, and for $H \geq \frac{1}{2}$.

We make use of Theorem 2.27, that we extended to hold under our current hypotheses. The remainder of the proof of this result can be carried out using exactly the same argument used in Theorem 4.3 of [BJQ17]. The argument translate without modifications to include the heat equation case, and the case $H \geq \frac{1}{2}$.

□

2.4 Weak continuity with respect to H

In this section we will state the main problem of our interest, that is the continuity of the solution u^H of (2.28) with respect to H.

Let $H \in (0,1)$ and let $\{W^H = W^H(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ be the random field defined in Proposition 2.11. We recall that

$$\mathrm{E}\Big[W^H(t,x)W^H(s,y)\Big] = \frac{1}{2}(t \wedge s)\Big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\Big),$$

so that it makes sense to say that W^H is a sBm in the time variable and a fBm of Hurst parameter $H \in (0,1)$ in the space variable.

We saw in the previous sections under which hypothesis there exists a unique mild solution (2.4) of a SPDE (2.1) driven by $X = W^H$.

We state informally the main problem of our interest: let u^H be a solution of (2.4) with $X = W^H$,

$$Lu^H(t,x) = b(u^H(t,x)) + \sigma(u^H(t,x))\dot{W}^H(t,x).$$

Is it true that $u^H \to u^{H_0}$, whenever $H \to H_0$? This is an analogous problem to the one we considered in Chapter 3, but in a 2-dimensional setting. We will give in Theorem 2.45 a positive answer to this question, where the convergence will be meant in the distributional sense.

2.4.1 Introduction

In any of the three settings (Hypothesis A, B, and C), consider equation (2.1) driven by W^H in its mild form (2.28), which we recall here

$$\begin{aligned} u^H(t,x) = I_0(t,x) + & \int_0^t \int_{\mathbb{R}} b(u^H(s,y)) G_{t-s}(x-y) dy\, ds \\ & + \int_0^t \int_{\mathbb{R}} \sigma(u^H(s,y)) G_{t-s}(x-y) W^H(ds,dy). \end{aligned}$$

We saw that for all sets of hypotheses A, B, C we have a result of existence and uniqueness for u^H. In the following, we have to slightly strengthen Hypothesis B. We introduce:

Hypothesis B1: [Semilinear Additive Case – Weak convergence]

We assume that $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, and $\sigma \equiv 1$. Let $H_0 \in (0,1)$ and assume $\{H_n,\, n \in \mathbb{N}\}$ is such that $H_n \to H_0$ as $n \to \infty$. Fix any $\alpha > H_0$; without loss of generality, we can assume $H_n \le \alpha$ for every $n \in \mathbb{N}$. For the initial conditions, we assume:

(a) *Wave equation*: u_0 and v_0 are α-Hölder continuous and bounded.

(b) *Heat equation*: u_0 is α-Hölder continuous and bounded.

We observe that, if we fix a single $n \in \mathbb{N}$ and we consider $H = H_n$, from the point of view of the regularity of the initial conditions Hypothesis B1 is strictly more restrictive that Hypothesis B. So all the results we obtained about existence and uniqueness of a mild solution under Hypothesis B still hold under Hypothesis B1.

The range of values of H for which such a solution exists depends on the hypotheses that we are considering. Under Hypothesis A and B1, we have that a solution exists, and has almost surely continuous paths, for every $H \in (0,1)$. Under Hypothesis C, the solution exists (and again has a.s. continuous paths) for every $H \in (\frac{1}{4}, 1)$. Indeed, speaking about the path regularity of the solutions, we know even more, namely that the paths are almost surely Hölder continuous, but for our purposes it is sufficient to exploit their bare continuity.

The fact that the solution u^H has almost surely continuous paths on $[0,T] \times \mathbb{R}$ means that, for every set of hypotheses, the solution u^H induce as a probability measure on $\mathcal{C}([0,T] \times \mathbb{R})$. We endow the space $\mathcal{C}([0,T] \times \mathbb{R})$ with the metric of uniform convergence on compact sets. Its topology is defined as follows: we have that $\{f_n,\, n \in \mathbb{N}\} \subset \mathcal{C}([0,T] \times \mathbb{R})$ converges to $f \in \mathcal{C}([0,T] \times \mathbb{R})$ as $n \to \infty$ if and only if, given any compact set $K \subset [0,T] \times \mathbb{R}$, we have that $f_n \to f$ uniformly on K. It is not immediate to show that this topology is metrizable. A reference for this fact can be found in page 68 of [McNt].

We recall also the concept of *weak convergence* (often named *convergence in distribution*, or *convergence in law*, in the probabilistic setting).

Definition 2.43. Let $\{P_n,\, n \in \mathbb{N}\}$ be a sequence of probability measures on some metric space $(S, \mathcal{S})$, where $\mathcal{S}$ is the σ-algebra generated by Borel sets of S. We say that P_n *converges weakly* to P (or, equivalently, *converges in distribution*) on S if for any bounded, continuous function $f : S \to \mathbb{R}$ one has

$$P_n(f) := \int_S f dP_n \xrightarrow{n\to\infty} P(f) := \int_S f dP,$$

and we denote it with $P_n \xrightarrow{d} P$.

Remark 2.44. With an abuse of notation, we will often speak about the weak convergence of sequences of S-valued random elements X_n, instead of the probability distributions P_{X_n} that they induce on S. In that case, by $X_n \xrightarrow{d} X$ we will mean $P_{X_n} \xrightarrow{d} P_X$.

We are ready to state the main result of this chapter

Theorem 2.45 ([GJQ20], [GJQ19])**.** *Consider the SPDE* (2.1) *driven by $X = W^H$, where L is either the heat operator or the wave operator. Fix any set of hypotheses: its mild formulation is given then by* (2.28)*, which translates to either* (2.29), (2.30) *or* (2.38) *respectively for Hypothesis A, B1 or C. Furthermore:*

- *i) Under Hypothesis A or B1, let $H_0 \in (0,1)$ and suppose $\{H_n,\, n \in \mathbb{N}\} \subset (0,1)$ such that $H_n \xrightarrow{n\to\infty} H_0$.*
- *i') Under Hypothesis C, let $H_0 \in (\frac{1}{4}, 1)$ and suppose $\{H_n,\, n \in \mathbb{N}\} \subset (\frac{1}{4}, 1)$ such that $H_n \xrightarrow{n\to\infty} H_0$.*

Then, it holds that $u^{H_n} \xrightarrow{d} u^{H_0}$ on $\mathcal{C}([0,T] \times \mathbb{R})$, as $n \to \infty$.

2.4.2 Auxiliary results

In the proof of the main result of the present chapter (Theorem 2.45) we need some technical results. We start with 4 lemmas, proved in [BJQ15], which provide explicit estimates, depending on H, of the norm in the space $L^2(\mathbb{R}; \mu^H)$ of the Fourier transforms of the fundamental solutions of the deterministic wave and heat equations. We recall that, respectively for the wave equation and for the heat equation, we have:

$$\mathcal{F}G_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|} \qquad \text{and} \qquad \mathcal{F}G_t(\xi) = \exp\Big(\frac{-t\xi^2}{2}\Big), \quad t > 0,\, \xi \in \mathbb{R}. \tag{2.44}$$

In the following three lemmas, we will denote either one of these two functions by $\mathcal{F}G_t(\xi)$. We recall that the spatial spectral measure is given by $\mu^H(d\xi) = c_H|\xi|^{1-2H}d\xi$ (see (2.9)).

Lemma 2.46 ([BJQ15], Lemma 3.1)**.** *Let $T > 0$. Then, the integral*

$$A_T(\alpha) := \int_0^T \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt$$

converges if and only if $\alpha \in (-1, 1)$. In this case, it holds:

$$A_T(\alpha) = \begin{cases} 2^{1-\alpha} C_\alpha \dfrac{1}{2-\alpha} T^{2-\alpha} & \text{for the wave equation,} \\[2ex] \dfrac{2}{1-\alpha}\Gamma\Big(\dfrac{\alpha+1}{2}\Big) T^{(1-\alpha)/2} & \text{for the heat equation,} \end{cases}$$

where the constant C_α is given by

$$C_\alpha = \begin{cases} \dfrac{\Gamma(\alpha)}{1-\alpha}\sin(\pi\alpha/2), & \alpha \in (-1,1) \setminus \{0\}, \\[2ex] \dfrac{\pi}{2}, & \alpha = 0. \end{cases}$$

Lemma 2.47 ([BJQ15], Lemma 3.4). *Let $T > 0$ and $\alpha \in (-1, 1)$. Then, for any $h > 0$, it holds:*

$$\int_0^T \int_{\mathbb{R}} (1 - \cos(\xi h)) \, |\mathcal{F}G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt \leq \begin{cases} C|h|^{1-\alpha} & \text{for the heat equation,} \\ CT|h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where $C = \int_{\mathbb{R}} (1 - \cos \eta)|\eta|^{\alpha-2} d\eta$.

Lemma 2.48 ([BJQ15], Lemma 3.5). *Let $T > 0$ and $\alpha \in (-1, 1)$. Then, for any $h > 0$, it holds:*

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t+h}(\xi) - \mathcal{F}G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt \leq \begin{cases} C_\alpha |h|^{(1-\alpha)/2} & \text{for the heat equation,} \\ C_\alpha T|h|^{1-\alpha} & \text{for the wave equation,} \end{cases}$$

where

$$C_\alpha = \int_{\mathbb{R}} \frac{(1 - e^{-\eta^2/2})^2}{|\eta|^{2-\alpha}} d\eta \quad \text{for the heat equation, and}$$

$$C_\alpha = 4 \int_{\mathbb{R}} \frac{\min(1, |\eta|^2)}{|\eta|^{2-\alpha}} d\eta \quad \text{for the wave equation.}$$

Lemma 2.49 ([BJQ15], Lemma D.2). *For any $H \in (0, \frac{1}{2})$ and for any $\xi \in \mathbb{R}$, we have:*

$$\int_{\mathbb{R}} \frac{|1 - e^{-i\xi x}|^2}{|x|^{2-2H}} dx = |\xi|^{1-2H} \frac{2\Gamma(2H + 1) \sin(\pi H)}{H(1 - 2H)}$$

We recall now briefly some basic probabilistic results about the tightness property of a set of measures $\{P_n, \, n \in \mathbb{N}\}$. Tightness is a useful tool when proving weak convergence, since it implies the relative compactness of the set of measures $\{P_n, \, n \in \mathbb{N}\}$ by the well-known Prohorov's Theorem. We make it more precise.

Definition 2.50. Let $(S, \mathcal{S})$ a metric space endowed with the σ-algebra $\mathcal{S}$ generated by Borel sets in S. Consider a sequence of probability measures $\{P_n, \, n \in \mathbb{N}\}$ on S. We say that the family $\{P_n, \, n \in \mathbb{N}\}$ on S is *tight* if, for any $\varepsilon > 0$, there exists a compact set $K \subset S$ such that

$$\sup_{n \in \mathbb{N}} P_n(K) < \varepsilon.$$

Definition 2.51. Let $(S, \mathcal{S})$ a metric space endowed with the σ-algebra $\mathcal{S}$ generated by Borel sets in S. Let $\{P_n, \, n \in \mathbb{N}\}$ be a family of probability measures on S. We say that the family $\{P_n, \, n \in \mathbb{N}\}$ is relatively compact if it contains a subsequence $\{P_{n_k}, \, k \in \mathbb{N}\}$ which converges weakly to some probability measure P on S, as $k \to \infty$.

Tightness is related to relative compactness by the following:

Theorem 2.52 (Prohorov's Theorem. [Bil], Theorem 5.1). *Let $(S, \mathcal{S})$ a metric space endowed with the σ-algebra $\mathcal{S}$ generated by Borel sets in S. Let $\{P_n, \, n \in \mathbb{N}\}$ be a family of probability measures on S. If $\{P_n, \, n \in \mathbb{N}\}$ is tight, then it is relatively compact.*

Remark 2.53. The converse implication in Prohorov's Theorem also holds, provided that S is separable and complete. Anyway, we do not need it and we will not take care of this fact in the following.

We remark that in our case we will have $S = \mathcal{C}([0, T] \times \mathbb{R})$. In this setting we have the following criterion for tightness (see [Yor83], Proposition 2.3):

Theorem 2.54. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of random functions indexed on the set Λ and taking values in the space $\mathcal{C}([0,T]\times\mathbb{R})$, in which we consider the metric of uniform convergence over compact sets. Then, the family $\{X_\lambda\}_{\lambda\in\Lambda}$ is tight if, for any compact set $J\subset\mathbb{R}$, there exist $p',p>0$, $\delta>2$, and a constant C such that the following holds for any $t',t\in[0,T]$ and $x',x\in J$:*

(i) $\sup_{\lambda\in\Lambda}\mathrm{E}\Big[|X_\lambda(0,0)|^{p'}\Big]<\infty$,

(ii) $\sup_{\lambda\in\Lambda}\mathrm{E}\Big[|X_\lambda(t',x')-X_\lambda(t,x)|^{p}\Big]\le C\Big(|t'-t|+|x'-x|\Big)^{\delta}$.

2.4.3 Linear additive case

In this section we prove Theorem 2.45 under Hypothesis A, i.e. in the linear additive case. We recall again the mild formulation (2.29) in this case:

$$u^H(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)W^H(ds,dy),$$

and we recall also that under Hypothesis A we can consider any $H\in(0,1)$. We have from Theorem 2.30 that a solution to (2.29) exists, it is unique and has continuous paths. We are ready to start the proof of our main result.

Proof (Theorem 2.45, linear additive case). We split the proof in two steps. In the first one, we prove that the sequence of stochastic convolutions $\tilde{u}^{H_n}:=u^{H_n}-I_0$ is tight in $\mathcal{C}([0,T]\times\mathbb{R})$. By Prohorov's Theorem 2.52, this implies that the sequence of convolutions $\tilde{u}^{H_n}$ possesses a weakly converging subsequence $\tilde{u}^{H_n}$ to some limit law Y. The second step is devoted to the identification of the limit law as $Y=\tilde{u}^{H_0}=u^{H_0}-I_0$. This is the typical strategy when showing convergence in distribution of a family of processes, and we will use it again in the linear multiplicative case in Section 2.4.5.

Step 1: Since $H_n\to H_0$, the sequence $\{H_n\}$ is contained in a compact set $K\subset(0,1)$. For a fixed $H\in(0,1)$, the solution u^H is expressed as

$$u^H(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)W^H(ds,dy).$$

We apply Theorem 2.54 to the family $\{\tilde{u}^H=u^H-I_0\}_{H\in K}$ of stochastic convolutions:

$$\tilde{u}^H(t,x)=u^H(t,x)-I_0(t,x)=\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)W^H(dy,ds).$$

We write then, supposing without loss of generality that $t'\ge t$ and $x'\ge x$:

$$\begin{aligned}\tilde{u}^H(t',x')-\tilde{u}^H(t,x)=&\int_t^{t'}\int_{\mathbb{R}}G_{t'-s}(x'-y)W^H(ds,dy)\\&+\int_0^t\int_{\mathbb{R}}[G_{t'-s}(x'-y)-G_{t-s}(x-y)]W^H(ds,dy).\end{aligned}$$

Thus, we have

$$\mathrm{E}\Big[|u^H(t,x)-u^H(t',x')|^p\Big]\le C_p(I_1+I_2),$$

where I_1,I_2 are defined as:

$$I_1:=\mathrm{E}\left[\Big|\int_t^{t'}\int_{\mathbb{R}}G_{t'-s}(x'-y)W^H(ds,dy)\Big|^p\right],$$

$$I_2 := \mathrm{E}\left[\Big|\int_0^t \int_{\mathbb{R}} [G_{t-s}(x-y) - G_{t'-s}(x'-y)] W^H(ds,dy)\Big|^p\right].$$

Since I_1 is the moment of order p of a centered Gaussian random variable, we have

$$\begin{aligned}
I_1 &= \mathrm{E}\left[\Big|\int_0^T \int_{\mathbb{R}} 1_{[t,t']}(s) G_{t'-s}(x'-y) W^H(ds,dy)\Big|^p\right] \\
&= z_p\, c_H^{p/2} \left[\int_0^T 1_{[t,t']}(s) \int_{\mathbb{R}} |\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi\, ds\right]^{p/2} \\
&= z_p\, c_H^{p/2} \left[\int_t^{t'} \int_{\mathbb{R}} |\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi\, ds\right]^{p/2} \\
&= z_p\, c_H^{p/2} \left[\int_0^{t'-t} \int_{\mathbb{R}} |\mathcal{F}G_{s'}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds'\right]^{p/2}.
\end{aligned} \tag{2.45}$$

Notice that we have used the standard properties of Fourier transform in the space variable, and we performed the change of variable $s' = t' - s$. The constant z_p is the p-order moment of a standard normal distribution and c_H is given by (2.9).

Now we apply Lemma 2.46 and obtain

$$I_1 \le \begin{cases} z_p\, c_H^{p/2} \left[2^{2H} \tilde{C}_{1-2H} \dfrac{1}{1+2H} (t'-t)^{1+2H}\right]^{p/2}, & \text{wave equation,} \\[2ex] z_p c_H^{p/2} \left[\frac{1}{H}\Gamma(1-H)(t'-t)^H\right]^{p/2}, & \text{heat equation.} \end{cases} \tag{2.46}$$

The above constant $\tilde{C}_{1-2H}$ is the one of Lemma 2.46:

$$\tilde{C}_{1-2H} = \begin{cases} \dfrac{\Gamma(1-2H)}{2H} \sin\left(\pi \dfrac{1-2H}{2}\right), & H \in (0,1),\ H \neq \dfrac{1}{2}, \\[2ex] \dfrac{\pi}{2}, & H = \dfrac{1}{2}. \end{cases}$$

First, we observe that z_p is independent of H and

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \le \frac{\Gamma(3)}{2\pi} = \frac{1}{\pi}.$$

Next, as far as estimate (2.46) for the wave equation is concerned, we note that $2^{2H} \le 4$ and $\frac{1}{1+2H} \le 1$, for any $H \in (0,1)$. Thus, we concentrate on the constant $\tilde{C}_{1-2H}$, which we show to be uniformly bounded in H. Clearly, the function $\tilde{C}_{1-2H} : (0,1) \to \mathbb{R}$ has, possibly, a singularity only in $H = \frac{1}{2}$, but since $\Gamma(x) \sim \frac{1}{x}$ as $x \to 0_+$, by simple calculations we have that the function $\tilde{C}_{1-2H}$ is continuous also at the point $H = \frac{1}{2}$. Therefore, $\tilde{C}_{1-2H}$ is bounded on the set K.

On the other hand, regarding estimate (2.46) for the heat equation, we have that $\dfrac{1}{H}\Gamma(1-H)$ defines a continuous function of H on the interval $(0,1)$, and thus it is bounded on K.

We now turn to the analysis of the term I_2. More precisely, we have

$$
\begin{aligned}
I_2 &= \mathrm{E}\left[\Big|\int_0^T\int_{\mathbb{R}} 1_{[0,t]}(s)[G_{t-s}(x-y)-G_{t'-s}(x'-y)]W^H(ds,dy)\Big|^p\right]\\
&= z_p\, c_H^{p/2}\left[\int_0^T 1_{[0,t]}(s)\int_{\mathbb{R}}\Big|\mathcal{F}\Big(G_{t-s}(x-\cdot)-G_{t'-s}(x'-\cdot)\Big)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds\right]^{p/2}\\
&= z_p\, c_H^{p/2}\left[\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}G_{t-s}(x-\cdot)(\xi)-\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds\right]^{p/2}\\
&\le z_p\, c_H^{p/2}C_p\Bigg(\left[\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)-\mathcal{F}G_{t-s}(x'-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds\right]^{p/2}\\
&\qquad+\left[\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}G_{t-s}(x'-\cdot)(\xi)-\mathcal{F}G_{t-s}(x-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds\right]^{p/2}\Bigg)\\
&= z_p c_H^{p/2}C_p\Big(J_1+J_2\Big),
\end{aligned}
$$

where C_p denotes some constant depending on p. We estimate J_1 and J_2 using similar techniques as those used for the term I_1. Hence, via the change of variable $s'=t-s$, we have:

$$
J_1=\left[\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}G_{s'+(t'-t)}(x'-\cdot)(\xi)-\mathcal{F}G_{s'}(x'-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds'\right]^{p/2}.
$$

Thus, by Lemma 2.48,

$$
J_1\le\begin{cases} M_H^{p/2}\, t^{p/2}(t'-t)^{pH}\le M_H^{p/2}\, T^{p/2}(t'-t)^{pH}, & \text{wave equation,}\\[2ex] N_H^{p/2}(t'-t)^{pH/2}, & \text{heat equation.}\end{cases}
$$

The above constants are the following:

$$
\begin{aligned}
\frac{1}{4}M_H &= \int_{\mathbb{R}}\frac{\min(1,|h|^2)}{|h|^{1+2H}}dh\\
&= \int_{|h|>1}\frac{1}{|h|^{1+2H}}dh+\int_{|h|<1}\frac{1}{|h|^{2H-1}}dh\\
&= \frac{1}{H}+\frac{1}{1-H},
\end{aligned}
$$

and

$$
\begin{aligned}
N_H &= \int_{\mathbb{R}}\frac{(1-e^{-\frac{h^2}{2}})^2}{|h|^{1+2H}}dh\le\int_{\mathbb{R}}\frac{1-e^{-\frac{h^2}{2}}}{|h|^{1+2H}}dh\\
&\le \int_{|h|>1}\frac{1}{|h|^{1+2H}}dh+\int_{|h|<1}\frac{1}{|h|^{2H-1}}dh\\
&= \frac{1}{H}+\frac{1}{1-H}.
\end{aligned}
$$

The function $H\mapsto\dfrac{1}{H}+\dfrac{1}{1-H}$ is again continuous in $(0,1)$, and thus bounded for $H\in K$.

For the term J_2, we have:

$$
\begin{aligned}
J_2 &= \left[\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}G_{t-s}(x'-\cdot)(\xi)-\mathcal{F}G_{t-s}(x-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds\right]^{p/2}\\
&= \left[\int_0^t\int_{\mathbb{R}}[1-\cos(\xi(x'-x))]\Big|\mathcal{F}G_{s'}(x-\cdot)(\xi)\Big|^2|\xi|^{1-2H}d\xi\, ds'\right]^{p/2},
\end{aligned}
$$

and applying Lemma 2.47 we end up with

$$J_2 \le \begin{cases} C_H^{p/2}\, t^{p/2}(x'-x)^{pH} \le C_H^{p/2}\, T^{p/2}(x'-x)^{pH}, & \text{wave equation,} \\ \\ C_H^{p/2}(x'-x)^{pH}, & \text{heat equation.} \end{cases}$$

Here, the constant C_H is

$$C_H = \int_{\mathbb{R}} \frac{1-cos(h)}{|h|^{1+2H}} dh \le \frac{1}{H} + \frac{1}{1-H},$$

which again is a bounded function on the set K.

To sum up, we have proved that

$$\mathrm{E}\Big[|\tilde{u}^H(t,x) - \tilde{u}^H(t',x')|^p\Big] \le C\Big((t'-t)^{\alpha p} + (x'-x)^{Hp}\Big),$$

where $\alpha = H$ for the wave equation and $\alpha = \frac{H}{2}$ for the heat equation, and the constant C depends only on p and T. Thus, choosing $p > \dfrac{4}{\min_{H\in K} H}$, we have that the hypotheses of Theorem 2.54 are fulfilled by the family $\{\tilde{u}^H\}_{H\in K}$, for both for (SWE) and (SHE), and thus the family is tight on $\mathcal{C}([0,T]\times\mathbb{R})$. This concludes the first step of the proof.

Step 2: In order to identify the limit law of the sequence $\{u^{H_n}\}_{n\ge 1}$, we proceed to prove the convergence of the finite dimensional distributions of $\tilde{u}^{H_n}$ when $n\to\infty$.

We recall that, for every $H \in (0,1)$, $\tilde{u}^H = u^H - I_0$ is a centered Gaussian process, so it suffices to analyze the convergence of the corresponding covariance functions.

Let $(t,x),(t',x') \in [0,T]\times\mathbb{R}$ and suppose without loss of generality that $t' \ge t$. Then,

$$\mathrm{E}\Big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}(t',x')\Big] = c_{H_n}\int_0^t\int_{\mathbb{R}} \mathcal{F}G_{t-s}(x-\cdot)(\xi)\overline{\mathcal{F}G_{t'-s}(x'-\cdot)(\xi)}\,|\xi|^{1-2H_n} d\xi\, ds.$$

Let us first consider the case of the wave equation. Taking into account the explicit form of $\mathcal{F}G_t(\xi)$ (see (2.44)), we have

$$\mathrm{E}\Big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}(t',x')\Big] = c_{H_n}\int_0^t\int_{\mathbb{R}} \frac{e^{-i\xi(x-x')}\sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_n}} d\xi\, ds.$$

We clearly have that $c_{H_n} \to c_{H_0}$. The integrand function in the latter integral converges, as $n\to\infty$, to

$$\frac{e^{-i\xi(x-x')}\sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_0}},$$

for almost every $(s,\xi)\in[0,t]\times\mathbb{R}$. Moreover, thanks to the fact that $|\sin(z)| \le z$ for all $z\in\mathbb{R}$, its modulus is dominated by the integrable function

$$\begin{cases} \dfrac{(t-s)(t'-s)}{|\xi|^{2\sup_n(H_n)-1}}, & s\in[0,t], |\xi|\le 1, \\ \\ \dfrac{1}{|\xi|^{2\inf_n(H_n)+1}}, & s\in[0,t], |\xi|>1. \end{cases}$$

Then, by the dominated convergence theorem, we obtain that

$$\begin{aligned} \lim_{n\to\infty}\mathrm{E}\Big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}(t',x')\Big] &= c_{H_0}\int_0^t\int_{\mathbb{R}} \frac{e^{-i\xi(x-x')}\sin((t-s)|\xi|)\sin((t'-s)|\xi|)}{|\xi|^{1+2H_0}} d\xi\, ds \\ &= \mathrm{E}\Big[\tilde{u}^{H_0}(t,x)\tilde{u}^{H_0}(t',x')\Big]. \end{aligned}$$

On the other hand, in the case of the heat equation, we have

$$\mathrm{E}\Big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}(t',x')\Big] = c_{H_n}\int_0^t\int_{\mathbb{R}} \frac{e^{-i\xi(x-x')}e^{-\frac{(t-s)\xi^2}{2}}e^{-\frac{(t'-s)\xi^2}{2}}}{|\xi|^{2H_n-1}}\,d\xi\,ds. \tag{2.47}$$

The pointwise limit of the above integrand is given by

$$\frac{e^{-i\xi(x-x')}e^{-\frac{(t-s)\xi^2}{2}}e^{-\frac{(t'-s)\xi^2}{2}}}{|\xi|^{2H_0-1}},$$

for all $s\in[0,t]$ and $\xi\in\mathbb{R}$, and its modulus reads

$$\frac{e^{-\frac{(t+t'-2s)\xi^2}{2}}}{|\xi|^{2H_n-1}}.$$

Now, we use the bound

$$e^{-ax^2} < \frac{1}{ax^2},\quad \text{if } a>0,$$

with $a=(t+t'-2s)/2$ (which is always positive provided that $s\in[0,t]$). Thus

$$\frac{e^{-\frac{(t+t'-2s)\xi^2}{2}}}{|\xi|^{2H_n-1}} \le \begin{cases} \dfrac{1}{|\xi|^{2\sup_n(H_n)-1}}, & |\xi|\le 1,\ s\in[0,t],\\[2ex] \dfrac{2}{(t'-t)|\xi|^{2\inf_n(H_n)+1}}, & |\xi|>1,\ s\in[0,t]. \end{cases}$$

This covers all cases except $t=t'$. In this latter case, the modulus of the integrand appearing in (2.47) becomes

$$\frac{e^{-(t-s)\xi^2}}{|\xi|^{2H_n-1}} \le \begin{cases} \dfrac{1}{|\xi|^{2\sup_n(H_n)-1}}, & |\xi|\le 1,\ s\in[0,t],\\[2ex] \dfrac{\exp\Big(-(t-s)\xi^2\Big)}{|\xi|^{2\inf_n(H_n)-1}}, & |\xi|>1,\ s\in[0,t], \end{cases}$$

and the integrability of this function is an easy consequence of Lemma 2.46. Therefore, by the dominated convergence theorem, we also obtain that

$$\lim_{n\to\infty}\mathrm{E}\Big[\tilde{u}^{H_n}(t,x)\tilde{u}^{H_n}(t',x')\Big] = \mathrm{E}\Big[\tilde{u}^{H_0}(t,x)\tilde{u}^{H_0}(t',x')\Big],$$

which concludes Step 2 of the proof.

To finish the proof of the theorem, it remains to observe that, since the translation by I_0 is clearly a continuous mapping from $\mathcal{C}([0,T]\times\mathbb{R})$ into itself, the convergence in distribution $\tilde{u}^{H_n}\xrightarrow{d}\tilde{u}^{H_0}$ implies the convergence in distribution $u^{H_n}\xrightarrow{d}u^{H_0}$, which was our statement. □

2.4.4 Semilinear additive case

We prove now Theorem 2.45 in the semilinear additive case. We are thus under Hypothesis B1, which allows us to consider any $H\in(0,1)$. The mild formulation in this case is given by (2.30), which we recall:

$$u^H(t,x) = I_0(t,x) + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u^H(s,y))dy\,ds + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^H(ds,dy).$$

Fix $H_0 \in (0,1)$ and consider any sequence $\{H_n,\, n \in \mathbb{N}\}$ such that $H_n \to H_0$ as $n \to \infty$. We will work under hypothesis B1 in the following.

With this hypotheses, we still have that the existence and uniqueness result for (2.30) holds, as we proved in Subsection 2.3.2. We prove our main result Theorem 2.45 in this framework now. We will divide the proof in three different cases. The first one is the wave equation case. The second is the heat equation case with the additional hypothesis that the function b is bounded and Lipschiz. The third one is the heat equation case with b only Lipschitz.

In the first two cases we will use the same technique in the proof, while in the third we will exploit the result for b bounded and a limiting argument to deduce the general result. Before entering into the proofs, we give two ad-hoc versions of Grönwall lemmas that will be used in the proof. The first one is relative to the wave equation, and has been already shown in [BeJo15]

Lemma 2.55. *Let $\{f_n,\, n \geq 0\}$ be a sequence of real-valued non-negative functions defined on $[0,T] \times [a-T, b+T]$, for some $a, b \in \mathbb{R}$ such that $a < b$, and $T > 0$. Suppose that there exist $\lambda, \mu > 0$ such that, for every $(t,x) \in [0,T] \times [a,b]$ and $n \geq 0$,*

$$f_{n+1}(t,x) \leq \lambda + \frac{\mu}{2} \int_0^t \int_{x-t+s}^{x+t-s} f_n(s,y)\, dyds,$$

and that f_0 is bounded. Then, for every $n \geq 0$ and $(t,x) \in [0,T] \times [a,b]$, it holds that

$$f_n(t,x) \leq \lambda \sum_{k=0}^{n-1} \frac{(\mu t^2)^k}{k!} + ||f_0||_\infty \frac{(\mu t^2)^n}{n!}, \tag{2.48}$$

which in particular implies that

$$\limsup_{n\to\infty} f_n(t,x) \leq \lambda \exp(\mu t^2).$$

Proof. We prove it by induction: the case $n = 1$ reduces to the inequality

$$f_1(t,x) \leq \lambda + \mu t^2 ||f_0||_\infty,$$

that is clearly satisfied. We go on with the inductive step: if (2.48) holds true, then

$$\begin{aligned}
f_{n+1}(t,x) &\leq \lambda + \frac{\mu}{2} \int_0^t \int_{x-t+s}^{x+t-s} \Big[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + ||f_0||_\infty \frac{(\mu s^2)^n}{n!}\Big] dsdy \\
&= \lambda + \frac{\mu}{2} \int_0^t 2(t-s) \Big[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + ||f_0||_\infty \frac{(\mu s^2)^n}{n!}\Big] ds \\
&\leq \lambda + \mu \int_0^t t \Big[\lambda \sum_{k=0}^{n-1} \frac{(\mu s^2)^k}{k!} + ||f_0||_\infty \frac{(\mu s^2)^n}{n!}\Big] ds \\
&= \lambda + \mu \Big[\lambda \sum_{k=0}^{n-1} \frac{\mu^k (t^2)^{k+1}}{k!(2k+1)} + ||f_0||_\infty \frac{\mu^n (t^2)^{n+1}}{n!(2n+1)}\Big] \\
&= \lambda + \lambda \sum_{k=0}^{n-1} \frac{\mu^{k+1} (t^2)^{k+1}}{k!(2k+1)} + ||f_0||_\infty \frac{\mu^{n+1} (t^2)^{n+1}}{n!(2n+1)} \\
&\leq \lambda \sum_{k=0}^{n} \frac{\mu^k (t^2)^k}{k!} + ||f_0||_\infty \frac{\mu^{n+1} (t^2)^{n+1}}{(n+1)!},
\end{aligned}$$

which is our thesis. In the last two inequalities, we shifted by one the index of the sum and we used the fact that $4k^2 + 6k + 2 > k + 1$, for every $k \in \mathbb{N}$. If we take the lim sup as $n \to \infty$ in both sides of the inequality we also obtain easily that

$$\limsup_{n\to\infty} f_n(t,x) \leq \lambda \exp(\mu t^2).$$

$\square$

Remark 2.56. Using Lemma 3.7 of [ChDa15], one could get a sharper version of this result.

We introduce now another Grönwall's type lemma that will be used in the case of the heat equation. This result is somehow analogous to Lemma 2.55, but its hypothesis are a bit more restrictive, since it asks that b is bounded and Lipschitz. This restriction on b will be the reason for which in the standing semilinear additive case we have to split the proof of Theorem 2.45, for the heat equation, in the cases with b bounded and b possibly unbounded.

Lemma 2.57. *Let $\{f_n\}_{n\geq 1}$, $f_n : [0,T]\times\mathbb{R}\to\mathbb{R}$, be a sequence of functions that satisfy, for every $(t,x)\in[0,T]\times\mathbb{R}$, the following inequality: for some $\mu,\lambda>0$,*

$$|f_{n+1}(t,x)-f_n(t,x)|\leq \mu\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}|b(f_n(s,y))-b(f_{n-1}(s,y))|dy\,ds+\lambda,$$

where $b:\mathbb{R}\to\mathbb{R}$ is bounded and Lipschitz continuous with Lipschitz constant C. Then, we have that, for any $n\geq 1$ and $(t,x)\in[0,T]\times\mathbb{R}$,

$$|f_{n+1}(t,x)-f_n(t,x)|\leq 2||b||_\infty\frac{C^{n-1}(\mu t)^n}{n!}+\sum_{k=0}^{n-1}\frac{\lambda t^k}{k!}.$$

As a consequence, we also have that

$$\limsup_{n\to\infty}\Big(\sup_{x\in\mathbb{R}}|f_{n+1}(t,x)-f_n(t,x)|\Big)\leq\lambda e^t.$$

Proof. We prove it by induction. First, we compute

$$\begin{aligned}|f_2(t,x)-f_1(t,x)|&\leq\mu\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}|b(f_1(s,y))-b(f_0(s,y))|dy\,ds+\lambda\\&\leq 2\mu||b||_\infty\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}dy\,ds+\lambda\\&\leq 2\mu||b||_\infty\int_0^t 1ds+\lambda\\&=2\mu t||b||_\infty+\lambda.\end{aligned}$$

For the inductive step, we have to exploit the Lipschitz continuity of b:

$$\begin{aligned}|f_{n+1}(t,x)-f_n(t,x)|&\leq\mu\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}|b(f_n(s,y))-b(f_{n-1}(s,y))|dy\,ds+\lambda\\&\leq\mu C\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}|f_n(s,y)-f_{n-1}(s,y)|dy\,ds+\lambda\\&\leq\mu C\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{|x-y|^2}{2(t-s)}}\Big[2||b||_\infty\frac{C^{n-2}(\mu s)^{n-1}}{(n-1)!}\\&\qquad+\sum_{k=0}^{n-2}\frac{\lambda s^k}{k!}\Big]dy\,ds+\lambda\\&=\int_0^t\Big[2||b||_\infty\frac{\mu^n C^{n-1}s^{n-1}}{(n-1)!}+\sum_{k=0}^{n-2}\frac{\lambda s^k}{k!}\Big]dy\,ds+\lambda\\&=2||b||_\infty C^{n-1}\frac{(\mu t)^n}{n!}+\sum_{k=1}^{n-1}\frac{\lambda t^k}{k!}+\lambda.\end{aligned}$$

A direct consequence of this fact is that

$$\limsup_{n\to\infty} |f_{n+1}(t,x) - f_n(t,x)| \le \lambda e^t,$$

which concludes the proof. □

Remark 2.58. The hypothesis that b is bounded is used only in the step $n = 1$ of the induction, to show that the integral is finite. However, it looks like there is no other way to prove the result for a general b without imposing some boundedness conditions on f_1 and f_0. The conditions on f_1, f_0 could be weaker than the one on b; for example, continuity and polynomial growth at $x \to \infty$ would be sufficient to prove the result.

Wave equation

Due to the difference between the two Grönwall lemmas we just proved, we provide two separate proofs for Theorem 2.45 for the wave and heat equation. We start from the wave equation.

Proof (Theorem 2.45, semilinear additive case, wave equation). We introduce the general argument that use to prove the result both in this case and in the following case. Let η be a deterministic function in $\mathcal{C}([0,T]\times\mathbb{R})$, and consider the (deterministic) integral equation

$$z(t,x) = \int_0^t \int_{\mathbb{R}} b(z(s,y))G_{t-s}(x-y)dsdy + \eta(t,x), \tag{2.49}$$

which is defined on the space $\mathcal{C}([0,T]\times\mathbb{R})$, endowed with the metric of uniform convergence on compact sets. Recall that here we are considering as G the fundamental solution of the wave equation, given by $G_t(x) = 1_{|x|\le t}(t,x)$.

We will prove that (2.49) admits a unique solution. This allows us to define the solution operator

$$F : \mathcal{C}([0,T]\times\mathbb{R}) \longrightarrow \mathcal{C}([0,T]\times\mathbb{R}) \tag{2.50}$$

by $(F\eta)(t,x) := z(t,x)$. We will show that this operator is continuous.

Denote now as $\bar{u}^{H_n}$ the solution of the linear additive mild equation (2.29) and with u^{H_n} the solution of the semilinear additive equation (2.30). Notice that, if we define (for any fixed $n\in\mathbb{N}$) $\eta(\cdot,\cdot) := \bar{u}^{H_n}(\cdot,\cdot)(\omega)$ and $z(\cdot,\cdot) := u^{H_n}(\cdot,\cdot)(\omega)$ we have that $u^{H_n} = F(\bar{u}^{H_n})$ (almost surely). Since we have proved already that Theorem 2.45 holds in the linear additive case, we know that $\bar{u}^{H_n}$ converges in law, in the space of continuous functions, to $\bar{u}^{H_0}$. Therefore, we can apply Theorem 2.7 of [Bil] to obtain the desired result.

We prove now that (2.49) is well-posed and defines a continuous operator from $\mathcal{C}([0,T]\times\mathbb{R}))$ into itself. We define the Picard iteration scheme

$$\begin{aligned} z_0(t,x) &:= \eta(t,x) \\ z_n(t,x) &:= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)b(z_{n-1}(s,y))dyds + \eta(t,x) \\ &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} b(z_{n-1}(s,y))dyds + \eta(t,x), \quad n \ge 1. \end{aligned} \tag{2.51}$$

Clearly, the above expressions of the Picard scheme are well-defined. Moreover, since b is Lipschitz continuous, if z_{n-1} is continuous then also $b \circ z_{n-1}$ is so. This gives by induction that z_n is a continuous function. Moreover, we will show that z_n converges uniformly on compact sets on $[0,T]\times\mathbb{R}$. More precisely, we prove that the sequence $\{z_n\}_{n\geq 0}$ is uniformly Cauchy on $[0,T]\times[-L,L]$, for every $L>0$. Indeed, for all $(t,x)\in[0,T]\times[-L,L]$, we have

$$\begin{aligned}
|z_{n+1}(t,x)-z_n(t,x)| &= \left|\frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s}[b(z_n(s,y))-b(z_{n-1}(s,y))]dy\,ds\right| \\
&\leq C\int_0^t\int_{x-t+s}^{x+t-s}\Big|z_n(s,y)-z_{n-1}(s,y)\Big|dy\,ds.
\end{aligned}$$

We can apply Lemma 2.55 to the sequence of functions $f_n := |z_{n+1}-z_n|$ and with $\lambda=0$ and $\mu=2C$, obtaining that

$$\begin{aligned}
|z_{n+1}(t,x)-z_n(t,x)| &\leq \Big(\sup_{(s,y)\in[0,T]\times[-L-T,L+T]}|z_1(s,y)-z_0(s,y)|\Big)\frac{(2Ct^2)^n}{n!} \\
&\leq \Big(\sup_{(s,y)\in[0,T]\times[-L-T,L+T]}|z_1(s,y)-z_0(s,y)|\Big)\frac{(2CL^2)^n}{n!}.
\end{aligned}$$

Notice that the latter bound does not depend on t and x. This remark, together with the fact that the function z_1-z_0 is bounded on any compact set, and that the sum $\sum_{k=0}^{\infty}\frac{(2CL^2)^n}{n!}$ is convergent, yield that the sequence $\{z_n(t,x)\}_{n\geq 0}$ is uniformly Cauchy on $[0,T]\times[-L,L]$. Let $z(t,x)$ denote its limit. Then, by the uniqueness of the pointwise limit, the fact that $\mathcal{C}([0,T]\times\mathbb{R})$ is a complete metric space (with the underlying metric) and that z_n, $n\geq 0$, are continuous functions, we have that z is also a continuous function in $\mathcal{C}([0,T]\times\mathbb{R})$.

Letting $n\to\infty$ in (2.51) and observing that $b\circ z_n\to b\circ z$ uniformly on compact sets, one immediately gets that z solves equation (2.49).

The uniqueness of the solution comes from a simple remark: suppose we have two solutions z_1, z_2 relative to the same η. Then, for a fixed $L>0$ and for any $(t,x)\in[0,T]\times[-L,L]$, we have

$$\begin{aligned}
|z_1(t,x)-z_2(t,x)| &\leq \frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s}|b(z_1(s,y))-b(z_2(s,y))|dy\,ds \\
&\leq C\int_0^t\int_{x-t+s}^{x+t-s}|z_1(s,y)-z_2(s,y)|dy\,ds.
\end{aligned}$$

It remains to apply Lemma 2.55 to obtain the uniqueness for every $L>0$, and thus for the equation on the whole space.

Let us now turn to the analysis of the solution operator $F:\mathcal{C}([0,T]\times\mathbb{R})\longrightarrow\mathcal{C}([0,T]\times\mathbb{R})$, which is defined by $F(\eta)(t,x):=z(t,x)$. We need to prove that this operator is continuous with respect to the metric of uniform convergence on compact sets. That is, we show the continuity of the restricted mapping

$$F_L:\mathcal{C}([0,T]\times\mathbb{R})\longrightarrow\mathcal{C}([0,T]\times\mathbb{R}),$$

for every $L>0$.

We denote by $||\cdot||_{\infty,L}$ the supremum norm on $\mathcal{C}([0,T]\times[-L,L])$. Let $z_1:=F(\eta_1)$ and $z_2:=F(\eta_2)$ for some $\eta_1,\eta_2\in\mathcal{C}([0,T]\times\mathbb{R})$. Then, for $(t,x)\in[0,T]\times[-L,L]$,

$$\begin{aligned}
|z_1(t,x)-z_2(t,x)| &\leq \int_0^t\int_{x-t+s}^{x+t-s}|b(z_1(s,y))-b(z_2(s,y))|dy\,ds+|\eta_1(t,x)-\eta_2(t,x)| \\
&\leq C\int_0^t\int_{x-t+s}^{x+t-s}|z_1(s,y)-z_2(s,y)|dy\,ds+||\eta_1-\eta_2||_{\infty,L}.
\end{aligned}$$

Here, we apply again Lemma 2.55 to obtain that

$$||z_1-z_2||_{\infty,L}\leq C||\eta_1-\eta_2||_{\infty,L}.$$

□

Heat equation, b bounded

Proof (Theorem 2.45, semilinear additive case, heat equation with b bounded). As in the previous case, it is sufficient to construct the operator (2.49), that we recall:

$$z(t,x) = \int_0^t \int_{\mathbb{R}} b(z(s,y)) G_{t-s}(x-y) ds dy + \eta(t,x),$$

where this time the fundamental solution is

$$G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}.$$

If we show that $F : \mathcal{C}([0,T]\times\mathbb{R}) \to \mathcal{C}([0,T]\times\mathbb{R})$ such that $F\eta(t,x) := z(t,x)$ is well-defined and continuous also in this case, we can repeat the argument we used for the wave equation to conclude that $u^{H_n} \to u^{H_0}$ in distribution on $\mathcal{C}([0,T]\times\mathbb{R})$ as $n\to\infty$, whenever $H_n \to H_0$.

We prove then the well-definiteness and continuity of F. As in the case of the wave equation, we consider the Picard iteration scheme

$$\begin{aligned}
z_0(t,x) &= \eta(t,x)\\
z_n(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b(z_{n-1}(s,y)) dy ds + \eta(t,x)\\
&= \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} b(z_{n-1}(s,y)) dy ds + \eta(t,x), \quad n \geq 1.
\end{aligned}$$

We clearly have that z_0 is continuous. Assume that z_{n-1} is well-defined and continuous, and we check that z_n is so. The well-definiteness of z_n follows from the fact that b is bounded, which implies that the integral defining $z_n(t,x)$ is convergent for every $(t,x)\in[0,T]\times\mathbb{R}$. Regarding the continuity of z_n, let $(t,x)\in[0,T]\times\mathbb{R}$ and pick a sequence $(t_m,x_m)\to(t,x)$ as $m\to\infty$. Then,

$$\begin{aligned}
z_n(t_m,x_m) &= \int_0^{t_m} \int_{\mathbb{R}} G_{t_m-s}(x_m-y) b(z_n(s,y)) dy\, ds + \eta(t_m,x_m)\\
&= \int_0^{t_m} \int_{\mathbb{R}} G_{s'}(y') b(z_{n-1}(t_m-s', x_m-y')) dy'\, ds' + \eta(t_m,x_m)\\
&= \int_0^{\sup_m t_m} \int_{\mathbb{R}} 1_{[0,t_m]\times\mathbb{R}}(s',y') G_{s'}(y') b(z_{n-1}(t_m-s', x_m-y')) dy'\, ds'\\
&\qquad + \eta(t_m,x_m).
\end{aligned}$$

Thanks to the continuity of b and z_{n-1}, the latter integrand converges point-wise to

$$1_{[0,t]\times\mathbb{R}}(s',y') G_{s'}(y') b(z_{n-1}(t-s', x-y')).$$

Since b is bounded and G has finite integral over $[0,\sup_m t_m]\times\mathbb{R}$, we can apply the dominated convergence theorem to obtain that

$$\lim_{m\to\infty} z_n(t_m,x_m) = z_n(t,x),$$

so z_n is continuous.

For every $(t,x)\in[0,T]\times\mathbb{R}$, we can infer that

$$|z_{n+1}(t,x) - z_n(t,x)| \leq \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} |b(z_n(s,y)) - b(z_{n-1}(s,y))| dy\, ds.$$

By Lemma 2.57, we get

$$|z_{n+1}(t,x) - z_n(t,x)| \leq 2||b||_\infty \frac{C^{n-1}t^n}{n!} \leq 2||b||_\infty \frac{C^{n-1}T^n}{n!}.$$

Since the rightmost term of this inequality is the general term of a converging series, and the series does not depend on (t,x), we can infer that the sequence $\{z_n(t,x)\}_{n\geq 0}$ is uniformly Cauchy in $\mathcal{C}([0,T]\times\mathbb{R})$. This means that a limit z exists and, since $z_n \to z$ uniformly, $z \in \mathcal{C}([0,T]\times\mathbb{R})$. Moreover, it is straightforward to verify that z is the solution to equation (2.49). Finally, uniqueness of solution can be easily checked by applying again Lemma 2.57, in the same way we did in the wave equation case.

As far as the continuity of the solution operator $F : \mathcal{C}([0,T]\times\mathbb{R}) \to \mathcal{C}([0,T]\times\mathbb{R})$ is concerned, where $F(\eta)(t,x) = z(t,x)$, this property can be verified in the same way as in the case of the wave equation, applying Lemma 2.57.

□

Heat equation, b general

Proof (Theorem 2.45, semilinear additive case, heat equation with b general). Recall that the initial condition u_0 is assumed to satisfy Hypothesis B1. In particular, u_0 is α-Hölder continuous for some $\alpha > H_0$.

We will use a truncation argument on the drift b: for every $m \geq 1$, set

$$b_m(x) := \begin{cases} b(x) \wedge m, & \text{if } b(x) \geq 0, \\ b(x) \vee -m, & \text{if } b(x) < 0. \end{cases}$$

We have that b_m is bounded and Lipschitz continuous, and converge pointwise to b, as $m \to \infty$. Moreover, a unique Lipschitz constant can be fixed for all functions b_m, $m \geq 1$, and b. We define $u_m^{H_n}$ to be the solution of (2.30) where b is replaced by b_m, and corresponding to the Hurst index H_n. An immediate consequence of Theorem 2.45 in the case b bounded is that, for any $m \geq 1$,

$$u_m^{H_n} \xrightarrow[n\to\infty]{d} u_m^{H_0} \tag{2.52}$$

on $\mathcal{C}([0,T]\times\mathbb{R})$.

The proof of Theorem 2.45 in our standing case is split in three steps.

Step 1: First, we check that the family of laws of $\{u^{H_n}\}_{n\geq 1}$ is tight in $\mathcal{C}([0,T]\times\mathbb{R})$. For this, we will apply again Centsov criterion stated in Theorem 2.54. We point out that, indeed, the computations of this step are valid for both heat and wave equations.

Notice that condition (i) of Theorem 2.54 is clearly satisfied, since $u^{H_n}(0,0)$ is deterministic and does not depend on n. Regarding condition (ii), let $t,t' \in [0,T]$ and $x,x' \in \mathbb{R}$ with $t' \geq t$ and $x' \geq x$, and we can suppose that $|x-x'| < 1$ and $|t-t'| < 1$. We aim to estimate

$$\begin{aligned} \mathrm{E}\Big[|u^{H_n}(t',x') - u^{H_n}(t,x)|^p\Big] &\leq C_p\Big(\mathrm{E}\Big[|u^{H_n}(t',x') - u^{H_n}(t,x')|^p\Big] \\ &\qquad\qquad + \mathrm{E}\Big[|u^{H_n}(t,x') - u^{H_n}(t,x)|^p\Big]\Big) \\ &=: C_p\Big(I + J\Big). \end{aligned} \tag{2.53}$$

We will see that

$$I \leq C_1|t'-t|^{\beta_I p}, \qquad J \leq C_2|x'-x|^{\beta_J p}, \tag{2.54}$$

where $\beta_I, \beta_J > 0$ are two positive constants.

To start with, we have that

$$
\begin{aligned}
I &\le C_p\Big(|I_0(t',x') - I_0(t,x')|^p \\
&\quad + \mathrm{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}} G_{t'-s}(x'-y)W^{H_n}(ds,dy) - \int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)W^{H_n}(ds,dy)\Big|^p\Big] \\
&\quad + \mathrm{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}} G_{t'-s}(x'-y)b(u^{H_n}(s,y))dy\,ds - \int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)b(u^{H_n}(s,y))dy\,ds\Big|^p\Big]\Big) \\
&=: C_p(I_1 + I_2 + I_3).
\end{aligned}
$$

Regarding I_1, it is known from [BJQ15], Theorem 3.7, that, for a α-Hölder continuous initial condition, it holds

$$
I_1 \le C|t'-t|^{\frac{\alpha p}{2}} \le C|t'-t|^{\frac{(\inf_n H_n)p}{2}}. \tag{2.55}
$$

Next, by step 1 in the proof of Theorem 2.45 in the linear additive case (Section 2.4.3), we clearly obtain that

$$
I_2 \le C|t'-t|^{\frac{H_n p}{2}} \le C|t'-t|^{\frac{(\inf_n H_n)p}{2}}. \tag{2.56}
$$

It remains to estimate I_3. First, in the first summand of I_3 we perform the change of variables $s' = s - (t'-t)$, so that we obtain $I_3 \le C_p(I_{3,1} + I_{3,2})$, where

$$
I_{3,1} := \mathrm{E}\Big[\Big|\int_{-(t'-t)}^0\int_{\mathbb{R}} G_{t-s'}(x'-y)b(u^{H_n}(s'+(t'-t),y))ds'dy\Big|^p\Big]
$$

and

$$
I_{3,2} := \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)\Big(b(u^{H_n}(s+(t'-t),y)) - b(u^{H_n}(s,y))\Big)dy\,ds\Big|^p\Big].
$$

Clearly, $I_{3,1} \le C|t'-t|^p$ by Hölder inequality, Lemma 2.36 and the linear growth of b. For $I_{3,2}$, we have that

$$
\begin{aligned}
I_{3,2} &= \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)\Big(b(u^{H_n}(s+(t'-t),y)) - b(u^{H_n}(s,y))\Big)dy\,ds\Big|^p\Big] \\
&\le C\,\mathrm{E}\Big[\int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)\Big|u^{H_n}(s+(t'-t),y)) - u^{H_n}(s,y)\Big|^p dy\,ds\Big] \\
&\le C\int_0^t\int_{\mathbb{R}} G_{t-s}(x'-y)\Big(\sup_{n\ge1}\sup_{y\in\mathbb{R}}\mathrm{E}\Big[\Big|u^{H_n}(s+(t'-t),y)) - u^{H_n}(s,y)\Big|^p\Big]\Big)dy\,ds \\
&= C\int_0^t\sup_{n\ge1}\sup_{y\in\mathbb{R}}\mathrm{E}\Big[\Big|u^{H_n}(s+(t'-t),y)) - u^{H_n}(s,y)\Big|^p\Big]ds.
\end{aligned}
$$

This latter estimate, together with (2.55) and (2.56) and the very definition of I, let us infer that

$$
\begin{aligned}
&\sup_{n\ge1}\sup_{x\in\mathbb{R}}\mathrm{E}\Big[|u^{H_n}(t+(t'-t),x) - u^{H_n}(t,x)|^p\Big] \\
&\qquad \le C_1|t'-t|^{\beta_I p} + C_2\int_0^t\sup_{n\ge1}\sup_{y\in\mathbb{R}}\mathrm{E}\Big[\Big|u^{H_n}(s+(t'-t),y)) - u^{H_n}(s,y)\Big|^p\Big]ds,
\end{aligned}
$$

where the constants C_1 and C_2 do not depend on H_n and $\beta_I = \frac{1}{2}\inf_n H_n$. Hence, by Grönwall lemma, we obtain the desired estimate for I (see (2.54)).

Let us now deal with the term J in (2.53). Assume that $x' = x+h$, for some $h>0$. We have

$$\begin{aligned}
\mathrm{E}\Big[|u^{H_n}(t,x+h)-u^{H_n}(t,x)|^p\Big] &\le C_p\Big(|I_0(t,x+h)-I_0(t,x)|^p \\
&+\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x+h-y)W^{H_n}(ds,dy)-\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W^{H_n}(ds,dy)\Big|^p\Big] \\
&+\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x+h-y)b(u^{H_n}(s,y))dy\,ds \\
&\quad-\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u^{H_n}(s,y))dy\,ds\Big|^p\Big]\Big) \\
&=: J_1+J_2+J_3.
\end{aligned} \tag{2.57}$$

By [BJQ15], Theorem 3.7, and step 1 in the proof of Theorem 2.45 in the linear additive case, we get, respectively,

$$J_1 \le C\,h^{(\inf_n H_n)p} \qquad \text{and} \qquad J_2 \le C\,h^{(\inf_n H_n)p}. \tag{2.58}$$

In order to tackle the term J_3, we perform the change of variable $y' = y-h$ in its first summand, yielding

$$J_3 = \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y')b(u^{H_n}(s,y'+h))dy'\,ds-\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)b(u^{H_n}(s,y))dy\,ds\Big|^p\Big].$$

Then, renaming the variable y' as y, we have

$$\begin{aligned}
J_3 &= \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}}\Big(b(u^{H_n}(s,y+h))-b(u^{H_n}(s,y))\Big)G_{t-s}(x-y)dy\,ds\Big|^p\Big] \\
&\le C\int_0^t \sup_{n\ge 1}\sup_{y\in\mathbb{R}}\mathrm{E}\Big[\Big|u^{H_n}(s,y+h))-u^{H_n}(s,y))\Big|^p\Big]ds.
\end{aligned}$$

Putting together this bound and those of (2.58), we get

$$\begin{aligned}
&\sup_{n\ge 1}\sup_{x\in\mathbb{R}}\mathrm{E}\Big[|u^{H_n}(t,x+h)-u^{H_n}(t,x)|^p\Big] \\
&\qquad\le C_1\,h^{\beta_J p}+C_2\int_0^t \sup_{n\ge 1}\sup_{y\in\mathbb{R}}\mathrm{E}\Big[\Big|u^{H_n}(s,y+h))-u^{H_n}(s,y)\Big|^p\Big]ds,
\end{aligned}$$

where $\beta_J = \inf_n H_n$. By Grönwall lemma, we conclude that estimates (2.54) hold. Therefore, by Theorem 2.54, the family of laws of $\{u^{H_n}\}_{n\ge 1}$ is tight in $\mathcal{C}([0,T]\times\mathbb{R})$.

Step 2: This part of the proof is devoted to show the following uniform $L^2(\Omega)$-convergence:

$$\sup_{H\in[a,b]}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x)-u^H(t,x)|^2\Big] \xrightarrow[m\to\infty]{} 0.$$

We remark that, indeed, the uniformity with respect to $(t,x)\in[0,T]\times\mathbb{R}$ will not be needed in step 3, but we obtain it for free thanks to our Grönwall-type argument exhibited below.

We argue as follows:

$$
\begin{aligned}
&\mathrm{E}\Big[|u_m^H(t,x) - u^H(t,x)|^2\Big] \\
&\quad \le C\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u_m^H(s,y)) - b(u^H(s,y))|^2\Big] dy\, ds \\
&\quad \le C\Big(\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u_m^H(s,y)) - b_m(u^H(s,y))|^2\Big] dy\, ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u^H(s,y)) - b(u^H(s,y))|^2\Big] dy\, ds\Big) \\
&\quad \le C\Big(\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|u_m^H(s,y) - u^H(s,y)|^2\Big] dy\, ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u^H(s,y)) - b(u^H(s,y))|^2 1_{\{|u^H(s,y)|>m\}}\Big] dy\, ds\Big). \\
&\quad \le C\Big(\int_0^t \sup_{H\in[a,b]} \sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(s',y) - u^H(s',y)|^2\Big] ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u^H(s,y)) - b(u^H(s,y))|^4\Big]^{\frac{1}{2}} \mathbb{P}(|u^H(s,y) > m|)^{\frac{1}{2}}\, dy\, ds\Big),
\end{aligned}
\tag{2.59}
$$

where in the progress we used the fact that $|b_m(u^H(s,y)) - b(u^H(s,y))| = 0$, whenever $|u^H(s,y)| \le m$.

A direct consequence of Lemma 2.36 is that u^H is uniformly bounded in $L^p(\Omega)$, with respect to $H \in [a,b]$ and $(t,x) \in [0,T]\times\mathbb{R}$, for any $p \ge 2$, which means that there exists a constant M_p which depends only on p and T such that

$$
\sup_{H\in[a,b]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] \le M_p. \tag{2.60}
$$

Hence, by Markov inequality,

$$
\mathbb{P}(|u^H(s,y)| > m) \le \frac{\mathrm{E}\Big[|u^H(s,y)|^2\Big]}{m^2} \le \frac{M_2}{m^2}.
$$

Note that the latter estimate is again uniform with respect to $H \in [a,b]$ and $(s,y) \in [0,T]\times\mathbb{R}$. Thus, going back to (2.59) and using the linear growth of b and (2.60), we get

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathrm{E}\Big[|b_m(u^H(s,y)) - b(u^H(s,y))|^4\Big]^{\frac{1}{2}} \mathbb{P}(|u^H(s,y) > m|)^{\frac{1}{2}} dy\, ds\Big) \\
&\quad \le \int_0^t \int_{\mathbb{R}} C\frac{M_2^{1/2}}{m} G_{t-s}(x-y) dy\, ds \le \int_0^t C\frac{M_2^{1/2}}{m} ds =: \frac{C}{m}.
\end{aligned}
\tag{2.61}
$$

We observe now that if on the left-hand side of (2.59) we replace t with any $t' \le t$, the inequality would still hold exactly in the same way (indeed, the integrand on the right-hand side is positive, so it is increasing as a function of t). Therefore, we can infer that

$$
\begin{aligned}
&\sup_{H\in[a,b]} \sup_{(t',x)\in[0,t]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t',x) - u^H(t',x)|^2\Big] \\
&\qquad \le \frac{C_1}{m} + C_2 \int_0^t \sup_{H\in[a,b]} \sup_{(s',y)\in[0,s]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(s',y) - u^H(s',y)|^2\Big] ds.
\end{aligned}
$$

Then, Grönwall lemma implies that

$$
\sup_{H\in[a,b]} \sup_{(t',x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t',x) - u^H(t',x)|^2\Big] \le \frac{C}{m} \xrightarrow[m\to\infty]{} 0,
$$

which is what we wanted to show.

Step 3: We prove that the finite dimensional distributions of u^{H_n} converge to those of u^{H_0}. Given a finite dimensional vector $\{(t_1,x_1),\dots,(t_k,x_k)\}$ and $f\in C_b(\mathbb{R}^k)$, we can write

$$\begin{aligned}
&\Big|\mathrm{E}\Big[f\big(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\big)-f\big(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\big)\Big]\Big|\\
&\quad\le\Big|\mathrm{E}\Big[f\big(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k)\big)-f\big(u^{H_n}_m(t_1,x_1),\dots,u^{H_n}_m(t_k,x_k)\big)\Big]\Big|\\
&\qquad+\Big|\mathrm{E}\Big[f\big(u^{H_n}_m(t_1,x_1),\dots,u^{H_n}_m(t_k,x_k)\big)-f\big(u^{H_0}_m(t_1,x_1),\dots,u^{H_0}_m(t_k,x_k)\big)\Big]\Big|\\
&\qquad+\Big|\mathrm{E}\Big[f\big(u^{H_0}_m(t_1,x_1),\dots,u^{H_0}_m(t_k,x_k)\big)-f\big(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k)\big)\Big]\Big|\\
&\quad=:I_1(m,n)+I_2(m,n)+I_3(m).
\end{aligned}$$

Assume that $f:\mathbb{R}^k\to\mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_f (we can always restrict to the class of Lipschitz continuous functions to verify weak convergence). Then, for all $H\in[a,b]$,

$$\begin{aligned}
&\sup_{H\in[a,b]}\Big|\mathrm{E}\Big[f(u^H(t_1,x_1),\dots,u^H(t_k,x_k))-f(u^H_m(t_1,x_1),\dots,u^H_m(t_k,x_k))\Big]\Big|\\
&\quad\le\sup_{H\in[a,b]}\mathrm{E}\Big[\Big|f(u^H(t_1,x_1),\dots,u^H(t_k,x_k))-f(u^H_m(t_1,x_1),\dots,u^H_m(t_k,x_k))\Big|\Big]\\
&\quad\le\sup_{H\in[a,b]}L_f\mathrm{E}\Big[\Big(\sum_{j=1}^k|u^H_m(t_j,x_j)-u^H(t_j,x_j)|^2\Big)^{1/2}\Big]\\
&\quad\le L_f\sup_{H\in[a,b]}\Big(\mathrm{E}\Big[\sum_{j=1}^k|u^H_m(t_j,x_j)-u^H(t_j,x_j)|^2\Big]\Big)^{1/2}\\
&\quad=L_f\sup_{H\in[a,b]}\Big(\sum_{j=1}^k\mathrm{E}\Big[|u^H_m(t_j,x_j)-u^H(t_j,x_j)|^2\Big]\Big)^{1/2}\\
&\quad\le L_fk^{\frac12}\Big(\sup_{H\in[a,b]}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathrm{E}\Big[|u^H_m(t,x)-u^H(t,x)|^2\Big]\Big)^{1/2},
\end{aligned}\tag{2.62}$$

where the last term converges to 0 as $m\to\infty$ thanks to step 2, and taking into account that we are considering an arbitrary but fixed number of terms k. Hence, for any $\varepsilon>0$, there exists $m_0\ge1$ such that, for all $m\ge m_0$, we have

$$\sup_{n\ge1}\Big(I_1(m,n)+I_3(m)\Big)\le\frac{\varepsilon}{2}.$$

In particular, we have

$$\Big|\mathrm{E}\Big[f(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k))-f(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k))\Big]\Big|\le I_2(m_0,n)+\frac{\varepsilon}{2}.$$

Finally, it is sufficient to observe that the convergence (2.52) implies the corresponding convergence of the finite dimensional distributions, and thus for some $n_0\ge1$ we have that, for all $n\ge n_0$, it holds $I_2(m_0,n)<\frac{\varepsilon}{2}$. Therefore,

$$\Big|\mathrm{E}\Big[f(u^{H_n}(t_1,x_1),\dots,u^{H_n}(t_k,x_k))-f(u^{H_0}(t_1,x_1),\dots,u^{H_0}(t_k,x_k))\Big]\Big|<\varepsilon,$$

where ε can be taken arbitrary small. This concludes the proof of Theorem 2.45 in the semilinear additive case for the stochastic heat equation in the case of a general Lipschitz continuous drift b.

□

2.4.5 Linear multiplicative case

We prove now Theorem 2.45 in the linear multiplicative case. This means that we are under Hypothesis C, and we can consider only $H \in (\frac{1}{4}, 1)$. The mild formulation in this case is given by (2.38), that is,

$$u^H(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u^H(s,y)W^H(ds,dy).$$

We remarked in Theorem 2.39 that a solution exists and it is unique for this problem. Moreover, we showed that also the Skorohod mild solution (2.41) exists and coincides with the Itô mild solution defined by (2.38), thanks to Theorem 2.42. From the latter result we also have that the Picard iteration scheme $\{u_m^H,\, m \in \mathbb{N}\}$, used in [BJQ15] and [DaQu11] to obtain a solution for (2.38) coincides with a sum of multiple Wiener integrals (up to order m) of a deterministic function g_m. We will recall this result precisely later and we will make use of it to prove our main result.

Let $H_0 \in (\frac{1}{4}, 1)$ and suppose that $\{H_n,\, n \in \mathbb{N}\} \subset (\frac{1}{4}, 1)$ is such that $H_n \to H_0$ as $n \to \infty$. We aim to prove that $u^{H_n} \to u^{H_0}$ in distribution on $\mathcal{C}([0,T] \times \mathbb{R}))$ as $n \to \infty$. We will split the proof in 3 parts: in the first two parts, we will show that the sequence of solutions $\{u^{H_n},\, n \in \mathbb{N}\}$ defines a tight family of probability measures on $\mathcal{C}([0,T] \times \mathbb{R})$. We split the computations in the case $H \in (\frac{1}{4}, \frac{1}{2})$, which has more involved calculations, and the case $H \in [\frac{1}{2}, 1)$, in which the calculations are more straightforward. We explain briefly why: in the case $H \in (\frac{1}{4}, \frac{1}{2})$, the Burkholder-Davis-Gundy inequality (2.25) forces us to consider the Fourier transform of the whole integrand process, while in the case $H \in [\frac{1}{2}, 1)$, when we use the Burkholder-Davis-Gundy inequality (2.23), we only have to compute the Fourier transform of the deterministic part of the integrand process, which will be explicit in our case. In the third part of the proof, we will identify the limit. To do it, we will exploit the representation result for multiple Wiener integrals Theorem 2.14 to compare the Picard iterations u_m^H relative to different values of H. This implies that we can compare also the solutions u^H relative to different values of H, and allows us to show the identification of the limit. With these facts in mind, we give the proof of Theorem 2.45 in the standing case.

Tightness in the case $H \in (\frac{1}{4}, \frac{1}{2})$

Let $\{u_m^{H_n},\, m \in \mathbb{N}\}$ be the sequence of Picard iterations defined by (2.42) in order to solve (2.38). Suppose for the moment that the limiting Hurst exponent $H_0 \in (\frac{1}{4}, \frac{1}{2}]$. If $H_0 \in (\frac{1}{4}, \frac{1}{2})$, we can assume without loss of generality that the whole sequence $\{H_n,\, n \in \mathbb{N}\} \subset [\eta_1, \eta_2] \subset (\frac{1}{4}, \frac{1}{2})$. If $H_0 = \frac{1}{2}$, we can assume at most that $\{H_n,\, n \in \mathbb{N}\} \subset [\eta_1, \frac{1}{2}) \subset (\frac{1}{4}, \frac{1}{2})$. From now on we will denote both type of sets as K, meaning that $K = [\eta_1, \eta_2]$ if $H_0 \in (\frac{1}{4}, \frac{1}{2})$ and $K = [\eta_1, \frac{1}{2})$ if $H_0 = \frac{1}{2}$. Clearly, if the limiting exponent $H_0 = \frac{1}{2}$ we cannot suppose that $H_n \to H_0$ always from below. In Section 2.4.5 we will handle also families of Hurst exponents of the type $K = (\frac{1}{2}, \eta_2]$, so that we complete our result (the union of a finite number of tight families is a tight family itself).

The solution u^H has been found in [BJQ15] as a limit of the Picard iterations (2.42), defined by

$$\begin{aligned} u_0^H(t,x) :=& I_0(t,x) \\ u_{m+1}^H(t,x) :=& I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u_m^H(s,y)W^H(ds,dy), \quad m \geq 0. \end{aligned}$$

The limit is found in the Banach space $(\mathcal{X}^H, ||\cdot||_{\mathcal{X}^H})$ of $L^2(\Omega)$-continuous, adapted and jointly measurable processes such that

$$||Y||_{\mathcal{X}^H} := ||Y||_{\mathcal{X}_1} + ||Y||_{\mathcal{X}_2^H} < \infty,$$

where, for a process $Y = \{Y(t,x),\ (t,x) \in [0,T] \times \mathbb{R}\}$ the (semi-)norms are defined by

$$||Y||_{\chi_1} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|Y(t,x)|^p\Big]^{1/p}$$

and

$$\begin{aligned} ||Y||_{\chi_2^H} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \Bigg(& \frac{H(1-2H)}{2} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \\ & \times \frac{\Big(\mathrm{E}\Big[|Y(s,y) - Y(s,z)|^p\Big]\Big)^{2/p}}{|y-z|^{2-2H}} dy\, dz\, ds \Bigg)^{1/2}. \end{aligned} \tag{2.63}$$

Notice that the L^p-part $||\cdot||_{\chi_1}$ of the norm $||\cdot||_{\chi^H}$ does not depend on H, as it is also pointed out by the notation itself, while the *Gagliardo-type* part $||\cdot||_{\chi_2^H}$ depends on H.

Remark 2.59. In [BJQ15], the norm $||\cdot||_{\chi_2^H}$ is defined without the constant $\frac{H(1-2H)}{2}$. Anyway, since the two definitions give rise to equivalent norms, the results about existence and uniqueness of a solution for equation (2.38) when $H \in (1/4, 1/2)$ are still holding true. On the other hand, we will see how adding this normalizing constant helps us in proving the uniform (in H) results that we need in the estimation process.

We prove an embedding lemma for the spaces χ^H, which could be of independent interest:

Lemma 2.60 (Sobolev-type embedding)**.** *Let $\frac{1}{4} < \alpha \leq \beta < \frac{1}{2}$. Then the following embedding holds:*

$$\chi^\alpha \hookrightarrow \chi^\beta$$

This means that there exists a constant C such that for every process Y which is adapted, jointly measurable and $L^2(\Omega)$-continuous it holds

$$||Y||_{\chi^\beta} \leq C||Y||_{\chi^\alpha}. \tag{2.64}$$

Moreover, it holds the following stronger property for the Gagliardo-type seminorms $||\cdot||_{\chi_2^\beta}$:

$$\sup_{\beta\in[\alpha,\frac{1}{2})} ||Y||_{\chi_2^\beta} \leq \tilde{C}||Y||_{\chi^\alpha} \tag{2.65}$$

where the constant $\tilde{C}$ depends only on p and T.

Proof. We follow the same reasoning of [DPV12]. Since for any $\gamma \in (1/4, 1/2)$ the norm $||\cdot||_\gamma = ||\cdot||_{\chi_1} + ||\cdot||_{\chi_2^\gamma}$ has a constant (with respect to γ) part, we only need to prove the inequality (2.64) for the $||\cdot||_{\chi_2^\gamma}$ part of the norm. We have

$$\begin{aligned} & \left(\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,z)|^p\Big]^{2/p}}{|y-z|^{2-2\beta}} dy\, dz\, ds \right)^{1/2} \\ & = \left(\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\beta}} dy\, d\overline{z}\, ds \right)^{1/2} \\ & \leq C(I_1 + I_2), \end{aligned} \tag{2.66}$$

where we label I_1 the term where we integrate in the variable $\overline{z}$ in the region $|\overline{z}| \geq 1$, and I_2 the term where we integrate in the region $|\overline{z}| < 1$. First we handle I_1

$$\begin{aligned}
I_1 =& \left(\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\beta}} d\overline{z}\, dy\, ds \right)^{1/2} \\
&\leq C_p \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|Y(t,x)|^p\Big]^{1/p} \left(\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{1}{|\overline{z}|^{2-2\beta}} d\overline{z}\, dy\, ds \right)^{1/2}
\end{aligned}$$

The integral $\int_{|\overline{z}|\geq 1} \frac{1}{|\overline{z}|^{2-2\beta}} d\overline{z} = \frac{2}{1-2\beta}$, therefore we have that

$$\begin{aligned}
&\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|\geq 1} G_{t-s}^2(x-y) \frac{1}{|\overline{z}|^{2-2\beta}} d\overline{z}\, dy\, ds \\
&\leq \beta \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) dy\, ds \leq \beta C_T \leq \frac{C_T}{2}
\end{aligned}$$

Thus we can conclude that

$$I_1 \leq C_{p,T} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|Y(t,x)|^p\Big]^{1/p}$$

Regarding I_2, we observe that

$$\begin{aligned}
I_2 =& \left(\frac{\beta(1-2\beta)}{2} \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|< 1} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\beta}} d\overline{z}\, dy\, ds \right)^{1/2} \\
&\leq \left(\frac{\alpha(1-2\alpha)}{2} \int_0^t \int_{\mathbb{R}} \int_{|\overline{z}|< 1} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\alpha}} d\overline{z}\, dy\, ds \right)^{1/2} \\
&\leq \left(\frac{\alpha(1-2\alpha)}{2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\alpha}} d\overline{z}\, dy\, ds \right)^{1/2} \\
&\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(\frac{\alpha(1-2\alpha)}{2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-s}^2(x-y) \frac{\mathrm{E}\Big[|Y(s,y) - Y(s,y-\overline{z})|^p\Big]^{2/p}}{|\overline{z}|^{2-2\alpha}} d\overline{z}\, dy\, ds \right)^{1/2} \\
&= ||Y||_{\chi_2^\alpha}.
\end{aligned}$$

Notice that both the estimate for I_1 and the estimate for I_2 are independent of $(t,x) \in [0,T]\times\mathbb{R}$ and of $\beta \in [\alpha, 1/2)$. Therefore, we can take the supremum for $(t,x) \in [0,T]\times\mathbb{R}$ and $\beta \in [\alpha, 1/2)$ in the left-hand side of (2.66) to conclude

$$\sup_{\beta\in[\alpha,\frac{1}{2})} ||Y||_{\chi_2^\beta} \leq C_{p,T}||Y||_{\chi_1} + ||Y||_{\chi_2^\alpha} \leq \tilde{C}||Y||_{\chi^\alpha}, \tag{2.67}$$

which obviously implies

$$||Y||_{\chi^\beta} \leq (C_{p,T}+1)||Y||_{\chi_1} + ||Y||_{\chi_2^\alpha} \leq C||Y||_{\chi^\alpha}.$$

for a suitable constant C. $\square$

We are ready now to state the main result of this subsection.

Proposition 2.61. *Let $\mathcal{U}_K := \{u^H,\ H \in K\}$ be the family of solutions of* (2.38)*, where K is either of the form $[\eta_1, \eta_2]$, with $\eta_1, \eta_2 \in (\frac{1}{4}, \frac{1}{2})$ and $\eta_1 \leq \eta_2$, or $K = [\eta_1, \frac{1}{2})$, where $\eta_1 \in (\frac{1}{4}, \frac{1}{2})$. Then, the family $\mathcal{U}_K$ is tight in $\mathcal{C}([0,T] \times \mathbb{R})$, endowed with the metric of uniform convergence on compact sets.*

We postpone the proof of this result, since we need some auxiliary results. The idea is to use again Centsov criterion (Theorem 2.54) in order to prove the tightness. We need to get an estimate of the type

(i) $\sup_{\lambda \in \Lambda} \mathrm{E}[|X_\lambda(0,0)|^{p'}] < \infty$,

(ii) $\sup_{\lambda \in \Lambda} \mathrm{E}\Big[|X_\lambda(t', x') - X_\lambda(t,x)|^p\Big] \leq C\Big(|t'-t| + |x'-x|\Big)^\delta$.

We will obtain such an estimate for u^H by obtaining first a similar estimate for the Picard iterations u^H_m. Since the estimation for the Picard iteration will be uniform with respect to H and satisfy some further conditions, we will be able to pass to the limit as $m \to \infty$ and obtain the estimate we need for the family u^H.

First of all, we show the following well-definiteness theorem for the Picard iterations $\{u^H_m\}_m$, which is an adaptation of Theorem 3.7 of [BJQ15], stated uniformly in H. We remark that also its proof is an adaptation of the proof presented in [BJQ15]. We only have to take care of the fact that we need those results uniformly in H.

Proposition 2.62. *Let $p \geq 2$ and $C_H = \frac{H(1-2H)}{2}$. For any $m \geq 0$, we have that*

$$\left.\begin{array}{ll} i) & \text{For any } H \in K,\ u^H_m(t,x) \text{ is well defined for any } (t,x) \in [0,T] \times \mathbb{R} \\ ii) & \displaystyle\sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathrm{E}\Big[|u^H_m(t,x)|^p\Big] < \infty \\ iii) & \displaystyle\sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} C_H \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \\ & \displaystyle\qquad \times \frac{\Big(\mathrm{E}\Big[|u^H_m(s,y) - u^H_m(s,z)|^p\Big]\Big)^{2/p}}{|y-z|^{2-2H}} dy\, dz\, ds < \infty \end{array}\right\} \qquad \text{(P)}$$

Proof. As we said, we adapt the proof of Theorem 3.7 of [BJQ15] to our case. The adaptation consists in taking care that all the estimates can be generalized in order to be uniform for $H \in K$. We prove the result by induction.

Step 1: we check that for $m = 0$ condition (P) holds.

First, notice that $u^H_0(t,x) = I_0(t,x)$ is the solution of the deterministic equation, which is well defined under our initial condition hypothesis, so that the first condition in (P) is satisfied for any $H \in K$.

Moreover, the explicit form of I_0, together with boundedness of the initial conditions (we are indeed considering a constant initial condition), easily implies that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |I_0(t,x)| < \infty,$$

which implies the condition *ii)* of (P) in the case $m = 0$. Regarding the third condition of (P), we use Lemma 2.60. Observe that

$$C_H \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\Big(\mathrm{E}\Big[|I_0(s,y) - I_0(s,z)|^p\Big]\Big)^{2/p}}{|y-z|^{2-2H}} dy\, dz\, ds = ||I_0||^2_{\mathcal{X}^H_2}.$$

Applying (2.67) with $\beta = H$, $\alpha = \eta_1$ we have that

$$\sup_{H\in K} ||I_0||^2_{\mathcal{X}_2^H} \leq \sup_{H\in[\eta_1,\frac{1}{2})} ||I_0||^2_{\mathcal{X}_2^H} \leq 2C^2_{p,T}||I_0||^2_{\mathcal{X}_1} + ||I_0||^2_{\mathcal{X}_2^{\eta_1}}$$

Since we already showed

$$||I_0||_{\mathcal{X}_1} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0(t,x)| < \infty,$$

we are left to prove that

$$||I_0||_{\mathcal{X}_2^{\eta_1}} < \infty,$$

which is an immediate consequence of the proof on page 17 of [BJQ15], since $\eta_1 \in (\frac{1}{4}, \frac{1}{2})$.

Step 2: we prove the first two conditions in (P) for $m+1$, assuming that they hold true for m. First, from Section 3.2 of [BJQ15] we have that the Picard iterations u^H_{m+1} are well-defined for every $H \in (\frac{1}{4}, \frac{1}{2})$. Regarding the condition *ii)* in (P), we have that

$$\begin{aligned}
\mathrm{E}\Big[|u^H_{m+1}(t,x)|^p\Big] =&\mathrm{E}\Big[\Big|I_0(t,x) + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u^H_m(s,y)W^H(ds,dy)\Big|^p\Big] \\
\leq& C\Big(|I_0(t,x)|^p + \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u^H_m(s,y)W^H(ds,dy)\Big|^p\Big]\Big).
\end{aligned} \tag{2.68}$$

We know from Step 1 that the first term is bounded uniformly in t, x, H. For the second term, we define $S^H_m(s,y) := G_{t-s}(x-y)u^H_m(s,y)$. We already know that S^H_m is integrable with respect to the noise W^H (from the proof of well definiteness of the Picard iterations u^H_{m+1}), so we are only left to show that the second term is also bounded. Extending the integration interval and using Theorem 2.19 we have that

$$\begin{aligned}
&\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u^H_m(s,y)W^H(ds,dy)\Big|^p\Big] \\
&\quad\leq z_p\mathrm{E}\Big[\Big|C_H\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|S^H_m(s,y)-S^H_m(s,z)|^2}{|y-z|^{2H-2}}dy\,dz\,ds\Big|^{p/2}\Big],
\end{aligned}$$

where z_p is the constant of the Burkholder-Davis-Gundy inequality and $C_H = H(1-2H)/2$ as before. If we now add and subtract the mixed term $G_{t-s}(x-y)u^H_n(s,z)$ we obtain

$$\begin{aligned}
&z_p\mathrm{E}\Big[\Big|C_H\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|S^H_m(s,y)-S^H_m(s,z)|^2}{|y-z|^{2H-2}}dy\,dz\,ds\Big|^{p/2}\Big] \\
&\leq Cz_p\Bigg(\mathrm{E}\Big[\Big|C_H\int_0^T\int_{\mathbb{R}^2} G^2_{t-s}(x-y)\frac{|u^H_m(s,y)-u^H_m(s,z)|^2}{|y-z|^{2-2H}}dy\,dz\,ds\Big|^{p/2}\Big] \\
&\qquad + \mathrm{E}\Big[\Big|C_H\int_0^T\int_{\mathbb{R}^2} |u^H_m(s,z)|^2\frac{|G_{t-s}(x-y)-G_{t-s}(x-z)|^2}{|y-z|^{2-2H}}dy\,dz\,ds\Big|^{p/2}\Big]\Bigg) \\
&=: C(I_1+I_2)
\end{aligned}$$

We estimate first I_1: thanks to the Minkowski inequality for integrals we have that

$$I_1 \leq \left(C_H\int_0^T\int_{\mathbb{R}^2} G^2_{t-s}(x-y)\frac{\mathrm{E}\Big[|u^H_m(s,y)-u^H_m(s,z)|^p\Big]^{2/p}}{|y-z|^{2-2H}}dy\,dz\,ds\right)^{p/2}$$

which is uniformly bounded for $H \in K$ and $(t,x) \in [0,T]\times\mathbb{R}$ thanks to the inductive hypothesis on the third condition in (P). Using the same inequality for I_2 we obtain:

$$\begin{aligned}
I_2 \leq & \left(C_H \int_0^T \int_{\mathbb{R}^2} \mathrm{E}\Big[|u_m^H(s,z)|^p\Big]^{2/p} \frac{|G_{t-s}(x-y) - G_{t-s}(x-z)|^2}{|y-z|^{2-2H}} dy\, dz\, ds \right)^{p/2} \\
\leq & \left(\sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x)|^p\Big] \right) \\
& \quad \times \left(C_H \int_0^T \int_{\mathbb{R}^2} \frac{|G_{t-s}(x-y) - G_{t-s}(x-z)|^2}{|y-z|^{2-2H}} dy\, dz\, ds \right)^{p/2}.
\end{aligned}$$

Thanks to the inductive hypothesis for condition *ii)* in (P) the first factor is bounded, while for the second factor we use Proposition 2.21 to obtain

$$\begin{aligned}
& \left(C_H \int_0^T \int_{\mathbb{R}^2} \frac{|G_{t-s}(x-y) - G_{t-s}(x-z)|^2}{|y-z|^{2-2H}} dy\, dz\, ds \right)^{p/2} \\
& \qquad = \left(c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds \right)^{p/2},
\end{aligned} \tag{2.69}$$

where we recall that the constant c_H is given by

$$c_H = \frac{\Gamma(1+2H)\sin(\pi H)}{2\pi}. \tag{2.70}$$

Notice that it holds $c_H \leq \frac{1}{2\pi}$ for $H \in (\frac{1}{4}, \frac{1}{2})$. Moreover, the last integral appearing in (2.69) is uniformly bounded for $H \in K$ and $(t,x) \in [0,T] \times \mathbb{R}$, thanks to Lemma 2.46. Indeed, it holds

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds \leq \begin{cases} 2^{2H} \tilde{C}_{1-2H} \frac{1}{1+2H} T^{1+2H} & \text{wave equation}, \\ \\ \frac{1}{H} \Gamma(1-H) T^H & \text{heat equation}. \end{cases} \tag{2.71}$$

The constants depending on H that appear can all be bounded uniformly when $H \in K$. Indeed, in Step 1 of the proof given in 2.4.3 we showed that this is true even when K is a compact interval in $(0,1)$. In particular, in that proof it is shown that the constant $\tilde{C}_{1-2H}$, which is defined in Lemma 2.46, defines a continuous function $\tilde{C} : (0,1) \to \mathbb{R}$, and then it is bounded on every compact interval. Thus, since also $I_2 < \infty$, we can conclude that

$$\sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathrm{E}\Big[|u_{m+1}^H(t,x)|^p\Big] < \infty.$$

Step 3: we prove that condition *iii)* in (P) holds for u_{m+1}^H. The proof is identical to the proof on pages 19-21 of [BJQ15], except for the fact that we have to take care that all the estimates are uniform with respect to $H \in K$. We prove it in detail: we need

$$\begin{aligned}
& \sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} C_H \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \\
& \qquad \times \frac{\left(\mathrm{E}\Big[|u_{m+1}^H(s,y+z) - u_{m+1}^H(s,y)|^p\Big] \right)^{2/p}}{|z|^{2-2H}} dy\, dz\, ds < \infty
\end{aligned} \tag{2.72}$$

We compute:

$$
\begin{aligned}
&\mathrm{E}\Big[|u^H_{m+1}(s,y+z)-u^H_{m+1}(s,y)|^p\Big]\\
&\quad\le C|I_0(s,y+z)-I_0(s,y)^p|\\
&\qquad + C\mathrm{E}\Big[\Big|\int_0^s\int_{\mathbb{R}}(G_{s-r}(y+z-v)-G_{s-r}(y-v))u^H_m(r,v)W^H(dr,dv)\Big|^p\Big]\\
&\quad\le C|I_0(s,y+z)-I_0(s,y)^p|\\
&\qquad + CC_H^{p/2}\mathrm{E}\Bigg(\Bigg|\int_0^s\int_{\mathbb{R}^2}\Big|\Big(G_{s-r}(y+z-v)-G_{s-r}(y-v)\Big)u^H_m(r,v)\\
&\qquad - \Big(G_{s-r}(y+z-\overline{v})-G_{s-r}(y-\overline{v})\Big)u^H_m(r,\overline{v})\Big|^2\frac{1}{|v-\overline{v}|^{2-2H}}dv\,d\overline{v}\,dr\Bigg|^{p/2}\Bigg).
\end{aligned}
$$

Plugging this computations into (2.72) we obtain two terms. The first term is the left-hand side of (2.72) itself, in the case $m=0$, which we already proved to be bounded in Step 1. The other term appearing is

$$
\begin{aligned}
&C_H^2 C\int_0^t\int_{\mathbb{R}^2}\frac{G^2_{t-s}(x-y)}{|z|^{2-2H}}\mathrm{E}\Bigg(\Bigg|\int_0^s\int_{\mathbb{R}^2}\Big|\Big(G_{s-r}(y+z-v)-G_{s-r}(y-v)\Big)u^H_m(r,v)\\
&\quad - \Big(G_{s-r}(y+z-\overline{v})-G_{s-r}(y-\overline{v})\Big)u^H_m(r,\overline{v})\Big|^2\frac{1}{|v-\overline{v}|^{2-2H}}dv\,d\overline{v}\,dr\Bigg|^{p/2}\Bigg)^{2/p}dz\,dy\,ds.
\end{aligned}
$$

We add and subtract the mixed term $(G_{s-r}(y+z-v)-G_{s-r}(y-v))u^H_m(r,\overline{v})$ and we have that the last integral is bounded by $C(C_H^2A_1+C_H^2A_2)$, where

$$
\begin{aligned}
C_H^2A_1 =&C_H^2\int_0^t\int_{\mathbb{R}^2}dz\,dy\,ds\frac{G^2_{t-s}(x-y)}{|z|^{2-2H}}\mathrm{E}\Bigg(\Bigg|\int_0^s\int_{\mathbb{R}^2}|G_{s-r}(y+z-v)-G_{s-r}(y-v)|^2\\
&\times|u^H_m(r,v)-u^H_m(r,\overline{v})|^2\frac{1}{|v-\overline{v}|^{2-2H}}dv\,d\overline{v}\,dr\Bigg|^{p/2}\Bigg)^{2/p},\\
C_H^2A_2 =&C_H^2\int_0^t\int_{\mathbb{R}^2}dz\,dy\,ds\frac{G^2_{t-s}(x-y)}{|z|^{2-2H}}\mathrm{E}\Bigg(\Bigg|\int_0^s\int_{\mathbb{R}^2}dv\,d\overline{v}\,dr|u^H_m(r,\overline{v})|^2\\
&\times\frac{|(G_{s-r}(y+z-v)-G_{s-r}(y-v))-(G_{s-r}(y+z-\overline{v})-G_{s-r}(y-\overline{v}))|^2}{|v-\overline{v}|^{2-2H}}\Bigg|^{p/2}\Bigg)^{2/p}.
\end{aligned}
$$

We start from A_1: by applying Minkowski inequality for integrals, then Fubini's theorem, together with some change of variables (see details on page 20-21 of [BJQ15]) one can see that

$$
\begin{aligned}
C_H^2A_1 \le&C_H^2\int_0^t dr\int_{\mathbb{R}}dz\int_{\mathbb{R}}dy\frac{|G_r(y+z)-G_r(y)|^2}{|z|^{2-2H}}\\
&\times\int_0^{t-r}ds\int_{\mathbb{R}}dv\int_{\mathbb{R}}d\overline{v}\,G^2_{t-r-s}(x-y-v)\frac{\mathrm{E}\Big[|u^H_m(s,v)-u^H_m(s,\overline{v})|^p\Big]^{2/p}}{|v-\overline{v}|^{2-2H}}\\
\le&C_H\int_0^t dr\int_{\mathbb{R}}dz\int_{\mathbb{R}}dy\frac{|G_r(y+z)-G_r(y)|^2}{|z|^{2-2H}}\\
&\times\sup_{H\in K}\sup_{(\nu,\mu)\in[0,T]\times\mathbb{R}}\Bigg[C_H\int_0^\nu ds\int_{\mathbb{R}}dv\int_{\mathbb{R}}d\overline{v}\,G^2_{\nu-s}(\mu-v)\frac{\mathrm{E}\Big[|u^H_m(s,v)-u^H_m(s,\overline{v})|^p\Big]^{2/p}}{|v-\overline{v}|^{2-2H}}\Bigg].
\end{aligned}
\tag{2.73}
$$

The supremum appearing in the last term is bounded thanks to the inductive hypothesis on condition *iii)* of (P). The remaining integral

$$C_H \int_0^t dr \int_{\mathbb{R}} dz \int_{\mathbb{R}} dy \frac{|G_r(y+z)-G_r(y)|^2}{|z|^{2-2H}} = c_H \int_0^t dr \int_{\mathbb{R}} |\mathcal{F}G_r(\xi)|^2 |\xi|^{1-2H} d\xi \tag{2.74}$$

by Proposition 2.21 The last term, after extending the integral in dr over $[0,T]$, is bounded thanks to (2.71), taking into account that again c_H, which is given by (2.70), satisfies $c_H \leq \frac{1}{2\pi}$.

This bound for the term $C_H^2 A_1$ is uniform with respect to $H \in K$ and to $(t,x) \in [0,T] \times \mathbb{R}$; indeed, the dependence on $x \in \mathbb{R}$ in the estimates was eliminated using the inductive hypothesis in the last term of (2.73). The dependence on $t \in [0,T]$ is handled by noticing that the last integral in (2.74) is monotone increasing in t, and bounded by (2.71) when $t = T$. Finally, we keep track of the dependence on $H \in K$ thanks to the constant $C_H = \frac{H(1-2H)}{2}$, which basically acts as a normalizer of the integral

$$\int_{|z|>1} \frac{1}{|z|^{2-2H}} dz \xrightarrow{H \to 1/2} \infty,$$

and gives a bound for the term $C_H^2 A_1$ which is also uniform for $H \in K$.

We handle now the term A_2. Using Minkowski inequality for integrals we have that

$$\begin{aligned}
C_H^2 A_2 \leq C_H^2 \int_0^t \int_{\mathbb{R}^2} dz\, dy\, ds \frac{G_{t-s}^2(x-y)}{|z|^{2-2H}} \Bigg(\int_0^s \int_{\mathbb{R}^2} dv\, d\overline{v}\, dr \mathrm{E}\Big[|u_m^H(r,\overline{v})|^p\Big]^{2/p} \\
\times \frac{|(G_{s-r}(y+z-v)-G_{s-r}(y-v)) - (G_{s-r}(y+z-\overline{v}) - G_{s-r}(y-\overline{v}))|^2}{|v-\overline{v}|^{2-2H}} \Bigg) \\
\leq C_H^2 \Bigg(\sup_{(t,x)} \mathrm{E}\Big[|u_m^H(t,x)|^p\Big]^{2/p} \Bigg) \int_0^t \int_{\mathbb{R}^2} dz\, dy\, ds \frac{G_{t-s}^2(x-y)}{|z|^{2-2H}} \int_0^s \int_{\mathbb{R}^2} dv\, d\overline{v}\, dr \\
\times \frac{|(G_{s-r}(y+z-v)-G_{s-r}(y-v)) - (G_{s-r}(y+z-\overline{v}) - G_{s-r}(y-\overline{v}))|^2}{|v-\overline{v}|^{2-2H}}.
\end{aligned} \tag{2.75}$$

Using again Proposition 2.21 and the induction hypothesis on the second condition in (P), we have that the last term is bounded by

$$C_H^2 A_2 \leq C_H c_H C \int_0^t \int_{\mathbb{R}^2} \frac{G_{t-s}^2(x-y)}{|z|^{2-2H}} dy\, ds \int_0^s \int_{\mathbb{R}} |1-e^{-i\xi z}|^2 |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{1-2H} d\xi\, dr.$$

Notice that, while using Proposition 2.21, one of the two constants C_H was "transformed" in c_H, in the same way as (2.74). Since c_H is bounded uniformly for $H \in K$ we include it in the constant C. Thanks to Lemma 2.49, we are able to compute explicitly the integral

$$\int_{\mathbb{R}} \frac{|1-e^{-i\xi z}|^2}{|z|^{2-2H}} dz = \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)} |\xi|^{1-2H},$$

which yields

$$\begin{aligned}
C_H^2 A_2 \leq & C_H \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)} C \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) dy\, ds \\
& \times \int_0^s \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dr \\
\leq & \Gamma(2H+1)\sin(\pi H) C \Big(\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) dy\, ds \Big) \\
& \times \Big(\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dr. \Big)
\end{aligned} \tag{2.76}$$

In the last inequality we used the fact that, since $C_H = \frac{H(1-2H)}{2}$,

$$C_H \frac{2\Gamma(2H+1)\sin(\pi H)}{H(1-2H)} = \Gamma(2H+1)\sin(\pi H).$$

Now we have to bound uniformly in t, x, H the last term of (2.76). First, we observe that $\Gamma(2H+1)\sin(\pi H) \le C$ for $H \in K$, since it is a continuous function of H. Regarding the first integral, we have that

$$\int_0^t \int_{\mathbb{R}} G^2_{t-s}(x-y) = \begin{cases} \frac{t^2}{4} & \text{wave equation,} \\ 4\sqrt{\pi}t^{1/2} & \text{heat equation,} \end{cases}$$

thus it can be bounded by its value at $t = T$, which is moreover finite. The last integral is the usual one appearing in Lemma 2.46, but this time with $\alpha = 2(1-2H)$. We rewrite (2.71) in this case:

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, ds = \begin{cases} 2^{4H-1}\tilde{C}_{2(1-2H)} \frac{1}{4H} T^{4H} & \text{wave equation,} \\ \\ \frac{1}{4H-1}\Gamma\left(\frac{3-4H}{2}\right) T^{\frac{4H-1}{2}} & \text{heat equation.} \end{cases} \tag{2.77}$$

As we said before, the constant $\tilde{C}_{1-2H}$ defines a continuous function from $(0,1)$ to $\mathbb{R}$. Via rescaling, it is easy to see that consequently $\tilde{C}_{2(1-2H)}$ is continuous on $(\frac{1}{4}, \frac{3}{4})$. Notice that the interval $(\frac{1}{4}, \frac{3}{4})$ strictly contains K, either in the form $K = [\eta_1, \eta_2]$ or in the form $K = [\eta_1, \frac{1}{2})$. Thus, $\tilde{C}_{2(1-2H)}$ is bounded uniformly in K. The other quantities appearing in (2.77) are obviously bounded on K. Indeed, $2^{4H-1} \le 2$, $\frac{1}{4H} \le 1$, $\frac{1}{4H-1} \le \frac{1}{4\eta_1 - 1}$, and $\Gamma\left(\frac{3-4H}{2}\right) \le \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Thus, we have bounded the three factors on the right-hand side of (2.76) independently of t, x, H.

Then, we can take the supremum over $(t, x) \in [0, T] \times \mathbb{R}$ and over $H \in K$ in (2.72) to obtain the third condition in (P) for the $(m+1)$-th Picard iteration u^H_{m+1}.

□

We have now that the Picard iterations are well-definite and that they belong to our space of functions χ^H. Moreover, for a fixed $m \in \mathbb{N}$, both their semi-norms are uniformly bounded with respect to H. This will be crucial to prove an estimate in the fashion of Centsov criterion for the Picard iterations $\{u^H_m\}_m$. This kind of estimate will be proved later in Proposition 2.66. Before that, we have to prove one more auxiliary result about the L^p norms of the Picard iterations: we need that they are bounded also uniformly with respect to $m \in \mathbb{N}$. Explicitly, we need:

$$\sup_{m \ge 0} \sup_{H \in K} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathrm{E}\Big[|u^H_m(t,x)|^p\Big] < \infty. \tag{2.78}$$

To obtain this result, we choose to adapt the proof of convergence of the Picard iterations given in [BJQ15] to show that this convergence is also uniform in H, when we consider only the L^p semi-norm. This will give the uniform boundedness (2.78) that we stated.

Define the auxiliary quantities:

$$\begin{aligned} V_m(t) &:= \sup_{H \in K} \sup_{x \in \mathbb{R}} \left(\mathrm{E}\Big[|u^H_m(t,x) - u^H_{m-1}(t,x)|^p\Big]\right)^{2/p} \\ W_m(t) &:= \sup_{H \in K} \sup_{x \in \mathbb{R}} C_H \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y)|y-z|^{2H-2} \\ &\quad \times \left(\mathrm{E}\Big[|u^H_m(s,y) - u^H_{m-1}(s,y) - u^H_m(s,z) + u^H_{m-1}(s,z)|^p\Big]\right)^{2/p} dy\, dz\, ds, \end{aligned}$$

where the constant C_H appearing is again $\frac{H(1-2H)}{2}$. Notice that thanks to Theorem 2.62 both quantities are bounded for any $m \in \mathbb{N}$. The bounds given in Theorem 3.8 of [BJQ15] for these two quantities have been used in Theorem 3.9 of [BJQ15] to show that the Picard iterations converge to a solution in the space χ^H. Here we are going to adapt those two results to the fact that we wish to show that the Picard iterations converge uniformly also with respect to $H \in K$. This will give us the fundamental property (2.78).

Proposition 2.63. *For any $m \geq 0$ and for any $t \in [0, T]$ we have that*

$$V_{m+1}(t) \leq \int_0^t V_m(s) J_1(t-s) ds + C W_m(t) \tag{2.79}$$

and

$$W_{m+1}(t) \leq \int_0^t V_m(s) J_2(t-s) ds + \int_0^t W_m(s) J_1(t-s) ds, \tag{2.80}$$

where J_1 and J_2 are non-negative integrable functions on $[0, T]$.

Proof. The proof just consists in adapting the proof of Theorem 3.8 of [BJQ15] to the fact that we are defining the quantities V_m and W_m taking the supremum also with respect to H. We check first (2.79): we have that

$$\mathrm{E}\Big[|u_{m+1}^H(t,x) - u_m^H(t,x)|^p\Big]^{2/p} \leq C(A_1 + A_2),$$

where

$$\begin{aligned} A_1 =& C_H \Bigg\{ \mathrm{E}\Big[\Big| \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y)|y-z|^{2H-2} \\ &\times |u_m^H(s,y) - u_{m-1}^H(s,y) - u_m^H(s,z) + u_{m-1}^H(s,z)|^2 dy\, dz\, ds \Big|^{p/2}\Big] \Bigg\}^{2/p} \\ A_2 =& C_H \Bigg\{ \mathrm{E}\Big[\Big| \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x-y) - G_{t-s}(x-z)|^2 \\ &\times |u_m^H(s,z) - u_{m-1}^H(s,z)|^2 |y-z|^{2H-2} dy\, dz\, ds \Big|^{p/2}\Big] \Bigg\}^{2/p}, \end{aligned}$$

Thanks to Minkowski's inequality for integrals, the term A_1 satisfies

$$A_1 \leq W_m(t).$$

The estimation of A_2 is a bit more involved: again by Minkowski's inequality for integral, it holds

$$\begin{aligned} A_2 \leq& C_H \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x-y) - G_{t-s}(x-z)|^2 \\ &\times \mathrm{E}\Big[|u_m^H(s,z) - u_{m-1}^H(s,z)|^p\Big]^{2/p} |y-z|^{2H-2} dy\, dz\, ds \\ \leq& C_H \int_0^t V_m(s) \int_{\mathbb{R}^2} |G_{t-s}(x-y) - G_{t-s}(x-z)|^2 |y-z|^{2H-2} dy\, dz\, ds, \end{aligned}$$

so that our candidate $J_1(t)$ is

$$J_1(t) = \sup_{H \in K} C_H \int_{\mathbb{R}^2} |G_t(x-y) - G_t(x-z)|^2 |y-z|^{2H-2} dy\, dz.$$

Here the dependence on $H \in K$ yields the first additional complication with respect to the proof given in Theorem 3.8 of [BJQ15], since we have a different form of J_1. To show that $J_1 \in L^1([0,T])$, first we note that by the change of variable $y' = x - y$ and $z' = x - z$ we can rewrite J_1 as

$$J_1(t) = \sup_{H \in K} C_H \int_{\mathbb{R}^2} |G_t(y) - G_t(z)|^2 |y - z|^{2H-2} dy\, dz \tag{2.81}$$

Then, we rely again on the normalizing constant C_H. We have that by Proposition 2.21

$$C_H \int_{\mathbb{R}^2} |G_t(y) - G_t(z)|^2 |y - z|^{2H-2} dy\, dz = c_H \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^{1-2H} d\xi.$$

We already showed that the constant c_H is bounded for $H \in K$, and thus we have that, by Lemma 2.46

$$\begin{aligned} J_1(t) = &\sup_{H \in K} c_H \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^{1-2H} d\xi \\ \leq &C \sup_{H \in K} \int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^{1-2H} d\xi \\ \leq &C \sup_{H \in K} \begin{cases} C'_H (2H+1) t^{2H} & \text{wave equation,} \\ C''_H H t^{H-1} & \text{heat equation.} \end{cases} \end{aligned} \tag{2.82}$$

Notice that, compared to Lemma 2.46, here we are only integrating in $d\xi$. We differentiated with respect to t that estimate in order to obtain the estimate that we needed. In Section 2.4.3 we showed that the constants C''_H and C''_H appearing are continuous for $H \in (0,1)$, and thus bounded for $H \in K$. The same is true obviously for the additional terms $2H + 1$ and H and thus the function J_1 can be bounded by

$$J_1(t) \leq C \begin{cases} t^{2\eta_1}, & t < 1, \\ t, & t \geq 1 \end{cases}$$

in the case of the wave equation and by

$$J_1(t) \leq C \begin{cases} t^{\eta_1 - 1}, & t < 1, \\ t^{-1/2}, & t \geq 1 \end{cases}$$

in the case of the heat equation, both of which are in $L^1([0,T])$. This concludes the proof of (2.79).

We are left to show that (2.80) holds. We follow again the proof of Theorem 3.8 of [BJQ15]. With a change of variable we rewrite the integral appearing in the definition of $W_{m+1}(t)$ as

$$\begin{aligned} B := C_H \int_0^t \int_{\mathbb{R}^2} &\frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \\ &\times \Big(\mathrm{E}\Big[|u^H_{m+1}(s, y+z) - u^H_m(s, y+z) - u^H_{m+1}(s, y) + u^H_m(s, y)|^p \Big] \Big)^{2/p} dy\, dz\, ds. \end{aligned}$$

Notice that

$$\begin{aligned} &\mathrm{E}\Big[|u^H_{m+1}(s, y+z) - u^H_m(s, y+z) - u^H_{m+1}(s, y) + u^H_m(s, y)|^p \Big] \\ =&\mathrm{E}\Big[\Big| \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}^2} (G_{s-r}(y+z-v) - G_{s-r}(y-v))(u^H_m(r,v) - u^H_{m-1}(r,v)) W^H(dr, dv) \Big|^p \Big]. \end{aligned}$$

If we apply Theorem 2.19 and then we add and subtract a mixed term (see [BJQ15] for the details) we have that $B \leq C(B_1 + B_2)$, where

$$B_1 = C_H^2 \int_0^t ds \int_{\mathbb{R}^2} dy\, dz \frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \Bigg\{ \mathrm{E}\Big[\Big| \int_0^s \int_{\mathbb{R}^2} |G_{s-r}(y+z-v) - G_{s-r}(y-v)|^2 \\ \times |u_m^H(r,v) - u_{m-1}^H(r,v) - u_m^H(r,\overline{v}) + u_{m-1}^H(r,\overline{v})|^2 \frac{1}{|v-\overline{v}|^{2-2H}} dv\, d\overline{v}\, dr \Big|^{p/2} \Big] \Bigg\}^{2/p},$$

$$B_2 = C_H^2 \int_0^t ds \int_{\mathbb{R}^2} dy\, dz \frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \Bigg\{ \mathrm{E}\Big[\Big| \int_0^s dr \int_{\mathbb{R}^2} dv\, d\overline{v} |u_m^H(r,\overline{v}) - u_{m-1}^H(r,\overline{v})|^2 \\ \times \frac{|G_{s-r}(y+z-v) - G_{s-r}(y-v) - G_{s-r}(y+z-\overline{v}) + G_{s-r}(y-\overline{v})|^2}{|v-\overline{v}|^{2-2H}} \Big|^{p/2} \Big] \Bigg\}^{2/p}.$$

Notice that, similarly to the proof of condition iii) of (P), we have that the constant C_H appears raised to the power 2. This is the key point in making our estimations uniform in H. For B_1, we proceed with the Minkowski inequality for integrals and several changes of variables (in the same way as the proof of condition iii) in (P)) to obtain that

$$B_1 \leq C_H^2 C \int_0^t \int_{\mathbb{R}^2} dr\, dz\, dy \frac{|G_r(y+z) - G_r(y)|^2}{|z|^{2-2H}} \Bigg(\int_0^{t-r} \int_{\mathbb{R}^2} ds\, dv\, d\overline{v} G_{t-r-s}(x-y-v)^2 \\ \times \frac{\mathrm{E}\Big[|u_m^H(s,v) - u_{m-1}^H(s,v) - u_m^H(s,\overline{v}) + u_{m-1}^H(s,\overline{v})|^p \Big]^{2/p}}{|v-\overline{v}|^{2-2H}} \Bigg).$$

Consider now the integral inside the parenthesis: if we multiply it by one of the two constants C_H, and we take the supremum with respect to $x-y$ and H we obtain $W_m(t-r)$, so we can write

$$B_1 \leq C_H C \int_0^t W_m(t-r) dr \int_{\mathbb{R}^2} \frac{|G_r(y+z) - G_r(y)|^2}{|z|^{2-2H}} dy\, dz.$$

If we now perform the change of variables $\tau = t - r$ and we take the supremum over H of the integral in $dy\, dz$ (using the other constant C_H), we obtain, by the definition of J_1,

$$B_1 \leq C \int_0^t W_m(\tau) J_1(t-\tau) d\tau$$

We point out that in (2.80) we do not have any constant C appearing in the rightmost addend. This is not a problem since we can re-define, in case $C > 1$, the function J_1 to be CJ_1, and the estimate (2.79) would still clearly hold true. Then we have that

$$B_1 \leq \int_0^t W_m(\tau) J_1(t-\tau) d\tau.$$

Regarding B_2, using again Minkowski inequality for integrals we obtain that

$$\begin{aligned}
B_2 \leq & C_H^2 \int_0^t ds \int_{\mathbb{R}^2} dy\, dz \frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \Big(\int_0^s dr \int_{\mathbb{R}^2} dv\, d\overline{v} \mathrm{E}\Big[|u_m^H(r,\overline{v}) - u_{m-1}^H(r,\overline{v})|^p \Big]^{2/p} \\
& \times \frac{|G_{s-r}(y+z-v) - G_{s-r}(y-v) - G_{s-r}(y+z-\overline{v}) + G_{s-r}(y-\overline{v})|^2}{|v-\overline{v}|^{2-2H}} \Big) \\
\leq & C_H^2 \int_0^t ds \int_{\mathbb{R}^2} dy\, dz \frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \int_0^s dr V_m(r) \\
& \times \Big(\int_{\mathbb{R}^2} dv\, d\overline{v} \frac{|G_{s-r}(y+z-v) - G_{s-r}(y-v) - G_{s-r}(y+z-\overline{v}) + G_{s-r}(y-\overline{v})|^2}{|v-\overline{v}|^{2-2H}} \Big) \\
= & C_H \int_0^t ds \int_{\mathbb{R}^2} dy\, dz \frac{G_{t-s}(x-y)^2}{|z|^{2-2H}} \int_0^s dr V_m(r) \\
& \qquad \times \Big(c_H \int_{\mathbb{R}} |1 - e^{-i\xi z}|^2 |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{1-2H} d\xi \Big).
\end{aligned}$$

In the last step we used again Proposition 2.21. Thanks to Lemma 2.49 we can compute explicitly the integral in dz

$$\int_{\mathbb{R}} \frac{|1-e^{-i\xi z}|}{|z|^{2-2H}} dz = |\xi|^{1-2H} \frac{\Gamma(2H+1)\sin(\pi H)}{C_H}.$$

If we plug this computation inside the last term of the inequality and we use Fubini's theorem we obtain, thanks to the fact that $\Gamma(2H+1)\sin(\pi H) \leq C$, independently from $H \in K$,

$$\begin{aligned}
B_2 \leq & C \int_0^t V_m(r) dr \Big(c_H \int_r^t \int_{\mathbb{R}} G_{t-s}(x-y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dy\, ds \Big). \\
= & C \int_0^t V_m(r) dr \Big(c_H \int_r^t \int_{\mathbb{R}} G_{t-s}(\overline{y})^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, d\overline{y}\, ds \Big).
\end{aligned}$$

Thus, our candidate for being J_2 is

$$J_2(t-r) := \sup_{H \in K} c_H \int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dy\, ds. \tag{2.83}$$

It is not evident that the function J_2 is a function only of the difference $t-r$. Anyway, Lemma 3.3 of [BJQ15] proves it. Indeed, it shows that

$$\int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dy\, ds = \begin{cases} C_{1,H}(t-r)^{4H+1} & \text{wave eq.,} \\ C_{2,H}(t-r)^{2H-1} & \text{heat eq.} \end{cases}$$

Thus we are only left to prove that the constants appearing are bounded uniformly in H, which in our case is crucial to be able to define J_2 as a supremum over $H \in K$. We evaluate directly the integral above, following [BJQ15]. First, we evaluate the integral in $d\xi$ using Lemma 2.46 with $\alpha = 2(1-2H)$

$$\int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi = \begin{cases} \frac{2^{4H-1}\Gamma(2-4H)\sin(\pi(1-2H))}{4H-1} (s-r)^{4H-1} & \text{wave eq.,} \\ \Gamma\Big(\frac{3-4H}{2}\Big)(s-r)^{2H-3/2} & \text{heat eq.} \end{cases} \tag{2.84}$$

We have also that, for any $t \geq 0$:

$$\int_{\mathbb{R}} G_t(y)^2 dy = \begin{cases} t/2, & \text{wave equation,} \\ 2\pi^{1/2} t^{-1/2} & \text{heat equation.} \end{cases} \tag{2.85}$$

We plug this two results together separately for the wave and for the heat equation.

For the wave equation, we have:

$$\begin{aligned}&\int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dz\, ds\\ &= \frac{2^{4H-2}\Gamma(2-4H)\sin(\pi(1-2H))}{4H-1} \int_r^t (t-s)(s-r)^{4H-1} ds.\end{aligned}$$

If we perform the change of variables $v = s-r$ and $v' = v/(t-r)$ we can compute

$$\begin{aligned}\int_r^t (t-s)(s-r)^{4H-1} ds = &\int_0^{t-r} (t-r-v) v^{4H-1} dv\\ =&(t-r)^{4H+1} \int_0^1 (1-v')(v')^{4H-1} dv'\\ =&(t-r)^{4H+1}\beta(2,4H)\end{aligned}$$

where the β function is also given by $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Then we have:

$$\begin{aligned}&\int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dz\, ds\\ &= \frac{2^{4H-2}\Gamma(2-4H)\sin(\pi(1-2H))}{4H-1}\beta(2,4H)(t-r)^{4H+1}\\ &= \hat{C}_H (t-r)^{4H+1}\end{aligned}$$

We want to assure that the constant $\hat{C}_H$ depending on H which appears is bounded for $H \in K$. Recall that the interval K is either of the form $K = [\eta_1, \eta_2]$ with $1/4 < \eta_1 < \eta_2 < 1/2$ or of the form $[\eta_1, 1/2)$ with $\eta_1 \in (1/4, 1/2)$.

Thus the lower bound for H is always a value η_1 strictly bigger than $1/4$, and we do not have any problem if the constant $\hat{C}_H$ is diverging for $H \to 1/4$. On the other hand, we allow our interval K to be extended up to $H = 1/2$, so we have to check that the factors are not diverging for $H \to 1/2$. This is the case only for $\Gamma(2-4H) \sim \frac{1}{2-4H}$ when $H \to 1/2$, which is balanced by the term $\sin(\pi(1-2H)) \sim \pi(1-2H)$ when $H \to 1/2$, giving to their product a finite limit equal to $\pi/2$ as $H \to 1/2$. Thus we can say that the constant $\hat{C}_H$ in the case of the wave equation is bounded by a constant C which does not depend on $H \in K$. Moreover, we already shown that also the constant c_H appearing in (2.83) is uniformly bounded for $H \in K$.

Then, we have:

$$\begin{aligned}J_2(t-r) &= \sup_{H\in K} c_H \int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi\, dy\, ds\\ &= \sup_{H\in K} c_H \hat{C}_H (t-r)^{4H+1},\\ &\le C \sup_{H\in K} (t-r)^{4H+1}\end{aligned}$$

which clearly belongs to $L^1([0,T])$, provided that we bound separately the case $t \le 1$ and $t > 1$ with different exponents, in the same way as we did for J_1.

Regarding the heat equation, the situation is simpler. Indeed, in this case when we plug together again (2.77) and (2.85) we obtain:

$$\begin{aligned}
&\int_r^t \int_{\mathbb{R}} G_{t-s}(y)^2 \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi \, dz \, ds \\
&= 2\pi^{1/2}\Gamma\Big(\frac{3-4H}{2}\Big) \int_r^t (t-s)^{-\frac{1}{2}}(s-r)^{2H-\frac{3}{2}} ds \\
&= 2\pi^{1/2}\Gamma\Big(\frac{3-4H}{2}\Big)\beta\Big(\frac{1}{2}, 2H-\frac{1}{2}\Big)(t-r)^{2H-1} \\
&= \hat{C}_H (t-r)^{2H-1}
\end{aligned}$$

Here the constant $\hat{C}_H$ is clearly bounded for $H \in K$, so that we have also in the case of heat equation that J_2 belongs to $L^1([0,T])$, reasoning as before.

Coming back to our estimation of the term B_2, we have then that it holds

$$B_2 \le \int_0^t V_m(r) J_2(t-r) dr,$$

as we wanted to show. Thus, we can conclude that it holds

$$W_{m+1}(t) \le C(B_1 + B_2) \le C\Big(\int_0^t V_m(s) J_2(t-s) ds + \int_0^t W_m(s) J_1(t-s) ds\Big),$$

which is equivalent to (2.80), provided that we rename the functions J_1 and J_2 after multiplying them by a constant. □

With Proposition 2.63 we just proved, we are able to show a more powerful version of Theorem 3.9 of [BJQ15]:

Theorem 2.64. *For any $H \in (1/4, 1/2)$, the sequence $\{u_m^H,\ m \ge 0\}$ of Picard iterations converges to a $L^2(\Omega)$-continuous process u^H in the space χ^H. This process is the unique mild solution of* (2.38)*.*

Moreover, the uniform L^p convergence is uniform also with respect to $H \in K$, i.e. it holds

$$\sup_{H \in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x) - u^H(t,x)|^p\Big] \xrightarrow{m\to\infty} 0$$

Proof. Most of the things stated in this result have been already proved in Theorem 3.9 of [BJQ15]. We have to check only one thing: that the modified definitions of V_m and W_m work to show that the Picard iterations converge uniformly also with respect to $H \in K$ to the solution u^H. There is no need to check that the solution is the same found in [BJQ15], since for a fixed value of H the norm $||\cdot||_{\chi^H}$ is equivalent to the one defined in Definition 3.6 of [BJQ15], as we noticed in Remark 2.59.

We define:

$$M_m(t) := V_m(t) + W_m(t)$$

and

$$J(t) := C(J_1(t) + J_2(t)).$$

It is immediate to verify that by applying two times recursively Proposition 2.63 we have

$$M_{m+1}(t) \le \int_0^t (M_m(s) + M_{m-1}(s)) J(t-s) ds.$$

We wish to apply a Gronwall lemma (Lemma 3.10 of [BJQ15]). We have to show that

$$\sup_{t\in[0,T]} M_1(t) < \infty,$$

$$\sup_{t\in[0,T]} M_2(t) < \infty.$$

Both requirements are met thanks to conditions ii) and iii) of (P). Thus, we have that

$$\sum_{m\geq 1} \sup_{t\in[0,T]} M_m(t) < \infty,$$

which implies that

$$\sum_{m\geq 1} \sup_{H\in K} ||u_m^H - u_{m-1}^H||_{\chi^H} < \infty.$$

This implies that $\{u_m^H\}_m$ is Cauchy in χ^H, uniformly with respect to $H \in K$, and then the sequence u_m^H converges uniformly in H to his limit u^H, which we already know to be existing and unique. This clearly implies that the convergence is uniform in H for the L^p semi-norm. □

This result yields an immediate useful corollary for our purposes

Corollary 2.65. *Since the sequence of Picard iterations $\{u_m^H\}_m$ converge uniformly in $L^p(\Omega)$ with respect to $t \in [0,T]$, $x \in \mathbb{R}$, $H \in K$, i.e.*

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x) - u^H(t,x)|^p\Big] \xrightarrow{n\to\infty} 0,$$

the family $\{u_m^H,\ m \geq 0, H \in K\}$ satisfies:

$$\sup_{H\in K} \sup_{m\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x)|^p\Big] < \infty.$$

Thanks to these results we are finally able to prove our main Hölder-type estimate for the Picard iterations $\{u_m^H, m \geq 0, H \in K\}$, which is an adaptation to our case of Proposition 2.2 of [BJQ16].

Proposition 2.66. *Fix any $h_0 \in (0,1)$. Then, for all $|h| \leq h_0$*

$$\left.\begin{aligned} \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \Big(\mathrm{E}\Big[|u_m^H(t,x+h) - u_m^H(t,x)|^p\Big]\Big)^{1/p} &\leq C_m|h|^{\eta_1} \\ \sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \Big(\mathrm{E}\Big[|u_m^H(t+h,x) - u_m^H(t,x)|^p\Big]\Big)^{1/p} &\leq C_m|h|^{\tilde{\eta}_1}, \end{aligned}\right\} \tag{Q}$$

where $\tilde{\eta}_1 = \eta_1$ for the wave equation, and $\tilde{\eta}_1 = \eta_1/2$ for the heat equation. The constant C_m satisfies

$$C_m \leq C(c(h_0) + \overline{c}(h_0)C_{m-1}), \tag{2.86}$$

where the functions $c, \overline{c} : \mathbb{R} \to \mathbb{R}$ are non-negative and $\lim_{h_0\to 0} \overline{c}(h_0) = 0$. We define $C_{-1} = 0$.

Proof. We adapt the proof of Proposition 2.2 of [BJQ16] to our case. Again, we have to take care that the proof can be adapted to the fact that here we need estimates which are uniform in H. As usual, when a constant does not depend on any significant quantity we will just denote it with C, and it may change from line to line.

Step 1: we check that for $m = 0$ condition (Q) holds.

We remark that the deterministic solution does not depend on H, since we fixed for all $H \in K$ the same initial conditions u_0, v_0 and we assumed that they are η_1-Hölder continuous functions.

In [BJQ15], the authors prove that for uniformly α-Hölder continuous initial conditions it holds that for the wave equation

$$
\begin{aligned}
\sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0(t,x+h)-I_0(t,x)| &\le C|h|^{\alpha},\\
\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} |I_0(t+h,x)-I_0(t,x)| &\le C(1+h_0^{1-H})|h|^{\alpha},
\end{aligned}
\tag{2.87}
$$

while for the heat equation it holds

$$
\begin{aligned}
\sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0(t,x+h)-I_0(t,x)| &\le C|h|^{\alpha},\\
\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} |I_0(t+h,x)-I_0(t,x)| &\le C|h|^{\alpha/2}.
\end{aligned}
\tag{2.88}
$$

In our case, we have the same estimates but with $\alpha=\eta_1$. Thus, we can take the supremum for $H\in K$ without changing the result, obtaining condition (Q) in the case $m=0$.

Step 2: we are left to show that the two conditions of (Q) hold true for u^H_{m+1}.

We follow the proof given in Proposition 2.2 of [BJQ16], adapting the argument to our necessities, i.e. that the estimation process is independent of H. We start computing the space increments of u^H_{m+1}:

$$
\begin{aligned}
&\mathrm{E}\Big[|u^H_{m+1}(t,x+h)-u^H_{m+1}(t,x)|^p\Big]^{1/p}\\
&\quad\le \mathrm{E}\Bigg[\bigg|I_0(t,x+h)-I_0(t,x)+\int_0^t\int_{\mathbb{R}}\Big(G_{t-s}(x+h-y)-G_{t-s}(x-y)\Big)u^H_m(s,y)W^H(ds,dy)\bigg|^p\Bigg]^{1/p}.
\end{aligned}
$$

To simplify a bit the notation we define:

$$
S^H_m(s,y):=\Big(G_{t-s}(x+h-y)-G_{t-s}(x-y)\Big)u^H_m(s,y).
$$

Thanks to Theorem 2.19, we can write

$$
\begin{aligned}
&\mathrm{E}\Big[|u^H_{m+1}(t,x+h)-u^H_{m+1}(t,x)|^p\Big]^{1/p}\\
&\quad\le C\Bigg(|I_0(t,x+h)-I_0(t,x)|+C_H^{1/2}\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}}|S^H_m(s,y)-S^H_m(s,z)|^2|y-z|^{2H-2}dz\,dy\,ds\Big|^{\frac{p}{2}}\Big]^{\frac{1}{p}}\Bigg)\\
&\quad=C(I_0+I_1).
\end{aligned}
$$

We already proved that $I_0\le|h|^{\eta_1}$ in Step 1. We compute I_1, following the proof of Proposition 2.2 of [BJQ16], together with Theorem 3.7 of [BJQ15]:

$$
I_1\le C(I_{11}+I_{12}),
$$

where

$$
\begin{aligned}
I_{11}=&C_H^{1/2}\Bigg(\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^2}|G_{t-s}(x+h-y)-G_{t-s}(x-y)|^2\frac{|u^H_m(s,y)-u^H_m(s,z)|^2}{|y-z|^{2-2H}}dy\,dz\,ds\Big|^{\frac{p}{2}}\Big]\Bigg)^{\frac{1}{p}}\\
I_{12}=&C_H^{1/2}\Bigg(\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^2}\frac{|u^H_m(s,z)|^2}{|y-z|^{2-2H}}|G_{t-s}(x+h-y)-G_{t-s}(x-y)\\
&\qquad-(G_{t-s}(x+h-z)-G_{t-s}(x-z))|^2dy\,dz\,ds\Big|^{\frac{p}{2}}\Big]\Bigg)^{\frac{1}{p}}
\end{aligned}
$$

We start from I_{11}: we use Minkowski inequality for integrals and we obtain

$$
\begin{aligned}
I_{11}^2 \leq & C_H \int_0^t \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \\
& \times \left(\int_{\mathbb{R}} \frac{\mathrm{E}\Big[|u_m^H(s,y+z) - u_m^H(s,y)|^p\Big]^{2/p}}{|z|^{2-2H}} dz \right) dy\, ds \\
\leq & I'_{11} + I''_{11}
\end{aligned}
$$

where I'_{11} is the integral in dz corresponding to the region $\{|z| > h_0\}$ and I''_{11} is the one corresponding to the region $\{|z| \leq h_0\}$.

We consider first I'_{11}: observe that from Corollary 2.65 we have that

$$
\sup_{H\in K} \sup_{m\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x)|^p\Big] < \infty.
$$

This implies that

$$
I'_{11} \leq C\, C_H \int_0^t \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \Big(\int_{|z|>h_0} \frac{1}{|z|^{2-2H}} dz \Big) dy\, ds.
$$

We compute explicitly the integral in dz:

$$
C_H \int_{|z|>h_0} \frac{1}{|z|^{2-2H}} dz = 2 \frac{H(1-2H)}{2} \frac{-h_0^{2H-1}}{2H-1} = H h_0^{2H-1}.
$$

Coming back to I'_{11}, we have:

$$
\begin{aligned}
I'_{11} \leq & C\, H\, h_0^{2H-1} \int_0^t \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 dy\, ds \\
\leq & C\, H\, h_0^{2H-1} |h| = C\, H \Big(\frac{|h|}{h_0} \Big)^{1-2H} |h|^{2H} \leq C\, H\, |h|^{2H} \leq C |h|^{2\eta_1},
\end{aligned}
$$

which completes the estimate for I'_{11}. Arguing on I''_{11}, we use the induction hypothesis to infer that

$$
\int_{|z|\leq h_0} \frac{\mathrm{E}\Big[|u_m^H(s,y) - u_m^H(s,z)|^p\Big]^{2/p}}{|z|^{2-2H}} dz \leq C_m^2 \int_{|z|\leq h_0} \frac{|z|^{2\eta_1}}{|z|^{2-2H}} dz.
$$

Since $h_0 < 1$, we have that $\frac{1}{|z|^{2-2H}} \leq \frac{1}{|z|^{2-2\eta_1}}$, and thus

$$
C_m^2 \int_{|z|\leq h_0} \frac{|z|^{2\eta_1}}{|z|^{2-2H}} dz \leq C_m^2 \int_{|z|\leq h_0} |z|^{4\eta_1 - 2} dz = 2 C_m^2 \frac{h_0^{4\eta_1-1}}{4\eta_1 - 1} = C\, C_m^2 h_0^{4\eta_1-1}.
$$

Then we have that

$$
\begin{aligned}
I''_{11} \leq & C_H\, C\, C_m^2 h_0^{4\eta_1-1} \int_0^t \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 dy\, ds \\
= & C_H\, C\, C_m^2 h_0^{4\eta_1-1} |h| \\
\leq & C C_m^2 h_0^{2\eta_1} \Big(\frac{|h|}{h_0} \Big)^{1-2\eta_1} |h|^{2\eta_1} \\
\leq & C C_m^2 h_0^{2\eta_1} |h|^{2\eta_1}.
\end{aligned}
$$

Then we managed to show that $I_{11}^2 \leq C|h|^{2\eta_1} + C\, C_m h_0^{2\eta_1} |h|^{2\eta_1}$, which implies

$$
I_{11} \leq C \Big(1 + C_m h_0^{\eta_1} \Big) |h|^{\eta_1}.
$$

We handle now the term I_{12}: we use again Minkowski inequality for integrals and Corollary 2.65 to obtain

$$I_{12}^2 \leq C_H\, C \int_0^t \int_{\mathbb{R}^2} \frac{1}{|y-z|^{2-2H}} |G_{t-s}(x+h-y) - G_{t-s}(x-y) \\ - G_{t-s}(x+h-z) + G_{t-s}(x-z)|^2 dz\, dy\, ds.$$

Thanks to Proposition 2.21, we have that

$$I_{12}^2 \leq C_H\, C \int_0^t \int_{\mathbb{R}} (1-\cos(h|\xi|)) |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds.$$

This last expression can be evaluated thanks to Lemma 2.47, with $\alpha = 1-2H$:

$$I_{12}^2 \leq C_H\, C\, \tilde{C}_H |h|^{2H}.$$

Here the constant $\tilde{C}_H$ appearing is defined as

$$\tilde{C}_H := \int_{\mathbb{R}} (1-\cos(x)) x^{-2H-1} dx,$$

which has been computed in Lemma D.1 of [BJQ15]. As usual, we have to take care that this constant can be bounded by a constant C which is independent from $H \in K$. The explicit values of $\tilde{C}_H$ are, in our setting:

$$\tilde{C}_H = \begin{cases} \frac{\Gamma(1-2H)\cos(\pi H)}{2H} & H \in (0,1/2), \\ \pi/2 & \text{H=1/2}. \end{cases}$$

This function of H is continuous for $H \in K$, since $\cos(\pi H) \sim \frac{\pi}{2}(1-2H)$ and $\Gamma(1-2H) \sim \frac{1}{1-2H}$ as $H \to 1/2$. Thus $\tilde{C}_H \to \pi/2$ as $H \to 1/2$, which implies that $\tilde{C}_H \leq C$, independently from $H \in K$. This means that we have:

$$I_{12} \leq C|h|^H \leq C|h|^{\eta_1}.$$

Putting together all the estimates for the space increments, we have proved that

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_{m+1}^H(t,x+h) - u_{m+1}^H(t,x)|^p\Big]^{1/p} \leq C\Big(C_0 + C_m h_0^{\eta_1}\Big)|h|^{\eta_1},$$

which is basically the same estimate obtained in Proposition 2.2 of [BJQ16], with η_1 replacing H.

We show now the result for the time increments. Again, we follow the steps of Proposition 2.2 of [BJQ16]:

$$\mathrm{E}\Big[|u_{m+1}^H(t+h,x) - u_{m+1}^H(t,x)|^p\Big]^{1/p} \leq C(J_0 + J_1 + J_2).$$

We compute the increments only for $h \geq 0$, since the case $h < 0$ is analogous. The term $J_0 = |I_0(t+h,x) - I_0(t,x)|$ has been estimated in Step 1. J_1 and J_2 are defined respectively by

$$J_1 = C_H^{1/2} \left(\mathrm{E}\Big[\Big| \int_t^{t+h} \int_{\mathbb{R}^2} \frac{|G_{t+h-s}(x-y)u_m^H(s,y) - G_{t+h-s}(x-z)u_m^H(s,z)|^2}{|y-z|^{2-2H}} dz\, dy\, ds\Big|^{p/2}\Big] \right)^{1/p},$$

$$J_2 = C_H^{1/2} \Bigg(\mathrm{E}\Big[\Big| \int_0^t \int_{\mathbb{R}^2} |(G_{t+h-s}(x-y) - G_{t-s}(x-y)u_m^H(s,y)) \\ - (G_{t+h-s}(x-z) - G_{t-s}(x-z)u_m^H(s,z))|^2 \frac{1}{|y-z|^{2-2H}} dz\, dy\, ds\Big|^{p/2}\Big] \Bigg)^{1/p}.$$

We start from J_1. First we apply Minkowski inequality for integrals to obtain

$$\begin{aligned}
J_1^2 \leq & C_H \int_t^{t+h} \int_{\mathbb{R}^2} \frac{\mathrm{E}\Big[|G_{t+h-s}(x-y)u_m^H(s,y) - G_{t+h-s}(x-z)u_m^H(s,z)|^p\Big]^{2/p}}{|y-z|^{2-2H}} dz\,dy\,ds \\
& \leq C(J_{11} + J_{12}),
\end{aligned}$$

where we obtain the two terms J_{11} and J_{12} by adding and subtracting the mixed term $G_{t+h-s}(x-y)u_n^H(s,z)$, so that:

$$\begin{aligned}
J_{11} =& C_H \int_t^{t+h} \int_{\mathbb{R}^2} G_{t+h-s}(x-y)^2 \frac{\mathrm{E}\Big[|u_m^H(s,y) - u_m^H(s,z)|^p\Big]^{2/p}}{|y-z|^{2-2H}} dz\,dy\,ds, \\
J_{12} =& C_H \int_t^{t+h} \int_{\mathbb{R}^2} \mathrm{E}\Big[|u_m^H(s,z)|^p\Big]^{2/p} \frac{|G_{t+h-s}(x-y) - G_{t+h-s}(x-z)|^2}{|y-z|^{2-2H}} dz\,dy\,ds.
\end{aligned}$$

We start with J_{11}: by a change of variables we have that

$$J_{11} = C_H \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y)^2 \left(\int_{\mathbb{R}} \frac{\mathrm{E}\Big[|u_m^H(s,y) - u_m^H(s,y+z)|^p\Big]^{2/p}}{|z|^{2-2H}} dz \right) dy\,ds.$$

We split the dz integral again into the regions $\{|z| \geq h_0\}$ and $\{|z| < h_0\}$, respectively, so that $J_{11} = J_{11}' + J_{11}''$. We already computed the integral

$$C_H \int_{|z|\geq h_0} \frac{1}{|z|^{2-2H}} dz = H\, h_0^{2H-1},$$

so that, thanks also to Corollary 2.65, we have that:

$$\begin{aligned}
J_{11}' \leq & C\, H h_0^{2H-1} \int_0^h \int_{\mathbb{R}} G_s^2(y) dy\,ds \\
\leq & C h_0^{2H-1} |h|^{\gamma/H} \leq C h_0^{2H-1+(\frac{1}{H}-2)\gamma} |h|^{2\gamma}.
\end{aligned}$$

We defined here $\gamma = H$ in the wave equation case and $\gamma = H/2$ in the heat equation case. The last estimation comes again from the fact that $|h|/h_0 \leq 1$ and that $\gamma/H - 2\gamma > 0$. Notice that we made again a key use of the normalizing constant C_H in the J_{11}' step.

To obtain the uniformity with respect to H, we have first to observe that the quantity $2H - 1 - (1/H - 2)\gamma = 0$ when $\gamma = H$ and it is equal to $H - 1/2$ when $\gamma = H/2$. In this second case we can bound $h_0^{H-1/2} \leq h_0^{\eta_1 - 1/2}$. We can conclude that

$$J_{11}' \leq \begin{cases} C h_0^{\eta_1 - 1/2} |h|^{\eta_1} & \text{heat equation,} \\ C |h|^{2\eta_1} & \text{wave equation.} \end{cases} \tag{2.89}$$

To estimate J_{11}'', we have to use the induction hypothesis into the dz integral:

$$\begin{aligned}
J_{11}'' \leq & C_H\, C_m^2 \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y)^2 \left(\int_{|z|\leq h_0} \frac{|z|^{2\eta_1}}{|z|^{2H-2}} dz \right) dy\,ds \\
\leq & C_H\, C_m^2 \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y)^2 \left(\int_{|z|\leq h_0} \frac{1}{|z|^{4\eta_1 - 2}} dz \right) dy\,ds \\
\leq & C\, C_H\, C_m^2 h_0^{4\eta_1 - 1 + \gamma/H - 2\gamma} |h|^{2\gamma} \\
\leq & \begin{cases} C\, C_m^2 h_0^{4\eta_1 - 1} |h|^{\eta_1} & \text{heat equation,} \\ C\, C_m^2 h_0^{4\eta_1 - 1} |h|^{2\eta_1} & \text{wave equation.} \end{cases}
\end{aligned}$$

We estimate now J_{12}: we use Corollary 2.65 and Proposition 2.21 to obtain

$$J_{12} \leq C\, c_H \int_0^h \int_{\mathbb{R}} |\mathcal{F}G_r(\xi)|^2 |\xi|^{1-2H} d\xi dr.$$

Notice that again we made use of the constant C_H, which is now replaced by $c_H \leq C$. We use now Lemma 2.46 to obtain

$$J_{12} \leq \begin{cases} C|h|^H \leq C|h|^{\eta_1} & \text{heat equation,} \\ C|h|^{2H+1} \leq C|h|^{2\eta_1+1} \leq C|h|^{2\eta_1} & \text{wave equation.} \end{cases}$$

This allows us to conclude that:

$$J_1 \leq \begin{cases} C(1 + h_0^{\frac{\eta_1}{2}-\frac{1}{4}} + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\frac{\eta_1}{2}} & \text{heat equation,} \\ C(1 + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\eta_1} & \text{wave equation} \end{cases} \tag{2.90}$$

which is a good estimate, since $2\eta_1 - \frac{1}{2} > 0$. e compute now J_2: we follow again Proposition 2.2 of [BJQ16], using again Minkowski inequality for integrals and obtaining that $J_1^2 \leq C(J_{21}+J_{22})$, where

$$\begin{aligned} J_{21} =& C_H \int_0^t \int_{\mathbb{R}^2} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 \\ & \times \left(\mathrm{E}\left[|u_m^H(s,y) - u_m^H(s,y+z)|^p\right]\right)^{2/p} |z|^{2H-2} dz\, dy\, ds, \\ J_{22} =& C_H \int_0^t \int_{\mathbb{R}^2} |G_{t+h-s}(x-y) - G_{t-s}(x-y) - G_{t+h-s}(x-z) + G_{t-s}(x-z)|^2 \\ & \times \left(\mathrm{E}\left[|u_m^H(s,z)|^p\right]\right)^{2/p} |y-z|^{2H-2} dz\, dy\, ds. \end{aligned}$$

As for I_{11} and J_{11}, we split the integral in the two regions $\{|z| \geq h_0\}$ and $\{|z| < h_0\}$, giving respectively the two terms J'_{21} and J''_{21}. Using Corollary 2.65 we have that

$$\begin{aligned} J'_{21} \leq & C\, H\, h_0^{2H-1} \int_0^t \int_{\mathbb{R}} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 dy\, ds \\ \leq & C\, h_0^{2H-1} |h|^{\gamma/H} \leq C\, h_0^{2H-1+(\frac{1}{H}-2)\gamma} |h|^{2\gamma} \\ \leq & \begin{cases} C h_0^{\eta_1 - 1/2} |h|^{\eta_1} & \text{heat equation,} \\ C|h|^{2\eta_1} & \text{wave equation.} \end{cases} \end{aligned}$$

The last step we made is identical to (2.89), while previously we used again the normalizing constant C_H as usual and we used the fact that the last deterministic integral

$$\int_0^t \int_{\mathbb{R}} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 dy\, ds$$

is bounded by $Ct|h|$ for the wave equation and by $Ch^{1/2}$ for the heat equation, as it is pointed out in page 24 of [BJQ15]. For J''_{21}, as usual we use the induction hypothesis to infer that, identically to I''_{11} and J''_{11}:

$$\begin{aligned} J''_{21} \leq & C\, C_H h_0^{4\eta_1 - 1} C_m^2 |h|^{\gamma/H} \\ \leq & \begin{cases} C\, C_m^2 h_0^{4\eta_1-1} |h|^{\eta_1} & \text{heat equation,} \\ C\, C_m^2 h_0^{4\eta_1-1} |h|^{2\eta_1} & \text{wave equation.} \end{cases} \end{aligned}$$

We are left to show that J_{22} can be bounded appropriately. We use Corollary 2.65 and Proposition 2.21 to obtain that

$$\begin{aligned}J_{22} \leq & C\, C_H \int_0^t \int_{\mathbb{R}^2} |G_{t+h-s}(x-y) - G_{t-s}(x-y) \\ & \qquad\qquad - G_{t+h-s}(x-z) + G_{t-s}(x-z)|^2 dz\, dy\, ds \\ = & C\, c_H \int_0^t \int_{\mathbb{R}} |\mathcal{F}G_{t+h-s}(\xi) - \mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds\end{aligned}$$

This can be estimated using Lemma 2.48 with $\alpha = 1 - 2H$. We have to take care that this estimate is independent from H. Looking at the proof of Lemma 2.48, given in [BJQ15], we see that it holds the following:

$$\int_0^t \int_{\mathbb{R}} |\mathcal{F}G_{t+h-s}(\xi) - \mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi\, ds \leq \begin{cases} CT|h|^{2H} & \text{wave equation,} \\ C|h|^H & \text{heat equation.} \end{cases}$$

Here the constants appearing are

$$C = C' \int_{\mathbb{R}} \frac{\min(1,|y|)^2}{|y|^{1+2H}} dy$$

(with C' independent from H) for the wave equation and

$$C = \int_{\mathbb{R}} \frac{(1-e^{-y^2/2})^2}{|y|^{1+2H}} dy$$

for the heat equation. We have to show that both are bounded uniformly with respect to $H \in K$. But this is clear since in both cases the integrand is summable for any $H \in K$ and it can be bounded by replacing $|y|^{1+2H}$ with $|y|^2$ when $|y| \leq 1$ and by $|y|^{1+2\eta_1}$ when $|y| > 1$, giving an estimation which is independent of $H \in K$.

This allows us to conclude for J_{22} that

$$J_{22} \leq \begin{cases} C|h|^H \leq C|h|^{\eta_1} & \text{heat equation,} \\ C|h|^{2H} \leq C|h|^{2\eta_1} & \text{wave equation.} \end{cases}$$

We have then, considering the whole J_2:

$$J_2 \leq \begin{cases} C(1 + h_0^{\frac{\eta_1}{2}-\frac{1}{4}} + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\frac{\eta_1}{2}} & \text{heat equation,} \\ C(1 + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\eta_1} & \text{wave equation.} \end{cases} \tag{2.91}$$

As we did for the space increments, we put now together all the estimates for the components J_0, J_1, J_2 of the time increments, obtaining

$$\begin{aligned}\mathrm{E}\Big[|u^H_{m+1}(t+h,x) - u^H_{m+1}(t,x)|^p\Big]^{1/p} & \\ \leq & C(J_0 + J_1 + J_2) \\ \leq & \begin{cases} C(1 + h_0^{\frac{\eta_1}{2}-\frac{1}{4}} + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\eta_1/2} & \text{heat equation,} \\ C(1 + C_m h_0^{2\eta_1-\frac{1}{2}})|h|^{\eta_1} & \text{wave equation.} \end{cases} \\ \leq & C\Big(c(h_0) + C_m \bar{c}(h_0)\Big)|h|^{\tilde{\eta}_1}.\end{aligned}$$

Notice that this is true because the estimates of J_1 and J_2 yield an identical result. We see clearly that the constant C_{m+1} can be defined as

$$C_{m+1} := C\Big(c(h_0) + C_m \bar{c}(h_0)\Big),$$

and that since $2\eta - \frac{1}{2} > 0$ the functions $c(h_0), \overline{c}(h_0)$ satisfy the our thesis, i.e. that they are non-negative and $\overline{c}(h_0) \to 0$ as $h\_0 \to 0$. We see that, as it happened for the space increments, those estimates are independent from $H \in K$ and $(t,x) \in [0,T] \times \mathbb{R}$, so that we can take the supremum with respect to those variables in the last inequality to obtain

$$\begin{aligned}\sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} & \mathrm{E}\Big[|u^H_{m+1}(t+h,x) - u^H_{m+1}(t,x)|^p\Big]^{1/p} \\ &\leq C\Big(c(h_0) + C_m\overline{c}(h_0)\Big)|h|^{\tilde{\eta}_1} \\ &= C_{m+1}|h|^{\tilde{\eta}_1},\end{aligned}$$

which concludes our proof. □

Combining Proposition 2.66 and Corollary 2.65 we finally have the following result:

Proposition 2.67. *There exists $h_0 > 0$ such that for every $|h| \leq h_0$ it holds*

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x+h) - u^H(t,x)|^p\Big] \leq C|h|^{p\eta_1}$$

and

$$\sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t+h,x) - u^H(t,x)|^p\Big] \leq C|h|^{p\tilde{\eta}_1},$$

where C is a constant depending only on p. Here $\tilde{\eta}_1 = \eta_1$ for the wave equation and $\tilde{\eta}_1 = \eta_1/2$ for the heat equation.

Proof. Recall that by Proposition 2.66 we have that

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H_m(t,x+h) - u^H_m(t,x)|^p\Big] \leq C_m|h|^{p\eta_1}$$

and

$$\sup_{H\in K} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathrm{E}\Big[|u^H_m(t+h,x) - u^H_m(t,x)|^p\Big] \leq C_m|h|^{p\tilde{\eta}_1},$$

where the sequence of constants $\{C_m\}_m$ satisfies (2.86). So we wish to take the limit for $m \to \infty$ on both sides on the inequalities and check that the same result still holds.

For the left-hand side, it is sufficient to notice that thanks to the uniform convergence in $L^p(\Omega)$ stated in Corollary 2.65 it holds:

$$\begin{aligned}&\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H_m(t,x+h) - u^H_m(t,x)|^p\Big] \\ &\xrightarrow{m\to\infty} \sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x+h) - u^H(t,x)|^p\Big].\end{aligned}$$

Thus we are only left to show that the limit can be taken also on the right-hand side, i.e. that the sequence of constants C_m is bounded, provided that we choose a suitable value for h_0. This has been already shown in Theorem 1.1 of [BJQ16], under the same hypothesis that we have here, so that the proof is complete. □

Now we have all that it is needed to prove our tightness result Proposition 2.61.

Proof. (Proposition 2.61). First, we notice that condition *i)* is clearly satisfied, since $u^H(0,0)$ is deterministic and independent from H, so that

$$\sup_{H\in K} \mathrm{E}[|u^H(0,0)|^p] = \sup_{H\in K} u^H(0,0) = \sup_{H\in K} u_0(0) = u_0(0) < \infty$$

We have to prove *ii)*. First, we rewrite

$$\mathrm{E}\Big[|u^H(t',x')-u^H(t,x)|^p\Big]^{1/p}=\mathrm{E}\Big[|u^H(t+h,x+k)-u^H(t,x)|^p\Big]^{1/p}$$

and we notice that

$$\begin{aligned}&\mathrm{E}\Big[|u^H(t+h,x+k)-u^H(t,x)|^p\Big]\\&\quad\le C\Big(\mathrm{E}\Big[|u^H(t+h,x+k)-u^H(t,x+k)|^p\Big]+\mathrm{E}\Big[|u^H(t,x+k)-u^H(t,x)|^p\Big]\Big).\end{aligned}$$

This two terms can be estimated separately using Proposition 2.67, uniformly in t,x,H, obtaining

$$\mathrm{E}\Big[|u^H(t+h,x+k)-u^H(t,x)|^p\Big]^{1/p}\le C(|h|^{p\tilde{\eta}_1}+|k|^{p\eta_1}).$$

Thanks to the uniformity with respect to t,x,H, we can conclude that for every t',t and x',x such that $|t'-t|<h_0$ and $|x'-x|<h_0$ it holds

$$\mathrm{E}\Big[|u^H(t',x')-u^H(t,x)|^p\Big]\le C(|t'-t|^{p\tilde{\eta}_1}+|x'-x|^{p\eta_1}).\tag{2.92}$$

As we already said, these estimates hold only for $|h|\le h_0$, but we need such an estimate to hold also when the increment $|h|>h_0$. This is not a problem, since whenever $|h|>h_0$ it is sufficient to update the constant C in order to have that

$$2^p\sup_{H\in K}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathrm{E}\Big[|u^H(t,x)|^p\Big]\le C\,h_0^{p\eta_1}.$$

□

Tightness in the case $H\in(\frac{1}{2},1)$

We wish to prove an analogous tightness result as Proposition 2.61 for the case $H>\frac{1}{2}$. We state it in Proposition 2.68 below.

Suppose now that the limiting exponent $H_0\in[\frac{1}{2},1)$, so whenever $H_n\to H_0$ we can suppose without loss of generality that $H_n\in K$, where K is either of the form $[\eta_1,\eta_2]$, with $\eta_1,\eta_2\in[\frac{1}{2},1)$ and $\eta_1\le\eta_2$. As we already observed at the beginning of Section 2.4.5, if we prove the tightness of the set of measures $\{u^H,\,H\in K\}$ also for K of the form considered here, this will include also the case in which $H_0=\frac{1}{2}$ and $H_n\to H_0$ neither from above nor from below.

We already pointed out before that, thanks to the fact that Theorem 2.17 has a stronger thesis than Theorem 2.19, we will be able to give a direct proof of the estimate needed for Centsov criterion, without having to pass through Picard iterations. We will prove the following:

Proposition 2.68. *Let $\mathcal{U}_K:=\{u^H,\,H\in K\}$ be the family of solutions of* (2.38)*, where K is of the form $[\eta_1,\eta_2]$, with $\eta_1,\eta_2\in[\frac{1}{2},1)$ and $\eta_1\le\eta_2$. Then, the family $\mathcal{U}_K$ is tight in $\mathcal{C}([0,T]\times\mathbb{R})$, endowed with the metric of uniform convergence on compact sets.*

Proof. We use again Centsov criterion Theorem 2.54. Thus, we have to prove again that, for every $t,t'\in[0,T]$ and for every $x,x'\in\mathbb{R}$:

$$\sup_{H\in[\eta_1,\eta_2]}\mathrm{E}\Big[|u^H(t',x')-u^H(t,x)|^p\Big]\le C(|t'-t|+|x'-x|)^{\delta}.\tag{2.93}$$

We split the proof in two steps: first, we prove a uniform $L^p(\Omega)$ boundedness result. Then, we prove (2.93).

Step 1: We show that

$$\sup_{H\in[\eta_1,\eta_2]}\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] < \infty. \tag{2.94}$$

To do this, we prove that the Picard iterations of our equation $\{u_n^H\}_{n,H}$ satisfy the following boundedness condition

$$\sup_{H\in[\eta_1,\eta_2]}\sup_{n\geq 0}\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(t,x)|^p\Big] < \infty. \tag{2.95}$$

Since for any $H \in [\eta_1, \eta_2]$ it holds that the Picard iteration scheme converges uniformly (in t, x) in $L^p(\Omega)$ to the relative solution u^H as $n \to \infty$ (Theorem 13 of [Dal99]), we have that for the set of solutions $\{u^H\}_H$ it holds (2.94). To obtain (2.95), we proceed as in page 22 of [Dal99]:

$$\begin{aligned}
\mathrm{E}\Big[|u_{n+1}^H(t,x)|^p\Big] \leq& C_p\Big(|I_0(t,x)|^p + \mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u_n^H(s,y)W^H(dy,ds)\Big|^p\Big]\Big)\\
\leq& C + C\,\mathrm{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)u_n^H(s,y)W^H(dy,ds)\Big|^p\Big]\\
=:& C + C\,J.
\end{aligned} \tag{2.96}$$

We apply now Theorem 2.17 to obtain

$$\begin{aligned}
J \leq& C\, c_H\,(\nu_{t,H})^{\frac{p}{2}-1}\int_0^t ds\Big(\sup_{x\in\mathbb{R}}\mathrm{E}\Big[|u_n^H(s,x)|^p\Big]\Big)\int_{\mathbb{R}}|\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2|\xi|^{1-2H}d\xi\\
\leq& C\, c_H(\nu_{t,H})^{\frac{p}{2}-1}\int_0^t ds\Big(\sup_{H\in[\eta_1,\eta_2]}\sup_{x\in\mathbb{R}}\mathrm{E}\Big[|u_n^H(s,x)|^p\Big]\Big)\int_{\mathbb{R}}|\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2|\xi|^{1-2H}d\xi.
\end{aligned} \tag{2.97}$$

Here, $\nu_{t,H}$ is defined as

$$\begin{aligned}
\nu_{t,H} :=& c_H\int_0^t\int_{\mathbb{R}}|\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2|\xi|^{1-2H}d\xi ds\\
=& c_H\int_0^t\int_{\mathbb{R}}|e^{i\xi x}\mathcal{F}G_{s'}(\xi)|^2|\xi|^{1-2H}d\xi ds'\\
=& c_H\int_0^t\int_{\mathbb{R}}|\mathcal{F}G_{s}(\xi)|^2|\xi|^{1-2H}d\xi ds.
\end{aligned}$$

Observe that for every $H \in (0,1)$ it holds $c_H := \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \leq \frac{1}{\pi},$. We notice moreover that $\nu_{t,H}$ is almost equal to the last integral appearing in (2.97), except for the fact that in $\nu_{t,H}$ we are also integrating with respect to the time variable. This difference does not allow us to consider both terms as a unique term. Therefore, we take them into account separately in our calculations.

We start with $\nu_{t,H}$: thanks to Lemma 2.46, this term, both in the case of the heat and the wave equations, is bounded uniformly in t and H, i.e.

$$\sup_{H\in[\eta_1,\eta_2]}\sup_{t\in[0,T]}\nu_{t,H} = C < \infty. \tag{2.98}$$

We compute now the $d\xi$ integral in (2.97). Here the estimation is more involved, at least for the case of the heat equation. In both cases we follow the explicit computations on page 13 of [BJQ15].

We start from the wave equation case:

$$
\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi = \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi \\
&= \int_{\mathbb{R}} \frac{\sin^2((t-s)|\xi|)}{|\xi|^2} |\xi|^{1-2H} d\xi = 2\int_0^\infty \frac{\sin^2((t-s)\xi)}{\xi^{1+2H}} d\xi \\
&= 2(t-s)^{2H} \int_0^\infty \frac{\sin^2(x)}{x^{1+2H}} dx = 2(t-s)^{2H} 2^{2H-1} C_{1-2H} \\
&= (t-s)^{2H} 2^{2H} C_{1-2H} \le T^{2H} 2^{2H} C_{1-2H},
\end{aligned}
\tag{2.99}
$$

where we used the change of variable $x = (t-s)\xi$ and the last integral is the same appearing in page 13 of [BJQ15]. The constant C_{1-2H} appearing is the same one appearing in Lemma 2.46. We already showed that C_{1-2H} defines a continuous function for $H \in (0,1)$, so it can be bounded by a constant C when $H \in [\eta_1, \eta_2]$. Thus we can conclude that in the case of the wave equation

$$
\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi = C < \infty.
$$

This implies that

$$
J \le C \int_0^t ds \Big(\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_n^H(s,x)|^p \Big] \Big).
$$

This quantity does not depend on H and x, so that we can infer from (2.96) that it holds

$$
\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_{n+1}^H(t,x)|^p \Big] \le C + C \int_0^t ds \Big(\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_n^H(s,x)|^p \Big] \Big).
$$

We can use now the classical Grönwall lemma with $f_n(s) := \sup_{H,x} \mathrm{E}\Big[|u_n^H(s,x)|^p \Big]$ and $g = C$ to conclude that

$$
\sup_{n\ge 0} \sup_{t\in[0,T]} \sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_n^H(t,x)|^p \Big] < \infty,
$$

which is what we wanted to show.

In the case of the heat equation, the computations are more involved, since in the estimate (2.99) the term $(t-s)$ appears raised to a negative power, and then it cannot be bounded uniformly by a constant for any $s \le t$. We have thus to use Lemma 15 of [Dal99] in his full generality. In detail, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi = \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi \\
&= \int_{\mathbb{R}} e^{-t|\xi|^2} |\xi|^{1-2H} d\xi = 2\int_0^\infty e^{-tx^2} x^{1-2H} dx \\
&= (t-s)^{H-1} \int_0^\infty e^{-y} y^{-H} dy = \Gamma(1-H)(t-s)^{H-1} =: g_H(t-s).
\end{aligned}
$$

Observe that for all $H \in [\eta_1, \eta_2]$ we have $g_H(t-s) \le g(t-s)$, where

$$
\tilde{g}(t-s) := \Gamma(1-\eta_2) \times \begin{cases} (t-s)^{\eta_1 - 1}, & s \in (t-1, t] \\ 1, & s \in [0, t-1]. \end{cases}
$$

Plugging this result into the estimation of J we have that

$$
J \le C \int_0^t ds \Big(\sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_n^H(s,x)|^p \Big] \Big) \tilde{g}(t-s).
$$

Again, this estimate is uniform with respect to H and x, so that we can take the supremum on the left-hand side of (2.96) as we did in the case of the wave equation, and conclude by Lemma 15 of [Dal99] (with f_n defined as before and $g(t-s) = C\tilde{g}(t-s)$) that (2.95) holds. Given (2.95), we have that (2.94) clearly follows

Step 2: We want to prove that (2.93) holds. We compute separately the time and the space increments, and then we deduce the general result. We start from the space increments.

We prove that, given $p > 2$, there exists C such that for every $|h| \leq 1$ it holds

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x+h) - u^H(t,x)|^p\Big] \leq C|h|^{\eta_1 p} \tag{2.100}$$

We start computing

$$\begin{aligned}
\mathrm{E}\Big[|u^H(t,x+h) - u^H(t,x)|^p\Big] \leq& C\Bigg(|I_0(t,x+h) - I_0(t,x)|^p \\
&+ \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h-y) - G_{t-s}(x-y)]u^H(s,y)W^H(dy,ds)\Big|^p\Big]\Bigg) \\
=:& \, C(A_1 + A_2)
\end{aligned}$$

Thanks to (2.88) and (2.87) we have that $\sup_{t,x} A_1 \leq C|h|^{\eta_1}$. Regarding A_2, we use Theorem 2.17 to obtain

$$\begin{aligned}
A_2 \leq& C(\nu_{t,H})^{\frac{p}{2}-1} \int_0^t ds \Big(\sup_{y\in\mathbb{R}} \mathrm{E}\Big[|u^H(s,y)|^p\Big]\Big) \\
&\times c_H \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t-s}(x-\cdot) + G_{t-s}(x+h-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi.
\end{aligned}$$

We point out that the quantity $\nu_{t,H}$ also depends of the integrand function, which in this case is $[G_{t-s}(x+h-y) - G_{t-s}(x-y)]u^H(s,y)$. We see how to handle this soon.

Thanks to Step 1, we can take the supremum of $\mathrm{E}\Big[|u^H(s,y)|^p\Big]$ also with respect to t and H, and obtain

$$\begin{aligned}
A_2 \leq& C(\nu_{t,H})^{\frac{p}{2}-1} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \sup_{H\in[\eta_1,\eta_2]} \mathrm{E}\Big[|u^H(t,x)|^p\Big] \\
&\times \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t-s}(x-\cdot) + G_{t-s}(x+h-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\
=& C(\nu_{t,H})^{\frac{p}{2}} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \sup_{H\in[\eta_1,\eta_2]} \mathrm{E}\Big[|u^H(t,x)|^p\Big] \\
\leq& C(\nu_{t,H})^{\frac{p}{2}},
\end{aligned}$$

where in the process we recognized that the integral appearing after taking the supremum was exactly equal to $\nu_{t,H}$. So we are only left to give an estimate of

$$(\nu_{t,H})^{\frac{p}{2}} := \Big(c_H \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t-s}(x-\cdot) + G_{t-s}(x+h-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds\Big)^{\frac{p}{2}}.$$

We will make use of Lemma 2.47:

$$
\begin{aligned}
(\nu_{t,H})^{\frac{p}{2}} =&\Big(c_H \int_0^t \int_{\mathbb{R}} |e^{i\xi x}(1-e^{-i\xi h})\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds\Big)^{\frac{p}{2}} \\
=&\Big(c_H \int_0^t \int_{\mathbb{R}} |1-e^{-i\xi h}|^2 |\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds\Big)^{\frac{p}{2}} \\
=&\Big(2c_H \int_0^t \int_{\mathbb{R}} (1-\cos(h\xi)) |\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds\Big)^{\frac{p}{2}} \\
\leq&\Big(2c_H \int_0^T \int_{\mathbb{R}} (1-\cos(h\xi)) |\mathcal{F}G_s(\xi)|^2 |\xi|^{1-2H} d\xi ds\Big)^{\frac{p}{2}} \\
\leq& C\Big(\tilde{C}_H |h|^{2H}\Big)^{\frac{p}{2}}.
\end{aligned}
$$

The constant $\tilde{C}_H$ is the same appearing in Lemma 3.4 of [BJQ15], and it is given by

$$
\tilde{C}_H := \int_{\mathbb{R}} (1-\cos(\eta))|\eta|^{-1-2H} d\eta < \frac{1}{H} + \frac{1}{1-H} \leq C,
$$

provided that $H \in [\eta_1, \eta_2]$. Thus, we have

$$
\sup_{H\in[\eta_1,\eta_2]} \sup_{t\in[0,T]} (\nu_{t,H})^{\frac{p}{2}} \leq C|h|^{Hp} \leq C|h|^{\eta_1 p},
$$

since $|h| \leq 1$. As an immediate consequence, we have that

$$
\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} A_2 \leq C|h|^{\eta_1 p}
$$

Since both the estimate for A_1 and for A_2 are independent from t, x, H we have that

$$
\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x+h) - u^H(t,x)|^p\Big] \leq C(A_1 + A_2) \leq C|h|^{\eta_1},
$$

which is exactly (2.100).

We need to estimate now the time increments: again, we want to prove that given $p \geq 2$ there exists a constant C such that for every $|h| \leq 1$ it holds

$$
\begin{aligned}
&\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0\vee(-h),T\wedge(T-h)]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t+h,x) - u^H(t,x)|^p\Big] \\
&\qquad \leq \begin{cases} C|h|^{\eta_1 p} & \text{wave equation,} \\ C|h|^{\frac{\eta_1}{2} p} & \text{heat equation.} \end{cases}
\end{aligned}
\tag{2.101}
$$

We compute, supposing $h > 0$ (the case $h < 0$ is analogous)

$$
\begin{aligned}
\mathrm{E}\Big[|u^H(t+h,x) - u^H(t,x)|^p\Big] \leq& C\Bigg(|I_0(t+h,x) - I_0(t,x)|^p \\
&+ \mathrm{E}\Bigg[\Bigg|\int_0^t \int_{\mathbb{R}} [G_{t+h-s}(x-y) - G_{t-s}(x-y)] u^H(s,y) W^H(dy,ds)\Bigg|^p\Bigg] \\
&+ \mathrm{E}\Bigg[\Bigg|\int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}(x-y) u^H(s,y) W^H(dy,ds)\Bigg|^p\Bigg]\Bigg) \\
=:& C(B_1 + B_2 + B_3).
\end{aligned}
$$

As for the space increments, from (2.88) and (2.87) we have that (recall that the deterministic solution I_0 is by construction independent from H)

$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} B_1 \leq \begin{cases} C|h|^{\frac{\eta_1}{2}p} & \text{heat equation,} \\ C|h|^{\eta_1 p} & \text{wave equation.} \end{cases}$$

Regarding B_2, we have by Theorem 2.17

$$\begin{aligned} B_2 \leq & C(\nu_{t,H})^{\frac{p}{2}-1} \int_0^t ds \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u^H(s,x)|^p\Big] \\ & \times c_H \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t+h-s}(x-\cdot) - G_{t-s}(x-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds. \\ \leq & C(\nu_{t,H})^{\frac{p}{2}-1} \sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x)|^p\Big] \\ & \times c_H \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t+h-s}(x-\cdot) - G_{t-s}(x-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\ \leq & C(\nu_{t,H})^{\frac{p}{2}}. \end{aligned}$$

We are then left to estimate $\nu_{t,H}$.

$$\begin{aligned} \nu_{t,H} = & c_H \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{t+h-s}(x-\cdot) - G_{t-s}(x-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\ = & c_H \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{s'+h}(x-\cdot) - G_s(x-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\ \leq & C \int_0^T \int_{\mathbb{R}} \Big|\mathcal{F}\Big(G_{s'+h}(x-\cdot) - G_s(x-\cdot)\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\ = & C \int_0^T \int_{\mathbb{R}} \Big|\mathcal{F}G_{s'+h}(\xi) - \mathcal{F}G_s(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds \\ \leq & C \begin{cases} C_{1,H} T|h|^{2H} & \text{wave equation,} \\ C_{2,H}|h|^{H} & \text{heat equation.} \end{cases} \end{aligned}$$

The last inequality holds thanks to Lemma 2.48.The constants $C_{1,H}, C_{2,H}$ are different, but we already showed in Section 2.4.3 that both can be bounded by $\frac{1}{H} + \frac{1}{1-H}$, which is a continuous function on $(0,1)$, and thus bounded on $[\eta_1, \eta_2]$ by a constant C. This means that

$$(\nu_{t,H})^{\frac{p}{2}} \leq \begin{cases} C|h|^{Hp} \leq C|h|^{\eta_1 p} & \text{wave equation,} \\ C|h|^{\frac{H}{2}p} \leq C|h|^{\frac{\eta_1}{2}p} & \text{heat equation.} \end{cases}$$

Since these estimates are now uniform with respect to t, x, H, we can infer that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} B_2 \leq \begin{cases} C|h|^{\eta_1 p} & \text{wave equation,} \\ C|h|^{\frac{\eta_1}{2}p} & \text{heat equation.} \end{cases}$$

We are left with estimating the term B_3:

$$\begin{aligned}
B_3 =&\mathrm{E}\Big[\Big|\int_0^T\int_{\mathbb{R}} 1_{[t,t+h]}(s)G_{t+h-s}(x-y)u^H(s,y)W^H(dy,ds)\Big|^p\Big]\\
\le&C(\nu_{t,H})^{\frac{p}{2}-1}\int_0^T ds\sup_{y\in\mathbb{R}}\mathrm{E}\Big[|u^H(s,y)|^p\Big]\\
&\times c_H\int_{\mathbb{R}}|1_{[t,t+h]}(s)\mathcal{F}G_{t+h-s}(x-\cdot)(\xi)|^2|\xi|^{1-2H}d\xi\\
\le&C(\nu_{t,H})^{\frac{p}{2}-1}\sup_{H\in[\eta_1,\eta_2]}\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathrm{E}\Big[|u^H(t,x)|^p\Big]\\
&\times c_H\int_0^T\int_{\mathbb{R}}1_{[t,t+h]}(s)|\mathcal{F}G_{t+h-s}(x-\cdot)(\xi)|^2|\xi|^{1-2H}d\xi ds\\
\le&C(\nu_{t,H})^{\frac{p}{2}}.
\end{aligned}$$

We have to estimate again $(\nu_{t,H})$: we perform the change of variables $s'=t+h-s$ and we use again Lemma 2.46 to obtain

$$\begin{aligned}
\nu_{t,H} =&c_H\int_0^T\int_{\mathbb{R}}1_{[0,h]}(s')|\mathcal{F}G_{s'}(\xi)|^2|\xi|^{1-2H}d\xi ds'\\
\le&C\int_0^h\int_{\mathbb{R}}|\mathcal{F}G_{s'}(\xi)|^2|\xi|^{1-2H}d\xi ds'\\
\le&C\begin{cases}\tilde{C}_{1,H}|h|^{1+2H} & \text{wave equation,}\\ \tilde{C}_{2,H}|h|^{H} & \text{heat equation.}\end{cases}
\end{aligned}$$

As we pointed out right after (2.71), the two constants $\tilde{C}_{1,H}$ and $\tilde{C}_{2,H}$ are continuous functions with respect to H on $(0,1)$. This means we can bound both with a constant C independent from $H\in[\eta_1,\eta_2]$. Thus we have

$$(\nu_{t,H})^{\frac{p}{2}}\le C\begin{cases}|h|^{p/2+Hp}\le|h|^{\eta_1 p} & \text{wave equation,}\\ |h|^{\frac{p}{2}H}\le|h|^{\frac{p}{2}\eta_1} & \text{heat equation.}\end{cases}$$

Since also in this case the estimate is independent from t,x,H, we can conclude that

$$\sup_{H\in[\eta_1,\eta_2]}\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}}B_3\le C\begin{cases}|h|^{\eta_1 p} & \text{wave equation,}\\ |h|^{\frac{p}{2}\eta_1} & \text{heat equation.}\end{cases}$$

Putting all the estimates together, we conclude that (2.101) holds.

Our final aim is to prove (2.93). We suppose that at least one of the two increments $|t'-t|$ and $|x'-x|$ is smaller than 1. Indeed, if both $|t'-t|\ge 1$ and $|x'-x|\ge 1$ we have, thanks to Step 1, the stronger estimate

$$\sup_{H\in[\eta_1,\eta_2]}\mathrm{E}\Big[|u^H(t',x')-u^H(t,x)|^p\Big]\le C.$$

Given t', t, x', x, we define $h = t' - t$ and $k = x' - x$ and we have

$$\begin{aligned}
\mathrm{E}\Big[|u^H(t',x') - u^H(t,x)|^p\Big] =& \mathrm{E}\Big[|u^H(t+h,x+k) - u^H(t,x)|^p\Big] \\
\leq& C\bigg(\mathrm{E}\Big[|u^H(t+h,x+k) - u^H(t,x+k)|^p\Big] + \mathrm{E}\Big[|u^H(t,x+k) - u^H(t,x)|^p\Big]\bigg) \\
\leq& C\bigg(\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t+h,x) - u^H(t,x)|^p\Big] \\
&+ \sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u^H(t,x+k) - u^H(t,x)|^p\Big]\bigg) \\
\leq& \begin{cases} C\Big(|h|^{\eta_1 p} + |k|^{\eta_1 p}\Big) = C\Big(|t'-t|^{\eta_1 p} + |x'-x|^{\eta_1 p}\Big) & \text{wave equation,} \\ C\Big(|h|^{\frac{\eta_1}{2} p} + |k|^{\eta_1 p}\Big) = C\Big(|t'-t|^{\frac{\eta_1}{2} p} + |x'-x|^{\eta_1 p}\Big) & \text{heat equation,} \end{cases}
\end{aligned}$$

so it suffices to take $p > \frac{4}{\eta_1}$ for the heat equation and $p > \frac{2}{\eta_1}$ to obtain (2.93) and conclude the proof.

□

We conclude this section by proving an easy consequence of Step 1 of the proof above, which extends Corollary 2.65 to the case $H \geq 1/2$. This result will be a key tool in the next section, when we identify the limit distribution.

Lemma 2.69. *Let $H \geq 1/2$ and $\{u_n^H\}_n$ be the sequence of Picard iterations* (2.42) *which converge to the solution of the mild formulation* (2.38)*. Then u_n^H converges in $L^p(\Omega)$ to the solution u^H uniformly also with respect to $H \in [\eta_1, \eta_2]$, i.e. it holds*

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(t,x) - u^H(t,x)|^p\Big] \xrightarrow{n\to\infty} 0.$$

Proof. It is sufficient to use the Gronwall lemma in the same fashion as Theorem 13 of [Dal99], with some care to the fact that the result has to be uniform also with respect to $H \in [\eta_1, \eta_2]$. We start by computing

$$\begin{aligned}
\mathrm{E}\Big[|u_{n+1}^H(t,x) - u_n^H(t,x)|^p\Big] =& \mathrm{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)[u_n^H(s,y) - u_{n-1}^H(s,y)]W^H(ds,dy)\Big|^p\Big] \\
\leq& c_p(\nu_{t,H})^{\frac{p}{2}-1} \int_0^t ds \Big(\sup_{x\in\mathbb{R}} \sup_{H\in[\eta_1,\eta_2]} \mathrm{E}\Big[|u_n^H(s,x) - u_{n-1}^H(s,x)|^p\Big]\Big) \\
&\times c_H \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 |\xi|^{1-2H} d\xi.
\end{aligned}$$

We have already noticed in (2.98) that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{t\in[0,T]} \nu_{t,H} < \infty.$$

Afterwards, we also proved that the $d\xi$ integral can be bounded by a constant C uniformly in $H, t,$ and x, All of these facts imply that

$$\mathrm{E}\Big[|u_{n+1}^H(t,x) - u_n^H(t,x)|^p\Big] \leq C \int_0^t \sup_{x\in\mathbb{R}} \sup_{H\in[\eta_1,\eta_2]} \mathrm{E}\Big[|u_n^H(s,x) - u_{n-1}^H(s,x)|^p\Big] ds.$$

If we define now

$$M_n(t) = \sup_{s\in[0,t]} \sup_{H\in[\eta_1,\eta_2]} \sup_{x\in\mathbb{R}} \mathrm{E}\Big[|u_{n+1}^H(s,x) - u_n^H(s,x)|^p\Big],$$

we have that it holds

$$M_n(t) \leq C \int_0^t M_{n-1}(s)ds,$$

which, together with the fact that

$$\sup_{t\in[0,T]} M_0(t) = \sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_1(t,x) - u_0(t,x)|^p\Big] < \infty,$$

implies by the classical Gronwall lemma that $\sum_{n\geq 0} M_n(T)$ converges, which implies that

$$\sup_{H\in[\eta_1,\eta_2]} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_n^H(t,x) - u^H(t,x)|^p\Big] \xrightarrow{n\to\infty} 0.$$

In fact, we already know that the (t,x)–uniform limit in $L^p(\Omega)$ of every u_n^H exists and it is equal to u^H. □

Identification of the limit

Let now $H_0 \in (\frac{1}{4}, 1)$ and consider $\{H_n,\, n \in \mathbb{N}\}$ such that $H_n \to H_0$ as $n \to \infty$. The tightness, that we showed in Proposition 2.61 and Proposition 2.68, of the set of probability measures induced by $\{u^{H_n}\}_n$ on $\mathcal{C}([0,T]\times\mathbb{R})$ implies that there exists a subsequence H_{n_k} such that

$$u^{H_{n_k}} \xrightarrow{d} Y, \qquad \text{as } k \to \infty$$

where Y is a stochastic process with a.s. continuous trajectories and the convergence is in the usual weak sense.

We want to identify this limit with u^{H_0}. To do this, we will check that given an arbitrary sequence H_n converging to H_0, the finite dimensional distributions of u^{H_n} will converge to those of u^{H_0}. This is sufficient to conclude the proof thanks to Theorem 2.6 of [Bil]: given an arbitrary subsequence $u^{H_{n_k}}$, thanks to our tightness results we would have that it has a further subsequence $u^{H_{n_{k_\ell}}}$ that converges to some process $\tilde{Y}$, and the identification of the finite dimensional convergence of $u^{H_{n_{k_\ell}}} \to u^{H_0}$ would allow us to use Theorem 2.6 of [Bil].

Before stating and proving the main result of this section, we need some preliminaries. The main idea is to show the convergence of the finite dimensional distributions of u^{H_n} to the ones of u^{H_0} by showing the $L^2(\Omega)$ convergence $u^{H_n}(t,x) \to u^{H_0}(t,x)$, for any fixed $(t,x) \in [0,T]\times\mathbb{R}$. This can be done thanks to the fact that, in Subsection 2.2.3, we defined all our noises $\{W^H,\, H\in(0,1)\}$ on the same probability space. The idea to show $u^{H_n}(t,x) \to u^{H_0}(t,x)$ is to show that $u_m^{H_n}(t,x) \to u_m^{H_0}(t,x)$ in $L^2(\Omega)$ as $n\to\infty$ for the m-th Picard iteration, and then to extend this to u^{H_n} thanks to a uniform limiting argument in m.

In order to show the convergence result for the Picard iterations, we will make use of Theorem 2.42, that allows us to see the m-th Picard iteration as

$$u_m^{H_n}(t,x) = \sum_{j=0}^{m} I_j^{H_n}(g_j(\cdot,t,x)),$$

where the latter is a finite sum of multiple Wiener integrals of order up to m and the g_j are given by (2.43). Then, the multiple Wiener integrals of order $j \leq m$ relative to different values of H can be compared explicitly thanks to the representation result for I_n^H given by Theorem 2.14.

We are now ready to prove our final result which, in turn, completes the proof of Theorem 2.45 in the linear multiplicative case.

Proposition 2.70. *Let $H_0 \in (\frac{1}{4}, 1)$, and $H_n \to H_0$. Let $\{u^{H_n}\}_n$ and u^{H_0} be the solutions of* (2.38) *with the correspondent noise. Then the finite dimensional distributions of u^{H_n} converge to those of u^{H_0}, as $n \to \infty$.*

Proof. In order to show the finite dimensional weak convergence we will make use of the uniform $L^p(\Omega)$ convergence of the Picard iterations u_m^H obtained in Corollary 2.65 for $H \le 1/2$ and in Lemma 2.69 for $H \ge 1/2$. It is sufficient to consider $p = 2$. We have:

$$\sup_{H\in K} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[|u_m^H(t,x) - u^H(t,x)|^2\Big] \xrightarrow{n\to\infty} 0.$$

As we already suggested, it is sufficient to show the stronger pointwise $L^2(\Omega)$ convergence. Thus, we show that, for any fixed $(t,x) \in [0,T] \times \mathbb{R}$:

$$\mathrm{E}\Big[|u^{H_n}(t,x) - u^{H_0}(t,x)|^2\Big] \xrightarrow{n\to\infty} 0. \tag{2.102}$$

We have that:

$$\begin{aligned}
\mathrm{E}\Big[|u^{H_n}(t,x) - u^{H_0}(t,x)|^2\Big] \le& C\Big(\mathrm{E}\Big[|u^{H_n}(t,x) - u_m^{H_n}(t,x)|^2\Big] \\
&+ \mathrm{E}\Big[|u_m^{H_n}(t,x) - u_m^{H_0}(t,x)|^2\Big] \\
&+ \mathrm{E}\Big[|u_m^{H_0}(t,x) - u^{H_0}(t,x)|^2\Big]\Big) \\
=:& I_1(m,n) + I_2(m,n) + I_3(m).
\end{aligned}$$

Thanks to Corollary 2.65 for $H \le 1/2$ and Lemma 2.69 for $H \ge \frac{1}{2}$, we can infer that, for a given $\varepsilon > 0$, we can choose m_0 big enough such that for every $m \ge m_0$ we have

$$\sup_{n\in\mathbb{N}} \Big[I_1(n,m) + I_3(m)\Big] < \varepsilon/2.$$

Thus we are left to show that $I_2(m_0, n) \to 0$ as $n \to \infty$. This, in particular, means that we need to show that the m_0-th Picard iteration is continuous in $L^2(\Omega)$ with respect to H.

By Theorem 2.42, we have that for any $H \in (\frac{1}{4}, 1)$ the Picard iterations $u_{m_0}^H$ can be represented as a sum of iterated integrals up to the order m_0. Precisely, we have that

$$u_m^H(t,x) = \sum_{j=0}^{m_0} I_j^H(g_j(\cdot, t, x)),$$

where the functions g_j are defined by (2.43). Since m_0 is fixed, we are considering a finite sum. Thus it suffices to show the the convergence in $L^2(\Omega)$, with respect to H, for a single iterated integral $I_j^H(g_j)$. We compute, using the representation result Theorem 2.14:

$$\begin{aligned}
I_j^{H_n}&(g_j)(t,x) - I_j^{H_0}(g_j)(t,x) \\
&= \int_{\{[0,T]\times\mathbb{R}\}^j} \Big((c_{H_n})^j |\xi_1|^{1/2-H_n} \cdots |\xi_j|^{1/2-H_n} - (c_{H_0})^j |\xi_1|^{1/2-H_0} \cdots |\xi_j|^{1/2-H_0}\Big) \\
&\qquad \times \mathcal{F}(g_j(t_1, \cdot, \dots, t_j, \cdot, t, x))(\xi_1, \dots, \xi_n) \tilde{W}(dt_1, d\xi_1) \cdots \tilde{W}(dt_j, d\xi_j)
\end{aligned} \tag{2.103}$$

Now, we use the classical Itô isometry to compute now

$$\begin{aligned}
\mathrm{E}\Big[|I_j^{H_n}&(g_j)(t,x) - I_j^{H_0}(g_j)(t,x)|^2\Big] \\
&= \int_{\{[0,T]\times\mathbb{R}\}^j} \Big|(c_{H_n})^j |\xi_1|^{1/2-H_n} \cdots |\xi_j|^{1/2-H_n} - (c_{H_0})^j |\xi_1|^{1/2-H_0} \cdots |\xi_j|^{1/2-H_0}\Big|^2 \\
&\qquad \times |\mathcal{F}(g_j(t_1, \cdot, \dots, t_j, \cdot, t, x))(\xi_1, \dots, \xi_n)|^2 d\xi_1 \cdots d\xi_j \, dt_1 \cdots dt_j.
\end{aligned}$$

We show that the last integral converges to 0 when $n \to \infty$. To do this, we have to compute explicitly the Fourier transform appearing in the above expression. Recall that

$$\mathcal{F}G_t(\xi) = e^{-\frac{t|\xi|^2}{2}}, \text{ for the heat equation,}$$

and

$$\mathcal{F}G_t(\xi) = \frac{\sin(t|\xi|)}{\xi}, \text{ for the wave equation .}$$

Moreover, in page 10 of [BJQ17] the authors computed explicitly the multiple Fourier transform

$$\begin{aligned}
&\mathcal{F}g_j(t_1, \cdot, \dots, t_j, \cdot, t, x)(\xi_1, \dots, \xi_j) \\
&\qquad = \eta e^{-i(\xi_1 + \cdots + \xi_j)x} \overline{\mathcal{F}G_{t_2 - t_1}(\xi_1)}\, \overline{\mathcal{F}G_{t_3 - t_2}(\xi_1 + \xi_2)} \cdots \\
&\qquad\quad \times \overline{\mathcal{F}G_{t - t_j}(\xi_1 + \cdots + \xi_j)} 1_{\{0 < t_1 < \cdots t_j < t\}}
\end{aligned}$$

Plugging in this expression in the last integral and performing basic computations (e.g. consider the integral on the simplex $T_j(t) := \{(t_1, \dots, t_j) | 0 < t_1 < \cdots < t_j < t\}$), we obtain

$$\begin{aligned}
&\mathrm{E}\Big[|I_j^{H_n}(g_j)(t,x) - I_j^{H_0}(g_j)(t,x)|^2\Big] \\
&\leq \int_{T_j(t)} \int_{\mathbb{R}^j} d\xi_1 \cdots d\xi_j\, dt_1 \cdots dt_j \Big| \eta \mathcal{F}G_{t_2 - t_1}(\xi_1)\, \mathcal{F}G_{t_3 - t_2}(\xi_1 + \xi_2) \cdots \\
&\qquad \times \mathcal{F}G_{t - t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 \\
&\qquad \times \Big| (c_{H_n})^j |\xi_1|^{1/2 - H_n} \cdots |\xi_j|^{1/2 - H_n} - C(H_0)^j |\xi_1|^{1/2 - H_0} \cdots |\xi_j|^{1/2 - H_0} \Big|^2 \\
&= \int_{T_j(t)} \int_{\mathbb{R}^j} d\xi_1 \cdots d\xi_j\, dt_1 \cdots dt_j \eta \Big(\prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1} - t_\ell}(\eta_\ell)|^2 \Big) \\
&\qquad \times \Big| (c_{H_n})^j |\eta_1|^{1/2 - H_n} |\eta_2 - \eta_1|^{1/2 - H_n} \cdots |\eta_j - \eta_{j-1}|^{1/2 - H_n} \\
&\qquad\qquad - (c_{H_0})^j |\eta_1|^{1/2 - H_0} |\eta_2 - \eta_1|^{1/2 - H_0} \cdots |\eta_j - \eta_{j-1}|^{1/2 - H_0} \Big|^2,
\end{aligned}$$

where we used the change of variables $\eta_\ell := \xi_1 + \cdots + \xi_\ell$, for $\ell = 1, \dots, j$. We wish to use dominated convergence to show that this integral converges to 0. It is clear that we have almost sure pointwise convergence to 0 of the integrand function on $T_j(t) \times \mathbb{R}^j$, for any fixed $t, x \in [0,T] \times \mathbb{R}$ (here t, x act as parameters of the integral, and moreover we do not need uniform estimates in this case). Indeed, for this it is sufficient to notice that the constant c_H is continuous as a function of H on $(0,1)$. Moreover, to show that an integrable upper bound exists, we observe first that c_H it is also bounded by a constant C independent from H, whenever $H \in K$, where K is any compact subset of $(0,1)$ (see its definition in Theorem 2.14).

Now, we show that the integrand can be bounded. We start by bounding it with the two terms:

$$\begin{aligned}
&\Big(\prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1} - t_\ell}(\eta_\ell)|^2 \Big) \Big| (c_{H_n})^j |\eta_1|^{\frac{1}{2} - H_n} |\eta_2 - \eta_1|^{\frac{1}{2} - H_n} \cdots |\eta_j - \eta_{j-1}|^{\frac{1}{2} - H_n} \\
&\qquad - (c_{H_0})^j |\eta_1|^{\frac{1}{2} - H_0} |\eta_2 - \eta_1|^{\frac{1}{2} - H_0} \cdots |\eta_j - \eta_{j-1}|^{\frac{1}{2} - H_0} \Big|^2, \\
&\leq \Big(\prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1} - t_\ell}(\eta_\ell)|^2 \Big) 2 \Bigg((c_{H_n})^{2j} |\eta_1|^{1 - 2H_n} |\eta_2 - \eta_1|^{1 - 2H_n} \cdots |\eta_j - \eta_{j-1}|^{1 - 2H_n} \\
&\qquad + (c_{H_0})^{2j} |\eta_1|^{1 - 2H_0} |\eta_2 - \eta_1|^{1 - 2H_0} \cdots |\eta_j - \eta_{j-1}|^{1 - 2H_0} \Bigg).
\end{aligned}$$

The two resulting terms are of the same type, except for the fact that the first one also depends on n, and they are equivalent to the integrands studied in [BJQ17], page 11-13 (in the case of wave equation only, and only for $H \in (\frac{1}{4}, \frac{1}{2})$). From now on we will only consider the part of the integrand function that depends on n; for the other part, its integrability will be an immediate consequence of the first case.

We study now separately the term

$$\Big(\prod_{\ell=1}^{j} |\mathcal{F}G_{t_{\ell+1}-t_\ell}(\eta_\ell)|^2\Big)(c_{H_n})^{2j}|\eta_1|^{1-2H_n}|\eta_2-\eta_1|^{1-2H_n}\cdots|\eta_j-\eta_{j-1}|^{1-2H_n} \tag{2.104}$$

in the cases $H_n \geq 1/2$ and $H_n \leq 1/2$. In the former, the bounding function can be produced quite simply, while in the latter we need some more effort.

Let us start from the latter case, i.e. $H_n \leq 1/2$: we want to produce an integrable function independent of n which bounds the integrand above. We use the following fact: whenever $H \in (0, \frac{1}{2})$,

$$\prod_{\ell=2}^{j} |\eta_\ell - \eta_{\ell-1}|^{1-2H} \leq \sum_{\alpha\in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_\ell}, \tag{2.105}$$

where D_j is a set with cardinality 2^{j-1}. Its elements are multi-indices $\alpha = (\alpha_1, \ldots, \alpha_j)$ whose sum equals $(j-1)(1-2H)$ and satisfy

$$\alpha_1 \in \{0, 1-2H\}, \text{ and } \alpha_\ell \in \{0, 1-2H, 2(1-2H)\}, \text{ for } \ell = 2, \ldots, j.$$

When $H = H_n$, we denote the α_ℓ as $\alpha_{\ell,n}$, replacing H with H_n in their definition. The fact that (2.105) holds true is based on the inequality, which holds for $a, b > 0$ and $p \in (0, 1]$:

$$(a+b)^p \leq a^p + b^p.$$

Using this inequality with $p = 1-2H > 0$ and doing some computations (page 12, [BJQ17]), we obtain (2.105). We can bound the integrand (2.104) (after removing the constant $c_{H_n} \leq C$) with:

$$\begin{aligned}\prod_{\ell=1}^{j} &\Big|\mathcal{F}G_{t_\ell-t_{\ell-1}}(\eta_\ell)\Big|^2 |\eta_1|^{1-2H_n}\cdots|\eta_j-\eta_{j-1}|^{1-2H_n} \\ &\leq \prod_{\ell=1}^{j} \Big|\mathcal{F}G_{t_\ell-t_{\ell-1}}(\eta_\ell)\Big|^2 |\eta_1|^{1-2H_n} \sum_{\alpha\in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_{\ell,n}}\end{aligned}$$

Let us remark that, for every $H \in (1/4, 1/2)$ and in the case of the wave equation, these integrands have already been shown, in pages 11-13 of [BJQ17], to be integrable on our domain. If we extend this result to the heat equation case, we can use the upper bound function defined in the following way: let $\beta_1 = \min_n H_n > 1/4$ and $\beta_2 = 1/2$. We define $f_0, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ as

$$f_0(|x|) = 1,$$

$$f_1(|x|) = \begin{cases} |x|^{1-2\beta_1} & |x| \geq 1, \\ |x|^{1-2\beta_2} = 1 & |x| < 1, \end{cases}$$

$$f_2(|x|) = \begin{cases} |x|^{2(1-2\beta_1)} & |x| \geq 1, \\ |x|^{2(1-2\beta_2)} = 1 & |x| < 1. \end{cases}$$

Define, for every $\alpha_{\ell,n}$, the quantity

$$N(\alpha_{\ell,n}) := \begin{cases} 0 & \alpha_{\ell,n} = 0, \\ 1 & \alpha_{\ell,n} = 1 - 2H_n, \\ 2 & \alpha_{\ell,n} = 2(1 - 2H_n). \end{cases}$$

We are now ready to bound the integrand with

$$\begin{aligned} &\prod_{\ell=1}^{j} \left| \mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell) \right|^2 |\eta_1|^{1-2H_n} \sum_{\alpha \in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_{\ell,n}} \\ \le &\prod_{\ell=1}^{j} \left| \mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell) \right|^2 f_1(|\eta_1|) \sum_{\alpha \in D_j} \prod_{\ell=1}^{j} f_{N(\alpha_{\ell,n})}(|\eta_\ell|). \end{aligned} \tag{2.106}$$

The last function is integrable; to see this it is sufficient to divide the space $T_j(t) \times \mathbb{R}^j$ in the 2^j regions generated by all the possible combinations $|\eta_\ell| \ge 1$ or $|\eta_\ell| < 1$, for $\ell = 1, \dots, j$. Then, in order to show that each of these reduced integrals is bounded, it suffices to bound it with the integral on the whole space, which we will show now to be bounded.

To check this last fact, it is sufficient to show it for a single integrand of the form

$$\prod_{\ell=1}^{j} \left| \mathcal{F}G_{t_\ell - t_{\ell-1}}(\eta_\ell) \right|^2 |\eta_1|^{\beta} \sum_{\alpha \in D_j} \prod_{\ell=1}^{j} |\eta_\ell|^{\alpha_\ell} \tag{2.107}$$

where, this time, the β, α_j do not take values into a discrete set, but they satisfy the weaker constraints

$$\beta \in K_0 \subset [0, 1/2), \ \alpha_1 \in K_1 \subset [0, 1/2), \text{ and } \alpha_\ell \in K_2 \subset [0, 1), \text{ for } \ell = 2, \dots, j,$$

where the sets K_1, K_2 are fixed compact and given by $K_1 = [0, 1 - 2\min H_n]$ and $K_2 = [0, 2(1 - 2\min H_n)]$ (we are assuming implicitly that $\min H_n < 1/2$; if this is not the case, then the entire sequence falls in the case $H_n \ge 1/2$, which we will study afterwards). It is important to notice that these sets K_0, K_1, K_2 are fixed, given the sequence H_n. The fact that $1 - 2\min H_n < 1/2$ and $2(1 - 2\min H_n) < 1$ turns out to be crucial for our estimates.

We write the integral of (2.107) on $T_j(t) \times \mathbb{R}^j$ explicitly (see [BJQ17], page 12)

$$\begin{aligned} &\int_{T_j(t)} \Big(\int_{\mathbb{R}} |\mathcal{F}G_{t_2 - t_1}(\eta_1)|^2 |\eta_1|^{\beta + \alpha_1} d\eta_1 \Big) \\ &\qquad \times \prod_{\ell=2}^{j} \Big(\int_{\mathbb{R}} |\mathcal{F}G_{t_{\ell+1} - t_\ell}(\eta_\ell)|^2 |\eta_\ell|^{\alpha_\ell} d\eta_\ell \Big) dt_1 \cdots dt_j. \end{aligned} \tag{2.108}$$

Notice that here we do not have any summation over D_j; in fact, we are considering only a single term. From now on, we have to consider separately the wave equation case and the heat equation case. As we already pointed out in (2.82), for any $\gamma \in (-1, 1)$ it holds

$$\int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^{\gamma} d\xi \le \begin{cases} C'_\gamma (2 - \gamma) t^{1-\gamma} & \text{wave equation,} \\ C''_\gamma \frac{1-\gamma}{2} t^{-\frac{(\gamma+1)}{2}} & \text{heat equation.} \end{cases}$$

We recall that the constants C'_γ and C''_γ are continuous with respect to $\gamma \in (-1, 1)$. This time we will use $\gamma = 1 - 2H$ and $\gamma = 2(1 - 2H)$, and still we can bound them uniformly with respect to $H \in K \subset (\frac{1}{4}, \frac{1}{2}]$, with K compact.

Using this result in (2.108) we obtain, for the heat equation,

$$\int_{T_j(t)} (t_2 - t_1)^{\frac{-\beta-\alpha_1}{2}} \prod_{\ell=2}^{j} (t_{\ell+1} - t_\ell)^{\frac{-\alpha_\ell - 1}{2}} dt_1 \cdots dt_j = C < \infty$$

The last integral is bounded thanks to the fact that the time intervals are finite and all the exponents are strictly greater than -1.

For the wave equation, we have

$$\int_{T_j(t)} (t_2 - t_1)^{1-\beta-\alpha_1} \prod_{\ell=2}^{j} (t_{\ell+1} - t_\ell)^{1-\alpha_\ell} dt_1 \cdots dt_j = C < \infty,$$

In this case, the exponents are even greater than 0. This concludes the proof in the case $H \in (\frac{1}{4}, \frac{1}{2}]$.

In the case $H \in [\frac{1}{2}, 1)$, the computations are much less involved. Recall that we essentially have to bound the integrand in the following integral

$$\int_{T_j(t)} \int_{\mathbb{R}^j} \Big(\prod_{\ell=1}^{j} \Big| \mathcal{F}G_{t_{\ell+1}-t_\ell}(\xi_1 + \cdots + \xi_\ell) \Big|^2 \Big) |\xi_1|^{1-2H_n} \cdots |\xi_j|^{1-2H_n} d\xi_1 \cdots d\xi_j dt_1 \cdots dt_j. \quad (2.109)$$

Here, the fact that $1 - 2H_n < 0$ is helping us. Indeed, we can define the bounding function in a quite straightforward way: let

$$g(|x|) := \begin{cases} 1 & |x| > 1, \\ |x|^{1-2(\max_n H_n)} & |x| < 1. \end{cases}$$

Clearly, the integrand function in (2.109) is bounded for any $n \in \mathbb{N}$ by

$$\Big(\prod_{\ell=1}^{j} \Big| \mathcal{F}G_{t_{\ell+1}-t_\ell}(\xi_1 + \cdots + \xi_\ell) \Big|^2 \Big) g(|\xi_1|) \cdots g(|\xi_j|).$$

We check that this upper bound function is integrable. We can compute

$$\begin{aligned} \int_{T_{j-1}(t_j)} \int_{\mathbb{R}^{j-1}} d\xi_1 \cdots d\xi_{j-1} dt_1 \cdots dt_{j-1} \Bigg(\Big(\prod_{\ell=1}^{j-1} \Big| \mathcal{F}G_{t_{\ell+1}-t_\ell}(\xi_1 + \cdots + \xi_\ell) \Big|^2 g(|\xi_\ell|) \Big) \\ \times \int_{t_{j-1}}^{t} \int_{\mathbb{R}} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 g(|\xi_j|) d\xi_j dt_j \Bigg). \end{aligned} \quad (2.110)$$

We compute:

$$\begin{aligned} &\int_{t_{j-1}}^{t} \int_{\mathbb{R}} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 g(|\xi_j|) d\xi_j dt_j \\ &= \int_{t_{j-1}}^{t} \int_{|\xi_j|>1} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 g(|\xi_j|) d\xi_j dt_j \\ &\quad + \int_{t_{j-1}}^{t} \int_{|\xi_j| \le 1} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 g(|\xi_j|) d\xi_j dt_j \\ &= \int_{t_{j-1}}^{t} \int_{|\xi_j|>1} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 d\xi_j dt_j \\ &\quad + \int_{t_{j-1}}^{t} \int_{|\xi_j| \le 1} \Big| \mathcal{F}G_{t-t_j}(\xi_1 + \cdots + \xi_j) \Big|^2 |\xi_j|^{1-2\min_n H_n} d\xi_j dt_j. \end{aligned}$$

We check separately the wave equation case and the heat equation case: for the wave equation, it holds that

$$|\mathcal{F}G_t(\xi)| = \Big|\frac{\sin(t|\xi|)}{|\xi|}\Big| \le t, \qquad \text{for every } \xi \in \mathbb{R}, \text{ for every } t \in [0,T]$$

Thus, we have

$$\int_{t_{j-1}}^{t}\int_{|\xi_j|\le 1}\Big|\mathcal{F}G_{t-t_j}(\xi_1+\cdots+\xi_j)\Big|^2|\xi_j|^{1-2\min_n H_n}d\xi_j dt_j$$
$$\le \int_{t_{j-1}}^{t}\int_{|\xi_j|\le 1}|t-t_j|^2|\xi_j|^{1-2\min_n H_n}d\xi_j dt_j \le \frac{CT^3}{1-\min_n H_n} < \infty,$$

and

$$\int_{t_{j-1}}^{t}\int_{|\xi_j|>1}\frac{\sin^2\Big[(t-t_j)|\xi_1+\cdots+\xi_j|\Big]}{|\xi_1+\cdots+\xi_j|^2}d\xi_j dt_j$$
$$\le \int_{t_{j-1}}^{t}\int_{\mathbb{R}}\frac{\sin^2\Big[(t-t_j)|\xi_1+\cdots+\xi_j|\Big]}{|\xi_1+\cdots+\xi_j|^2}d\xi_j dt_j = \int_{t_{j-1}}^{t}C(t-t_j)dt_j < \infty,$$

since $\int_{\mathbb{R}}\frac{\sin^2(t|x|)}{|x|^2}dx = \pi t$.

We can now repeat exactly the same computations, starting from (2.110) in the case $j-1, j-2, \ldots, 1$ to obtain easily the integrability of our upper bound for the wave equation case.

For the heat equation, we have

$$|\mathcal{F}G_t(\xi)| = |e^{-\frac{t|\xi|^2}{2}}| \le C, \qquad \text{for every } \xi \in \mathbb{R}, \text{ for every } t \in [0,T].$$

We have

$$\int_{t_j}^{t}\int_{|\xi_j|\le 1}\Big|\mathcal{F}G_{t-t_j}(\xi_1+\cdots+\xi_j)\Big|^2|\xi_j|^{1-2\min_n H_n}d\xi_j dt_j$$
$$\le \int_{t_j}^{t}\int_{|\xi_j|\le 1}C|\xi_j|^{1-2\min_n H_n}d\xi_j dt_j \le \frac{CT}{1-\min_n H_n} = C,$$

and

$$\int_{t_j}^{t}\int_{|\xi_j|>1}\exp\Big(-\frac{(t-t_j)|\xi_1+\cdots+\xi_j|^2}{2}\Big)d\xi_j dt_j$$
$$\le \int_{t_j}^{t}\int_{\mathbb{R}}\exp\Big(-\frac{(t-t_j)|\xi_1+\cdots+\xi_j|^2}{2}\Big)d\xi_j dt_j = \int_{t_j}^{t}C\sqrt{t-t_j}dt_j < \infty,$$

which, again by iterating this computation, shows that the upper bound function is bounded also in the heat equation case. This completes the proof. □

We put together the pieces of the proof of Theorem 2.45 in our standing case.

Proof (Theorem 2.45, linear multiplicative case). It is sufficient to notice that thanks to Proposition 2.61 and Proposition 2.68 we have that the sequence of probability measures induced by $\{u^{H_n},\, n \in \mathbb{N}\}$ is tight on $\mathcal{C}([0,T]\times\mathbb{R})$.

This fact, together with the finite dimensional $L^2(\Omega)$ convergence proven in Proposition 2.70, shows that $u^{H_n} \xrightarrow{d} u^{H_0}$ on $\mathcal{C}([0,T]\times\mathbb{R})$, when $n \to \infty$.

□

3 | A rough paths approach to SDEs with fractional noise

Our main objective in this chapter is to study the continuity properties with respect to H of the SDE, defined for $t \in [0, T]$:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^H, \tag{3.1}$$

where the driving noise W^H is a fBm of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$, here interpreted in the framework of rough paths theory.

The interest for such a problem comes from the natural translation of the results of Chapter 2 to the framework, developed very recently in literature, of the regularity structures. The first step towards this direction is to consider the one-dimensional SDE problem in the setting of rough paths theory.

Our main problem in this chapter will be the continuity of the SDE solution $\{X_t,\, t \in [0, T]\}$ with respect to the parameter H. This problem has been already investigated by [RiTa16], [RiTa17] in the case $H \to (\frac{1}{2})_+$ (that is, when the convergence is from above). We give a proof of the continuity result when the limit value $H_0 \in (\frac{1}{3}, \frac{1}{2})$. Such convergence result has been proved for a general class of noises in [FrVi] (see Theorem 15.51 therein). Due to a change in the definition of ρ-variation, introduced in the errata corrige of [FrVi], the above result has been subsequently slightly modified. With the new definition, carefully studied in [FrVi11], it is not entirely clear whether one can apply Theorem 15.51 of [FrVi] to the noise W^H, in order to prove the weak convergence result for (3.1). The main result of this chapter is a refined version of this proof, which uses a slightly weaker hypothesis instead of the one assumed in [FrVi], and which is specialized to the case of the noise W^H. This work will appear in the forthcoming paper [DGMU20].

3.1 Rough paths theory

In this section we will give a brief introduction to rough paths theory, giving the basic notions necessary to motivate and define the objects that we will use in the following. We follow closely the rather direct approach of [FrHa].

3.1.1 Motivation

Rough paths theory originates in the 90s from a very natural problem. Let us consider the classical stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{3.2}$$

where μ, σ are regular functions (for example, Lipschitz continuous), and W_t is a sBm defined of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\, t \in [0, \infty)\}, \mathbb{P})$. In the context of probability theory, this equation has a well-established solution theory, through Itô (or Stratonovich) integration theory. Anyway, if, for a fixed $\omega \in \Omega$, we consider this equation as a deterministic problem, the resulting ODE is ill-posed.

We will briefly explain why. For a complete and very clear insight on this (and much more), we refer to [FrHa], Chapter 1. In order to give meaning from an analytical point of view to (3.2) one needs to be able to define, for $f, g : [0, T] \to \mathbb{R}$ at least continuous, the integral

$$\int_0^T f(t)dg(t),$$

where f and g play the role of X and W, respectively. This problem is well-known, and the most classical theory of such integrals is the *Riemann-Stieltjes* integration theory. In this framework, supposing that f is continuous and g is of bounded variation, one has that the integral $\int f dg$ is well-defined in the sense that

$$\lim_{|\mathcal{P}|\to 0} \sum_{t_j \in \mathcal{P}} f(c_j)(g(t_{j+1}) - g(t_j)) =: \int_0^T f(t)dg(t) \tag{3.3}$$

is a good definition of integral. Here, we are taking the limit over partitions $\mathcal{P}$ of $[0, T]$ such that the mesh $|\mathcal{P}| := \max_j |t_{j+1} - t_j| \to 0$ and the evaluation points $c_j \in [t_j, t_{j+1}]$ can be chosen arbitrarily. The limit is remarkably independent from the way in which the partitions tend to 0 and from the choice of the evaluation points c_j.

This classical integration result has a natural extension in the framework of Hölder continuous functions, called *Young integral.* The intuitive idea is to trade some of the regularity of g to f, while the definition (3.3) remains well-posed.

Definition 3.1. Let (S, d) be a metric space and let $\alpha \in (0, 1)$ We say that a function $f : \mathbb{R} \to S$ is *α-Hölder continuous* if there exists a constant $C < \infty$ such that

$$\sup_{s\neq t} \frac{d(f(s), f(t))}{|s - t|^\alpha} = C. \tag{3.4}$$

We denote the vector space of such functions $C^\alpha(\mathbb{R}; S)$. The definition naturally extends to functions defined on any interval $[a, b] \subset \mathbb{R}$.

With this definition in mind, we have that the limit in (3.3) is well-posed also if $f \in \mathcal{C}^\alpha([0, T]; \mathbb{R})$ and $g \in \mathcal{C}^\beta([0, T]; \mathbb{R})$, provided that $\alpha + \beta > 1$. Anyway, in order to give a meaning to (3.2) one should be able at least to give a meaning to the integral

$$\int_{\mathbb{R}} W_t dW_t,$$

which is impossible even in this framework. Indeed, by Proposition 1.7, the trajectories of sBm only belong to $\mathcal{C}^\alpha([0, T]; \mathbb{R})$, for every $\alpha < \frac{1}{2}$, giving $2\alpha < 1$.

We explain more clearly why this limitation is really a problem for SDEs. As we already pointed out, there exist probabilistic ways to find a solution to (3.2). Suppose that we are exploiting Itô integration theory to give meaning to (3.2). Denote by $S : W \mapsto X$ the solution map (or Itô map) that associates to the sBm W the solution X of (3.2). This map cannot be made continuous in the sense of paths, whatever (reasonable) space we use to define it on. We make this precise:

Proposition 3.2 ([FrHa], Proposition 1.1)**.** *Let $\mathcal{C}([0, T]; \mathbb{R})$ be the space of real-valued continuous functions endowed with the uniform convergence norm, and let W be a sBm. There exists no separable Banach space $\mathcal{B} \subset \mathcal{C}([0, T]; \mathbb{R})$ such that* *i) The paths of W lie almost surely in $\mathcal{B}$*

ii) The map

$$(f,g) \to \int_0^{\cdot} f(t)\dot{g}(t)dt,$$

which is well-defined for smooth functions, can be extended to a continuous map from $\mathcal{B} \times \mathcal{B} \to \mathcal{C}([0,T];\mathbb{R})$.

This result shows that it is impossible to make the solution map S continuous with the structure that we imposed up to now. The main idea of rough paths theory is to overcome this problem by enhancing the process W with its *iterated integral*

$$\mathbb{W}_{s,t} = \int_s^t (W_\tau - W_s)dW_\tau, \tag{3.5}$$

defined in a convenient way. The main contribution of rough paths theory is the following: if, instead of the process W, we consider the pair $(W, \mathbb{W})$ on a suitable space, the solution map to equation (3.2) given by $(W, \mathbb{W}) \mapsto X$ can be made continuous.

A first interesting fact is that the lift map $\Phi : W \mapsto (W, \mathbb{W})$ is universal, in the sense that it does not depend from the form of equation (3.2). In general, there are various possible choices for the lift map Φ. For example, in the case of sBm one can define (3.5) in the Itô or in the Stratonovich sense, and this will lead to different notions of solution.

There is a second remarkable fact, that shows how the present construction is quite general. Denote as $S^i : W \mapsto X$ the solution map in the Itô sense and $S^s : W \mapsto X$ the solution map in the Stratonovich sense. Given our construction, in both cases the solution map factorizes in $S^i = \hat{S}^i \circ \Phi^i$ and $S^s = \hat{S}^s \circ \Phi^s$., where we denoted as Φ^i and Φ^s the lift map respectively of Itô and Stratonovich integration. We have that the lifted solution map $\hat{S}^i = \hat{S}^s =: \hat{S}$, giving

$$S^i = \hat{S} \circ \Phi^i, \qquad S^s = \hat{S} \circ \Phi^s,$$

so that the map $\hat{S}$ is independent from the choice of the lift.

3.1.2 Hölder spaces and lifted paths spaces

We introduce now the main objects that we need to define rough paths spaces. The natural construction of rough paths spaces is done for V-valued processes X_t, where V is a Banach space, but, for simplicity, in the following we restrict for simplicity to the case $V = \mathbb{R}$.

Let now $X = \{X_t,\, t \in [0,T]\}$ be a continuous function from $[0,T]$ to $\mathbb{R}$. We wish to define a rough path for X. Until now, we only considered analytical properties of a path. If we want to give a general construction of a rough path over X we have to identify the set of algebraic and analytical properties that a notion of iterated integral $\mathbb{X} : [0,T]^2 \to \mathbb{R}$ has to satisfy in order to be a good definition of integral. From now on, given a function $X : [0,T] \to \mathbb{R}$ we denote th increment of X over $[s,t]$ as $X_{s,t} := X_t - X_s$, not to be confused with $\mathbb{X}_{s,t}$ which is defined on $[0,T]^2$.

Definition 3.3. Let $X : [0,T] \to \mathbb{R}$ and $\mathbb{X} : [0,T]^2 \to \mathbb{R}$ be continuous functions. We say that $\mathbb{X}$ satisfies the *Chen's relation* if for any $s,u,t \in [0,T]$ it holds

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u}X_{u,t} \tag{3.6}$$

Relation (3.6) is very natural in the context of integration: it is immediate to verify that if $\mathbb{W}$ is a lift defined by (3.5), where we interpret the right-hand side as a Itô integral, we have that (3.6) is satisfied. Moreover, since $X_{t,t} = 0$, it follows from (3.6) with $s = u = t$ that $\mathbb{X}_{t,t} = 0$ too (this is another natural property, if we think of $\mathbb{X}$ as an integral).

Another crucial consequence of (3.6), that we do not prove here, is the fact that the knowledge of a path $t \to (X_{0,t}, \mathbb{X}_{0,t})$ entirely determines the form of $\mathbb{X}$. Thus, we can say that the couple $(X, \mathbb{X})$ is really a path, that is, a one-parameter object.

We come now to the analytical properties that it is necessary to impose on X and $\mathbb{X}$, recalling that our general goal is to extend the theory of Young integrals to functions which are rougher, i.e. less regular.

Definition 3.4. Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. We define the space of *α-Hölder rough paths* $\mathscr{C}^\alpha = \mathscr{C}^\alpha([0,T])$ as the space of pairs $\mathbf{X} = (X, \mathbb{X})$ that satisfy (3.6) and for which it holds

$$||X||_\alpha := \sup_{s \neq t} \frac{|X_{s,t}|}{|s-t|^\alpha}, \qquad \text{and} \qquad ||\mathbb{X}||_{2\alpha} := \sup_{s \neq t} \frac{|\mathbb{X}_{s,t}|}{|s-t|^{2\alpha}}. \tag{3.7}$$

Remark 3.5. The space $\mathscr{C}^\alpha$ can be seen as the subset of the vector space $\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}$ of the pairs $\mathbf{X} = (X, \mathbb{X})$ which satisfy (3.7) but not necessarily (3.6). This space, endowed with the natural norm

$$||\mathbf{X}||_{\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}} := X_0 + ||X||_\alpha + ||\mathbb{X}||_{2\alpha}$$

is a Banach space. Somewhat uncomfortably, $\mathscr{C}^\alpha$ is only a subset of this space and it is not a linear subspace, due to the non-linear scaling of (3.6). In detail, we have for $\mathbf{X} \in \mathscr{C}^\alpha$ and $\lambda \in \mathbb{R}$ that

$$\lambda \mathbb{X}_{s,t} - \lambda \mathbb{X}_{s,u} - \lambda \mathbb{X}_{u,t} = \lambda(\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t}) \neq (\lambda X_t - \lambda X_u)(\lambda X_u - \lambda X_s) = \lambda^2 X_{s,u} X_{u,t},$$

except for $\lambda = 0, 1$.

By Remark 3.5, we cannot see $\mathscr{C}^\alpha$ as a linear subspace of $\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}$. Anyway, the norm which makes $\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}$ a Banach still induces a good notion of distance on $\mathcal{C}^\alpha$:

Definition 3.6. Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. On the space $\mathscr{C}^\alpha$ we define the *α-Hölder rough path metric* as

$$\rho_\alpha(\mathbf{X}, \mathbf{Y}) := ||X - Y||_\alpha + ||\mathbb{X} - \mathbb{Y}||_{2\alpha} = \sup_{s \neq t} \frac{|X_{s,t} - Y_{s,t}|}{|s-t|^\alpha} + \sup_{s \neq t} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|s-t|^{2\alpha}} \tag{3.8}$$

Remark 3.7. Notice that (3.8) does not correspond exactly to the distance induced by the norm of $\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}$ on $\mathscr{C}^\alpha$, since it lacks the initial condition term $|X_0 - Y_0|$. Anyway, it is convenient for our purposes to consider Definition 3.6.

The non-linear scaling property of (3.6) given by $(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$ suggests the definition of the following quantity, which is homogeneous with respect to (3.6)

Definition 3.8. We define on $\mathscr{C}^\alpha$ the *α-Hölder rough path norm* as the quantity given by

$$||\mathbf{X}||_{\mathscr{C}^\alpha} := ||X||_\alpha + \sqrt{||\mathbb{X}||_{2\alpha}}. \tag{3.9}$$

Remark 3.9. The quantity $||\mathbf{X}||_{\mathscr{C}^\alpha}$ is not a norm in the usual sense, because $||\lambda \mathbf{X}||_{\mathscr{C}^\alpha} \neq |\lambda| \cdot ||\mathbf{X}||_{\mathscr{C}^\alpha}$, but scales correctly with respect to the (3.6)-preserving transformation $(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$. Indeed, introducing the notation $\mathbf{X}_\lambda := (\lambda X, \lambda^2 \mathbb{X})$ we have that

$$||\mathbf{X}_\lambda||_{\mathscr{C}^\alpha} = |\lambda| \cdot ||\mathbf{X}||_{\mathscr{C}^\alpha}.$$

Having defined the algebraic well-posedness relation (3.6) and the spaces with suitable regularity, we may ask ourselves if they permit a good definition of rough paths. First, we observe that neither (3.6) nor the definition of $\mathscr{C}^\alpha$ imply any type of chain rule or integration by parts formula.

This is not surprising. Considering the basic sBm setting $X = W$, we know that there are several choices (in fact, infinite) for the definition of the integral with respect to W, and only the Stratonovich integral preserves the classical rules of integration. Anyway, the Itô integral is a very popular choice, thanks to its non-anticipativity property, which is considered natural for many applications.

Keeping these considerations in mind, we introduce now the concept of *geometric rough path*. The idea is to encode the chain rule as an algebraic property in the rough path space. If X is a smooth function, we have that

$$\begin{aligned}\mathbb{X}_{s,t} =& \int_s^t (X_\tau - X_s) dX_\tau = \int_s^t X_\tau dX_\tau - X_t X_s + (X_s)^2 \\ =& \frac{(X_t)^2 - (X_s)^2}{2} - X_t X_s + (X_s)^2 \\ =& \frac{1}{2}(X_t - X_s)^2.\end{aligned} \tag{3.10}$$

Definition 3.10. We define the space $\mathscr{C}_g^\alpha$ of *geometric rough paths* as the space of rough paths in $\mathscr{C}^\alpha$ which moreover satisfy condition (3.10).

Remark 3.11. We note that (3.10) completely determines the form of $\mathbb{X}$. This is true only in dimension one. If we consider paths with values in $\mathbb{R}^d$, the function $\mathbb{X}$ becomes $\mathbb{R}^d \otimes \mathbb{R}^d$-valued (matrix-valued) and condition (3.10) becomes $\mathrm{Sym}(\mathbb{X}) = \frac{1}{2}(X_{s,t} \otimes X_{s,t})$. In the latter case we would still have some freedom on the non-diagonal terms $\mathbb{X}^{ij}_{s,t}$, which translates to some freedom in the definition of $\int X^i dX^j$, whenever $i \neq j$ (we will not go into details here). We refer to [FrHa] for a precise description of the $\mathbb{R}^d$ case.

Remark 3.12. In both [FrVi] and [FrHa], the authors introduce an useful construction, which permits to see rough paths as Lie group valued functions, for a suitable group $G^N(\mathbb{R}^d)$. Due to the fact that we are in the case $d = 1$, in our case this construction does not help us and we will not present it. Anyway, some of the results from [FrVi] that we will use are originally stated following this notation. We will translate them to be consistent with the notation that we are using. In particular, the distance denoted with $d(\mathbf{X}_s, \mathbf{X}_t)$ in [FrVi] and with $d_C(\mathbf{X}_s, \mathbf{X}_t)$ in [FrHa] is simply $|X_{s,t}| + |\mathbb{X}_{s,t}|^{1/2}$ in our case. From now on, we will denote

$$d(\mathbf{X}_s, \mathbf{X}_t) := |X_{s,t}| + |\mathbb{X}_{s,t}|^{\frac{1}{2}}$$

3.1.3 Gaussian processes as rough paths

We report here some useful results about the theory of Gaussian rough paths. Following [FrVi] and [FrHa], we wish to construct a canonical rough path structure for a class of continuous Gaussian processes which satisfy a certain condition on their covariance structure. This will include as a special case the fBm, which we will then study in detail in Section 3.2.

Let $\{X_t, t \in [0,T]\}$ be a real-valued centred continuous Gaussian process with covariance structure given, for $s, t \in [0,T]$, by

$$\mathrm{E}[X_t X_s] = K(s,t).$$

We recall that a Gaussian process is completely determined by its mean and covariance. We define now some notions that we will use throughout the rest of this chapter. First, a bit of notation; given a function $f : [0,T]^2 \to \mathbb{R}$, we denote with

$$f\begin{pmatrix} s,t \\ u,v \end{pmatrix} = f(t,v) - f(t,u) - f(s,v) + f(s,u)$$

the rectangular increment of f.

Definition 3.13. Let $\Delta := \{0 \le s \le t \le T\}$ and $\omega : \Delta \times \Delta \to [0, \infty)$. We say that ω is a *2D control* if it is super-additive in the following way: given a rectangle $R \subset [0,T]^2$ and any finite partition $\{R_j, 1 \le j \le n\}$ of R, we have

$$\omega(R) \ge \sum_{j \le N} \omega(R_j).$$

Given a function f defined on rectangles, we say that f is *controlled by* the control ω if for any rectangle $R \subset [0,T]^2$ it holds

$$|f(R)| \le \omega(R)$$

Definition 3.14. Let $f : [0,T]^2 \to \mathbb{R}$ and let $p \in [1, \infty)$. For an interval $[a,b]$, such that $a \le b$, let $\mathcal{D}([a,b])$ be the family of finite partitions $\{a_i, i \le n\}$ of $[s,t]$. We define for every rectangle $R = [s,t] \times [u,v] \subset [0,T]^2$, with $0 \le s \le t \le T$ and $0 \le u \le v \le T$ the *p-variation*

$$V_p(f, [s,t] \times [u,v]) := \sup_{\substack{\{t_i\} \in \mathcal{D}([s,t]) \\ \{t'_j\} \in \mathcal{D}([u,v])}} \left(\sum_{i,j} \left| f \begin{pmatrix} t_i, t_{i+1} \\ t_j, t_{j+1} \end{pmatrix} \right|^p \right)^{\frac{1}{p}} \tag{3.11}$$

and we say that f has *finite p-variation* if it holds that $V_p(f, [0,T]^2) < \infty$.

Definition 3.15. Let $f : [0,T]^2 \to \mathbb{R}$ and let $p \in [1, \infty)$. For every rectangle $R \subset [0,T]^2$ we define the *controlled p-variation* as

$$|f|_{p\text{-var},R} := \sup_{\pi \in \mathcal{P}(R)} \left(\sum_{A \in \pi} \left| f(A) \right|^p \right)^{\frac{1}{p}}, \tag{3.12}$$

where we denoted as $\mathcal{P}(R)$ the family of partitions of R made by rectangles.

Remark 3.16. Let us remark that the family of grid-like partitions used to define (3.11) is actually smaller than the family of general rectangular partitions used to define (3.12). This implies that trivially, for any f, for any R and for any $p \ge 1$ one has

$$V_p(f, R) \le |f|_{p\text{-var},R}.$$

Less trivially, one has that this inequality is strict, whenever $p > 1$. We will see an example of this behaviour in the case of the fBm W^H in Proposition 3.24.

We will see how, given a continuous and centred Gaussian process X with covariance K, it is possible to construct a canonical rough path $\mathbf{X}$, provided that the covariance function K has some p-variation regularity, and that the p-variation of K is controlled by some 2D control ω. We make it more precise.

Theorem 3.17 ([FrVi], Theorem 15.33)**.** *Let X_t, for $t \in [0,T]$, be a centred continuous Gaussian process with values in $\mathbb{R}$. Suppose that there exists a $\rho \in [1,2)$ such that the covariance K of X has finite controlled ρ-variation dominated by a 2D control ω such that $\omega([0,T]^2) < \infty$.*

Then, there exists a unique process $\mathbf{X}$ in $\mathscr{C}^\alpha$ such that $\mathbf{X}$ lifts X, in the sense that $\pi_1(\mathbf{X}_t) = X_t - X_0$.

Moreover, there exists a constant $C = C(\rho)$ such that for every $s \le t$ and for every $q \ge 1$ it holds

$$\mathrm{E}\Big[d(\mathbf{X}_s, \mathbf{X}_t)^q\Big]^{\frac{1}{q}} := \mathrm{E}\Big[\Big(|X_{s,t}| + |\mathbb{X}_{s,t}|^{1/2}\Big)^q\Big]^{\frac{1}{q}} \le C(\rho)\sqrt{q}\,\omega([s,t]^2)^{\frac{1}{2\rho}} \tag{3.13}$$

The lift $\mathbf{X}$ is unique and natural in the sense that it is the limit in the space of rough paths $\mathscr{C}_g^\alpha$ of any sequence X_n of piecewise linear or mollified approximations to X such that $||X_n - X||_\infty \to 0$ almost surely.

Remark 3.18. Regarding the approximations to a rough path $\mathbf{X}$ via regular functions, we refer to Chapter 15 fo [FrVi], in which there is a large discussion about piecewise linear and mollified approximations of a Gaussian process. A complete discussion about this topic would exceed the scope of this work.

3.1.4 Rough differential equations

Given a rough path $\mathbf{X} = (X, \mathbb{X})$, one wishes to construct a rough differential equations (RDEs) theory dealing with equations driven by a rough path $\mathbf{X}$. Since we already pointed out that a rough path $\mathbf{X}$ is really a one-dimensional object, in principle there is no problem in trying to consider equations of the form

$$dY_t = f(Y_t)d\mathbf{X}_t. \tag{3.14}$$

We want to understand, as usual, this equation in the integral form

$$Y_t = Y_0 + \int_0^t f(Y_s)d\mathbf{X}_s. \tag{3.15}$$

The problem in (3.15) is that we have to make sense of the integral with respect to a rough path $\mathbf{X}$. This is a very rich problem in rough paths theory. Lyons' original idea (see [Lyo98]) was to define the integral $\int F(X_t)d\mathbf{X}_t$ for functions F of X which were suitably regular. Then, Gubinelli in [Gub04] extended this approach to include integrands Y which were *controlled* (we will not explore the concept) by X, giving rise to a Banach space of integrands. We refer to Chapter 4 of [FrHa] for an overview of the topic.

To our purposes, we will not enter into the details of the construction of the integral. We just explain the main idea that connects the construction of the integral for a large enough class of integrands with the existence and uniqueness results for equations like (3.15). The idea is the following: if one is able to define the integral $\int F(X)d\mathbf{X}$ for a suitable large class of functions F, one may be able to reformulate the existence for a solution to (3.15) as a fixed point problem, and solve it through some iterative method.

In general, there is a huge literature regarding the construction of a solution theory for equations of the type (3.15) (see the already mentioned [FrVi], [FrHa], [Gub04], [Lyo98]).

Another important feature, that we will use in the following, is the fact that the solution map is continuous, when seen as a map from rough path spaces to function spaces (actually, to even richer spaces). Again, we will not enter into details, apart from remarking, as we already did before, that the lack of continuity of the solution map in the framework of classical SDEs was one of the motivating reasons for the introduction of rough paths.

We move now to our specific case: we recall that we want to study an equation of the form (3.1), which we rewrite here

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^H.$$

One may notice that (3.1) and (3.14) differ for the presence of a drift term. Anyway, it is possible to see any d-dimensional RDE with drift as a $d+1$-dimensional RDE without drift, by seeing the dt integral as a "regular rough path" and adding it as an additional dimension of the rough path. The details of this construction are given in [FrVi], Chapter 12.

The solution X_t^H, as always, is interpreted in the sense

$$X_t^H = X_0^H + \int_0^t \mu(X_t^H)dt + \int_0^t \sigma(X_t^H)dW_t^H, \tag{3.16}$$

where $X_0 \in L^2(\Omega)$ is the initial condition of the problem. The integral appearing will be interpreted as an integral with respect to a rough path $\mathbf{W}^H = (W^H, \mathbb{W}^H)$ defined over the process W^H.

We are left to show that a canonical rough path lift $\mathbf{W}^H$ exists for W^H. We follow the construction of a rough path for Gaussian processes given in Subsection 3.1.3. In particular, we want to use Theorem 3.17 to obtain a canonical geometric rough path for W^H, for every $H \in (\frac{1}{3}, \frac{1}{2}]$.

We need to find a $\rho \in [1, 2)$ such that the ρ-variation of K^H is finite and bounded by a control ω_H. It turns out that, thanks to Theorem 1 of [FrVi11] and to Proposition 3.24 (which we state and prove in the next section), the function ω_H, defined for some $\varepsilon > 0$ small enough,

$$\omega_H(R) := |K^H|_{\frac{1}{2H}+\varepsilon\text{-var},R}^{\frac{1}{2H}+\varepsilon}$$

is a good control for the $(\frac{1}{2H} + \varepsilon)$-variation of K^H (notice that $\frac{1}{2H} < \frac{3}{2}$ for our range of values of H, thus there exists an $\varepsilon > 0$ such that $\frac{1}{2H} + \varepsilon < 2$). This construction will be clarified later, when we will also prove some stronger results.

We summarize all the previous considerations in a minimal existence/uniqueness and continuity result. The result we report is minimal in the sense that we put the minimal structure that we are going to use in the following section. We refer to [FrVi], Section 12.1.2 for a complete study of the problem.

Theorem 3.19. *Let $H \in (\frac{1}{3}, \frac{1}{2}]$, let $X_0^H = x_0 \in \mathbb{R}$ be a constant and let $\mu, \sigma \in \mathcal{C}_b^3(\mathbb{R})$ (bounded functions which are three times differentiable). Then, there exists a unique solution $X^H = (X_t^H)_{t\in[0,T]}$ of equation (3.1) with initial condition x_0. Moreover, the solution X^H is a continuous function of $\mathbf{W}^H = (W_t^H, \mathbb{W}_{s,t}^H)$, in the sense that the solution map S, given by*

$$\begin{aligned} S : \mathscr{C}^\alpha &\longrightarrow \mathcal{C}^\alpha([0, T]) \\ \mathbf{W}^H &\longmapsto X^H, \end{aligned} \tag{3.17}$$

is continuous, for any $0 < \alpha < H$.

3.2 Weak continuity with respect to the noise

In this section we study our main problem in this Chapter, that is, the continuity in law of the solution to (3.1) with respect to H. Let us write again equation (3.1)

$$dX_t^H = \mu(X_t^H)dt + \sigma(X_t^H)dW_t^H,$$

where W^H is a fBm of Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$, and we highlighted the dependence of the solution X_t from H denoting it with X_t^H. We restrict to $H \leq \frac{1}{2}$ in order to use the rough paths techniques in a non-trivial way. Indeed, when $H > \frac{1}{2}$ the regularity of the noise allows for a classical solution theory in the sense of Young integration. By Theorem 3.19 we have that a solution to (3.1) exists and it is unique. Moreover, the solution operator is continuous from $\mathscr{C}^\alpha$ to $\mathcal{C}^\alpha([0, T])$, for any $0 < \alpha < H$.

Remark 3.20. When $H = \frac{1}{2}$, the solution $X^{\frac{1}{2}}$ to (3.1) becomes a Stratonovich solution of an SDE driven by a sBm. This is a direct consequence of the fact that when we lift a sBm $W^{\frac{1}{2}}$ to a geometric rough path, we obtain the Stratonovich integral.

Since we have existence and uniqueness of a solution X^H for every $H \in (\frac{1}{3}, \frac{1}{2}]$, a natural question that can be addressed is: does the solution X^H change continuously with respect to H? Looking to the equation through the glasses of modelling, this question can be reformulated as: can I say that if I get a small error on the estimate of H, I also get a small error in my model prediction X^H?

We prove the following result:

Theorem 3.21. *let us consider equation (3.1), for $t \in [0, 1]$, with $\mu, \sigma \in C_b^3(\mathbb{R})$. Let X_0 be an $L^2(\Omega)$ random variable independent on W^{H_n} and let us denote by X^{H_n,X_0} the solution of the equation (3.1) with $H = H_n \in \left(\frac{1}{3}, \frac{1}{2}\right]$. If $H_n \to H_0 \in \left(\frac{1}{3}, \frac{1}{2}\right]$, then X^{H_n,X_0} converges to X^{H_0,X_0} in distribution in the space $C^{\frac{1}{3}}([0, 1])$.*

Remark 3.22. The fact that we consider only $t \in [0,1]$ is not a true restriction, since one can always reformulate an equation on $[0,T]$ as an equation on $[0,1]$ reparametrizing the noise by considering W^H_{tT}, for $t = [0,1]$ (see also [FrVi], Chapter 15). Anyway, this assumption simplifies a bit the estimates in the proof of Theorem 3.21.

In order to prove the theorem, we reason similarly to the following result of [FrVi], adapting the first part of their proof to work with a more general assumption and writing carefully the details for the second part.

Theorem 3.23 ([FrVi], Theorem 15.51). *Let X^n, for $1 \le n \le \infty$ be a sequence of centred $\mathbb{R}$-valued continuous Gaussian processes in $[0,1]$. Assume:*

i) There exists a $\rho \in [1,2)$ such that the covariances K^n of X^n are of finite ρ-variation, and they are controlled uniformly by a 2-dimensional control ω, in the sense that

$$\sup_{n\in\mathbb{N}} |K^n|^{\rho}_{\rho\text{-}var,[s,t]^2} \le \omega([s,t]^2).$$

ii) $\mathbf{X}^n$ is the natural lift of X^n with paths in $C_o^{0,p\text{-}var}([0,T], G^3(\mathbb{R}))$, for some $p > 2\rho$.

iii) The covariances K^n converge pointwise to K^∞ on $[0,1]^2$.

Then, for every $p > 2\rho$, X^n converges weakly to X^∞ in the p-variation topology. Moreover, if ω is Hölder dominated, the convergence holds also with respect to the $\frac{1}{p}$-Hölder topology.

The space $G^3(\mathbb{R})$ that appears is the space of geometric paths, introduced in Chapter 7 of [FrVi]. Another useful tool is the following result, which is a slight generalization of a result in [FrVi11]

Proposition 3.24 ([FrVi11], Examples 1-2). *The covariance K^H of a fBm of parameter $H \in (0, \frac{1}{2}]$ has bounded $\frac{1}{2H}$-variation $V_{\frac{1}{2H}}(K^H, [0,T]^2)$, which moreover satisfies for every $s < t$*

$$V_{\frac{1}{2H}}(K^H, [s,t]^2) \le c_H |t-s|^{2H}. \tag{3.18}$$

Moreover, if one considers $H \in [\eta_1, \frac{1}{2}]$, with $\eta_1 > 0$ fixed, the constant c_H can be chosen uniformly with respect to H.

Finally, one has that the controlled $\frac{1}{2H}$-variation is infinite, that is,

$$|K^H|_{\frac{1}{2H}\text{-}var,R} = \infty.$$

Remark 3.25. Proposition 3.24 shows that the ρ-variation and the controlled ρ-variation are really two different concepts.

Proof (Proposition 3.24). The proof is almost the same given in [FrVi11], Example 1. In our result we take care explicitly in the estimates of the dependence from H. Regarding the second part of the statement, i.e. that $|K^H|_{\frac{1}{2H}\text{-var},R} = \infty$, we refer to [FrVi11], Example 2.

We scrutinize the computations in Example 1 of [FrVi11]. Fix $0 \le s \le t \le 1$. Consider two partitions $\{t_i\}, \{t'_j\} \in \mathcal{D}([s,t])$. Following [FrVi11], we have

$$\sum_j \left| \mathrm{E}\Big[W^H_{t_i,t_{i+1}} W^H_{t_j,t_{j+1}} \Big] \right|^{\frac{1}{2H}} \le 6^{\frac{1}{2H}} |t_i - t_{i+1}|. \tag{3.19}$$

We remark that this is true because all the non-explicit constants $C(H)$ that appear in the proof given by [FrVi11] (denoted there as c_H) can be chosen to be equal to 1, thus we are only left with the explicit constant $6^{\frac{1}{2H}}$. This is true because $C(H)$'s arises in a term of the type $\left|\mathrm{E}[W^H_{s,t}W^H_{u,v}]\right|$, with $0 \le s \le u \le v \le t \le 1$. We write explicitly

$$
\begin{aligned}
\left|\mathrm{E}[W^H_{s,t}W^H_{u,v}]\right| &\stackrel{\text{def}}{=} \mathrm{E}\Big[(W^H_t - W^H_s)(W^H_v - W^H_u)\Big]\Big| \\
&= \Big|\mathrm{E}\Big[(W^H_t - W^H_v + W^H_v - W^H_u + W^H_u - W^H_s)(W^H_v - W^H_u)\Big]\Big| \\
&= \Big|\mathrm{E}\Big[(W^H_t - W^H_v)(W^H_v - W^H_u)\Big] \\
&\quad + \mathrm{E}\Big[(W^H_v - W^H_u)^2\Big] \\
&\quad + \mathrm{E}\Big[(W^H_u - W^H_s)(W^H_v - W^H_u)\Big]\Big|. \\
&= \frac{1}{2}\Big||t-u|^{2H} - |t-v|^{2H} + |v-s|^{2H} - |u-s|^{2H}\Big| \\
&\le |u-v|^{2H},
\end{aligned}
$$

where in the last step we used the fact that $|t-u|^{2H} = |t-v+v-u|^{2H} \le |t-v|^{2H} + |v-u|^{2H}$, which is true since $0 < 2H \le 1$. This implies obviously that $|t-u|^{2H} - |t-v|^{2H} \le |v-u|^{2H}$, which is what we used. This estimate appears several times in the proof of [FrVi11], but does not add any dependence of the constants from H.

Coming back to (3.19), we are only left to sum over i obtaining

$$
\sum_{i,j}\left|\mathrm{E}\Big[W^H_{t_i,t_{i+1}}W^H_{t_j,t_{j+1}}\Big]\right|^{\frac{1}{2H}} \le 6^{\frac{1}{2H}}|t-s|,
$$

and we can take the sup over all partitions of $\mathcal{D}([s,t])$ to obtain our result. The final constant $c_H = 6^{\frac{1}{2H}}$ is clearly bounded whenever $H > \eta_1 > 0$, if η_1 is fixed, giving also the second part of the statement, i.e. the independence from H of the constant.

□

Proof (Theorem 3.21). The idea of the proof is to exploit the continuity of the solution map stated in Theorem 3.19. It is sufficient to show that

$$
\mathbf{W}^{H_n} = (W^{H_n}_\tau, \mathbb{W}^{H_n}_{s,t}) \xrightarrow{n\to\infty} \mathbf{W}^{H_0} = (W^{H_0}_\tau, \mathbb{W}^{H_0}_{s,t})
$$

in the space $\mathscr{C}^{\frac{1}{3}}$. Indeed, by the continuity of the solution map stated in Theorem 3.19, this implies that

$$
X^{H_n,X_0} \xrightarrow{n\to\infty} X^{H_0,X_0}
$$

in $\mathcal{C}^{\frac{1}{3}}([0,T])$. We are then left to show that $\mathbf{W}^{H_n} \to \mathbf{W}^H$ holds in the space $\mathscr{C}^{\frac{1}{3}}$.

To prove this, we follow the same reasoning as in the general convergence result Theorem 3.23. However, in our proof, instead of *i)*, we use the slightly weaker assumption

i') There exist a $\rho \in [1,2)$ such that the covariances K^n of X^n are of finite ρ-variation, and each of them is controlled by a 2D control ω_n, which satisfy the uniform Hölder bound

$$
\sup_{n\in\mathbb{N}} |K^n|^\rho_{\rho\text{-var},[s,t]^2} = \sup_{n\in\mathbb{N}} \omega_n([s,t]^2) \le C|t-s|.
$$

The stronger version *i)* was used in the proof of Theorem 3.23 to obtain the tightness of the set of measures. In our case we obtain it directly via the Kolmogorov-Lamperti criterion (Corollary A.11, [FrVi]).

Our proof is structured in the following way:

Step 1) We state some useful properties of the p-variation and controlled p-variation for functions $f:[0,T]^2 \to \mathbb{R}$

Step 2) We use these properties together with Proposition 3.24 and Corollary A.11 of [FrVi] and Theorem 15.33 of [FrVi] to obtain the tightness of the set of measures induced by $(\mathbf{W}^{H_n})_{n\in\mathbb{N}}$

Step 3) We identify the limit in the same fashion as in Theorem 15.51, [FrVi], using linear approximations of $\mathbf{W}^{H_n}$ which converge uniformly with respect to $n \in \mathbb{N}$.

Step 1

In [FrVi11], Theorem 1, the authors showed that even if the ρ-variation and the controlled ρ-variation are different concepts, they are ε-close concepts. Precisely, we have that for $\varepsilon > 0$, $p \geq 1$, there exists an explicit constant $C(p,\varepsilon) \geq 1$ such that for every $f:[0,T]^2 \to \mathbb{R}$ and for every R rectangle in $[0,T]^2$ it holds

$$|f|_{p+\varepsilon\text{-var},R} \leq C(p,\varepsilon) V_p(f,R). \tag{3.20}$$

This of course implies that

$$|f|_{p+\varepsilon\text{-var},R} \leq C(p,\varepsilon)|f|_{p\text{-var},R}$$

and that

$$V_{p+\varepsilon}(f,R) \leq C(p,\varepsilon) V_p(f,R) \tag{3.21}$$

The constant $C(p,\varepsilon)$ is given by (see [FrVi11], Theorem 3 with $\theta = 1+\frac{1}{p} - \frac{1}{p+\varepsilon} = 1 + \frac{\varepsilon}{p(p+\varepsilon)}$ and the arbitrary $\alpha \in (1,\theta)$ that we fix as $\alpha := \frac{\theta-1}{2} + 1 = 1 + \frac{\varepsilon}{2p(p+\varepsilon)}$)

$$\begin{aligned} C(p,\varepsilon) = \Big\{ \Big[1 + \zeta\Big(1 + \frac{\varepsilon}{2p^2 + 2\varepsilon p + \varepsilon}\Big)\Big]^{1+\frac{\varepsilon}{2p(p+\varepsilon)}} \\ \times \zeta\Big(1 + \frac{\varepsilon}{2p(p+\varepsilon)}\Big) + \Big[1 + \zeta\Big(1+\frac{\varepsilon}{p(p+\varepsilon)}\Big)\Big]\Big\}, \end{aligned} \tag{3.22}$$

where ζ denotes the Riemann zeta function. We notice that this quantity, for any fixed $\varepsilon > 0$, is continuous with respect to $p \in [1,\infty)$. Indeed, it only diverges when $p \to \infty$, since $\zeta(x) \to \infty$ when $x \to 1^+$.

Step 2

In order to obtain a uniform Kolmogorov-type estimate for the $\mathbf{W}^{H_n}$ we want to apply Theorem 15.33 of [FrVi] (Theorem 3.17). The key point is that the constant C appearing in the estimate (3.13), provided that the process X has finite controlled ρ-variation, depends only on ρ,

If we show that there exists a unique $\rho \in [1,2)$, independent from $n \in \mathbb{N}$, such that all processes W^{H_n} have bounded ρ-variation, we could then give a uniform estimate of

$$\sup_{n\in\mathbb{N}} \mathrm{E}\Big[d(\mathbf{W}^{H_n}_t, \mathbf{W}^{H_n}_s)^q\Big]^{\frac{1}{q}}$$

We prove that there exists such a $\rho \in [1,2)$. Since we are considering $H_0 > \frac{1}{3}$ as our limiting value, then there exists a $\delta > 0$ such that it holds definitively $H_n > \frac{1}{3} + \delta$, and thus

$$\sup_{n \geq n_0(\delta)} \frac{1}{2H_n} = \rho_0 < \frac{3}{2}.$$

Let $\varepsilon_1 := \frac{\frac{3}{2}-\rho_0}{2}$ be fixed in the following considerations. By Step 1, we can define for every $n \in \mathbb{N}$ a 2D control ω_{H_n} in the following way:

$$\begin{aligned}\omega_{H_n}(R) :=& |K^{H_n}|^{\frac{1}{2H_n}+\varepsilon_1}_{\frac{1}{2H_n}+\varepsilon_1\text{-var},R} \\ \leq& C\Big(\frac{1}{2H_n},\varepsilon_1\Big)^{\frac{1}{2H_n}+\varepsilon_1} V^{\frac{1}{2H_n}+\varepsilon_1}_{\frac{1}{2H_n}}(K^{H_n},R) \\ \leq& C\, V^{\frac{1}{2H_n}+\varepsilon_1}_{\frac{1}{2H_n}}(K^{H_n},R),\end{aligned} \tag{3.23}$$

where in the last inequality we used the fact that, for our fixed $\varepsilon_1 > 0$, the quantity $C(\frac{1}{2H_n},\varepsilon_1)$ is continuous (and thus bounded) for $H_n \in (0,\frac{1}{2}]$ (that corresponds to the region $p \in [1,\infty)$ in (3.22)).

Moreover, by [FrVi11], Example 2 we have that

$$\begin{aligned}V^{\frac{1}{2H_n}+\varepsilon_1}_{\frac{1}{2H_n}}(K^{H_n},[s,t]^2) \leq& c_H|t-s|^{2H_n\left(\frac{1}{2H_n}+\varepsilon_1\right)} \\ =& c_H|t-s|^{1+2H_n\varepsilon_1} \\ \leq& c|t-s|,\end{aligned} \tag{3.24}$$

where in the last inequality we used Proposition 3.24 (specifically, the fact that c_H can be chosen independently from $H \in (\frac{1}{3},\frac{1}{2}])$ and the fact that $|t-s| \leq 1$, together with $2H_n\varepsilon_1 > 0$.

As a consequence, for every $n \in \mathbb{N}$ the control ω_{H_n} satisfies

$$\omega_{H_n}([s,t]^2) \leq C|t-s|, \tag{3.25}$$

which means that it is an Hölder dominated control. Moreover, the bound on the control does not depend on n and therefore, by Theorem 15.33 of [FrVi11] with $\rho = \rho_0$ there exists a constant $C = C(\rho_0)$ such that for every $q \in [1,\infty)$ and for every $s,t \in [0,1]$

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathrm{E}\Big[d(\mathbf{W}^{H_n}_t,\mathbf{W}^{H_n}_s)^q\Big]^{\frac{1}{q}} \leq& \sup_{n\in\mathbb{N}} C\sqrt{q}\,\omega_{H_n}([s,t]^2)^{\frac{1}{2\rho_0}} \\ \leq& C\sqrt{q}|t-s|^{\min_n H_n}.\end{aligned} \tag{3.26}$$

We use this uniform estimate in the Kolmogorov-Lamperti tightness criterion (see, for example, Corollary A.11 of [FrVi]), with $r := 2\rho_0 = \frac{1}{\min H_n} \in [1,3)$ and we can choose $q >> 1$ such that $\frac{1}{r} - \frac{1}{q} \geq \frac{1}{3}$. This is possible because $\frac{1}{r} > \frac{1}{3}$.

This means that our sequence $\mathbf{W}^{H_n}$ is tight in $\mathscr{C}^{\frac{1}{3}}$, and thus it possesses a subsequence converging to some limit $\mathbf{Y}$.

Step 3

We identify the limit $\mathbf{Y}$ as $\mathbf{W}^{H_0}$.by following the same strategy as in the proof of Theorem 3.23 in [FrVi].

The fundamental idea is to show that, for every $n \in \mathbb{N}$, the (lifted) piecewise linear approximations $S_3(W^{H_n,D})$ of $\mathbf{W}^{H_n}$ converge to $\mathbf{W}^{H_n}$ in $L^q(\mathbb{P})$, uniformly with respect to $n \in \mathbb{N}$, whenever the amplitude of the dissection D of $[0,1]$ tends to 0. In order to prove this, it suffices to use the uniform estimates of Theorem 15.42 in [FrVi]. We do not go into the details of the construction of $S_3(W^{H_n,D})$, we only remark that they are a natural approximation of the fBm W^H and of his lift $\mathbf{W}^H$.

Summarizing: we already shown that the entire sequence $(K^{H_n})_n$ has finite controlled ρ_0-variation, for some $\rho_0 < \frac{3}{2}$, and the controls ω_{H_n} satisfy (by (3.25))

$$\sup_{n\in\mathbb{N}} \omega_{H_n}([0,1]^2) = M < \infty.$$

We now fix $\rho = \rho_0$ and $p = 3 > 2\rho_0$ in Theorem 15.42 of [FrVi] to obtain that, for every $\eta \in (0, \frac{1}{2\rho_0} - \frac{1}{p})$ there exists a constant $C_1(\rho_0, p, M, \eta)$ such that for every dissection D of $[0,1]$ it holds

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mathrm{E}\Big[||\mathbf{W}^{H_n} - S_3(W^{H_n,D}))||^q_{\mathscr{C}^{1/3}}\Big]^{\frac{1}{q}} \leq & \sup_{n\in\mathbb{N}} C_1\sqrt{q} \max_{t_i\in D} \omega_{H_n}([t_i, t_{i+1}]^2)^{\frac{\eta}{3}} \\
\leq & C_1\sqrt{q} \sup_{n\in\mathbb{N}} \max_{t_i\in D} c_{H_n}|t_i - t_{i+1}|^{\frac{\eta}{3}} \\
\leq & C_1 C\sqrt{q} \sup_{n\in\mathbb{N}} \operatorname{diam}(D)^{\frac{\eta}{3}} \\
= & C_1 C\sqrt{q} \operatorname{diam}(D)^{\frac{\eta}{3}}.
\end{aligned} \tag{3.27}$$

Thus, it holds that the piecewise linear approximations $S_3(W^{H_n,D_m})$ converge to $\mathbf{W}^{H_n}$, uniformly with respect to n, whenever the amplitude of the dissections D_m tends to zero for $m \to \infty$.

Now it suffices to apply Lemma 15.50 of [FrVi] with $Z^{m,n} = W^{H_n,D_m}$, where D_m is a sequence of dissections of $[0,1]$ whose amplitudes tend to zero. The lemma is basically a careful application of triangle inequality in a double convergence, where one of these convergences is uniform (the one in m, uniform with respect to n). Since for the linear approximations, which are essentially finite dimensional processes, the weak convergence $S_3(W^{H_n,D_m}) \to S_3(W^{H_0,D_m})$ as $n \to \infty$ holds trivially, for every $m \in \mathbb{N}$, this is sufficient to conclude the proof.

□

Remark 3.26. When $H > \frac{1}{2}$, the result can be proven following the same steps, but without the need of rough paths theory. Indeed, the solution map $W^H \to X^H$ is continuous, since we are in the framework of Young integration theory. This means that whenever $H > \frac{1}{2}$ it is sufficient to show that, for some $\alpha > \frac{1}{2}$, it holds $W^H \to W^{H_0}$ in $\mathcal{C}^\alpha([0,T])$ when $H \to H_0$. This can be shown again via Kolmogorov-Lamperti criterion (Corollary A.11, [FrVi]).

4 | A fractional-Hawkes model for electricity prices

The last part of the thesis is devoted to a modelling and computational application of the noises we studied. Our aim is to develop a model for the description and the forecast of the gross prices of electricity in the liberalized Italian energy market. This will be also the subject of our forthcoming paper [GiMo19].

The common thread with the previous chapters is the use of a fractional noise. In this case, this is represented by the choice of a SDE driven by a fBm B^H as one the components of our model. Apart from this component, our model will contain a jump process, modelled through a self-exciting Hawkes process, which aims to model the *shock formation* in the markets, and a deterministic component, which aims to reproduce the seasonal trends of the prices.

We will finally validate the model via the analysis of our data time series, and we will compute and evaluate the forecast produced by our model, computing the prediction intervals (PI) estimated by our model, and evaluating their quality by using adequate evaluation metrics like the Winkler score and the Pinball loss function.

4.1 Mathematical modelling of electricity markets

In the last decades the electricity market has been liberalized in a growing number of countries, especially in the European Union. Liberalized markets have been introduced for example in Germany, Italy, Spain, UK, as well as in all nordic countries. The introduction of competitive markets has been reshaping the landscape of the power sectors. Electricity price now undergoes to market rules, but it is very peculiar. Indeed, electricity is a non-storable commodity, hence the need of having a particular organization in the market emerged. This has usually resulted in the creation of a *day-ahead market*: a market in which every day there are some auctions regarding the delivery of energy at a fixed time of the following day. The price of the electricity is determined by crossing the supply curve and the demand curve, for the hour for which the auction is taking place (see e.g. [RFC]). The steepness of supply and demand curve can be regarded as the cause of one of the main characteristics of the electricity price market, i.e. *shock formation* in the prices, which is one of the more important aspects that distinguish the electricity market from the other similar financial markets, and also one of the most difficult to model. A *shock*, or *spike* is a sudden large rise in the price followed by a rapid drop to its regular level. Power prices may soar during short periods of time, and then fall back to more normal levels shortly after (see e.g. [BBK]).

The distinction between spiky and "standard" behavior turns out to be crucial in the modelling of the electricity price: the need to obtain a good reproduction of the spikes in the models represents one of the main differences between other financial markets and the electricity market. Another important difference with respect to other markets is the seasonality that can be observed. It is mainly due to a clear weekly periodicity, caused by the fluctuations in consumptions during different days of the week [RFC]. There is also a long-term seasonal effect on the prices, which appears over time lengths of approximately 3-4 months.

Two of the first models for electricity price are due to Schwartz [Sch97, LuSc02]. In [Sch97] the authors introduced an Ornstein-Uhlenbeck model for the spot price dynamics which included a mean-reversion component, and later on, in [LuSc02], a deterministic component describing the seasonality was added. Since this works, a widespread literature has been proposed in order to model the basic features of this market, especially about the formation of spikes, which were not covered by the aforementioned papers [Sch97],[LuSc02]. An interesting review of the state of the art has been given by Weron in [Wer14]. The interested reader may also refer to the huge amount of papers therein. Weron proposes a classification of the models in five main classes: i) the multi-agent models, which simulate the operation of a system of heterogeneous agents interacting with each other, and build the price process by matching the demand and supply in the market; ii) the fundamental (structural) methods, which describe the price dynamics by modelling the impacts of important physical and economic factors on the price of electricity; iii) the reduced-form (quantitative, stochastic) models, which characterize the statistical properties of electricity prices over time, with the ultimate objective of derivatives evaluation and risk management; iv) the statistical approaches, which are either direct applications of the statistical techniques of load forecasting or power market implementations of econometric models; v) the computational intelligence techniques, which combine elements of learning, evolution and fuzziness to create approaches that are capable of adapting to complex dynamic systems,

Here we are interested in the third class and, partially, in the second class. In our stochastic model the time dynamics of the spot price is described by modelling the drift via a deterministic function which models the long–term seasonality, while the noises responsible of the standard fluctuations and the extreme spikes are described by the solutions of two independent SDE's, one of them modelling the *standard behavior* and the other one modelling the *spiky behavior*. Our model can be summarized as

$$S(t) = f(t) + \sum_{i=1}^{2} X_i(t), \quad t \geq t_0 \in \mathbb{R}_+$$

where f is a deterministic function and the X_i, for $i = 1, 2$ are two stochastic processes subject to a mean reversion term, responsible for the randomness in the base component and in the spiky regime, respectively. Different examples of these kind of models may be found for example in [KMS10, MeTa08, Wer14, JMSS18].

From now on, we denote with $Y_t,\ t \in \{1, \dots, 3287\}$ the time series of the spot prices under study. Often, among the characteristics of the spot prices, one that is not taken into account is the presence of self-correlations in the price increments $Y_t - Y_{t-1}$. The presence of this feature suggests, when trying to model these kind of markets, to modify the structure of the existing models to include the self-correlations. One of the possible choices that have been used in literature so far is to consider a fractionally integrated ARFIMA model, a generalisation of the classical ARIMA model, as it has been done in [GiGr13], and in other cases reported in the review [Wer14]. In particular, in [GiGr13] this has been done for the Italian electricity market.

From the point of view of reduced-form models, the natural adaptation might be to consider a fBm as the driving noise of the base component instead of the usual sBm. This is the direction of this work. In particular, the process X_1 will be a fractional Ornstein-Uhlenbeck process.

In literature there have been several attempts of using a fractional Brownian motion in financial market modelling. Its relatively simple nature, combined with its flexibility in modelling data whose increments are self–correlated, gave rise to a growing number of models involving fBm. Anyway, it was pointed out quite early by Rogers in [Rog97] that a model involving fBm would result in admitting the presence of some kind of arbitrage in the market.

More into details, Cheridito [Che02] proved that there are strong arbitrage opportunities for the fractional models of the form

$$\begin{aligned} X(t) =& \nu(t) + \sigma B^H(t) \\ X(t) =& \exp(\nu(t) + \sigma B^H(t)), \end{aligned} \tag{4.1}$$

where $\nu(t)$ is a measurable bounded deterministic function and B^H is a fractional Brownian motion of *Hurst parameter* $H \in (0,1)$. This arbitrage opportunities can be built provided that we are allowed to use the typical set of admissible trading strategies (see [Che02] for the complete definitions). This set of admissible strategies in particular allows to buy and sell the stock continuously in time, which is a questionable assumption in many frameworks. In [Che02] the author proved that for the models (4.1) the arbitrage opportunities disappear, provided that we restrict the set of strategies to the ones that impose an arbitrary (but fixed) waiting time $h > 0$ between a transaction and the following one. In the present work we consider a fractional Ornstein–Uhlenbeck process. We cannot use directly the results in [Che02], but the extension to this family of processes should be straightforward and may be subject for future work. About the restriction of a waiting time $h > 0$, we point out that it is meaningful in the case of day–ahead markets, like electricity market is. Indeed, the price is established only once per day, and thus we can observe realisations of the process X_1 it only at discrete times. Obviously, in this case there would be no possibility of considering a strategy which needs arbitrarily quick (in time) modifications.

A striking empirical feature of electricity spot prices is the presence of spikes, that can be described by a jump in the price process immediately followed by a fast reversion towards the mean. It is interesting to notice that in the case of the Italian electricity market the presence of several jumps is shown, many of which appearing clustered over short time periods. As a consequence, the second component X_2 will be defined as the solution of a mean reverting processes driven by a self-exciting Hawkes process, which is a jump process whose jumps frequency depends upon the previous history of the jump times. In particular, right after a jump has occurred, the probability of observing a subsequent jump is higher than usual. The interested reader may refer to [BMM15, Haw18] fo an excellent survey on the introduction, the relevant mathematical theory and overview of applications of Hawkes processes in finance and for more recent financial applications.

The second part of the chapter is devoted to a complete computational study. We apply the model to the study case of the time series of the Italian MGP, the data of the day-ahead market (see [Prices]) from January 1, 2009 to December 31, 2017. The first two years are the sample considered for the estimation and validation of the model. We carry out the difficult task of separating the components of the raw prices into our main components (weekly component, long-term seasonal component, standard behaviour, spiky behaviour). Then we deal with the problem of the estimation the parameters of the model and we test the forecasting performance of our model on forecasting horizons from one to thirty days. The parameters are estimated in a *rolling window* fashion. We construct prediction intervals (PI) and quantile forecasts (QF) and evaluate them via a class of adequate evaluation metrics like the Winkler score and the Pinball loss function.

We conclude that the analysis shows some quantitative evidence that both the fractional Brownian motion and the Hawkes process are adequate to model the electricity price markets.

4.2 A model driven by jumps and a fractional Brownian motion

In this section, we introduce the structure of the model, we plot some of his paths for different values of the parameters, and we discuss in detail the techniques of parameter estimation that we will use to calibrate the model to the real data that we will consider.

4.2.1 The equations

The model we propose extends in different ways some relevant models already available in the literature. In particular, we consider a modification of the model proposed in [BKM07, MeTa08] and then modified for example in [JMSS18], by including some self–exciting features, via Hawkes-type processes.

We adopt an arithmetic model as in [BBV13, BKN12, BKM07, JMSS18] in which the power price dynamics is assumed to be the sum of several factors. We suppose that the spot price process $S = \{S(t), t \in \mathbb{R}_+\}$ evolves according to the following dynamics

$$S(t) = f(t) + X(t). \tag{4.2}$$

The function $f(t)$ describes the deterministic trend of the evolution, while the process $X = \{X(t), t \in \mathbb{R}_+\}$ describes the stochastic part. The latter is a superposition of two factors: X_1, known in literature as the *base component*, which is continuous a.e. and aims to model the standard behavior of the electricity price, and X_2 which is the *jump component*, describing the spiky behavior of the electricity prices, overlapped to the base signal. This means that, for any $t \in \mathbb{R}_+$,

$$X(t) = X_1(t) + X_2(t). \tag{4.3}$$

First of all, we consider a mean-reverting model: despite the possible noise, the price tends to a specific level. In particular, our starting assumption is that both in the basic and spiky regimes prices tend to revert towards their mean, even if with different strengths. This is because we expect that whether the price strongly deviates from the mean value, as during a spike, then it returns to the average level with a stronger force than usual.

Regarding the base component, let us note that in many time series of the electricity markets, an evidence of correlation among price increments is clear. For example, see Figure 4.11. In order to capture better such a correlation within different returns, we consider an additive noise driven by a fractional Brownian motion.

Furthermore, the Italian market is rather peculiar since clearly identifiable spikes are rare; as a consequence the intensity of the spike process is small and becomes difficult to be estimated. Moreover, despite the small number of spikes, a clustering effect seems to be present; so one might better include the effect of a self-exciting stochastic process. Hence, by following recent literature [BaMu14, BMM15, BCZ13, CHST18, DaZa14, Hai17, JMSS18], we model the jump component X_2 via a Hawkes marked process.

To be more precise, let us consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$. We suppose that X_1 follows a stochastic differential equation driven by a fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}_+\}$ with Hurst parameter $H \in (0, 1)$ and diffusion coefficient $\sigma \in \mathbb{R}_+$, subject to mean reversion around a level zero, with strength $\alpha_1 \in \mathbb{R}_+$.

For any $t \in \mathbb{R}_+$, we define $X_1(t)$ as the solution of the following equation

$$dX_1(t) \quad = \quad -\alpha_1 X_1(t)dt + \sigma dB^H(t). \tag{4.4}$$

Proposition 4.1 ([KMR]). *Given $\alpha_1, \sigma \in \mathbb{R}_+$ and $H \in (0, 1)$, Equation (4.4) admits the unique solution*

$$\begin{aligned} X_1(t) &= X_1(0)\, e^{-\alpha_1 t} - \alpha_1\, \sigma e^{-\alpha_1 t} \int_0^t e^{-\alpha_1 s} dB^H(s) + \sigma B^H(t) \qquad (4.5)\\ &= X_1(0)\, e^{-\alpha_1 t} + \sigma \int_0^t e^{-\alpha_1 (t-s)} dB^H(s)). \end{aligned}$$

X_1 is called a fractional Ornstein-Uhlenbeck process.

The covariance structure of such a process is rather complex (see Theorem 1.43 in [KMR], simplified in the case of the variance of the 1-dimensional marginals.

Proposition 4.2 ([KMR], Theorem 1.43). *Given $\alpha_1, \sigma \in \mathbb{R}_+$ and $H \in (0,1)$, the following properties hold.*

i) For any $t \in \mathbb{R}_+$, the variable $X_1(t)$ has a Gaussian distribution, i.e.

$$X_1(t) \sim \mathcal{N}\left(X_1(0)e^{-\alpha_1 t}, V_{\alpha_1}(t)\right), \tag{4.6}$$

where the variance $V_{\alpha_1}(t)$ is given by

$$V_{\alpha_1}(t) = H\sigma^2 \int_0^t x^{2H-1}\left(e^{-\alpha_1 s} + e^{-\alpha_1(2t-s)}\right) ds. \tag{4.7}$$

ii) The variance has the following time asymptotic behavior

$$\lim_{t\to+\infty} V_{\alpha_1}(t) = \alpha_1^{-2H}\, H\sigma^2\, \Gamma(2H),$$

where $\Gamma : \mathbb{R} \to \mathbb{R}$ is the classical Γ function.

We move now to the X_2 component: we wish to define it as the solution of a mean-reverting SDE driven by a Hawkes marked process π, i.e. as

$$X_2(t) = X_2(0) - \int_0^t \alpha_2 X_2(s)ds + \int_0^t \int_0^{\lambda_s} \int_0^\infty z\ \pi\left(ds, d\eta, dz\right). \tag{4.8}$$

We introduce its components in detail. Consider a marked point process

$$\{(T_i, Z_i)\}_{i\in\mathbb{N}}, \tag{4.9}$$

where, for any $i \in \mathbb{N}$, T_i is the random time at which the i-th jump occurs and Z_i is the relative random jump size. So we may express the counting measure J of the jumps via the marked process (4.9) as

$$J(dt) = \sum_{i=1}^\infty Z_i\ \epsilon_{T_i}(dt) = \int_{\mathbb{R}} zQ(dt, dz), \tag{4.10}$$

where ϵ_x is the Dirac measure localized in x and Q is the following marked counting measure on $\mathbb{R}_+ \times \mathbb{R}$

$$Q(dt, dz) = \sum_{i=1}^\infty \epsilon_{(T_i, Z_i)}(dt, dz). \tag{4.11}$$

The counting process $N = \{N_t\}_{t\in\mathbb{R}_+}$ associated to the marked point process (4.9) is such that, for any $t \in \mathbb{R}_+$

$$N_t = \sum_{i=1}^\infty \epsilon_{T_i}([0,t]) = Q([0,t] \times \mathbb{R}). \tag{4.12}$$

The process N is characterized by its time dependent conditional intensity λ_t, $t \in \mathbb{R}_+$, which is the quantity such that:

$$\lambda_t = \lim_{dt\to 0} \frac{\mathbb{E}\left[N_{t+dt} - N_t | \mathcal{F}_t\right]}{dt},$$

and

$$prob\left(N_{t+dt} - N_t = k | \mathcal{F}_t\right) = \begin{cases} 1 - \lambda_t\, dt + o(dt), & k = 0; \\ \lambda_t\, dt + o(dt), & k = 1; \\ o(dt), & k > 1. \end{cases}$$

In our case, we suppose that, for any $t \in \mathbb{R}_+$, λ_t is a function of past jumps of the process, i.e.

$$\lambda_t = \lambda + \int_0^t \Phi(t-s) dN_s, \tag{4.13}$$

with background intensity $\lambda > 0$ and excitation function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Whenever $\Phi(\cdot) \neq 0$, the resulting process is different from a homogeneous Poisson process, and if

$$\|\Phi\|_1 = \int_0^\infty \Phi(t)dt < 1, \tag{4.14}$$

the existence of a unique process is implied. Condition (4.14) also implies the stationarity of the process, that is that its distributions are invariant under translations [BrMa96, BaMu14, BMM15]. Equation (4.13) states that the random times of the jumps are governed by a constant intensity λ and that any time a jump occurs, it excites the process in the sense that it changes the rate of the arrival of subsequent jumps, by means of a kernel Φ. Usually, the latter decreases to 0, so that the influence of a jump upon future jumps decreases and tends to 0 for larger time increments. We say in this case that N is a univariate Hawkes processes [Haw71(1), Haw71(2)]. Note that we may make explicit the dependence of the intensity process upon the random jump times $\{T_i\}_{i\in\mathbb{N}}$ by the following

$$\lambda_t = \lambda + \int_0^t \Phi(t-s) \sum_{i\in\mathbb{N}} \epsilon_{T_i}(dx) = \lambda + \sum_{i\in\mathbb{N}\,:\ T_i \le t} \Phi(t - T_i).$$

As it happens in many examples in modelling (see [BCZ13, BaMu14, BMM15]), we consider an exponential model for the excitation function, that is

$$\Phi(t) = \gamma e^{-\beta t}, \tag{4.15}$$

where $\gamma, \beta \in \mathbb{R}_+$ represent the instantaneous increase after a jump and the speed of the reversion to λ of the excitation intensity. As a consequence, the intensity (4.13) becomes

$$\lambda_t = \lambda + \int_0^t \gamma e^{-\beta(t-s)} dN_s = \lambda + \gamma \sum_{i\in\mathbb{N}\,:\ T_i\le t} e^{-\beta(t-T_i)}. \tag{4.16}$$

It may be seen as a solution of the following stochastic differential equation

$$d\lambda_t = \beta\left(\lambda - \lambda_t\right) + \gamma dN_s. \tag{4.17}$$

Notice that (4.16) is the solution of the equation (4.17) when the process starts in λ_0 infinitely in the past and it is at its stationary regime. Otherwise, in order to model a process from some time after it is started and setting an initial condition $\lambda_0 = \lambda^*$ the conditional intensity, solution of (4.17) would be

$$\lambda_t = e^{-\beta t}\left(\lambda^* - \lambda\right) + \lambda + \int_0^t \gamma e^{-\beta(t-s)} dN_s. \tag{4.18}$$

As mentioned above, for t large enough the impact of the initial condition vanishes, since the first term would die out. Note that a new jump of N_t increases the intensity, which increases the probability of new jump, but the process does not necessarily blow up because the drift is negative if $\lambda_t > \lambda$. Furthermore, while the process $\{N_t\}_{t\in\mathbb{R}_+}$ is non Markovian, the bidimensional process $\{(N_t, \lambda_t)\}_{t\in\mathbb{R}_+}$ is a Markov process [BMM15], such that

$$\begin{aligned} d\mathbb{E}\left[N_t\right] &= \mathbb{E}\left[\lambda_t\right] dt, && (4.19)\\ d\mathbb{E}\left[\lambda_t\right] &= \left(\beta\lambda + (\gamma-\beta)\mathbb{E}\left[\lambda_t\right]\right) dt. && (4.20) \end{aligned}$$

Since the solution of equation (4.20) is

$$\mathbb{E}\left[\lambda_t\right] = \mathbb{E}\left[\lambda_0\right] e^{(\gamma-\beta)t} + \beta\gamma \int_0^t e^{-(\gamma-\beta)(t-s)} ds,$$

if $\gamma > \beta$, then the intensity would explode in the average, and so it would happen for the process N_t. This is not the case in the stationary regime, since in the exponential case, assumption (4.14) becomes

$$1 > \nu = \|\Phi\|_1 = \int_0^\infty \gamma e^{-\beta t} dt = \frac{\gamma}{\beta},$$

i.e.

$$\gamma < \beta. \tag{4.21}$$

With this definition of N in mind, we introduce the noise term π appearing in the stochastic differential equation (4.8) that defines the process X_2. Let π be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with intensity measure $\Lambda = \nu_+ \times \nu_+ \times \mu$, where ν_+ is a Lebesgue measure on $\mathbb{R}_+$. The measure μ is the distribution of the size of the jumps that satisfies condition (4.22). We suppose that the size distribution is given by a Borel measure μ on $\mathbb{R}_+$, satisfying the condition

$$\int_0^\infty (\eta \wedge \eta^2)\mu(d\eta) < \infty. \tag{4.22}$$

If we suppose $\mu(d\eta) = \epsilon_1(d\eta)$, the jumps are of size one. In [DFH16], in a more general setting in which they consider multidimensional non linear Hawkes process, the author prove that the Hawkes process (4.12) with conditional intensity given by (4.16) may be written as

$$N_t = \int_0^t \int_0^{\lambda_t} \int_0^\infty \pi\left(ds, d\eta, dz\right). \tag{4.23}$$

In conclusion, the process $X = X_1 + X_2$ is given by the solution of the following system, for $t \in \mathbb{R}_+$,

$$X_1(t) = X_1(0) - \int_0^t \alpha_1 X_1(s) ds + \sigma \int_0^t dB^H(s); \tag{4.24}$$

$$X_2(t) = X_2(0) - \int_0^t \alpha_2 X_2(s) ds + \int_0^t \int_0^{\lambda_s} \int_0^\infty z\, \pi\left(ds, d\eta, dz\right), \tag{4.25}$$

coupled with equation (4.17), with $\gamma, \beta, \lambda \in \mathbb{R}_+$.

Finally, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the natural filtration generated by the processes. System (4.24)-(4.25) admits a unique solution thanks to classical results.

4.2.2 Path simulations

In the following we consider the simulation results showing the macroscopic behaviour of the model, by considering some fixed set of parameters in Tables 4.1–4.3. For the jump size distribution μ we choose to consider the Generalized Extreme value distribution, that is a probability measure depending from 3 parameters $\tilde{\mu}, \xi \in \mathbb{R}$ and $\tilde{\sigma} > 0$ with density function given by

$$f(x) = \frac{1}{\tilde{\sigma}}\, t(x)^{\xi+1} e^{-t(x)}, \tag{4.26}$$

where

$$t(x) = \begin{cases} \left(1 + \xi\left(\frac{x-\tilde{\mu}}{\tilde{\sigma}}\right)\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ e^{-(x-\tilde{\mu})/\tilde{\sigma}} & \text{if } \xi = 0. \end{cases}$$

α_1	α_2	σ	λ	$\tilde{\mu}$	ξ	$\tilde{\sigma}$
0.1	0.5	6	0.01	18	0.7	2

Table 4.1: Fixed parameters used in the simulations: mean-reverting parameters α_1 and α_2, diffusion coefficient σ, basic Poisson point process parameter λ and the parameters $\tilde{\mu}, \xi \in \mathbb{R}$ and $\tilde{\sigma} > 0$ in the Generalized Extreme Value distribution (4.26).

Some of the parameters we considered for these simulaitons are fixed, namely the set of parameters appearing in Table 4.1. On the other hand, for some of the parameters of our model, we consider a changing set of parameters in order to evaluate the impact of some of the important features that we introduce with our model: the fBm depending from the parameter H and the parameters γ, β of the self-exciting part of the Hawkes process

Parameter	**a.1.**	**a.2.**	**a.3.**
H	0.2	0.5	0.7

Table 4.2: Set of simulation parameters: values for the Hurst parameter H in the diffusion term in (4.24).

Parameter	**(a)**	**(b)**	**(c)**	**(d)**
γ	0	0.05	0.15	0.3
β	0	0.08	0.2	0.5

Table 4.3: Set of simulation parameters for the Hawkes excitation function: γ and β in (4.17) satisfying stationarity condition (4.21).

We consider, only in this section, a fixed deterministic function

$$f(t) = 130 \cdot 1_{[0,\infty)}(t),$$

and the following deterministic initial condition for the processes X_1 and X_2

$$X_1(0) \quad = \quad X_2(0) = 0.$$

Stochastic simulation are carried out by generating exact paths of Fractional Gaussian Noise by using circulant embedding (for $1/2 < H < 1$) and Lowen's method (for $0 < H < 1/2$), while the Hawkes process is generated by a thinning procedure for inhomogeneous Poisson process as in [Oga781].

We see in Figures 4.1–4.8 some simulations of a path of $X = X_1 + X_2$ for the different values of the parameters chosen in Tables 4.1–4.3. Even in some cases this meant that some parts of the path are not visible, we chose to keep the same scale in all figures. This makes us see very clearly the differences caused in the nature of the process X_1 by the changes in the values of H. We see in particular that, keeping α_1, σ fixed we get a much more variable process as long as H increases.

Regarding the jump component X_2, which is independent from X_1, we see that the cluster effect is clearly visible for the sets of parameters (b)–(d). It seems that the set of parameters (b) is producing more clusters than the others. This may seem strange, since in this case the parameter γ is lower than in (c) and (d), but we remark that in all cases the parameter β, which models the speed of mean reversion of λ_t towards λ, varies consistently with γ.

We make a remark about the relation of this simulations with the real data. If we compare Figures 4.1–4.8 with Figure 4.9, in which we plot the entire dataset that we will analyse, we can make some qualitative considerations. Our simulations of $X_1 + X_2$ do not include any seasonal

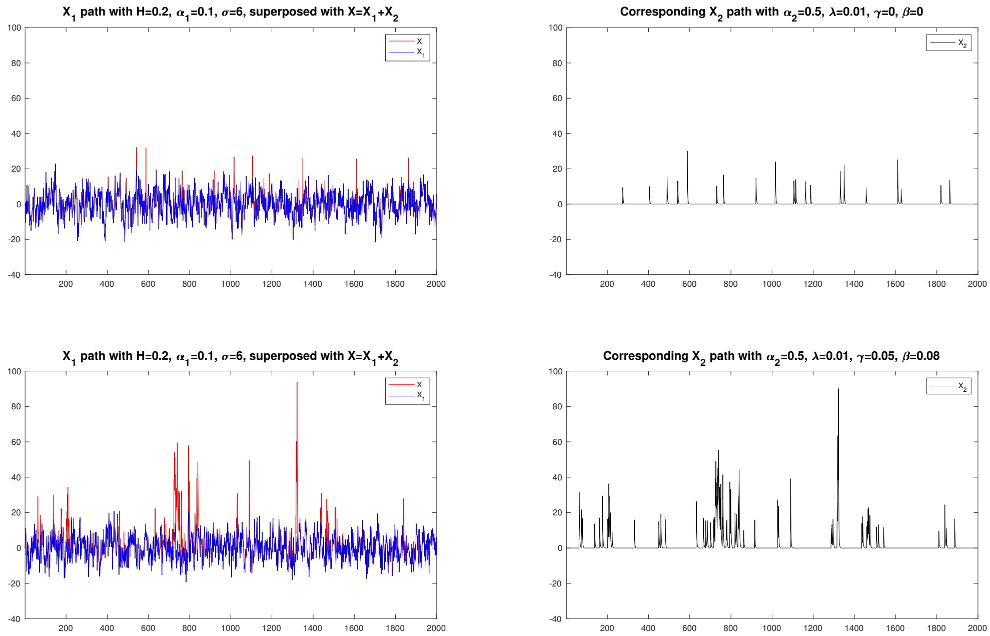

Figure 4.1: Path of the process $X = X_1 + X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.2$ and (a) $\lambda = 0, \beta = 0$; (b) $\lambda = 0.05, \beta = 0.08$

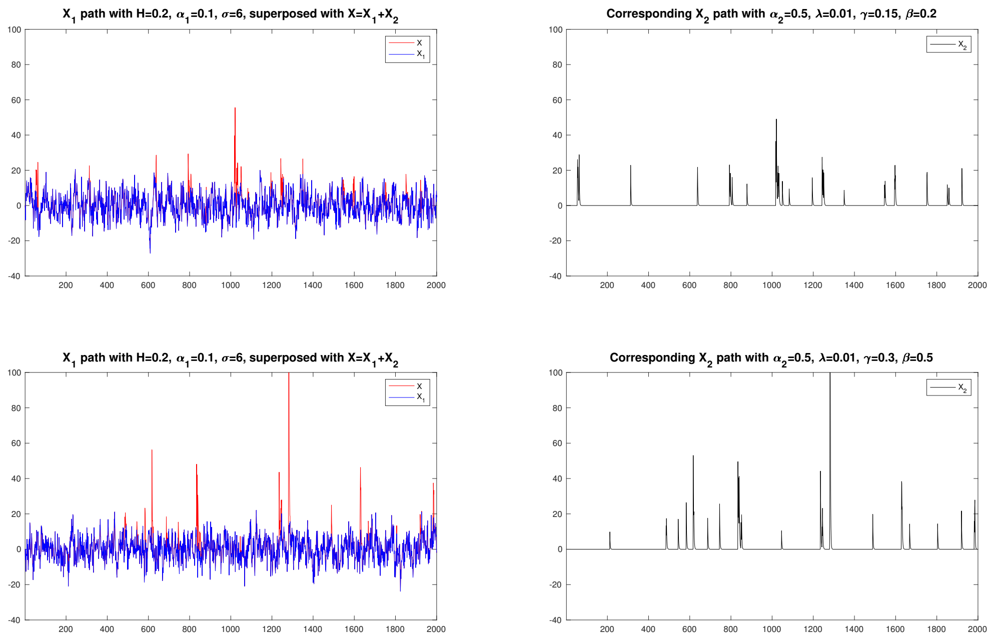

Figure 4.2: Path of the process $X = X_1 + X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.2$ and (c) $\lambda = 0.15, \beta = 0.2$; (d) $\lambda = 0.3, \beta = 0.5$.

component, and this is clearly visible. Anyway, from the point of view of the appearance of the paths, we see some similarities between Figure 4.9, the bottom plot of Figure 4.3 and the top plot of Figure 4.4, which are relative to $H = 0.3$ and the set of parameters (b) and (c) for the

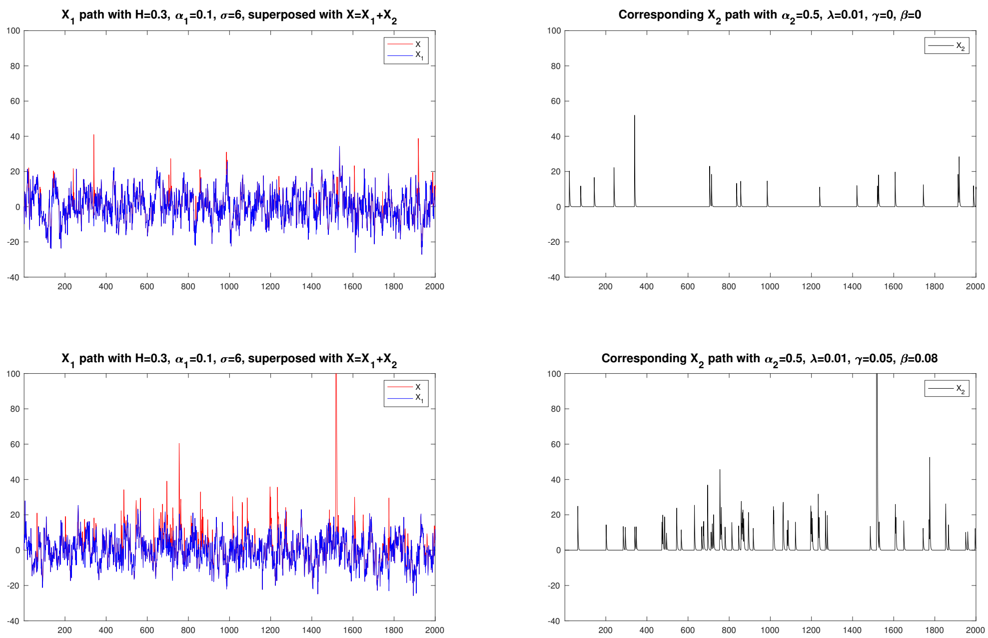

Figure 4.3: Path of the process $X = X_1+X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.3$ and (a) $\lambda = 0, \beta = 0$; (b) $\lambda = 0.05, \beta = 0.08$

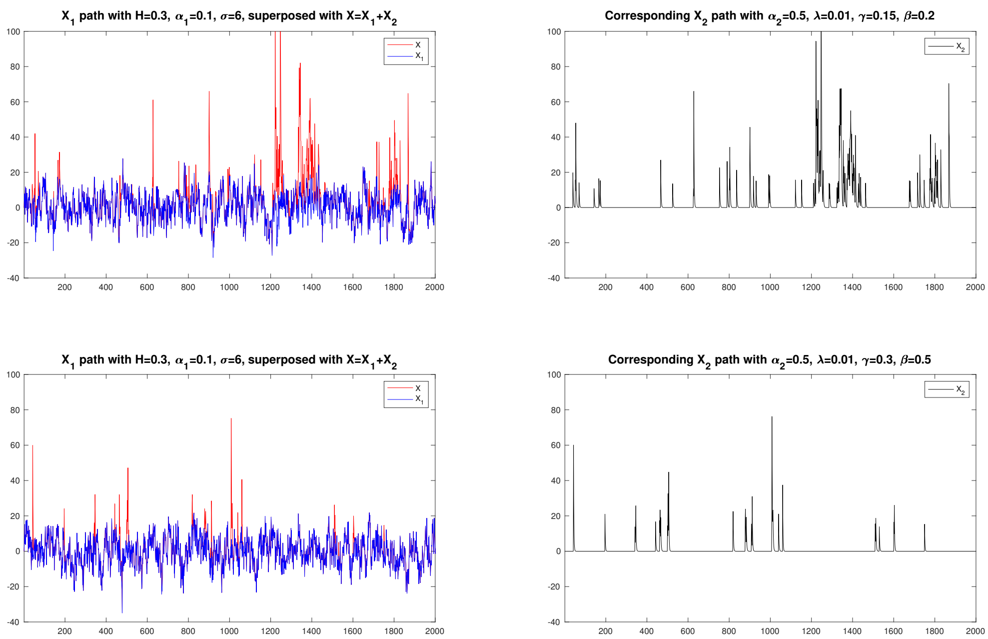

Figure 4.4: Path of the process $X = X_1+X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.3$ and (c) $\lambda = 0.15, \beta = 0.2$; (d) $\lambda = 0.3, \beta = 0.5$.

Hawkes process, corresponding to $\gamma = 0.05$, $\beta = 0.08$ and $\gamma = 0.15$, $\beta = 0.2$. In both of these figures the standard volatility seems quite similar to the one of the real data, and moreover the jump behaviour (both amplitude and clusters) is quite similar to the one of Figure 4.9.

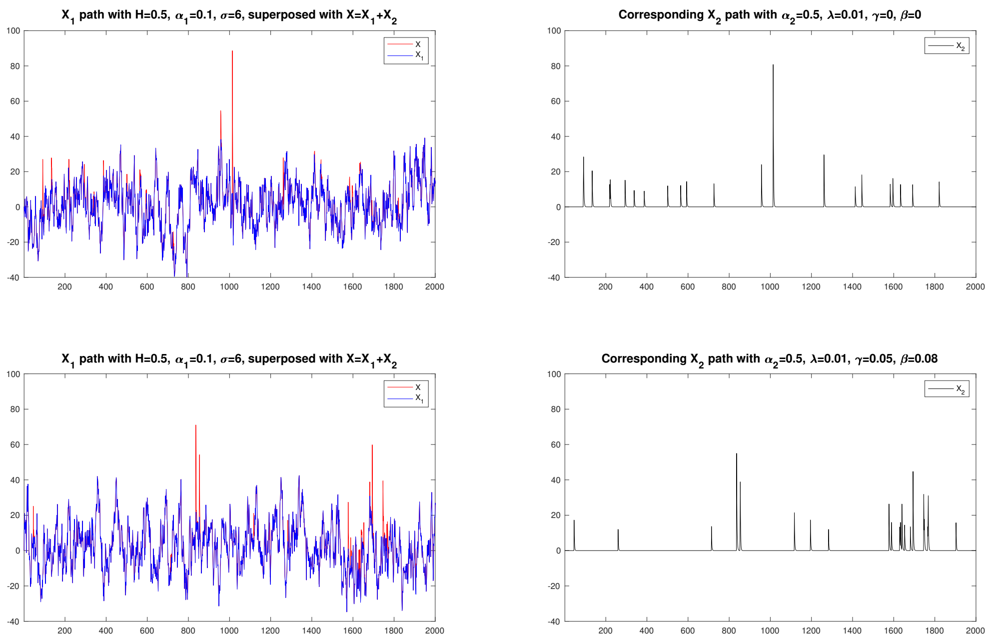

Figure 4.5: Path of the process $X = X_1+X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.5$ and (a) $\lambda = 0, \beta = 0$; (b) $\lambda = 0.05, \beta = 0.08$

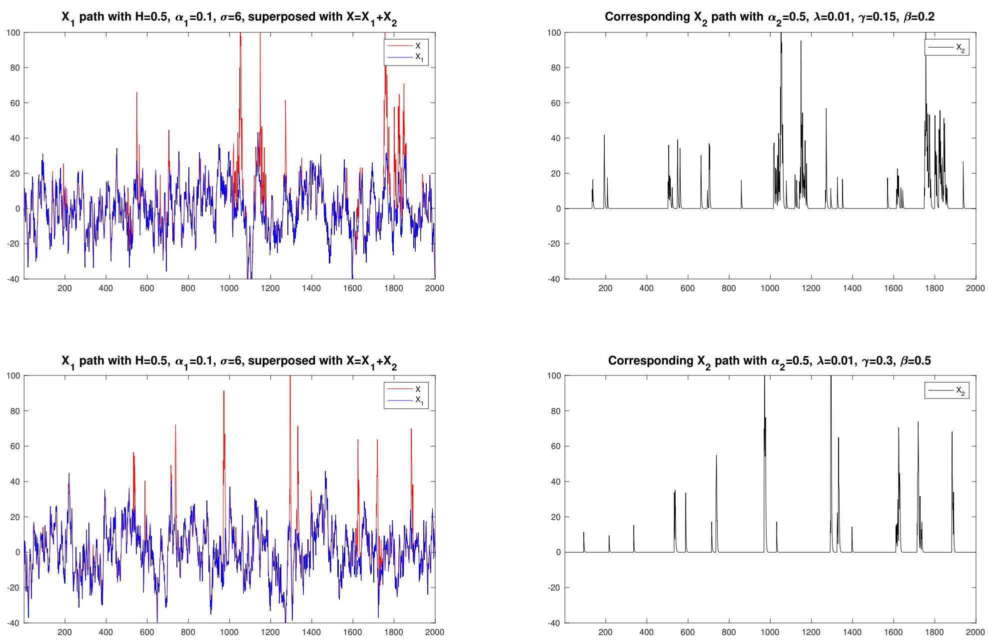

Figure 4.6: Path of the process $X = X_1+X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.5$ and (c) $\lambda = 0.15, \beta = 0.2$; (d) $\lambda = 0.3, \beta = 0.5$.

These considerations are coherent with the estimates that we will get in Section 4.3.3.

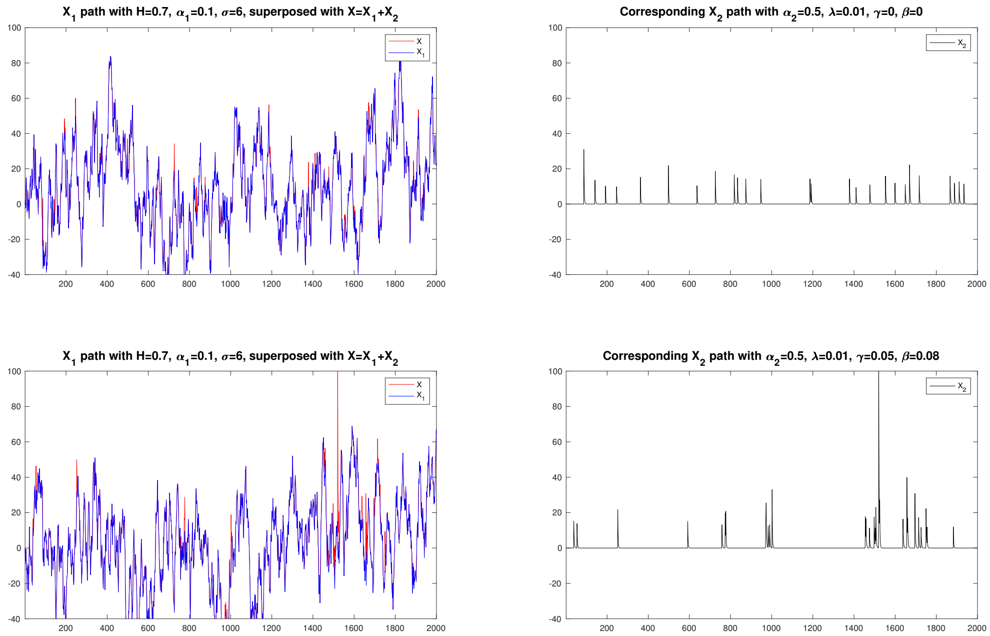

Figure 4.7: Path of the process $X = X_1{+}X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.7$ and (a) $\lambda = 0, \beta = 0$; (b) $\lambda = 0.05, \beta = 0.08$

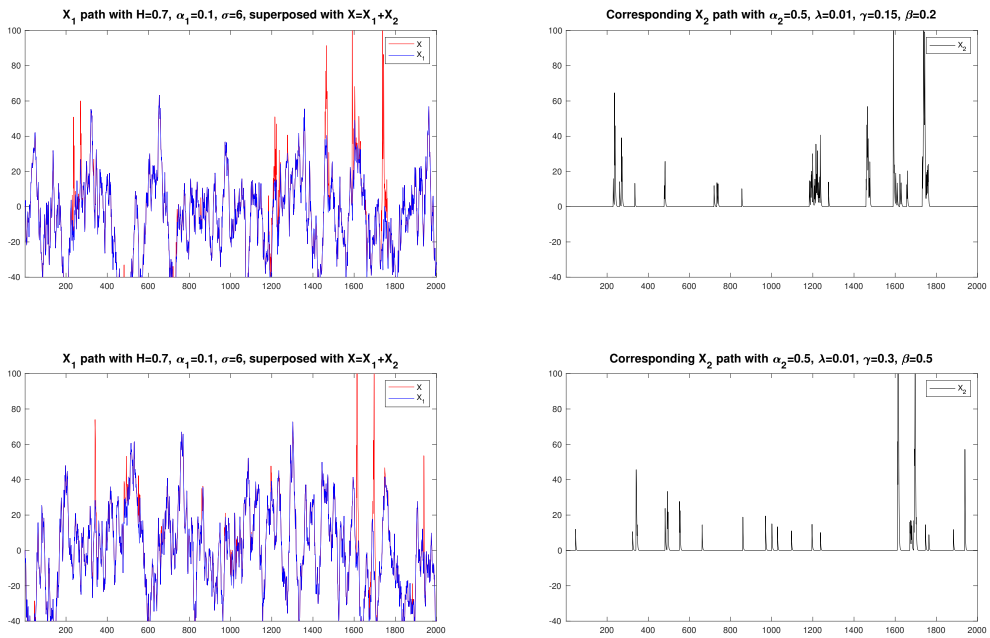

Figure 4.8: Path of the process $X = X_1{+}X_2$ (red) and of X_1 alone (blue), with the corresponding X_2 in black on the right. Case $H = 0.7$ and (c) $\lambda = 0.15, \beta = 0.2$; (d) $\lambda = 0.3, \beta = 0.5$.

4.2.3 Parameter estimation

This section is devoted to the methods of estimation for the parameters of the two stochastic components X_1 and X_2 of our model. We recall that we have the following set of parameters to be estimated from the dataset Y

Equation	Parameter
$\mathbf{X_1}$	α_1
$\mathbf{X_1}$	σ
$\mathbf{X_1}$	H
$\mathbf{X_2}$	α_2
$\mathbf{X_2}$	λ_0
$\mathbf{X_2}$	γ
$\mathbf{X_2}$	β
$\mathbf{X_2}$	parameters for jump distribution

Table 4.4: Parameters of the model

Estimations for X_1

For the base component X_1, we have to estimate three parameters: α_1, which is the rate of mean-reversion of the process to the zero level, σ, which is the variance parameter of the fractional noise B^H, and the parameter of fractionality of the noise H itself, which determines whether the contributions of the noise term are positively correlated (if $H > 1/2$), negatively correlated (if $H < 1/2$) or non-correlated (if $H = 1/2$, which is the case in which the noise is a classical Brownian motion).

We discuss first the estimation of the Hurst exponent H, which is the coefficient of fractionality of our driving noise B^H. The estimation of this parameter is very important for practical purposes, since it determines the magnitude of the self-correlation of the noise of our model. There are various techniques that can be used to infer the Hurst coefficient from a discrete signal. In [MiWo07] there is a good review of some of these techniques. Anyway, these techniques can be used to estimate the Hurst coefficient only supposing that we are observing a path of a fractional Brownian motion $\{B_t^H(\omega)\}_t$ alone, without any drift. Since we are using the solution of an Ornstein-Uhlenbeck SDE to model the base component, in principle we could not use these techniques. Indeed, we tried to implement some of those techniques to estimate the Hurst coefficient from realized paths of the process X_1, and they performed poorly.

We decide to use, instead, the estimator presented in Theorem 3.4 of [KMR], which is a consistent estimator of the Hurst coefficient given the observations of a wide class of SDEs driven by fractional Brownian motion, including the case of an Ornstein-Uhlenbeck model.

Suppose that we can observe our signal on the interval $[0, T]$ partitioned as $\{\frac{k}{n}T\}_k$, for $k \in \{0, \dots n\}$. Given a stochastic process $\{V, t \in [0, T]\}$, define the quantity

$$\Delta_{n,k}^{(2)} X_1 := X_1\Big(\frac{k+1}{n}T\Big) - 2X_1\Big(\frac{k}{n}T\Big) + X_1\Big(\frac{k-1}{n}T\Big).$$

By Theorem 3.4 of [KMR], we have that

$$\hat{H}_n := \frac{1}{2} - \frac{1}{2\log 2} \log\left(\frac{\sum_{k=1}^{2n-1}(\Delta_{2n,k}^{(2)} X_1)^2}{\sum_{k=1}^{n-1}(\Delta_{n,k}^{(2)} X_1)^2}\right)$$

is a strongly consistent estimator for the value of H, i.e. it holds

$$\hat{H}_n \xrightarrow{n\to\infty} H.$$

In our case, our dataset will be discretized in time steps with minimum length equal to one, so we will have to stop at a specific step of the discretization, which is exactly the maximum value of n such that $\frac{T}{2n} > 1$.

In order to estimate σ, we consider the estimator introduced in Proposition 4.2 of [HNZ17]. Suppose we observe the whole path of a solution $X_1(t)$ of a fractional Ornstein-Uhlenbeck equation. We define the estimator, for any $k \geq 2$ and for any $p \geq 1$, as

$$\hat{\sigma}_T(n) := \frac{n^{-1+pH} V^n_{k,p} X_1(T)}{c_{k,p} T}, \tag{4.27}$$

where $V^n_{k,p} X_1(t)$ is the k-th order p-th variation process defined by

$$V^n_{k,p} X_1(t) := \sum_{i=1}^{[nt]-k+1} \left| \Delta_k X_1 \Big(\frac{i-1}{n} \Big) \right|^p := \sum_{i=1}^{[nt]-k+1} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} X_1 \Big(\frac{i+j-1}{n} \Big) \right|^p.$$

and the constant $c_{k,p}$ appearing in (4.27) is given by

$$c_{k,p} := \frac{2^{p/2} \Gamma\Big(\frac{p+1}{2}\Big)}{\Gamma\Big(\frac{1}{2}\Big)} \rho^{p/2}_{k,H},$$

where $\rho_{k,H}$ is

$$\rho_{k,H} := \sum_{j=-k}^{k} (-1)^{1-j} \binom{2k}{k-j} |j|^{2H}.$$

With a discrete dataset of time lenght N and with minimum discretization length equal to 1, we can only compute the estimator $\hat{\sigma}_T(n)$ for $n = 1$ and $T = N$. We choose to use $k = 2$ and $p = 2$.

We estimate then α_1 by using an ergodic estimator that is introduced again in [HNZ17]. We first define its continuous version. Given a solution $X_1(t)$ defined for $t \in [0, T]$ of an Ornstein-Uhlenbeck equation, we define it as

$$\hat{\alpha}_1(T) := \Big(\frac{1}{\sigma^2 H \Gamma(2H) T} \int_0^T X_1(t)^2 dt \Big)^{-\frac{1}{2H}}. \tag{4.28}$$

It holds that $\hat{\alpha}_1(T) \to \alpha_1$ a.s. when $T \to \infty$. This estimator can be easily discretized. Suppose that we are observing our process X_1 in the time points $\{kh\}$, for $h = 0, \dots, n$. Here h is the amplitude of the time discretization, and we suppose that $h = h(n)$ is such that $hn \to \infty$ and $h(n) \to 0$ when $n \to \infty$. In [HNZ17] the authors define the discretized estimator as

$$\hat{\alpha}_1(n) := \Big(\frac{1}{\sigma^2 H \Gamma(2H) T} \sum_{k=1}^{n} X_1(kh)^2 \Big)^{-\frac{1}{2H}}. \tag{4.29}$$

The only difference with the continuous version is the discretization of the integral appearing in (4.28). The following result holds:

Theorem 4.3 ([HNZ17], Theorem 5.6)**.** *Let X_1 be the solution of an Ornstein-Uhlenbeck process observed at times $\{kh, k = 0, \dots, n\}$, and such that $h = h(n)$ satisfies $hn \to \infty$ and $h(n) \to 0$ when $n \to \infty$. Moreover, if we suppose that*

i) if $H \in (0, \frac{3}{4})$, $nh^p \to 0$ as $n \to \infty$ for some $p \in \Big(1, \frac{3+2H}{1+2H} \wedge (1+2H)\Big)$

ii) if $H = \frac{3}{4}$, $\frac{h^p n}{\log(hn)} \to 0$ as $n \to \infty$ for some $p \in (1, \frac{9}{5})$

iii) if $H \in (\frac{3}{4}, 1)$, $h^p n \to 0$ as $n \to \infty$ for some $p \in \Big(1, \frac{3-H}{2-H}\Big)$

Then the estimator $\hat{\alpha}_1 \xrightarrow{n\to\infty} \alpha_1$ *a.s.*

We observe that in our case we are not able to extend the total length of the observation interval, since only a single time series is given. In addition, our dataset consists in daily observations, so that the minimum time increment h that we can consider is $h = 1$. We aim to define a proper function $h = h(n)$ such that it satisfies the conditions of Theorem 4.3 for every value of $H \in (0, 1)$ (which is in principle unknown).

Let N be the length of our dataset. As a fundamental condition, we want that $h(N) = 1$, so that we are able to compute the N-th approximation of our estimator with our data. We look for our candidate h within the class of functions

$$h(n) = \left(\frac{N}{n}\right)^{\delta},$$

for some positive δ to be determined. All h in this class satisfy that $h(N) = 1$. We need also need to have that $nh \to \infty$, as $n \to \infty$, which imposes the condition $\delta < 1$. Moreover, we have to impose on h conditions *i), ii), iii)* of Theorem 4.3. We compute

$$nh(n)^p = \frac{N^{p\delta}}{n^{p\delta-1}}.$$

In condition *i)* and *iii)* we need $nh^p \to 0$ as $n \to \infty$, while in *ii)* we need $\frac{nh^p}{\log(nh)} \to 0$. Since $\log(nh) \to \infty$ by hypothesis, the condition $nh^p \to 0$ is more restrictive and we impose it also in *ii)*. Since we do not need a priori which is the value of H of our fractional Ornstein-Uhlenbeck process X_1, we find a $p = p(H)$ that is a good choice for any value of $H \in (0, 1)$. One can easily verify that $p(H) = 1 + H$ lies in all the admissible intervals for p in *i), ii), iii)*. With this choice of p, the expression of nh^p reads

$$nh(n)^p = \frac{N^{(1+H)\delta}}{n^{\delta(1+H)-1}}.$$

In order for the right-hand side to converge to zero we must have $\delta > \frac{1}{(1+H)}$. So we are left with the pair of conditions

$$\frac{1}{1+H} < \delta < 1,$$

which are both satisfied if we define

$$\delta = \delta(H) := \frac{1}{(1+H)^{\frac{1}{2}}},$$

for any $H \in (0, 1)$. With this choice of h, we are able to calculate the N-th step of the approximation of α_1, regardless of the estimation $\hat{H}$ of H that we obtained.

Still, in the definition of $\hat{\alpha}_1(n)$ there is a clear dependence on σ, which in our case is unknown. We remark that anyway, since the estimator $\hat{\sigma}$ converges a.s. to its respective true value as the order of the approximation increases, we have that the estimator $\hat{\alpha}_1$ converges a.s. to α_1 also if we substitute $\hat{\sigma}$ to σ in its definition.

Case	$\hat{\alpha}_1$	$q_{5\%}(\hat{\alpha}_1)$	$q_{95\%}(\hat{\alpha}_1)$	$\hat{\sigma}$	$q_{5\%}(\hat{\sigma})$	$q_{95\%}(\hat{\sigma})$	$\hat{H}$	$q_{5\%}(\hat{H})$	$q_{95\%}(\hat{H})$
a.1	**0.1069**	0.0050	0.2523	**6.2334**	5.7834	6.7926	**0.1940**	0.0885	0.2969
a.2	**0.1030**	0.0220	0.2098	**6.2313**	5.7705	6.8007	**0.2927**	0.1902	0.3931
a.3	**0.1017**	0.0445	0.1737	**6.2233**	5.7024	6.8636	**0.4909**	0.3982	0.5819
a.4	**0.1019**	0.0531	0.1641	**6.1976**	5.4920	7.0783	**0.6882**	0.6024	0.7715

Table 4.5: Estimated parameters of the component X_1, given $M = 20000$ realizations of the component X_1 itself for each of the parameters set a.1 . . . a.4.

Estimations for X_2

For the jump component X_2, there are three separated tasks to carry out in order to estimate the parameters of the model. First, one has to estimate the parameters of the self-exciting intensity of the Hawkes process. Second, one has to choose and fit an adequate distribution for the jump magnitude. Third, one has to estimate the mean-reverting parameter α_2 appearing in (4.25).

We start with the parameters of the jump intensity λ_t defined in (4.16). In [Oza79] the author gives an explicit formula for the log-likelihood for the observed jump times T_i, given the three parameters λ, γ, β of the intensity function λ_t. The log-likelihood takes the form

$$\begin{aligned} L(T_1, \dots T_n | \lambda, \gamma, \beta) &= -\lambda T_n + \sum_{j=1}^{n} \frac{\gamma}{\beta} \left(e^{-\beta(T_n - T_j)} - 1 \right) \\ &\quad + \sum_{j=1}^{n} \log \Big(\lambda + \gamma A(j) \Big), \end{aligned}$$

where $A(1) = 0, A(j) = \sum_{i=1}^{j-1} e^{-\beta(T_j - T_i)}, j \geq 2$. In order to have a more efficient maximization process, one can immediately compute the partial derivatives of L. One has

$$\begin{aligned} \frac{\partial \log L}{\partial \gamma} &= \sum_{j=1}^{n} \frac{1}{\beta} \left(e^{-\beta(T_n - T_j)} - 1 \right) + \sum_{i=j}^{n} \left[\frac{A(i)}{\lambda + \gamma A(i)} \right]; \\ \frac{\partial \log L}{\partial \beta} &= -\gamma \sum_{j=1}^{n} \left[\frac{1}{\beta} (T_n - T_j) e^{-\beta(T_n - T_j)} + \frac{1}{\beta^2} e^{-\beta(T_n - T_j)} \right] \\ &\quad - \sum_{j=1}^{n} \left[\frac{\gamma B(i)}{\lambda + \gamma A(i)} \right]; \\ \frac{\partial \log L}{\partial \lambda} &= -T_n + \sum_{j=1}^{n} \left[\frac{1}{\lambda + \gamma A(i)} \right]; \end{aligned}$$

with

$$\begin{aligned} \frac{\partial^2 \log L}{\partial \gamma^2} &= -\sum_{i=j}^{n} \left[\frac{A(i)}{\lambda + \gamma A(i)} \right]^2; \\ \frac{\partial^2 \log L}{\partial \beta^2} &= \gamma \sum_{j=1}^{n} \left[\frac{1}{\beta} (T_n - T_j)^2 e^{-\beta(T_n - T_j)} + \frac{2}{\beta^2} (T_n - T_j) e^{-\beta(T_n - T_j)} \right. \\ &\quad \left. + \frac{2}{\beta^3} \left(e^{-\beta(T_n - T_j)} - 1 \right) \right] \\ &\quad + \sum_{i=j}^{n} \left[\frac{\gamma C(i)}{\lambda + \gamma A(i)} - \left(\frac{\gamma B(i)}{\lambda + \gamma A(i)} \right)^2 \right]; \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \lambda^2} &= -\sum_{j=1}^{n}\left[\frac{1}{(\lambda+\gamma A(i))^2}\right];\\
\frac{\partial^2 \log L}{\partial \beta \partial \gamma} &= -\sum_{j=1}^{n}\left[\frac{1}{\beta}(T_n - T_j)e^{-\beta(T_n-T_j)} + \frac{1}{\beta^2}\left(e^{-\beta(T_n-T_j)} - 1\right)\right]\\
&\quad +\sum_{i=j}^{n}\left[\frac{\gamma A(i)B(i)}{(\lambda+\gamma A(i))^2} - \frac{B(i)}{\lambda+\gamma A(i)}\right];\\
\frac{\partial^2 \log L}{\partial \gamma \partial \lambda} &= -\sum_{i=j}^{n}\left[\frac{A(i)}{(\lambda+\gamma A(i))^2}\right];\\
\frac{\partial^2 \log L}{\partial \beta \partial \gamma} &= \sum_{i=j}^{n}\left[\frac{\gamma B(i)}{(\lambda+\gamma A(i))^2}\right].
\end{aligned}
$$

In the previous equation the functions B and C are defined as $B(1) = 0, B(j) = \sum_{i=1}^{j-1}(T_j - T_i)e^{-\beta(T_j-T_i)}, j \geq 2$ and $C(1) = 0, C(j) = \sum_{i=1}^{j-1}(T_j - T_i)^2 e^{-\beta(T_j-T_i)}$, $j \geq 2$. Since the log-likelihood is non-linear with respect to the parameters, the maximization if performed by using nonlinear optimazation tecniques [Oza79].

Case	$\hat{\lambda}_0$	$q_{5\%}(\hat{\lambda}_0)$	$q_{95\%}(\hat{\lambda}_0)$
b.1 $\lambda_0 = 0.1$	**0.0101**	0.0061	0.0141
b.2 $\lambda_0 = 0.1$	**0.0105**	0.0059	0.0162
b.3 $\lambda_0 = 0.1$	**0.0095**	0.0057	0.0139
b.4 $\lambda_0 = 0.1$	**0.0088**	0.0053	0.0126
Case	$\hat{\gamma}$	$q_{5\%}(\hat{\gamma})$	$q_{95\%}(\hat{\gamma})$
b.1 $\gamma = 0$	**0.0046**	$3.84{\cdot}10^{-9}$	0.0267
b.2 $\gamma = 0.05$	**0.0366**	0.0146	0.0606
b.3 $\gamma = 0.15$	**0.0840**	0.0472	0.1217
b.4 $\gamma = 0.3$	**0.1163**	0.0479	0.1850
Case	$\hat{\beta}$	$q_{5\%}(\hat{\beta})$	$q_{95\%}(\hat{\beta})$
b.1 $\beta = 0$	**0.6097**	0.0407	0.8025
b.2 $\beta = 0.08$	**0.0814**	0.0318	0.1379
b.3 $\beta = 0.2$	**0.1414**	0.0846	0.2113
b.4 $\beta = 0.5$	**0.2782**	0.1393	0.4421

Table 4.6: Estimated parameters of the component X_2, given $M = 20000$ realizations of the component X_2 itself for each of the parameters set b.1 –b.4.

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We see that the estimated values are below the true values, especially for big values of γ, β. For the set of parameters (a), the value of β is largely overestimated, but this is not a problem since the corresponding value of γ are very small, and thus there is no observable self-excitement.

Regarding the jump magnitude distribution, we fit the data via an MLE procedure by considering a Generalized Extreme Value (GEV) distribution. It is a continuous distribution which may seen as the approximation of the maxima of sequences of of independent and identically distributed random variables. It depends upon three parameters which allow to fit properly the data.

Finally, the estimation of the mean-reverting parameter α_2 of the jump component X_2 can be done by using the estimator defined in [KMS10]. Given a dataset Y_2 which we aim to model with our jump process X_2, a consistent estimator for the mean-reversion parameter α_2 is

$$\hat{\alpha}_2 = \log\Big(\max_{1\leq j\leq N} \frac{Y_2(j-1)}{Y_2(j)}\Big). \tag{4.30}$$

We will need an approximation of this estimator in our estimation process, which is very closely related to one made in [KMS10]. The details will be discussed during the data filtering process in the next section.

4.3 A study case: Italian data

Here we describe the time series that we are sudying, that is, the Italian electricity spot prices, that inspired the choice of the model that we made in Section 4.2. We first perform some explorative analysis on the time series, and after that, we discuss the problem of data filtering that we need to solve in order to obtain from rough data the different components of out model. In the end, we perform out-of-sample simulations to try to predict future prices of electricity, discussing the results with some evaluation metric like Winkler score and Pinball loss function.

4.3.1 The dataset

We will consider the time series of the Italian *Mercato del giorno prima* (MGP, the day-ahead market), available at [Prices]. Figure 4.9 shows a plot of the daily price time series $Y(t)$ from January, 1$^{\text{st}}$ 2009 ($t = 1$) to December, 31$^{\text{st}}$ 2017 ($t = 3287$).

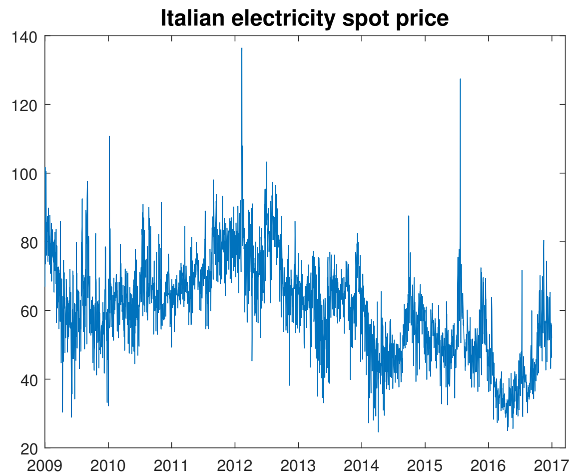

Figure 4.9: The time series $Y(t)$ of the the daily electricity price in the MGP from January, 01 2009 ($t = 1$) to December, 31 2017 ($t = 3287$).

The MGP market is a day-ahead market, i.e. a market in which the price is established via an auction for the blocks of electricity to be delivered the following day. The agents that operate as buyers in the market have to submit their offers between 8:00 a.m. and 12:00 noon of the previous day. The offers regard the hourly blocks of electricity which will be delivered the following day. This means that an agent will submit 24 different offers (with different prices and quantities) for the electricity of the following day, and he will do it all at the same time. Also the sellers submit their offers, by telling the quantity of energy that they are willing to sell and the price at which they want to sell it. The *market price* is then established before 12:55 p.m., and it is an hourly price, determined by finding the intersection of the demand and the offer curve relative to the specific hour of the day. After the determination of the market price, all the electricity bought and sold for that hour is traded at the market price.

We choose to model the daily average of the hourly price. This is a quite common choice in the literature, especially for reduced-form models [Wer14]. Indeed, the main aim of reduced–form models is to be able to capture more the medium term (days/weeks) distribution of prices than the hourly price, and to use these estimated distributions of prices to pricing electricity future contracts, which are very useful and used financial instruments in the electricity market.

We are aware that this averaging filters out many extreme behaviours of the market. Indeed, a single hourly extreme price is unlikely to produce a significant variation in the average daily price. Anyway, we are also aware that reduced–form models usually perform poorly on the hourly scale [Wer14]. We remark that all the analysis that follows has been carried out also using on–peak (08:00–20:00) and off–peak (20:00–08:00) data separately, without obtaining a significant difference from the entire day averages.

The data available start from April, 2004, which is the moment in which the liberalised electricity market started in Italy, but we chose to focus on more recent data, from 2009 onwards, to make the model more tight to the present nature of the electricity market. This does not prevent the performance evaluation of the model from being sufficiently robust, since the dimension of the sample is $N = 3287$.

We use the data in the following way: the first 730 days have been used for the study of the dataset and for the validation of the model. Then, we evaluated the performance on forecasting future prices for time horizons of length $h = 1, \dots, 30$, using *rolling windows*: at time t, we use the data from $t - 730$ to t for the calibration of the parameters of the model, and we simulate the future price at time $t+h$ using those parameters. Then, we move ahead from time t to $t+h$ and we repeat the previous steps, starting from parameter estimation.

The first task that has to be performed on the price time series Y is the separation, or *filtering*, of the different signals. It is clear that this is not an easy task and it might not be done in a unique way. In literature a lot of effort as been done for this purpose (see [JTWW13, NTW13, Wer14, MeTa08, KMS10, NoWe18]). Moreover, we think that in our case the relatively small presence of clearly recognisable spikes in our dataset makes the spike identification task even more difficult than usual.

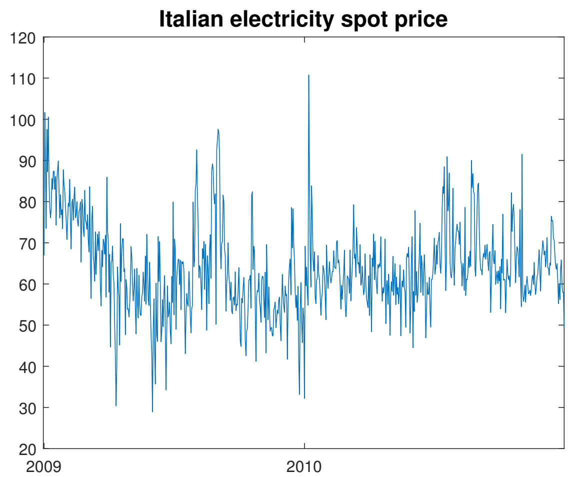

Figure 4.10: Calibration window for the time series $Y(t)$ of the the daily electricity price in the MGP from January, 01 2009 ($t = 1$) to December, 31 2010 ($t = 730$).

Note that from now on we consider as the window for the calibration of the model the one corresponding to the first two years, 2009 and 2010. The reduced time series $Y(t), t \in [1, 730]$ is shown in Figure 4.10.

4.3.2 Data filtering

Weekly seasonal component. The first component that should be identified and estimated is the one dealing with trends and seasonality in the data. As stated in [JTWW13] and in the literature therein, the estimation routines are usually quite sensitive to extreme observations, i.e. the electricity price spikes. Hence, one should first filter out the spikes, that often are identified by the outliers. Actually, whether to filter out the spikes before or after the identification of the deterministic trends is still an open question in general.

Furthermore, the deseasonalizing methods used in literature are very different: some authors suggest to considered sums of sinusoidal functions with different intensities [CaFi05, GeRo06, KMS10], others consider piecewise constant functions (or dummies) for the month [FHS11, LuSc02], or the day [DeJ06] or remove the weekly periodicity by subtracting the *average week* [JTWW13]. It turns out than an interesting and more robust technique is the method of wavelet decomposition and smoothing, applied among others in [JTWW13, NTW13, Wer06, Wer08].

In Figure 4.11 we show the autocorrelation function both for the whole series and within the calibration window. We see how for any lag multiple of seven the correlation is statistically significant. In the calibration windows only for the lag=14 there is not a significant correlation at the level 95%. In any case the presence of weekly periodicity is clear.

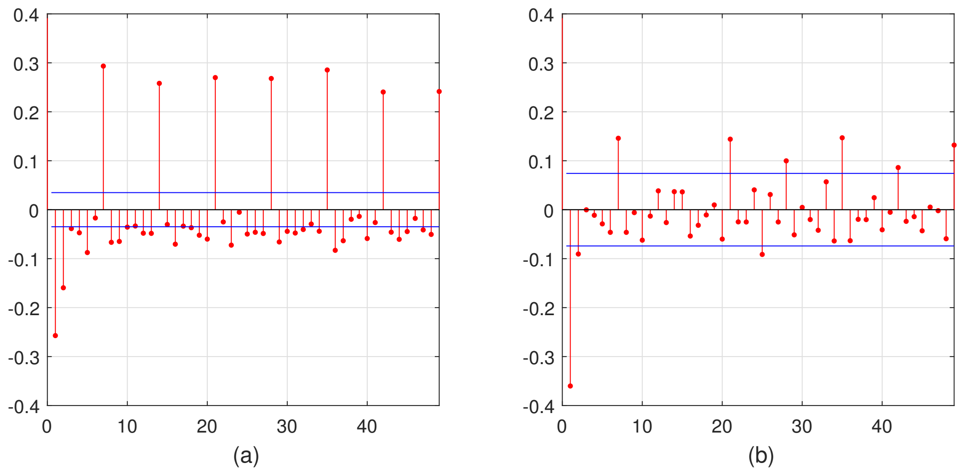

Figure 4.11: Autocorrelation function of the daily electricity price returns: (a) January, 01 2009 ($t = 1$) to December, 31 2017 ($t = 3288$)(b) the calibration window January, 01 2009 ($t = 1$) to December, 31 2010 ($t = 730$).

As a consequence, the first step that we chose to perform on the data is to remove the weekly periodicity. As in [DeJ06, Wer14] we do it via the use of dummy variables, which take constant value for each different weekday. Hence, we subtract the *average week* calculated as the sample mean of the sub-samples of the prices corresponding to each day of the week, as in [JTWW13]. Public holidays are treated in this study as the eighth day of the week. Hence, the total number of dummies is eight.

Formally, we define a function $D = D(t)$ which determines which label the t-th day has, i.e. $D(t) = i, i = 1, , \dots, 7, 8$, if day t correspond to Monday (and not a festivity), ..., Sunday (and not a festivity) and a festivity, respectively.

We define the restricted time series $Y_{D=j}$ as the time series formed only by the price values labelled with the day j, and with $\overline{Y_{D=j}}$ its arithmetic mean. Then we define the whole dummy variables function Y_D as

$$Y_D(t) = \sum_{j=1}^{8} 1_{D(t)=j}(t)\overline{Y_{D=j}}. \tag{4.31}$$

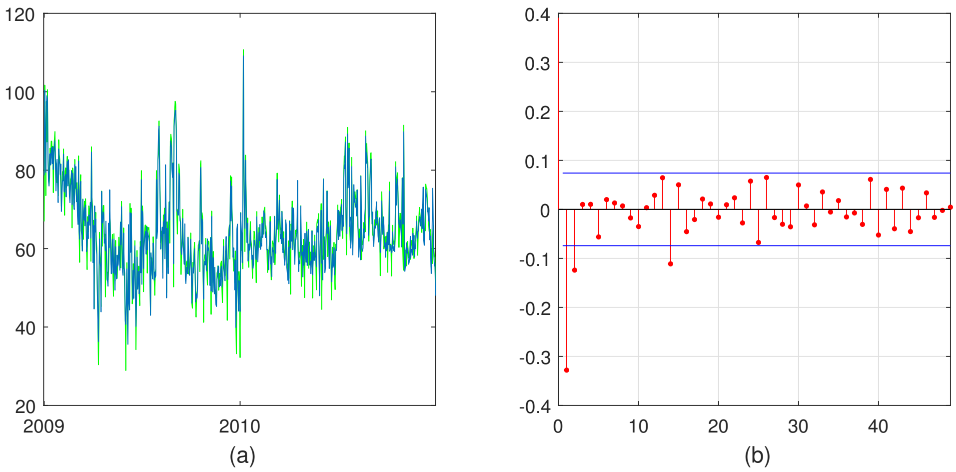

Figure 4.12: (a) Original prices Y (green) and prices after dummies removal Y_w (blue); (b) Autocorrelation of the returns of the prices Y_w

As a consequence, the new time series we are going to consider is the one defined as follows

$$Y_w(t) := Y(t) - \hat{f}_s(t) =: Y(t) - (Y_D(t) - \overline{Y}), \tag{4.32}$$

where $\overline{Y}$ is the arithmetic mean of $Y(t), t = 1, ..., 730$. The result of (4.31)-(4.32) is that the arithmetic mean of the prices Y_w corresponding to a specific day of the week coincides with the mean of the all prices within the calibration window. As one can see in Figure 4.12-(a), the resulting series Y_w after removing the dummy variables is barely distinguishable from the original time series Y; it is only a bit more regular. On the other hand, Y_w does not show weekly correlation, as we notice from Figure 4.12-(b).

Jump component. Before filtering the long-term seasonal component, we chose to filter out the jump component. The reason for this order lies in the following consideration: at a small scale, the presence of a slowly moving seasonal trend does not affect the recognition of a price spike. On the other hand, if we chose to filter the long-term seasonal component before filtering out the spikes, the presence of one of more spikes could affect the form of the seasonal component, which is something that we intuitively regard as incorrect. Indeed, we tend to consider the spikes as an "external event" in our setting, and we do not want the seasonal term to be affected by the presence of a price spike.

As in [KMS10], the idea is to obtain the filtered time series, denoted Y_J, as the series that contains the jumps and their paths of reversion towards their mean. We first estimate the mean-reversion speed α_2 by the estimator $\hat{\alpha}_2$ given by (4.30); by performing the estimate along the entire time series, not only in the jump times: this is not restrictive, since the strongest rates of reversion towards the mean happen right after a jump has occurred. Afterwards we identify the jump times. The idea is to consider as jumps the price increments that exceed k standard deviations of the price increments time series. This cannot be implemented to the time series Y_w directly, because, in case two spikes appear one after the other, the second one would not be considered as a spike. In order to avoid this effect, we define the modified time series $\tilde{Y}_w$ as

$$\widetilde{Y}_w(t) := (1 - \alpha_2)Y_w(t) + \hat{\alpha}_2\bar{Y}_{30}(t),$$

where $\bar{Y}_{30}(t)$ is the moving average of the time series Y_w over periods of 30 days. Then, we defined the times series

$$\left\{Y_w(t) - \widetilde{Y}_w(t-1)\right\} - t = 2, 3, \ldots \tag{4.33}$$

of the modified increments, which takes into account a reversion effect towards the moving average Y_{30}. It performs very well also when the spikes appear in clusters. Then, denoted by $\widetilde{\sigma}$ the standard deviation of the series (4.33), we say that a spike occurs at time τ if

$$\left|Y_w(\tau) - \widetilde{Y}_w(\tau - 1)\right| > 2.5\widetilde{\sigma}. \tag{4.34}$$

If N is the number of detected spikes, i.s. if $\{\tau_j\}_{j=1,\dots,N}$ are the times satisfying condition (4.34), the corresponding jumps are defined, for $j = 1, \dots, N$ as

$$\hat{\mu}_j = Y_w(\tau_j) - \widetilde{Y}_w(\tau_j - 1).$$

Once we have the estimates $\{(\tau_j, \hat{\mu}_j)\}_{j=1,\dots,N}$ of the times and magnitudes (4.9) of the jumps, we obtain the estimation $Y_J(t)$ of the solution X_2 of (4.25) as follows

$$Y_J(t) = \sum_{j=1}^{N} \mu_j e^{-\hat{\alpha}_2(t-\tau_j)} \epsilon_{\tau_j}([-\infty, t]) \tag{4.35}$$

Given Y_J as in (4.35), we denote the filtered time series as

$$Y_s(t) = Y_w(t) - Y_J(t).$$

Long-term seasonal component:
We explain now how we identify the long-term seasonality component $\hat{f}_l$, which will be so that the total deterministic component will be given by $\hat{f} = \hat{f}_s + \hat{f}_l$. There is a lot of literature on the subject (see, for example, [BKN12, CaFi05, DeJ06, GeRo06, JTWW13, LuSc02, Wer08]). These references try to explain such a component by means of sinusoidal functions or sums of sinusoidal functions of different frequencies. In the case of our price series it seems that there is no statistically significant dependence upon such periodic function, both in the case of month, half-year or a year periodicity.

As a consequence, we chose to use the method of wavelet decomposition and smoothing, applied among others in [JTWW13, NTW13, Wer06, Wer08]. The idea is to consider the time series Y_s and convolving it repeatedly with a family of wavelets, which have the effect of smoothing the series Y_s. If we manage to smooth Y_s enough to remove the effect of stochastic oscillation, but not too much to remove the long-term trend, then we can subtract this smoothed version of Y_s from Y_s itself, obtaining a centred time series with almost no long-term oscillation.

We go more into details: we refer to [NTW13] and the literature therein. We use wavelets belonging to the Daubechies family, of order 24, denoted by (F–db24). Wavelets of different families and orders make different trade-offs between how compactly they are localized in time and their smoothness. Any function or signal (here, Y_s) can be built up as a sequence of projections onto one father (W) wavelet and a sequence of mother wavelets (D): $Y_S = W_k + D_k + D_{k-1} + ... + D_1$, where 2^k is the maximum scale sustainable by the number of observations. At the coarsest scale the signal can be estimated by W_k. At a higher level of refinement the signal can be approximated by $W_{k-1} = W_k + D_k$. At each step, by adding a mother wavelet D_j of a lower scale $j = k-1, k-2, ...$, one obtain a better estimate of the original signal. Here we use $k = 8$, which corresponds a quasi-annual ($2^8 = 256$ days) smoothing. Then, the estimator $\hat{f}_l$ of the long-term deterministic part $f - \hat{f}_s$ is given by

$$\hat{f}_l(t) = W_8(t), \tag{4.36}$$

the Daubechies wavelets of order 24 at level 8.

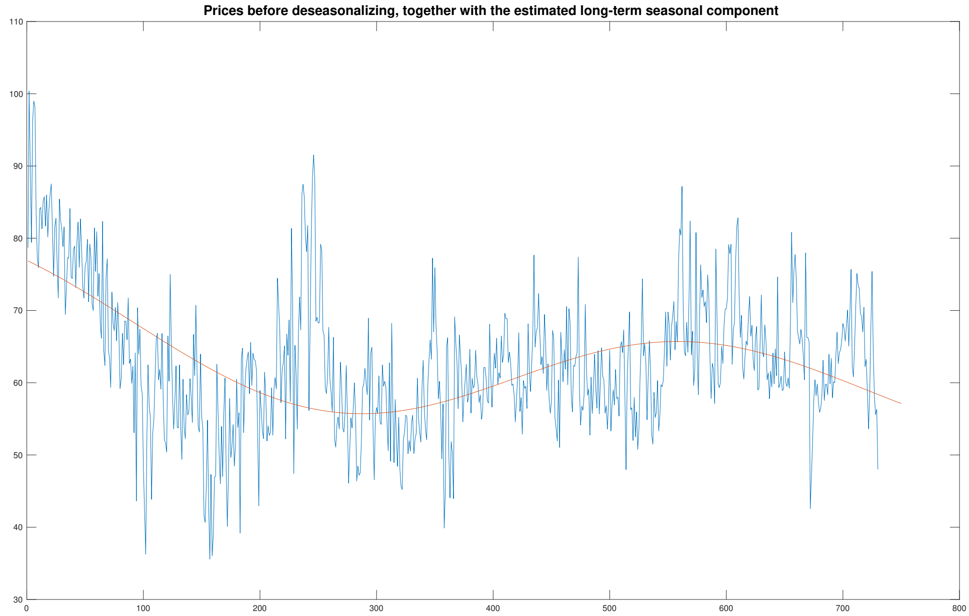

Figure 4.13: Price series Y_S (blue) and estimated $\hat{f}$ (red) via Daubechies of order 24 wavelets at level 8.

The resulting time series Y_f given by

$$Y_f(t) = Y_s(t) - \hat{f}_l(t), \tag{4.37}$$

represents finally a realization of the base component X_1.

Figure 4.13 show the price series Y_S and the overlapped estimated $\hat{f}_l$ via Daubechies wavelets of order 24 at level 8, and the final signal Y_f, which is we recall to be a sample of the process X_1. The wavelet interpolation is here extended outside the calibration window. This is not automatic in the case of wavelet decomposition, since the wavelets are compactly supported and we are convolving only up to the final time of our dataset. To obtain this prolongation, we prolonged the time series in the forecasting window by using the technique of exponential reversion to the median, thoroughly studied in [NTW13], before applying the wavelet de-noising. In this way we have been able to obtain a function $\hat{f}_l$ which extends also to times t in the forecasting windows.

4.3.3 Out of sample simulations

We will perform and assess here the forecasts of future electricity prices through our model. We first outline the simulation scheme and make some considerations about the parameters in the rolling windows. After this, we define the metrics which we will subsequently use to evaluate our results.

Parameter estimation in the rolling windows

We recall that for each time t, when forecasting the price distribution at time $t + k_h$ (where $k_h \in \{1, \dots, 30\}$ is the forecasting horizon), we carry out a new calibration of all the parameters of the model, including the Hurst coefficient H. We think that in this way, if there is a change in the data coming as input, the model is able to change its fine structure coherently with these changes. For example, if the self–correlations changes, or disappears, at some point, we expect the parameter H to change consequently, and possibly approach $\frac{1}{2}$.

We start analysing H: as long as our rolling window for estimation is advancing,the values of H are on average increasing. The first estimate is $\hat{H} = 0.2909$. The mean value across the whole dataset is 0.3735. In general, the values are such that

$$0.2128 \leq \hat{H} \leq 0.6234.$$

Notice that the maximum value is above the $H = 1/2$ threshold and this shows that a positive correlation between the increments may occur. We show the averaged behaviour of H across all our time lengths in Figure 4.14. In general, as we already pointed out, we think that this

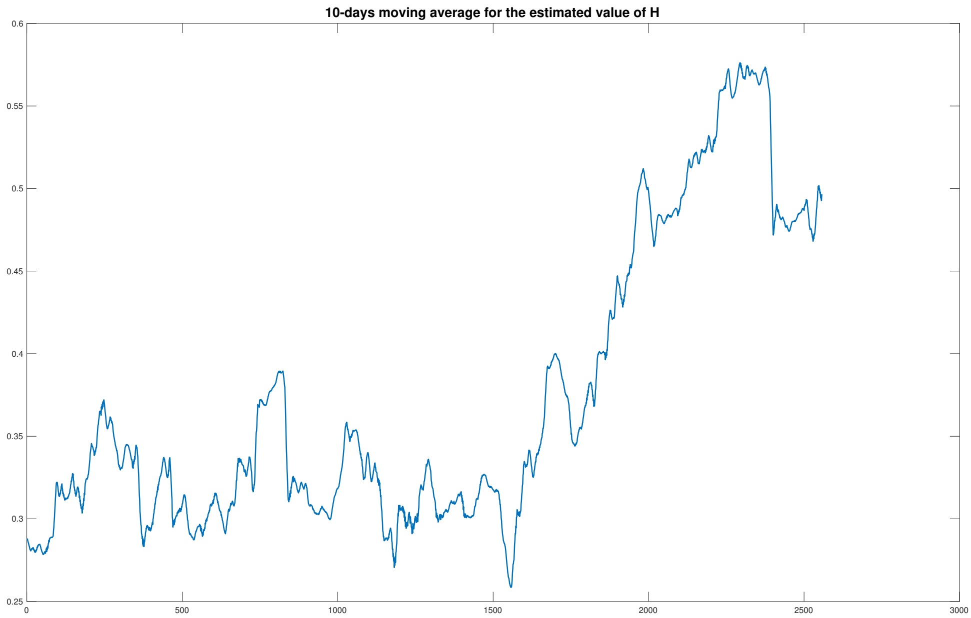

Figure 4.14: 10-days average of the estimated value $\hat{H}$ of H. We chose the 10-day average in order to smooth out some irregular behaviour at shorter scale.

moving identification of the parameter H is useful to update the fractional structure of the model when the input data are suggesting to do so, giving a better modelling flexibility, also in case of future changes in the market nature.

Regarding the parameters of the fractional Ornstein–Uhlenbeck process X_1, α_1 and σ, progressing in our rolling window, we found a change in the estimated values of the parameters; summarizing, we obtained that the range of values for $\hat{\alpha}_1$ and $\hat{\sigma}$

$$0.0453 \leq \hat{\alpha}_1 \leq 0.6823$$
$$4.2069 \leq \hat{\sigma} \leq 8.8972.$$

We see that there is a large variation in the parameters, especially for $\hat{\alpha}_1$, but we remark that this variation is gradual, since the mean reversion rate decreases while moving through our forecasting dataset, together with the volatility $\hat{\sigma}$. Looking at Figure 4.9, this can be observed, at least for the volatility, also by a macroscopic observation of the data.

Looking at the estimation of the parameters of the jump process X_2, a bit of variability through the dataset is shown, but there is not an evidence of a particular pattern. Moreover, the jump observations are relatively rare (20-40), so based on the available data, it would be even more difficult to draw conclusions on their long term behaviour. The estimated values of the mean–reversion coefficient α_2 are such that $\hat{\alpha}_2 \in [0.3564, 0.6211]$, with an the mean value

0.4358. As the Hawkes process parameter estimation regards, $\hat{\lambda}_0 \in [0.0101, 0.0284]$ with mean value 0.0169, $\hat{\gamma} \in [1.12 \cdot 10^{-9}, 0.1574]$, with mean their mean 0.0625 and $\hat{\beta} \in [0.0012, 0.9993]$ with mean 0.3662. There is a great variation in such estimates. This, in our opinion, is due to the low dimension of the dataset, because not many spikes are present. Thus, the MLE method is finding sometimes a good evidence of a self-excitement (when γ is bigger and relatively close to β) and sometimes no evidence of self-excitement (when γ is very small and/or β is much bigger than γ). To show this fact, we plot in Figure 4.15 an histogram of the ration γ/β, which is a very good indicator of the presence of self-excitement. From the histogram, we can see that roughly half of the time the ratio is below 0.25 (showing little or no self-excitement), and half of the time above (showing a significant self-excitement). We think this is another evidence of the flexibility of our model, similarly to the estimations of H. If some self-excitement seems to be present, then the model is including it. Otherwise, the model will simply produce a classical point process with constant intensity.

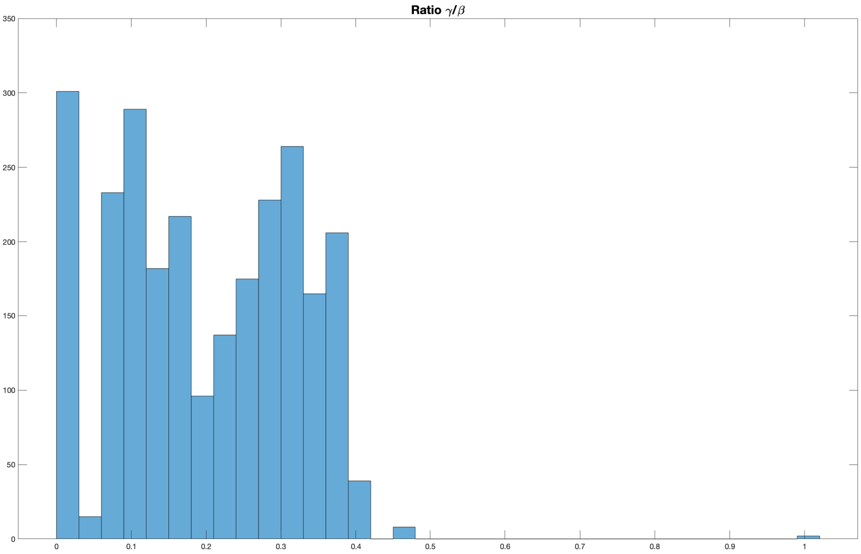

Figure 4.15: Ratio between $\hat{\gamma}$ and $\hat{\beta}$ throughout the entire dataset.

As the estimation of the parameters $(\tilde{\mu}, \xi, \tilde{\sigma})$ of the Generalized Extreme Value distribution we have that $\hat{\tilde{\mu}} \in [11.2506, 19.4064]$ with a mean value 15.6687, $\hat{\xi} \in [-0.4809, 4.2591]$ with mean 0.4125 and $\hat{\tilde{\sigma}} \in [0.3248, 3.8260]$ with mean 2.2331. We only remark that even if there is a great variability in the parameters, the median value of the jump size is not varying from one estimate to the other. We recall that the median (the mean is not always defined) of a GEV distribution is given by

$$\text{Median} = \mu + \sigma \frac{\log(2)^{-\xi} - 1}{\xi}.$$

In Figure 4.16, we see that this value is oscillating between 11 and 22, which are reasonable values for the jumps in our dataset.

Forecasting performance

In this section we will evaluate the performance of the model described in Section 4.2 when it is used to forecast future values of the electricity prices. As pointed out in [Wer14], there is no universal standard for evaluating the forecasts.

The most widely used technique is to obtain point forecasts, i.e. single forecast values, and evaluate them using some error function. The most common error function for this type

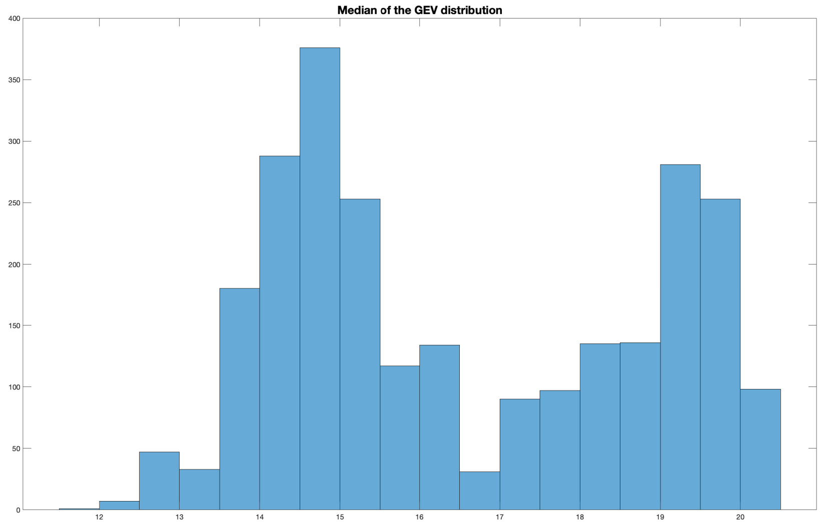

Figure 4.16: Median of the GEV distribution throughout the entire dataset.

of forecasts is Mean Absolute Percentage Error (MAPE), together with its refinement Mean Absolute Scaled Error (MASE). Another frequently used measure is the Root Mean Square Variation (RMSV), which is simply the estimated standard variation of the forecast error. In our model, the SDE-type structure is not particularly suitable for giving short-term point forecasts, as it is also pointed out in [Wer14]. So in the following we will not concentrate our analysis on point forecasts, as we do not expect our model to be able to outperform more sophisticated and parameter-rich model in this task. Instead, we focus on the relatively novel concept of Interval Forecast, that we introduce now

Interval Forecast (prediction interval, PI)

More recently, as was already suggested in [Wer14] and as it has been more thoroughly analized in the very recent review [NoWe18], the driving interest in forecast evaluation has been put in interval forecasts and density forecasts. Interval forecast have also been used as the official evaluating system in 2014's Global Energy Forecasting Competition (GEFCom2014). For this and other reasons that we will point out, in this paper we will concentrate mainly on interval forecast. As it is said in [NoWe18], there is a close relation between interval forecasts and density forecasts.

Interval forecasts (also called Prediction intervals, shortly PI) are a method for evaluating forecasts which consists in constructing the intervals in which the actual price is going to lie with estimated probability α, for $\alpha \in (0,1)$. There are many ways to build the interval, depending also on the type of model that one is using. After an interval forecast is obtained, its performance can be evaluated in different ways, see [NoWe18] for a complete review of the existing techniques. Here we will evaluate interval forecasts for different time lags h, using 3 different techniques: Unconditional Coverage (UC), Pinball loss function (PLF) and Winkler Score (WS).

Unconditional Coverage (UC). Establishing the UC just means that we evaluate nominal rate of coverage of the Prediction intervals; as stated in [NoWe18], one can simply evaluate this quantity, or consider the average deviation from the expected rate α. As pointed out in [NoWe18], if we call P_t the actual price, $[\hat{L}_t, \hat{U}_t]$ the Prediction interval at level $\alpha \in (0,1)$, we are checking the fact that the random variable

$$I_t = \begin{cases} 1, & \text{if } P_t \in [\hat{L}_t, \hat{U}_t], \\ 0, & \text{if } P_t \notin [\hat{L}_t, \hat{U}_t], \end{cases}$$

has a Bernoulli $B(1, \alpha)$ distribution. This works clearly under the assumption that the violations are independent, which may not always be the case.

Pinball loss function (PLF). The Pinball loss function is the function that has been used for evaluating the models participating in the GEFCom2014, and it is a scoring function which can be calculated for every quantile q. If we denote with P_t the actual price, with $Q_q(\hat{P}_t)$ the q-th quantile of the estimated prices $\hat{P}_t$ obtained with the model, the Pinball loss function is defined as

$$\text{Pin}(Q_q(\hat{P}_t), P_t, q) := \begin{cases} (1-q)(Q_q(\hat{P}_t) - P_t), & \text{if } P_t < Q_q(\hat{P}_t), \\ q(P_t - Q_q(\hat{P}_t)), & \text{if } P_t \geq Q_q(\hat{P}_t). \end{cases}$$

Winkler Score (WS). The Winkler score is a scoring rule which is similar to the Pinball loss function, with the aim of rewarding both reliability (the property of having the right share of actual data inside the α-th interval) and sharpness (having smaller intervals). For a central α-th interval $[\hat{L}_t, \hat{U}_t]$, $\delta_t := \hat{U}_t - \hat{L}_t$, and for a true price P_t, the WS is defined as

$$\text{WS}([\hat{L}_t, \hat{U}_t], P_t) = \begin{cases} \delta_t, & \text{if } P_t \in [\hat{L}_t, \hat{U}_t], \\ \delta_t + \frac{2}{\alpha}(P_t - \hat{U}_t), & \text{if } P_t > \hat{U}_t, \\ \delta_t + \frac{2}{\alpha}(\hat{L}_t - P_t), & \text{if } P_t < \hat{L}_t. \end{cases}$$

As it can be seen, the WS has a fixed part which depends only on the dimension of the Prediction intervals.

The models. We checked the performance of 3 different models:

- The 2 SDE models described in Section 4.2, one with fBm, and one with sBm.
- A naive model, built as Naive(t)=D(t)+H, where D is the dummy variables function and H is randomly sampled historical noise ([NoWe18]) coming from the relative calibration window.

Forecasting horizons. As already mentioned, used as the calibration window a rolling window with fixed dimension of 730 prices, corresponding to the 730 days of past observations. In this framework, we will consider the following forecasting horizons k_h:

$$k_h = \{1, 2, \ldots, 29, 30\}$$

For each forecasting horizon we make a new estimate of the parameters at a distance k_h from the previous one, in order to have the sampled prices coming from disjoint time intervals.

Results

We start analysing the performance of the models by their observed UC. In Figure 4.17 we report their performance in a plot which spans across all the forecasting horizons k_h that we are considering. The dotted black line represents the relative level of coverage that we should attain. The closer we are to the dotted line, the more accurate a model is in covering that specific interval.

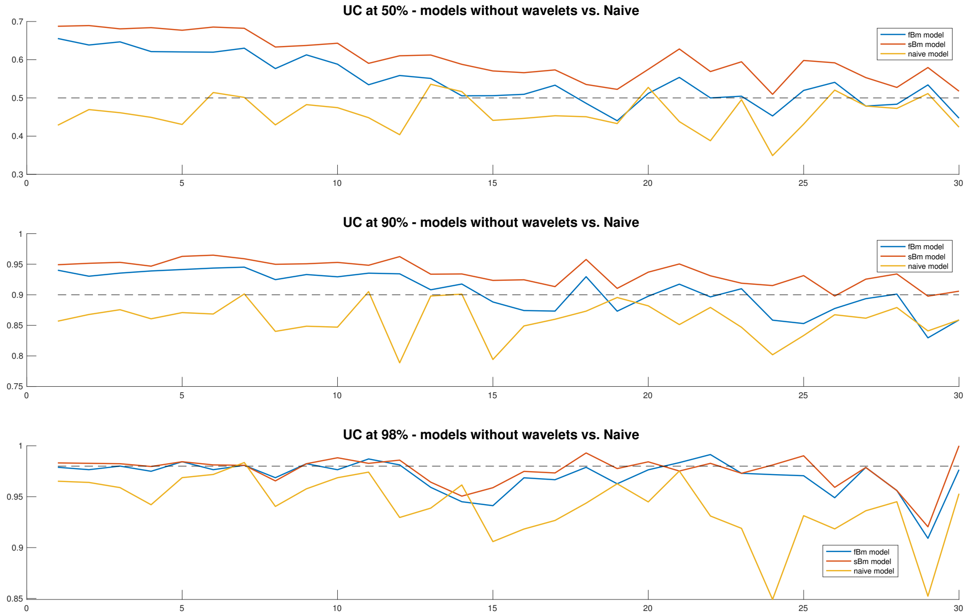

Figure 4.17: Observed Unconditional Coverage of the model with the fractional Brownian motion (blue), with the standard Brownian motion (red) and the naive one (yellow)) for $50\%, 90\%$ and 98% coverage intervals. On the horizontal axis, we represent the length k_h of the forecasting window we are considering, while on the vertical axis we represent the UC value.

In the 50% interval, the naïve model seems to be more stable, even if it is almost always under-covering the interval. Among our models, the fBm model, except for the shorter forecasting horizons, is performing remarkably well. The sBm model is over-covering the interval, for almost all forecasting horizons.

Moving to the 90% (mid plot) and 98% (bottom plot) prediction intervals, the fBm model performs in general better than the other ones, except for a slight excess in coverage for the 90% PI with small forecasting windows. In the 98% PI, also the sBm model has a very good performance, which will be confirmed by the numerical data for the UC reported in Table 4.7.

Avg. score\Model	fBm	sBm	Naïve
$UC_{50\%}$	54.54%	60.37%	**46.02%**
$UC_{50\%}$ error	+4.54%	+10.37%	**−3.98%**
$UC_{50\%}$ abs. error	5.95%	10.37%	**4.83%**
$UC_{90\%}$	**90.63%**	93.65%	86.02%
$UC_{90\%}$ error	**−0.93%**	−9.10%	+3.79%
$UC_{90\%}$ abs. error	**2.81%**	3.67%	4.04%
$UC_{98\%}$	97.02%	**97.58%**	94.13%
$UC_{98\%}$ error	−0.98%	**−0.42%**	3.87%
$UC_{98\%}$ abs. error	1.19%	**0.99%**	3.90%

Table 4.7: Coverage rate for the estimated PI, averaged over all forecasting horizons $k_h = 1, \ldots, 30$. The average error, as it can be seen, is just the difference of the average coverage from the nominal value.

We analysed then the WS and the PLF of the different models. In Figure 4.18 and in Figure 4.19 we reported again the results spanning along all forecasting horizons. We note that the PLF is a function of the quantile we are evaluating, so that in principle we would have to evaluate it separately for every quantile $q = 1, \ldots, 99$. As was also did in the GEFCom2014 competition, we averaged first over all quantiles, in order to obtain a single value and make comparisons easier.

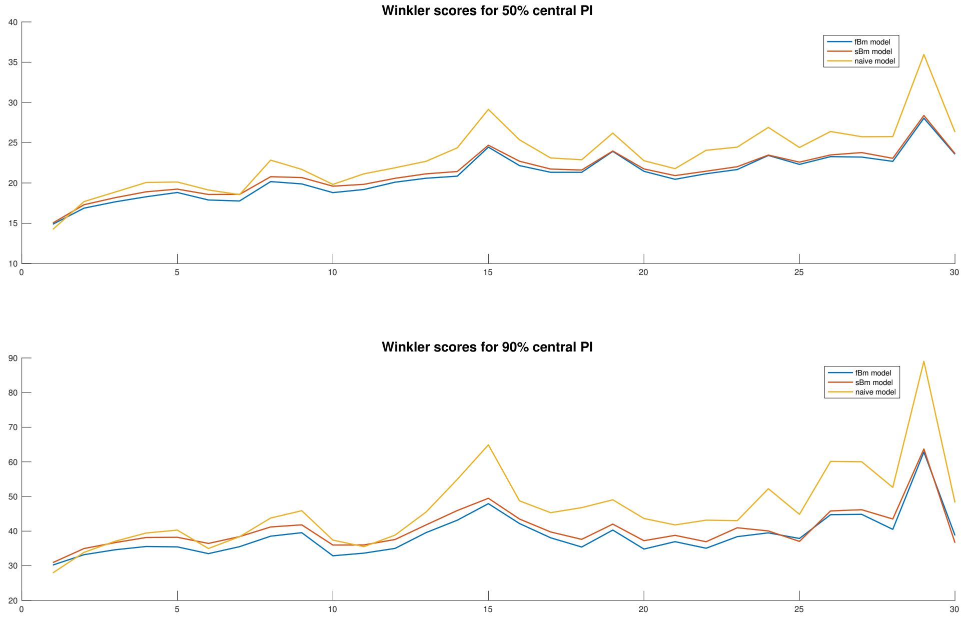

Figure 4.18: Winkler scores for 50% and 90% PI.

In terms of WS (Figure 4.18), the fBm and the sBm models outperform the the naïve benchmark. The difference between the fBm model and the sBm model is very small in general, but the fBm model performs better than the sBm one in almost every prediction interval. This is true especially if we consider the 90% interval WS.

There seems to be a sort of contradiction in our results: indeed, when it comes to the UC at level 50%, the naïve model seemed to be slightly better than the fBm model, while in terms of WS for the 50% PI the fBm is clearly superior to the naïve benchmark. This is possible because the WS is a metric which not only evaluates the share of coverage of a prediction interval, but also gives a penalty for missed values, and this penalty depends on the magnitude of the error made. Thus, it may seem reasonable to suppose that the naïve model, while performing quite good in terms of coverage at the 50% PI, makes bigger errors than the fBm model.

The results about the PLF are quite similar to the ones of the WS. Again, the fBm and the sBm model outperform the naïve model, while being very close one to each other. Again, the fBm model performs slightly better than the sBm model.

We make anyway a remark: both in terms of WS and PLF, the naïve model is performing better than the fBm and the sBm model when the forecasting horizon $k_h = 1$. This is somewhat consistent with a fact mentioned in [Wer14], which we already reported: the reduced-form models, like ours, usually have a quite poor performance in very short-term forecasts.

Score \Model	fBm	sBm	Naïve
$\text{WS}_{50\%}$	**20.87**	21.30	23.14
$\text{WS}_{90\%}$	**38.62**	40.44	46.25
PLF	**2.3484**	2.3920	2.6164

Table 4.8: Winkler scores and Pinball loss function values, averaged over all forecasting horizons $k_h = 1, \ldots, 30$.

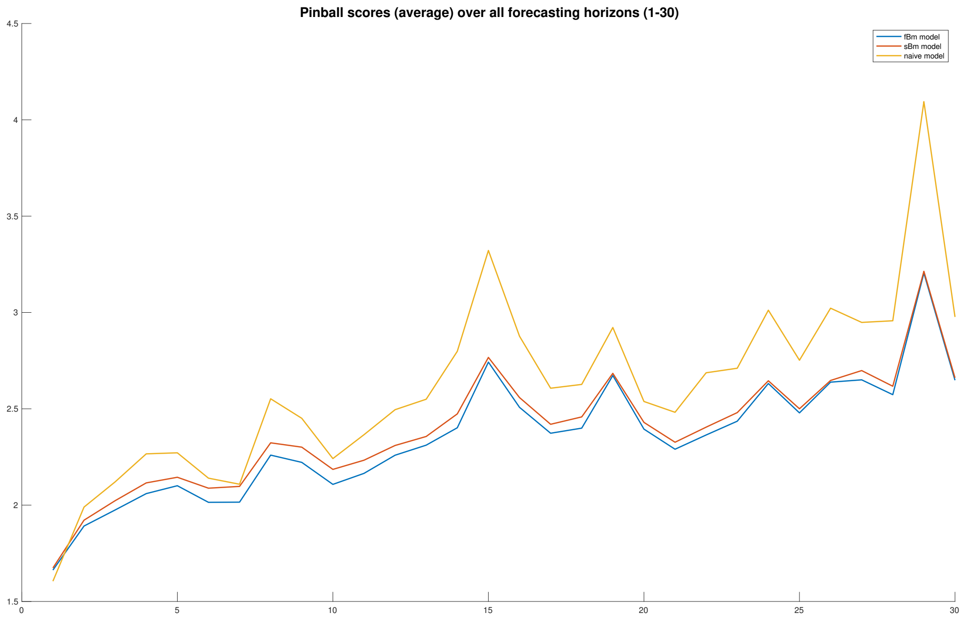

Figure 4.19: Pinball loss function (average).

In Table 4.7 and Table 4.8 we reported the above discussed values, averaged over all different forecasting horizons. In terms of the UC, each of the three models is performing better than the others for a certain PI, while both for the WS and the PLF we see the better performance of the fBm model also from these numerical data.

4.3.4 Conclusions

Drawing some conclusions from the results analyzed above, there are some evidences that a fBm-driven model may be more adequate to model the electricity prices than a sBm-driven model.

Regarding the forecasting performance (QF and PI), the fBm methods have better performance than the sBm ones in terms of WS and PLF, while both the sBm and the naïve model enjoy some success when evaluating the UC.

To understand this apparent contradiction, we remark (as we already did) that WS and PLF are scoring rules which give a penalty for missed forecasts (while UC does not), and these penalties depend also on the magnitude of the error. The fact that fBm models outperform sBm models in this evaluation may mean that the QF and the PI given by the fBm models are in some sense more robust than the sBm ones (and also than the ones of the naive benchmark).

Concerning the model structure, we remark that we found very satisfactory the fact that the parameter estimation for the Hawkes process gave roughly half of the times a very significant value, meaning that the clustering effect is not only visible on a macroscopic scale, but is also captured by the numerical methods.

This was not assured in principle, since the Italian market is rather peculiar, having only a small number of real spikes. This gives, as a consequence, that the intensity of the spike process is small and could become difficult to estimate, even if this was not the case for our data.

Regarding the role of the fractional Brownian motion in the model, we remark that we found some very interesting informations from the estimation procedure. The fact, shown in

Figure 4.14, that the parameter H is tending to 0.5 in more recent times may mean that the market is finding automatically a way towards the "independence of increments", which would be implied by the fact that $H = 0.5$. This is remarkable also for the fact that the independence of increments is closely related, for these models, with the absence of arbitrage. Even if we pointed out that arbitrage is usually not possible, for this kind of models, when trading only once per day, a good question for future developments in this sense may be: are electricity markets, which are "young" financial markets, finding their own stability with the passing of time, or are our findings specific to the italian market? In any case, are these changes going to last in the future or we may see a return of a fractional effect in the next years?

A | Appendix

A.1 Basic probability results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We recall that given a stochastic process $\{X_t, t \in \mathbb{R}\}$ with values in some metric space (S, d), we say that $\tilde{X}$ is a modification of X if it is a stochastic process such that for every $t \in \mathbb{R}$

$$\mathbb{P}(X_t = \tilde{X}_t) = 1.$$

Definition A.1. Let $f : \mathbb{R} \to S$, where (S, d) is a metric space. Let $\alpha \in (0, 1)$. We say that f is *Hölder continuous* of exponent α if it satisfies

$$\sup_{s \neq t} \frac{|f(s) - f(t)|}{|t - s|^\alpha} < \infty$$

Theorem A.2 (Kolmogorov continuity theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (S, d) be a complete metric space. Let $X : \mathbb{R} \times \Omega \to S$ be a stochastic process. Assume that there exist $\alpha, \beta > 0$ such that for every $s, t \in \mathbb{R}$ it holds*

$$\mathrm{E}\Big[|X_t - X_s|^\alpha\Big] \leq C|t - s|^{1+\beta}.$$

Then, there exists a modification $\tilde{X}$ of X with continuous sample paths $X_\cdot(\omega)$, for all $\omega \in \Omega$. Moerover, for every $\gamma \in (0, \frac{\beta}{\alpha})$, the modification can be chosen such that the sample paths are γ-Hölder continuous.

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