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VISCOSITY SOLUTIONS FOR CONTROLLED MCKEAN–VLASOV JUMP-DIFFUSIONS*

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Abstract. We study a class of nonlinear integrodifferential equations on a subspace of all probability measures on the real line related to the optimal control of McKean–Vlasov jump-diffusions. We develop an intrinsic notion of viscosity solutions that does not rely on the lifting to a Hilbert space and prove a comparison theorem for these solutions. We also show that the value function is the unique viscosity solution.

Key words. viscosity solutions, McKean–Vlasov control, Wasserstein space, optimal control

AMS subject classifications. 49L25, 35Q91

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1. Introduction.

The main goal of this paper is to develop a viscosity theory for integrodifferential equations on a subspace of all probability measures on the real line related to the optimal control of McKean–Vlasov jump-diffusions. These control problems are motivated by the mean field games theory developed by Lasry and Lions [29, 30, 31] (see also the videos of the College de France lectures of Lions [33]) and by Huang, Caines, and Malhamé [25, 26, 27]. Although the mean-field games and McKean–Vlasov control problems are related, there are subtle differences between these problems, and a thorough introduction is given by Carmona, Delarue, and Lachapelle [14]. Indeed, for both problems the master equations share many common properties as initially derived by Bensoussan, Freshe, and Yam [6, 8, 7]. We refer to the videos of Lions [33], the lecture notes of Cardaliguet [12], and the exhaustive book of Carmona and Delarue [13] for more information on both problems and also for further references.

The state space of these problems is the set of probability measures, and in most applications the Wasserstein space of probability measures with finite second moments is used. Since the space of probability measures is not linear, one encounters some difficulties in differentiation, and Lions [33] observed that one can naturally lift functions defined on the Wasserstein space to functions on an appropriate $\mathcal{L}^2$ space, which allows for standard differentiation and more importantly an immediate use of Itô's calculus. This approach is then used by Cardaliguet et al. [11] to obtain the regularity of the solutions to the master equation of a mean-field game. This very strong regularity result implies in particular a classical interpretation of the master equations on the Wasserstein space. On the other hand, in the absence of such strong regularity, one needs to develop the notion of viscosity solutions for McKean–Vlasov

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control problems. Gangbo, Nguyen, and Tudorascu [23] also observe that the tangent bundle of the Wasserstein space is given by appropriate $\mathcal{L}^2$ spaces and develop a viscosity theory for a classical mechanics problem in this space. More recently, Pham and Wei [34, 35] initiated the study of viscosity solutions by using Lions' lifting for controlled diffusion processes. Bandini et al. [3, 4] further developed this theory for the dynamic programming equations for the partially observed systems which also have the same structure. An important advantage of this approach to viscosity theory, in addition to the Hilbert structure of $\mathcal{L}^2$, is its ability to utilize the existing results for viscosity solutions on Hilbert spaces [32, 19]. An intrinsic approach to viscosity solutions without lifting could also have advantages, and Wu and Zhang [37] study this approach for diffusion process using the techniques developed for path-dependent viscosity solution [17, 18].

Our goal is to develop a viscosity theory for jump-diffusion processes. For the standard control problems, the corresponding dynamic programming equations contain nonlocal integral terms related to the infinitesimal generator of the jump-Markov processes. Still these equations have maximum principle, and a viscosity theory is appropriate. Starting from [36, 20, 16, 2] definitions, stability and comparison results for nonlinear integrodifferential equations of this type have been developed. We refer to more recent paper by Barles and Imbert [5] for more information.

The jump terms in these equations introduce several new aspects. In particular, for the McKean–Vlasov control problems, the operator appearing in the dynamic programming equations does not act on the Lions derivative (i.e., the derivative in the $\mathcal{L}^2$ space of the lifted function) but rather on the standard (sometimes called linear) derivative. Indeed, when all functions are smooth, it is immediate that the Lions derivative is an $\mathcal{L}^2$ function and it is equal to the space derivative of the linear derivative (see section 5.4 in [13]). For the diffusion problems, only the space derivatives of the linear derivative appear in the dynamic programming equation, and therefore one can simply replace them by the Lions derivative. For the integrodifferential equations, however, one needs to recover the linear derivative from the Lions derivative even to state the equations. Unfortunately the required regularity (to immediately connect these two derivatives) is not readily available when one is working in the viscosity structure.

We choose to work directly on the space of probability measures with the linear derivative to develop an intrinsic theory. Although this approach has several advantages, the dynamic programming equations on the space of probability measures are not as well studied as the lifted equation on the $\mathcal{L}^2$ spaces, and parts of the viscosity theory have to be revisited. Indeed, we first provide appropriate definitions of viscosity sub- and supersolutions for a class of integrodifferential equations in this space. We then show that the value function is a viscosity solution in this sense. Several properties of the dynamics are used to construct the framework that is appropriate for this problem. In particular, we consider the equation only on the subset of the measures that have exponential moments.

One of the main contributions of this paper is a comparison result for the viscosity solutions. An important ingredient of our approach is a distance-like function d given for two probability measures μ, ν by

$$d(\mu,\nu) = \sum_{j=1}^{\infty} c_j \langle \mu - \nu, f_j \rangle^2,$$

where the countable set $\{f_j\}_{j\in\mathbb{N}}$ is carefully constructed to have several important invariance-type properties. In the standard doubling-variables argument, we penalize the two points using d. Then the subtle properties of f_j allow us to estimate its linear derivative of d by itself.

The paper is organized as follows. We first introduce a class of optimal control problems of McKean–Vlasov type in the next section. A guiding example for this class is a model of technological innovation [28, 1]. We discuss this problem in section 3. The natural state space for this study is the set of measures with exponential moments, and under mild assumptions, the corresponding dynamical system lives in this space. In section 5 we define this space, prove its functional analytic properties, and show its connection to the controlled dynamics. In section 6 we give the definition of a viscosity solution and in section 7 show that the value function is a viscosity solution. Section 8 provides the construction of the functions f_j and the comparison result. We prove several technical results in the appendix.

Notation. For a random variable X, defined on a probability space $(\Omega, \mathcal{F}, P)$, we denote by $\mathcal{L}(X)$ the distribution of X under P. We denote by $\mathcal{P}(\mathbb{R})$ the space of probability measures on $\mathbb{R}$ and by $ca(\mathbb{R})$ the linear space of countably additive measures. For any $\mu \in \mathcal{P}(\mathbb{R})$ and for any integrable function $f : \mathbb{R} \to \mathbb{R}$, we use the standard compact notation $\langle \mu, f \rangle := \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x)$. If f is smooth, $f^{(i)}$ denotes the ith order derivative of f with $f^{(0)} = f$. We endow the space of probability measures $\mathcal{P}(\mathbb{R})$ with the weak* topology $\sigma(\mathcal{P}(\mathbb{R}), \mathcal{C}_b(\mathbb{R}))$, where $\mathcal{C}_b(\mathbb{R})$ is the space of continuous and bounded functions on $\mathbb{R}$. We denote by $\mu_n \to \mu$ the $\sigma(\mathcal{P}(\mathbb{R}), \mathcal{C}_b(\mathbb{R}))$-convergence of μ_n to μ, i.e., $\langle \mu_n, f \rangle$ converges to $\langle \mu, f \rangle$ for every $f \in \mathcal{C}_b(\mathbb{R})$.

2. The control problem and the assumption.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, P)$ be a given filtered probability space supporting the following class of controlled McKean–Vlasov stochastic differential equations (SDEs) with initial condition $\mathcal{L}(X_t) = \mu \in \mathcal{P}(\mathbb{R})$ and

$$\mathrm{d}X_s = b(s, \mathcal{L}(X_s), \alpha_s)\,\mathrm{d}s + \sigma(s, \mathcal{L}(X_s), \alpha_s)\,\mathrm{d}W_s + \mathrm{d}J_s, \quad s > t, \tag{2.1}$$

where J_s is a purely discontinuous process with controlled intensity $\lambda(s, \mathcal{L}(X_s), \alpha_s)$ and jump size given by an independent random variable ξ with distribution $\gamma \in \mathcal{P}(\mathbb{R})$. The class of admissible controls $\mathcal{A}$ is the set of all measurable deterministic functions of time with values in a prescribed measurable space A. Theorem 4.1 below proves that under suitable assumptions, (2.1) has a unique solution for any given (t, μ, α). We denote this solution by $(X_u^{t,\mu,\alpha})_{u\in[t,T]}$, but to ease the notation, we also use the notation X^α when the initial condition is clear from the context. The value function is then given by

$$V(t, \mu) := \inf_{\alpha\in\mathcal{A}} \left[\int_t^T L(s, \mathcal{L}(X_s^{t,\mu,\alpha}), \alpha_s)\,\mathrm{d}s + G(\mathcal{L}(X_T)) \right]$$

with given functions L and G. The optimal control problem consists of finding the value V and a minimizer (if it exists).

We close this section by stating a set of conditions assumed to hold throughout the paper, and they will not always be stated explicitly later on.

ASSUMPTION 2.1. *There exist constants $C_0, \kappa_0, \delta > 0$ such that the coefficients $b, \sigma, \lambda, L : [0, T] \times \mathcal{P}(\mathbb{R}) \times A \to \mathbb{R}$ satisfy the following conditons:*

(H1) *For any $\mu \in \mathcal{P}(\mathbb{R})$, $a \in A$, $s \in [0, T]$,*

$$|b(s, \mu, a)| + |\sigma(s, \mu, a)| + |\lambda(s, \mu, a)| \le C_0.$$

(H2) *There exists a finite set $\mathcal{I} \subset \mathbb{N}$ such that for any $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, $a \in A$, $t, s \in [0,T]$,*

$$|b(t,\mu,a) - b(s,\mu',a)| + |\sigma(t,\mu,a) - \sigma(s,\mu',a)| \leq \kappa_0\Big(|t-s| + \sum_{i\in\mathcal{I}} |\langle \mu - \mu', x^i\rangle|\Big),$$
$$|\lambda(t,\mu,a) - \lambda(s,\mu',a)| \leq \kappa_0\Big(|t-s| + \sum_{i\in\mathcal{I}} |\langle \mu - \mu', x^i\rangle|\Big).$$

(H3) *γ has δ-*exponential moment*:*

$$\int_{\mathbb{R}} \exp(\delta|x|)\gamma(\mathrm{d}x) < \infty.$$

(H4) *L is of the form $L_1(t,\mu,a) + L_2(a)\langle \mu, L_3(\cdot)\rangle$, where $L_1 : [0,T]\times\mathcal{P}(\mathbb{R})\times A \to \mathbb{R}$ is continuous in (t,μ), uniformly in a, $L_2 : A \to \mathbb{R}$ with $\sup_{a\in A} L_2(a) < \infty$, and $L_3 : \mathbb{R} \to \mathbb{R}$ satisfies $|L_3(x)x| \leq C_0 \exp(\delta|x|)$ for every $x \in \mathbb{R}$. The terminal cost G is continuous.*

In what follows, the constants $C_0, \kappa_0, \delta > 0$ are always as in the above assumption.

Remark 2.2. The condition (H2) is a form of Lipschitz continuity with respect to cylindrical functions of the measure arguments. This condition also allows us to show the Lipschitz continuity of the above functions with respect to a function d. Indeed, in the comparison below, we construct a distance-like function $d(\mu,\mu') := \sum_{j=1}^{\infty} c_j|\langle \mu - \nu, f_j\rangle|$ which, restricted to a suitable compact set, is a metric compatible with weak* convergence. As the class $\{f_j\}_{j\in\mathbb{N}}$ contains all monomials, the condition (H2) implies the Lipschitz continuity with respect to d, restricted to the chosen compact set.

3. A model of technological innovation.

We briefly present here an example of a McKean–Vlasov control problem where the underlying process is a jump-diffusion. The controlled equations represent a model of knowledge diffusion which appeared in the macroeconomic literature in the area of search-theoretic models of technological change, e.g. [1],[1] [28]. With controls $\alpha = (\bar{\alpha}, \hat{\alpha})$, a social planner aims at promoting technological innovation in the society by controlling the process

$$\mathrm{d}X_s^\alpha = b\big(\mathcal{L}(X_s^\alpha), \bar{\alpha}_s\big)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s + \mathrm{d}J_s, \tag{3.1}$$

where $X_0 \sim \mu_0$ and J_s is a purely discontinuous process with controlled intensity $\lambda(\mathcal{L}(X_s^\alpha), \hat{\alpha}_s)$ and jump size given by a nonnegative independent random variable ξ with distribution γ. The value $\exp(X_s^\alpha)$ represents the efficiency of the production of a continuum of consumption goods (technological frontier), and the initial (logarithmic) efficiency is represented by the distribution μ_0. The aim is to maximize the average efficiency of the production of goods in order to foster the growth of the economy:

$$\text{maximize!} \quad \mathbb{E}\left[\int_0^T (1-\hat{\alpha}_s)\exp(X_s^\alpha) - (\bar{\alpha}_s)^2\,\mathrm{d}s\right],$$

where $\alpha = (\bar{\alpha}, \hat{\alpha})$ is chosen from an appropriate class of deterministic processes.

[1]We thank Rama Cont for bringing this paper to our attention.

The social planner can promote innovation by issuing research funds (exercising the control $\bar{\alpha}$). On the other hand, she can promote exchange of ideas by setting up meetings at a controlled Poisson rate. Meetings have the effect of inducing a nonnegative jump in the technological frontier, according to a random variable with distribution γ. The functions λ and b are bounded since meetings cannot happen too frequently and research funds have a limited impact on the technological frontier. These functions also depends on the distribution of X^α. This aspect can represent a positive feedback effect of a productive economy provided that the dependences on the distribution are appropriately monotone. Finally, the random Brownian component incorporates fluctuations in the efficiency of the production due to external contingent factors.

This model satisfies Assumption 2.1 under some appropriate regularity conditions on the parameters and initial distribution. Indeed, it is clear that (3.1) has the same structure as (2.1). Moreover, because controls are deterministic,

$$\mathbb{E}\left[\int_0^T (1-\hat{\alpha}_s)\exp(X_s^\alpha)\,\mathrm{d}s\right] = \int_0^T (1-\hat{\alpha}_s)\mathbb{E}\left[\exp(X_s^\alpha)\right]\mathrm{d}s$$

and

$$\mathbb{E}\left[\exp(X_s^\alpha)\right] = \int \exp(x)\mathcal{L}(X_s^\alpha)(dx) = \langle \mathcal{L}(X_s^\alpha), \exp(\cdot)\rangle.$$

Hence, the running cost L is given by

$$L(t,\mu,(\bar{\alpha},\hat{\alpha})) = (1-\hat{\alpha})\langle \mu, \exp(\cdot)\rangle - \bar{\alpha}^2.$$

In particular, L has the form L_2 in hypothesis (H4).

We refer to [1] for further examples of problems where the controlled process is only a diffusion without jump terms.

4. State space and dynamic programming. Since the Brownian motion has exponential moments, Assumption 2.1, in particular (H3), implies that the solutions of the state equation (2.1) has also exponential moments. Therefore it is natural to study the optimal control problem in $\mathcal{O} := [0,T) \times \mathcal{M}$, where $\mathcal{M}$ is the subset of probability measures with δ-exponential moments, i.e.,

$$\mu \in \mathcal{M} \quad \Leftrightarrow \quad \langle \mu, \exp(\delta|\cdot|)\rangle = \int_{\mathbb{R}} \exp(\delta|x|)\mu(\mathrm{d}x) < \infty,$$

where δ is as in (H3). Our first result is the well-posedness of the problem, and its straightforward proof is given in Appendix A.

Theorem 4.1. *Under Assumption* 2.1*, the SDE* (2.1) *has a unique solution,* $(X_u^{t,\mu,\alpha})_{u\in[t,T]}$*, for any* $(t,\mu,\alpha) \in \mathcal{O}\times\mathcal{A}$*.*

The described McKean–Vlasov control problem is deterministic, and therefore, it is classical that the dynamic programming principle holds [21]

$$V(t,\mu) = \inf_{\alpha\in\mathcal{A}} \left[\int_t^\theta L(s,\mathcal{L}(X_s^{t,\mu,\alpha}),\alpha_s)\,\mathrm{d}s + V(\theta,\mathcal{L}(X_\theta^{t,\mu,\alpha}))\right] \quad \forall\theta\in[t,T]. \tag{4.1}$$

We need several definitions to formally state the corresponding dynamic programming equation.

DEFINITION 4.2. *For* $\varphi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$*, the* linear derivative *of* φ *at* $\mu \in \mathcal{P}(\mathbb{R})$*, when exists, is a function* $D_m\varphi : \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *such that for every* $\mu, \mu' \in \mathcal{P}(\mathbb{R})$*,*

$$\varphi(\mu) - \varphi(\mu') = \int_0^1 \int_{\mathbb{R}} D_m\varphi(r\mu + (1-r)\mu', x)\ (\mu - \mu')(dx)\, \mathrm{d}r.$$

When $\varphi : [0,T] \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$*, with an abuse of notation, we denote the linear derivative with respect to the* μ*-variable still by* $D_m\varphi : [0,T] \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$*.*

This derivative was used by Fleming and Viot [22] to study a martingale problem in populations dynamics. Also recently Cuchiero, Larsson, and Svaluto-Ferro [15] provided several of its properties in the context of polynomial diffusions. For a detailed comparison of different notions of differentiability on spaces of measures we refer to the recent paper by Gangbo and Tudorascu [24] and to the recent book by Carmona and Delarue [13, section I.5].

Remark 4.3. Consider the linear function $\varphi(\mu) = \langle \mu, f \rangle$ with some $f : \mathbb{R} \to \mathbb{R}$. It is immediate that $D_m\varphi(\mu, x) = f(x)$ for any $(\mu, x) \in \mathcal{P}(\mathbb{R}) \times \mathbb{R}$. Moreover, suppose that $\varphi : ca(\mathbb{R}) \to \mathbb{R}$ is Frechet differentiable and such that $D\varphi : ca(\mathbb{R}) \to \mathbb{R}$ can be represented as $D\varphi[\mu] = \langle \mu, f \rangle$ for some $f : \mathbb{R} \to \mathbb{R}$. Then $f = D_m\varphi$, namely, $D\varphi[\mu] = \langle \mu, D_m\varphi \rangle$.

By the chain rule, the linear derivative of $\varphi(\mu) = F(\langle \mu, f \rangle)$ with some smooth function F is equal to $D_m\varphi(\mu, x) = F'(\langle \mu, f \rangle) f(x)$.

For a given input function $v = v(t, \mu, x)$, the operator $\mathcal{L}_t^{a,\mu}$ acting on the x-variable is given by

$$\begin{aligned}\mathcal{L}_t^{a,\mu}[v](x) := b(t,\mu,a)\frac{\partial v}{\partial x}(t,\mu,x) &+ \frac{1}{2}\sigma^2(t,\mu,a)\frac{\partial^2 v}{\partial x^2}(t,\mu,x)\\ &+ \lambda(t,\mu,a)\int_{\mathbb{R}}(v(t,\mu,x+y) - v(t,\mu,x))\gamma(dy).\end{aligned}$$

Using the above definitions, classical considerations starting from (4.1) formally lead to the following dynamic programming equation:

$$-\partial_t V(t,\mu) + \sup_{a \in A} H^a(t, \mu, D_m V) = 0, \tag{4.2}$$

where

$$H^a(t,\mu,v) := -L(t,\mu,a) - \langle \mu, \mathcal{L}_t^{a,\mu}[v] \rangle.$$

Indeed, as in the finite-dimensional optimal control theory, if the value function is smooth and cylindrical (i.e., if V has the form

$$V(t,\mu) = F(t, \langle \mu, f_1 \rangle, \ldots, \langle \mu, f_n \rangle)$$

for some smooth functions F and $f_1, \ldots, f_n$, then it is possible to derive (4.2) rigorously. Importantly, in this case, the classical Itô's formula can be applied to $V(u, \mathcal{L}(X_u^{t,\mu,\alpha})) = F(u, \mathbb{E}[f_1(X_u^{t,\mu,\alpha})], \ldots, \mathbb{E}[f_n(X_u^{t,\mu,\alpha})])$ for any given $(t,\mu,\alpha) \in \mathcal{O} \times \mathcal{A}$ and $u \in [t,T]$. However, this assumption on the value function is not expected to hold and also is not needed. In section 7, we prove that the value function is the unique viscosity solution to (4.2) even when it is neither smooth nor cylindrical.

5. σ-compactness of the state space. Recall that $\mathcal{O} := [0,T) \times \mathcal{M}$, and $\mathcal{M}$ is the set of probability measures μ satisfying $\langle \mu, \exp(\delta|\cdot|)\rangle < \infty$, where δ is as in (H3). We endow this space with the subspace topology induced by $\mathcal{P}$, i.e., weak* convergence. We use the product topology on $\overline{\mathcal{O}} := [0,T] \times \mathcal{M}$, and emphasize that $\overline{\mathcal{O}}$ is not the topological closure of $\mathcal{O}$ but simply includes the final time.

The space $\mathcal{O}$ has a suitable σ-compact structure which is compatible with the McKean–Vlasov dynamics. This representation of $\mathcal{O}$ is instrumental to obtain uniform integrability of the viscosity test functions as well as some continuity properties of the Hamiltonian. We continue by constructing this structure.

For δ as in (H3), set

$$e_\delta(x) := \exp\left(\delta\left[\sqrt{x^2+1}-1\right]\right), \quad x \in \mathbb{R}.$$

We note that e_δ is twice continuously differentiable and

$$\exp(\delta[|x|-1]) \le e_\delta(x) \le \exp(\delta|x|) \le e^\delta e_\delta(x) \quad \forall x \in \mathbb{R}.$$

For $N \in \mathbb{N}$ and C_0, δ as in Assumption 2.1, let

$$\mathcal{O}_N := \left\{(t,\mu) \in [0,T) \times \mathcal{P}(\mathbb{R}) \mid \langle \mu, e_\delta\rangle \le N e^{K^* t}\right\},$$

where

$$K^* = K^*(C_0,\delta) := \frac{\delta C_0}{2}(2 + C_0 + \delta C_0) + C_0\left(\int_{\mathbb{R}} e^{\delta|x|}\gamma(\mathrm{d}x) - 1\right). \tag{5.1}$$

The exact definition of K^* is not important for the functional analytic properties of $\mathcal{O}_N$ but is used centrally in the next lemma to prove an invariance property.

It is clear that $\mathcal{O} = [0,T) \times \mathcal{M} = \cup_{N=1}^\infty \mathcal{O}_N$ and $\overline{\mathcal{O}} = \cup_{N=1}^\infty \overline{\mathcal{O}}_N$, where

$$\overline{\mathcal{O}}_N := \left\{(t,\mu) \in [0,T] \times \mathcal{P}(\mathbb{R}) \mid \langle \mu, e_\delta\rangle \le N e^{K^* t}\right\}.$$

We also use the following notation for a constant $b > 0$:

$$\mathcal{M}_b := \left\{\mu \in \mathcal{P}(\mathbb{R}) \mid \langle \mu, e_\delta\rangle \le b\right\}.$$

The following lemma shows that for each N, $\mathcal{O}_N$ and thus also $\mathcal{O}$ remain invariant under the controlled dynamics (2.1) for any control. In particular, this means that for any given initial law $\mu \in \mathcal{O}_N$, we may restrict the dynamic programming equation (4.2) to $\mathcal{O}_N$.

Lemma 5.1. *Under Assumption* 2.1, *for any $N \in \mathbb{N}$, the set $\mathcal{O}_N$ is invariant for the SDE* (2.1), *namely,*

$$(t,\mu) \in \mathcal{O}_N \Rightarrow (u, \mathcal{L}(X_u^{t,\mu,\alpha})) \in \mathcal{O}_N \quad \forall (u,\alpha) \in [t,T] \times \mathcal{A},$$

where $(X_u^{t,\mu,\alpha})_{u\in[t,T]}$ is the solution to (2.1) *with initial condition $\mathcal{L}(X_t^{t,\mu,\alpha}) = \mu$.*

Proof. Set $\varphi(x) := \sqrt{x^2+1} - 1$ so that

$$e_\delta(x) = e^{\delta\varphi(x)}, \quad x \in \mathbb{R}.$$

It is clear that φ is twice continuously differentiable and both $|\varphi'|$ and $\varphi'' > 0$ are bounded by 1.

Fix $(t,\mu) \in \mathcal{O}_N$ and $\alpha \in \mathcal{A}$. For $u \in [t,T]$, set $Y_u := e^{\delta\varphi(X_u)}$, $\mu_u := \mathcal{L}(X_u)$, where $X_u := X_u^{t,\mu,\alpha}$. In particular, $\mu_t = \mu$ for any control α, and by Itô's formula,

$$\begin{aligned} Y_u = Y_t &+ \int_t^u b(s,\mu_s,\alpha_s)\delta\varphi'(X_s)Y_s\,\mathrm{d}s \\ &+ \frac{1}{2}\int_t^u \sigma^2(s,\mu_s,\alpha_s)\big[\delta\varphi''(X_s) + \delta^2(\varphi'(X_s))^2\big]Y_s\,\mathrm{d}s \\ &+ \int_t^u \sigma(s,\mu_s,\alpha_s)\delta\varphi'(X_s)Y_s\,\mathrm{d}W_s + \sum_{t\le s\le u}\Delta Y_s. \end{aligned}$$

In view of assumption (H1), the stochastic integral in the above formula is a local martingale. We take expectation on both sides up to a localizing sequence of stopping times $\{\tau_n\}_n$. We also use assumption (H1) to estimate that the expectation of the second and third terms of the above sum is bounded by

$$C_1\mathbb{E}\bigg[\int_t^u Y_{s\wedge\tau_n}\,\mathrm{d}s\bigg],$$

where $C_1 := \frac{\delta C_0}{2}(2 + C_0 + \delta C_0)$ and C_0 is as in assumption (H1).

We next estimate $e_J := \mathbb{E}[\sum_{t\le s\le u\wedge\tau_n}\Delta Y_s]$. First observe that for any $x,y \in \mathbb{R}$, $|\varphi(y+x) - \varphi(y)| \le |x|$. We then estimate, by using assumption (H3),

$$\begin{aligned} e_J &= \mathbb{E}\int_t^{u\wedge\tau_n}\lambda(s,\mu_s,\alpha_s)\int_{\mathbb{R}} e^{\delta\varphi(X_{s\wedge\tau_n}+x)} - e^{\delta\varphi(X_{s\wedge\tau_n})}\gamma(\mathrm{d}x)\,\mathrm{d}s \\ &\le C_0\mathbb{E}\int_t^{u\wedge\tau_n} Y_{s\wedge\tau_n}\int_{\mathbb{R}}\big(e^{\delta|x|} - 1\big)\gamma(\mathrm{d}x)\,\mathrm{d}s \\ &\le C_2\mathbb{E}\int_t^u Y_{s\wedge\tau_n}\,\mathrm{d}s, \end{aligned}$$

where $C_2 := C_0\big(\int_{\mathbb{R}} e^{\delta|x|}\gamma(\mathrm{d}x) - 1\big)$. These and Fubini's theorem imply that

$$\mathbb{E}[Y_{u\wedge\tau_n}] \le \mathbb{E}[Y_t] + K^*\int_t^u \mathbb{E}\big[Y_{s\wedge\tau_n}\big]\,\mathrm{d}s,$$

where K^* is as in (5.1). By Gronwall's lemma and Fatou's lemma,

$$\mathbb{E}[Y_u] \le e^{K^*(u-t)}\mathbb{E}[Y_t] = e^{K^*(u-t)}\ \langle\mu, e^{\delta\varphi}\rangle = e^{K^*(u-t)}\langle\mu, e_\delta\rangle.$$

As $(t,\mu) \in \mathcal{O}_N$, $\langle\mu, e_\delta\rangle \le Ne^{K^*t}$. Hence,

$$\mathbb{E}[Y_u] \le e^{K^*(u-t)}\langle\mu, e_\delta\rangle \le e^{K^*u}. \qquad \square$$

We provide the proof of the following simple result for completeness.

LEMMA 5.2. *For $N \in \mathbb{N}$, $\overline{\mathcal{O}}_N$ is a compact subset of $[0,T] \times \mathcal{P}(\mathbb{R})$.*

Proof. Fix $b > 0$. For R sufficiently large, we have $e_\delta(x)/|x| \ge 1$ for any $|x| \ge R$, which implies

$$\begin{aligned} \sup_{\mu\in\mathcal{M}_b}\mu([-R,R]^c) &= \sup_{\mu\in\mathcal{M}_b}\int_{|x|\ge R}\mu(\mathrm{d}x) \\ &\le \sup_{\mu\in\mathcal{M}_b}\int_{|x|\ge R}\frac{e_\delta(x)}{|x|}\mu(\mathrm{d}x) \le \frac{b}{R}, \end{aligned}$$

and the last term converges to 0 as $R \to \infty$. Hence $\mathcal{M}_b$ is tight, and by Prokhorov's theorem, it is relatively compact. We next show that it is also closed. Consider a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_b$ such that $\mu_n \to \mu$. Set $f_m(x) := e_\delta(x) \wedge m$. Since $f_m \in \mathcal{C}_b(\mathbb{R})$, $f_m \le e_\delta(x)$, and $\mu_n \in \mathcal{M}_b$,

$$\langle \mu, f_m \rangle = \lim_{n\to\infty} \langle \mu_n, f_m \rangle \le b \quad \forall m > 0.$$

By the monotone convergence theorem,

$$\langle \mu, e_\delta \rangle = \lim_{m\to\infty} \langle \mu, f_m \rangle \le b.$$

Hence, $\mu \in \mathcal{M}_b$, and consequently, $\mathcal{M}_b$ is compact.

For every N, $\overline{\mathcal{O}}_N$ is a subset of $[0,T] \times \mathcal{M}_{Ne^{K^*t}}$; hence, it is relatively compact. Consider a sequence $\{(t_n, \mu_n)\}_{n\in\mathbb{N}} \subset \overline{\mathcal{O}}_N$ such that $(t_n, \mu_n) \to (t, \mu)$. Proceeding exactly as above, we can show that

$$\langle \mu, e_\delta \rangle = \lim_{n\to\infty} \lim_{m\to\infty} \langle \mu_n, f_m \rangle \le N e^{K^* t}.$$

Hence, $(t,\mu) \in \overline{\mathcal{O}}_N$, and consequently, $\overline{\mathcal{O}}_N$ is compact. ∎

We close this section by recalling a well-known result; see [9, Theorem 30.1]. Suppose $\mu, \nu \in \mathcal{M}$. Then,

$$\mu = \nu \quad \Leftrightarrow \quad \langle \mu - \nu, x^j \rangle = 0 \ \ \forall\, j = 1, 2, \dots \tag{5.2}$$

6. Viscosity solutions and test functions. In this section, we define viscosity sub- and supersolutions to the dynamic programming equation (4.2). As is standard in the viscosity theory, one has to first specify the class of test functions. We continue by this selection.

DEFINITION 6.1. *A* cylindrical function *is a map of the form* $(t,\mu) \mapsto F(t, \langle \mu, f \rangle)$ *for some function* $f : \mathbb{R} \to \mathbb{R}$ *and* $F : [0,T] \times \mathbb{R} \to \mathbb{R}$*. This function is called* cylindrical polynomial *if* f *is a polynomial and* F *is continuously differentiable.*

The above class is not large enough, and we extend it to its linear span. For any polynomial f, $\deg(f)$ denotes the degree of f.

DEFINITION 6.2. *For* $E \subseteq \overline{\mathcal{O}}$*, a* viscosity test function on E *is a function of the form*

$$\varphi(t,\mu) = \sum_{j=1}^{\infty} \varphi_j(t,\mu), \quad (t,\mu) \in E,$$

where $\{\varphi_j\}_j$ *is a sequence of cylindrical polynomials that are absolutely convergent at every* (t,μ) *and for every* $N \in \mathbb{N}$*,*

$$\lim_{M\to\infty} \sum_{j=M}^{\infty} \sup_{(t,\mu)\in E} \sum_{i=0}^{\deg(D_m\varphi_j)} \left| \langle \mu, (D_m\varphi_j)^{(i)} \rangle \right| = 0. \tag{6.1}$$

We let Φ_E *be the set of all viscosity test functions on* E*.*

Lemma 6.8 below shows that for a cylindrical polynomial φ, $\langle \mu, (D_m\varphi)^{(i)} \rangle$ is uniformly bounded on $(t,\mu,a) \in \overline{\mathcal{O}}_N \times A$ for every $i = 0, \dots, \deg(D_m\varphi)$. Therefore, all cylindrical polynomials are test functions on every $\overline{\mathcal{O}}_N$.

Remark 6.3. There are several other choices for test functions. In particular, we could even restrict F to be quadratic or extend it to be more general with some integrability properties. They all would yield equivalent definitions, and we do not pursue this equivalence here.

When μ_t is the law of a stochastic process X_t and φ is a cylindrical function, $\varphi(t,\mu_t) = F(t,\langle \mu_t, f\rangle) = F(t,\mathbb{E}[f(X_t)])$. Then, one can employ the standard Itô formula; see Proposition 6.9 below.

DEFINITION 6.4. *For $E \subseteq \overline{\mathcal{O}}$ and $(t,\mu) \in E$ with $t < T$, the* superjet *of u at (t,μ) is given by*

$$J_E^{1,+}u(t,\mu) := \big\{(\partial_t\varphi(t,\mu), D_m\varphi(t,\mu,\cdot)) \mid \varphi \in \Phi_E, (u-\varphi)(t,\mu) = \max_E(u-\varphi)\big\}.$$

The subjet *of u at (t,μ) is defined as $J_E^{1,-}u(t,\mu) := -J_E^{1,+}(-u)(t,\mu)$.*

DEFINITION 6.5. *On a subspace $E \subseteq \overline{\mathcal{O}}$, the* (sequential) upper semicontinuous envelope *of u on E is defined by*[2]

$$u_E^*(t,\mu) := \limsup_{E \ni (t',\mu') \to (t,\mu)} u(t,\mu),$$

where the lim sup *is taken over all sequences in E converging to (t,μ). The* lower semicontinuous envelope u_*^E *is defined analogously.*

We use the compact notations

$$u^* := u^*_{\overline{\mathcal{O}}}, \quad u_* := u_*^{\overline{\mathcal{O}}}, \quad u_N^* := u^*_{\overline{\mathcal{O}}_N}, \quad u_*^N := u_*^{\overline{\mathcal{O}}_N}.$$

We note that as opposed to the finite-dimensional cases, when u is not continuous, the dependence of u_N^* and u_*^N on N is nontrivial. This emanates from the fact that the interiors of all $\mathcal{O}_N$ are empty.

To simplify the notation, we write $H = \sup_{a\in A} H^a$.

DEFINITION 6.6. *We say that a function $u : \mathcal{O}_N \to \mathbb{R}$ is a* viscosity subsolution *of* (4.2) *on $\mathcal{O}_N$ if, for every $(t,\mu) \in \mathcal{O}_N$,*

$$-\pi_t + H(t,\mu,\pi_\mu) \le 0 \qquad \forall(\pi_t,\pi_\mu) \in J^{1,+}_{\mathcal{O}_N} u_N^*(t,\mu).$$

We say that a function $v : \mathcal{O}_N \to \mathbb{R}$ is a viscosity supersolution *of* (4.2) *on $\mathcal{O}_N$ if for every $(t,\mu) \in \mathcal{O}_N$,*

$$-\pi_t + H(t,\mu,\pi_\mu) \ge 0 \qquad \forall(\pi_t,\pi_\mu) \in J^{1,-}_{\mathcal{O}_N} u_*^N(t,\mu).$$

A viscosity solution *of* (4.2) *is a function on $\mathcal{O}$ that is both a subsolution and a supersolution of* (4.2) *on $\mathcal{O}_N$ for every $N \in \mathbb{N}$.*

We continue with several technical results. Ultimately, we want to show some continuity properties of H.

DEFINITION 6.7. *We say that g has δ-subexponential growth if $|g(x)x| \le \hat{C}e^{\delta|x|}$ for some $\hat{C} > 0$ and every $x \in \mathbb{R}$.*

Note that any polynomial has δ-subexponential growth.

[2]As $\mathcal{O}$ is first countable, semicontinuity coincides with sequential semicontinuity.

LEMMA 6.8. *Let $\delta > 0$ be as in* (H3)*. For any continuous g with δ-subexponential growth,*

$$\sup_{\mu\in\mathcal{M}_b}\langle\mu,|g|\rangle<\infty \quad \textit{and} \quad \lim_{R\to\infty}\sup_{\mu\in\mathcal{M}_b}\int_{|x|\geq R}|g(x)|\mu(\mathrm{d}x)=0.$$

Moreover, there is a constant C, depending only on the constants appearing in Assumption 2.1*, such that for any cylindrical polynomial φ and $N\in\mathbb{N}$,*

$$\sup_{a\in A,(t,\mu)\in\overline{\mathcal{O}}_N}\left|\langle\mu,\mathcal{L}_t^{a,\mu}[D_m\varphi]\rangle\right|\leq C\sup_{(t,\mu)\in\overline{\mathcal{O}}_N}\sum_{i=0}^{\deg(D_m\varphi)}\left|\langle\mu,(D_m\varphi)^{(i)}\rangle\right|<\infty. \tag{6.2}$$

Proof. Since g has δ-subexponential growth,

$$|g(x)x|\leq\hat{C}\exp(\delta|x|)\leq\hat{C}e^{\delta}e_\delta(x)=:\tilde{C}e_\delta(x),\quad x\in\mathbb{R}.$$

By definition of $\mathcal{M}_b$, $\sup_{\mu\in\mathcal{M}_b}\langle\mu,e_\delta\rangle\leq b$ and since g is bounded on compact sets the estimate $\sup_{\mu\in\mathcal{M}_b}\langle\mu,|g|\rangle<\infty$ follows. Moreover, for $R\geq 1$,

$$\begin{aligned}\sup_{\mu\in\mathcal{M}_b}\int_{|x|\geq R}|g(x)|\mu(\mathrm{d}x)&\leq\tilde{C}\sup_{\mu\in\mathcal{M}_b}\int_{|x|\geq R}\frac{e_\delta(x)}{|x|}\mu(\mathrm{d}x)\\&\leq\frac{\tilde{C}}{R}\sup_{\mu\in\mathcal{M}_b}\int_{|x|\geq R}e_\delta(x)\mu(\mathrm{d}x)\\&\leq\frac{b\tilde{C}}{R}.\end{aligned}$$

Let f be a polynomial. Then,

$$\begin{aligned}\langle\mu,\mathcal{L}_t^{a,\mu}[f]\rangle&=b(t,\mu,a)\langle\mu,f'\rangle+\frac{1}{2}\sigma^2(t,\mu,a)\langle\mu,f''\rangle\\&\quad+\lambda(t,\mu,a)\langle\mu,\int_{\mathbb{R}}(f(x+y)-f(x))\gamma(dy)\rangle.\end{aligned}$$

We rewrite the last term by Taylor expansion of the polynomial f as follows:

$$\langle\mu,\int_{\mathbb{R}}(f(x+y)-f(x))\gamma(dy)\rangle=\sum_{i=1}^{\deg(f)}\frac{\langle\mu,f^{(i)}\rangle}{i!}\int_{\mathbb{R}}y^i\gamma(dy).$$

The above equations, together with Assumption 2.1 and the fact that all derivatives of f have δ-subexponential growth, imply (6.2). The result for a cylindrical polynomial follows similarly. ∎

PROPOSITION 6.9. *For every $\varphi\in\Phi_{\mathcal{O}_N}$, $(t,\mu)\in\mathcal{O}_N$, and $\alpha\in\mathcal{A}$,*

$$\varphi(u,\mu_u)=\varphi(t,\mu)+\int_t^u\left[\partial_t\varphi(s,\mu_s)+\langle\mu_s,\mathcal{L}_s^{\alpha_s,\mu_s}[D_m\varphi]\rangle\right]\mathrm{d}s,\quad u\in[t,T], \tag{6.3}$$

where $\mu_s=\mathcal{L}(X_s^{t,\mu,\alpha})$ and $(X_s^{t,\mu,\alpha})_{s\in[t,T]}$ is the solution to (2.1) *with initial distribution μ. Moreover, the map $(t,\mu)\mapsto H(t,\mu,D_m\varphi)$ is continuous on $\mathcal{O}_N$.*

Proof. Fix $\varphi\in\Phi_{\mathcal{O}_N}$, $(t,\mu)\in\mathcal{O}_N$, and $\alpha\in\mathcal{A}$, and let μ_s be as in the statement. In view of Lemma 5.1, $\mu_s\in\mathcal{O}_N$ for all $s\in[t,T]$.

Let first $\varphi(\mu) = \langle \mu, f\rangle$ with f polynomial, so that $D_m\varphi = f$ and $\langle \mu_s, f\rangle = \mathbb{E}f(X_s^{t,\mu,\alpha})$. By stochastic calculus

$$\langle \mu_u, f\rangle = \langle \mu, f\rangle + \int_t^u \langle \mu_s, \mathcal{L}_s^{\alpha_s,\mu_s}[f]\rangle \,\mathrm{d}s.$$

Moreover, this derivative is uniformly bounded on $\mathcal{O}_N$ by the previous lemma. Now consider a cylindrical polynomial $\varphi(t,\mu) = F(t, \langle \mu, f\rangle)$. By calculus,

$$\varphi(u, \mu_u) = \varphi(t,\mu) + \int_t^u \left[\partial_t \varphi(s,\mu_s) + F_x(s, \langle \mu_s, f\rangle)\langle \mu_s, \mathcal{L}_s^{\alpha_s,\mu_s}[f]\rangle\right] \mathrm{d}s.$$

Since $D_m\varphi(s,\mu) = F_x(s, \langle \mu, s\rangle)f$, the above proves (6.3) for cylindrical polynomials. For a general $\varphi \in \Phi_{\mathcal{O}_N}$, (6.3) follows directly from above, the condition (6.1), and the fact that $\mu_s \in \mathcal{O}_N$ for all $s \in [t,T]$.

We now show continuity of H. Since all derivatives of f have δ-subexponential growth, Lemma 6.8 and the fact that φ is a smooth function imply $\langle \mu, (D_m\varphi)^{(i)}\rangle$ is continuous on every $\mathcal{O}_N$ for any $i \in \mathbb{N}$. In particular the uniform continuity of $(t,\mu) \mapsto \langle \mu, \mathcal{L}_t^{a,\mu}[D_m\varphi(t,\mu)]\rangle$ follows from (H1) and (H2), and for L it is assumption (H4). Hence, $H(t,\mu,D_m\varphi)$ is continuous for all cylindrical polynomials. This continuity extends directly to all functions of the type $\varphi^M := \sum_{j=1}^M F_j(\langle \mu, f_j\rangle)$.

Now consider a general test function $\varphi = \sum_{j=1}^\infty F_j(\langle \mu, f_j\rangle)$, and for $M \in \mathbb{N}$ set $\varphi^M := \sum_{j=1}^M F_j(\langle \mu, f_j\rangle)$. Since $\varphi \in \Phi_{\mathcal{O}_N}$, it satisfies (6.1). This together with (6.2) imply that

$$\lim_{M\to\infty} \sup_{a\in A, (t,\mu)\in\overline{\mathcal{O}}_N} \sum_{j=M}^\infty \left|\langle \mu, \mathcal{L}_t^{a,\mu}[D_m\varphi_j]\rangle\right| = 0.$$

The above uniform limit enables us to conclude that $H(t,\mu,D_m\varphi^M)$ converges uniformly to $H(t,\mu,D_m\varphi)$ as M tends to infinity. Hence, $H(t,\mu,D_m\varphi)$ is also continuous. ∎

7. Value function. In this section we show that the value function V is a viscosity solution to (4.2). We start with two technical lemmata.

LEMMA 7.1. *For every $N \in \mathbb{N}$, $(t_0,\mu_0) \in \mathcal{O}_N$, there exists a viscosity test function $\phi \in \Phi_{\mathcal{O}_N}$ such that $\phi(t,\mu) \geq 0$, with equality only in (t_0,μ_0), and*

$$(\phi(t_0,\mu_0), \partial_t\phi(t_0,\mu_0), D_m\phi(t_0,\mu_0,\cdot)) = (0,0,0).$$

In particular, in the definition of viscosity sub- and supersolutions, without loss of generality, we may assume that the extrema are strict.

Proof. Fix $(t_0,\mu_0) \in \mathcal{O}_N$, and set

$$\phi(t,\mu) = \phi(t,\mu;t_0,\mu_0) := (t-t_0)^2 + \sum_{j=1}^\infty \frac{1}{(j+1)2^j}\langle \mu - \mu_0, x^j\rangle^2.$$

By (5.2) $\phi(t,\mu) > 0$ when $(t,\mu) \neq (t_0,\mu_0)$. For any $j \in \mathbb{N}$, let $\varphi_j(\mu) = \frac{1}{(j+1)2^j}\langle \mu - \mu_0, x^j\rangle^2$ and observe that

$$\sup_{(t,\mu)\in\mathcal{O}_N} \sum_{i=0}^{\deg(D_m\varphi_j)} \left|\langle \mu, (D_m\varphi_j)^{(i)}\rangle\right| \leq \frac{1}{2^j}K_N$$

for some constant K_N which only depend on $\mathcal{O}_N$. It follows that ϕ satisfies (6.1). It is clear that ϕ has all the claimed properties. ∎

LEMMA 7.2. *For each N, V, V_N^*, and V_*^N are bounded on $\mathcal{O}_N$.*

Proof. Let $(t,\mu) \in \mathcal{O}_N$. From Lemma 5.1, $\mathcal{O}_N$ is invariant for (2.1), and recall that $\overline{\mathcal{O}}_N$ is compact. Assumption (H4) and Lemma 6.8 imply that $|L| + |G|$ is uniformly bounded on $\overline{\mathcal{O}}_N$ by a constant K_N. It follows that $|V(t,\mu)| \leq (1+T)K_N$ on $\mathcal{O}_N$. ◻

The proof of the next result is standard [10, 21].

THEOREM 7.3. *Assume* (4.1) *holds. For any $N \in \mathbb{N}$, the value function V is both a viscosity sub- and a supersolution to* (4.2) *on $\mathcal{O}_N$ and*

$$V_N^*(T,\cdot) = V_*^N(T,\cdot) = G \quad \textit{on } \mathcal{M}_{Ne^{K^*T}}.$$

Proof. Fix $N \in \mathbb{N}$, and note that both envelopes V_N^*, V_*^N are finite by Lemma 7.2.

Step 1. *V_N^* is a viscosity subsolution for $t < T$.* Suppose that for $\varphi \in \Phi_{\mathcal{O}_N}$ and $(t,\mu) \in \mathcal{O}_N$,

$$0 = (V_N^* - \varphi)(t,\mu) = \max_{\mathcal{O}_N} (V_N^* - \varphi).$$

Let (t_n,μ_n) be a sequence in $\mathcal{O}_N$ such that $(t_n,\mu_n,V(t_n,\mu_n)) \to (t,\mu,V_N^*(t,\mu))$. Fix $a \in A$, and let $(X_s^{t_n,\mu_n,a})_{s\in[t_n,T]}$ denote the solution to (2.1) with constant control a and distribution μ_n at the initial time t_n. For ease of notation, we set $\mu_s^{n,a} := \mathcal{L}(X_s^{t_n,\mu_n,a})$. We use the dynamic programming (4.1) with $\theta_n := t_n + h$ for $0 < h < T - t$ to obtain

$$V(t_n,\mu_n) \leq \int_{t_n}^{\theta_n} L(s,\mu_s^{n,a},a)\,\mathrm{d}s + V(\theta_n,\mu_{\theta_n}^{n,a}) \leq \int_{t_n}^{\theta_n} L(s,\mu_s^{n,a},a)\,\mathrm{d}s + \varphi(\theta_n,\mu_{\theta_n}^{n,a}).$$

We pass to the limit to arrive at

$$V_N^*(t,\mu) = \varphi(t,\mu) \leq \int_t^{t+h} L(s,\mu_s^a,a)\,\mathrm{d}s + \varphi(t+h,\mu_{t+h}^a),$$

where μ_s^a is the distribution of the solution to (2.1) with initial data μ at time t and constant control a. We now use (6.3) to obtain

$$0 \leq \int_t^{t+h} [\partial_t\varphi(s,\mu_s^a) - H^a(s,\mu_s^a,D_m\varphi)]\,\mathrm{d}s.$$

Since this holds for every $h > 0$ and $a \in A$, we conclude that

$$-\partial_t\varphi(t,\mu) + \sup_{a\in A} H^a(t,\mu,D_m\varphi) \leq 0.$$

Step 2. *V_*^N is a viscosity supersolution for $t < T$.* Suppose that there exist $(t,\mu) \in \mathcal{O}_N$ and $\varphi \in \Phi_{\mathcal{O}_N}$ such that

$$0 = (V_*^N - \varphi)(t,\mu) = \min_{\mathcal{O}_N} (V_N^* - \varphi).$$

In view of Lemma 7.1, without loss of generality we assume that above minimum is strict. Towards a counterposition assume that

$$-\partial_t\varphi(t,\mu) + H(t,\mu,D_m\varphi) < 0.$$

By the continuity of H proved in Proposition 6.9, there exists a neighbourhood B of (t,μ) such that

$$-\partial_t\varphi(t,\mu) - \langle \mu, \mathcal{L}_t^{a,\mu}[D_m\varphi]\rangle \le L(t,\mu,a) \quad \forall (t,\mu)\in B_N := B\cap \mathcal{O}_N,\ \forall a \in A.$$

Let (t_n,μ_n) be a sequence in $\mathcal{O}_N$ such that $(t_n,\mu_n,V(t_n,\mu_n)) \to (t,\mu,V_*^N(t,\mu))$. It is clear that for all large n, $(t_n,\mu_n)\in B_N$. Fix an arbitrary control $\alpha\in\mathcal{A}$, and let $(X_s^{t_n,\mu_n,\alpha})_{s\in[t_n,T]}$ denote the solution to (2.1) with distribution μ_n at the initial time t_n. For ease of notation, we set $\mu_s^{n,\alpha} := \mathcal{L}(X_s^{t_n,\mu_n,\alpha})$. Consider the deterministic times

$$\theta_n := \inf\{s \ge t_n : (s,\mu_s^{n,\alpha}) \notin B_N\} \wedge T.$$

By (6.3),

$$\begin{aligned}\varphi(t_n,\mu_n) &= \varphi(\theta_n,\mu_{\theta_n}^{n,\alpha}) - \int_{t_n}^{\theta_n}\left[\partial_t\varphi(s,\mu_s^{n,\alpha}) + \langle \mu_s^{n,\alpha}, \mathcal{L}_s^{\alpha_s,\mu_s^{n,\alpha}}[D_m\varphi]\rangle\right]\mathrm{d}s\\ &\le \varphi(\theta_n,\mu_{\theta_n}^{n,\alpha}) + \int_{t_n}^{\theta_n} L(s,\mu_s^{n,\alpha},\alpha)\,\mathrm{d}s.\end{aligned}$$

Since $\overline{\mathcal{O}}_N\setminus B_N = \overline{\mathcal{O}}_N\setminus B$ is compact and $V_*^N-\varphi$ has a strict minimum at (t,μ), there exists $\eta>0$, independent of α such that $\varphi\le V_*^N-\eta\le V-\eta$ on $\overline{\mathcal{O}}_N\setminus B$. Hence, the above inequality implies that

$$\varphi(t_n,\mu_n) \le V(\theta_n,\mu_{\theta_n}^{n,\alpha}) + \int_{t_n}^{\theta_n} L(s,\mu_s^{n,\alpha},\alpha)\,\mathrm{d}s - \eta.$$

Since the $(\varphi - V)(t_n,\mu_n)\to 0$, for n large enough,

$$V(t_n,\mu_n) \le \int_{t_n}^{\theta_n} L(s,\mu_s^{n,\alpha},\alpha)\,\mathrm{d}s + V(\theta_n,\mu_{\theta_n}^{n,\alpha}) - \frac{\eta}{2}.$$

As the above inequality holds with $\eta>0$ independent of $\alpha\in\mathcal{A}$, it is in contradiction with (4.1). Hence, V_*^N is a viscosity supersolution to (4.2).

Step 3. $V_N^* = G$ *on* $\mathcal{M}_{Ne^{K^*T}}$. Consider a sequence $\overline{\mathcal{O}}_N \ni (t_n,\mu_n)\to(T,\mu)$ such that $V_N^*(T,\mu) = \lim_{n\to\infty} V(t_n,\mu_n)$. By assumption (H4), the uniform continuity of L_1 implies $\int_{t_n}^T L_1(s,\mu_s^{n,\alpha},\alpha_s)\to 0$. Also, by Lemma 6.8, the integral $\int_{t_n}^T L_2(\alpha_s)\langle\mu_s^{n,\alpha},L_3\rangle \le \tilde{C}(T-t_n)$ converges to zero. We next show that $\mu_T^{n,\alpha}\to\mu$. By the compactness of $\overline{\mathcal{O}}_N$, there exists $\hat{\mu}\in\mathcal{M}_N$ such that $\mu_T^{n,\alpha}\to\hat{\mu}$ (up to a subsequence). Itô's formula and Lemma 6.8 imply that $|\langle\mu_T^{n,\alpha}-\mu_n,x^j\rangle|\to 0$ for every $j\in\mathbb{N}$. This implies that $\hat{\mu}=\mu$. Hence, for an arbitrary $\alpha\in\mathcal{A}$, we have,

$$V_N^*(T,\mu) = \lim_{n\to\infty} V(t_n,\mu_n) \le \lim_{n\to\infty}\left[\int_{t_n}^T L(s,\mu_s^{n,\alpha},\alpha_s) + G(\mu_T^{n,\alpha})\right] = G(\mu).$$

As $V_N^*(T,\mu)\ge V(T,\mu) = G(\mu)$, we conclude that $V_N^*(T,\mu) = G(\mu)$.

Step 4. $V_*^N = G$ *on* $\mathcal{M}_{Ne^{K^*T}}$. Again consider $\overline{\mathcal{O}}_N\ni(t_n,\mu_n)\to(T,\mu)$ satisfying $V_*^N(T,\mu) = \lim_{n\to\infty}V(t_n,\mu_n)$. As in the previous step $\int_{t_n}^T L(s,\mu_s^{\alpha,n},\alpha_s)\to 0$ uniformly in α and $G(\mu_T^{n,\alpha})\to G(\mu)$ as $n\to\infty$. For any $n\in\mathbb{N}$, choose $\alpha^n\in\mathcal{A}$ so that $V(t_n,\mu_n)\ge\int_{t_n}^T L(s,\mu_s^{n,\alpha^n},\alpha_s^n) + G(\mu_T^{n,\alpha^n}) - 1/n$. This implies that

$$V_*^N(T,\mu) = \lim_{n\to\infty} V(t_n,\mu_n) \ge \lim_{n\to\infty}\left[\int_{t_n}^T L(s,\mu_s^{n,\alpha^n},\alpha_s^n) + G(\mu_T^{n,\alpha^n})\right] = G(\mu). \qquad \square$$

8. A comparison result. The following is the main comparison result.

THEOREM 8.1 (comparison). *Let u be a u.s.c. subsolution to* (4.2) *on $\mathcal{O}_N$ and v an l.s.c. supersolution to* (4.2) *on $\mathcal{O}_N$, satisfying $u(T,\mu) \le v(T,\mu)$ for any $(T,\mu) \in \overline{\mathcal{O}}_N$. Then $u \le v$ on $\overline{\mathcal{O}}_N$.*

The following corollary is the unique characterization of the value function. Recall, for any function u, we use the notation u^* to denote the upper semicontinuous envelope of u restricted to $\overline{\mathcal{O}}$ and u_* is the lower semicontinuous envelope of u restricted to $\overline{\mathcal{O}}$.

COROLLARY 8.2. *The value function V is the unique viscosity solution to* (4.2) *on $\mathcal{O}$ satisfying $V^*(T,\mu) = V_*(T,\mu) = G$ for $(T,\mu) \in \overline{\mathcal{O}}$. Moreover, V restricted to $\overline{\mathcal{O}}$ is continuous, i.e., $V^* = V_*$.*

Proof. We apply the above comparison result to V_N^*, V_*^N and use Theorem 7.3 to conclude that the subsolution V_N^* is less than the supersolution V_*^N. Since the opposite inequality is immediate from their definitions, $V_N^* = V_*^N =: V_N$. In view of Lemma B.1 proved in the appendix, this implies that $V^* = V_* = V$.

Let v be a viscosity solution to (4.2) and $v^*(T,\mu) = v_*(T,\mu) = G$ for $(T,\mu) \in \overline{\mathcal{O}}$. Since $v_* \le v_*^N \le v_N^* \le v^*$, we also have $v_N^*(T,\mu) = v_*^N(T,\mu) = G$ for $(T,\mu) \in \overline{\mathcal{O}}_N$. Then, the comparison result implies that $v_N^* \le V_*^N = V_N = V_N^* \le v_*^N \le v_N^*$. Hence, $v_N^* = v_*^N = V_N$. This proves the uniqueness. ∎

The remainder of this section is devoted to the proof of Theorem 8.1. We begin by constructing a specific class of polynomials that is central to the comparison proof. Recall that, for any polynomial f, $\deg(f)$ is the degree of f.

DEFINITION 8.3. *We say that a set of polynomials χ has the* $(*)$-property *if it satisfies*

1. *for any $g \in \chi$, $g^{(i)} \in \chi$ for all $i = 0, \dots, \deg(g)$;*
2. *for any $g \in \chi$, $\sum_{i=1}^{\deg(g)} m_i g^{(i)} \in \chi$ with $m_i := \frac{1}{i!}\int_{\mathbb{R}} y^i \gamma(dy)$.*

Let Σ be the collection of all sets of polynomials that have the $(*)$-property.

Set
$$\chi(f) := \bigcap_{\chi \in \Sigma, f \in \chi} \chi.$$
One can directly show that $\chi(f)$ has the $(*)$-property, and hence it is the smallest set of polynomials with the $(*)$-property that includes f. It is also clear that for every $g \in \chi(f)$, $\chi(g) \subset \chi(f)$.

Example 8.4. The following are a few examples of the above sets.
$$\begin{aligned}
\chi(x) &= \{0, 1, m_1, x\},\\
\chi(x^2) &= \{0, 2, 2m_1, 2m_1^2, 2x, 2m_1 x + 2m_2, x^2\},\\
\chi(x^3) &= \{0, 6, 6m_1, 6m_1^2, 6m_1^3, 6x, 6m_1 x + 6m_2, 6m_1^2 x + 12 m_1 m_2, 3x^2,\\
&\qquad 3m_1 x^2 + 6m_2 x + 6m_3, x^3\}.
\end{aligned}$$

LEMMA 8.5. *For any polynomial f, $\chi(f)$ is finite.*

Proof. We show this by induction on the degree of the polynomial. Indeed if $\deg(f) = 0$, $\chi(f) = \{f, 0\}$ and hence is finite. Towards an induction proof, assume that we have shown that $\chi(h)$ is finite for every polynomial h with $\deg h \le n$ for some integer $n \ge 0$. Let f be a polynomial with $\deg(f) = n+1$. Set $\hat{g} := \sum_{i=1}^{\deg(f)} m_i f^{(i)}$.

Then, $\deg(\hat{g}) = n$, and consequently by our assumption $\chi(\hat{g})$ is finite. Moreover,

$$\chi(f) = \{f\} \cup \chi(\hat{g}) \cup \bigcup_{i=1}^{\deg(f)} \chi(f^{(i)}).$$

As $\deg(f^{(i)}) \le n$ for every $i \ge 1$, $\chi(f^{(i)})$ are finite by the induction hypothesis, and therefore, $\chi(f)$ is also finite. ∎

Set $\Theta := \cup_{j=1}^{\infty}\chi(x^j)$. Then, Θ contains all monomials $\{x^j\}_{j=1}^{\infty}$, it is countable, and $\chi(f) \subset \Theta$ for every $f \in \Theta$. Let $\{f_j\}_{j=1}^{\infty}$ be an enumeration of Θ.

The definition of $\mathcal{M}_b$ and Lemma 6.8 imply that

$$s_j(b) := 1 + \sup_{\mu \in \mathcal{M}_b} \langle \mu, f_j \rangle^2 < \infty \quad \forall j = 1, 2, \dots .$$

As $\chi(f) \subset \Theta$ for every $f \in \Theta$, there exists a finite index set I_j satisfying

$$\chi(f_j) = \left\{ f_i \mid i \in I_j \right\}, \quad j = 1, 2, \dots$$

Fix $b > 0$, and set

$$c_j(b) := \left(\sum_{k \in I_j} 2^k \right)^{-1} \left(\sum_{k \in I_j} s_k(b) \right)^{-2}. \tag{8.1}$$

Since $f_j \in \chi(f_j)$, $j \in I_j$, and therefore, $c_j(b) \le 2^{-j}$. Hence, $\sum_{j=1}^{\infty} c_j(b) \le 1$. Also, for each $i \in I_j$, $f_i \in \chi(f_j)$, and consequently, $\chi(f_i) \subset \chi(f_j)$. This implies that $I_i \subset I_j$. Moreover, $s_j(b) \ge 1$. Hence, the definition (8.1) implies that

$$c_j(b) \le c_i(b) \qquad \forall i \in I_j. \tag{8.2}$$

Finally, observe that, by the definitions of $s_j(b)$ and $c_j(b)$,

$$\sum_{j=1}^{\infty} c_j(b) \langle \mu, f_j \rangle^2 \le 1 \quad \forall \mu \in \mathcal{M}_b. \tag{8.3}$$

Proof of Theorem 8.1. Fix $N \in \mathbb{N}$.

To simplify the notation we write c_j for $c_j(Ne^{K^*T})$. In particular, for any $(t, \mu) \in \overline{\mathcal{O}}_N$, $\mu \in \mathcal{M}_{Ne^{K^*t}} \subset \mathcal{M}_{Ne^{K^*T}}$, and therefore, by (8.2)

$$\sup_{(t,\mu)\in\overline{\mathcal{O}}_N} \sum_{j=1}^{\infty} c_j \langle \mu, f_j \rangle^2 \le 1.$$

Towards a counterposition, suppose that $\sup_{\overline{\mathcal{O}}_N}(u - v) > 0$. Since $u - v$ is u.s.c. and $\overline{\mathcal{O}}_N$ is weak* compact, the maximum

$$\ell := \max_{(t,\mu)\in\overline{\mathcal{O}}_N} \big((u - v)(t, \mu) - 2\eta(T - t) \big)$$

is achieved and $\ell > 0$ for all sufficiently small $\eta \in (0, \eta_0]$.

Step 1. *Doubling of variables.* Recall $\Theta = \{f_j\}_{j=1}^{\infty}$ and the constants $\{c_j\}$ in (8.1) with $b = Ne^{K^*T}$. For $n \in \mathbb{N}$, $\varepsilon > 0$, $\eta \in (0, \eta_0]$ set

$$\phi_\varepsilon(t, \mu, s, \nu) := u(t, \mu) - v(s, \nu) - \frac{1}{\varepsilon} \sum_{j=1}^{\infty} c_j \langle \mu - \nu, f_j \rangle^2 - \beta_{\eta,\varepsilon}(t, s),$$

$$\beta_{\eta,\varepsilon}(t, s) := \eta(T - t + T - s) + \frac{1}{\varepsilon}(t - s)^2.$$

By our assumptions, ϕ_ε admits a maximizer $(t^*_\varepsilon, \mu^*_\varepsilon, s^*_\varepsilon, \nu^*_\varepsilon)$ satisfying

$$\phi_\varepsilon(t^*_\varepsilon, \mu^*_\varepsilon, s^*_\varepsilon, \nu^*_\varepsilon) = \max_{\overline{\mathcal{O}}_N} \phi_\varepsilon \geq \ell > 0. \tag{8.4}$$

Since $\overline{\mathcal{O}}_N$ is compact and u is u.s.c., $M := \max_{\overline{\mathcal{O}}_N} u \in \mathbb{R}$. As v is l.s.c., similarly $m := \min_{\overline{\mathcal{O}}_N} v \in \mathbb{R}$. In view of (8.4),

$$0 \leq \frac{\zeta_\epsilon}{\varepsilon} := \frac{1}{\varepsilon}\left[\sum_{j=1}^\infty c_j \langle \mu^*_\varepsilon - \nu^*_\varepsilon, f_j\rangle^2 + (t^*_\varepsilon - s^*_\varepsilon)^2\right] \leq M - m - \ell =: C < \infty.$$

As $\overline{\mathcal{O}}_N$ is compact, there exist subsequences $\{(t^*_{\varepsilon_i}, \mu^*_{\varepsilon_i}), (s^*_{\varepsilon_i}, \nu^*_{\varepsilon_i})\}_{i\in\mathbb{N}}$ such that $\mu^*_{\varepsilon_i}$ and $\nu^*_{\varepsilon_i}$ converge to μ^* and ν^*, respectively, and $t^*_{\varepsilon_i}$ and $s^*_{\varepsilon_i}$ both converge to t^*.

Step 2. $\nu^* = \mu^*$. Since ζ_ϵ converges to zero, $\langle \mu^*_\varepsilon - \nu^*_\varepsilon, f_j\rangle$ converges to zero for each j. As $\Theta = \{f_j\}_{j=1}^\infty$ contains all the monomials, $\lim_{\varepsilon\to 0}\langle \mu^*_\varepsilon - \nu^*_\varepsilon, x^j\rangle = 0$ for any $j \in \mathbb{N}$. In view of Lemma 6.8, the map $\mu \mapsto \langle \mu, x^j\rangle$ is continuous on $\overline{\mathcal{O}}_N$. Hence,

$$\langle \mu^* - \nu^*, x^j\rangle = \lim_i \langle \mu^*_{\varepsilon_i} - \nu^*_{\varepsilon_i}, x^j\rangle = 0, \quad j = 1, 2, \ldots$$

By (5.2), we conclude that $\nu^* = \mu^*$.

Step 3. $t^* < T$. Towards a counterposition, assume that $t^* = T$. Since by hypothesis $(u - v)(T, \cdot) \leq 0$, v is l.s.c., and u is u.s.c.,

$$\begin{aligned} 0 \geq (u - v)(T, \mu^*) &\geq \limsup_i u(t_{\varepsilon_i}, \mu^*_{\varepsilon_i}) - v(s_{\varepsilon_i}, \nu^*_{\varepsilon_i}) \\ &\geq \limsup_i \phi_{\varepsilon_i}(t^*_{\varepsilon_i}, \mu^*_{\varepsilon_i}, s^*_{\varepsilon_i}, \nu^*_{\varepsilon_i}) \geq \ell > 0. \end{aligned}$$

Step 4. *We claim that* $\limsup_{i\to\infty} \frac{\zeta_{\varepsilon_i}}{\varepsilon_i} = 0$. Indeed,

$$\begin{aligned} \ell &\geq \phi_\epsilon(t^*, \mu^*, t^*, \mu^*) \\ &= u(t^*, \mu^*) - v(t^*, \mu^*) - 2\eta(T - t^*) \\ &\geq \limsup_{i\to\infty} \left(u(t^*_{\varepsilon_i}, \mu^*_{\varepsilon_i}) - v(s^*_{\varepsilon_i}, \nu^*_{\varepsilon_i}) - \eta(T - t^*_{\varepsilon_i} + T - s^*_{\varepsilon_i})\right) \\ &\geq \ell + \limsup_{i\to\infty} \frac{1}{\varepsilon_i}\left(\sum_{j=1}^\infty c_j\langle \mu^*_{\varepsilon_i} - \nu^*_{\varepsilon_i}, f_j\rangle^2 + (t^*_{\varepsilon_i} - s^*_{\varepsilon_i})^2\right) \\ &= \ell + \limsup_{i\to\infty} \frac{\zeta_{\varepsilon_i}}{\varepsilon_i}. \end{aligned}$$

Hence we conclude that

$$\limsup_{i\to\infty} \frac{\zeta_{\varepsilon_i}}{\varepsilon_i} = 0. \tag{8.5}$$

Step 5. *Initial estimate.* Let $\{\mu^*_\varepsilon\}$, $\{\nu^*_\varepsilon\}$ as in Step 1, and set

$$\pi^*_\varepsilon(\cdot) := \frac{2}{\varepsilon}\sum_{j=1}^\infty c_j\langle \mu^*_\varepsilon - \nu^*_\varepsilon, f_j\rangle f_j(\cdot).$$

Note that

$$\pi^*_\varepsilon(\cdot) = D_m\varphi_1(\mu^*_\varepsilon, \cdot) = -D_m\varphi_2(\nu^*_\varepsilon, \cdot),$$

where $\varphi_1(\mu) := \frac{1}{\varepsilon}\sum_{j=1}^{\infty} c_j\langle \mu - \nu_\varepsilon^*, f_j\rangle^2$, respectively, $\varphi_2(\mu) := \frac{1}{\varepsilon}\sum_{j=1}^{\infty} c_j\langle \mu_\varepsilon^* - \mu, f_j\rangle^2$. One can directly verify that φ_1 and φ_2 are test functions on $\overline{\mathcal{O}}_N$, i.e., $\varphi_1, \varphi_2 \in \Phi_{\overline{\mathcal{O}}_N}$. We thus have

$$(\partial_t\beta_{\eta,\varepsilon}(t_\varepsilon^*, s_\varepsilon^*), \pi_\varepsilon^*) \in J^{1,+}u(t_\varepsilon^*, \mu_\varepsilon^*), \quad (-\partial_s\beta_{\eta,\varepsilon}(t_\varepsilon^*, s_\varepsilon^*), \pi_\varepsilon^*) \in J^{1,-}v(s_\varepsilon^*, \nu_\varepsilon^*).$$

Then, by the viscosity properties of u and v,

$$-\partial_t\beta_{\eta,\varepsilon}(t_\varepsilon^*, s_\varepsilon^*) + H(t_\varepsilon^*, \mu_\varepsilon^*, \pi_\varepsilon^*) \le 0, \quad \partial_s\beta_{\eta,\varepsilon}(t_\varepsilon^*, s_\varepsilon^*) + H(s_\varepsilon^*, \nu_\varepsilon^*, \pi_\varepsilon^*) \ge 0.$$

We combine and use the definition of $\beta_{\eta,\varepsilon}$ to arrive at

$$\begin{aligned}0 < 2\eta &\le H(s_\varepsilon^*, \nu_\varepsilon^*, \pi_\varepsilon^*) - H(t_\varepsilon^*, \mu_\varepsilon^*, \pi_\varepsilon^*)\\ &= \sup_{a\in A} H^a(s_\varepsilon^*, \nu_\varepsilon^*, \pi_\varepsilon^*) - \sup_{a\in A} H^a(t_\varepsilon^*, \mu_\varepsilon^*, \pi_\varepsilon^*)\\ &\le \sup_{a\in A}(H^a(s_\varepsilon^*, \nu_\varepsilon^*, \pi_\varepsilon^*) - H^a(t_\varepsilon^*, \mu_\varepsilon^*, \pi_\varepsilon^*)) =: \sup_{a\in A} I^a.\end{aligned}$$

Moreover,

$$\begin{aligned}I^a &:= L(t_\varepsilon^*, \mu_\varepsilon^*, a) - L(s_\varepsilon^*, \nu_\varepsilon^*, a) + \langle \mu_\varepsilon^*, \mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}[\pi_\varepsilon^*]\rangle - \langle \nu_\varepsilon^*, \mathcal{L}_{s_\varepsilon^*}^{a,\nu_\varepsilon^*}[\pi_\varepsilon^*]\rangle\\ &= \underbrace{L(t_\varepsilon^*, \mu_\varepsilon^*, a) - L(s_\varepsilon^*, \nu_\varepsilon^*, a)}_{I_1^a} + \underbrace{\langle \mu_\varepsilon^* - \nu_\varepsilon^*, \mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}[\pi_\varepsilon^*]\rangle}_{I_2^a} + \underbrace{\langle \nu_\varepsilon^*, \mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}[\pi_\varepsilon^*] - \mathcal{L}_{s_\varepsilon^*}^{a,\nu_\varepsilon^*}[\pi_\varepsilon^*]\rangle}_{I_3^a}.\end{aligned}$$

By assumption (H4) and Lemma 6.8, $\lim_{\varepsilon\to 0}\sup_{a\in A} I_1^a \to 0$.

Step 6. *Estimate of* I_2. We rewrite the second term as

$$I_2 := \sup_{a\in A} I_2^a \le \sup_{a\in A}\frac{2}{\varepsilon}\sum_{j=1}^{\infty} c_j|\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle\langle \mu_\varepsilon^* - \nu_\varepsilon^*, \mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}[f_j]\rangle| \le I_2^b + I_2^\sigma + I_2^\gamma,$$

related to the three terms appearing in the generator $\mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}$, which appear explicitly below.

By construction, for every $j \in \mathbb{N}$, there exists an index $k_1(j)$ such that $f_j' = f_{k_1(j)}$. Also, as $f_j' = f_{k_1(j)} \in \chi(f_j)$, $\chi(f_{k_1(j)}) \subset \chi(f_j)$, and consequently, $I_{k_1(j)} \subset I_j$. Therefore, the definition (8.1) yields that $c_j \le c_{k_1(j)}$. We now directly estimate using these and (H1) to obtain

$$\begin{aligned}I_2^b &= \sup_{a\in A}\frac{2}{\varepsilon}\sum_{j=1}^{\infty} c_j|\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle\langle \mu_\varepsilon^* - \nu_\varepsilon^*, b(t_\varepsilon^*, \mu_\varepsilon^*, a)f_j'\rangle|\\ &\le C\frac{2}{\varepsilon}\sum_{j=1}^{\infty} c_j|\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j'\rangle|\\ &\le C\frac{2}{\varepsilon}\left(\sum_{j=1}^{\infty} c_j\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2 + \sum_{j=1}^{\infty} c_{k_1(j)}\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_{k_1(j)}\rangle^2\right)\\ &\le C\frac{4}{\varepsilon}\sum_{j=1}^{\infty} c_j\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2,\end{aligned}$$

which converges to 0 by (8.5).

We estimate I_2^σ similarly. Indeed, for every $j \in \mathbb{N}$, there exists an index $k_2(j)$ such that $f_j'' = f_{k_2(j)}$ and $c_j \le c_{k_2(j)}$. Then,

$$\begin{aligned}
I_2^\sigma &= \sup_{a\in A} \frac{2}{\varepsilon} \sum_{j=1}^\infty c_j |\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle \langle \mu_\varepsilon^* - \nu_\varepsilon^*, \sigma(t_\varepsilon^*, \mu_\varepsilon^*, a) f_j''\rangle| \\
&\le C\frac{2}{\varepsilon} \sum_{j=1}^\infty c_j |\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j''\rangle| \\
&\le C\frac{2}{\varepsilon} \left(\sum_{j=1}^\infty c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2 + \sum_{j=1}^\infty c_{k_2(j)} \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_{k_2(j)}\rangle^2 \right),
\end{aligned}$$

which also converges to 0 by (8.5).

We analyze I_2^γ next. By the Taylor expansion of f_j,

$$\begin{aligned}
g_j(x) &:= \int_{\mathbb{R}} [f_j(x+y) - f_j(x)]\gamma(dy) \\
&= \sum_{i=1}^{\deg(f_j)} \frac{f_j^{(i)}(x)}{i!} \int_{\mathbb{R}} y^i \gamma(dy) = \sum_{i=1}^{\deg(f_j)} m_i f_j^{(i)}(x).
\end{aligned}$$

Again, by the construction of $\{f_j\}$, for all $j \in \mathbb{N}$, there exists $k_\lambda(j)$ such that $g_j = f_{k_\lambda(j)}$ and $c_j \le c_{k_\lambda(j)}$. Hence,

$$\begin{aligned}
I_2^\lambda &= \sup_{a\in A} \frac{2}{\varepsilon} \sum_{j=1}^\infty c_j |\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle \langle \mu_\varepsilon^* - \nu_\varepsilon^*, \lambda(t_\varepsilon^*, \mu_\varepsilon^*, a) g_j\rangle| \\
&\le C\frac{2}{\varepsilon} \sum_{j=1}^\infty c_j |\langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle \langle \mu_\varepsilon^* - \nu_\varepsilon^*, g_j\rangle| \\
&\le C\frac{2}{\varepsilon} \left(\sum_{j=1}^\infty c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2 + \sum_{j=1}^\infty c_{k_\lambda(j)} \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_{k_\lambda(j)}\rangle^2 \right).
\end{aligned}$$

As this quantity also vanishes as $\varepsilon \to 0$, we conclude that $I_2 \to 0$ as ε goes to zero.

Step 7. *Estimating* I_3. As in the previous step, we write

$$I_3 = \sup_{a\in A} \langle \nu_\varepsilon^*, \mathcal{L}_{t_\varepsilon^*}^{a,\mu_\varepsilon^*}[\pi_\varepsilon^*] - \mathcal{L}_{s_\varepsilon^*}^{a,\nu_\varepsilon^*}[\pi_\varepsilon^*]\rangle \le I_3^b + I_3^\sigma + I_3^\gamma$$

related to the three terms appearing in the generator. Since the estimates of each term are very similar to each other, we provide the details of only the first one.

By (H2), there exists C_1 such that

$$\big(b(t_\varepsilon^*, \mu_\varepsilon^*, a) - b(s_\varepsilon^*, \nu_\varepsilon^*, a)\big)^2 \le C_1 (t_\varepsilon^* - s_\varepsilon^*)^2 + C_1 \sum_{j=1}^\infty c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2.$$

It follows that

$$\begin{aligned}
|I_3^b| &\le \sup_{a\in A} \frac{2}{\varepsilon} \sum_{j=1}^\infty c_j \left| \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle \ \langle \nu_\varepsilon^*, (b(t_\varepsilon^*, \mu_\varepsilon^*, a) - b(s_\varepsilon^*, \nu_\varepsilon^*, a)) f_j'\rangle \right| \\
&\le \frac{2}{\varepsilon} \sum_{j=1}^\infty c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2 \\
&\quad + \frac{2C_1}{\varepsilon} \left((t_\varepsilon^* - s_\varepsilon^*)^2 + \sum_{j=1}^\infty c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j\rangle^2 \right) \sum_{j=1}^\infty c_j \langle \nu_\varepsilon^*, f_j'\rangle^2.
\end{aligned}$$

Note that by (8.3),

$$\sum_{j=1}^{\infty} c_j \langle \nu_\varepsilon^*, f_j' \rangle^2 \le \sum_{j=1}^{\infty} c_j \langle \nu_\varepsilon^*, f_j \rangle^2 \le 1.$$

Hence,

$$|I_3^b| \le \frac{4C_1}{\varepsilon} \left((t_\varepsilon^* - s_\varepsilon^*)^2 + \sum_{j=1}^{\infty} c_j \langle \mu_\varepsilon^* - \nu_\varepsilon^*, f_j \rangle^2 \right).$$

In view of (8.5), we conclude that I_3^b goes to zero as $\varepsilon \to 0$. Repeating the same argument for I_3^σ and I_3^λ, we conclude that I_3 also converges to zero.

Step 8. *Conclusion* . In Step 5 we have shown that

$$0 < 2\eta \le \sup_{a \in A} I^a = I_1 + I_2 + I_3.$$

In the preceding steps we have shown that each of the three terms converges to zero as ε tends to zero. Clearly this contradicts with the fact that $\eta > 0$. ∎

Appendix A. Solutions of controlled McKean–Vlasov SDEs. For completeness, we provide here an existence result for the McKean–Vlasov SDE (2.1).

Using the functions and coefficients of section 8, we fix $b > 0$ and start by proving functional analytic properties of $\mathcal{M}_b$. Set

$$d(\mu, \nu; b) := \sum_{j=1}^{\infty} c_j(b) |\langle \mu - \nu, f_j \rangle|, \quad \mu, \nu \in \mathcal{M}_b.$$

LEMMA A.1. *A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_b$ converges weakly to $\mu \in \mathcal{M}_b$ if and only if $\lim_{n\to\infty} d(\mu_n, \mu; b) = 0$.*

Proof. As Θ contains all monomials, in view of (5.2), $d(\mu, \nu; b) = 0$ if and only if $\mu = \nu$, and one can then directly verify that d is a metric on $\mathcal{M}_b$. Moreover, since $\sum_{j=1}^{\infty} c_j(b) \le 1$, by (8.3), $d \le 1$ on $\mathcal{M}_b$. Suppose $\mu_n \to \mu$ as $n \to \infty$. By dominated convergence,

$$\lim_{n\to\infty} d(\mu_n, \mu; b) = \sum_{j=1}^{\infty} c_j(b) \lim_{n\to\infty} |\langle \mu_n - \mu, f_j \rangle| = 0,$$

where the last equality follows from Lemma 6.8. Now suppose $d(\mu_n, \mu; b) \to 0$ as $n \to \infty$. Since $\mathcal{M}_b$ is compact, the sequence $\{\mu_n\}$ has limit points, and since d is a metric, we conclude that it can only have one limit point μ. ∎

We next fix $t \in [0, T]$ and consider the space

$$\mathcal{X}_t(b) := \left\{ \bar{\mu} = (\mu_s)_{s\in[t,T]} \mid \mu_s \in \mathcal{M}_b \;\; \forall s \in [t, T] \right\}$$

and the function

$$d_T(\mu, \nu; b) = \sup_{t \le s \le T} d(\mu_s, \nu_s; b).$$

It is straightforward to see that d_T is a metric on $\mathcal{X}_t(b)$.

LEMMA A.2. *$(\mathcal{X}_t(b), d_T)$ is a complete metric space.*

Proof. Let $\{\bar{\mu}^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. In particular $\{\mu_s^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{M}_b, d)$ for any $s \in [t, T]$, and by Lemma A.1, there exists $\mu_s \in \mathcal{M}_b$ such that $\mu_s^n \to \mu_s$ as $n \to \infty$. We claim that $\bar{\mu} := (\mu_s)_{s\in[t,T]}$ is the limit of $\{\bar{\mu}^n\}_{n\in\mathbb{N}}$.

Indeed, for $\varepsilon > 0$, there is $\bar{n}$ such that $d(\mu_s^n, \mu_s^m; b) \leq \varepsilon$ for every $n, m \geq \bar{n}$ and $s \in [t, T]$. By letting m tend to infinity and by using the previous lemma, we conclude that $d(\mu_s^n, \mu_s; b) \leq \varepsilon$ for any $s \in [t, T]$. The result follows after taking the supremum over $s \in [t, T]$. ∎

This structure allows us to study the McKean–Vlasov equation (2.1). For similar results we refer to the book of Carmona and Delarue [13] and the references therein.

THEOREM A.3. *Under Assumption* 2.1, *for any* $(t, \mu) \in \mathcal{O}$ *and control* $\alpha \in \mathcal{A}$, (2.1) *with initial data* $X_t \sim \mu$ *has a unique solution.*

Proof. Fix $(t, \mu) \in \mathcal{O}$ and control $\alpha \in \mathcal{A}$. There is $N \in \mathbb{N}$ such that $\mu \in \mathcal{M}_{Ne^{K^* t}}$. Let $\mathcal{X} := \mathcal{X}_t(Ne^{K^*(T-t)})$ and $c_j := c_j((Ne^{K^*(T-t)})$.

For any $\bar{\mu} = (\mu_s)_{s\in[t,T]} \in \mathcal{X}$, set

$$X_s^{t,\bar{\mu},\alpha} := \int_t^s b(r, \mu_r, \alpha_r) dr + \int_t^s \sigma(r, \mu_r, \alpha_r) \, \mathrm{d}W_r + \sum_{t \leq r \leq s} \Delta J_r$$

with distribution μ at time t and

$$\Phi : \mathcal{X} \to \mathcal{X}, \quad \bar{\mu} \mapsto \Phi(\bar{\mu}) := (\mathcal{L}(X_s^{t,\bar{\mu},\alpha}))_{s\in[t,T]}.$$

Recall that the set $\overline{\mathcal{O}}_N$ in Lemma 5.1 is invariant for (2.1). Therefore, $\Phi(\bar{\mu}) \in \mathcal{X}$. Moreover, the law of any solution to (2.1) is a fixed point of Φ.

To simplify the notation for $\bar{\mu}, \bar{\mu}' \in \mathcal{X}$, let $\bar{\nu} = \Phi(\bar{\mu}), \bar{\nu}' = \Phi(\bar{\mu}')$. Consider now $f_j \in \Theta$. We now apply Itô's formula to arrive at

$$\begin{aligned} f_j(X_s^{t,\bar{\mu},\alpha}) = X_t^{t,\bar{\mu},\alpha} &+ \int_t^s b(r, \mu_r, \alpha_r) f_j'(X_r^{t,\bar{\mu},\alpha}) dr \\ &+ \frac{1}{2} \int_t^s \sigma^2(r, \mu_r, \alpha_r) f_j''(X_r^{t,\bar{\mu},\alpha}) dr \\ &+ \int_t^s \sigma(r, \mu_r, \alpha_r) f_j'(X_r^{t,\bar{\mu},\alpha}) \, \mathrm{d}W_r + \sum_{t \leq r \leq s} f_j(\Delta J_r). \end{aligned}$$

From assumption (H1), the stochastic integral in the above formula is a local martingale. Denote by $\{\tau_n\}_{n\in\mathbb{N}}$ a localizing sequence, and take expectation on both sides. Recalling that α is deterministic, we obtain

$$\begin{aligned} \mathbb{E}[f_j(X_{s\wedge\tau_n}^{t,\bar{\mu},\alpha})] = X_t^{t,\bar{\mu},\alpha} &+ \int_t^s b(r, \mu_r, \alpha_r) \mathbb{E}[f_j'(X_r^{t,\bar{\mu},\alpha}) 1_{t \leq r \leq \tau_n}] dr \\ &+ \frac{1}{2} \int_t^s \sigma^2(r, \mu_r, \alpha_r) \mathbb{E}[f_j''(X_r^{t,\bar{\mu},\alpha}) 1_{t \leq r \leq \tau_n}] dr + \mathbb{E}\Big[\sum_{t \leq r \leq s\wedge\tau_n} f_j(\Delta J_r)\Big]. \end{aligned}$$

By dominated convergence, the equality passes to the limit as $n \to \infty$. For ease of notation, denote $\Delta b(r) := b(r, \mu_r, \alpha_r) - b(r, \mu_r', \alpha_r)$ and similarly $\Delta\sigma^2(r)$ and $\Delta\lambda(r)$. From $\langle \nu_s, f_j \rangle = \mathbb{E}[f_j(X_s^{t,\bar{\mu},\alpha})]$, we deduce

$$\begin{aligned} \langle \nu_s - \nu_s', f_j \rangle = &\int_t^s \Delta b(r) \langle \nu, f_j' \rangle dr + \int_t^s b(r, \mu_r', \alpha_r) \langle \nu - \nu', f_j' \rangle dr \\ &+ \frac{1}{2} \int_t^s \Delta\sigma^2(r) \langle \nu, f_j'' \rangle dr + \frac{1}{2} \int_t^s \sigma^2(r, \mu_r', \alpha_r) \langle \nu - \nu', f_j'' \rangle dr \\ &+ \int_t^s \Delta\lambda(r) \langle \nu, g_j \rangle dr + \int_t^s \lambda(r, \mu_r', \alpha_r) \langle \nu - \nu', g_j \rangle dr, \end{aligned}$$

where $g_j = \sum_{i=1}^{\deg(f_j)} m_i f_j^{(i)}$ with $m_i := \frac{1}{i!}\int_{\mathbb{R}} y^i \gamma(dy)$. Recall now that the collection of coefficients $\{c_j\}_{j\in\mathbb{N}}$ satisfies (8.2), so that

$$c_j \le k_j := \min\{c_{j_1}, c_{j_2}, c_{j_g}\},$$

where c_{j_1}, c_{j_2}, and c_{j_g} are the coefficients of f_j', f_j'', and g_j, respectively. We can therefore multiply by k_j both sides of the above equality to get, using also assumption (H1) and (H2),

$$\begin{aligned} c_j|\langle \nu_s - \nu_s', f_j\rangle| &\le k_j|\langle \nu_s - \nu_s', f_j\rangle| \\ &\le \bar{C}\int_t^s d(\mu_r, \mu_r')\big(c_{j_1}|\langle \nu, f_j'\rangle| + c_{j_2}|\langle \nu, f_j''\rangle| + c_{g_j}|\langle \nu, g_j\rangle|\big)dr \\ &\quad + \int_t^s c_{j_1}|\langle \nu - \nu', f_j'\rangle| + c_{j_2}|\langle \nu - \nu', f_j''\rangle| dr + \int_t^s c_{g_j}|\langle \nu - \nu', g_j\rangle| dr \end{aligned}$$

for some constant $\bar{C}$ which depends only on the coefficients of (2.1). By summing up over $j \in \mathbb{N}$ and recalling (8.3), we obtain

$$d(\nu_s, \nu_s') \le 3\bar{C}\left(\int_t^s d(\mu_r, \mu_r') + \int_t^s d(\nu_r, \nu_r')\right).$$

Using Gronwall's lemma, we obtain

$$d_s(\Phi(\bar{\mu}), \Phi(\bar{\mu}')) \le e^{3\bar{C}s}\int_t^s d_r(\bar{\mu}, \bar{\mu}')$$

for any $t \le s \le T$. Denoting now $C(s) := e^{3\bar{C}s}$ and Φ^k the composition of k times the map Φ, it can be verified, by induction, that

$$d_T(\Phi^k(\bar{\mu}), \Phi^k(\bar{\mu}')) \le \frac{C(T)^k T^k}{k!} d_T(\bar{\mu}, \bar{\mu}').$$

For k large enough Φ^k is a contraction on $\mathcal{X}$, which is a complete metric space in view of Lemma A.2. Thus, the map Φ admits a unique fixed point. ∎

Appendix B. Semicontinuous envelopes. In this section, we show that the semicontinuous envelopes defined on $\mathcal{O}_N$ converge to the envelopes defined on $\mathcal{O}$.

LEMMA B.1. *Let (E, τ) be a topological space and $(E_N, \tau_N)_{N\in\mathbb{N}}$ a sequence of topological spaces with $(E_N)_{N\in\mathbb{N}}$ increasing to E, i.e., $\cup_{n\in\mathbb{N}} E_N = E$ and $E_N \subset E_{N+1}$ for any N. Let τ_N the subspace topology induced by τ. Denote by $u^* : E \to \mathbb{R}\cup\{\infty\}$ the upper semicontinuous envelope on (E, τ) and by $u_N^* : E_N \to \mathbb{R}\cup\{\infty\}$ the upper semicontinuous envelope on (E_N, τ_N). Then, $\lim_{N\to\infty} u_N^* = u^*$. Similarly, if u_*^N is the lower semicontinuous envelope on (E_N, τ_N), then $\lim_{N\to\infty} u_*^N = u_*$*

Proof. Consider the following representations of the semicontinuous envelopes. Let $U(\mu)$ be the collection of τ-neighborhoods of μ. Then, since E_N is endowed with the subspace topology, for any $N \in \mathbb{N}$,

$$\begin{aligned} u^*(\mu) &= \inf_{W\in U(\mu)} \sup_W u \qquad \text{for } \mu \in E, \\ u_N^*(\mu) &= \inf_{W\in U(\mu)} \sup_{W\cap E_N} u \quad \text{for } \mu \in E_N. \end{aligned}$$

Clearly $u_N^* \leq u_{N+1}^* \leq u^*$. Suppose first $u^*(\mu) < \infty$. For $W \in U(\mu)$, choose a sequence μ_n such that $\sup_W u \leq u(\mu_n) + 1/n$. Let $M : \mathbb{N} \to \mathbb{N}$ be a function such that $\mu_n \in E_{M(n)}$. Without loss of generality, we may choose M to be strictly increasing. Thus,

$$\sup_n \sup_{W \cap E_{M(n)}} u = \sup_W u.$$

Since above holds for every $W \in U(\mu)$, $\lim_{N\to\infty} u_N^*(\mu) = u^*(\mu)$. If $u^*(\mu) = \infty$, we repeat the same argument with a sequence μ_n such that $u(\mu_n) > n$. ∎

REFERENCES

[1] Y. Achdou, F. J. Buera, J.-M. Lasry, P.-L. Lions, and B. Moll, *Partial differential equation models in macroeconomics*, Philos. Trans. Roy. Soc. A, 372 (2014), p. 20130397.

[2] S. Awatif, *Equations d'hamilton-jacobi du premier ordre avec termes intégro-différentiels: Partie 1: Unicité des solutions de viscosité*, Comm. Partial Differential Equations, 16 (1991), pp. 1057–1074.

[3] E. Bandini, A. Cosso, M. Fuhrman, and H. Pham, *Randomized filtering and bellman equation in Wasserstein space for partial observation control problem*, Stochastic Process. Appl., 129 (2019), pp. 674–711.

[4] E. Bandini, A. Cosso, M. Fuhrman, and H. Pham, *Backward for optimal control of partially observed path-dependent stochastic systems: A control randomization approach*, Ann. Appl. Prbab., 28 (2018), pp. 1634–1678.

[5] G. Barles and C. Imbert, *Second-order elliptic integro-differential equations: Viscosity solutions' theory revisited*, Ann. Inst. N. Poincaré Anal. Non Linéaire, 25 (2008), pp. 567–585.

[6] A. Bensoussan, J. Frehse, and S. P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer, Cham, 2013.

[7] A. Bensoussan, J. Frehse, and S. P. Yam, *The master equation in mean field theory*, J. Math. Pures Appl (9), 103 (2015), pp. 1441–1474.

[8] A. Bensoussan, J. Frehse, and S. P. Yam, *On the interpretation of the master equation*, Stochastic Process. Appl., 127 (2017), pp. 2093–2137.

[9] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, 2013.

[10] B. Bouchard and N. Touzi, *Weak dynamic programming principle for viscosity solutions*, SIAM J. Control Optim., 49 (2011), pp. 948–962.

[11] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions, *The Master Equation and the Convergence Problem in Mean Field Games*, Princeton University Press, Princeton, NJ, 2019.

[12] P. Cardialaguet, *Notes on Mean-Field Games (from P-L. Lions' Lectures at de Collége France)*, 2013.

[13] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications* I-II, Springer, Cham, 2018.

[14] R. Carmona, F. Delarue, and A. Lachapelle, *Control of Mckean–Vlasov dynamics versus mean field games*, Math. Financ. Econ., 7 (2013), pp. 131–166.

[15] C. Cuchiero, M. Larsson, and S. Svaluto-Ferro, *Probability measure-valued polynomial diffusions*, Electron. J. Probab., 24 (2019), 30.

[16] P. Dupuis, H. Ishii, and H. M. Soner, *A viscosity solution approach to the asymptotic analysis of queueing systems*, Ann. Probab., 18 (1990), pp. 226–255.

[17] I. Ekren, N. Touzi, and J. Zhang, *Viscosity solutions of fully nonlinear parabolic path dependent PDE's: Part* I, Ann. Probab., 44 (2016), pp. 1212–1253.

[18] I. Ekren, N. Touzi, and J. Zhang, *Viscosity solutions of fully nonlinear parabolic path dependent PDE's: Part* II, Ann. Probab., 44 (2016), pp. 2507–2553.

[19] G. Fabbri, F. Gozzi, and A. Swiech, *Stochastic Optimal Control in Infinite Dimension*, Springer, Cham, 2017.

[20] W. H. Fleming and H. M. Soner, *Asymptotic expansions for Markov processes with Lévy generators*, Appl. Math. Optim., 19 (1989), pp. 203–223.

[21] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer Science & Business Media, New York, NY, 2006.

[22] W. H. Fleming and M. Viot, *Some measure-valued Markov processes in population genetics theory*, Indiana Univ. Math. J., 28 (1979), pp. 817–843.

[23] W. Gangbo, T. Nguyen, and A. Tudorascu, *Hamilton-Jacobi equations in the Wasserstein space*, Methods Appl. Anal., 15 (2008), pp. 155–184.

[24] W. Gangbo and A. Tudorascu, *On differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations*, J. Math. Pures Appl. (9), 125 (2019), pp. 119 – 174.

[25] M. Huang, P. E. Caines, and R. P. Malhamé, *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–252.

[26] M. Huang, P. E. Caines, and R. P. Malhamé, *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–252.

[27] M. Huang, P. E. Caines, and R. P. Malhamé, *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Automat. Control, 52 (2007), pp. 1560–1571.

[28] S. S. Kortum, *Research, patenting, and technological change*, Econometrica, (1997), pp. 1389–1419.

[29] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* I*–Le cas stationnaire*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 619–625.

[30] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* II*–Horizon fini et contrôle optimal*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 679–684.

[31] J.-M. Lasry and P.-L. Lions, *Mean field games*, Jpn. J. Math., 2 (2007), pp. 229–260.

[32] P. Lions, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions.* III*. Uniqueness of viscosity solutions for general second-order equations*, J. Funct. Anal., 86 (1989), pp. 1–18.

[33] P.-L. Lions, *Thèorie des jeux de champ moyen et applications*, 2006–2012, http://www.college-de-france.fr/default/EN/all/equder/audiovideo.jsp.

[34] H. Pham and X. Wei, *Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics*, SIAM J. Control Optim., 55 (2017), pp. 1069–1101.

[35] H. Pham and X. Wei, *Bellman equation and viscosity solutions for mean-field stochastic control problem*, ESAIM Control Optim. Calc. Var. 24 (2018), pp. 437–461.

[36] H. M. Soner, *Optimal control of jump-Markov processes and viscosity solutions*, in Stochastic differential Systems, Stochastic Control Theory and Applications, W. Fleming and P.-L. Lions, eds., Springer, Cham, 1988, pp. 501–511.

[37] C. Wu and J. Zhang, *Viscosity Solutions to Parabolic Master Equations and McKean-Vlasov SDEs with Closed-Loop Controls*, arXiv preprint, 2019, https://arxiv.org/abs/1805.02639.