TRIPLE PRODUCT p-ADIC L-FUNCTIONS FOR BALANCED WEIGHTS

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Abstract. We construct p-adic triple product L-functions that interpolate (square roots of) central critical L-values in the balanced region. Thus, our construction complements that of M. Harris and J. Tilouine.

There are four central critical regions for the triple product L-functions and two opposite settings, according to the sign of the functional equation. In the first case, three of these regions are of interpolation, having positive sign; they are called the unbalanced regions and one gets three p-adic L-functions, one for each region of interpolation (this is the Harris-Tilouine setting). In the other setting there is only one region of interpolation, called the balanced region: we produce the corresponding p-adic L-function. Our triple product p-adic L-function arises as p-adic period integrals interpolating normalizations of the local archimedean period integrals. The latter encode information about classical representation theoretic branching laws. The main step in our construction of p-adic period integrals is showing that these branching laws vary in a p-adic analytic fashion. This relies crucially on the Ash-Stevens theory of highest weight representations over affinoid algebras.

Contents

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1. Introduction

Consider three finite slope cuspidal p-adic Coleman eigenfamilies $\underline{\mathbf{f}} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ new of tame levels (N_1, N_2, N_3), nebetypes $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and eigenvalues $(\alpha_1, \alpha_2, \alpha_3)$ for the U_p-operator of slope (h_1, h_2, h_3), parametrized by the product of three connected affinoid subdomains $\underline{U} = U_1 \times U_2 \times U_3 \subset \mathcal{X}^3$, where $\mathcal{X}$ denotes the weight space and we suppose these rigid analytic objects to be defined over a p-adic field F. By an integer point we mean $\underline{k} \in \mathbb{N}^3 \cap \underline{U}$, say $\underline{k} = (k_1, k_2, k_3)$, such that $k_i > h_i - 1$. We also call k_i an integer point. If $\underline{k}$ is an integer point, we write $\mathbf{f}_{\underline{k}} = (\mathbf{f}_{1,k_1}, \mathbf{f}_{2,k_2}, \mathbf{f}_{3,k_3})$ for the specialization of $\underline{\mathbf{f}}$, which is a triple of classical modular forms with $\mathbf{f}_{i,k_i}$ of weight $k_i + 2$. Because $\alpha_{i,k_i} = \pm p^{k_i/2}$ if $\mathbf{f}_{i,k_i}$ is p-new, for every $k_i \neq 2h_i$ we know that $\mathbf{f}_{i,k_i} = f_{i,k_i}$ is old at p and it is the p-stabilization of some newform

$$\mathbf{f}_{i,k_i}^{\#} = f_{i,k_i}^{\#} \in S_{k_i+2}\left(\Gamma_0\left(N_i\right), \varepsilon_i\right)^{N_i-\mathrm{new}}.$$

We say that k_i is a generic integer point in this case and we say that $\underline{k}$ is a generic integer point if k_i is such a point for $i = 1, 2, 3$. We refer the reader to Remark 8.1 below for more details.

The problem we are interested in is about interpolating the function

$$\underline{k} := (k_1, k_2, k_3) \mapsto L\left(f_{1,k_1}^{\#} \times f_{2,k_2}^{\#} \times f_{3,k_3}^{\#}, c_{\underline{k}}\right)$$

for the central critical value $c_{\underline{k}} := \frac{k_1+k_2+k_3+4}{2}$. Here $L\left(f_{1,k_1}^{\#} \times f_{2,k_2}^{\#} \times f_{3,k_3}^{\#}, s\right)$ is the triple product complex L-function (see for example [24, §1] for its definition). Let us write $\pi_i := \pi_{f_{i,k_i}}$ for the automorphic representation attached to $f_i := f_{i,k_i}^{\#}$. If we want $\Pi_{\underline{k}} := \pi_1 \otimes \pi_2 \otimes \pi_3$ to be selfdual, the following condition $(CC)_{k_1,k_2,k_3}$ needs to be imposed:

$$(CC)_{k_1,k_2,k_3} : \varepsilon_1\varepsilon_2\varepsilon_3 = 1, \text{ so that } k_1 + k_2 + k_3 \in 2\mathbb{N}.$$

There are four central critical regions, namely

$$\begin{aligned}
\Sigma_1 &:= \left\{(k_1, k_2, k_3) : k_1 > k_2 + k_3 \text{ and } (CC)_{k_1,k_2,k_3} \text{ holds}\right\}, \\
\Sigma_2 &:= \left\{(k_1, k_2, k_3) : k_2 > k_1 + k_3 \text{ and } (CC)_{k_1,k_2,k_3} \text{ holds}\right\}, \\
\Sigma_3 &:= \left\{(k_1, k_2, k_3) : k_3 > k_1 + k_2 \text{ and } (CC)_{k_1,k_2,k_3} \text{ holds}\right\}, \\
\Sigma_{123} &:= \left\{(k_1, k_2, k_3) : k_1 \leq k_2 + k_3, k_2 \leq k_1 + k_3, k_3 \leq k_1 + k_2 \text{ and } (CC)_{k_1,k_2,k_3} \text{ holds}\right\}.
\end{aligned}$$

The transcendental nature of the Deligne's period Ω depends on the critical region. We have, up to powers of π,

$$\Omega = (f_i, f_i)^2 \text{ on } \Sigma_i \text{ and } \Omega = (f_1, f_1)(f_2, f_2)(f_3, f_3) \text{ on } \Sigma_{123}.$$

Here $(f, g) = (f, g)_k$ is the Petersson inner product, that we normalized by the volume of the corresponding modular curve:

$$(f, g)_k := \frac{1}{\mu\left(\mathcal{H}/\Gamma_0(N)\right)} \int_{\mathcal{H}/\Gamma_0(N)} f(z)\, \overline{g}(z) \operatorname{Im}(z)^k \mu(z), \; \mu = \frac{dxdy}{y^2}. \tag{1}$$

Let S_i be the set of places such that $\pi_{i,v}$ admits a Jacquet-Langlands lift $\pi_{i,v}^D$ to the group of units of the division quaternion $\mathbb{Q}_v$-algebra. Set $S := S_1 \cap S_2 \cap S_3$ and, for every $v \in S$, let d_v (resp. d_v^D) be the dimension of the space of trilinear forms on $\pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$ (resp. $\pi_{1,v}^D \otimes \pi_{2,v}^D \otimes \pi_{3,v}^D$). Define, for every $v \in S$,

$$\varepsilon_v\left(f_1 \times f_2 \times f_3\right) = \begin{cases} 1 & \text{if } d_v = 1 \text{ and } d_v^D = 0 \\ -1 & \text{if } d_v = 0 \text{ and } d_v^D = 1. \end{cases}$$

It is a theorem of Prasad (see [34]) that the above function is indeed well defined, i.e. only one of the above two possibilities occurs. Write $S = S^+ \sqcup S^-$, where $S^{\pm} := \{v : \varepsilon_v(f_1 \times f_2 \times f_3) = \pm 1\}$, set $D = D_{JL}^- := \prod_{l \in S^- - \{\infty\}} l$ and let $B_D = B_{\Pi_{\underline{k}}}$ be the quaternion algebra ramified at the finite primes dividing D. If Π is an irreducible cuspidal automorphic representation of $\mathbf{GL}_2^3$, we let B_Π be the quaternion algebra obtained by the above recipe and say that it is the one predicted by [34]. Recalling the dependence of these considerations from the weight, so that $S^- = S^-_{\underline{k}}$ (resp. $D = D_{\underline{k}}$), the sign of the function equation at $\underline{k}$ is given by the formula

$$\varepsilon\left(\mathbf{f}^\#_{\underline{k}}\right) = \prod_{v\in S}\varepsilon_v\left(\mathbf{f}^\#_{\underline{k}}\right) := \prod_{v\in S}\varepsilon_v\left(f_1\times f_2\times f_3\right) = (-1)^{\# S^-_{\underline{k}}}.$$

Let $\varepsilon_{\text{fin}}\left(\mathbf{f}^\#_{\underline{k}}\right)$ be the product of the finite local signs, so that $\varepsilon\left(\mathbf{f}^\#_{\underline{k}}\right) = \varepsilon_{\text{fin}}\left(\mathbf{f}^\#_{\underline{k}}\right)\varepsilon_\infty\left(\mathbf{f}^\#_{\underline{k}}\right)$. We remark that the nature of the local sign at infinity depends on the critical region: we have $\varepsilon_\infty\left(\mathbf{f}^\#_{\underline{k}}\right) = 1$ if $\underline{k}\in\Sigma_i$ for $i = 1$, 2 or 3, while $\varepsilon_\infty\left(\mathbf{f}^\#_{\underline{k}}\right) = -1$ if $\underline{k}\in\Sigma_{123}$. Let us assume that N_i is squarefree for $i = 1,2,3$. It is easy to see that $\varepsilon_{\text{fin}}\left(\mathbf{f}^\#_{\underline{k}}\right)\equiv 1$ or $\varepsilon_{\text{fin}}\left(\mathbf{f}^\#_{\underline{k}}\right)\equiv -1$ for every generic integer $\underline{k}$. Indeed, under this assumption we have, for every such $\underline{k}$ and every finite $v = l\in S^-_{\mathbf{f}^\#_{\underline{k}}}$,

$$\varepsilon_l\left(\mathbf{f}^\#_{\underline{k}}\right) = -a_l\left(f_{1,k_1}\right)a_l\left(f_{2,k_2}\right)a_l\left(f_{3,k_3}\right)l^{-\frac{k_1+k_2+k_3}{2}},$$

a function which can be p-adically interpolated and then needs to be constant for all weights (by connectedness of the U_i's). Hence, having fixed $\underline{\mathbf{f}}$ there is a well defined finite "generic sign" $\varepsilon_{\text{fin}}\left(\underline{\mathbf{f}}\right)$ of the family, $B_D = B_{\Pi_{\underline{k}}}$ does not depend on the generic integer point $\underline{k}$ and we have a well-posed interpolation problem. Of course, we expect $\varepsilon_{\text{fin}}\left(\underline{\mathbf{f}}\right)$ and $B_D = B_{\Pi_{\underline{k}}}$ to be defined in general, i.e. independent of the generic integer point $\underline{k}$ (as explained below, we can give evidences). At this point the consideration splits in two cases.

If $\varepsilon_{\text{fin}}\left(\underline{\mathbf{f}}\right) = 1$ (hence D is the product of an *even* number of primes), then $\varepsilon\left(\mathbf{f}^\#_{\underline{k}}\right) = 1$ for every generic $\underline{k}\in\Sigma_1$, Σ_2 or Σ_3. One gets three (square root) p-adic L-function $\mathcal{L}_p^{\Sigma_i}\left(\underline{\mathbf{f}}\right)$, one for every region Σ_i, with the property that

$$\mathcal{L}_p^{\Sigma_i}\left(\underline{\mathbf{f}}\right)\left(\underline{k}\right)^2 \doteq L\left(f^\#_{1,k_1}\times f^\#_{2,k_2}\times f^\#_{3,k_3}, c_{\underline{k}}\right)\text{ for }\underline{k}\in\Sigma_i\text{ generic.}$$

Here we write $\doteq$ to mean equality up to Euler factors, periods and local constants. On the other hand, $L\left(f_{k_1}\times f_{k_2}\times f_{k_3}, c_{\underline{k}}\right) = 0$ when $\underline{k}\in\Sigma_{123}$ is generic because of the sign of the functional equation and the interpolation problem on Σ_{123} is trivial. This is the Harris-Tilouine setting studied in [25], under some ordinariness assumption and supposing $N_1 = N_2 = N_3 = D = 1$. These p-adic L-functions have recently found interesting applications in [15] and [16]. See also [1] for an extension of Harris-Tilouine construction to Coleman families and [14] for related constructions.

When $\varepsilon_{\text{fin}}\left(\underline{\mathbf{f}}\right) = -1$ (hence D is the product of an *odd* number of primes), then $\varepsilon\left(\mathbf{f}^\#_{\underline{k}}\right) = 1$ for every generic $\underline{k}\in\Sigma_{123}$. The interpolation problem is therefore non-trivial only in the balanced region. We get a (square root) p-adic L-functions $\mathcal{L}_p^{\Sigma_{123}}\left(\underline{\mathbf{f}}\right)$ which interpolates in the region Σ_{123}:

$$\mathcal{L}_p^{\Sigma_{123}}\left(\underline{\mathbf{f}}\right)\left(\underline{k}\right)^2 \doteq L\left(f^\#_{1,k_1}\times f^\#_{2,k_2}\times f^\#_{3,k_3}, c_{\underline{k}}\right)\text{ for }\underline{k}\in\Sigma_{123}\text{ generic.}$$

In order to formulate the result in a simpler form, let us assume that $M = N_1 = N_2 = N_3$ and that the nebetypes ε_i are trivial (but we will keep track of the characters in order to state (5) below with the correct Euler factors). Then we get a formula

$$\mathcal{L}_p^{\Sigma_{123}}\left(\underline{\mathbf{f}}\right)\left(\underline{k}\right)^2 = \mathcal{E}_p\left(\underline{\alpha},\underline{k}\right)^2\frac{\left(\varphi^\#_{\underline{\mathbf{f}},\underline{k}},\varphi^\#_{\underline{\mathbf{f}},\underline{k}}\right)_{\underline{k}}}{2L\left(1,\Pi_{\underline{k}},\mathrm{Ad}\right)}L\left(1/2,\Pi_{\underline{k}}\right)\prod_{l\mid M}\frac{2}{l}\left(1+\frac{1}{l}\right),\tag{2}$$

where the quantities appearing in the right hand side are the following. Define $\underline{k}_1^* := \frac{-k_1+k_2+k_3}{2}$, $\underline{k}_2^* := \frac{k_1-k_2+k_3}{2}$, $\underline{k}_3^* := \frac{k_1+k_2-k_3}{2}$ and $\underline{k}^* := \frac{k_1+k_2+k_3}{2}$, which are integers in view of the definition of Σ_{123}. We have the Euler factor

$$\mathcal{E}_p\left(\underline{\alpha},\underline{k}\right) = \mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)\left(1-\frac{1}{\alpha_1\alpha_2\alpha_3}p^{\underline{k}^*+1}\right),$$

where $\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right) := 1-\varepsilon_1(p)^{-1}\frac{\alpha_1}{\alpha_2\alpha_3}p^{\underline{k}_1^*}$ and $\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)$ and $\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)$ are defined in a similar way. By the p-adic Jacquet-Langlands correspondence (see [10]), the three Coleman families $\underline{\mathbf{f}}$ lift to three eigenfamilies $\underline{\varphi}_{\underline{\mathbf{f}}} = \left(\varphi_{\mathbf{f}_1},\varphi_{\mathbf{f}_2},\varphi_{\mathbf{f}_3}\right)$ on B_D, whose specialization $\varphi_{\underline{\mathbf{f}},\underline{k}} = \left(\varphi_{\mathbf{f}_1,k_1},\varphi_{\mathbf{f}_2,k_2},\varphi_{\mathbf{f}_3,k_3}\right)$ at a generic $\underline{k}$ is the p-stabilization of three newforms $\varphi^\#_{\underline{\mathbf{f}},\underline{k}} = \left(\varphi^\#_{\mathbf{f}_1,k_1},\varphi^\#_{\mathbf{f}_2,k_2},\varphi^\#_{\mathbf{f}_3,k_3}\right)$ on B_D of level M/D; then $\left(\varphi^\#_{\underline{\mathbf{f}},\underline{k}},\varphi^\#_{\underline{\mathbf{f}},\underline{k}}\right)_{\underline{k}}$ is the product of quaternionic Petersson products $\left(\varphi_{i,k_i}^{\#},\varphi_{i,k_i}^{\#}\right)_{k_i}$ (see (19) and the definition before Lemma 3.2). Finally, setting $\Gamma_{\mathbb{C}}(s):=2(2\pi)^{-s}\Gamma(s)$, we have

$$L\left(1/2,\Pi_{\underline{k}}\right)=L_{\infty}\left(1/2,\Pi_{\underline{k}}\right)L\left(f_{1,k_1}^{\#}\times f_{2,k_2}^{\#}\times f_{3,k_3}^{\#},c_{\underline{k}}\right),$$

where

$$\begin{aligned}L_{\infty}\left(1/2,\Pi_{\underline{k}}\right) &= \Gamma_{\mathbb{C}}\left(\underline{k}^*+2\right)\Gamma_{\mathbb{C}}\left(\underline{k}_1^*+1\right)\Gamma_{\mathbb{C}}\left(\underline{k}_2^*+1\right)\Gamma_{\mathbb{C}}\left(\underline{k}_3^*+1\right)\\ &= 2^4(2\pi)^{-2\underline{k}^*-5}\left(\underline{k}^*+1\right)!\underline{k}_1^*!\underline{k}_2^*!\underline{k}_3^*!\end{aligned}$$

and $L\left(s,\Pi_{\underline{k}},\mathrm{Ad}\right)$ is a adjoint L-function.

Remark 1.1. It follows from Deligne's proof of the generalized Ramanujan conjecture that $\mathcal{E}_p\left(\underline{\alpha},\underline{k}\right)\neq 0$ for every generic integer point $\underline{k}$.

Before explaining the idea of the proof, let us remark that the ratio $\frac{\left(\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#},\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#}\right)_{\underline{k}}}{L\left(1,\Pi_{\underline{k}},\mathrm{Ad}\right)}$ can be given a different arrangement as follows (see (73)). First, a result of Shimura and Hida relates $L\left(1,\Pi_{\underline{k}},\mathrm{Ad}\right)$ to the period $\left(\mathbf{f}_{\underline{k}}^{\#},\mathbf{f}_{\underline{k}}^{\#}\right)_{\underline{k}}:=\left(f_{1,k_1,}^{\#},f_{1,k_1,}^{\#}\right)_{k_1}\left(f_{2,k_2,}^{\#},f_{2,k_2,}^{\#}\right)_{k_2}\left(f_{3,k_3,}^{\#},f_{3,k_3,}^{\#}\right)_{k_3}$. Second we define, using Proposition 7.8, a rigid analytic function $\left(\underline{\varphi_{\mathbf{f}}},\underline{\varphi_{\mathbf{f}}}\right)_p$ uniquely characterized by the interpolation formula

$$\left(\underline{\varphi_{\mathbf{f}}},\underline{\varphi_{\mathbf{f}}}\right)_p\left(\underline{k}\right)=\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\left(\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#},\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#}\right)_{\underline{k}},$$

where $\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)=\prod_{i=1,2,3}\mathcal{E}_p^{\Omega}\left(\alpha_i,k_i\right)$ and $\mathcal{E}_p^{\Omega}\left(\alpha_i,k_i\right):=\left(1-\varepsilon_i(p)\,\alpha_i^{-2}p^{k_i}\right)\left(1-\varepsilon_i(p)\,\alpha_i^{-2}p^{k_i+1}\right)$.

Remark 1.2. It follows from Deligne's proof of the generalized Ramanujan conjecture that, for every generic integer point $\underline{k}$, $1-\varepsilon_i(p)\,\alpha_i^{-2}p^{k_i}\neq 0$. Hence, we have $\mathcal{E}_p^{\Omega}\left(\alpha_i,k_i\right)=0$ if and only if $\alpha_i=\pm\sqrt{\varepsilon_i(p)}p^{\frac{k_i+1}{2}}$. In particular, assuming that the slope of α_i is $\leq h_i$, we see that the condition $\mathcal{E}_p^{\Omega}\left(\alpha_i,k_i\right)=0$ implies $k_i\leq 2h_i-1$. Consequently, for all but finitely many generic integer point $\underline{k}$, we have $\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\neq 0$.

Then one finds, assuming that $\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\neq 0$,

$$\frac{\left(\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#},\varphi_{\underline{\mathbf{f}},\underline{k}}^{\#}\right)_{\underline{k}}}{L\left(1,\Pi_{\underline{k}},\mathrm{Ad}\right)}=\frac{27}{4^{\underline{k}^*}\pi^3}\prod_{l|M}\left(1+\frac{1}{l}\right)^3\frac{\left(\underline{\varphi_{\mathbf{f}}},\underline{\varphi_{\mathbf{f}}}\right)_p\left(\underline{k}\right)}{\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\left(\mathbf{f}_{\underline{k}}^{\#},\mathbf{f}_{\underline{k}}^{\#}\right)_{\underline{k}}}$$

(see (74)). Hence (2) becomes

$$\mathcal{L}_p^{\Sigma_{123}}\left(\underline{\mathbf{f}}\right)\left(\underline{k}\right)^2=\frac{27}{4^{\underline{k}^*+1}\pi^3}\frac{\mathcal{E}_p\left(\underline{\alpha},\underline{k}\right)^2}{\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)}\frac{\left(\underline{\varphi_{\mathbf{f}}},\underline{\varphi_{\mathbf{f}}}\right)_p\left(\underline{k}\right)}{\left(\mathbf{f}_{\underline{k}}^{\#},\mathbf{f}_{\underline{k}}^{\#}\right)_{\underline{k}}}L\left(1/2,\Pi_{\underline{k}}\right)\prod_{l|M}\frac{2}{l}\left(1+\frac{1}{l}\right)^4.$$

Remark 1.3. The p-adic period function $\left(\underline{\varphi_{\mathbf{f}}},\underline{\varphi_{\mathbf{f}}}\right)_p$ is the tensor product of the three functions $\left(\varphi_{\mathbf{f}_i},\varphi_{\mathbf{f}_i}\right)_p:=\frac{p+1}{\alpha_i}\left(\varphi_{\mathbf{f}_i},\varphi_{\mathbf{f}_i}\right)$ characterized by the interpolation property $\left(\varphi_{\mathbf{f}_i},\varphi_{\mathbf{f}_i}\right)_p(k)=\mathcal{E}_p^{\Omega}\left(\alpha_i,k\right)\left(\varphi_{i,k}^{\#},\varphi_{i,k}^{\#}\right)_k$ (see Proposition 7.8). In the ordinary case, one can normalize the families φ_i and relate them to Hida's canonical periods and congruence ideals (see [35, §2.1 and §2.2 and Proposition 6.4], [28, Theorem 0.1, Conjecture 0.2 (*iii*)], [27, Corollary 10.6], [26, Theorem 5.1] and [32, Proposition 10.1.1]).

Let us now explain how (2) is proved and the relevance of the assumption that we have done on the Coleman family. First, as explained, one problem is that there a priori no well defined interpolation problem because of the lack of a generic sign fixing a region of interpolation; second, as we will see, the special value formula requires test vectors and it is not clear that they move p-adically in general. In order to circumvent this issue, we are inspired by Ichino's special value formula. Let us fix a (definite for our purposes) quaternion algebra B, let $\mathbf{B}^{\times}$ be the algebraic group associated to its invertible elements with center $\mathbf{Z}_{\mathbf{B}^{\times}}$ and set $[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}_{\mathbf{B}^{\times}}} := \mathbf{Z}_{\mathbf{B}^{\times}}(\mathbb{A})\backslash\mathbf{B}^{\times}(\mathbb{A})/B^{\times}$. Then Ichino's formula takes the form

$$(3)\quad I_B(\psi)^2 = \frac{C}{2^3}\frac{\zeta_{\mathbb{Q}}^2(2)\,L(1/2,\Pi'(\psi))}{L(1,\Pi'(\psi),\mathrm{Ad})}\prod\nolimits_v I_v(\psi_v) = \frac{\left\langle\psi^\flat,\psi^{\flat\vee}\right\rangle_{L^2}}{2^3}\frac{\zeta_{\mathbb{Q}}^2(2)\,L(1/2,\Pi'(\psi))}{L(1,\Pi'(\psi),\mathrm{Ad})}\prod\nolimits_v C_v^{\psi^\flat,\psi^{\flat\vee}}(\psi_v),$$

where the notations are as follows. In the first equality we have that $\psi=\otimes_v\psi_v\in\Pi(\psi)$ is an L^2-automorphic form on $B^{\times 3}$, that we assume to be in an irreducible representation $\Pi(\psi)$ of $B^{\times 3}$, $\Pi'(\psi)$ is the automorphic representation of $\mathbf{GL}_2^3$ which corresponds to $\Pi(\psi)$ via the Jacquet-Langlands correspondence, $I_B(\psi) = \int_{[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}_{\mathbf{B}^{\times}}}}\psi(x)\,d\mu_{[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}_{\mathbf{B}^{\times}}}}(x)$, C is a non-zero constant defined in (22) below (which depends on the choice of local pairings) and $I_v(\psi_v)$ is defined in (25) (see (17) and the lines after (18) for the definition of $\check{\psi}\in\Pi(\psi)^\vee$ appearing in (25)). In the second equality we have determined $C=\frac{\langle\psi^\flat,\psi^{\flat\vee}\rangle_{L^2}}{\prod_v\langle\psi_v^\flat,\psi_v^{\flat\vee}\rangle_v}$ and defined $C_v^{\psi^\flat,\psi^{\flat\vee}}(\psi_v):=\frac{I_v(\varphi)}{\langle\psi_v^\flat,\psi_v^{\flat\vee}\rangle_v}$ using auxiliary $\psi^\flat\in\Pi(\psi)$ and $\psi^{\flat\vee}\in\Pi(\psi)^\vee$ such that $\left\langle\psi^\flat,\psi^{\flat\vee}\right\rangle_{L^2}\neq 0$. Let us remark that, as explained in the proof of Theorem 3.4, formula (3) is a special case of a more general Ichino's formula obtained choosing the dual vector $\psi^\vee=\check{\psi}$ in order to get the square $I_B(\psi)^2$. Though qualitatively equivalent, the local constants $C_v^{\psi^\flat,\psi^{\flat\vee}}(\psi_v)$ that appear in the second expression are more convenient to work with (see Remark 3.5). Although by a result of Prasad (see [34]) $I_B=0$ on $\Pi(\psi)$ except in case $B=B_{\Pi'(\psi)}$, a problem which is always meaningful is to try to interpolate the function $\underline{k}\mapsto I_B\left(\psi_{\underline{k}}\right)$ if $\psi_{\underline{k}}$ is an L^2-automorphic form canonically attached to a vector valued modular form $\varphi_{\underline{k}}=\left(\varphi_{1,k_1},\varphi_{2,k_2},\varphi_{3,k_3}\right)$ on $B^{\times 3}$ (as explained below) which comes as the specialization of a p-adic family $\underline{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ on $B^{\times 3}$. Also, the Jacquet conjecture proved by Harris and Kudla (see [24] and Theorem 3.4 (2) below) tells us that, when $B=B_{\Pi'\left(\psi_{\underline{k}}\right)}$, there exists $\psi'_{\underline{k}}\in\Pi'\left(\psi_{\underline{k}}\right)$ such that the associated local constants $I_v\left(\psi'_{\underline{k},v}\right)$ are non-zero and, hence

$$(4)\qquad I_B\left(\psi'_{\underline{k}}\right)\neq 0\Longleftrightarrow L\left(1/2,\Pi'\left(\psi_{\underline{k}}\right)\right)\neq 0.$$

In this case, we say that $\psi'_{\underline{k}}$ is a test vector. Formulated in this way our interpolation problem, we can *remove all the assumptions* that was done on the three Coleman families $\underline{\mathbf{f}}$ and give an unconditional p-adic Ichino's formula analogous to (3). In order to state our main result, let us recall that $\varphi_i\in M_p^{\diamond}(\mathcal{D}_{\mathbf{k}_i}(W),\omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$ where the notation is the following. Let $\Sigma_0(p\mathbb{Z}_p)\subset\mathbf{M}_2(\mathbb{Z}_p)$ be the subsemigroup of matrices having non-zero determinant, upper left entry $a\in\mathbb{Z}_p^\times$ and lower left entry $c\in p\mathbb{Z}_p$ and set $\Gamma_0(p\mathbb{Z}_p):=\Sigma_0(p\mathbb{Z}_p)\cap\mathbf{GL}_2(\mathbb{Z}_p)$. The inclusion $U_i\subset\mathcal{X}$ corresponds to a continuous character $\mathbf{k}_i:\mathbb{Z}_p^\times\to\mathcal{O}(U_i)$ and one may consider the space $\mathcal{D}_{\mathbf{k}_i}(W)$ of locally analytic distributions on $W:=\mathbb{Z}_p^\times\times\mathbb{Z}_p$ that are "homogeneous of weight $\mathbf{k}_i$" (see §4.1 for the precise definition). Also, $\omega_{0,p}^{\mathbf{k}_i}:\mathbf{Z}_{\mathbf{B}^{\times}}(\mathbb{A}_{\mathrm{f}})\to\mathcal{O}(U_i)$ is the character defined by the formula $\omega_{0,p}^{\mathbf{k}_i}(z):=\omega_{\mathrm{f},i}(z)\left(\frac{z}{\mathrm{N}_{\mathrm{f}}(z)}\right)_p^{-\mathbf{k}_i}$, where $\mathrm{N}_{\mathrm{f}}(z)$ is defined by the formula $z=\mathrm{N}_{\mathrm{f}}(z)\frac{z}{\mathrm{N}_{\mathrm{f}}(z)}$ with $\mathrm{N}_{\mathrm{f}}(z)\in\mathbb{Q}_+^\times$ and $\frac{z}{\mathrm{N}_{\mathrm{f}}(z)}\in\widehat{\mathbb{Z}}$ (see also §2.1), $\omega_{\mathrm{f},i}$ is the inverse of the adelization $\omega_{\mathrm{f},i}^{-1}(z):=\varepsilon_i\left(\frac{z}{\mathrm{N}_{\mathrm{f}}(z)}\right)$ of ε_i[1] and $(-)_p$ means that we take the p-component (which is indeed an element of $\mathbb{Z}_p^\times$). Because $\Sigma_0(p\mathbb{Z}_p)$ acts on $\mathcal{D}_{\mathbf{k}_i}(W)$ from the right (by right multiplication on the row vectors in W) one may form the space $M_p^{\diamond}(\mathcal{D}_{\mathbf{k}_i}(W),\omega_{0,p}^{\mathbf{k}_i})$, which is a subspace of the space of those functions $\varphi_i:\mathbf{B}^{\times}(\mathbb{A}_{\mathrm{f}})\to\mathcal{D}_{\mathbf{k}_i}(W)$ with the property that there exists an open and compact subgroup $K^p\subset\mathbf{B}^{\times}(\mathbb{A}_{\mathrm{f}}^p)$ (here $\mathbb{A}_{\mathrm{f}}^p$ is the prime to p part of $\mathbb{A}_{\mathrm{f}}$) such that $\varphi_i(zuxg_{\mathrm{f}})=\omega_{0,p}^{\mathbf{k}_i}(z)\,\varphi_i(x)\,u_p^{-1}$ for every $z\in\mathbf{Z}_{\mathbf{B}^{\times}}(\mathbb{A}_{\mathrm{f}})$, $u\in K^p\Gamma_0(p\mathbb{Z}_p)$, $x\in\mathbf{B}^{\times}(\mathbb{A}_{\mathrm{f}})$ and $g\in B^\times$. These spaces are naturally $\mathbf{B}^{\times}(\mathbb{A}_{\mathrm{f}}^p)$-modules and they are further endowed with the action of a U_p-operator (see the lines before Proposition 5.4): the superscript $(-)^{\alpha_i}$ refers to the α_i-eigenspace for the U_p-operator. Also, if $k_i\in U_i$ is an integer, we may consider the space of two variables polynomials that are homogeneous of degree

[1]The fact that the central characters are the inverse of the usual ones is due to the fact that the L^2-automorphic forms on B that appear in §3.2 enjoy the equivariance property $f(xg)=f(x)$, as opposite to the usual convention $f(gx)=f(x)$. Thus we consider right $\mathbf{B}^{\times}(\mathbb{A}_f)$ action $(fu)(x):=f(ux)$ on them, rather than the usual left action $(uf)(x):=f(xu^{-1})$. The rule $f^*(x):=f(x^{-1})$, which satisfies $(fu)^*=uf^*$, exchange the two spaces, but the central characters of the corresponding spaces are reversed.

k_i and we write $\mathbf{V}_{k_i,F}$ for its F-dual; next, let $\omega_0^{k_i} : \mathbf{Z}_{\mathbf{B}^\times}(\mathbb{A}_\mathrm{f}) \to F^\times$ be defined by the formula $\omega_0^{k_i}(z) := \omega_{\mathrm{f},i}(z)\,\mathrm{N}_\mathrm{f}^{k_i}(z)$ and let $M^\diamond\left(\mathbf{V}_{k_i,F}, \omega_0^{k_i}\right)$ be the space of functions $\varphi_{i,k_i} : \mathbf{B}^\times(\mathbb{A}_\mathrm{f}) \to \mathbf{V}_{k_i,F}$ with the property that there exists an open and compact subgroup $K^p \subset \mathbf{B}^\times(\mathbb{A}_\mathrm{f}^p)$ such that $\varphi_{i,k_i}(zuxg_\mathrm{f}) = \omega_0^{k_i}(z)\,\varphi_{i,k_i}(x)\,g_\infty$ for every z, u, x and g as above, which is again a $\mathbf{B}^\times(\mathbb{A}_\mathrm{f}^p)$-module endowed with a U_p-operator. There is a specialization map $\varphi_i \mapsto \varphi_{i,k_i}$ from $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})$ to $M^\diamond\left(\mathbf{V}_{k_i,F}, \omega_0^{k_i}\right)$ (see (41)) satisfying the following properties (easily proved by means of the Ash-Stevens machinery [4], see also [9]): it respects the above actions, both $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{K^p}$ and $M^\diamond\left(\mathbf{V}_{k_i,F}, \omega_0^{k_i}\right)^{K^p}$ admit slope $\leq h$ decompositions for the U_p-operator (as defined in [4]) for every open and compact subgroup K^p as above and the slope $\leq h$ parts $(-)^{\leq h}$ satisfy the control theorem $\frac{M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{K^p,\leq h}}{I_{k_i} M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{K^p,\leq h}} \overset{\sim}{\to} M^\diamond\left(\mathbf{V}_{k_i,F}, \omega_0^{k_i}\right)^{K^p,\leq h}$ for every $k+1 > h$ (here $I_{k_i} \subset \mathcal{O}(U_i)$ is the ideal of functions vanishing at k_i). Let us write

$$M_p^{\underline{\alpha}}(\underline{U}) := M_p^\diamond(\mathcal{D}_{\mathbf{k}_1}(W), \omega_{0,p}^{\mathbf{k}_1})^{\alpha_1} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_2}(W), \omega_{0,p}^{\mathbf{k}_2})^{\alpha_2} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_3}(W), \omega_{0,p}^{\mathbf{k}_3})^{\alpha_3},$$

so that we focus on elements $\underline{\varphi} \in M_p^{\underline{\alpha}}(\underline{U})$. Our main result is that there is a (unique up to sign) $\mathcal{O}(\underline{U})$-valued $\mathcal{O}(\underline{U})$-linear functional $\mathcal{L}_p^{\underline{\alpha}} = \mathcal{L}_{p,\underline{U}}^{\underline{\alpha}} : M_p^{\underline{\alpha}}(\underline{U}) \to \mathcal{O}(\underline{U})$ such that, for every $\underline{\varphi} \in M_p^{\underline{\alpha}}$ which is the tensor product of three families, if $\varphi_{\underline{k}}$ belongs to an irreducible representation $\Pi\left(\varphi_{\underline{k}}\right)$, then

$$\begin{aligned}
\mathcal{L}_p^{\underline{\alpha}}(\underline{\varphi})(\underline{k})^2 &= \mathcal{E}_p(\underline{\alpha}, \underline{k})^2 \frac{C_{\underline{k}}}{2^9 3^2} \frac{\zeta_\mathbb{Q}^2(2)\, L\left(1/2, \Pi'\left(\varphi_{\underline{k}}^\#\right)\right)}{L\left(1, \Pi'\left(\varphi_{\underline{k}}^\#\right), \mathrm{Ad}\right)} \prod\nolimits_v I_v(\varphi_{\underline{k}}^\#) \\
&= \mathcal{E}_p(\underline{\alpha}, \underline{k})^2 \frac{\left(\varphi_{\underline{k}}^{\flat\#}, \varphi_{\underline{k}}^{\flat\flat\#}\right)_{\underline{k}}}{2L\left(1, \Pi'\left(\varphi_{\underline{k}}^\#\right), \mathrm{Ad}\right)} L\left(1/2, \Pi'\left(\varphi_{\underline{k}}^\#\right)\right) \prod\nolimits_{v \neq \infty, p} C_v^{\varphi_{\underline{k}}^{\flat\#}, \varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^\#\right),
\end{aligned} \tag{5}$$

where $\varphi_{\underline{k}}^\# = \left(\varphi_{k_1}^\#, \varphi_{k_2}^\#, \varphi_{k_3}^\#\right)$ is the unique triple which componentwisely has p-stabilization $\varphi_{\underline{k}}$, $\Pi'\left(\varphi_{\underline{k}}^\#\right) = \Pi'\left(\varphi_{\underline{k}}\right) := \Pi'\left(\psi_{\underline{k}}\right)$, $C_{\underline{k}} = C$ is defined in (22) below, $I_v(\varphi_{\underline{k}}^\#)$ and $C_v^{\varphi_{\underline{k}}^{\flat\#}, \varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^\#\right)$ are again defined in (25) and, as in (3) above, the second equality holds with the auxiliary choice of vectors $\varphi_{\underline{k}}^{\flat\#}$ and $\varphi_{\underline{k}}^{\flat\flat\#}$ such that $\left(\varphi_{\underline{k}}^{\flat\#}, \varphi_{\underline{k}}^{\flat\flat\#}\right)_{\underline{k}} \neq 0$ and satisfying a local condition at p (see the lines before Theorem 7.3). This is proved in Theorem 7.3 and Theorem 8.3, from which (2) is deduced as a special case (see also §8.3), is obtained by providing conditions on the Coleman families under which one knows a priori that $B = B_{D_{\underline{k}}}$ for every generic integer $\underline{k}$, an explicit test vector $\underline{\varphi}$ moving in families can be written down for which the corresponding local constants $C_v^{\varphi_{\underline{k}}^\#, \varphi_{\underline{k}}^\#}\left(\varphi_{\underline{k}}^\#\right)$ has been computed (that is, we take $\varphi_{\underline{k}}^\# = \varphi_{\underline{k}}^{\flat\#} = \varphi_{\underline{k}}^{\flat\flat\#}$) and then relating $\left(\varphi_{\underline{k}}^\#, \varphi_{\underline{k}}^\#\right)_{\underline{k}}$ to $\left(\varphi_{\underline{\mathbf{f}},\underline{k}}^\#, \varphi_{\underline{\mathbf{f}},\underline{k}}^\#\right)_{\underline{k}}$ (see (76)).

Let us write $M_p^\diamond\left(\underline{U}, \underline{\varphi}_\mathbf{f}\right) \subset M_p^{\underline{\alpha}}(\underline{U})$ for the $\mathbf{B}^\times(\mathbb{A}_\mathrm{f}^p)$-representation generated by $\underline{\varphi}_\mathbf{f}$ over $\mathcal{O}(\underline{U})$ and suppose that there is some generic integer point $\underline{k}^0$ such that $L\left(f_{1,k_1^0}^\# \times f_{2,k_2^0}^\# \times f_{3,k_3^0}^\#, c_{\underline{k}^0}\right) \neq 0$ for $B = B_{D_{\underline{k}^0}}$. Then we see from (5), (4) and Remark 1.1 that $\mathcal{L}_p^{\underline{\alpha}} \neq 0$ as a functional on $M_p^\diamond\left(\underline{U}, \underline{\varphi}_\mathbf{f}\right)$ and, hence, there is some $\underline{\varphi} \in M_p^\diamond\left(\underline{U}, \underline{\varphi}_\mathbf{f}\right)$ such that $\mathcal{L}_p^{\underline{\alpha}}(\underline{\varphi}) \neq 0$. In particular, we see that $B = B_{D_{\underline{k}}}$ and that $\underline{\varphi}_{\underline{k}}$ is a test vector for every generic integer point $\underline{k}$ in a Zariski open subset of $\underline{U}$, by (5) and Remark 1.1.

Let us briefly explain how Theorem 7.3 is proved. First, the L^2-automorphic forms to which ψ belongs are related to the vector valued modular forms to which $\varphi_{\underline{k}}$ described above belongs; via this identification, the integral $I_B(\psi)$ that appear in the left hand side of the Ichino's formula is related to a functional $t_{\underline{k}}$ on vector valued modular forms (see §3 and Theorem 3.4). This is done by appealing to the results of [22], which set up a general formalism for getting such a kind of results in the setting of Gan-Gross-Prasad conjectures when the real points of the algebraic group are compact modulo the center. This linear functional $t_{\underline{k}}$ is obtained by evaluating the vector valued forms at the product $\Delta_{\underline{k}/E} \in \mathbf{P}_{k_1,F} \otimes \mathbf{P}_{k_2,F} \otimes \mathbf{P}_{k_3,F}$ of certain powers of determinants (see (13)): the resulting formula could be viewed as an analogous of the Hatcher's formula in our setting (which indeed can be deduced from Waldspurger's formula via the method of [22]). It turns out that $t_{\underline{k}}$ can not be deformed p-adically, but it is closely related to three linear forms $t^{\circ}_{i,\underline{k}}$ (only defined on distribution valued modular forms) which can be easily moved p-adically. The relationship between $t_{\underline{k}}$ and $t^{\circ}_{i,\underline{k}}$ is that they differs by the Euler factor $\mathcal{E}_{p,i}\left(\underline{\alpha},\underline{k}\right)$ and the action of an operator W_p (closely related to the Atkin-Lehner operator). For example (see Corollary 5.6 and (46)):

$$t^{\circ}_{3,\underline{k}}(\varphi_1\otimes\varphi_2\otimes\varphi_3\mid W_p)=\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)t_{\underline{k}}(\varphi_1\otimes\varphi_2\otimes\varphi_3\mid W_p).$$

Expressing $t_{\underline{k}}(\varphi_1\otimes\varphi_2\otimes\varphi_3\mid W_p)$ in terms of $t_{\underline{k}}\left(\varphi^{\#}_{\underline{k}}\right)$ gives rise to another Euler factor $\alpha_i\widehat{\mathcal{E}}_{p,i}\left(\underline{\alpha},\underline{k}\right)$ such that $\mathcal{E}_{p,i}\left(\underline{\alpha},\underline{k}\right)\widehat{\mathcal{E}}_{p,i}\left(\underline{\alpha},\underline{k}\right)=\mathcal{E}_{p}\left(\underline{\alpha},\underline{k}\right)$ does not depend on the choice of $i=1,2,3$ (see Proposition 6.5). The result is that, writing $\mathcal{L}^{\underline{\alpha}}_{p}\left(\underline{\varphi}\right)\left(\underline{k}\right)$ for any one of $\frac{p+1}{\alpha_1}t^{\circ}_{1,\underline{k}}\left(\varphi_1\mid W_3\otimes\varphi_2\otimes\varphi_3\right)$, $\frac{p+1}{\alpha_2}t^{\circ}_{2,\underline{k}}\left(\varphi_1\otimes\varphi_2\mid W_3\otimes\varphi_3\right)$ or $\frac{p+1}{\alpha_3}t^{\circ}_{2,\underline{k}}\left(\varphi_1\otimes\varphi_2\otimes\varphi_3\mid W_3\right)$, all of them satisfy the same interpolation property

$$\mathcal{L}^{\underline{\alpha}}_{p}\left(\underline{\varphi}\right)\left(\underline{k}\right):=\mathcal{E}_{p}\left(\underline{\alpha},\underline{k}\right)t_{\underline{k}}\left(\varphi^{\#}_{\underline{k}}\right); \tag{6}$$

by Zariski density of the integer points, these p-adically moving quantities needs to be the same and this interpolation formula uniquely characterize $\mathcal{L}^{\underline{\alpha}}_{p}\left(\underline{\varphi}\right)$ (thus fixing the sign). Applying our vector valued version of Ichino's formula (Theorem 3.4), (5) is deduced from (6). Let us remark that one feature of our asymmetric construction is that one gets in a natural way improved p-adic L-functions defined on appropriate improving planes (see Proposition 7.7).

Remark 1.4. The local constants $C_v^{\psi^\flat,\psi^{\flat\vee}}\left(\psi_v,\psi_v^\vee\right)$ that appear in (24) below have been quite largely studied in the literature about subconvexity problems when $\psi_v=\psi_v^\flat$ and $\psi_v^\vee=\psi^{\flat\vee}=\overline{\psi}_v$ (see [42] and [30]). They appear when one specializes Ichino's formula (27) to the case $\psi^\vee=\overline{\psi}$: then $I_B\left(\psi\right)^2$ (of (3)) is replaced by $\left|I_B\left(\psi\right)\right|^2$ and $\left\langle\psi,\check{\psi}\right\rangle_{L^2}$ by $\left\langle\psi,\overline{\psi}\right\rangle_{L^2}$, which is always non-zero. Our assumption that the nebetype are trivial in Theorem 8.3 allows us to appeal to this existing literature. Time after our works was completed, Hsieh has successfully completed the computation of the local constants $C_v^{\psi^\flat,\psi^{\flat\vee}}\left(\psi_v,\check{\psi}_v\right)\neq 0$ taking $\psi_v^\flat$ and $\psi^{\flat\vee}$ newvectors as defined in Example 3.3 and explicit vectors ψ_v linearly depending on $\psi_v^\flat$ (see [29, §6.1]). Rather than appealing to his results, we illustrate the method for specifying the local constants in (5) in order to get Theorem 8.3 in a simplified setting, allowing us to appeal to the previously existing literature and make an easier choice of test vectors. In the ordinary case, Hsieh has (also) given a construction of the balanced triple product p-adic L-functions based on our vector valued version of Ichino's formula (Theorem 3.4) involving the trilinear form $t_{\underline{k}}$ (see [29, Proposition 4.10]). He was able, in this case, to give a very nice Gross' style interpretation of our p-adically moving trilinear form $t^{\circ}_{3,\underline{k}}$ (cfr. the proof of [29, Proposition 4.9] with our Proposition 5.4 and Corollary 5.6).

Remark 1.5. Suppose that $\mathbf{k}:\mathbb{Z}_p^\times\to\mathcal{O}^\times$ and $\mathbf{k}':\mathbb{Z}_p^\times\to\mathcal{O}'^\times$ valued in (the invertible elements of) F-Banach algebras and are two continuous homomorphisms and that $\phi:\mathcal{O}\to\mathcal{O}'$ is a continuous homomorphism of F-algebras with the property that $\mathbf{k}'=\phi\circ\mathbf{k}$. Then it is easy to see that the canonical map $M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W),\omega^{\mathbf{k}}_{0,p})\widehat{\otimes}_{\mathcal{O},\phi}\mathcal{O}'\to M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W),\omega^{\mathbf{k}}_{0,p})$ is an isomorphism. In particular, the definition of $M_p^{\underline{\alpha}}\left(\underline{U}\right)$ can be uniquely extended to arbitrary admissible open subsets of $\mathcal{X}$. Correspondingly, it follows from the fact that our construction of $\mathcal{L}^{\underline{\alpha}}_{p,\underline{U}}$ behaves well with respect to base changes that we can uniquely extend its definition to arbitrary admissible open subsets of $\mathcal{X}$ and that the resulting functional is uniquely characterized by the interpolation property (6) and, up to sign, by (5).

Suppose that $\underline{k}$ is an integer point which is not generic: then at least one of the forms φ_{i,k_i} is new at p. We compute the Euler factors in this case in Proposition 7.5. Our interest is motivated by the forthcoming work [6] in which exceptional zero phenomena of these p-adic L-functions are investigated, an analogue of those discovered in [33] and studied in [23] (see Remark 7.6). We will give an algebraic interpretation of these result in the framework of Nekovar-style weight pairings as defined in [39] and [40]. Particularly interesting is the case where a local change of sign at p produces an extra vanishing due to the complex L-function (see Remark 7.6): we relate the derivatives of our p-adic L-function to the Abel-Jacobi image of diagonal cycles.

Let B be a definite quaternion division $\mathbb{Q}$-algebra which is split at the prime p and let $\mathbf{B}$ (resp. $\mathbf{B}^{\times}$) be the associated ring scheme (resp. algebraic group). We write $\mathbb{A}=\mathbb{A}_{\mathrm{f}}\times\mathbb{R}$ for the adele ring of $\mathbb{Q}$ and define $\mathbb{A}_{\mathrm{f}}^{p}$ by the rule $\mathbb{A}_{\mathrm{f}}=\mathbb{A}_{\mathrm{f}}^{p}\times\mathbb{Q}_{p}$. We set $B_{\mathrm{f}}:=\mathbf{B}(\mathbb{A}_{\mathrm{f}})$ (resp. $B_{\mathrm{f}}^{\times}:=\mathbf{B}^{\times}(\mathbb{A}_{\mathrm{f}})$), $B_{\mathrm{f}}^{\times,p}:=\mathbf{B}(\mathbb{A}_{\mathrm{f}}^{p})$ and $B_{v}=\mathbf{B}(\mathbb{Q}_{v})$ (resp. $B_{v}^{\times}:=\mathbf{B}^{\times}(\mathbb{Q}_{v})$) if v is either a finite place or $v=\infty$, so that $B_{\mathrm{f}}^{\times}=B_{\mathrm{f}}^{\times,p}\times B_{p}^{\times}$. We write $b\mapsto b^{\iota}$ for the main involution and $\mathrm{nrd}:\mathbf{B}^{\times}\to\mathbf{G}_{m}$ for the reduced norm.

If $\mathbf{Z}\subset\mathbf{Z}_{\mathbf{B}^{\times}}=\mathbf{G}_{m}$ is a closed subgroup (such as the trivial subgroup or the whole center), we define $Z_{\mathrm{f}}:=\mathbf{Z}(\mathbb{A}_{\mathrm{f}})$, $Z_{v}:=\mathbf{Z}(\mathbb{Q}_{v})$ and $Z_{\mathrm{f}}^{p}:=\mathbf{Z}(\mathbb{A}_{\mathrm{f}}^{p})$, so that $Z_{\mathrm{f}}=Z_{\mathrm{f}}^{p}\times Z_{p}$. We will need to consider double cosets of the form

$$\left[\mathbf{B}^{\times}(\mathbb{A})\right]_{\mathbf{Z}}:=\mathbf{Z}(\mathbb{A})\backslash\mathbf{B}^{\times}(\mathbb{A})/B^{\times}\text{ and }\left[B_{\mathrm{f}}^{\times}\right]_{\mathbf{Z}}:=Z_{\mathrm{f}}\backslash B_{\mathrm{f}}^{\times}/B^{\times}.$$

In order to later apply the results from [22], we fix measures as follows. We take the Tamagawa measure $\mu_{\mathbf{Z}(\mathbb{A})\backslash\mathbf{B}^{\times}(\mathbb{A})}$ on $\mathbf{Z}(\mathbb{A})\backslash\mathbf{B}^{\times}(\mathbb{A})$ and write $\mu_{[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}}}$ for the quotient measure (normalized in the usual way). Next we choose $\mu_{\mathbf{Z}\backslash\mathbf{B}^{\times},\infty}$ on $\mathbf{Z}(\mathbb{R})\backslash\mathbf{B}^{\times}(\mathbb{R})$ and $\mu:=\mu_{B_{\mathrm{f}}^{\times}}$ on $B_{\mathrm{f}}^{\times}$ such that $\mu(K)\in\mathbb{Q}$ for some (and hence every) open and compact subgroup $K\subset B_{\mathrm{f}}^{\times}$ and such that, writing $\mu_{B_{\mathrm{f}}^{\times}/B^{\times}}$ for the induced quotient measure on $B_{\mathrm{f}}^{\times}/B^{\times}$ (normalized in the usual way), which restricts to an invariant measure $\mu_{[B_{\mathrm{f}}^{\times}]_{\mathbf{Z}}}$ on $C(B_{\mathrm{f}}\backslash B_{\mathrm{f}}^{\times}/B^{\times})\subset C(B_{\mathrm{f}}^{\times}/B^{\times})$,

$$\int_{[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}}}f(x)\,d\mu_{[\mathbf{B}^{\times}(\mathbb{A})]_{\mathbf{Z}}}(x)=\int_{[B_{\mathrm{f}}^{\times}]_{\mathbf{Z}}}\left(\int_{\mathbf{Z}(\mathbb{R})\backslash\mathbf{B}^{\times}(\mathbb{R})}f(x_{\mathrm{f}}x_{\infty})\,d\mu_{\mathbf{Z}\backslash\mathbf{B},\infty}(x_{\infty})\right)d\mu_{[B_{\mathrm{f}}^{\times}]_{\mathbf{Z}}}(x_{\mathrm{f}})\tag{7}$$

is satisfied. We let $m_{\mathbf{Z}\backslash\mathbf{B}^{\times},\infty}$ be the total measure of $\mathbf{Z}(\mathbb{R})\backslash\mathbf{B}^{\times}(\mathbb{R})$.

Let $\Sigma_{0}(p\mathbb{Z}_{p})\subset\mathbf{M}_{2}(\mathbb{Z}_{p})$ be the subsemigroup of matrices having non-zero determinant, upper left entry $a\in\mathbb{Z}_{p}^{\times}$ and lower left entry $c\in p\mathbb{Z}_{p}$ and set $\Gamma_{0}(p\mathbb{Z}_{p}):=\Sigma_{0}(p\mathbb{Z}_{p})\cap\mathbf{GL}_{2}(\mathbb{Z}_{p})$. Consider an open and compact subgroup $K_{p}^{\diamond}\subset B_{p}^{\times}$ (it will be $\Gamma_{0}(p\mathbb{Z}_{p})$ in our applications). We will also need to consider a subsemigroup $K_{p}^{\diamond}\subset\Sigma_{p}\subset B_{p}^{\times}$ and to define $\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right):=B_{\mathrm{f}}^{\times,p}\times\Sigma_{p}$ (we will take $\Sigma_{p}=K_{p}^{\diamond}$, $\Sigma_{0}(p\mathbb{Z}_{p})$ or $B_{p}^{\times}$).

Let $\mathcal{K}:=\mathcal{K}\left(B_{\mathrm{f}}^{\times}\right)$ (resp. $\mathcal{K}^{\diamond}:=\mathcal{K}\left(B_{\mathrm{f}}^{\times},K_{p}^{\diamond}\right)$) be the set of open and compact subgroups $K\subset B_{\mathrm{f}}^{\times}$ (resp. $K=K^{p}\times K_{p}$ with $K^{p}\subset B_{\mathrm{f}}^{\times,p}$ and $K_{p}\subset K_{p}^{\diamond}$ open and compact). If S is a $B_{\mathrm{f}}^{\times}$-module (resp. a $\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)$-module), then we define

$$S^{\mathcal{K}}:=\bigcup_{K\in\mathcal{K}}S^{K}\text{ (resp. }S^{\mathcal{K}^{\diamond}}:=\bigcup_{K\in\mathcal{K}^{\diamond}}S^{K}).$$

We note that the Hecke operators $\mathcal{H}\left(B_{\mathrm{f}}^{\times}\right)$ (resp. $\mathcal{H}\left(\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)\right)$) act on $S^{\mathcal{K}}$ (resp. $S^{\mathcal{K}^{\diamond}}$) by double cosets of elements of $B_{\mathrm{f}}^{\times}$ (resp. $\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)$). We describe the action on $S^{\mathcal{K}^{\diamond}}$ for a $\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)$-module S (the action of $S^{\mathcal{K}}$ is similar). If $K_{1},K_{2}\in\mathcal{K}^{\diamond}$ and $\pi\in\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)$, the space $K_{1}\backslash K_{1}\pi K_{2}$ is finite[2] and we may write

$$K_{1}\pi K_{2}=\bigsqcup_{x\in K_{1}\backslash K_{1}\pi K_{2}}K_{1}x.$$

As usual, we may define

$$\cdot\mid K_{1}\pi K_{2}:S^{K_{1}}\to S^{K_{2}}$$

by the rule

$$v\mid K_{1}\pi K_{2}=\sum_{x\in K_{1}\backslash K_{1}\pi K_{2}}vx.\tag{8}$$

The mapping $u\mapsto\pi u$ induces a bijection $\left(K_{2}\cap\pi^{-1}K_{1}\pi\right)\backslash K_{2}\to K_{1}\backslash K_{1}\pi K_{2}$, so that we may take $x=\pi u$ in the above expression:

$$v\mid K_{1}\pi K_{2}=\sum_{u\in(K_{2}\cap\pi^{-1}K_{1}\pi)\backslash K_{2}}v\pi u.\tag{9}$$

We can define in this way an action of the Hecke algebra $\mathcal{H}\left(\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)\right)$ of elements of $\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)$. When $K_{p}^{\diamond}=B_{p}^{\times}$, we have $V^{\mathcal{K}^{\diamond}}=V^{\mathcal{K}}$ and we have an action of $\mathcal{H}\left(\Sigma_{p}\left(B_{\mathrm{f}}^{\times}\right)\right)=\mathcal{H}\left(B_{\mathrm{f}}^{\times}\right)$. Let $\mathcal{K}^{\diamond\diamond}\subset\mathcal{K}^{\diamond}$ be the subset of those groups such that $K_{p}=K_{p}^{\diamond}$ and write $\mathcal{H}(\Sigma_{p})$ for the Hecke algebra of double cosets $K\pi K$ with π concentrated in $\pi_{p}\in\Sigma_{p}$ and $K\in\mathcal{K}^{\diamond\diamond}$. Then (8) defines an operator on $V^{\mathcal{K}^{\diamond\diamond}}=\left(V^{\mathcal{K}^{\diamond}}\right)^{K_{p}^{\diamond}}$ by means of the formula $vU_{\pi}:=v\mid K\pi K$ if $v\in V^{K}$ where $K\in\mathcal{K}^{\diamond\diamond}$, i.e. it does not depend on $K\in\mathcal{K}^{\diamond\diamond}$. It follows

[2] Indeed note that $K_{1}\pi K_{2}$ is compact, being the image of $K_{1}\times K_{2}$ by means of the continuous map given by $(x,y)\mapsto x\pi y$. Since K_{1} is open, $K_{1}\pi K_{2}=\bigsqcup_{i}K_{1}\pi_{i}$ is an open covering which, by compactness, admits a finite refinement.

that $V^{\mathcal{K}^{\diamond\diamond}}$ is endowed with an action of $B_{\mathrm{f}}^{\times,p} \times \mathcal{H}(\Sigma_p)$: write $\widehat{\pi}_p$ for the idele concentrated at p, where we have $(\widehat{\pi}_p)_p = \pi_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$; then $V^{\mathcal{K}^{\diamond\diamond}}$ is endowed with an action of the operator $U_p := U_{\widehat{\pi}_p}$.

Let (V, ρ) be a right representation of $G_\infty \in \{B^\times, B_\infty^\times\}$ (resp. Σ_p) with coefficients in some commutative unitary ring R. If $g \in \mathbf{B}^\times(\mathbb{A})$, we will write $g_v \in B_v^\times$ for its v-component. When ρ is understood, we simply write vg_∞ (resp. vg_p) for $v\rho(g_\infty)$ (resp. $v\rho(g_p)$). Fix a character $\omega_0 : Z_{\mathrm{f}} \longrightarrow R^\times$ (resp. $\omega_{0,p} : Z_{\mathrm{f}} \longrightarrow R^\times$). Define $S(B_{\mathrm{f}}^\times, \rho)$ (resp. $S_p(B_{\mathrm{f}}^\times, \rho)$) to be the space of maps $\varphi : B_{\mathrm{f}}^\times \to V$ endowed with the $(B^\times, B_{\mathrm{f}}^\times)$-action (resp. $(B^\times, \Sigma_p(B_{\mathrm{f}}^\times))$-action) given by

$$(g\varphi u)(x) := \varphi(uxg_{\mathrm{f}})\rho(g_\infty^{-1}), \quad \text{where } g \in B^\times \text{ and } u \in B_{\mathrm{f}}^\times$$
$$(\text{resp. } (g\varphi u)(x) := \varphi(uxg_{\mathrm{f}})\rho(u_p), \text{ where } g \in B^\times \text{ and } u \in \Sigma_p(B_{\mathrm{f}}^\times)).$$

Then

$$S(B_{\mathrm{f}}^\times, \rho, \omega_0) := \{\varphi \in S(B_{\mathrm{f}}^\times, \rho) : \varphi(zx) = \omega_0(z)\varphi(z) \text{ for all } z \in Z_{\mathrm{f}}\}$$
$$(\text{resp. } S_p(B_{\mathrm{f}}^\times, \rho, \omega_{0,p}) := \{\varphi \in S_p(B_{\mathrm{f}}^\times, \rho) : \varphi(zx) = \omega_{0,p}(z)\varphi(x) \text{ for all } z \in Z_{\mathrm{f}}\})$$

is a sub $(B^\times, B_{\mathrm{f}}^\times)$-module (resp. sub $(B^\times, \Sigma_p(B_{\mathrm{f}}^\times))$-module). We also write

$$S\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_0\right) := S(B_{\mathrm{f}}^\times, \rho, \omega_0)^{(B^\times,1)} \ (\text{resp. } S_p\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_{0,p}\right) := S_p(B_{\mathrm{f}}^\times, \rho, \omega_{0,p})^{(B^\times,1)})$$

and

$$M(B_{\mathrm{f}}^\times, \rho, \omega_0) := S\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_0\right)^{(1,\mathcal{K})} \ (\text{resp. } M_p(B_{\mathrm{f}}^\times, \rho, \omega_{0,p}) := S_p\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_{0,p}\right)^{(1,\mathcal{K}^\diamond)}).$$

The former is called the *space of ρ-valued modular forms* and the latter the *space of ρ-valued p-adic modular forms*; they are Hecke modules as explained above. Also, setting

$$M_p^\diamond(B_{\mathrm{f}}^\times, \rho, \omega_{0,p}) := S_p\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_{0,p}\right)^{(1,\mathcal{K}^{\diamond\diamond})} = M_p(B_{\mathrm{f}}^\times, \rho, \omega_{0,p})^{K_p^\diamond},$$
$$M^\diamond(B_{\mathrm{f}}^\times, \rho, \omega_0) := S\left(B_{\mathrm{f}}^\times/B^\times, \rho_{/B^\times}, \omega_0\right)^{(1,\mathcal{K}^{\diamond\diamond})} = M(B_{\mathrm{f}}^\times, \rho, \omega_0)^{K_p^\diamond},$$

we get a $B_{\mathrm{f}}^{\times,p} \times \mathcal{H}(\Sigma_p)$-module, as explained above. We omit ω_0 from the notation when $Z_{\mathrm{f}} = 1$ and write $M(Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^\times, \rho) := M(B_{\mathrm{f}}^\times, \rho, \omega_{0,p})$ when ω_0 is the trivial character of Z_{f}. Sometimes we will abusively replace ρ with the underlying subspace V in the notation. The same shorthands apply in the p-adic case.

The following remarks are easily checked.

Remark 2.1. Suppose that $\chi_0 : B_{\mathrm{f}}^\times \to R^\times$ is a character with the property that $\chi_0(K) = 1$ for some $K \in \mathcal{K}$ and that $\chi_\infty : G_\infty \to R^\times$ is a character with the property that $\chi_{0|B^\times} = \chi_{\infty|B^\times}$.

(1) If $\varphi \in M(B_{\mathrm{f}}^\times, \rho, \omega_0)$, then the rule $(\chi_0\varphi)(x) := \chi_0(x)\varphi(x)$ defines an element $\chi_0\varphi \in M\left(B_{\mathrm{f}}^\times, \rho(\chi_\infty), \chi_{0|Z}\omega_0\right)$.
(2) We have $\chi_0 \in M\left(B_{\mathrm{f}}^\times, R(\chi_\infty), \chi_{0|Z}\right)$.

Remark 2.2. Suppose that $\chi_{0,p} : B_{\mathrm{f}}^\times \to R_p^\times$ and $\chi_p : \Sigma_p \to R_p^\times$ are a characters such that there is some $K \in \mathcal{K}^\diamond$ such that $\chi_{0,p}(u) = \chi_p(u_p)^{-1}$ every $u \in K$ and $\chi_{0,p|B^\times} = 1$.

(1) If $\varphi \in M_p(B_{\mathrm{f}}^\times, \rho, \omega_{0,p})$, then the rule $(\chi_{0,p}\varphi)(x) := \chi_{0,p}(x)\varphi(x)$ defines an element $\chi_0\varphi \in M_p\left(B_{\mathrm{f}}^\times, \chi_p\rho, \chi_{0,p|Z}\omega_{0,p}\right)$.
(2) We have $\chi_{0,p} \in M\left(B_{\mathrm{f}}^\times, R(\chi_p), \chi_{0,p|Z}\right)$.

The connection between modular forms and p-adic modular forms is the content of the following proposition. We suppose that we are given $\omega_0 : Z_{\mathrm{f}} \to R^\times$ and coefficient rings $i_\infty : R \subset R_\infty$ and $i_p : R \subset R_p$. For a character χ of some group with values in $R^\times$, we let $i_{p*}(\chi) := i_p \circ \chi$ and $i_{\infty*}(\chi) := i_\infty \circ \chi$. We also assume that we are given a representation ρ_p (resp. ρ_∞) of $B_p^\times$ (resp. $B_\infty^\times$) with coefficients in R_p (resp. R_∞) with the property that

$$\rho := \rho_{p|B^\times} = \rho_{\infty|B^\times} \subset \rho_p, \rho_\infty$$

takes coefficients in R: we distinguish between the R-valued representation ρ of $B^\times$ and the R_p-valued representation $\rho_{p|B^\times}$ of $B^\times$.

Lemma 2.3. *The rules*

$$\begin{array}{cc} M\left(B_{\mathrm{f}}^\times,\rho_{p|B^\times}\right)\to M_p\left(B_{\mathrm{f}}^\times,\rho_p\right) & M_p\left(B_{\mathrm{f}}^\times,\rho_{p|B^\times}\right)\to M\left(B_{\mathrm{f}}^\times,\rho_p\right) \\ \varphi\mapsto\psi_\varphi:\psi_\varphi(x):=\varphi(x)\,x_p^{-1} & \psi\mapsto\varphi_\psi:\varphi_\psi(x):=\psi(x)\,x_p \end{array}$$

set up a right $\Sigma_p\left(B_{\mathrm{f}}^\times\right)$-equivariant bijection and $M\left(B_{\mathrm{f}}^\times,\rho\right)\subset M\left(B_{\mathrm{f}}^\times,\rho_{p|B^\times}\right)$ is identified with the submodule of those $\psi\in M_p\left(B_{\mathrm{f}}^\times,\rho_p\right)$ such that $\psi(x)\in\rho\subset\rho_p$ for every $x\in B_{\mathrm{f}}^\times$. Furthermore, if ρ_p has central character ω_{ρ_p} and $(-)_p:Z_{\mathrm{f}}\to Z_p$ is the projection induced by $B_{\mathrm{f}}^\times\to B_p^\times$, then the bijection induces

$$M\left(B_{\mathrm{f}}^\times,\rho,\omega_0\right)\subset M\left(B_{\mathrm{f}}^\times,\rho_{p|B^\times},i_{p*}(\omega_0)\right)\simeq M_p\left(B_{\mathrm{f}}^\times,\rho_p,\omega_{0,p}\right)$$

with $\omega_{0,p}:=i_{p*}(\omega_0)\,\omega_{\rho_p}^{-1}\left((-)_p\right)$. These identifications and inclusions are $\mathcal{H}\left(\Sigma_p\left(G_{\mathrm{f}}\right)\right)$-equivariant.

Proof. Indeed the above rules induce a $\left(B^\times,\Sigma_p\left(B_{\mathrm{f}}^\times\right)\right)$-equivariant identification $S\left(B_{\mathrm{f}}^\times,\rho_{p|B^\times}\right)\simeq S_p\left(B_{\mathrm{f}}^\times,\rho_p\right)$. Since $B_{\mathrm{f}}=B_{\mathrm{f}}^{\times,p}\times B_p^\times$ topologically, $\mathcal{K}^\diamond\subset\mathcal{K}$ is a cofinal family and we have $S^{\mathcal{K}}=S^{\mathcal{K}^\diamond}$ for every $B_{\mathrm{f}}^\times$-module. Hence, taking $(B^\times,\mathcal{K})$-invariant on the left and $(B^\times,\mathcal{K}^\diamond)$-invariants on the right yields the $\Sigma_p\left(B_{\mathrm{f}}^\times\right)$-equivariant identification. Then one checks that the correspondence has the required properties. □

Example 2.4. The above lemma notably applies in the following setting: let $\mathbf{V}$ be an algebraic representation of $B^\times$ over $R=\mathbb{Q}\subset\mathbb{C},\mathbb{Q}_p$ (or a quadratic field $R=K\subset\mathbb{C},\mathbb{Q}_p$ which splits B) and set $(V,\rho):=\mathbf{V}(\mathbb{Q})$ (or $\mathbf{V}(K)$), $(V_\infty,\rho_\infty):=\mathbf{V}(\mathbb{C})$ (with the action restricted to $B_\infty^\times\subset\mathbf{B}^\times(\mathbb{C})$) and $\left(V_p,\rho_p\right):=\mathbf{V}(\mathbb{Q}_p)$. We can also take R large enough for the values of the characters ω_{ρ_p} and ω_0 to take values in it (and replace $\mathbb{Q}_p$ by a finite extension F and consider the action restricted to $B_p^\times\subset\mathbf{B}^\times(F)$).

Suppose that we are given $\omega_0:Z_{\mathrm{f}}\to R^\times$ (resp. $\omega_{0,p}:Z_{\mathrm{f}}\to R_p^\times$) and write $X\left(B^\times,\omega_0\right)$ (resp. $X_p\left(B^\times,\omega_{0,p}\right)$) to denote the set of couples (χ_0,χ_∞) as in Remark 2.1 (resp. Remark 2.2) such that $\chi_{0|Z}=\omega_0$ (resp. $\chi_{0,p|Z}=\omega_{0,p}$). We also suppose, in the following remark, that we are given characters $\chi_0:B_{\mathrm{f}}^\times\to R^\times$, $\chi_p:B_p^\times\to R_p^\times$ and $\chi_\infty:G_\infty\to R_\infty^\times$ such that $\chi:=\chi_{p|B^\times}=\chi_{\infty|B^\times}:B^\times\to R^\times$ and that χ_p and χ_∞ are continuous with respect to topologies on R_p and, respectively, R_∞. Then the condition $\chi_{p|B^\times}=\chi_{\infty|B^\times}$ implies that χ_p and χ_∞ determine each other.

Remark 2.5. We have $(\chi_0,\chi)\in X\left(B^\times,\omega_0\right)$ (equivalently, $\left(i_{\infty*}(\chi_0),\chi_\infty\right)\in X\left(B^\times,i_{\infty*}(\omega_0)\right)$) if and only if $\left(\chi_{0,p},\chi_p\right)\in X_p\left(B^\times,\omega_{0,p}\right)$, where $\chi_{0,p}(x):=i_{p*}(\chi_0)(x)\,\chi_p^{-1}(x_p)$ and $\omega_{0,p}(z):=i_{p*}(\omega_0)(z)\,\chi_p^{-1}(z_p)$. In this case, regarding χ_0 (resp. $\chi_{0,p}$) as modular forms via Remark 2.1 (2) (resp. Remark 2.2 (1)), we have that χ_0 corresponds to $i_{p*}\left(\chi_p\right)\simeq\chi_{0,p}$ via the inclusion inclusions/identifications

$$\chi_0\in M\left(G_0,R(\chi),\omega_0\right)\subset M\left(G_0,R_p\left(\chi_p\right),i_{p*}(\omega_0)\right)\simeq M_p\left(G_0,R_p\left(\chi_p\right),\omega_{0,p}\right)$$

provided by Lemma 2.3. Furthermore, via the inclusions/identifications provided by Lemma 2.3, twisting by χ_0 (or $i_{p*}\left(\chi_p\right)$) as in Remark 2.1 (1) corresponds to twisting by $\chi_{0,p}$ as in Remark 2.2 (2).

2.1. **The norm forms.** Here is a key example of modular form. Consider the (normalized) absolute value functions $|-|_v:\mathbb{Q}_v^\times\to\mathbb{R}_+^\times$, $|-|_{\mathbb{A}_{\mathrm{f}}}:\mathbb{A}_{\mathrm{f}}^\times\to\mathbb{Q}_+^\times$ and $|-|_{\mathbb{A}}:\mathbb{A}^\times\to\mathbb{R}_+^\times$. Setting

$$\mathrm{N}:=|-|_{\mathbb{A}_{\mathrm{f}}}^{-1}|-|_\infty:\mathbb{A}^\times=\mathbf{G}_m(\mathbb{A})\longrightarrow\mathbb{C}^\times$$

gives a function such that $\mathrm{N}_{\mathrm{f}}\mathrm{N}_\infty^{-1}=|-|_{\mathbb{A}}^{-1}$ is trivial on $\mathbb{Q}^\times=\mathbf{G}_m(\mathbb{Q})$ by the product formula. Suppose that $\chi:\mathbf{B}^\times\to\mathbf{G}_m$ is an algebraic character and that $\tau:R^\times\to G$ is a character. Then we define $\tau_\chi:\mathbf{B}^\times(R)\overset{\chi_R}{\to}R^\times\overset{\tau}{\to}G$. In particular, we have the continuous character

$$\mathrm{N}_\chi:\mathbf{B}^\times(\mathbb{A})\xrightarrow{\chi_{\mathbb{A}}}\mathbb{A}^\times\xrightarrow{\mathrm{N}}\mathbb{R}_+^\times$$

and, recalling that $\mathrm{N_f} = |-|_{\mathbb{A}_\mathrm{f}}^{-1}$ and $\mathrm{N}_\infty = |-|_\infty$,

$$\mathrm{N}_{\chi,\mathrm{f}} : \mathbf{B}^\times(\mathbb{A}_\mathrm{f}) \xrightarrow{\chi_{\mathbb{A}_\mathrm{f}}} \mathbb{A}_\mathrm{f}^\times \xrightarrow{|-|_{\mathbb{A}_\mathrm{f}}^{-1}} \mathbb{Q}_+^\times \text{ and } \mathrm{N}_{\chi,\infty} : \mathbf{B}^\times(\mathbb{R}) \xrightarrow{\chi_\infty} \mathbb{R}^\times \xrightarrow{|-|_\infty} \mathbb{R}_+^\times.$$

Of course $\mathrm{N}_{\chi,\mathrm{f}}$ (resp. $\mathrm{N}_{\chi,\infty}$) is the finite adele (resp. ∞) component of N_χ, as suggested by the notation. If $\kappa : \mathbb{Q}_+^\times \to R^\times$ is a character (that we usually write exponentially $r \mapsto r^\kappa$), we can also define

$$\mathrm{N}_{\chi,\mathrm{f}}^\kappa : \mathbf{G}(\mathbb{A}_\mathrm{f}) \xrightarrow{\mathrm{N}_{\chi,\mathrm{f}}} \mathbb{Q}_+^\times \xrightarrow{\kappa} R^\times$$

Note that $\chi_\infty(\mathbf{B}^\times(\mathbb{R})) = \chi_\infty\left(\mathbf{B}^\times(\mathbb{R})^\circ\right) \subset \mathbb{R}_+^\times$ (because B is definite), implying that $\chi_\mathbb{Q}(\mathbf{B}^\times(\mathbb{Q})) \subset \mathbb{Q}_+^\times$ and we may consider $\kappa_\chi := \kappa \circ \chi_\mathbb{Q}$. If $V = (V, \rho)$ is a representation of $\mathbf{G}(\mathbb{R})$ with coefficients in R, we write $V(\kappa_\chi) = (V, \rho(\kappa_\chi))$ for the representation $\rho(\kappa_\chi)(g)(v) := \kappa_\chi(g)\rho(g)v$.

Remark 2.6. The continuous character N_χ is such that $\mathrm{N}_{\chi,\mathrm{f}}\mathrm{N}_{\chi,\infty}^{-1}$ is trivial on $\mathbf{B}^\times(\mathbb{Q})$ and we have

$$\mathrm{N}_{\chi,\mathrm{f}}^\kappa \in M\left(\mathbf{B}^\times(\mathbb{A}_\mathrm{f}), R(\kappa_\chi), \mathrm{N}_{\chi,\mathrm{f}|Z_\mathrm{f}}^\kappa\right)^K$$

for every open and compact $K \in \mathcal{K}$.

Proof. This is an application of the product formula and the fact that $\chi_\mathbb{Q}(\mathbf{B}^\times(\mathbb{Q})) \subset \mathbb{Q}_+^\times$, implying that Remark 2.1 (2) applies with $(\chi_0, \chi_\infty) = \left(\mathrm{N}_{\chi,\mathrm{f}}^\kappa, \mathrm{N}_{\chi,\infty}^\kappa\right)$ and $\mathrm{N}_{\chi,\infty}^\kappa := \kappa \circ \mathrm{N}_{\chi,\infty} = \kappa \circ \chi_\mathbb{Q}$. □

Taking

$$\chi = \mathrm{nrd} : \mathbf{B}^\times \to \mathbf{G}_m$$

yields, for every $\kappa = k \in \mathbb{Z}$ (viewed as the character $k : \mathbb{Q}^\times \to R$ via $r \mapsto r^k$), the norm form

$$\mathrm{Nrd}_\mathrm{f}^k := \mathrm{N}_{\chi,\mathrm{f}}^k \in M\left(\mathbf{G}(\mathbb{A}_\mathrm{f}), \mathbb{Q}(k), \mathrm{N}_{\mathrm{f}|Z_\mathrm{f}}^{2k}\right)^K, \text{ for every } K \in \mathcal{K}.$$

We also write $\mathrm{Nrd}_\infty^k := \mathrm{N}_{\chi,\infty}^k$ in this case. Applying Lemma 2.3 with $\rho = \mathbb{Q}(k)$, $\rho_p = \mathbb{Q}_p(k)$ and $\varphi = \mathrm{Nrd}_\mathrm{f}^k(-)_p \in M(\mathbf{G}(\mathbb{A}_\mathrm{f}), \mathbb{Q}_p(k), 2k)^K$ yields the p-adic modular form

$$\mathrm{Nrd}_p^k := \psi_\varphi \in M_p\left(\mathbf{G}(\mathbb{A}_\mathrm{f}), \mathbb{Q}_p(k), \mathrm{N}_{p|Z_\mathrm{f}}^{2k}\right)^K, \text{ for every } K \in \mathcal{K}.$$

We have, explicitly, writing $(-)_p$ for the p-component of an adelic element (and viewing the rational numbers diagonally in $\mathbb{A}_\mathrm{f}$):

$$\mathrm{Nrd}_p^k(x) = \mathrm{Nrd}_\mathrm{f}^k(x)_p\, x_p^{-1} = \left(\frac{\mathrm{Nrd}_\mathrm{f}(x)_p}{\mathrm{nrd}_p(x_p)}\right)^k = \left(\frac{\mathrm{Nrd}_\mathrm{f}(x)}{\mathrm{nrd}_{\mathbb{A}_\mathrm{f}}(x)}\right)_p^k \text{ and } \mathrm{N}_p(z) = \left(\frac{\mathrm{N_f}(z)}{z}\right)_p.$$

We now remark that $\frac{\mathrm{Nrd_f}(x)_p}{\mathrm{nrd}_p(x_p)} \in \mathbb{Z}_p^\times$ for any $x \in B_\mathrm{f}^\times$. Suppose now that we are given $\mathbf{k} : \mathbb{Z}_p^\times \to \mathcal{O}^\times$ which is a continuous group homomorphism, with $\mathcal{O}$ a locally convex $\mathbb{Q}_p$-algebra. Since $K_p^\diamond \subset B_p^\times$ is a compact subgroup, nrd_p maps it into the maximal open compact subgroup $\mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times$:

$$\mathrm{nrd}_p : K_p^\diamond \to \mathbb{Z}_p^\times.$$

If D is a $K_p^\diamond$-module with coefficients in $\mathcal{O}$, it makes sense to consider $D(\mathbf{k}) := D\left(\mathrm{nrd}_p^\mathbf{k}\right)$, the same representation with action $v \cdot_\mathbf{k} g := \mathrm{nrd}_p^\mathbf{k}(g)vg$. With this notation, we have

$$\mathrm{Nrd}_p^\mathbf{k} \in M_p\left(\mathbf{G}(\mathbb{A}_\mathrm{f}), \mathcal{O}(\mathbf{k}), \mathrm{N}_{p|Z_\mathrm{f}}^{2\mathbf{k}}\right)^K, \text{ for every } K \in \mathcal{K}^\diamond$$

which *interpolates* the norm forms $\mathrm{Nrd}_\mathrm{f}^k \simeq \mathrm{Nrd}_p^k$ with $k \in \mathbb{Z}$.

2.2. **Multilinear forms.** For $x \in B_{\mathrm{f}}^{\times}$ and $K \in \mathcal{K}$, define $\Gamma_K(x) = B^{\times} \cap x^{-1}Kx$. Being discrete (as $B^{\times}$ is) and compact (as K is), the set $\Gamma_K(x)$ is finite. For each $K \in \mathcal{K}$ and each set $R_K \subset B_{\mathrm{f}}^{\times}$ of representatives of $K \backslash B_{\mathrm{f}} / B^{\times}$, define

$$T_{R_K} : M\left(B_{\mathrm{f}}^{\times}, R\right)^K \longrightarrow R \quad \text{by} \quad T_{R_K}(f) := \mu(K) \sum_{x \in R_K} \frac{f(x)}{|\Gamma_K(x)|}.$$

It is easy to see that this is a well defined quantity which is independent from the choice of K (see [22, §3.1.1] for details), implying that this family defines

$$T_{B_{\mathrm{f}}^{\times}/B^{\times}} : M\left(B_{\mathrm{f}}^{\times}, R\right) \to R \text{ and } T_{Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}/B^{\times}} : M\left(Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}, R\right) \to R$$

where $T_{B_{\mathrm{f}}/B^{\times}} = T_K$ on $M\left(B_{\mathrm{f}}^{\times}, R\right)^K$ and $T_{Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}/B^{\times}} := T_{B_{\mathrm{f}}^{\times}/B^{\times} | M\left(Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}, R\right)}$.

Suppose that we are given a right representation (V, ρ) of $G_{\infty} \in \{B^{\times}, B_{\infty}^{\times}\}$ (resp. Σ_p) with coefficients in some commutative unitary ring R and group homomorphisms $k : \mathbb{Q}^{\times} \to R^{\times}$ (resp. $\mathbf{k} : \mathbb{Z}_p^{\times} \to R^{\times}$). If

$$\Lambda \in Hom_{R\left[B_{\infty}^{\times}\right]}(\rho, R(k)) \ \text{ (resp. } \in Hom_{R\left[K_p^{\diamond}\right]}(\rho, R(\mathbf{k}))\text{)},$$

Then we may define the R-linear morphisms

$$M(\Lambda) : M\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right) \to R \text{ (resp. } M_p(\Lambda) : M_p\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{p|Z_{\mathrm{f}}}^{2k}\right) \to R\text{)}$$

by the rule

$$M(\Lambda)(\varphi) := \mu(K) \textstyle\sum_{x \in K \backslash B^{\times}/B^{\times}} \dfrac{\Lambda(\varphi(x))}{|\Gamma_K(x)| \operatorname{Nrd}_{\mathrm{f}}^{k}(x)} \text{ if } \varphi \in M\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right)^K$$

$$\text{(resp. } M(\Lambda)(\varphi) := \mu(K) \textstyle\sum_{x \in K \backslash B^{\times}/B^{\times}} \dfrac{\Lambda(\varphi(x))}{|\Gamma_K(x)| \operatorname{Nrd}_{p}^{\mathbf{k}}(x)} \text{ if } \varphi \in M_p\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{p|Z_{\mathrm{f}}}^{2\mathbf{k}}\right)^K \text{)}$$

Alternatively, we have

$$M(\Lambda) : M\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right) \overset{\Lambda_*}{\to} M\left(B_{\mathrm{f}}^{\times}, R(k), \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right) \overset{\left\langle \cdot, \operatorname{Nrd}_{\mathrm{f}}^{-k} \right\rangle}{\to} R$$

$$\text{(resp. } M_p(\Lambda) : M_p\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{p|Z_{\mathrm{f}}}^{2\mathbf{k}}\right) \overset{\Lambda_*}{\to} M_p\left(B_{\mathrm{f}}^{\times}, R(k), \mathrm{N}_{p|Z_{\mathrm{f}}}^{2\mathbf{k}}\right) \overset{\left\langle \cdot, \operatorname{Nrd}_{p}^{-\mathbf{k}} \right\rangle}{\to} R\text{)},$$

where:

- Λ_* is the morphism induced by functoriality and Λ, i.e. $\Lambda_*(\varphi)(x) := \Lambda(\varphi(x))$;
- $\langle \cdot, \cdot \rangle$ is the natural pairing

$$\langle \cdot, \cdot \rangle : M\left(B_{\mathrm{f}}^{\times}, R(k), \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right) \otimes_R M\left(B_{\mathrm{f}}^{\times}, R(-k), \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{-2k}\right) \overset{\otimes}{\to} M\left(Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}, R\right) \overset{T_{Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}/B^{\times}}}{\to} R$$

$$\text{(resp. } \langle \cdot, \cdot \rangle : M_p\left(B_{\mathrm{f}}^{\times}, R(\mathbf{k}), \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2\mathbf{k}}\right) \otimes_R M_p\left(B_{\mathrm{f}}^{\times}, R(-\mathbf{k}), \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{-2\mathbf{k}}\right) \overset{\otimes}{\to} M_p\left(Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}, R\right) \overset{T_{Z_{\mathrm{f}} \backslash B_{\mathrm{f}}^{\times}/B^{\times}}}{\to} R\text{)},$$

  with $(\varphi_1 \otimes \varphi_2)(x) := \varphi_1(x) \varphi_2(x)$.

It follows from this description that the quantity is well defined. Finally, when $\rho = \rho_1 \otimes_R \ldots \otimes_R \rho_n$ and $\omega_{0,i}$ (resp. $\omega_{0,p,i}$) are such that

$$\omega_{0,1} \ldots \omega_{0,n} = \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k} \text{ (resp. } \omega_{0,p,1} \ldots \omega_{0,p,n} = \mathrm{N}_{p|Z_{\mathrm{f}}}^{2\mathbf{k}}\text{)}, \tag{10}$$

we can define the R-linear morphism

$$J(\Lambda) : M\left(B_{\mathrm{f}}^{\times}, \rho_1, \omega_{0,1}\right) \otimes_R \ldots \otimes_R M\left(B_{\mathrm{f}}^{\times}, \rho_n, \omega_{0,n}\right) \overset{\otimes}{\to} M\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2k}\right) \overset{M(\Lambda)}{\to} R$$

$$\text{(resp. } J_p(\Lambda) : M_p\left(B_{\mathrm{f}}^{\times}, \rho_1, \omega_{0,1}\right) \otimes_R \ldots \otimes_R M_p\left(B_{\mathrm{f}}^{\times}, \rho_n, \omega_{0,n}\right) \overset{\otimes}{\to} M_p\left(B_{\mathrm{f}}^{\times}, \rho, \mathrm{N}_{\mathrm{f}|Z_{\mathrm{f}}}^{2\mathbf{k}}\right) \overset{M_p(\Lambda)}{\to} R\text{)} \tag{11}$$

where $\otimes$ is obtained by iteration of $(\varphi_1 \otimes \varphi_2)(x) := \varphi_1(x) \otimes_R \varphi_2(x)$ in case $n = 2$.

Let us now assume that we are in the setting of Lemma 2.3, with representations $\rho_{i,p}$ (resp. $\rho_{i,\infty}$) of $B_p^{\times}$ (resp. $B_{\infty}^{\times}$) having coefficients in R_p (resp. R_{∞}), the property that $\rho_i := \rho_{i,p|B^{\times}} = \rho_{i,\infty|B^{\times}} \subset \rho_{i,p}, \rho_{i,\infty}$ has coefficients in R and suppose that (10) satisfied. Furthermore, suppose that $(\Lambda_p, \Lambda_\infty)$ is a couple of elements $\Lambda_p \in Hom_{R_p[B_p^\times]}(\rho_p, R_p(k))$ and $\Lambda_\infty \in Hom_{R_\infty[B_\infty^\times]}(\rho_\infty, R_\infty(k))$ with the property that

$$\Lambda := \Lambda_{p|\rho} = \Lambda_{\infty|\rho} \in Hom_{R[B^\times]}(\rho, R(k))$$

(Here we assume that $k : \mathbb{Q}^\times \to R$ and identify it with $i_{p*}(\chi) := i_p \circ \chi$ and $i_{\infty *}(\chi) := i_\infty \circ \chi$).

Proposition 2.7. *Via the inclusions/identifications provided by Lemma 2.3, we have*

$$J_p(\Lambda_p)_{|\otimes_{i=1}^n M\left(B_{\mathrm{f}}^\times, \rho_i, \omega_{0,i}\right)} = J(\Lambda)_{|\otimes_{i=1}^n M\left(B_{\mathrm{f}}^\times, \rho_i, \omega_{0,i}\right)}$$

on

$$\otimes_{i=1}^n M\left(B_{\mathrm{f}}^\times, \rho_i, \omega_{0,i}\right) \subset \otimes_{i=1}^n M_p\left(B_{\mathrm{f}}^\times, \rho_{i,p}, \omega_{0,p,i}\right), \otimes_{i=1}^n M\left(B_{\mathrm{f}}^\times, \rho_{i,\infty}, i_{\infty *}(\omega_{0,i})\right).$$

Proof. It is easily checked that all the canonical morphisms involved in the definition of $J_p(\Lambda_p)$ and $J(\Lambda_\infty)$ match: the non canonical ones, namely $\left\langle \cdot, \mathrm{Nrd}_{\mathrm{f}}^{-k} \right\rangle$ and $\left\langle \cdot, \mathrm{Nrd}_p^{-k} \right\rangle$ match because $\mathrm{Nrd}_{\mathrm{f}}^{-k}$ corresponds to Nrd_p^{-k} via Lemma 2.3. $\square$

2.3. **Pairings and adjointness.** Suppose that D (resp. E) is a Σ_D (resp. Σ_E) module, where Σ_D (resp. Σ_E) satisfies the assumption that was done on Σ_p, and we let $\omega_{0,p,D}, \omega_{0,p,E} : Z_{\mathrm{f}} \to R^\times$ be characters such that $\omega_{0,p,D}\omega_{0,p,E} = \omega_{0,p}$. We assume that we are given a group homomorphism $\mathbf{k} : \mathbb{Z}_p^\times \to R^\times$ and a pairing

$$\langle -, - \rangle \in Hom_{R[K_p^\diamond]}(D \otimes_R E, R(\mathbf{k})).$$

Then (11) gives

$$\langle -, - \rangle_{M_p} : M_p\left(B_{\mathrm{f}}^\times, D, \omega_{0,p,D}\right) \otimes_R M_p\left(B_{\mathrm{f}}^\times, E, \omega_{0,p,E}\right) \to R.$$

We suppose $\Sigma_D = \Sigma_p$, $\Sigma_D = \Sigma_p^\iota$ and $\left(K_p^\diamond\right)^\iota = K_p^\diamond \subset \Sigma_p \cap \Sigma_p^\iota$(as in case $K_p^\diamond = \Gamma_0(p\mathbb{Z}_p)$ and $\Sigma_p = \Sigma_0(p\mathbb{Z}_p)$) and that $\mathbf{Z} = \mathbf{Z}_{\mathbf{B}^\times} = \mathbf{G}_m$. Assuming that E has central character $\kappa_E : \mathbb{Z}_p^\times \to R^\times$, we can consider the second of the following compositions:

$$\mathrm{nrd}_{\mathrm{f}}^{\omega_{0,p,E}} : B_{\mathrm{f}}^\times \overset{\mathrm{nrd_f}}{\to} \mathbb{A}_{\mathrm{f}}^\times = Z_{\mathrm{f}} \overset{\omega_{0,p,E}}{\to} R^\times \text{ and } \mathrm{nrd}_p^{\kappa_E} : K_p^\diamond \overset{\mathrm{nrd}_p}{\to} \mathbb{Z}_p^\times \overset{\kappa_E}{\to} R^\times.$$

Suppose that $\mathbf{k}, \kappa_E : \mathbb{Z}_p^\times \to R^\times$ extends to a character $\widetilde{\mathbf{k}}, \widetilde{\kappa_E} : \mathbb{Q}_p^\times \to R^\times$. Then

$$\mathrm{nrd}_p^{\widetilde{\kappa_E}} : \Sigma_p \overset{\mathrm{nrd}_p}{\to} \mathbb{Q}_p^\times \overset{\widetilde{\kappa_E}}{\to} R^\times$$

is an extension of $\mathrm{nrd}_p^{\kappa_E}$ to Σ_p and we let $Hom_{R[\Sigma_p, \Sigma_p^\iota]}\left(D \otimes E, R\left(\widetilde{\mathbf{k}}\right)\right)$ be the set of those pairings such that

$$\langle v\sigma, w \rangle = \mathrm{nrd}_p^{\widetilde{\mathbf{k}} - \widetilde{\kappa_E}}(\sigma) \langle v, w\sigma^\iota \rangle \text{ for every } \sigma \in \Sigma_p.$$

We remark that, for every element $u \in K_p^\diamond$,

$$\langle vu, wu \rangle = \mathrm{nrd}_p^{\mathbf{k} - \kappa_E}(u) \langle v, wuu^\iota \rangle = \mathrm{nrd}_p^{\mathbf{k}}(u) \langle v, w \rangle,$$

so that $Hom_{R[\Sigma_p, \Sigma_p^\iota]}\left(D \otimes E, R\left(\widetilde{\mathbf{k}}\right)\right) \subset Hom_{R[K_p^\diamond]}(D \otimes E, R(\mathbf{k}))$.

Remark 2.8. Suppose now that $D \subset \widetilde{D}$ and $E \subset \widetilde{E}$, where $\widetilde{D}$ and $\widetilde{E}$ are $B_p^\times$-modules, the above inclusions are Σ_p and, respectively, Σ_p^ι-equivariant and that $\widetilde{E}$ has central character $\kappa_{\widetilde{E}} = \widetilde{\kappa_E}$ extending κ_E. If $\langle \cdot, \cdot \rangle \in Hom_{\mathcal{O}[K_p^\diamond]}(D \otimes E, R(\mathbf{k}))$ extends to $\langle \cdot, \cdot \rangle^\sim \in Hom_{\mathcal{O}[B_p^\times]}\left(\widetilde{D} \otimes \widetilde{E}, R\left(\widetilde{\mathbf{k}}\right)\right)$ then $\langle \cdot, \cdot \rangle \in Hom_{\mathcal{O}[\Sigma_p, \Sigma_p^\iota]}\left(D \otimes E, R\left(\widetilde{\mathbf{k}}\right)\right)$:

$$\langle v\sigma, w \rangle = \left\langle v\sigma, w\sigma^{-1}\sigma \right\rangle^\sim = \mathrm{nrd}_p^{\widetilde{\mathbf{k}}}(\sigma) \left\langle v, w\sigma^{-1} \right\rangle^\sim = \mathrm{nrd}_p^{\widetilde{\mathbf{k}} - \widetilde{\kappa_E}}(\sigma) \langle v, w\sigma^\iota \rangle.$$

In the following proposition, we suppose that $f \in M_p\left(B_{\mathrm{f}}^\times, D, \omega_{0,p,D}\right)^{K_1}$ and $g \in M_p\left(B_{\mathrm{f}}^\times, E, \omega_{0,p,E}\right)^{K_2}$ (and make a similar assumption for classical, i.e. non p-adic, modular forms in the M's spaces). Finally, we assume that

$$\langle -, - \rangle \in Hom_{R[\Sigma_p, \Sigma_p^\iota]}\left(D \otimes E, R\left(\widetilde{\mathbf{k}}\right)\right)$$

(but for classical modular forms, we suppose $\langle -,-\rangle \in Hom_{R[B^{\times}]}\left(D\otimes_R E, R\left(\widetilde{\mathbf{k}}\right)\right)$ where $\widetilde{\mathbf{k}}:\mathbb{Q}^{\times}\to R^{\times}$ and does not require E to have central character $\kappa_E:\mathbb{Q}^{\times}\to R^{\times}$). We write $T_{\pi}:=K_1\pi K_2$, $T_{\pi^{\iota}}:=K_2\pi^{\iota}K_1$ and $T_{\pi^{-1}}:=K_2\pi^{-1}K_1$.

Proposition 2.9. *We have the following formulas, in the p-adic case:*

$$\mu\left(K_2\right)^{-1}\langle f\mid T_{\pi},g\rangle=\mathrm{Nrd}_{\mathrm{f}}^{\widetilde{\mathbf{k}}}\left(\pi\right)_p\mathrm{nrd}_p^{-\widetilde{\kappa_E}}\left(\pi_p\right)\mathrm{nrd}_{\mathrm{f}}^{-\omega_{0,p,E}}\left(\pi\right)\mu\left(K_1\right)^{-1}\langle f,g\mid T_{\pi^{\iota}}\rangle .$$

For classical modular forms, $\mu\left(K_2\right)^{-1}\langle f\mid T_{\pi},g\rangle=\mathrm{Nrd}_{\mathrm{f}}^{\widetilde{\mathbf{k}}}\left(\pi\right)\mu\left(K_1\right)^{-1}\langle f,g\mid T_{\pi^{-1}}\rangle$ *and, whenever E has central character κ_E,* $\mu\left(K_2\right)^{-1}\langle f\mid T_{\pi},g\rangle=\mathrm{Nrd}_{\mathrm{f}}^{\mathbf{k}}\left(\pi\right)\mathrm{nrd}_{\mathrm{f}}^{-\kappa_E}\left(\pi\right)\mu\left(K_1\right)^{-1}\langle f,g\mid T_{\pi^{\iota}}\rangle$.

Proof. Note that K_i always contains a decomposable open and compact subgroup K_i' and, because $\pi\in B_{\mathrm{f}}^{\times}$, we see that $\pi_l\in K_{i,l}'\subset K_i$ for all but finitely may l's (the inclusion viewing $B_l^{\times}\subset B_{\mathrm{f}}^{\times}$ as the l-component of $B_{\mathrm{f}}^{\times}$). It follows that we may assume that π is concentrated at a finite number of components; then, we leave to the reader to check that we may assume that $\left|\Gamma_{K_i}\left(x\right)\right|=1$ for all $x\in B_{\mathrm{f}}^{\times}$ and $i=1,2$. Having made this reduction, we compute, for p-adic modular forms,

$$\begin{aligned}
\mu\left(K_2\right)^{-1}\langle f\mid K_1\pi K_2,g\rangle &= \sum\nolimits_{x\in K_2\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle\left(f\mid K_1\pi K_2\right)\left(x\right),g\left(x\right)\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(x\right)}\\
&= \sum\nolimits_{u\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash K_2,x\in K_2\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f\left(\pi ux\right)\pi_pu_p,g\left(x\right)\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(x\right)}\\
&= \sum\nolimits_{u\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash K_2,x\in K_2\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f\left(\pi ux\right)\pi_pu_p,g\left(ux\right)u_p\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(ux\right)\mathrm{nrd}_p^{\mathbf{k}}\left(u\right)}\\
&= \sum\nolimits_{u\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash K_2,x\in K_2\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f\left(\pi ux\right)\pi_p,g\left(ux\right)\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(ux\right)}\\
&= \sum\nolimits_{y\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f\left(\pi y\right)\pi_p,g\left(y\right)\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(y\right)}.
\end{aligned}$$

Here we have employed (9) in the second equality, the K_2-invariance of g and $\mathrm{Nrd}_p^{\mathbf{k}}$ in the third equality and the $K_p^{\diamond}$-equivariance of $\langle -,-\rangle$ in the fourth equality. Letting g, f, K_2, K_1 and π^{ι} play the roles of f, g, K_1, K_2 and π respectively, we also see that

$$\mu\left(K_1\right)^{-1}\langle f,g\mid K_2\pi^{\iota}K_1\rangle=\sum\nolimits_{z\in\left(K_1\cap\pi^{-\iota}K_2\pi^{\iota}\right)\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\left\langle f\left(z\right),g\left(\pi^{\iota}z\right)\pi_p^{\iota}\right\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(z\right)}.$$

Note, however, that $y\mapsto\pi y$ induces a well defined map $H\backslash B_{\mathrm{f}}^{\times}/B^{\times}\to\pi H\pi^{-1}\backslash B_{\mathrm{f}}^{\times}/B^{\times}$ for any subgroup H and we have $\pi^{\iota}H\pi^{-\iota}=\pi^{-1}H\pi$. Taking $H=K_2\cap\pi^{-1}K_1\pi$ we see that $\pi H\pi^{-1}=K_1\cap\pi^{-\iota}K_2\pi^{\iota}$. Making the change of variables $z=\pi y$, we have

$$\begin{aligned}
\mu\left(K_1\right)^{-1}\langle f,g\mid K_2\pi^{\iota}K_1\rangle &= \sum\nolimits_{y\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\left\langle f\left(\pi y\right),g\left(\pi^{\iota}\pi y\right)\pi_p^{\iota}\right\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(\pi y\right)}\\
&= \mathrm{nrd}_p^{\widetilde{\kappa_E}-\widetilde{\mathbf{k}}}\left(\pi_p\right)\mathrm{Nrd}_p^{-\mathbf{k}}\left(\pi\right)\sum\nolimits_{y\in\left(K_2\cap\pi^{-1}K_1\pi\right)\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f\left(\pi y\right)\pi_p,g\left(\pi^{\iota}\pi y\right)\rangle}{\mathrm{Nrd}_p^{\mathbf{k}}\left(y\right)}.
\end{aligned}$$

Here we have used $\left\langle v,w\pi_p^{\iota}\right\rangle=\mathrm{nrd}_p^{\widetilde{\kappa_E}-\widetilde{\mathbf{k}}}\left(\pi_p\right)\langle v\pi_p,w\rangle$. We now remark that $\pi^{\iota}\pi=\mathrm{nrd}\left(\pi\right)\in Z_{\mathrm{f}}$, so that $g\left(\pi^{\iota}\pi y\right)=\mathrm{nrd}_{\mathrm{f}}^{\omega_{0,p,E}}\left(\pi\right)g\left(y\right)$. It follows that

$$\mu\left(K_1\right)^{-1}\langle f,g\mid K_2\pi^{\iota}K_1\rangle=\mathrm{nrd}_p^{\widetilde{\kappa_E}-\widetilde{\mathbf{k}}}\left(\pi_p\right)\mathrm{Nrd}_p^{-\mathbf{k}}\left(\pi\right)\mathrm{nrd}_{\mathrm{f}}^{\omega_{0,p,E}}\left(\pi\right)\mu\left(K_2\right)^{-1}\langle f\mid K_1\pi K_2,g\rangle .$$

The relation $\mathrm{Nrd}_p^{\kappa}\left(x\right)=\left(\frac{\mathrm{Nrd}_{\mathrm{f}}\left(x\right)_p}{\mathrm{nrd}_p\left(x_p\right)}\right)^{\kappa}$ gives the claim:

$$\mathrm{nrd}_p^{\widetilde{\kappa_E}-\widetilde{\mathbf{k}}}\left(\pi_p\right)\mathrm{Nrd}_p^{-\mathbf{k}}\left(\pi\right)=\mathrm{nrd}_p^{\widetilde{\kappa_E}-\widetilde{\mathbf{k}}}\left(\pi_p\right)\frac{\mathrm{Nrd}_{\mathrm{f}}^{-\mathbf{k}}\left(\pi\right)_p}{\mathrm{nrd}_p^{-\mathbf{k}}\left(\pi_p\right)}=\mathrm{nrd}_p^{\widetilde{\kappa_E}}\left(\pi_p\right)\mathrm{Nrd}_{\mathrm{f}}^{-\widetilde{\mathbf{k}}}\left(\pi\right)_p .$$

For modular forms one finds, by a similar computation,

$$\mu(K_2)^{-1}\langle f\mid K_1\pi K_2,g\rangle_{K_2}=\sum\nolimits_{y\in(K_2\cap\pi^{-1}K_1\pi)\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f(\pi y),g(y)\rangle}{\mathrm{Nrd}_{\mathrm{f}}^{\mathbf{k}}(y)},$$
$$\mu(K_1)^{-1}\langle f,g\mid K_2\pi^{-1}K_1\rangle_{K_1}=\sum\nolimits_{z\in(K_1\cap\pi K_2\pi^{-1})\backslash B_{\mathrm{f}}^{\times}/B^{\times}}\frac{\langle f(z),g(\pi z)\rangle}{\mathrm{Nrd}_{\mathrm{f}}^{\mathbf{k}}(z)}.$$

The first equality in this case follows and the second, which can also be proved by a similar computation as above, is actually a consequence of the first in this setting, since one checks $g\mid K_2\pi^{-1}K_1=\mathrm{nrd}_{\mathrm{f}}^{-\kappa_E}(\pi)\cdot g\mid K_2\pi^{\iota}K_1$ (because $\mathrm{nrd}(\pi)\in Z_{\mathrm{f}}=Z_{B_{\mathrm{f}}^{\times}}$). □

3. The special value formula and its p-adic avatar

We are now going to recall the special value formula proved in [22], specialized to the triple product case, which can be regarded as a vector valued version of Ichino's formula [31] and a generalization of [7].

Let $E/\mathbb{Q}$ be a Galois splitting field for B and fix $\mathbf{B}_{/E}\simeq\mathbf{M}_{2/E}$ inducing $\mathbf{B}_{/E}^{\times}\simeq\mathbf{GL}_{2/E}$. If $k\in\mathbb{N}$, we let $\mathbf{P}_{k/E}$ be the left $\mathbf{GL}_{2/E}$-representation on two variables polynomials of degree k, the action being defined by the rule $(gP)(X,Y)=P((X,Y)g)$. We write $\mathbf{V}_k$ for the dual right representation. If $\underline{k}:=(k_1,...,k_r)\in\mathbb{N}^r$, we may identify $\mathbf{P}_{k_1/E}\otimes...\otimes\mathbf{P}_{k_r/E}$ with the space of $2r$-variable polynomials $\mathbf{P}_{\underline{k}/E}$ which are homogeneous of degree k_i in the i-th couple of variables $W_i:=(X_i,Y_i)$. Then $\mathbf{V}_{k_1/E}\otimes...\otimes\mathbf{V}_{k_r/E}$ is identified with the dual $\mathbf{V}_{\underline{k}/E}$ of $\mathbf{P}_{\underline{k}/E}$ and any $P\in\mathbf{P}_{\underline{k}/E}(-r)^{\mathbf{GL}_{2/E}}$, i.e. such that $gP=\det(g)^rP$, induces

$$\Lambda_P\in Hom_{\mathbf{GL}_{2/E}}\left(\mathbf{V}_{\underline{k}/E},\mathbf{1}_{/E}(r)\right)$$

by the rule $\Lambda_P(l):=l(P)$. Note also that, if $P\neq0$ then there is l such that $l(P)=1$ and we see that $\Lambda_P\neq0$. Setting $0\neq\delta^k(X_1,Y_1,X_2,Y_2):=\begin{vmatrix}X_1&Y_1\\X_2&Y_2\end{vmatrix}^k$, we have $\delta^1(W_1g,W_2g)=\det(g)\,\delta^1(W_1,W_2)$, from which it follows that $\delta_k\in\mathbf{P}_{k,k/E}$ and $g\delta^k=\det(g)^k\delta^k$. We deduce that $\langle-,-\rangle_{k/E}:=\Lambda_{\delta^k}\neq0$ satisfies the above requirement and, hence, defines

$$\langle-,-\rangle_{k/E}\in Hom_{\mathbf{GL}_{2/E}}\left(\mathbf{V}_{k/E}\otimes\mathbf{V}_{k/E},\mathbf{1}_{/E}(k)\right);\tag{12}$$

then the irreducibility of the $\mathbf{GL}_{2/E}$ representation $\mathbf{V}_{k/E}$ implies that this non-zero pairing is perfect and symmetric. Next, if $\underline{k}:=(k_1,k_2,k_3)\in\mathbb{N}^3$, we define

$$\langle-,-\rangle_{\underline{k}/E}:=\langle-,-\rangle_{k_1/E}\otimes\langle-,-\rangle_{k_2/E}\otimes\langle-,-\rangle_{k_3/E}\in Hom_{\mathbf{GL}_{2/E}^3}\left(\mathbf{V}_{\underline{k}/E}\otimes\mathbf{V}_{\underline{k}/E},\mathbf{1}_{/E}(\underline{k})\right).$$

We remark that, viewing a left representation as a right representation by means of the inversion we have $\mathbf{P}_{k/E}=\mathbf{V}_{k/E}^{\vee}$ (resp. $\mathbf{P}_{\underline{k}/E}=\mathbf{V}_{\underline{k}/E}^{\vee}$) and, hence, $\langle-,-\rangle_{k/E}$ (resp. $\langle-,-\rangle_{\underline{k}/E}$) induces $\mathbf{P}_{k/E}=\mathbf{V}_{k/E}^{\vee}\simeq\mathbf{V}_{k/E}(-k)$ (resp. $\mathbf{P}_{\underline{k}/E}=\mathbf{V}_{\underline{k}/E}^{\vee}\simeq\mathbf{V}_{\underline{k}/E}(-\underline{k}^*)$), which in turn induces

$$\langle-,-\rangle_{k/E}\in Hom_{\mathbf{GL}_{2/E}}\left(\mathbf{P}_{k/E}\otimes\mathbf{P}_{k/E},\mathbf{1}_{/E}(-k)\right)\text{ and }\langle-,-\rangle_{\underline{k}/E}\in Hom_{\mathbf{GL}_{2/E}^3}\left(\mathbf{P}_{\underline{k}/E}\otimes\mathbf{P}_{\underline{k}/E},\mathbf{1}_{/E}(-\underline{k})\right)$$

(and the latter is the tensor product of the former pairings).

If $\underline{k}:=(k_1,k_2,k_3)\in\mathbb{N}^3$, we define the quantities $\underline{k}^*:=\frac{k_1+k_2+k_3}{2}$, $\underline{k}_1^*:=\frac{-k_1+k_2+k_3}{2}$, $\underline{k}_2^*:=\frac{k_1-k_2+k_3}{2}$ and $\underline{k}_3^*:=\frac{k_1+k_2-k_3}{2}$. With a slight abuse of notation, we write $\mathbf{P}_{\underline{k}/E}$ and $\mathbf{V}_{\underline{k}/E}$ to denote the external tensor product, which is a representation of $\mathbf{GL}_{2/E}^3$. When $\underline{k}$ is balanced, we can also define

$$\Lambda_{\underline{k}/E}\in Hom_{\mathbf{GL}_{2/E}}\left(\mathbf{V}_{\underline{k}/E},\mathbf{1}_{/E}(\underline{k}^*)\right)$$

as follows. The balanced condition precisely means that $\underline{k}_i^*\in\mathbb{N}$ for $i=1,2,3$, so that we can consider

$$0\neq\Delta_{\underline{k}/E}:=\delta^{\underline{k}_1^*}(W_2,W_3)\,\delta^{\underline{k}_2^*}(W_1,W_3)\,\delta^{\underline{k}_3^*}(W_1,W_2)\in\mathbf{P}_{\underline{k}/E}.\tag{13}$$

We have $g\Delta_{\underline{k}/E}=\det(g)^{\underline{k}^*}\Delta_{\underline{k}/E}$. Hence $\Delta_{\underline{k}/E}\in\mathbf{P}_{\underline{k}/E}(-\underline{k}^*)^{\mathbf{GL}_{2/E}}$ and we may set $\Lambda_{\underline{k}/E}:=\Lambda_{\Delta_{\underline{k}/E}}\neq0$.

The following result is an application of the Clebsch-Gordan decomposition that we leave to the reader.

Lemma 3.1. *Suppose that $2\underline{k}^*=k_1+k_2+k_3\in2\mathbb{N}$ and $\underline{k}$ is balanced.*

(1) *There is a representation $\mathbf{V}_{\underline{k}}$ of $\mathbf{B}^{\times}$ (with the diagonal action) such that $E\otimes\mathbf{V}_{\underline{k}}\simeq\mathbf{V}_{\underline{k}/E}$ via $\mathbf{B}_{/E}^{\times}\simeq\mathbf{GL}_{2/E}$.*

(2) *We have, setting* $\mathbf{B}_1^\times := \ker(\mathrm{nrd})$,

$$\dim\left(Hom_{\mathbf{B}_1^\times}\left(\mathbf{V}_{\underline{k}},\mathbf{1}\right)\right)=\dim\left(Hom_{\mathbf{SL}_{2/E}}\left(\mathbf{V}_{\underline{k}/E},\mathbf{1}_{/E}\right)\right)=1.$$

For $i=1,2,3$, let ω_i be an unitary Hecke character of the form $\omega_i=\omega_{\mathrm{f},i}\otimes sgn\,(-)^{k_i}$ and set $\omega_{0,i}:=\omega_{\mathrm{f},i}\mathrm{N}_{\mathrm{f}}^{k_i}$. Assuming that $\omega_1\omega_2\omega_3=1$, we see that

$$\underline{k}^*\in\mathbb{N}\text{ and }\mathrm{N}_{\mathrm{f}}^{2\underline{k}^*}=\omega_{1,0}\omega_{2,0}\omega_{3,0}. \tag{14}$$

It follows from an adelic version of the Peter-Weyl Theorem (see [22, Proposition 6.1]) that, if $\pi_i=\pi_{i,\mathrm{f}}\otimes\mathbf{V}_{k_i,\mathbb{C}}^u$ is an irreducible unitary automorphic form with central character ω_i (and $\mathbf{V}_{k_i,\mathbb{C}}^u$ the unitary twist of $\mathbf{V}_{k_i,\mathbb{C}}$), the rule $f_i\left(\Lambda\otimes\varphi\right)(x):=\mathrm{Nrd}_{\mathrm{f}}^{-k_i/2}\left(x_{\mathrm{f}}\right)\mathrm{Nrd}_\infty^{\underline{k}/2}\left(x_\infty\right)\Lambda\left(\varphi\left(x_f\right)x_\infty^{-1}\right)$ defines a canonical $\mathbf{B}^\times(\mathbb{A})$-equivariant identification:

$$f_i:\mathbf{V}_{k_i,\mathbb{C}}^{\vee,u}\otimes_{\mathbb{C}}M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i,\mathbb{C}},\omega_{0,i}\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-k_i/2}\pi_{i,\mathrm{f}}\right]\simeq A\left(\mathbf{B}^\times(\mathbb{A}),\omega_i\right)[\pi_i], \tag{15}$$

where $(-)[\theta]$ means taking the θ-component and $A\left(\mathbf{B}^\times(\mathbb{A}),\omega_i\right)$ is the space of K-finite automorphic forms. We remark the we could have considered automorphic forms for the algebraic group $\mathbf{B}^{\times 3}$ and, with $\Pi:=\pi_1\otimes\pi_2\otimes\pi_3$, so that $\Pi=\Pi_{\mathrm{f}}\otimes\mathbf{V}_{\underline{k},\mathbb{C}}^u$, we have the canonical $\mathbf{B}^\times(\mathbb{A})$-equivariant identification:

$$f:\mathbf{V}_{\underline{k},\mathbb{C}}^{\vee,u}\otimes_{\mathbb{C}}M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},\mathbb{C}},\omega_0\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}\Pi_{\mathrm{f}}\right]\simeq A\left(\mathbf{B}^{\times 3}(\mathbb{A}),\omega\right)[\Pi], \tag{16}$$

where $\omega=(\omega_1,\omega_2,\omega_3)$, $\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}:=\left(\mathrm{Nrd}_{\mathrm{f}}^{-k_1/2},\mathrm{Nrd}_{\mathrm{f}}^{-k_2/2},\mathrm{Nrd}_{\mathrm{f}}^{-k_3/2}\right)$, $\mathrm{Nrd}_\infty^{\underline{k}/2}$ and $\mathrm{N}_{\mathrm{f}}^{\underline{k}}=(\mathrm{N}_{\mathrm{f}}^{k_1},\mathrm{N}_{\mathrm{f}}^{k_2},\mathrm{N}_{\mathrm{f}}^{k_3})$ are defined in a similar way, $\omega_0:=\omega_{\mathrm{f}}\mathrm{N}_{\mathrm{f}}^{\underline{k}}$ and $f\left(\Lambda\otimes\varphi\right)(x):=\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}\left(x_{\mathrm{f}}\right)\mathrm{Nrd}_\infty^{\underline{k}/2}\left(x_\infty\right)\Lambda\left(\varphi\left(x_f\right)x_\infty^{-1}\right)$.

3.1. **Periods.** From now on, we will need to fix an embedding $E\subset\mathbb{C}$, which allows us to regard $\mathbf{V}_{k_i,E}$-valued (resp. $\mathbf{V}_{\underline{k},E}$-valued) modular forms as $\mathbf{V}_{k_i,\mathbb{C}}$-valued (resp. $\mathbf{V}_{\underline{k},\mathbb{C}}$-valued) and suppose that E is large enough to contain the values of the characters $\omega_{i,0}$. Let us remark that we have a morphism

$$M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i,E},\omega_{0,i}\right)\longrightarrow M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i,E},\mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}\right) \tag{17}$$

defined by the rule $\varphi\mapsto\check{\varphi}$, where $\check{\varphi}(x):=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}}(x)\varphi(x)=\mathrm{Nrd}_{\mathrm{f}}^{k_i}(x)\mathrm{nrd}_{\mathrm{f}}^{-\omega_{0,i}}(x)\varphi(x)$ with our usual shorthand $\mathrm{nrd}_{\mathrm{f}}^{-\chi}:=\chi^{-1}\circ\mathrm{nrd}_{\mathrm{f}}$ (the equality because $\omega_{i,\mathrm{f}}=\omega_0\mathrm{N}_{\mathrm{f}}^{-k_i}$). Indeed, Remark 2.1 applies as follows. Since ω_i is a Hecke character, we have $\mathrm{nrd}_{\mathrm{f}|B^\times}^{-\omega_{\mathrm{f},i}}=\mathrm{nrd}_{\infty|B^\times}^{\omega_{i,\infty}}$ and we see that $\left(\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}},\mathrm{nrd}_\infty^{\omega_{i,\infty}}\right)\in X\left(B^\times,\mathrm{nrd}_{\mathrm{f}|\mathbf{Z}_{\mathbf{B}^\times}(\mathbb{A}_{\mathrm{f}})}^{-\omega_{\mathrm{f},i}}\right)$. Finally, because B is definite, $\mathrm{nrd}_\infty(\mathbf{B}^\times(\mathbb{R}))\subset\mathbb{R}_+^\times$, so that $\mathrm{nrd}_\infty^{\omega_{\infty,i}}=1$ (because $\omega_{\infty,i}=sgn(-)^{k_i}$), and $\mathrm{nrd}_{\mathrm{f}|\mathbf{Z}_{\mathbf{B}^\times}(\mathbb{A}_{\mathrm{f}})}^{-\omega_{\mathrm{f},i}}=\omega_{\mathrm{f},i}^{-2}$, so that $\mathrm{nrd}_{\mathrm{f}|\mathbf{Z}_{\mathbf{B}^\times}(\mathbb{A}_{\mathrm{f}})}^{-\omega_{\mathrm{f},i}}\omega_{0,i}=\mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}$: we have checked that

$$\left(\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}},1\right)\in X\left(B^\times,\omega_{\mathrm{f},i}^{-2}\right). \tag{18}$$

A similar twist works with the modular forms on $\mathbf{B}^{\times 3}$, with $\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}}$ replaced by $\mathrm{nrd}_{\mathrm{f}}^{-\underline{\omega}_{\mathrm{f}}}:=\left(\mathrm{nrd}^{-\omega_{\mathrm{f},1}},\mathrm{nrd}^{-\omega_{\mathrm{f},2}},\mathrm{nrd}^{-\omega_{\mathrm{f},3}}\right)$ and $\mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}$ replaced by $\mathrm{N}_{\mathrm{f}}^{2\underline{k}}\omega_0^{-1}$, where $\mathrm{N}_{\mathrm{f}}^{2\underline{k}}=(\mathrm{N}_{\mathrm{f}}^{2k_1},\mathrm{N}_{\mathrm{f}}^{2k_2},\mathrm{N}_{\mathrm{f}}^{2k_3})$. Then, of course, $\varphi\mapsto\check{\varphi}$ commutes with the tensor product.

It follows that we can consider

$$(-,-)_{k_i}:M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i/E},\omega_{0,i}\right)\otimes_{\mathbb{C}}M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i/E},\omega_{0,i}\right)\longrightarrow E \tag{19}$$

and

$$(-,-)_{\underline{k}}:M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},/E},\omega_0\right)\otimes_{\mathbb{C}}M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},E},\omega_0\right)\longrightarrow E$$

defined as follows. Let us write again

$$\langle -,-\rangle_{k_i/E}:M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i/E},\omega_{0,i}\right)\otimes_E M\left(B_{\mathrm{f}}^\times,\mathbf{V}_{k_i/E},\mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}\right)\longrightarrow E$$

for the morphism (11) induced by $\langle -,-\rangle_{k_i/E}$ (see (12)) on modular forms (defined because $\omega_{0,i}\mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}=\mathrm{N}_{\mathrm{f}}^{2k_i}$). Then we define

$$(\varphi_1,\varphi_2)_{k_i}:=\langle\varphi_1,\check{\varphi}_2\rangle_{k_i/E}=\mu(K)\textstyle\sum_{x\in K\backslash B^\times/B^\times}\dfrac{\langle\varphi_1(x),\varphi_2(x)\rangle_{k_i/E}}{|\Gamma_K(x)|\,\mathrm{nrd}_{\mathrm{f}}^{\omega_{0,i}}(x)},$$

if φ_1 and $\check{\varphi}_2$ are K-invariant. Working in a similar way with the algebraic group $\mathbf{B}^{\times 3}$ we get $(-,-)_{\underline{k}}$. Alternatively, the right hand side of the equality

$$(-,-)_{\underline{k}} = (-,-)_{k_1} \otimes_E (-,-)_{k_2} \otimes_E (-,-)_{k_3}$$

gives an alternative definition.

Lemma 3.2. *We have that $(-,-)_{\underline{k}}$ is a perfect pairing.*

Proof. Let us first remark that, from the definition of the adelic Peter-Weyl theorem [22, Proposition 6.1], the fact that $\check{\varphi}(x) := \mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}}(x)\varphi(x)$ and the fact that $\mathrm{nrd}_{\infty}^{-\omega_{\infty,i}} = 1$ there is a commutative diagram

$$\begin{array}{ccc} \mathbf{V}_{k_i,\mathbb{C}}^{\vee,u} \otimes_{\mathbb{C}} M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k_i,\mathbb{C}}, \omega_{0,i}\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-k_i/2}\pi_{i,\mathrm{f}}\right] & \longrightarrow & \mathbf{V}_{k_i,\mathbb{C}}^{\vee,u} \otimes_{\mathbb{C}} M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k_i,\mathbb{C}}, \mathrm{N}_{\mathrm{f}}^{2k_i}\omega_{0,i}^{-1}\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-k_i/2}\pi_{i,\mathrm{f}}^{\vee}\right] \\ f_i \downarrow & & \downarrow f_i \\ A\left(\mathbf{B}^{\times}(\mathbb{A}), \omega_i\right)[\pi_i] & \longrightarrow & A\left(\mathbf{B}^{\times}(\mathbb{A}), \omega_i\right)[\pi_i^{\vee}] \end{array}$$

if we define the lower arrow via $\psi \mapsto \check{\psi}$, where $\check{\psi}(x) := \mathrm{nrd}_{\mathbb{A}}^{-\omega_i}(x)\psi(x)$. There is also a similar commutative diagram for automorphic forms on $\mathbf{B}^{\times 3}$, which is the tensor product of the above commutative diagrams. Then, arguing as in the proof of [22, Proposition 6.1 (2)] (based on the Schur orthogonality relations, but for linear pairings, to which [22, Lemma 3.7] and (7) are applied), we see that

$$\left\langle \psi_1, \check{\psi}_2 \right\rangle_{L_2} := \int_{[\mathbf{B}^{\times 3}(\mathbb{A})]_{\mathbf{Z}_{\mathbf{B}^{\times 3}}}} \psi_1(x)\check{\psi}_2(x)\mu_{[\mathbf{B}^{\times 3}(\mathbb{A})]_{\mathbf{Z}_{\mathbf{B}^{\times 3}}}}(x) = \frac{\langle \Delta_1, \Delta_2 \rangle_{\underline{k}}}{d_{\underline{k}}} \langle \varphi_1, \check{\varphi}_2 \rangle_{\underline{k}} = \frac{\langle \Delta_1, \Delta_2 \rangle_{\underline{k}}}{d_{\underline{k}}} (\varphi_1, \varphi_2)_{\underline{k}}$$

if $\psi_h = f(\Delta_h \otimes_{\mathbb{C}} \varphi_h)$ is in $A\left(\mathbf{B}^{\times 3}(\mathbb{A}), \omega\right)[\Pi]$ and $\left[\mathbf{B}^{\times 3}(\mathbb{A})\right]_{\mathbf{Z}_{\mathbf{B}^{\times 3}}} = \mathbf{Z}_{\mathbf{B}^{\times 3}}(\mathbb{A})\backslash\mathbf{B}^{\times 3}(\mathbb{A})/\mathbf{B}^{\times 3}(\mathbb{Q})$ with the product measure. Here $d_{\underline{k}}$ is the formal degree of $\mathbf{V}_{\underline{k},\mathbb{C}}$ (which only depends on the Haar measure, once $\langle -,- \rangle_{\underline{k}}$ on $\mathbf{P}_{\underline{k}/E} = \mathbf{V}_{\underline{k}/E}^{\vee}$ is obtained from $\langle -,- \rangle_{\underline{k}}$ on $\mathbf{V}_{\underline{k}/E}$ tautologically as above). Recalling that the formal degree of a representation of $\mathbf{B}^{\times 3}(\mathbb{R})$ equals the dimension times the inverse of the total measure $m_{\mathbf{Z_B}\backslash\mathbf{B}^{\times},\infty}^{3}$ of $\mathbf{Z}_{\mathbf{B}^{\times 3}}(\mathbb{R})\backslash\mathbf{B}^{\times 3}(\mathbb{R})$ (by compactness of $\mathbf{Z}_{\mathbf{B}^{\times 3}}(\mathbb{R})\backslash\mathbf{B}^{\times 3}(\mathbb{R})$), we find

$$\left\langle \psi_1, \check{\psi}_2 \right\rangle_{L^2} = \frac{m_{\mathbf{Z_B}\backslash\mathbf{B}^{\times},\infty}^{3} \langle \Delta_1, \Delta_2 \rangle_{\underline{k}}}{(k_1+1)(k_2+1)(k_3+1)} (\varphi_1, \varphi_2)_{\underline{k}}. \tag{20}$$

Because the left hand side is a perfect pairing, our claim follows. (It is not difficult to see that $\langle -,- \rangle_{\underline{k}/E}$ on modular forms is perfect because it is induced by the perfect pairing $\langle -,- \rangle_{\underline{k}/E}$ on the finite dimensional vector spaces $\mathbf{V}_{\underline{k}}$; next, noticing that $\varphi \mapsto \check{\varphi}$ is a linear bijection, gives a direct proof of the perfectness of $(-,-)_{\underline{k}}$. However, we will need (20) below). □

Fix an identification $\mathbf{B}\left(\mathbb{A}_{\mathrm{f}}^{\mathrm{Disc}(B)}\right) \simeq \mathbf{M}_2\left(\mathbb{A}_{\mathrm{f}}^{\mathrm{Disc}(B)}\right)$ and, for an integer N such that $(N, \mathrm{Disc}(B)) = 1$, write $K_0^{\mathrm{Disc}(B)}(N) \subset \mathbf{B}^{\times}\left(\mathbb{A}_{\mathrm{f}}^{\mathrm{Disc}(B)}\right)$ (resp. $K_1^{\mathrm{Disc}(B)}(N) \subset K_0^{\mathrm{Disc}(B)}(N)$) for the subgroup which corresponds to matrices with integral coefficients having lower left entry $c \equiv 0 \bmod (N)$ (resp. $c \equiv 0 \bmod (N)$ and upper left entry $a = 1$). Setting $\mathcal{O}_{\mathrm{Disc}(B)} := \prod_{l|\mathrm{Disc}(B)} \mathcal{O}_{B_v}^{\times}$ we can define

$$K_0(N) := K_0^{\mathrm{Disc}(B)}(N) \times \mathcal{O}_{\mathrm{Disc}(B)}^{\times} \text{ and } K_1(N) := K_1^{\mathrm{Disc}(B)}(N) \times \mathcal{O}_{\mathrm{Disc}(B)}^{\times}.$$

Assuming that $\mu_{\varphi(N)} \subset E$, we can decompose

$$M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k,E}\right)^{K_1(N)} = \bigoplus\nolimits_{\varepsilon:\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^{\times} \to \mathcal{O}^{\times}} M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k,E}\right)^{K_1(N)}(\varepsilon),$$

where $M(\varepsilon)$ is the submodule of elements $x \in M$ such that $xu = \varepsilon(u)x$ if we define $\varepsilon(u) := \varepsilon(a_u)$ for a_u the upper left entry of $u \in K_0(N)$. We also have

$$M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k,E}\right)^{K_1(N)}(\varepsilon) = M\left(B_{\mathrm{f}}^{\times}, \mathbf{V}_{k,E}, \omega_0^{\varepsilon,k}\right)^{K_0(N)}.$$

Example 3.3. Suppose that $\varphi=\varphi_1\otimes\varphi_2\otimes\varphi_3$ and that

$$\varphi_i\in M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E}\right)^{K_1(N_i)}(\varepsilon_i)=M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E},\omega_0^{\varepsilon_i,k_i}\right)^{K_0(N_i)}.$$

Then we see that $\varphi\mapsto\check{\varphi}$ is a map

$$M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E},\omega_0^{\varepsilon_i,k_i}\right)^{K_0(N_i)}\longrightarrow M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E},\omega_0^{\varepsilon_i^{-1},k_i}\right)^{K_0(N_i)}.$$

Indeed, $\omega_0^{\varepsilon_i,k_i}\omega_0^{\varepsilon_i^{-1},k_i}=\mathrm{N}_{\mathrm{f}}^{2k_i}$, so that we have

$$\langle -,-\rangle_{k_i/E}:M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E},\omega_0^{\varepsilon,k_i}\right)\otimes_E M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_i,E},\omega_0^{\varepsilon_i^{-1},k_i}\right)\longrightarrow E$$

and $(\varphi_i,\psi_i)_{k_i}=\langle\varphi_i,\check{\psi}_i\rangle_{k_i/E}$ by definition. Write $\widehat{\omega}_{N_i}\in B_{\mathrm{f}}^{\times}$ for the matrix concentrated at the primes l such that $l^{e_l}\parallel N_i$, where we have $(\widehat{\omega}_{N_i})_l=\omega_{l^{e_l}}:=\begin{pmatrix}0&-1\\ l^{e_l}&0\end{pmatrix}$ and write W_{N_i} for the Hecke operator it induces. If $\varphi_i^{\flat}\in M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k,E},\omega_0^{\varepsilon_i,k}\right)^{K_0(N_i)}$ is a newvector, then setting $\varphi_i^{\flat\flat}:=\varphi_i^{\flat}\mid W_{N_i}$ it is easily checked that $(\check{\varphi_i^{\flat\flat}})$ is a newvector in the dual representation, hence a scalar multiple of $\overline{\varphi_i^{\flat}}$ and then the proof of Lemma 3.2 shows that $\left(\varphi_i^{\flat},\varphi_i^{\flat\flat}\right)_{k_i}\neq 0$; in particular, setting $\varphi^{\flat}:=\varphi_1^{\flat}\otimes\varphi_2^{\flat}\otimes\varphi_3^{\flat}$ and $\varphi^{\flat\flat}:=\varphi_1^{\flat\flat}\otimes\varphi_2^{\flat\flat}\otimes\varphi_3^{\flat\flat}$ we see that

$$\Omega\left(\varphi^{\flat}\right):=\left(\varphi^{\flat},\varphi^{\flat\flat}\right)_{\underline{k}}\neq 0. \tag{21}$$

3.2. **The special value formula.** It follows from (14), that we can consider the quantity $t_{\underline{k}}:=J\left(\Lambda_{\underline{k}/E}\right)$ defined by (11):

$$t_{\underline{k}}:M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},E},\omega_0\right)=M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_1,E},\omega_{0,1}\right)\otimes_E M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_2,E},\omega_{0,2}\right)\otimes_E M\left(B_{\mathrm{f}}^{\times},\mathbf{V}_{k_3,E},\omega_{0,3}\right)\to E.$$

The choice of $\Lambda_{\underline{k}/E}\in\mathbf{V}_{\underline{k},E}^{\vee}$ (and $E\in\mathbb{C}$) yields, via (16), the embedding

$$f_{\Lambda_{\underline{k}/E}}:M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},\mathbb{C}},\omega_0\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}\Pi_{\mathrm{f}}\right]\hookrightarrow A\left(\mathbf{B}^{\times 3}\left(\mathbb{A}\right),\omega\right)[\Pi]$$

for every irreducible automorphic representation Π of $\mathbf{B}^{\times 3}$. Let us write Π' for the automorphic representation of $\mathbf{GL}_2^3$ which corresponds to Π under the Jacquet-Langlands correspondence.

Before stating our next result, we need to recall the definitions of some relevant quantities. Let $\langle -,-\rangle_{L^2}$ be the pairing defined before (20) and fix non-zero $\mathbf{B}^{\times 3}(\mathbb{Q}_v)$-invariant pairings $\langle -,-\rangle_v$ between Π_v and its dual representation. The irreducibility of Π_v implies that there is a non-zero constant C such that

$$\left\langle\psi^{\flat},\psi^{\flat\vee}\right\rangle_{L^2}=C\prod\nolimits_v\left\langle\psi_v^{\flat},\psi_v^{\flat\vee}\right\rangle_v. \tag{22}$$

Then one defines the bilinear form on $\Pi_v\times\Pi_v^{\vee}$ via the formula

$$I_v(\psi_v\otimes\psi_v^{\vee}):=\frac{L_v\left(1,\Pi_v',\mathrm{Ad}\right)}{\zeta_{\mathbb{Q}_v}^2\left(2\right)L_v\left(1/2,\Pi_v'\right)}\int_{\mathbf{Z_B}(\mathbb{Q}_v)\backslash\mathbf{B}^{\times}(\mathbb{Q}_v)}\left\langle\psi_v\Pi_v(x)^{-1},\psi_v^{\vee}\right\rangle_v\mu_{\mathbf{Z_B}\backslash\mathbf{B}^{\times},v}(x), \tag{23}$$

depending on the choice of the local measure, and set

$$C_v^{\psi_v^{\flat},\psi_v^{\flat\vee}}(\psi_v,\psi_v^{\vee}):=\frac{I_v(\psi_v\otimes\psi_v^{\vee})}{\left\langle\psi_v^{\flat},\psi_v^{\flat\vee}\right\rangle_v}. \tag{24}$$

Note also that (16) shows that

$$f_{\Lambda_{\underline{k}/E}}:\mathbb{C}\Lambda_{\underline{k}/E}\otimes M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},\mathbb{C}},\omega_0\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}\Pi_{\mathrm{f}}\right]\xrightarrow{\sim}\Pi_{\mathrm{f}}.$$

Suppose that φ, $\varphi^{\flat}$ and $\varphi^{\flat\flat}$ in $M\left(B_{\mathrm{f}}^{\times 3},\mathbf{V}_{\underline{k},\mathbb{C}},\omega_0\right)\left[\mathrm{Nrd}_{\mathrm{f}}^{-\underline{k}/2}\Pi_{\mathrm{f}}\right]$ corresponds to pure tensors $\psi:=f_{\Lambda_{\underline{k}/E}}(\varphi)=\otimes_v\psi_v$, $\psi^{\flat}:=f_{\Lambda_{\underline{k}/E}}\left(\varphi^{\flat}\right)=\otimes_v\psi_v^{\flat}$ and $\psi^{\flat\flat}:=f_{\Lambda_{\underline{k}/E}}\left(\varphi^{\flat\flat}\right)=\otimes_v\psi_v^{\flat\flat}$. Setting $\psi^{\flat\vee}:=(\check{\psi^{\flat\flat}})$ (where $\psi^{\flat\flat}\mapsto(\check{\psi^{\flat\flat}})$ is the operation in the lower horizontal row in the proof of Lemma 3.2), we define

$$I_v(\varphi)=I_v(\psi_v):=I_v(\psi_v,\check{\psi}_v)\text{ and }C_v^{\varphi^{\flat},\varphi^{\flat\flat}}(\varphi)=C_v^{\varphi^{\flat},\varphi^{\flat\flat}}(\psi_v):=C_v^{\psi_v^{\flat},\psi_v^{\flat\vee}}(\psi_v,\check{\psi}_v). \tag{25}$$

(Note that $(\check{\psi}^{\flat\flat})$ is again a pure tensor, being the product of $\psi^{\flat\flat}$ by a pure tensor).

The following result is deduced in [22, Theorem 8.2] from [31] or [24] and the Jacquet conjecture proved in [24].

Theorem 3.4. *Suppose that $\underline{k}$ is balanced and that $\omega_i = \omega_{i,\mathrm{f}} \otimes sgn(-)^{k_i}$ are unitary Hecke characters such that $\omega_1\omega_2\omega_3 = 1$, implying $\underline{k}^* \in \mathbb{N}$. Consider the quantity*

$$t_{\underline{k}}(\varphi) = \mu(K_\varphi) \sum_{x \in K_\varphi \backslash B_{\mathrm{f}}^\times / B^\times} \frac{\Lambda_{\underline{k}}(\varphi_1(x) \otimes \varphi_2(x) \otimes \varphi_3(x))}{|\Gamma_{K_\varphi}(x)| \operatorname{Nrd}_{\mathrm{f}}^{\underline{k}^*}(x)},$$

where $K_\varphi \in \mathcal{K}$ is such that $K_\varphi \subset K_{\varphi_1} \cap K_{\varphi_2} \cap K_{\varphi_3}$ and

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \bigotimes_{i=1}^3 M\left(B_{\mathrm{f}}^\times, \mathbf{V}_{k_i,\mathbb{C}}, \omega_{i,0}\right)^{K_{\varphi_i}} = M\left(B_{\mathrm{f}}^{\times 3}, \mathbf{V}_{\underline{k},E}, \omega_0\right)^{K_{\varphi_1} \times K_{\varphi_2} \times K_{\varphi_3}}.$$

(1) *We have the equality*

$$t_{\underline{k}}^2(\varphi) = \frac{C}{2^9 3^2} \frac{\zeta_{\mathbb{Q}}^2(2) L(1/2, \Pi')}{L(1, \Pi', \mathrm{Ad})} \prod_v I_v(\varphi) = \frac{(\varphi^\flat, \varphi^{\flat\flat})_{\underline{k}}}{2L(1, \Pi', \mathrm{Ad})} L(1/2, \Pi') \prod_{v \neq \infty} C_v^{\varphi^\flat, \varphi^{\flat\flat}}(\varphi) \tag{26}$$

as quadratic forms on

$$f_{\Lambda_{\underline{k}/E}} : M\left(B_{\mathrm{f}}^{\times 3}, \mathbf{V}_{\underline{k},E}, \omega_0\right)\left[\operatorname{Nrd}_{\mathrm{f}}^{-\underline{k}/2} \Pi_{\mathrm{f}}\right] \hookrightarrow A\left(\mathbf{B}^{\times 3}(\mathbb{A}), \omega\right)[\Pi],$$

where $C \neq 0$ is the constant defined in (22) below, $I_v(\varphi)$ and $C_v^{\varphi^\flat, \varphi^{\flat\flat}}(\varphi)$ are defined in (25) and the second equality depends on the choice of vectors $\varphi^\flat, \varphi^{\flat\flat} \in \operatorname{Nrd}_{\mathrm{f}}^{\underline{k}/2} \Pi_{\mathrm{f}} \subset M\left(B_{\mathrm{f}}^{\times 3}, \mathbf{V}_{\underline{k},E}, \omega_0\right)$ such that $(\varphi^\flat, \varphi^{\flat\flat})_{\underline{k}} \neq 0$ (see Lemma 3.2 for their existence and (21) for a specific choice).

(2) *Suppose that $B = B_{\Pi'}$ is the quaternion algebra predicted by [34]. Then there exists φ whose associated local constants I_v are all non-zero and, hence, $L(\Pi', 1/2) \neq 0$ if and only if $t_{\underline{k}} \neq 0$ on $M\left(B_{\mathrm{f}}^{\times 3}, \mathbf{V}_{\underline{k},E}, \omega_0\right)\left[\operatorname{Nrd}_{\mathrm{f}}^{-\underline{k}/2} \Pi_{\mathrm{f}}\right]$.*

Proof. (1) Let us explain how to deduce the result from [22, Theorem 8.2] in the form we need here. There, it is taken the normalization from [31, Theorem 1.1], which requires [31, (1.3) and (1.4)]. We will explicitly fix our local measures in §3.2.1 below in such a way that [31, (1.4)] is in force and (7) and the conditions before it are satisfied (so that [22, Theorem 7.2] is in force).

If φ (resp. $\varphi^\vee$ in the dual representation) correspond to a pure tensor $\psi := f_{\Lambda_{\underline{k}/E}}(\varphi) = \otimes_v \psi_v$ (resp. $\psi^\vee := f_{\Lambda_{\underline{k}/E}}(\varphi^\vee) = \otimes_v \psi_v^\vee$), then [31, Theorem 1.1] (but where we allow C to be arbitrary, i.e. [31, (1.4)] is in force but [31, (1.3)] may be not) gives, thanks to [22, Theorem 7.2 and §8.1]:

$$\begin{aligned} t_{\underline{k}}(\varphi)\, t_{\underline{k}}(\varphi^\vee) &= \frac{C}{2^3 m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}^2} \frac{\zeta_{\mathbb{Q}}^2(2) L(1/2, \Pi')}{L(1, \Pi', \mathrm{Ad})} \prod_v I_v(\psi_v \otimes \psi_v^\vee) \\ &= \frac{\left\langle \psi^\flat, \psi^{\flat\vee} \right\rangle_{L^2}}{2^3 m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}^2} \frac{\zeta_{\mathbb{Q}}^2(2) L(1/2, \Pi')}{L(1, \Pi', \mathrm{Ad})} \prod_v C_v^{\psi_v^\flat, \psi_v^{\flat\vee}}(\psi_v, \psi_v^\vee), \end{aligned} \tag{27}$$

where the latter equality holds when $\left\langle \psi^\flat, \psi^{\flat\vee} \right\rangle_{L^2} \neq 0$. If we specialize to the case where $\varphi^\vee := \check{\varphi}$, then we see that $t_{\underline{k}}(\varphi) = t_{\underline{k}}(\check{\varphi})$ because we are twisting by $\operatorname{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}}$, which restricts on the diagonally embedded center of $\mathbf{B}^\times$ to 1 and $t_{\underline{k}}$ only depend on its diagonal restriction. Define $\psi^\flat := f_{\Lambda_{\underline{k}/E}}(\varphi^\flat)$, $\psi^{\flat\flat} := f_{\Lambda_{\underline{k}/E}}(\varphi^{\flat\flat})$ and $\psi^{\flat\vee} := (\check{\psi}^{\flat\flat})$. The first formula is obtained from the definition (25), once we recall that $\check{\psi} = f_{\Lambda_{\underline{k}/E}}(\check{\varphi}) = \otimes_v \check{\psi}_v$ (from the commutative diagram for automorphic forms on $\mathbf{B}^{\times 3}$ analogous to those displayed in the proof of Lemma 3.2) and we will fix the measures in such a way that $m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty} = 24$.

It follows from (16) and the definition of $f_{\Lambda_{\underline{k}/E}}$ that we have $\psi_\infty = \psi_\infty^\flat = \psi_\infty^{\flat\flat} = \Lambda_{\underline{k}} = \Delta_{\underline{k}}$ (via $\mathbf{V}_{\underline{k},\mathbb{C}}^{\vee,u} = \mathbf{P}_{\underline{k},\mathbb{C}}^u$ provided by the tautological evaluation pairing) and $\check{\psi}_\infty = (\check{\psi}^{\flat\flat})_\infty = \psi_\infty$ because there is no twist at infinity in the definition of $\check{\psi}$ (as remarked in the proof of Lemma 3.2, $\operatorname{nrd}_\infty^{-\omega_{\infty,i}} = 1$). Then, the invariance property of $\Delta_{\underline{k}}$ under the (unitarized) diagonal action implies that the integral that appears in (24) for $v=\infty$ is $m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}\left\langle \Delta_{\underline{k}},\Delta_{\underline{k}}\right\rangle_\infty$. Hence we find

$$C_\infty^{\varphi^\flat,\varphi^{\flat\flat}}(\psi_\infty)=m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}\frac{L_\infty(1,\Pi'_v,\mathrm{Ad})}{\zeta_{\mathbb{R}}^2(2)\,L_\infty(1/2,\Pi'_v)}.$$

Also, (20) specializes to

$$\left\langle \psi^\flat,\psi^{\flat\vee}\right\rangle_{L^2}=\frac{m^3_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}\left\langle \Delta_{\underline{k}},\Delta_{\underline{k}}\right\rangle_{\underline{k}}}{(k_1+1)(k_2+1)(k_3+1)}\left(\varphi^\flat,\varphi^{\flat\flat}\right)_{\underline{k}}.$$

Inserting these last two equations in (27) gives the formula:

$$t^2_{\underline{k}}(\varphi)=\gamma_{\infty,\underline{k}}\zeta^2_{\mathbb{Q},\mathrm{f}}(2)\frac{(\varphi^\flat,\varphi^{\flat\flat})_{\underline{k}}}{L(1,\Pi',\mathrm{Ad})}L(1/2,\Pi')\prod\nolimits_{v\neq\infty}C_v^{\varphi^\flat,\varphi^{\flat\flat}}(\psi_v)$$

where

$$\gamma_{\infty,\underline{k}}=\frac{1}{2^3m^2_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}}\frac{\langle\psi,\check{\psi}\rangle_{L^2}}{(\varphi^\flat,\varphi^{\flat\flat})_{\underline{k}}}C_\infty^{\varphi^\flat,\varphi^{\flat\flat}}(\psi_\infty)\zeta^2_{\mathbb{R}}(2)=\frac{m^2_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}}{2^3}\frac{\left\langle \Delta_{\underline{k}},\Delta_{\underline{k}}\right\rangle_{\underline{k}}L_\infty(1,\Pi'_v,\mathrm{Ad})}{(k_1+1)(k_2+1)(k_3+1)\,L_\infty(1/2,\Pi'_v)}.$$

The claim is proved, once we fix local measures which in turn fix the local constants $C_v^{\varphi^\flat}(-)$ in such a way that $m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}=24$, make explicit the ratio $\frac{\langle\Delta_\infty,\Delta_\infty\rangle_{\underline{k}}L_\infty(1,\Pi'_v,\mathrm{Ad})}{(k_1+1)(k_2+1)(k_3+1)L_\infty(1/2,\Pi'_v)}=\frac{1}{4\pi^4}$ and insert the value $\zeta_{\mathbb{Q},\mathrm{f}}(2)=\frac{\pi^2}{6}$: this is detailed in 3.2.1 below.

(2) This is proved in [22, Theorem 8.1 (3)], as a consequence of the Jacquet's conjecture proved in [24], using [22, Theorem 8.1 (2)] and (27). $\square$

Remark 3.5. If one wants to make (26) explicit, the first equality is poorly useful because I_v depends on the choice of $\langle -,-\rangle_v$ and, hence, on the fixed local model. On the other hand, $C_v^{\varphi^\flat,\varphi^{\flat\flat}}$ is much more canonical: in fact, we see from (23) and (24) that $C_v^{\psi_v^\flat,\psi_v^{\flat\vee}}(\psi_v,\psi_v^\vee)$ does not depend on the choice of the local pairing $\langle -,-\rangle_v$; also, if ψ_v and $\psi_v^\vee$ linearly depends on $\psi_v^\flat$ and, respectively, $\psi_v^{\flat\vee}$, we see that $C_v^{\psi_v^\flat,\psi_v^{\flat\vee}}(\psi_v,\psi_v^\vee)$ does not depend on the line spanned by either ψ_v and $\psi_v^\flat$ or $\psi_v^\vee$ and $\psi_v^{\flat\vee}$. This is very convenient in order to "transport" calculations from abstract local models.

Regarding $t^2_{\underline{k}}$ as the algebraic part of $L(1/2,\Pi)$ (see [22] for a justification), it follows from Theorem 3.4 that the relevant part to be interpolated is $t_{\underline{k}}$. Applying Proposition 2.7, we place $t_{\underline{k}}$ in a p-adic setting, making it correspond to $t_{\underline{k}}:=J_p\left(\Lambda_{\underline{k}/\mathbb{Q}_p}\right)$

$$t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3)=\mu(K_\varphi)\sum_{x\in K_\varphi\backslash B_{\mathrm{f}}^\times/B^\times}\frac{\Lambda_{\underline{k}/\mathbb{Q}_p}(\varphi_1(x)\otimes\varphi_2(x)\otimes\varphi_3(x))}{\left|\Gamma_{K_\varphi}(x)\right|\mathrm{Nrd}_p^{\underline{k}^*}(x)}. \tag{28}$$

We have already interpolated the association $\underline{k}\mapsto\mathrm{Nrd}_p^{\underline{k}^*}(x)$ in §2.1 and we will now proceed to interpolate the association $\underline{k}\mapsto\Lambda_{\underline{k}/\mathbb{Q}_p}$. To this end, we first review and prove some facts on distribution modules, by means of which p-adic families of modular forms are defined.

3.2.1. *Choice of measures and further computations.* Let us fix local measures in a such a way that the condition $\mu_{B_{\mathrm{f}}^\times}(K)\in\mathbb{Q}$ for some (and hence every) open and compact subgroup $K\subset B_{\mathrm{f}}^\times$ and the integration formula (7) are satisfied: at the same time this fix the constant $m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}$, that we have to determine, and show that [31, (1.4)] is in force. To this end, we first record the following lemma, whose proof is easy and left to the reader. Suppose that $\Gamma\subset G=G_0\times G_\infty$ is a discrete subgroup of a product of Hausdorff and locally compact topological groups G and that composition with the projection makes $\Gamma\subset G\to G_0$ a discrete subgroup such that G_0/Γ is compact. Let us fix Haar measures μ_G (resp. μ_{G_0}) on G (resp. G_0) and let $\mu_{G/\Gamma}$ (resp. $\mu_{G_0/\Gamma}$) be the G-invariant (resp. G_0-invariant) quotient measures on G/Γ (resp. G_0/Γ), normalized as usual. Let $Z=Z_0\times Z_\infty\subset G_0\times G_\infty=G$ be a fixed closed subgroup in the center. Using the compactness of G_0/Γ, it is not difficult to see that there is a (unique up to non-zero scalar) non-zero left G_0-invariant Radon measure $\mu_{Z_0\backslash G_0/\Gamma}$ on $Z_0\backslash G_0/\Gamma$ with the property that $\mu_{Z_0\backslash G_0/\Gamma}=\mu_{G_0/\Gamma|C_c(Z_0\backslash G_0/\Gamma)}$ if

$C_c\left(Z_0\backslash G_0/\Gamma\right) \subset C_c\left(G_0/\Gamma\right)$ by means of the pull-back induced by $\pi : G_0/\Gamma \to Z\backslash G_0/\Gamma$. On the other hand, we set $\mu_{Z\backslash G/\Gamma} := \mu_{\frac{G}{Z}/\frac{\Gamma Z}{Z}}$, where once again the subquotient measures on the right hand side are obtained from μ_G with the usual normalizations.

We normalize the Haar measure μ_{G_∞} on G_∞ so that $\mu_G = \mu_{G_0} \times \mu_{G_\infty}$ is satisfied:

$$\int_G f(g)\, d\mu_G(g) = \int_{G_0} \left(\int_{G_\infty} f(g_0 g_\infty)\, d\mu_{G_\infty}(g_\infty) \right) d\mu_{G_0}(g_0).$$

Assuming that $Z_\infty\backslash G_\infty$ is compact, we get the formula

$$\int_{Z\backslash G/\Gamma} f(x)\, d\mu_{Z\backslash G/\Gamma}(x) = c \int_{Z_0\backslash G_0/\Gamma} \left(\int_{Z_\infty\backslash G_\infty} f(x_0 x_\infty)\, d\mu_{Z_\infty/G_\infty}(x_\infty) \right) d\mu_{Z_0\backslash G_0/\Gamma}(x_0)$$

for some $c \in \mathbb{R}_+^\times$.

Lemma 3.6. *Suppose that G_0 is locally profinite and let $\mathcal{K} = \mathcal{K}(G_0)$ be the set of its open and compact subgroups.*

(1) *We have $c = 1$.*
(2) $\mu_{Z\backslash G/\Gamma}(Z\backslash G/\Gamma) = \mu_{Z_0\backslash G_0/\Gamma}(Z_0\backslash G_0/\Gamma)\, \mu_{Z_\infty/G_\infty}(Z_\infty/G_\infty)$.
(3) *If $\mu_{G_0}(K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}$, we have $\mu_{Z_0\backslash G_0/\Gamma}(Z_0\backslash G_0/\Gamma) \in \mathbb{Q}$.*

Let us apply this lemma with $\Gamma = B^\times$, $G_0 = B_{\mathrm{f}}^\times$ and $G_\infty = \mathbf{B}^\times(\mathbb{R})$. Let us write D for the discriminant of our quaternion algebras and fix an identification $\mathbf{B}\left(\mathbb{A}_{\mathrm{f}}^D\right) \simeq \mathbf{M}_2\left(\mathbb{A}_{\mathrm{f}}^D\right)$. Fix local measures $\mu_{\mathbf{B}^\times,l}$ (resp. $\mu_{\mathbf{Z_B}\backslash\mathbf{B}^\times,l}$) of $\mathbf{B}^\times(\mathbb{Q}_l)$ (resp. $\mathbf{Z_B}(\mathbb{Q}_l)\backslash\mathbf{B}^\times(\mathbb{Q}_l)$) at the finite primes l as in [42, 2.2]: if $l \nmid D$ (resp. $l \mid D$), $\mu_{\mathbf{B}^\times,l}$ is the Haar measure such that $\mu_{\mathbf{B}^\times,l}(\mathbf{GL}_2(\mathbb{Z}_l)) = 1$ (resp. $\mu_{\mathbf{B}^\times,l}\left(\mathcal{O}_{B_l}^\times\right) = (p-1)^{-1}$, where $\mathcal{O}_{B_l}$ is a maximal order in B_l) and $\mu_{\mathbf{Z_B}\backslash\mathbf{B}^\times,l}$ the quotient measure, normalized in the usual way. Then $\mu_{B_{\mathrm{f}}^\times}$ satisfies our first requirement. Next, we take $\mu_{\mathbf{B}^\times(\mathbb{A})}$ on $\mathbf{B}^\times(\mathbb{A})$ in such a way that the induced quotient measure $\mu_{\mathbf{Z_B}(\mathbb{A})\backslash\mathbf{B}^\times(\mathbb{A})}$ on $\mathbf{Z_B}(\mathbb{A})\backslash\mathbf{B}^\times(\mathbb{A})$ is the Tamagawa measure, so that [31, (1.4)] is in force. Finally, we fix $\mu_{\mathbf{B}^\times,\infty}$ via the formula $\mu_{\mathbf{B}^\times(\mathbb{A})} = \mu_{B_{\mathrm{f}}^\times} \times \mu_{\mathbf{B}^\times,\infty}$. It follows from Lemma 3.6 (1) that (7) is in force and from Lemma 3.6 (2) that we have:

$$\mu_{[\mathbf{B}^\times(\mathbb{A})]_{\mathbf{Z_B}}}\left(\left[\mathbf{B}^\times(\mathbb{A})\right]_{\mathbf{Z_B}}\right) = m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty}\, \mu_{[B_{\mathrm{f}}^\times]_{\mathbf{Z_B}}}\left(\left[B_{\mathrm{f}}^\times\right]_{\mathbf{Z_B}}\right).$$

The left hand side is the Tamagawa number of $\mathbf{Z_B}\backslash\mathbf{B}^\times$, which is known to be 2 (see [41, Theorem 3.2.1]). On the other hand, choosing the local measures at the finite primes $l \mid D$ in such a way that $\mu_{\mathbf{B}^\times,l}\left(\mathcal{O}_{B_l}^\times\right) = 1$, the total measure of $\left[B_{\mathrm{f}}^\times\right]_{\mathbf{Z_B}}$ is known to be $\frac{1}{12}\prod_{l\mid D}(l-1)$ by the Eichler's mass formula (see [43, Lemma 2.2 and Theorem 3.6, (3.17)], for example, where $\zeta_{\mathbb{Q},\mathrm{f}}(-1) = -1/12$). We deduce that $\mu_{[B_{\mathrm{f}}^\times]_{\mathbf{Z_B}}}\left(\left[B_{\mathrm{f}}^\times\right]_{\mathbf{Z_B}}\right) = \frac{1}{12}$ and, hence,

$$m_{\mathbf{Z_B}\backslash\mathbf{B}^\times,\infty} = 24.$$

Next, we recall that we have, setting $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$,

$$\begin{aligned} L_\infty(s, \Pi', \mathrm{Ad}) &= \pi^{-3}\Gamma_{\mathbb{C}}(s+k_1+2)\Gamma_{\mathbb{C}}(s+k_2+2)\Gamma_{\mathbb{C}}(s+k_3+2), \\ L_\infty(s, \Pi') &= \Gamma_{\mathbb{C}}\left(s+\underline{k}^*+\frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s+\underline{k}_1^*+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\underline{k}_2^*+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\underline{k}_3^*+\frac{1}{2}\right). \end{aligned}$$

Then we see that

$$\frac{L_\infty(1, \Pi', \mathrm{Ad})}{L_\infty(1/2, \Pi')} = \frac{1}{4\pi^4}\frac{\Gamma(k_1+2)\Gamma(\underline{k}^*+2)\Gamma(\underline{k}^*+2)}{\Gamma(\underline{k}^*+2)\Gamma(\underline{k}_1^*+1)\Gamma(\underline{k}_2^*+1)\Gamma(\underline{k}_3^*+1)}.$$

We claim that

$$\left\langle \Delta_{\underline{k}}, \Delta_{\underline{k}} \right\rangle_{\underline{k}} = \frac{(\underline{k}^*+1)!\underline{k}_1^*!\underline{k}_2^*!\underline{k}_3^*!}{k_1!k_2!k_3!} = \frac{\Gamma(\underline{k}^*+2)\,\Gamma(\underline{k}_1^*+1)\,\Gamma(\underline{k}_2^*+1)\,\Gamma(\underline{k}_3^*+1)}{\Gamma(k_1+1)\,\Gamma(k_2+1)\,\Gamma(k_3+1)},$$

from which the equality $\frac{\langle\Delta_{\underline{k}},\Delta_{\underline{k}}\rangle_{\underline{k}}L_\infty\left(1,\Pi'_v,\mathrm{Ad}\right)}{(k_1+1)(k_2+1)(k_3+1)L_\infty(1/2,\Pi'_v)}=\frac{1}{4\pi^4}$ follows. The pairing $\langle -,-\rangle_{k_i/E}$ on the polynomials is explicitly given by

$$\left\langle X_i^rY_i^{k_i-r},X_i^sY_i^{k_i-s}\right\rangle_{k_i/E}=\begin{cases}(-1)^{k_i-r}\binom{k_i}{r}^{-1} & \text{if } r+s=k_i,\\ 0 & \text{if } r+s\neq k_i.\end{cases}$$

Let us write $\delta_3:\mathbf{P}_{k_1,k_2,k_3}\to\mathbf{P}_{k_1+1,k_2+1,k_3}$ for the multiplication by $\delta^1(W_1,W_2)$ map and let $\delta_3^*:\mathbf{P}_{k_1+1,k_2+1,k_3}\to\mathbf{P}_{k_1,k_2,k_3}$ be its adjoint with respect to the perfect pairings $\langle -,-\rangle_{(k_1,k_2,k_3)}$ and $\langle -,-\rangle_{(k_1+1,k_2+1,k_3)}$ (we do not write the subscript $/E$ for brevity). Then one checks that $\delta_3(\Delta_{k_1,k_2,k_3})=\Delta_{k_1+1,k_2+1,k_3}$ and $\delta_3^*(\Delta_{k_1+1,k_2+1,k_3})=\frac{(\underline{k}^*+2)(\underline{k}_3^*+1)}{(k_1+1)(k_2+1)}\Delta_{k_1,k_2,k_3}$ (note that δ_3 and hence δ_3^* are $\mathbf{SL}_2$-equivariant, from which the equality $\delta_3^*(\Delta_{k_1+1,k_2+1,k_3})=\lambda\Delta_{k_1,k_2,k_3}$ is known a priori taking the $\mathbf{SL}_2$-invariants for a scalar factor λ[3]). Assuming by induction that we have proved our claim for $\underline{k}=(k_1,k_2,k_3)$, we find the claim for (k_1+1,k_2+1,k_3):

$$\begin{aligned}\langle\Delta_{k_1+1,k_2+1,k_3},\Delta_{k_1+1,k_2+1,k_3}\rangle_{k_1+1,k_2+1,k_3}&=\langle\delta_3(\Delta_{k_1,k_2,k_3}),\Delta_{k_1+1,k_2+1,k_3}\rangle_{k_1+1,k_2+1,k_3}\\&=\langle\Delta_{k_1,k_2,k_3},\delta_3^*(\Delta_{k_1+1,k_2+1,k_3})\rangle_{k_1,k_2,k_3}=\frac{(\underline{k}^*+2)(\underline{k}_3^*+1)}{(k_1+1)(k_2+1)}\langle\Delta_{k_1,k_2,k_3},\Delta_{k_1,k_2,k_3}\rangle_{k_1,k_2,k_3}\\&=\frac{(\underline{k}^*+2)(\underline{k}_3^*+1)}{(k_1+1)(k_2+1)}\frac{(\underline{k}^*+1)!\underline{k}_1^*!\underline{k}_2^*!\underline{k}_3^*!}{k_1!k_2!k_3!}=\frac{(\underline{k}^*+2)!\underline{k}_1^*!\underline{k}_2^*!(\underline{k}_3^*+1)!}{(k_1+1)!(k_2+1)!k_3!}.\end{aligned}$$

4. Spaces of homogeneous p-adic distribution spaces

4.1. **Locally analytic homogeneous distributions.** By a p-adic manifold X we always mean a locally compact and paracompact manifold over a fixed spherically complete non-archimedean p-adic field. For a Banach algebra $\mathcal{O}$, we let $\mathcal{A}(X,\mathcal{O})$ be the space of $\mathcal{O}$-valued locally analytic functions on X and set $\mathcal{D}(X,\mathcal{O}):=\mathcal{L}_{\mathcal{O}}(\mathcal{A}(X,\mathcal{O}),\mathcal{O})\subset\mathcal{L}(\mathcal{A}(X,\mathcal{O}),\mathcal{O})$, the strong $\mathcal{O}$-dual of $\mathcal{A}(X,\mathcal{O})$. If $f:X\to Y$ is a morphism of p-adic manifolds, we have

$$f_{\mathcal{O}}^*:\mathcal{A}(Y,\mathcal{O})\longrightarrow\mathcal{A}(X,\mathcal{O})\text{ and }f_*^{\mathcal{O}}:\mathcal{D}(X,\mathcal{O})\longrightarrow\mathcal{D}(Y,\mathcal{O}),$$

the first being the pull-back of functions $f_{\mathcal{O}}^*(F):=F\circ f$ and the second operation being the strong $\mathcal{O}$-dual of the first. We note that

$$f_*^{\mathcal{O}}\left(\delta_x^{\mathcal{O}}\right)=\delta_{f(x)}^{\mathcal{O}},\text{ for every } x\in X \tag{29}$$

if $\delta_\cdot^{\mathcal{O}}:X\to\mathcal{D}(X,\mathcal{O})$ denotes the Dirac distribution map. It is useful to remark that the $\mathcal{O}$-linear span of $\left\{\delta_x^{\mathcal{O}}:x\in X\right\}$ is dense in $\mathcal{D}(X,\mathcal{O})$ (see [21, (52)]): we refer to this fact using the set phrase "by density of Dirac distributions". It can be shown that there are topological identifications[4]

$$\mathbf{T}_{\mathcal{D}(X)}^{\mathcal{O}}:\mathcal{O}\widehat{\otimes}\mathcal{D}(X)\xrightarrow{\sim}\mathcal{D}(X,\mathcal{O})$$

and

$$\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}^{\mathcal{O}_1,\mathcal{O}_2}:\mathcal{D}(X_1,\mathcal{O}_1)\widehat{\otimes}_\iota\mathcal{D}(X_2,\mathcal{O}_2)\xrightarrow{\sim}\mathcal{D}\left(X_1\times X_2,\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2\right).$$

They are characterized by the equalities:

$$\mathbf{T}_{\mathcal{D}(X)}^{\mathcal{O}}\left(1_{\mathcal{O}}\widehat{\otimes}\delta_x\right)=\delta_x^{\mathcal{O}}\text{ and }\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}^{\mathcal{O}_1,\mathcal{O}_2}\left(\delta_{x_1}^{\mathcal{O}_1}\widehat{\otimes}_\iota\delta_{x_2}^{\mathcal{O}_2}\right)=\delta_{(x_1,x_2)}^{\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2}. \tag{30}$$

We will usually suppress the reference to the Banach algebra when this is the fixed p-adic field.

Suppose from now on that X is endowed with the action of a p-adic Lie group T, meaning that the action is given by a locally analytic map $a:T\times X\to X$. Then T naturally acts from the right on $\mathcal{A}(X,\mathcal{O})$ and

[3] In order to determine λ, note that

$$\lambda\langle P,\Delta_{k_1,k_2,k_3}\rangle_{k_1,k_2,k_3}=\langle P,\delta_3^*(\Delta_{k_1+1,k_2+1,k_3})\rangle_{k_1+1,k_2+1,k_3}=\langle\delta_3P,\Delta_{k_1+1,k_2+1,k_3}\rangle_{k_1+1,k_2+1,k_3}.$$

A good choice is to take $P=Y_1^{k_1}\otimes X_2^{\underline{k}_3^*}Y_2^{\underline{k}_1^*}\otimes X_3^{k_3}$.

[4] We write $V\otimes_\iota W$ (resp. $V\otimes W$) to denote $V\otimes W$ with the inductive (resp. projective) tensor topology.

from the left on $\mathcal{D}(X,\mathcal{O})$. The left action of T on $\mathcal{D}(X)$ can be extended, with respect to $\delta_{\cdot}:T\to\mathcal{D}(T)$, to a left action of $\mathcal{D}(T)$ making $\mathcal{D}(X)$ a $\mathcal{D}(T)$-module by the convolution product:

$$\mathcal{D}(T)\otimes_{\iota}\mathcal{D}(X)\overset{\mathbf{P}_{\mathcal{D}(T),\mathcal{D}(X)}}{\longrightarrow}\mathcal{D}(T\times X)\xrightarrow{a_*}\mathcal{D}(X). \tag{31}$$

We note the formula

$$\delta_t\cdot\delta_x=\delta_{tx}\text{ for }t\in T\text{ and }x\in X, \tag{32}$$

which indeed characterizes the multiplication law by density of the Dirac distributions. Also, we remark that the multiplication map is in general separately continuous, while it is continuous if we assume that T and X are compact. In particular, one checks that $\mathcal{D}(T)$ becomes an algebra in this way. We write $Hom_{\mathcal{A}}(T,\mathcal{O}^{\times})$ to denote the group of those group homomorphisms such that their composition with the inclusion $\mathcal{O}^{\times}\subset\mathcal{O}$ belongs to $\mathcal{A}(T,\mathcal{O})$. We also write $Hom_{\mathcal{L}}(\mathcal{D}(T),\mathcal{O})$ to denote the space of those morphisms of locally convex spaces that are morphisms of algebras. Then there is a bijection (see [21, Lemma 7.2])

$$C^{\mathcal{O}}:Hom_{\mathcal{L}}(\mathcal{D}(T),\mathcal{O})\overset{\sim}{\to}Hom_{\mathcal{A}}\left(T,\mathcal{O}^{\times}\right),\text{ via }C^{\mathcal{O}}(\mathbf{k})(t):=\mathbf{k}(\delta_t). \tag{33}$$

We will abuse of notations, when there will be no risk of confusion, and identify these two sets, deserving the exponential notation to the group homomorphisms and calling the elements of these sets weights.

If $\mathbf{k}$ is a weight, we may consider the space of locally analytic homogeneous functions:

$$\mathcal{A}_{\mathbf{k}}(X)=\mathcal{A}(X,\mathbf{k})=\left\{F\in\mathcal{A}(X,\mathcal{O}):F(tx)=t^{\mathbf{k}}F(x)\right\}.$$

It is indeed a closed $\mathcal{O}$-submodule of $\mathcal{A}(X,\mathcal{O})$. Viewing both $\mathcal{O}$ and $\mathcal{D}(X)$ as $\mathcal{D}(T)$-modules by means of $\mathbf{k}$ and, respectively, the convolution product, we may define

$$\mathcal{D}_{\mathbf{k}}(X):=\mathcal{O}\widehat{\otimes}_{\mathbf{k}}\mathcal{D}(X)\text{ and }\mathcal{D}(X,\mathbf{k}):=\mathcal{L}_{\mathcal{O}}(\mathcal{A}_{\mathbf{k}}(X),\mathcal{O}).$$

We also assume from now on that X is endowed with a right action by a semigroup Σ such that $\sigma:\Sigma\to\Sigma$ is locally analytic for every $\sigma\in\Sigma$, which is compatible with the left T-action in the sense that $t(x\sigma)=(tx)\sigma$ for all $t\in T$, $x\in X$, and $\sigma\in\Sigma$. It follows that σ induces a well defined action on $\mathcal{A}_{\mathbf{k}}(X)$, $\mathcal{D}_{\mathbf{k}}(X)$ and $\mathcal{D}(X,\mathbf{k})$. The relation between the space $\mathcal{D}_{\mathbf{k}}(X)$ and $\mathcal{D}(X,\mathbf{k})$ is expressed by means of an $(\mathcal{O},\Sigma)$-equivariant morphism of locally convex spaces

$$\mathbf{T}^{\mathbf{k}}_{\mathcal{D}(X)}:\mathcal{D}_{\mathbf{k}}(X)\to\mathcal{D}(X,\mathbf{k}) \tag{34}$$

which is an isomorphism when X is a trivial (equivalently locally trivial) T-bundle. It is characterized by the property that

$$\mathbf{T}^{\mathbf{k}}_{\mathcal{D}(X)}\left(1\widehat{\otimes}_{\mathbf{k}}\delta_x\right)=\delta_x^{\mathbf{k}}\text{ for every }x\in X,$$

if $\delta_x^{\mathbf{k}}$ is the image of $\delta_x^{\mathcal{O}}$. We refer the reader to [21, Lemma 7.3 and Proposition 7.6] for details.

It follows from (34) that the elements of $\mathcal{D}_{\mathbf{k}}(X)$ naturally integrates functions in $\mathcal{A}_{\mathbf{k}}(X)$. Furthermore they are endowed with natural specialization maps, not possessed by the spaces $\mathcal{D}(X,\mathbf{k})$, defined as follows. If we are given $\mathbf{k}_i\in Hom_{\mathcal{L}}(\mathcal{D}(T),\mathcal{O}_i)$, we say that $\mathbf{k}_1$ specializes via ϕ to $\mathbf{k}_2$, and we write $\mathbf{k}_1\overset{\phi}{\to}\mathbf{k}_2$, if $\phi\in Hom_{\mathcal{L}}(\mathcal{O}_1,\mathcal{O}_2)$ and $\mathbf{k}_2=\phi\circ\mathbf{k}_1$. Then we have an induced specialization map

$$\phi_*:\mathcal{D}_{\mathbf{k}_1}(X)\to\mathcal{D}_{\mathbf{k}_2}(X)\text{ via }\phi_*\left(\alpha\widehat{\otimes}_{\mathbf{k}_1}\mu\right):=\phi(\alpha)\widehat{\otimes}_{\mathbf{k}_2}\mu. \tag{35}$$

4.2. **Multiplying locally analytic homogeneous distributions.** Now suppose that we are given two p-adic locally compact and paracompact manifolds X_i endowed with analytic actions of T_i for $i=1,2$, so that $T_1\times T_2$ act on $X_1\times X_2$ in the obvious way. Let us be given $\mathbf{k}_i\in Hom_{\mathcal{L}}(\mathcal{D}(T_i),\mathcal{O}_i)$. We define the continuous morphism of locally convex spaces

$$\mathbf{k}_1\boxplus\mathbf{k}_2:\mathcal{D}(T_1\times T_2)\overset{\mathbf{P}^{-1}_{\mathcal{D}(T_1),\mathcal{D}(T_2)}}{\to}\mathcal{D}(T_1)\widehat{\otimes}_{\iota}\mathcal{D}(T_2)\overset{\mathbf{k}_1\widehat{\otimes}_{\iota}\mathbf{k}_2}{\to}\mathcal{O}_1\widehat{\otimes}_{\iota}\mathcal{O}_2\overset{\widehat{1}}{\to}\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2. \tag{36}$$

Exploiting the effect on Dirac distributions and noticing that the multiplications laws are separately continuous by (31), it is not difficult to deduce from the density of Dirac distributions that $\mathbf{k}_1\boxplus\mathbf{k}_2$ is a morphism of algebras, hence

$$\mathbf{k}_1\boxplus\mathbf{k}_2\in Hom_{\mathcal{L}}\left(\mathcal{D}(T_1\times T_2),\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2\right).$$

We assume that X_i is further endowed with a right action by a semigroup Σ_i having the same properties of the Σ-action considered above.

Lemma 4.1. *There is a unique morphism of locally convex spaces* $\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ *making the following diagram commutative, which is* $\left(\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2,\Sigma_1\times\Sigma_2\right)$*-equivariant:*

$$\begin{array}{ccc}
\mathcal{O}_1\widehat{\otimes}\mathcal{D}\left(X_1\right)\widehat{\otimes}_\iota\mathcal{O}_2\widehat{\otimes}\mathcal{D}\left(X_2\right) & \overset{1_{\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2}\widehat{\otimes}\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}}{\to} & \mathcal{O}_1\widehat{\otimes}\mathcal{O}_2\widehat{\otimes}\mathcal{D}\left(X_1\times X_2\right)\\
\downarrow & & \downarrow\\
\mathcal{D}_{\mathbf{k}_1}\left(X_1\right)\widehat{\otimes}_\iota\mathcal{D}_{\mathbf{k}_2}\left(X_2\right) & \overset{\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}}{\to} & \mathcal{D}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left(X_1\times X_2\right).
\end{array} \tag{37}$$

Proof. Let B be the composition of $1_{\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2}\widehat{\otimes}\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ with the right vertical morphism. Since we know that B is continuous and $\mathcal{D}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left(X_1\times X_2\right)$ is Hausdorff and complete, we first need to show that, for every $\alpha_i\in\mathcal{O}_i$, $\mu_i\in\mathcal{D}\left(X_i\right)$ and $\nu_i\in\mathcal{D}\left(T_i\right)$

$$B\left(\alpha_1\mathbf{k}_1\left(\nu_1\right)\widehat{\otimes}\mu_1\widehat{\otimes}_\iota\alpha_2\mathbf{k}_1\left(\nu_2\right)\widehat{\otimes}\mu_2\right)=B\left(\alpha_1\widehat{\otimes}\left(\nu_1\cdot\mu_1\right)\widehat{\otimes}_\iota\alpha_2\widehat{\otimes}\left(\nu_2\cdot\mu_2\right)\right).$$

It turns out that this is equivalent to checking the equalities

$$\mathbf{P}_{\mathcal{D}(T_1),\mathcal{D}(T_2)}\left(\nu_1\widehat{\otimes}\nu_2\right)\cdot\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\mu_1\widehat{\otimes}\mu_2\right)=\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\left(\nu_1\cdot\mu_1\right)\widehat{\otimes}\left(\nu_2\cdot\mu_2\right)\right)\text{ in }\mathcal{D}\left(X_1\times X_2\right). \tag{38}$$

When $\mu_i=\delta_{x_i}$ and $\nu_i=\delta_{t_i}$ we have indeed, by (30) and (32)

$$\begin{aligned}
&\mathbf{P}_{\mathcal{D}(T_1),\mathcal{D}(T_2)}\left(\nu_1\widehat{\otimes}\nu_2\right)\cdot\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\mu_1\widehat{\otimes}\mu_2\right)=\delta_{(t_1,t_2)}\cdot\delta_{(x_1,x_2)}=\delta_{(t_1x_1,t_2x_2)},\\
&\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\left(\nu_1\cdot\mu_1\right)\widehat{\otimes}\left(\nu_2\cdot\mu_2\right)\right)=\mathbf{P}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\delta_{t_1x_1}\widehat{\otimes}\delta_{t_2x_2}\right)=\delta_{(t_1x_1,t_2x_2)}.
\end{aligned}$$

We note that both the left and the right hand sides of (38) are linear in the variables μ_i and ν_i. Furthermore, if we fix three of these variables, the two resulting functions are continuous in the remaining variable thanks to (31) showing that the multiplication laws are separately continuous. Hence the claimed equality (38) follows from the density of Dirac distributions. The existence and uniqueness of $\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ follows and, since $\mathcal{O}_1\widehat{\otimes}\mathcal{D}\left(X_1\right)\widehat{\otimes}_\iota\mathcal{O}_2\widehat{\otimes}\mathcal{D}\left(X_2\right)\to\mathcal{D}_{\mathbf{k}_1}\left(X_1\right)\widehat{\otimes}_\iota\mathcal{D}_{\mathbf{k}_1}\left(X_2\right)$ is surjective and all the arrows other than $\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ in (37) are $\left(\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2,\Sigma_1\times\Sigma_2\right)$-equivariant (by (30)), implying that $\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ is equivariant as well. ☐

In particular we may define

$$\overline{\mathbf{P}}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}:\mathcal{D}_{\mathbf{k}_1}\left(X_1\right)\widehat{\otimes}_\iota\mathcal{D}_{\mathbf{k}_1}\left(X_2\right)\overset{\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}}{\to}\mathcal{D}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left(X_1\times X_2\right)\overset{\mathbf{T}^{\mathbf{k}_1\boxplus\mathbf{k}_2}_{\mathcal{D}(X_1\times X_2)}}{\to}\mathcal{D}\left(X_1\times X_2,\mathbf{k}_1\boxplus\mathbf{k}_2\right).$$

If $\mu_i\in\mathcal{D}_{\mathbf{k}_i}\left(X_i\right)$ for $i=1,2$, we set

$$\mu_1\boxtimes\mu_2:=\overline{\mathbf{P}}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(\mu_1\widehat{\otimes}_\iota\mu_2\right)\in\mathcal{D}\left(X_1\times X_2,\mathbf{k}_1\boxplus\mathbf{k}_2\right).$$

Of course, the formation of $\mathbf{k}_1\boxplus\mathbf{k}_2$, $\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ and $\overline{\mathbf{P}}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}$ extends to a finite number of indices and the usual associativity constraints are satisfied, as well as the compatibility with the commutativity constraints in the sources and the targets of these maps. We finally remark that the equations

$$\begin{aligned}
\mathbf{P}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(1_{\mathcal{O}_1}\widehat{\otimes}_{\mathbf{k}_1}\delta_{x_1}\widehat{\otimes}_\iota 1_{\mathcal{O}_2}\widehat{\otimes}_{\mathbf{k}_2}\delta_{x_2}\right) &= 1_{\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2}\widehat{\otimes}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\delta_{(x_1,x_2)},\\
\overline{\mathbf{P}}^{\mathbf{k}_1,\mathbf{k}_2}_{\mathcal{D}(X_1),\mathcal{D}(X_2)}\left(1_{\mathcal{O}_1}\widehat{\otimes}_{\mathbf{k}_1}\delta_{x_1}\widehat{\otimes}_\iota 1_{\mathcal{O}_2}\widehat{\otimes}_{\mathbf{k}_2}\delta_{x_2}\right) &= \delta^{\mathbf{k}_1\boxplus\mathbf{k}_2}_{(x_1,x_2)}
\end{aligned} \tag{39}$$

characterize these maps.

4.3. **Algebraic operations on weights.** Setting $\mathcal{X}_T\left(\mathcal{O}\right):=Hom_{\mathcal{A}}\left(T,\mathcal{O}^\times\right)$ defines a group functor on Banach algebras, so that we have

$$+:\mathcal{X}_T\left(\mathcal{O}\right)\times\mathcal{X}_T\left(\mathcal{O}\right)\longrightarrow\mathcal{X}_T\left(\mathcal{O}\right)\text{ and }-:\mathcal{X}_T\left(\mathcal{O}\right)\longrightarrow\mathcal{X}_T\left(\mathcal{O}\right).$$

It follows from (33) that we may transport these operations getting

$$+:Hom_{\mathcal{L}}\left(\mathcal{D}\left(T\right),\mathcal{O}_i\right)\times Hom_{\mathcal{L}}\left(\mathcal{D}\left(T\right),\mathcal{O}_i\right)\longrightarrow Hom_{\mathcal{L}}\left(\mathcal{D}\left(T\right),\mathcal{O}_i\right)$$

and

$$-:Hom_{\mathcal{L}}\left(\mathcal{D}\left(T\right),\mathcal{O}_i\right)\longrightarrow Hom_{\mathcal{L}}\left(\mathcal{D}\left(T\right),\mathcal{O}_i\right).$$

Our next task it to interpolate these operations.

If we are given $\mathbf{k}_i \in Hom_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O}_i)$, then we define

$$\mathbf{k}_1 \oplus \mathbf{k}_2 : \mathcal{D}(T) \xrightarrow{\Delta_*} \mathcal{D}(T \times T) \xrightarrow{\mathbf{k}_1 \boxplus \mathbf{k}_2} \mathcal{O}_1 \widehat{\otimes} \mathcal{O}_2,$$

where $\Delta : T \to T \times T$ is the diagonal map and $\mathbf{k}_1 \boxplus \mathbf{k}_2$ is given by (36).

If $\mathbf{k} \in Hom_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O})$, then we define

$$\ominus \mathbf{k} : \mathcal{D}(T) \xrightarrow{i_*} \mathcal{D}(T) \xrightarrow{\mathbf{k}} \mathcal{O},$$

where $i : T \to T$ is the inversion, and set

$$\mathbf{k}_1 \ominus \mathbf{k}_2 := \mathbf{k}_1 \oplus (\ominus \mathbf{k}_2)$$

We note that these operations are obviously functorial and compatible with specialization.

Exploiting the definitions and making (36) explicit it is easy to check the following result.

Lemma 4.2. *Suppose that* $\mathbf{k}, \mathbf{k}_i \in Hom_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O})$ *and write*

$$m_{\mathcal{O}} : \mathcal{O} \widehat{\otimes} \mathcal{O} \to \mathcal{O}$$

for the multiplication map. Then $-\mathbf{k} = \ominus \mathbf{k}$ *and*

$$\mathbf{k}_1 + \mathbf{k}_2 : \mathcal{D}(T) \xrightarrow{\mathbf{k}_1 \oplus \mathbf{k}_2} \mathcal{O} \widehat{\otimes} \mathcal{O} \xrightarrow{m_{\mathcal{O}}} \mathcal{O}.$$

We now illustrate why $\mathbf{k}_1 \oplus \mathbf{k}_2$ interpolates the $+$ operation. Suppose that F is our p-adic working field and that $k_i \in Hom_{\mathcal{L}}(\mathcal{D}(T), F)$ are such that $\mathbf{k}_i \overset{\phi_i}{\to} k_i$. Then

$$\mathbf{k}_1 \oplus \mathbf{k}_2 \overset{\phi_1 \widehat{\otimes} \phi_2}{\to} k_1 \oplus k_2$$

by the compatibility of the $\oplus$-operation with specializations. But we have $F \widehat{\otimes} F = F$ canonically and the identification is given by m_F. Hence $\mathbf{k}_1 \oplus \mathbf{k}_2$ specializes via $\phi_1 \widehat{\otimes} \phi_2$ to $k_1 + k_2$, thanks to Lemma 4.2. In particular, suppose that $\mathcal{X}_T$ is representable by a rigid analytic space (for example because T is compact) and that $\mathbf{k}_i \overset{\phi_i}{\to} k_i$ corresponds to $k_i \in U_i$, with $U_i \subset \mathcal{X}_T$ an affinoid neighbourhood of k. Then $\mathbf{k}_1 \oplus \mathbf{k}_2$ corresponds to

$$U_1 \times U_2 \subset \mathcal{X}_T \times \mathcal{X}_T \overset{+}{\to} \mathcal{X}_T.$$

We finally remark that, as a consequence of the associativity of the operation in T, we have

$$(\mathbf{k}_1 \oplus \mathbf{k}_2) \oplus \mathbf{k}_3 \simeq \mathbf{k}_1 \oplus (\mathbf{k}_2 \oplus \mathbf{k}_3)$$

up to

$$(\mathcal{O}_1 \otimes \mathcal{O}_2) \otimes \mathcal{O}_3 \simeq \mathcal{O}_1 \otimes (\mathcal{O}_2 \otimes \mathcal{O}_3).$$

A similar compatibility holds true for the commutativity, when T is commutative as in our applications.

Suppose now $T \simeq \Delta \times (1 + p\mathbb{Z}_p)^r$, where Δ is the torsion part of T, and consider the multiplication by 2 map $t \mapsto t^2$ (we write T multiplicatively). We say that $\mathbf{k} \in \mathcal{X}_T(\mathcal{O})$ is even if it is in the image of $2^* : \mathcal{X}_T(\mathcal{O}) \to \mathcal{X}_T(\mathcal{O})$ and set $\frac{\mathbf{k}}{2}$ for an element in the inverse image of $\mathbf{k}$. For example, suppose that $p \neq 2$ and $T = \mathbb{Z}_p^\times \simeq \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$. We can decompose every $\mathbf{k} \in \mathcal{X}_T(\mathcal{O})$ in the form $\mathbf{k} = ([\mathbf{k}], \langle \mathbf{k} \rangle)$, where $[\mathbf{k}] \in \mathbb{F}_p^\times$ and $\langle \mathbf{k} \rangle \in \mathcal{X}_{1+p\mathbb{Z}_p}(\mathcal{O})$. Since $t \mapsto t^2$ is invertible on $1 + p\mathbb{Z}_p$, $\mathbf{k} = ([\mathbf{k}], \langle \mathbf{k} \rangle)$ is even if and only if $[\mathbf{k}] \in \mathbb{F}_p^{\times 2}$ and then $\frac{\mathbf{k}}{2} \in \left\{ \left(\frac{[\mathbf{k}]}{2}, \frac{\langle \mathbf{k} \rangle}{2} \right), \left(-\frac{[\mathbf{k}]}{2}, \frac{\langle \mathbf{k} \rangle}{2} \right) \right\}$; if $[\mathbf{k}] = [k_0]$ for some integer k_0 our convention is to choose $\frac{\mathbf{k}}{2} = \left(\frac{[k_0]}{2}, \frac{\langle \mathbf{k} \rangle}{2} \right)$. Then $\mathbf{k} \overset{\phi}{\to} k \in \mathbb{N}$ implies $k \in 2\mathbb{N}$ and $\frac{\mathbf{k}}{2} \overset{\phi}{\to} \frac{k}{2}$. The elements of $Hom_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O}) \simeq \mathcal{X}_T(\mathcal{O})$ are called $\mathcal{O}$-weights; we will freely identify $\mathbf{k}_1 \boxplus \mathbf{k}_2 \simeq (\mathbf{k}_1, \mathbf{k}_2)$. We will write $\mathcal{X} := \mathcal{X}_{\mathbb{Z}_p^\times}$.

5. The p-adic trilinear form

The semigroup $\Sigma_0(p\mathbb{Z}_p) \subset \mathbf{M}_2(\mathbb{Z}_p)$ acts from the right on the set $W := \mathbb{Z}_p^\times \times \mathbb{Z}_p$. Setting $\omega_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, we have $\widehat{W} := W\omega_p = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$, on which $\Sigma_0(p\mathbb{Z}_p)^\iota = \omega_p^{-1}\Sigma_0(p\mathbb{Z}_p)\omega_p$ acts from the right. Hence, for a $\mathcal{O}$-weight $\mathbf{k}$, we may form the right $\Sigma_0(p\mathbb{Z}_p)$-module (resp. $\Sigma_0(p\mathbb{Z}_p)^\iota$-module) $\mathcal{D}_{\mathbf{k}}(W)$ (resp. $\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right)$). Taking $K_p^\diamond = \Gamma_0(p\mathbb{Z}_p) := \Sigma_0(p\mathbb{Z}_p) \cap \mathbf{GL}_2(\mathbb{Z}_p)$, we may form the spaces of p-adic families of modular forms on $\mathbf{B}^\times$:

$$M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}) := M_p^\diamond\left(B_{\mathrm{f}}^\times, \mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}\right).$$

Recall we work over a p-adic field F and consider Banach F-algebras: we set $M_p^\diamond(\mathbf{V}_{k,F}, \omega_{0,p}) := M_p^\diamond\left(B_{\mathrm{f}}^\times, \mathbf{V}_{k,F}, \omega_{0,p}\right)$ and $M^\diamond(\mathbf{V}_{k,F}, \omega_0) := M^\diamond\left(B_{\mathrm{f}}^\times, \mathbf{V}_{k,F}, \omega_0\right)$. As explained after (9), they are naturally $B_{\mathrm{f}}^{\times,p} \times \mathcal{H}(\Sigma_0(p\mathbb{Z}_p))$-modules and, in particular, they are endowed with a U_p-operator.

Example 5.1. Recall the open and compact subgroups $K_0(N), K_1(N) \subset B_{\mathrm{f}}^\times$ defined before Example 3.3. Assuming that $\mu_{\varphi(N)} \subset F$, we can decompose

$$M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W))^{K_1(N)} = \bigoplus\nolimits_{\varepsilon:\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^\times \to \mathcal{O}^\times} M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W))^{K_1(N)}(\varepsilon),$$

where $M(\varepsilon)$ is the submodule of elements $x \in M$ such that $xu = \varepsilon(u)x$ if we define $\varepsilon(u) := \varepsilon(a_u)$ for a_u the upper left entry of $u \in K_0(N)$. Setting $\omega_{0,p}^{\varepsilon,\mathbf{k}}(z) := \varepsilon\left(\frac{z}{\mathrm{N_f}(z)}\right)\mathrm{N}_p^{\mathbf{k}}(z) = \varepsilon\left(\frac{z}{\mathrm{N_f}(z)}\right)\left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-\mathbf{k}}$, we have

$$M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W))^{K_1(N)}(\varepsilon) = M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}^{\varepsilon,\mathbf{k}}\right)^{K_0(N)} \subset M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}^{\varepsilon,\mathbf{k}}\right).$$

If $\mathbf{k} \overset{\phi}{\to} k \in \mathbb{N}$, there is a specialization map

$$\phi_*^{\mathrm{alg}} : M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W)) \xrightarrow{\phi_*} M_p^\diamond(\mathcal{D}_k(W)) \xrightarrow{\nu_k} M_p^\diamond(\mathbf{V}_{k,F}) \tag{40}$$

where the first arrow is induced by (35) and the second is the restriction to polynomials map (regarded as functions on W). We will usually write $\varphi_k := \phi_*^{\mathrm{alg}}(\varphi)$ when $\varphi \in M_p^\diamond(\mathcal{D}_{\mathbf{k}}(W))$.

Example 5.2. Suppose that we are in the setting of Example 5.1, so that $\mu_{\varphi(N)} \subset F$. Then we can decompose $M^\diamond(\mathbf{V}_{k,F})^{K_1(N)}$ as we did for p-adic forms. Setting $\omega_0^{\varepsilon,k}(z) := \varepsilon\left(\frac{z}{\mathrm{N_f}(z)}\right)\mathrm{N}_{\mathrm{f}}^k(z)$, we have

$$M^\diamond(\mathbf{V}_{k,F})^{K_1(N)}(\varepsilon) = M^\diamond\left(\mathbf{V}_{k,F}, \omega_0^{\varepsilon,k}\right)^{K_0(N)} \subset M^\diamond\left(\mathbf{V}_{k,F}, \omega_0^{\varepsilon,k}\right).$$

Furthermore, when $\varepsilon : \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^\times \to \mu_{\varphi(N)} \subset F^\times$, the specialization map induces (see Lemma 2.3 for the isomorphism)

$$\phi_*^{\mathrm{alg}} : M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}^{\varepsilon,\mathbf{k}}\right) \to M_p^\diamond\left(\mathbf{V}_{k,F}, \omega_{0,p}^{\varepsilon,k}\right) \simeq M^\diamond\left(\mathbf{V}_{k,F}, \omega_0^{\varepsilon,k}\right).$$

Justified by the above example and changing a bit the notation to make it consistent with that of §3, we will consider characters of the form $\omega_{0,p}^{\mathbf{k}}(z) = \omega_{\mathrm{f}}(z)\mathrm{N}_p^{\mathbf{k}}(z) = \omega_{\mathrm{f}}(z)\left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-\mathbf{k}}$, where ω_{f} is the finite part of a unitary Hecke character which is unramified outside p. As explained above, setting $\omega_0^k(z) := \omega_{\mathrm{f}}(z)\mathrm{N}_{\mathrm{f}}^k(z)$ we see that (40) induces

$$M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W), \omega_{0,p}^{\mathbf{k}}\right) \longrightarrow M_p^\diamond\left(\mathbf{V}_{k,F}, \omega_{0,p}^k\right) \simeq M^\diamond\left(\mathbf{V}_{k,F}, \omega_0^k\right). \tag{41}$$

We identify $\psi : \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \overset{\sim}{\to} \mathbf{M}_2(\mathbb{Q}_p)$ by the rule $\psi(x_1, y_1, x_2, y_2) := \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ and, for a subset $S \subset \mathbb{Q}_p^2 \times \mathbb{Q}_p^2$, we define

$$\delta_S : S \to \mathbb{Q}_p,\ \delta_S(s) := \det(\psi(s)) \text{ and } S_n := \delta_S^{-1}\left(p^n\mathbb{Z}_p^\times\right).$$

For a continuous group homomorphism $\mathbf{k}:\mathbb{Z}_p^\times\to\mathcal{O}^\times$ valued in a $\mathbb{Q}_p$-Banach algebra $\mathcal{O}$, we may consider

$$\delta_{S_0}^{\mathbf{k}}:S_0\to\mathbb{Z}_p^\times\xrightarrow{\mathbf{k}}\mathcal{O}^\times,\ \delta_S^{\mathbf{k}}(s):=\delta_S(s)^{\mathbf{k}}.$$

It is a locally analytic function, when $S\subset\mathbb{Q}_p^2\times\mathbb{Q}_p^2$ is a submanifold, because $\mathbf{k}$ is locally analytic and $S_0\subset S$ is a submanifold.

Since $\psi(w_1g,w_2g)=\psi(w_1,w_2)g$ for any $w_i=(x_i,y_i)$ with $i=1,2$ and $g\in\mathbf{GL}_2(\mathbb{Q}_p)$, we have $\delta_{Sg}(sg)=\delta_S(s)\det(g)$ for any $s\in S$ and $(Sg)_n=S_{n-\nu_p(g)}g$ where $\nu_p:=\mathrm{ord}_p\circ\det$. In particular, if $\Sigma\subset\mathbf{GL}_2(\mathbb{Q}_p)$ is a subsemigroup acting on S, then $\Gamma_\Sigma:=\Sigma\cap\det^{-1}\left(\mathbb{Z}_p^\times\right)$ acts on S_n for every n. We have

$$\delta_{S_0}^{\mathbf{k}}(sg)=\det(g)^{\mathbf{k}}\,\delta_{S_0}^{\mathbf{k}}(s)\tag{42}$$

and, in particular,

$$\delta_{S_0}^{\mathbf{k}}(t_1s_1,t_2s_2)=t_1^{\mathbf{k}}t_2^{\mathbf{k}}\delta_{S_0}^{\mathbf{k}}(s).\tag{43}$$

Noticing that $\left(W\times\widehat{W}\right)_0=W\times\widehat{W}$ and $\left(\widehat{W}\times W\right)_0=\widehat{W}\times W$, we may consider the locally analytic functions

$$\delta_{W\times\widehat{W}}^{\mathbf{k}}:W\times\widehat{W}\to\mathcal{O}^\times,\ \delta_{\widehat{W}\times W}^{\mathbf{k}}:\widehat{W}\times W\to\mathcal{O}^\times\text{ and }\delta_{(W\times W)_0}^{\mathbf{k}}:W\times W\to\mathcal{O}^\times.$$

Recall our notation for the twists by the norm. Since $\Gamma_0(p\mathbb{Z}_p)\subset\mathbf{GL}_2(\mathbb{Q}_p)$ is compact, nrd_p maps it into the maximal open compact subgroup $\mathbb{Z}_p^\times\subset\mathbb{Q}_p^\times$. Hence, if D is a $\Gamma_0(p\mathbb{Z}_p)$-module with coefficients in $\mathcal{O}$, it makes sense to consider $D(\mathbf{k}):=D\left(\mathrm{nrd}_p^{\mathbf{k}}\right)$, the same representation with action $v\cdot_{\mathbf{k}}g:=\mathrm{nrd}_p^{\mathbf{k}}(g)\,vg$. Also, recall we have $\mathrm{Nrd}_p^{\mathbf{k}}\in M_p^\diamond\left(\mathcal{O}(\mathbf{k}),\mathrm{N}_p^{2\mathbf{k}}\right)^K$ for every $K\in\mathcal{K}^\diamond$ (indeed $\mathrm{N}_p^{2\mathbf{k}}=\mathrm{Nrd}_p^{\mathbf{k}}$ on $Z_{\mathrm{f}}=\mathbb{A}_{\mathrm{f}}^\times$).

If $\underline{\mathbf{k}}=(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ where $\mathbf{k}_i:\mathbb{Z}_p^\times\to\mathcal{O}_i^\times$ are $\mathcal{O}_i$-valued weights such that $\mathbf{k}_1\oplus\mathbf{k}_2\oplus\mathbf{k}_3$ is even, set $\underline{\mathbf{k}}^*:=\frac{\mathbf{k}_1\oplus\mathbf{k}_2\oplus\mathbf{k}_3}{2}$, $\underline{\mathbf{k}}_1^*:=\frac{\ominus\mathbf{k}_1\oplus\mathbf{k}_2\oplus\mathbf{k}_3}{2}$, $\underline{\mathbf{k}}_2^*:=\frac{\mathbf{k}_1\ominus\mathbf{k}_2\oplus\mathbf{k}_3}{2}$ and $\underline{\mathbf{k}}_3^*:=\frac{\mathbf{k}_1\oplus\mathbf{k}_2\ominus\mathbf{k}_3}{2}$, so that $\underline{\mathbf{k}}_i^*:\mathbb{Z}_p^\times\to\mathcal{O}_{\underline{\mathbf{k}}}^\times$ for $\mathcal{O}_{\underline{\mathbf{k}}}:=\mathcal{O}_1\widehat{\otimes}\mathcal{O}_2\widehat{\otimes}\mathcal{O}_3$. We define $\underline{W}:=W\times W\times W$, $\underline{W}_1:=\widehat{W}\times W\times W$, $\underline{W}_2:=W\times\widehat{W}\times W$ and $\underline{W}_3:=W\times W\times\widehat{W}$. Also, if $p_i:\underline{W}_i\to W\times W$ denotes the projection onto the components which are different from i, we define $\underline{W}_i^\circ:=p_i^{-1}\left((W\times W)_0\right)$ (for example, $\underline{W}_3^\circ:=(W\times W)_0\times\widehat{W}$). Then we define the locally analytic functions

$$\Delta_{i,\underline{\mathbf{k}}}^\circ:\underline{W}_i^\circ\to\mathcal{O}_{\underline{\mathbf{k}}}^\times$$

by the rule

$$\begin{aligned}\Delta_{1,\underline{\mathbf{k}}}^\circ(w_1,w_2,w_3)\ &:\ =\delta_{(W\times W)_0}^{\underline{\mathbf{k}}_1^*}(w_2,w_3)\,\delta_{\widehat{W}\times W}^{\underline{\mathbf{k}}_2^*}(w_1,w_3)\,\delta_{\widehat{W}\times W}^{\underline{\mathbf{k}}_3^*}(w_1,w_2),\\ \Delta_{2,\underline{\mathbf{k}}}^\circ(w_1,w_2,w_3)\ &:\ =\delta_{\widehat{W}\times W}^{\underline{\mathbf{k}}_1^*}(w_2,w_3)\,\delta_{(W\times W)_0}^{\underline{\mathbf{k}}_2^*}(w_1,w_3)\,\delta_{W\times\widehat{W}}^{\underline{\mathbf{k}}_3^*}(w_1,w_2),\\ \Delta_{3,\underline{\mathbf{k}}}^\circ(w_1,w_2,w_3)\ &:\ =\delta_{W\times\widehat{W}}^{\underline{\mathbf{k}}_1^*}(w_2,w_3)\,\delta_{W\times\widehat{W}}^{\underline{\mathbf{k}}_2^*}(w_1,w_3)\,\delta_{(W\times W)_0}^{\underline{\mathbf{k}}_3^*}(w_1,w_2).\end{aligned}$$

We remark that $\Gamma_0(p\mathbb{Z}_p)=\Gamma_0(p\mathbb{Z}_p)^\iota=\Sigma_0(p\mathbb{Z}_p)\cap\Sigma_0(p\mathbb{Z}_p)^\iota$ acts diagonally on $\underline{W}_i^\circ$. The following lemma is an application of (42), (43) and the definitions of §4.3.

Lemma 5.3. *We have* $\Delta_{i,\underline{\mathbf{k}}}^\circ\in\mathcal{A}_{\mathbf{k}_1\boxplus\mathbf{k}_2\boxplus\mathbf{k}_3}\left(\underline{W}_i^\circ\right)(-\underline{\mathbf{k}}^*)^{\Gamma_0(p\mathbb{Z}_p)}$.

We will now focus on the $i=3$ index, the other cases being similar. Since $\underline{W}_3=\underline{W}_3^\circ\sqcup\left(\underline{W}_3-\underline{W}_3^\circ\right)$ (resp. $W^2=\left(W^2\right)_0\sqcup\left(W^2-\left(W^2\right)_0\right)$) is a disjoint decomposition in open subsets, we have an extension by zero map $\cdot_\circ:\mathcal{A}_{\underline{\mathbf{k}}}\left(\underline{W}_3^\circ\right)\to\mathcal{A}_{\underline{\mathbf{k}}}\left(\underline{W}_3\right)$ (resp. $\cdot_0:\mathcal{A}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left(\left(W^2\right)_0\right)\to\mathcal{A}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left(W^2\right)$). By duality, we obtain a map

$$\cdot^\circ:\mathcal{D}\left(\underline{W}_3,\underline{\mathbf{k}}\right)\to\mathcal{D}\left(\underline{W}_3^\circ,\underline{\mathbf{k}}\right)\ \text{(resp. }\cdot^0:\mathcal{D}_{\mathbf{k}_1\boxplus\mathbf{k}_2}(W\times W)\to\mathcal{D}_{\mathbf{k}_1\boxplus\mathbf{k}_2}\left((W\times W)_0\right)\text{)}.$$

It follows from Lemma 5.3 that we may consider

$$\Lambda_{3,\underline{\mathbf{k}}}^\circ(\mu_1,\mu_2,\mu_3):=(\mu_1\boxtimes\mu_2\boxtimes\mu_3)^\circ\left(\Delta_{3,\underline{\mathbf{k}}}^\circ\right)\in\mathcal{O}_{\underline{\mathbf{k}}}\ \left(\mu_i\in\mathcal{D}_{\mathbf{k}_i}(W)\text{ for }i=1,2\text{ and }\mu_3\in\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right)\right)$$

and that we have

$$\Lambda_{3,\underline{\mathbf{k}}}^\circ\in Hom_{\mathcal{O}[\Gamma_0(p\mathbb{Z}_p)]}\left(\mathcal{D}_{\mathbf{k}_1}(W)\otimes_{\mathcal{O}}\mathcal{D}_{\mathbf{k}_2}(W)\otimes_{\mathcal{O}}\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right),\mathcal{O}_{\underline{\mathbf{k}}}\left(\underline{\mathbf{k}}^*\right)\right).$$

Suppose that we are given characters $\omega_{0,p}^{\mathbf{k}_i}$ for $i = 1,2,3$ such that $\mathrm{N}_p^{2\underline{\mathbf{k}}^*} = \omega_{0,p}^{\mathbf{k}_1}\omega_{0,p}^{\mathbf{k}_2}\omega_{0,p}^{\mathbf{k}_3}$. Taking $\Lambda = \Lambda_{3,\underline{\mathbf{k}}}^{\circ}$ in (11)) gives the trilinear form

$$t_{3,\underline{\mathbf{k}}}^{\circ} : M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_1}(W), \omega_{0,p}^{\mathbf{k}_1}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_2}(W), \omega_{0,p}^{\mathbf{k}_2}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right), \omega_{0,p}^{\mathbf{k}_3}\right) \to \mathcal{O}_{\underline{\mathbf{k}}}.$$

Suppose, for example, that we may write $\omega_{0,p}^{\mathbf{k}_i}(z) = \omega_{\mathrm{f},i}(z)\,\mathrm{N}_p^{\mathbf{k}_i}(z) = \omega_{\mathrm{f},i}(z)\left(\frac{z}{\mathrm{N}_{\mathrm{f}}(z)}\right)_p^{-\mathbf{k}_i}$ with $\omega_{\mathrm{f},i}$ taking values in F with $\omega_{\mathrm{f},1}\omega_{\mathrm{f},2}\omega_{\mathrm{f},3} = 1$; then we see that $\omega_{0,p}^{\mathbf{k}_1}\omega_{0,p}^{\mathbf{k}_2}\omega_{0,p}^{\mathbf{k}_3} = \mathrm{N}_p^{2\underline{\mathbf{k}}^*}$ and the above definition applies. Let us remark that, when $\underline{\mathbf{k}}$ is such that $\underline{\mathbf{k}}_i^* = c \in \mathbb{N}$, the expression (13) defines an element $\Delta_{3,\underline{\mathbf{k}}} \in \mathcal{A}_{\mathbf{k}_1 \boxplus \mathbf{k}_2 \boxplus \mathbf{k}_3}(\underline{W}_3)(-\underline{\mathbf{k}}^*)^{\Gamma_0(p\mathbb{Z}_p)}$. We can therefore integrate this function without first applying $\cdot^0$ to the measures involved. The result is a trilinear form

$$t_{3,\underline{\mathbf{k}}} : M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_1}(W), \omega_{0,p}^{\mathbf{k}_1}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_2}(W), \omega_{0,p}^{\mathbf{k}_2}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right), \omega_{0,p}^{\mathbf{k}_3}\right) \to \mathcal{O}_{\underline{\mathbf{k}}}. \tag{44}$$

Let $\widehat{\omega}_p$ be the idele concentrated at p, where we have $(\widehat{\omega}_p)_p = \omega_p$. Because $\omega_p : \mathcal{D}_{\mathbf{k}_3}(W) \to \mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right)$, the formula $(\varphi_3 \mid W_p)(x) := (\varphi_3\widehat{\omega}_p)(x) = \varphi_3(\widehat{\omega}_p x)\,\omega_p$ defines

$$W_p : M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}(W), \omega_{0,p}^{\mathbf{k}_3}\right) \longrightarrow M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right), \omega_{0,p}^{\mathbf{k}_3}\right) \tag{45}$$

It follows from (39) that, if $\underline{\mathbf{k}}_i^* = c \in \mathbb{N}$ and $\varphi_{i,k_i} = \phi_{i*}^{\mathrm{alg}}(\varphi_i)$ where $\phi : \mathbf{k}_i \to k_i$, then

$$\phi\left(t_{3,\underline{\mathbf{k}}}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mid W_3)\right) = t_{\underline{k}}\left(\varphi_{1,k_1}, \varphi_{2,k_2}, \varphi_{3,k_3} \mid W_p\right). \tag{46}$$

This applies, in particular, when ϕ is the identity and $\underline{\mathbf{k}} = \underline{k} \in \mathbb{N}^3$ is balanced: then $\varphi_{i,k_i} = \phi_{i*}^{\mathrm{alg}}(\varphi_i) = \nu_k(\varphi_i)$ is just the restriction to polynomials map (see (40)) and ϕ that appears in (46) is the identity.

The following key calculation relates the trilinear form $t_{3,\underline{k}}^{\circ}$ to $t_{3,\underline{k}}$ under the running assumption that $\underline{\mathbf{k}} = \underline{k} \in \mathbb{N}^3$ is balanced (to be in force from now until the end of Corollary 5.6 below). Write $\widehat{\pi}_p$ for the idele concentrated at p, where we have $(\widehat{\pi}_p)_p = \pi_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. If $\varphi \in M_p^{\diamond}(\mathcal{D}_k(W))$, recall the U_p-operator defined by the double coset $K\widehat{\pi}_pK$, where $K = K^p\Gamma_0(p\mathbb{Z}_p)$ and $\varphi \in M_p^{\diamond}(\mathcal{D}_k(W))^{K^p}$ (see the discussion after (9)).

Proposition 5.4. *For $i = 1,2$, suppose $\varphi_i \in M_p^{\diamond}(\mathcal{D}_{k_i}(W))$ is a U_p-eigenvector with $\varphi_i|U_p = a_i\varphi_i$, and view $\varphi_1 \otimes \varphi_2$ as an element of $M_p^{\diamond}(\mathcal{D}_{k_1 \boxplus k_2}(W \times W))$. Then*

$$(\varphi_1 \otimes \varphi_2)|U_p = a_1a_2\big(\varphi_1 \otimes \varphi_2 - i_*(\varphi_1 \otimes \varphi_2)^0\big),$$

where $i_* : M_p^{\diamond}(\mathcal{D}_{k_1 \boxplus k_2}((W \times W)_0)) \to M_p^{\diamond}(\mathcal{D}_{k_1 \boxplus k_2}(W \times W))$ is induced by the map $i_* : \mathcal{D}_{k_1 \boxplus k_2}((W \times W)_0) \to \mathcal{D}_{k_1 \boxplus k_2}(W \times W)$ obtained from the inclusion $i : (W \times W)_0 \hookrightarrow W \times W$.

Proof. Consider the decomposition

$$W = \bigsqcup_{i=0}^{p-1} W_i, \quad \text{where} \quad W_i = W\pi_i = \{w = (x,y) \in W : y \equiv ix \pmod{p}\}.$$

Then $K\widehat{\pi}_pK = \bigsqcup_{i=0}^{p-1} K\pi_i$ and we can compute:

$$\begin{aligned}
a_1a_2(\varphi_1 \otimes \varphi_2)(x) &= (\varphi_1|U_p \otimes \varphi_2|U_p)(x) \\
&= \sum_{i,j=0}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j \\
&= \sum_{i=0}^{p-1} \varphi_1(\pi x)\pi_i \otimes \varphi_3(\pi_i x)\pi_i + \sum_{\substack{i,j=0 \\ i \neq j}}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j \\
&= ((\varphi_1 \otimes \varphi_2)|U_p)(x) + A.
\end{aligned}$$

It remains to show that $A = a_1 a_2 i_*((\varphi_1 \otimes \varphi_2)^0(x))$. To this end, note that we may write

$$W^2 = \bigsqcup_{i,j=0}^{p-1} W_i \times W_j = \bigsqcup_{i=0}^{p-1} W_i \times W_i \sqcup \bigsqcup_{\substack{i,j=0 \\ i\neq j}}^{p-1} W_i \times W_j.$$

Subordinate to this decomposition of spaces, we have a corresponding decomposition of $\mathcal{D}_{k_1 \boxplus k_2}(W^2)$:

$$\mathcal{D}_{k_1 \boxplus k_2}(W^2) = \bigoplus_{i=0}^{p-1} \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_i) \oplus \bigoplus_{\substack{i,j=0 \\ i\neq j}}^{p-1} \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j).$$

Note that the spaces W_i are all $\mathbb{Z}_p^\times$-stable, so that these spaces of distributions are defined. Writing $\mathrm{proj}_{i,j} : \mathcal{D}_{k_1 \boxplus k_2}(W^2) \to \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j)$ for the associated projections, we have

$$a_1 a_2 \sum_{i,j=0}^{p-1} \mathrm{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)) = a_1 a_2 (\varphi_1(x) \otimes \varphi_2(x)) = \sum_{i,j=0}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j.$$

Since

$$\mu_1 \pi_i \otimes \mu_2 \pi_j \in \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j),$$

for every $\mu_1 \in \mathcal{D}_{k_1}(W)$ and $\mu_2 \in \mathcal{D}_{k_2}(W)$ (as it can be checked on Dirac distributions), taking $\mu_1 = \varphi_1(\pi_i x)$ and $\mu_2 = \varphi_2(\pi_j x)$ it follows that

$$a_1 a_2 \,\mathrm{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)) = \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)$$

for all i, j. Therefore,

$$A = a_1 a_2 \sum_{\substack{i,j=0 \\ i\neq j}}^{p-1} \mathrm{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)). \tag{47}$$

One easily verifies the equality

$$(W \times W)_0 = \bigsqcup_{\substack{i,j=0 \\ i\neq j}}^{p-1} W_i \times W_j,$$

implying that

$$\sum_{\substack{i,j=0 \\ i\neq j}}^{p-1} \mathrm{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)) = i_*((\varphi_1 \otimes \varphi_2)^0(x)). \tag{48}$$

Now substitute (48) into (47). □

We are going to apply the results of §2.3. We take $\Sigma_p = \Sigma_0(p\mathbb{Z}_p)$, $D = \mathcal{D}_{k_1 \boxplus k_2}(W \times W)$ and $E = \mathcal{D}_{k_3}(\widehat{W})$. Then $\mathbf{k} = \underline{k}^*$ (resp. the central character $\kappa_E = k_3$ of $E = \mathcal{D}_{k_3}(\widehat{W})$) extends to the character $\widetilde{\mathbf{k}} = \underline{k}^*$ of $\mathbb{Q}_p^\times$ (resp. the character $\widetilde{\kappa_E} = k_3$ of $\mathbb{Q}_p^\times$). Finally, we suppose that we may write $\omega_{0,p}^{k_i}(z) = \omega_{\mathrm{f},i}(z) \left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-k_i}$ with $\omega_{\mathrm{f},1}\omega_{\mathrm{f},2}\omega_{\mathrm{f},3} = 1$. Then $\omega_{0,p,D}(z) = \omega_{\mathrm{f},1}(z)\omega_{\mathrm{f},2}(z)\left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-k_1-k_2}$, $\omega_{0,p,E}(z) = \omega_{\mathrm{f},3}(z)\left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-k_3}$ and we have $\omega_{0,p,D}\omega_{0,p,E} = \omega_{0,p}$ with $\omega_{0,p}(z) = \left(\frac{z}{\mathrm{N_f}(z)}\right)_p^{-k_1-k_2-k_3} = \mathrm{Nrd}_p^{\underline{k}^*}(z)$.

Lemma 5.5. *With these notations the trilinear form $t_{3,\underline{k}}$ defines an element of $\mathrm{Hom}_{\mathcal{O}[\Sigma_p, \Sigma_p^\iota]}(D \otimes E, \mathcal{O}(\underline{k}^*))$. Furthermore, we have $\mathrm{Nrd}_{\mathrm{f}}^{\widetilde{\mathbf{k}}}(\pi)_p \,\mathrm{nrd}_p^{-\widetilde{\kappa_E}}(\pi_p)\,\mathrm{nrd}_{\mathrm{f}}^{-\omega_{0,p,E}}(\pi) = \omega_{\mathrm{f},3}\left(\frac{\mathrm{Nrd_f}(\pi)}{\mathrm{nrd}(\pi)}\right)\mathrm{Nrd}_{\mathrm{f}}^{\underline{k}_3^*}(\pi)_p$ in Proposition 2.9.*

Proof. Note that $\Delta_{i,\underline{k}}$ defines indeed $\widetilde{\Delta}_{\underline{k}} \in \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \times \mathbb{Q}_p^2)$ such that $\widetilde{\Delta}_{\underline{k}|\underline{W}_3} = \Delta_{i,\underline{k}}$. We take $\widetilde{D} := \mathcal{D}_{k_1 \boxplus k_2}(\mathbb{Q}_p^2 \times \mathbb{Q}_p^2)$ and $\widetilde{E} := \mathcal{D}_{k_3}(\mathbb{Q}_p^2)$, so that $\mathbf{k}_{\widetilde{E}} = \widetilde{\mathbf{k}}_E = k_3$. The pairing associated to $t_{3,\underline{k}}$ is given by $\langle \mu_{12}, \mu_3 \rangle := (\mu_{12} \widehat{\otimes} \mu_3)(\Delta_{\underline{k}})$ and we define $\langle \mu_{12}, \mu_3 \rangle^{\sim} := (\mu_{12} \widehat{\otimes} \mu_3)(\widetilde{\Delta}_{\underline{k}})$. Since $\mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 = (W^2 \times \widehat{W}) \sqcup Z$ with Z an open subset,

$$\begin{aligned}
&\mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \times \mathbb{Q}_p^2) = \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(W^2 \times \widehat{W}) \oplus \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(Z) \\
\text{and} \quad &\mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \times \mathbb{Q}_p^2) = \mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(W^2 \times \widehat{W}) \oplus \mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(Z).
\end{aligned}$$

For elements $\mu_{12} \in \mathcal{D}_{k_1 \boxplus k_2}(W^2)$ and $\mu_3 \in \mathcal{D}_{k_3}(\widehat{W})$, the distribution $\mu_{12} \widehat{\otimes} \mu_3$ is supported on $\mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(W^2 \times \widehat{W})$ and we see that $\langle \mu_{12}, \mu_3 \rangle^{\sim} = \langle \mu_{12}, \mu_3 \rangle$. We note that the relation $\sigma \widetilde{\Delta}_{\underline{k}} = \det(\sigma)^{\underline{k}} \widetilde{\Delta}_{\underline{k}}$ for every $\sigma \in \Sigma_p$ implies

$$\langle \mu_{12}\sigma, \mu_3\sigma \rangle^{\sim} := (\mu_{12}\sigma \widehat{\otimes} \mu_3\sigma)(\widetilde{\Delta}_{\underline{k}}) = \det(\sigma)^{\underline{k}} (\mu_{12} \widehat{\otimes} \mu_3)(\widetilde{\Delta}_{\underline{k}}) = \det(\sigma)^{\underline{k}} \langle \mu_{12}, \mu_3 \rangle^{\sim}.$$

Now apply Remark 2.8 in order to get the first statement. Finally, the second statement follows by a simple computation. □

Write $\mathbf{p}' \in \mathbb{A}_{\mathrm{f}}^{\times}$ for the finite idele $(\mathbf{p}')_v = p$ for every $v \neq p$ and $(\mathbf{p}')_p = 1$.

Corollary 5.6. *For $i = 1, 2$, let $\varphi_i \in M_p^{\diamond}(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})$ be a U_p-eigenvector with $\varphi_i | U_p = \alpha_i \varphi_i$ and let $\varphi_3 \in M_p^{\diamond}(\mathcal{D}_{k_3}(\widehat{W}), \omega_{0,p}^{k_3})$ be a U_p^{ι}-eigenvector with $\varphi_3 | U_p^{\iota} = \alpha_3 \varphi_3$. Then*

$$t_{3,\underline{k}}^{\circ}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \left(1 - \omega_{\mathrm{f},3}(\mathbf{p}') \frac{\alpha_3}{\alpha_1 \alpha_2} p^{k_3^*}\right) t_{3,\underline{k}}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3).$$

Proof. Recall the morphism $\cdot^0 : \mathcal{D}_{\mathbf{k}_1 \boxplus \mathbf{k}_2}(W \times W) \to \mathcal{D}_{k_1 \boxplus k_2}((W \times W)_0)$. Given $\mu_i \in \mathcal{D}_{\mathbf{k}_i}(W)$ for $i = 1, 2$ and $\mu_3 \in \mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right)$, we may therefore consider

$$\langle \mu_{12}, \mu_3 \rangle_t^{\circ} := \left(\mu_{12}^0 \boxtimes \mu_3\right) \left(\Delta_{\underline{\mathbf{k}}}^{\circ}\right).$$

This is granted by Lemma 5.3, which also implies $\langle -, - \rangle_t^{\circ} \in Hom_{\mathcal{O}[\Gamma_0(p\mathbb{Z}_p)]}\left(\mathcal{D}_{\mathbf{k}_1 \boxplus \mathbf{k}_2}\left((W^2)_0\right) \otimes \mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right), \mathcal{O}\left(\underline{\mathbf{k}}^*\right)\right)$. Taking $\Lambda = \langle -, - \rangle_t^{\circ}$ in (11) gives the bilinear form

$$\langle -, - \rangle_t^{\circ} : M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_1 \boxplus \mathbf{k}_2}\left((W^2)_0\right), \omega_{0,p,D}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}\left(\widehat{W}\right), \omega_{0,p,E}\right) \to \mathcal{O},$$

It is clear that $\left\langle \mathbf{P}^{\mathbf{k}_1, \mathbf{k}_2}\left(\mu_1 \widehat{\otimes}_{\iota} \mu_2\right)^0, \mu_3 \right\rangle_t^{\circ} = t_{3,\underline{\mathbf{k}}}^{\circ}(\mu_1, \mu_2, \mu_3)$ (we just need to check the equality on Dirac distributions), from which we see that

$$t_{3,\underline{\mathbf{k}}}^{\circ}(\varphi_1, \varphi_2, \varphi_3) = \left\langle (\varphi_1 \otimes \varphi_2)^0, \varphi_3 \right\rangle_t^{\circ},$$

if $\varphi_1 \otimes \varphi_2$ is viewed as an element of $M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_1 \boxplus \mathbf{k}_2}\left(W^2\right), \omega_{0,p,D}\right)$. A similar result holds true for $t_{3,\underline{k}}$, namely we may define as above

$$\langle -, - \rangle_t : M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_1 \boxplus \mathbf{k}_2}\left(W^2\right), \omega_{0,p,D}\right) \otimes M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}_3}(W), \omega_{0,p,E}\right) \to \mathcal{O}$$

for which

$$t_{3,\underline{k}}(\varphi_1, \varphi_2, \varphi_3) = \langle \varphi_1 \otimes \varphi_2, \varphi_3 \rangle_t.$$

By Proposition 5.4, we have

$$\langle \varphi_1 \otimes \varphi_2, \varphi_3 \rangle_t = \frac{1}{\alpha_1 \alpha_2} \langle (\varphi_1 \otimes \varphi_2) | U_p, \varphi_3 \rangle_t + \langle i_*(\varphi_1 \otimes \varphi_2)^0, \varphi_3 \rangle_t \tag{49}$$

Proposition 2.9, which applies thanks to Lemma 5.5, implies that

$$\langle (\varphi_1 \otimes \varphi_2) | U_p, \varphi_3 \rangle_t = \omega_{\mathrm{f},3}(\mathbf{p}') p^{k_3^*} \langle \varphi_1 \otimes \varphi_2, \varphi_3 | U_p^{\iota} \rangle_t = \omega_{\mathrm{f},3}(\mathbf{p}') p^{k_3^*} \alpha_3 \langle \varphi_1 \otimes \varphi_2, \varphi_3 \rangle_t. \tag{50}$$

Finally, it is easy to see that $\left\langle i_*\left(\mu_1\otimes\mu_2\right)^0,\mu_3\right\rangle_t=\left\langle\left(\mu_1\otimes\mu_2\right)^0,\mu_3\right\rangle_t^\circ$ (once again checking the equality on Dirac distributions), from which we see that

$$\left\langle i_*\left(\varphi_1\otimes\varphi_2\right)^0,\varphi_3\right\rangle_t=\left\langle\left(\varphi_1\otimes\varphi_2\right)^0,\varphi_3\right\rangle_t^\circ. \tag{51}$$

The result follows by combining (49), (50), and (51). □

5.1. **p-adic periods.** Writing $\mathbf{k}=\mathbf{k}_i$ and $\mathcal{O}=\mathcal{O}_i$, as remarked after (43), we have $\delta^{\mathbf{k}}_{W\times\widehat{W}}:W\times\widehat{W}\to\mathcal{O}^\times$: it follows from (42) that we have $\delta^{\mathbf{k}}_{W\times\widehat{W}}\in\mathcal{A}_{\mathbf{k}\boxplus\mathbf{k}}\left(W\times\widehat{W}\right)(-\mathbf{k})^{\Gamma_0(p\mathbb{Z}_p)}$ and we can consider

$$B_{\mathbf{k}}\in Hom_{\mathcal{O}[\Gamma_0(p\mathbb{Z}_p)]}\left(\mathcal{D}_{\mathbf{k}}(W)\otimes_{\mathcal{O}}\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\mathcal{O}(\mathbf{k})\right)$$

defined by $B_{\mathbf{k}}(\mu_1\otimes\mu_2):=(\mu_1\boxtimes\mu_2)\left(\delta^{\mathbf{k}}_{W\times\widehat{W}}\right)$. Taking $\Lambda=B_{\mathbf{k}}$ in (11)) gives the bilinear form

$$\langle-,-\rangle_{\mathbf{k}}:M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W),\omega^{\mathbf{k}}_{0,p}\right)\otimes M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\mathrm{N}_p^{2\mathbf{k}}\left(\omega^{\mathbf{k}}_{0,p}\right)^{-1}\right)\to\mathcal{O}.$$

Suppose now that $\mathbf{k}\xrightarrow{\phi}k\in\mathbb{N}$ and let us remark that there are specialization maps

$$\begin{aligned}\phi^{\mathrm{alg}}_*&:M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\omega^{\mathbf{k}}_{0,p}\right)\longrightarrow M_p^\diamond\left(\mathbf{V}_{k,E},\omega^k_{0,p}\right)\simeq M^\diamond\left(\mathbf{V}_{k,E},\omega^k_0\right)\text{ and}\\ \phi^{\mathrm{alg}}_*&:M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\mathrm{N}_p^{2\mathbf{k}}\left(\omega^{\mathbf{k}}_{0,p}\right)^{-1}\right)\longrightarrow M_p^\diamond\left(\mathbf{V}_{k,E},\mathrm{N}_p^{2k}\left(\omega^k_{0,p}\right)^{-1}\right)\simeq M^\diamond\left(\mathbf{V}_{k,E},\mathrm{N}_{\mathrm{f}}^{2k}\left(\omega^k_0\right)^{-1}\right),\end{aligned}\tag{52}$$

defined in the same way as (41) was defined, i.e. via the morphism induced by (35) and the restriction to polynomials (now regarded as functions on $\widehat{W}$). Let us apply Remark 2.5 to (18). First, it tells use that $\left(\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}},1\right)\in X\left(B^\times,\omega_{\mathrm{f}}^{-2}\right)$ corresponds to $\left(\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}},1\right)\in X_p\left(B^\times,\omega_{\mathrm{f}}^{-2}\right)$ (taking $\chi_0=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}}$ and $\chi_\infty=\chi_p=1$ in loc.cit. we see that $\chi_{0,p}=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}}$). Second, it tells us that (17) (for the central character $\omega^k_0(z)=\omega_{\mathrm{f}}(z)\mathrm{N}^k_{\mathrm{f}}(z)$) corresponds to

$$M_p^\diamond\left(\mathbf{V}_{k,E},\omega^k_{0,p}\right)\longrightarrow M_p^\diamond\left(\mathbf{V}_{k,E},\mathrm{N}_p^{2k}\left(\omega^k_{0,p}\right)^{-1}\right)\tag{53}$$

defined via Remark 2.2 (1), i.e. given by $\varphi\mapsto\check{\varphi}$, where $\check{\varphi}(x):=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}}(x)\varphi(x)=\mathrm{Nrd}_p^{k_i}(x)\mathrm{nrd}_{\mathrm{f}}^{-\omega^k_{0,p}}(x)\varphi(x)$ (the equality because $\omega^k_{0,p}(z)=\omega_{\mathrm{f}}(z)\mathrm{N}^k_p(z)$, implying that $\omega_{\mathrm{f}}=\omega^k_{0,p}\mathrm{N}_p^{-k}$). Hence, the same formula $\varphi\mapsto\check{\varphi}$, where $\check{\varphi}(x):=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f},i}}(x)\varphi(x)$, defines

$$M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\omega^{\mathbf{k}}_{0,p}\right)\longrightarrow M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\mathrm{N}_p^{2\mathbf{k}}\left(\omega^{\mathbf{k}}_{0,p}\right)^{-1}\right)\tag{54}$$

which interpolates (53) $\simeq$ (17) via (52). Recall the W_p-operator $M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W),\omega^{\mathbf{k}}_{0,p}\right)\to M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}\left(\widehat{W}\right),\omega^{\mathbf{k}}_{0,p}\right)$ (see (45)) which interpolates the W_p-operator $M_p^\diamond\left(\mathbf{V}_{k,E},\omega^k_{0,p}\right)\to M_p^\diamond\left(\mathbf{V}_{k,E},\omega^k_{0,p}\right)$ (defined by the same formula) via the specialization maps (41) and (52). Finally, it is clear from its definition that $\langle-,-\rangle_{\mathbf{k}}$ interpolates $\langle-,-\rangle_k$ via the specialization maps (41) and (52). Hence, setting $(\varphi,\psi):=\left\langle\varphi,(\psi\mid W_p)^{\vee}\right\rangle_{\mathbf{k}}$ we have proved the following result (the uniqueness follows from the fact that the weight space is reduced and $\mathbb{N}$ is Zariski dense in the open affinoid subdomain $U\subset\mathcal{X}$).

Lemma 5.7. *Suppose that $\mathbf{k}:\mathbb{Z}_p^\times\to\mathcal{O}$ corresponds to an open affinoid subdomain $U\subset\mathcal{X}$ and that $\varphi,\psi\in M_p^\diamond\left(\mathcal{D}_{\mathbf{k}}(W),\omega^{\mathbf{k}}_{0,p}\right)$. There is a unique $(\varphi,\psi)\in\mathcal{O}$ such that for every $k\in U\cap\mathbb{N}$ which corresponds to $\mathbf{k}\xrightarrow{\phi}k\in\mathbb{N}$ we have, setting $\varphi_k:=\phi^{alg}_*(\varphi)$ and $\psi_k:=\phi^{alg}_*(\psi)$:*

$$(\varphi,\psi)(k):=\phi((\varphi,\psi))=(\varphi_k,\psi_k\mid W_p)_k.$$

6. Degeneracy maps and p-stabilizations

If $g \in \mathbf{GL}_2(\mathbb{Q}_p)$, we let $\widehat{g}$ be the idele concentrated in p, where we have $\widehat{g}_p = g$. In particular, we write $\widehat{\pi}_p$ (resp. $\widehat{\omega}_p$) for the idele concentrated at p, where we have

$$(\widehat{\pi}_p)_p = \pi_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \ (\widehat{\omega}_p)_p = \omega_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$

We fix levels $K \subset K^{\#}$ of the form $K = K^p \Gamma_0(p\mathbb{Z}_p)$ and $K^{\#} = K^p \mathbf{GL}_2(\mathbb{Z}_p)$. Let us record the following fact.

Lemma 6.1. *We have* $K^{\#} = \bigsqcup_{i=0,\dots,p-1,\infty} K\widehat{\gamma}_i$ *with* $\gamma_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$ *for* $i = 0, \dots, p-1$ *and* $\gamma_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $K\widehat{\pi}_p K^{\#} = \bigsqcup_{i=0}^{p-1} K\widehat{\pi}_i \sqcup K\widehat{\omega}_p$ *with* $\pi_i = \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}$*. Also,*

$$K^{\#}\widehat{\pi}_p K^{\#} = \bigsqcup_{i=0,\dots,p-1,\infty} K^{\#}\widehat{\pi}_p^{\iota}\gamma_i = \bigsqcup_{i=0,\dots,p-1} K^{\#}\widehat{\pi}_i \sqcup K^{\#}\widehat{\pi}_p^{\iota}.$$

Proof. Indeed, a direct computation shows that $\bigsqcup_i K\widehat{\gamma}_i \subset K^{\#}$ (resp. $\bigsqcup_{i=0}^{p-1} K\widehat{\pi}_i \sqcup K\widehat{\omega}_p \subset K\widehat{\pi}_p K^{\#}$, $\bigsqcup_i K^{\#}\widehat{\pi}_p^{\iota}\gamma_i \subset K^{\#}\widehat{\pi}_p K^{\#}$ and $\bigsqcup_{i=0,\dots,p-1} K^{\#}\widehat{\pi}_i \sqcup K^{\#}\widehat{\pi}_p^{\iota} \subset K^{\#}\widehat{\pi}_p K^{\#}$). The first equality is then easily checked (and equivalent to $\left[K_p^{\#} : K_p\right] = p+1$ or the fact that there are $p+1$ index p $\mathbb{Z}_p$-sublattices in $\mathbb{Z}_p^2$). To see the other equalities, we may fix a left and right invariant Haar measure μ and check that both sides have the same measure as follows. First we remark that, because $\widehat{\omega}_p\widehat{\gamma}_\infty = \widehat{\pi}_p$,

$$K\widehat{\pi}_p K^{\#} = K\widehat{\omega}_p\widehat{\gamma}_\infty K^{\#} = K\widehat{\omega}_p K^{\#} = \widehat{\omega}_p K K^{\#} = \widehat{\omega}_p K^{\#}. \tag{55}$$

Then we see that

$$\mu\left(K\widehat{\pi}_p K^{\#}\right) = \mu\left(\widehat{\omega}_p K^{\#}\right) = \mu\left(K^{\#}\right) = (p+1)\,\mu(K),$$

proving the second equality because

$$\mu\left(\bigsqcup_{i=0}^{p-1} K\widehat{\pi}_i \sqcup K\widehat{\omega}_p\right) = (p+1)\,\mu(K).$$

Using the first equality, one checks that

$$K^{\#}\widehat{\pi}_p K^{\#} = \bigcup_{i=0,\dots,p-1,\infty} \widehat{\gamma}_i^{-1} K\widehat{\pi}_p K^{\#}.$$

Then we see that

$$\mu\left(K^{\#}\widehat{\pi}_p K^{\#}\right) \leqslant (p+1)\mu\left(K\widehat{\pi}_p K^{\#}\right) = (p+1)\mu\left(K^{\#}\right),$$

proving the third and the fourth equalities because

$$\mu\left(\bigsqcup_{i=0,\dots,p-1,\infty} K^{\#}\widehat{\pi}_p^{\iota}\widehat{\gamma}_i\right) = \mu\left(\bigsqcup_{i=0,\dots,p-1} K^{\#}\widehat{\pi}_i \sqcup K^{\#}\widehat{\pi}_p^{\iota}\right) = (p+1)\mu\left(K^{\#}\right).$$

$\square$

We suppose in this §6 that $\widehat{\mathbb{Z}}^{\times} \subset K$ and that we may write $\omega_0(z) = \omega_{\mathrm{f}}(z)\,\mathrm{N}_{\mathrm{f}}^{k}(z)$. We have have two degeneracy maps

$$K^{\#}1K = K^{\#}1, K^{\#}\widehat{\pi}_p^{\iota}K : M\left(\mathbf{V}_{k,F},\omega_0\right)^{K^{\#}} \to M\left(\mathbf{V}_{k,F},\omega_0\right)^{K}.$$

Define

$$\varphi^{(p)} := \varphi \mid K^{\#}\widehat{\pi}_p^{\iota}K \in M\left(\mathbf{V}_{k,F},\omega_0\right)^{K}.$$

Let us now fix $0 \neq \varphi \in M\left(\mathbf{V}_{k,F},\omega_0\right)^{K^{\#}}$ such that $\varphi \mid T_p = a_p(\varphi)\,\varphi$ and define $M\left(\mathbf{V}_{k,F},\omega_0\right)^{K,\varphi\text{-old}} \subset M\left(\mathbf{V}_{k,F},\omega_0\right)^{K}$ to be the span of $\left\{\varphi,\varphi^{(p)}\right\}$. We define the Hecke polynomial at p and the quantities $\alpha_p(\varphi)$ and $\beta_p(\varphi)$ to via the formula

$$X^2 - a_p(\varphi)\,X + \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^{k+1} = \left(X - \alpha_p(\varphi)\right)\left(X - \beta_p(\varphi)\right). \tag{56}$$

Next, let us set

$$\begin{aligned} \varphi^{\alpha} &= \varphi^{\alpha_p(\varphi)} := \varphi - \alpha_p(\varphi)^{-1}\varphi^{(p)} \in M\left(\mathbf{V}_{k,F},\omega_0\right)^{K,\varphi\text{-old}}, \\ \varphi^{\beta} &= \varphi^{\beta_p(\varphi)} := \varphi - \beta_p(\varphi)^{-1}\varphi^{(p)} \in M\left(\mathbf{V}_{k,F},\omega_0\right)^{K,\varphi\text{-old}}. \end{aligned}$$

We say that φ is semisimple (resp. not semisimple) if $\alpha_p(\varphi) \neq \beta_p(\varphi)$ (resp. $\alpha_p(\varphi) = \beta_p(\varphi)$). Conjecturally, φ is always semisimple, as shown in [12, Corollary 3.2 and Remark 3.3].

Remark 6.2. Suppose that $\widehat{\mathbb{Z}}^\times \subset K'$ and that $\psi \in M(\mathbf{V}_{k,F},\omega_0)^{K'}$. Write $\mathbf{p} := \widehat{p}$ (resp. $\mathbf{p}' \in \widehat{\mathbb{Z}}^\times \subset K'$) for the idele concentrated at p, where we have $\mathbf{p}_p = p$ (resp. the finite idele defined by the conditions $(\mathbf{p}')_v = p$ for every $v \neq p$ and $(\mathbf{p}')_p = 1$). Then $\varphi \mid K'\mathbf{p} = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi$.

Proof. Indeed, we may write $p = \mathbf{p}\mathbf{p}'$ and we see that:

$$(\varphi \mid K'\mathbf{p})(x) = \varphi(\mathbf{p}x) = \varphi\left(\mathbf{p}'^{-1}xp\right) = \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\varphi(x)\,p = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi(x).$$

□

Set $T_p := K^{\#}\widehat{\pi}_p K^{\#}$ (acting on $M(\mathbf{V}_{k,F},\omega_0)^{K^{\#}}$), $U_p := K\widehat{\pi}_p K$ and $W_p := K\widehat{\omega}_p$ (both acting on $M(\mathbf{V}_{k,F},\omega_0)^{K}$).

Corollary 6.3. *If $\varphi \in M(\mathbf{V}_{k,F},\omega_0)^{K^{\#}}$ we have*

$$\begin{aligned} \varphi \quad &| \quad U_p = \varphi \mid T_p - \varphi^{(p)},\ \varphi^{(p)} \mid U_p = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^{k+1}\varphi \\ \varphi \quad &| \quad W_p = \varphi^{(p)} \text{ and } \varphi^{(p)} \mid W_p = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi. \end{aligned}$$

Proof. Noticing that $K^{\#}\widehat{\pi}_p^{\iota}K = K^{\#}\widehat{\pi}_p^{\iota}$ (because $\widehat{\gamma}_\infty\widehat{\omega}_p = \widehat{\pi}_p^{\iota}$, arguing as in (55)), the first equality is a direct consequence of the last decomposition of Lemma 6.1 and the definition $\varphi^{(p)} := \varphi \mid K^{\#}\widehat{\pi}_p^{\iota}K$. Since $\varphi^{(p)} = \varphi \mid K^{\#}\widehat{\pi}_p^{\iota}$ we find

$$\varphi^{(p)} \mid U_p = \textstyle\sum_{i=0}^{p-1}\left(\varphi \mid K^{\#}\widehat{\pi}_p^{\iota}\right)\widehat{\pi}_i = \sum_{i=0}^{p-1}\varphi\widehat{\pi}_p^{\iota}\widehat{\pi}_i.$$

We now remark that

$$\pi_p^{\iota}\pi_i = \begin{pmatrix} p & ip \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} p,$$

so that $\widehat{\pi}_p^{\iota}\widehat{\pi}_i \in K\mathbf{p}$. It follows from Remark 6.2 and the K-invariance of φ that we have $\varphi\widehat{\pi}_p^{\iota}\widehat{\pi}_i = \varphi\mathbf{p} = \varphi \mid K\mathbf{p} = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi$. Hence we find

$$\varphi^{(p)} \mid U_p = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \textstyle\sum_{i=0}^{p-1}\varphi = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^{k+1}\varphi.$$

The equality $\varphi \mid W_p = \varphi^{(p)}$ follows from $K^{\#}\widehat{\pi}_p^{\iota}K = K^{\#}\widehat{\omega}_p$ (because we remarked that $K^{\#}\widehat{\pi}_p^{\iota}K = K^{\#}\widehat{\pi}_p^{\iota}$ and $K^{\#}\widehat{\pi}_p^{\iota} = K^{\#}\widehat{\omega}_p$ in view of $\widehat{\pi}_p^{\iota} = \widehat{\gamma}_\infty\widehat{\omega}_p$) and the fact that $\varphi \mid K\widehat{\omega}_p = \varphi \mid K^{\#}\widehat{\omega}_p$ because $\varphi \in M(\mathbf{V}_{k,F},\omega_0)^{K^{\#}}$. Finally, since $\omega_p^2 = -p$, once again noticing that $K^{\#}\widehat{\pi}_p^{\iota}K = K^{\#}\widehat{\omega}_p$, we find

$$\varphi^{(p)} \mid W_p = \left(\varphi \mid K^{\#}\widehat{\omega}_p\right)\widehat{\omega}_p = \varphi\widehat{\omega}_p^2 = \varphi \mid K^{\#}\mathbf{p} = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi.$$

□

The following result, whose proof is left to the reader, can now be deduced from Corollary 6.3 by standard linear algebra and the well-known fact that $\mathrm{Im}\left(K^{\#}1K\right) \cap \mathrm{Im}\left(K^{\#}\widehat{\pi}_p^{\iota}K\right) = 0$.

Proposition 6.4. *The following facts are true, assuming that F is a field such that $\alpha_p(\varphi), \beta_p(\varphi) \in F$ for the statements* $(2)-(5)$.

1. *The space $M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$ is two dimensional with basis $\left\{\varphi, \varphi^{(p)}\right\}$, stable under the action of the U_p and W_p operators.*
2. *We have $\varphi^\alpha \mid U_p = \alpha_p(\varphi)\varphi$, $\varphi^\beta \mid U_p = \beta_p(\varphi)\varphi$ and, if $\psi \in M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$ is such that $\psi \mid U_p = \rho\psi$, then $\psi = \varphi^\alpha$ or $\psi = \varphi^\beta$ up to a scalar factor.*
3. *We have*

$$F(\varphi^\alpha \mid W_p) \cap F\varphi^\alpha = F(\varphi^\alpha \mid W_p) \cap F\varphi^\beta = 0 \text{ (resp. } F\left(\varphi^\beta \mid W_p\right) \cap F\varphi^\alpha = F\left(\varphi^\beta \mid W_p\right) \cap F\varphi^\beta = 0\text{)}$$

unless $\alpha_p(\varphi)^2 = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k$ (resp. $\beta_p(\varphi)^2 = \omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k$) and, in this case, $\varphi^\alpha \mid W_p = -\alpha_p(\varphi)\varphi^\alpha$ (resp. $\varphi^\beta \mid W_p = -\beta_p(\varphi)\varphi^\beta$). In general,

$$\varphi^\alpha \mid W_p = \varphi^{(p)} - \alpha_p(\varphi)^{-1}\omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi \text{ (resp. } \varphi^\beta \mid W_p = \varphi^{(p)} - \beta_p(\varphi)^{-1}\omega_{\mathrm{f}}(\mathbf{p}')^{-1} p^k \varphi\text{).}$$

(4) *If φ is semisimple, then U_p is diagonalizable on $M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$ and we have*
$$M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}=F\varphi^{\alpha}\oplus F\varphi^{\beta}.$$
(5) *If φ is not semisimple, then*
$$0\neq F\varphi^{\alpha}=F\varphi^{\beta}\subset M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$$
is a one dimensional subspace of $M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$ and U_p is not diagonalizable on $M(\mathbf{V}_{k,F},\omega_0)^{K,\varphi\text{-}old}$.

6.1. **The case of three p-old forms.** Let us assume that $\varphi_i\in M(\mathbf{V}_{k_i,F},\omega_{0,i})^{K^{\#}}$ are such that $\varphi_i\mid T_p=a_p(\varphi_i)\varphi_i$ for $i=1,2,3$ and let us write $\alpha_i:=\alpha(\varphi_i)$ and $\beta_i:=\beta(\varphi_i)$. If $\underline{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ and $\underline{k}=(k_1,k_2,k_3)$, we define
$$\mathcal{E}_{p,1}(\underline{\alpha},\underline{k}):=1-\omega_{\mathrm{f},1}(\mathbf{p}')\frac{\alpha_1}{\alpha_2\alpha_3}p^{\underline{k}_1^*},\ \mathcal{E}_{p,2}(\underline{\alpha},\underline{k}):=1-\omega_{\mathrm{f},2}(\mathbf{p}')\frac{\alpha_2}{\alpha_1\alpha_3}p^{\underline{k}_2^*},\ \mathcal{E}_{p,3}(\underline{\alpha},\underline{k}):=1-\omega_{\mathrm{f},3}(\mathbf{p}')\frac{\alpha_3}{\alpha_1\alpha_2}p^{\underline{k}_3^*},$$
$$\mathcal{E}_{p}(\underline{\alpha},\underline{k}):=\mathcal{E}_{p,1}(\underline{\alpha},\underline{k})\mathcal{E}_{p,2}(\underline{\alpha},\underline{k})\mathcal{E}_{p,3}(\underline{\alpha},\underline{k})\left(1-\frac{1}{\alpha_1\alpha_2\alpha_3}p^{\underline{k}^*+1}\right)$$
and then (the last equality with the convention that it simply means "formally remove the $\mathcal{E}_{p,i}(\underline{\alpha},\underline{k})$ factor"):
$$\widehat{\mathcal{E}}_{p,i}(\underline{\alpha},\underline{k}):=\textstyle\prod_{j\neq i}\mathcal{E}_{p,j}(\underline{\alpha},\underline{k})\left(1-\frac{1}{\alpha_1\alpha_2\alpha_3}p^{\underline{k}^*+1}\right)=\frac{\mathcal{E}_{p}(\underline{\alpha},\underline{k})}{\mathcal{E}_{p,i}(\underline{\alpha},\underline{k})}.$$

Proposition 6.5. *With the above notations, the following formulas hold:*
$$t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}\mid W_p,\varphi_2^{(\alpha_2)},\varphi_3^{(\alpha_3)}\right)=\frac{\alpha_1}{p+1}\widehat{\mathcal{E}}_{p,1}(\underline{\alpha},\underline{k})\,t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3),$$
$$t_{\underline{k}}\left(\varphi_1^{(\alpha_1)},\varphi_2^{(\alpha_2)}\mid W_p,\varphi_3^{(\alpha_3)}\right)=\frac{\alpha_2}{p+1}\widehat{\mathcal{E}}_{p,2}(\underline{\alpha},\underline{k})\,t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3),$$
$$t_{\underline{k}}\left(\varphi_1^{(\alpha_1)},\varphi_2^{(\alpha_2)},\varphi_3^{(\alpha_3)}\mid W_p\right)=\frac{\alpha_3}{p+1}\widehat{\mathcal{E}}_{p,3}(\underline{\alpha},\underline{k})\,t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3).$$

Proof. We have, by definition and Proposition 6.4 (3),
$$\begin{aligned}t_{\underline{k}}\left(\varphi_1^{(\alpha_1)},\varphi_2^{(\alpha_2)},\varphi_3^{(\alpha_3)}\mid W_p\right)&=-t_{\underline{k}}\left(\varphi_1-\alpha_1^{-1}\varphi_1^{(p)},\varphi_2-\alpha_2^{-1}\varphi_2^{(p)},\alpha_3^{-1}p^{k_3}\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\varphi_3-\varphi_3^{(p)}\right)\\&=-\left(A^{(3)}-B^{(3)}+C^{(3)}-D^{(3)}\right)\end{aligned}$$
where
$$\begin{aligned}A^{(3)}&=\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\alpha_3^{-1}p^{k_3}t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3),\\B^{(3)}&=\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\alpha_1^{-1}\alpha_3^{-1}p^{k_3}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2,\varphi_3\right)+\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\alpha_2^{-1}\alpha_3^{-1}p^{k_3}t_{\underline{k}}\left(\varphi_1,\varphi_2^{(p)},\varphi_3\right)\\&\quad+t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3^{(p)}\right)\\C^{(3)}&=\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}p^{k_3}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2^{(p)},\varphi_3\right)+\alpha_1^{-1}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2,\varphi_3^{(p)}\right)+\alpha_2^{-1}t_{\underline{k}}\left(\varphi_1,\varphi_2^{(p)},\varphi_3^{(p)}\right)\\D^{(3)}&=\alpha_1^{-1}\alpha_2^{-1}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2^{(p)},\varphi_3^{(p)}\right).\end{aligned}$$

Regarding $t_{\underline{k}}$ as a pairing as we did in the proof of Corollary 5.6 (for $t_{3,\underline{k}}$), we compute
$$\begin{aligned}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2,\varphi_3\right)&=\left\langle\varphi_1\mid K^{\#}\widehat{\pi}_p^{\iota}K,\varphi_2\otimes\varphi_3\right\rangle_t\ \text{(the pairing does not depend on the level)}\\&=\left\langle\varphi_1\mid K^{\#}\widehat{\pi}_p^{\iota}K,\varphi_2\otimes\varphi_3\mid K^{\#}1K\right\rangle_t\ \text{(by Proposition 2.9)}\\&=(p+1)^{-1}\left\langle\left(\varphi_1\mid K^{\#}\widehat{\pi}_p^{\iota}K\right)\mid K1K^{\#},\varphi_2\otimes\varphi_3\right\rangle_t\ \text{(we have } K^{\#}\widehat{\pi}_p^{\iota}K=K^{\#}\widehat{\pi}_p^{\iota})\\&=(p+1)^{-1}\textstyle\sum_{\gamma_i\in K\backslash K^{\#}}\left\langle\varphi_1\widehat{\pi}_p^{\iota}\gamma_i,\varphi_2\otimes\varphi_3\right\rangle_t.\end{aligned}\tag{57}$$
It follows from Lemma 6.1 that, if $K^{\#}=\bigsqcup_i K\gamma_i$, then $K^{\#}\widehat{\pi}_pK^{\#}=\bigsqcup_i K^{\#}\widehat{\pi}_p^{\iota}\gamma_i$. Therefore,
$$\begin{aligned}\textstyle\sum_{\gamma_i\in K\backslash K^{\#}}\left\langle\varphi_1\widehat{\pi}_p^{\iota}\gamma_i,\varphi_2\otimes\varphi_3\right\rangle_t&=\left\langle\varphi_1\mid T_p,\varphi_2\otimes\varphi_3\right\rangle_t=a_p(\varphi_1)\left\langle\varphi_1,\varphi_2\otimes\varphi_3\right\rangle_t\\&=a_p(\varphi_1)\,t_{\underline{k}}(\varphi_1,\varphi_2,\varphi_3).\end{aligned}\tag{58}$$

We have proved that

$$t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2,\varphi_3\right)=(p+1)^{-1}\,a_p\left(\varphi_1\right)t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).$$

Working in a similar way for the other two terms of $B^{(3)}$ we deduce (recall $\alpha_3 p^{-k_3}=\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)\beta_3^{-1}p$):

$$\begin{aligned}(p+1)B^{(3)} &= \left\{\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_1^{-1}a_p\left(\varphi_1\right)\alpha_3^{-1}p^{k_3}+\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_2^{-1}a_p\left(\varphi_2\right)\alpha_3^{-1}p^{k_3}+a_p\left(\varphi_3\right)\right\}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right)\\ &= \{\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3}+\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_1^{-1}\alpha_3^{-1}\beta_1p^{k_3}+\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3}\\ &\quad +\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_2^{-1}\alpha_3^{-1}\beta_2p^{k_3}+\alpha_3+\beta_3\}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).\end{aligned}$$

Noticing that we have $K^{\#}\widehat{\pi}_p^{\iota}K=K^{\#}\widehat{\pi}_p^{\iota}$, we find

$$\varphi_1^{(p)}\otimes\varphi_2^{(p)}=\left(\varphi_1\otimes\varphi_2\right)\mid K^{\#}\widehat{\pi}_p^{\iota}K.$$

Hence we find, using the adjointness property of Proposition 2.9,

$$\begin{aligned}t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2^{(p)},\varphi_3\right) &= \left\langle\left(\varphi_1\otimes\varphi_2\right)\mid K^{\#}\widehat{\pi}_p^{\iota}K,\varphi_3\mid K^{\#}1K\right\rangle_t\\ &= (p+1)^{-1}\,\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)p^{\underline{k}_3^*}\left\langle\varphi_1\otimes\varphi_2,\left(\varphi_3\mid K^{\#}1K\right)\mid K\widehat{\pi}_pK^{\#}\right\rangle_t\\ &= (p+1)^{-1}\,\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)p^{\underline{k}_3^*}a_p(\varphi_3)\left\langle\varphi_1\otimes\varphi_2,\varphi_3\right\rangle_t\\ &= (p+1)^{-1}\,\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)p^{\underline{k}_3^*}a_p(\varphi_3)t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).\end{aligned}$$

Working in a similar way for the other two terms of $C^{(3)}$ we deduce (recall $\alpha_3p^{-k_3}=\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)\beta_3^{-1}p$)

$$\begin{aligned}(p+1)C^{(3)} &= \{\alpha_1^{-1}\alpha_2^{-1}a_p\left(\varphi_3\right)\alpha_3^{-1}p^{k_3+\underline{k}_3^*}+\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)\alpha_1^{-1}a_p\left(\varphi_2\right)p^{\underline{k}_2^*}\\ &\quad +\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)\alpha_2^{-1}a_p\left(\varphi_1\right)p^{\underline{k}_1^*}\}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right)\\ &= \{\alpha_1^{-1}\alpha_2^{-1}p^{k_3+\underline{k}_3^*}+\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}\beta_3p^{k_3+\underline{k}_3^*}\\ &\quad +\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)\alpha_1^{-1}\alpha_2p^{\underline{k}_2^*}+\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)\alpha_1^{-1}\beta_2p^{\underline{k}_2^*}\\ &\quad +\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)\alpha_2^{-1}\alpha_1p^{\underline{k}_1^*}+\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)\alpha_2^{-1}\beta_1p^{\underline{k}_1^*}\}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).\end{aligned}$$

Finally, once again using $K^{\#}\widehat{\pi}_p^{\iota}K=K^{\#}\widehat{\pi}_p^{\iota}$, we find

$$\varphi_1^{(p)}\otimes\varphi_2^{(p)}\otimes\varphi_3^{(p)}=\varphi_1\otimes\varphi_2\otimes\varphi_3\mid K^{\#}\widehat{\pi}_p^{\iota}$$

and $t_{\underline{k}}\left(\varphi_1^{(p)},\varphi_2^{(p)},\varphi_3^{(p)}\right)=p^{\underline{k}^*}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right)$. Hence we find

$$D^{(3)} = \alpha_1^{-1}\alpha_2^{-1}p^{\underline{k}^*}t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).$$

Putting everything together, we have computed that

$$t_{\underline{k}}\left(\varphi_1^{(\alpha_1)},\varphi_2^{(\alpha_2)},\varphi_3^{(\alpha_3)}\mid W_p\right)=-\left(p+1\right)^{-1}E\cdot t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right),\tag{59}$$

where (using $\beta_i=\omega_{\mathrm{f},i}\left(\mathbf{p}'\right)^{-1}\alpha_i^{-1}p^{k_i+1}$):

$$\begin{aligned}E &= \omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3}+\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3+1}\\ &\quad -\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3}-\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)^{-1}\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_1^{-2}\alpha_3^{-1}p^{k_1+k_3+1}-\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3}\\ &\quad -\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)^{-1}\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_2^{-2}\alpha_3^{-1}p^{k_2+k_3+1}-\alpha_3-\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_3^{-1}p^{k_3+1}\\ &\quad +\alpha_1^{-1}\alpha_2^{-1}p^{k_3+\underline{k}_3^*}+\omega_{\mathrm{f},3}\left(\mathbf{p}'\right)^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-2}p^{2k_3+\underline{k}_3^*+1}\\ &\quad +\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)\alpha_1^{-1}\alpha_2p^{\underline{k}_2^*}+\alpha_1^{-1}\alpha_2^{-1}p^{k_2+\underline{k}_2^*+1}\\ &\quad +\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)\alpha_2^{-1}\alpha_1p^{\underline{k}_1^*}+\alpha_2^{-1}\alpha_1^{-1}p^{k_1+\underline{k}_1^*+1}\\ &\quad -\alpha_1^{-1}\alpha_2^{-1}p^{\underline{k}^*}-\alpha_1^{-1}\alpha_2^{-1}p^{\underline{k}^*+1}.\end{aligned}$$

Let us now remark that, writing (i,j) for the j-term of the i-line, we have the following simplifications: $(1,1)$ with $(2,1)$, $(1,2)$ with $(3,3)$, $(4,1)$ with $(7,1)$ (because $k_3+\underline{k}_3^*=\underline{k}$) and $(6,2)$ with $(7,2)$ (because $k_1+\underline{k}_1^*=\underline{k}$). Hence, we find (recalling that $\omega_{\mathrm{f},1}\omega_{\mathrm{f},2}\omega_{\mathrm{f},3}=1$ on $\widehat{\mathbb{Z}}^{\times}$ in the first equality and noticing that

$k_1+k_3+1=\underline{k}_2^*+\underline{k}^*+1$, $k_3=\underline{k}_1^*+\underline{k}_2^*$, $k_2+k_3+1=\underline{k}_1^*+\underline{k}^*+1$, $2k_3+\underline{k}_3^*+1=\underline{k}_1^*+\underline{k}_2^*+\underline{k}^*+1$ and $k_2+\underline{k}_2^*+1=\underline{k}^*+1$ to get the factorization):

$$\begin{aligned}E &= -\alpha_3(\omega_{\mathrm{f},2}(\mathbf{p}')\,\alpha_1^{-2}\alpha_3^{-2}p^{k_1+k_3+1}+\omega_{\mathrm{f},1}(\mathbf{p}')\,\omega_{\mathrm{f},2}(\mathbf{p}')\,\alpha_3^{-2}p^{k_3}\\ &\quad+\omega_{\mathrm{f},1}(\mathbf{p}')\,\alpha_2^{-2}\alpha_3^{-2}p^{k_2+k_3+1}+1-\omega_{\mathrm{f},3}(\mathbf{p}')^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-3}p^{2k_3+\underline{k}_3^*+1}\\ &\quad-\omega_{\mathrm{f},2}(\mathbf{p}')\,\alpha_1^{-1}\alpha_2\alpha_3^{-1}p^{\underline{k}_2^*}-\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}p^{k_2+\underline{k}_2^*+1}-\omega_{\mathrm{f},1}(\mathbf{p}')\,\alpha_2^{-1}\alpha_1\alpha_3^{-1}p^{\underline{k}_1^*})\\ &= -\alpha_3\left(1-\omega_{\mathrm{f},1}(\mathbf{p}')\frac{\alpha_1}{\alpha_2\alpha_3}p^{\underline{k}_1^*}\right)\left(1-\omega_{\mathrm{f},2}(\mathbf{p}')\frac{\alpha_2}{\alpha_1\alpha_3}p^{\underline{k}_2^*}\right)\left(1-\frac{1}{\alpha_1\alpha_2\alpha_3}p^{\underline{k}^*+1}\right).\end{aligned}$$

Inserting this computation of E in (59) gives the third equation. The first two equations are proved in a similar way. $\square$

Let us discuss the p-adic periods.

Lemma 6.6. *Suppose that $0\neq\varphi_1,\varphi_2\in M(\mathbf{V}_{k,F},\omega_0)^{K^\#}$ are such that $\varphi_i\mid T_p=a_p(\varphi_i)\,\varphi_i$ and let $\alpha_i=\alpha_p(\varphi_i)$ be a root of the Hecke polynomial at p of φ_i (see (56)). If $a_p(\varphi_1)=a_p(\varphi_2)$ and $\alpha:=\alpha_1=\alpha_2$, then we have*

$$\left(\varphi^{(\alpha)},\varphi^{(\alpha)}\mid W_p\right)_k=\frac{\alpha\left(1-\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^k\right)\left(1-\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^{k+1}\right)}{p+1}(\varphi,\varphi)_k.$$

Proof. We have, by Proposition 6.4 (3),

$$\begin{aligned}\left(\varphi_1^{(\alpha)},\varphi_2^{(\alpha)}\mid W_p\right)_k &= \left(\varphi_1-\alpha^{-1}\varphi_1^{(p)},\varphi_2^{(p)}-\alpha^{-1}\omega_{\mathrm{f}}(\mathbf{p}')^{-1}p^k\varphi_2\right)_k\\ &= -\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^k(\varphi_1,\varphi_2)_k+\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^k\left(\varphi_1^{(p)},\varphi_2\right)_k+\left(\varphi_1,\varphi_2^{(p)}\right)_k-\alpha^{-1}\left(\varphi_1^{(p)},\varphi_2^{(p)}\right)_k.\end{aligned}\tag{60}$$

If $\varphi,\psi\in M(\mathbf{V}_{k,F},\omega_0)^{K^\#}$ and $\psi^\vee\in M\left(\mathbf{V}_{k,F},\mathrm{N}_{\mathrm{f}}^{2k}\left(\omega_0^k\right)^{-1}\right)^{K^\#}$ are such that $\varphi\mid T_p=a_p(\varphi)\,\varphi$, the adjointness property of Proposition 2.9 gives, arguing as in (57) and (58):

$$(p+1)\left\langle\varphi^{(p)},\psi^\vee\right\rangle_k=\left\langle\varphi\mid T_p,\psi^\vee\right\rangle_k=a_p(\varphi)\left\langle\varphi,\psi^\vee\right\rangle_k\overset{\text{take }\psi^\vee=\check{\psi}}{\Longrightarrow}(p+1)\left(\varphi^{(p)},\psi\right)_k=a_p(\varphi)(\varphi,\psi)_k.\tag{61}$$

In order to get a symmetrical relation, suppose now we have $\psi^\vee\mid T_p=a_p\left(\psi^\vee\right)\psi^\vee$ and apply once again Proposition 2.9 arguing as in (57) and (58) in order to get

$$(p+1)\left\langle\varphi,\left(\psi^\vee\right)^{(p)}\right\rangle_k=a_p\left(\psi^\vee\right)\left\langle\varphi,\psi^\vee\right\rangle_k.$$

Next, note that twisting does not exactly commutes with the right $B_{\mathrm{f}}^\times$-actions: rather we have $\check{\varphi}g=\mathrm{nrd}_{\mathrm{f}}^{-\omega_{\mathrm{f}}}(g)(\check{\varphi g})$. It follows that $\check{\psi}^{(p)}=\omega_{\mathrm{f}}^{-1}(\widehat{p})(\check{\psi^{(p)}})$ and $a_p\left(\check{\psi}\right)=\omega_{\mathrm{f}}^{-1}(\widehat{p})a_p(\psi)$. Then, taking $\psi^\vee=\check{\psi}$ in the above relation gives

$$\begin{aligned}&(p+1)\,\omega_{\mathrm{f}}^{-1}(\widehat{p})\left\langle\varphi,(\check{\psi^{(p)}})\right\rangle_k=(p+1)\left\langle\varphi,\check{\psi}^{(p)}\right\rangle_k=a_p\left(\check{\psi}\right)\left\langle\varphi,\check{\psi}\right\rangle_k=\omega_{\mathrm{f}}^{-1}(\widehat{p})a_p(\psi)\left\langle\varphi,\check{\psi}\right\rangle_k\\ &\Leftrightarrow(p+1)\left\langle\varphi,(\check{\varphi^{(p)}})\right\rangle_k=a_p(\psi)\left\langle\varphi,\check{\psi}\right\rangle_k\Leftrightarrow(p+1)\left(\varphi,\psi^{(p)}\right)_k=a_p(\psi)(\varphi,\psi)_k.\end{aligned}\tag{62}$$

Finally, the invariance property of $\langle-,-\rangle_k$ gives $\left\langle\varphi^{(p)},\left(\psi^\vee\right)^{(p)}\right\rangle_k=p^k\left\langle\varphi,\psi^\vee\right\rangle_k$ and then we find (because $\omega_{\mathrm{f}}^{-1}(\widehat{p})=\omega_{\mathrm{f}}(\mathbf{p}')$):

$$\omega_{\mathrm{f}}^{-1}(\widehat{p})\left\langle\varphi^{(p)},(\check{\psi^{(p)}})\right\rangle_k=\left\langle\varphi^{(p)},\check{\psi}^{(p)}\right\rangle_k=p^k\left\langle\varphi,\check{\psi}\right\rangle_k\Leftrightarrow\left(\varphi^{(p)},\psi^{(p)}\right)_k=\omega_{\mathrm{f}}(\mathbf{p}')^{-1}p^k(\varphi,\psi)_k.\tag{63}$$

Inserting (61), (62) and (63) in (60) yields

$$\left(\varphi_1^{(\alpha)},\varphi_2^{(\alpha)}\mid W_p\right)_k=(p+1)^{-1}E\cdot(\varphi_1,\varphi_1)_k,$$

where (using $\beta = \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1}$):

$$\begin{aligned}
E &= -\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} + \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^{k}a_p(\varphi) \\
&\quad + a_p(\psi) - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} \\
&= -\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} + \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} + \omega_{\mathrm{f}}(\mathbf{p}')^{-2}\alpha^{-3}p^{2k+1} \\
&\quad + \alpha + \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} \\
&= -\omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k+1} + \omega_{\mathrm{f}}(\mathbf{p}')^{-2}\alpha^{-3}p^{2k+1} + \alpha - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-1}p^{k} \\
&= \alpha\left(1 - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^{k}\right)\left(1 - \omega_{\mathrm{f}}(\mathbf{p}')^{-1}\alpha^{-2}p^{k+1}\right).
\end{aligned}$$

□

6.2. **The case of two p-old forms.** Let us assume that $\varphi_i \in M(\mathbf{V}_{k_i,F}, \omega_{0,i})^{K^\#}$ are such that $\varphi_i \mid T_p = a_p(\varphi_i)\varphi_i$ for $i = 1, 2$ and that $\varphi_3 \in M(\mathbf{V}_{k_3,F}, 1)^K$ is p-new, has even weight and trivial central character, i.e. it is such that $\varphi_3 \mid U_p = -w_{p,3}p^{k_3/2}\varphi_3$ and $\varphi_3 \mid W_p = w_{p,3}p^{k_3/2}\varphi_3$ with $w_{p,3} \in \{\pm 1\}$. To make the notation uniform, we define $\alpha_3 := -w_{p,3}p^{k_3/2}$ and $\varphi_3^{(\alpha_3)} := \varphi_3$. Then

$$\mathcal{E}_{p,1}(\underline{\alpha}, \underline{k}) = 1 + w_{p,3}\omega_{\mathrm{f},1}(\mathbf{p}')\frac{p^{(k_2-k_1)/2}\alpha_1}{\alpha_2},\ \mathcal{E}_{p,2}(\underline{\alpha}, \underline{k}) = 1 + w_{p,3}\omega_{\mathrm{f},2}(\mathbf{p}')\frac{p^{(k_1-k_2)/2}\alpha_2}{\alpha_1}$$

$$\text{and } \mathcal{E}_{p,3}(\underline{\alpha}, \underline{k}) = 1 + w_{p,3}\frac{p^{(k_1+k_2)/2}}{\alpha_1\alpha_2}.$$

Proposition 6.7. *With the above notations, the following formulas hold:*

$$\begin{aligned}
t_{\underline{k}}\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}\right) &= \mathcal{E}_{p,3}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) \\
&= w_{p,3}\omega_{\mathrm{f},1}(\mathbf{p}')^{-1}p^{(k_1-k_2)/2}\mathcal{E}_{p,3}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right), \\
t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)}\right) &= \mathcal{E}_{p,3}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right) \\
&= w_{p,3}\omega_{\mathrm{f},2}(\mathbf{p}')^{-1}p^{(k_2-k_1)/2}\mathcal{E}_{p,3}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right), \\
t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p\right) &= \frac{\alpha_3}{\alpha_2}\mathcal{E}_{p,2}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right) \\
&= \frac{\alpha_3}{\alpha_1}\mathcal{E}_{p,1}(\underline{\alpha}, \underline{k})\, t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right).
\end{aligned}$$

Proof. We begin with the following remark from [17, discussion after Theorem 2]: because $t_{\underline{k}}$ is $K^\#$-invariant (resp. $\omega_p^{-1}K^\#\omega_p$-invariant)[5] and $\varphi_i \in M(\mathbf{V}_{k_i,F}, \omega_{0,i})^{K^\#}$ (resp. $\varphi_i^{(p)} \in M(\mathbf{V}_{k_i,F}, \omega_{0,i})^{\omega_p^{-1}K^\#\omega_p}$) for $i = 1, 2$, $\varphi \mapsto t_{\underline{k}}(\varphi_1, \varphi_2, \varphi)$ (resp. $\varphi \mapsto t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2^{(p)}, \varphi\right)$) is $K^\#$-invariant (resp. $\omega_p^{-1}K^\#\omega_p$-invariant) and, hence, it is zero on the irreducible representation $V_{\varphi_3} \subset M(\mathbf{V}_{k_3,F}, \omega_{0,3})$ generated by φ_3, whose dual representation does not have non-zero $K^\#$-invariant (resp. $\omega_p^{-1}K^\#\omega_p$-invariant) vectors. In particular,

$$t_{\underline{k}}(\varphi_1, \varphi_2, \varphi) = t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2^{(p)}, \varphi\right) = 0. \tag{64}$$

The computations of $t_{\underline{k}}\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3\right)$ and $t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3\right)$ are the same, so that we can work with the second. We have, by Proposition 6.4 (3) and (64):

$$\begin{aligned}
t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3\right) &= t_{\underline{k}}\left(\varphi_1 - \alpha_1^{-1}\varphi_1^{(p)}, \varphi_2^{(p)} - \alpha_2^{-1}p^{k_2}\omega_{\mathrm{f},2}(\mathbf{p}')^{-1}\varphi_2, \varphi_3\right) \\
&= t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right) + \omega_{\mathrm{f},2}(\mathbf{p}')^{-1}\alpha_1^{-1}\alpha_2^{-1}p^{k_2}t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) \qquad (65)
\end{aligned}$$

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[5]The trilinear form $t_{\underline{k}}$ satisfies the invariance formula

$$t_{\underline{k}}(\varphi_1 u, \varphi_2 u, \varphi_3 u) = \mathrm{Nrd}_f(u)^{\underline{k}^*}\, t_{\underline{k}}(g_1, \varphi_2, \varphi_3)$$

and we have $\mathrm{Nrd}_f(u) = 1$ for $u \in K^\#$ or $u \in \omega_p^{-1}K^\#\omega_p$.

Similarly, because $\varphi_3 \mid W_p = w_{p,3}p^{k_3/2}\varphi_3$ and by (64):

$$\begin{aligned} t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3 \mid W_p\right) &= w_{p,3}p^{k_3/2}\left(\varphi_1 - \alpha_1^{-1}\varphi_1^{(p)}, \varphi_2 - \alpha_2^{-1}\varphi_2^{(p)}, \varphi_3\right) \\ &= -w_{p,3}p^{k_3/2}\left(\alpha_1^{-1}t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) + \alpha_2^{-1}t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right)\right). \end{aligned} \tag{66}$$

Because $K^{\#}\widehat{\pi}_p^{\iota}K = K^{\#}\widehat{\omega}_p$, we have $\varphi_i^{(p)} = \varphi_i \mid K^{\#}\widehat{\omega}_p$; also, noticing that $\widehat{\omega}_p^2 = \widehat{-p}$, we see that $\widehat{\omega}_p^{-1} = \widehat{-p}^{-1}\widehat{\omega}_p = \widehat{-1}\widehat{p}^{-1}\widehat{\omega}_p$, implying that $K'\widehat{\omega}_p^{-1} = K'\mathbf{p}^{-1}\widehat{\omega}_p$ for $K' \in \left\{K^{\#}, K\right\}$. Applying Remark 6.2 gives $\varphi_i \mid K'\widehat{\omega}_p^{-1} = \omega_{\mathrm{f},i}\left(\mathbf{p}'\right)p^{-k_i}\varphi_i \mid K'\widehat{\omega}_p$. Hence we find (using the invariance property of $t_{\underline{k}}$ in the second equality[6] and $\varphi_3 \mid W_p = w_{p,3}p^{k_3/2}\varphi_3$ in the last equality):

$$\begin{aligned} t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) &= t_{\underline{k}}\left(\varphi_i\widehat{\omega}_p, \varphi_2, \varphi_3\right) = p^{\underline{k}^*}t_{\underline{k}}\left(\varphi_1, \varphi_2\widehat{\omega}_p^{-1}, \varphi_3\widehat{\omega}_p^{-1}\right) \\ &= \frac{\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)}{p^{k_2+k_3}}p^{\underline{k}^*}t_{\underline{k}}\left(\varphi_1, \varphi_2\widehat{\omega}_p, \varphi_3\widehat{\omega}_p\right) \\ &= \omega_{\mathrm{f},2}\left(\mathbf{p}'\right)w_{p,3}p^{(k_1-k_2)/2}t_{\underline{k}}\left(\varphi_1, \varphi_2^{(p)}, \varphi_3\right). \end{aligned} \tag{67}$$

Inserting (67) in (65) and (66) yields the claimed formulas. □

6.3. **The case of one p-old form.** Let us assume that $\varphi_i \in M\left(\mathbf{V}_{k_i,F}, \omega_{0,i}\right)^{K^{\#}}$ are such that $\varphi_i \mid T_p = a_p\left(\varphi_i\right)\varphi_i$ for $i = 1$ and that $\varphi_i \in M\left(\mathbf{V}_{k_i,F}, 1\right)^K$ are p-new, have even weight and trivial central character for $i = 2, 3$ (implying $\omega_{\mathrm{f},1}\left(\mathbf{p}'\right) = 1$). As in the setting of Proposition 6.7, we write $\alpha_i := -w_{p,i}p^{k_i/2}$ and $\varphi_i^{(\alpha_3)} := \varphi_i$ for $i = 2, 3$. Then

$$\mathcal{E}_{p,1}\left(\underline{\alpha}, \underline{k}\right) = 1 - w_{p,2}w_{p,3}\alpha_1 p^{-k_1/2} \text{ and } \mathcal{E}_{p,2}\left(\underline{\alpha}, \underline{k}\right) = \mathcal{E}_{p,3}\left(\underline{\alpha}, \underline{k}\right) = 1 - w_{p,2}w_{p,3}\alpha_1^{-1}p^{k_1/2}.$$

Proposition 6.8. *With the above notations, the following formulas hold:*

$$\begin{aligned} t_{\underline{k}}\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}\right) &= -\alpha_1^{-1}p^{k_1}\mathcal{E}_{p,1}\left(\underline{\alpha}, \underline{k}\right)t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right), \\ t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)}\right) &= -\alpha_2\mathcal{E}_{p,2}\left(\underline{\alpha}, \underline{k}\right)t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right), \\ t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p\right) &= -\alpha_3\mathcal{E}_{p,3}\left(\underline{\alpha}, \underline{k}\right)t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right). \end{aligned}$$

Proof. The computations of $t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2 \mid W_p, \varphi_3\right)$ and $t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2, \varphi_3 \mid W_p\right)$ are the same, so that we can work with the second. We have, by Proposition 6.4 (3):

$$\begin{aligned} t_{\underline{k}}\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2, \varphi_3\right) &= t_{\underline{k}}\left(\varphi_1^{(p)} - \alpha_1^{-1}p^{k_1}\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)^{-1}\varphi_1, \varphi_2, \varphi_3\right) \\ &= t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) - \alpha_1^{-1}\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)^{-1}p^{k_1}t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right), \end{aligned} \tag{68}$$

Also, because $\varphi_3 \mid W_p = w_{p,3}p^{k_3/2}\varphi_3$:

$$\begin{aligned} t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2, \varphi_3 \mid W_p\right) &= w_{p,3}p^{k_3/2}t_{\underline{k}}\left(\varphi_1 - \alpha_1^{-1}\varphi_1^{(p)}, \varphi_2, \varphi_3\right) \\ &= w_{p,3}p^{k_3/2}\left(t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right) - \alpha_1^{-1}t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right)\right). \end{aligned} \tag{69}$$

Arguing similarly as we did in (67) we find

$$t_{\underline{k}}\left(\varphi_1^{(p)}, \varphi_2, \varphi_3\right) = w_{p,2}w_{p,3}p^{k_1/2}t_{\underline{k}}\left(\varphi_1, \varphi_2, \varphi_3\right) \tag{70}$$

Inserting (70) in (68) and (70) yields the claimed formulas (recall $\omega_{\mathrm{f},1}\left(\mathbf{p}'\right) = 1$). □

[6] We have $\mathrm{Nrd}_f\left(\widehat{\omega}_p\right) = \left|\mathrm{nrd}\left(\widehat{\omega}_p\right)\right|_{\mathbb{A}_f}^{-1} = |p|_p^{-1} = p$.

6.4. **The case of three p-new forms.** Let us assume that $\varphi_i \in M\left(\mathbf{V}_{k_i,F},1\right)^K$ are p-new, have even weight and trivial central character for $i=1,2,3$. As usual, we write $\alpha_i := -w_{p,i}p^{k_i/2}$ and $\varphi_i^{(\alpha_3)} := \varphi_i$ for $i=1,2,3$. Then

$$\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right) = \mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right) = \mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right) = 1 + w_{1,p}w_{2,p}w_{3,p}.$$

Proposition 6.9. *With the above notations, the following formulas hold:*

$$t_{\underline{k}}\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}\right) = -\alpha_1 t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right),\ t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)}\right) = -\alpha_2 t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right)$$

$$\text{and } t_{\underline{k}}\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p\right) = -\alpha_3 t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right).$$

Furthermore, we have $t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right) = 0$ when $w_{1,p}w_{2,p}w_{3,p} = -1$.

Proof. Just use $\varphi_i \mid W_p = w_{p,i}p^{k_i/2}\varphi_i$ to get the first formulas. The last assertion is a consequence of the invariance property of $t_{\underline{k}}$, which gives the first of the following equalities, and again the relation $\varphi_i \mid W_p = w_{p,i}p^{k_i/2}\varphi_i$, which gives second equality below:

$$p^{\underline{k}^*} t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right) = t_{\underline{k}}\left(\varphi_1 \mid W_p, \varphi_2 \mid W_p, \varphi_3 \mid W_p\right) = w_{1,p}w_{2,p}w_{3,p}p^{\underline{k}^*} t_{\underline{k}}\left(\varphi_1,\varphi_2,\varphi_3\right)$$

$\square$

7. Proof of the main result

7.1. **Interpolation property of the p-adic trilinear form.** Recall our given $\underline{\mathbf{k}} = (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ such that $\mathbf{k}_1 \oplus \mathbf{k}_2 \oplus \mathbf{k}_3$ is even, where $\mathbf{k}_i : \mathbb{Z}_p^\times \to \mathcal{O}_i$ and $\mathcal{O}_{\underline{\mathbf{k}}} := \mathcal{O}_1 \widehat{\otimes} \mathcal{O}_2 \widehat{\otimes} \mathcal{O}_3$. Consider the spaces $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})$, where $\omega_{0,p}^{\mathbf{k}_i}(z) = \omega_{\mathrm{f},i}(z)\,\mathrm{N}_p^{\mathbf{k}_i}(z)$ with $\omega_{\mathrm{f},i}$ the finite part of a unitary Hecke character taking values in F that are unramified outside p and such that $\omega_{\mathrm{f},1}\omega_{\mathrm{f},2}\omega_{\mathrm{f},3} = 1$. Recall that specialization maps attached to $\phi_i : \mathbf{k}_i \to k_i \in \mathbb{N}$:

$$\phi_{i,*}^{\mathrm{alg}} : M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i}) \longrightarrow M_p^\diamond(\mathbf{V}_{k_i,F}, \omega_{0,p}^{k_i}) \simeq M^\diamond(\mathbf{V}_{k_i,F}, \omega_0^{k_i}),$$

where $\omega_{0,p}^{k_i}(z) = \omega_{\mathrm{f},i}(z)\,\mathrm{N}_p^{k}(z)$ and $\omega_0^{k_i}(z) = \omega_{\mathrm{f},i}(z)\,\mathrm{N}_{\mathrm{f}}^{k}(z)$. Let set up the following notation in order to precisely give our statement. Let us fix $\underline{\alpha} := (\alpha_1,\alpha_2,\alpha_3)$, where $\alpha_i \in \mathcal{O}_i^\times$ and write $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$ for the subspace of $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})$ which is the kernel of $U_p - \alpha_i$.

Remark 7.1. There are plenty of examples of non-zero eigenvectors with associated invertible eigenvalue because the U_p-operator acts on these spaces and the Ash-Stevens theory of [4] applies to show that they have slope $\leq h \in \mathbb{R}$ decompositions (as defined in [4]): writing $(-)^{\leq h}$ for the slope $\leq h$ part, any eigenvector in $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\leq h}$ has eigenvalue in $\mathcal{O}_i^\times$. Furthermore, the Ash-Stevens theory of [4] applies to show that we have the control theorem in our setting, from which one can easily deduce that the U_p-eigenvectors of slope $\leq h < k+1$ on $M_p^\diamond(\mathbf{V}_{k_i,F}, \omega_{0,p}^{k_i})^{\leq h}$ lifts to eigenfamilies belonging to $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\leq h}$ in an essential unique way, when $\phi_i : \mathbf{k}_i \to k_i$ is obtained from $k_i \in U_i \subset \mathcal{X}$ (see also [9] and [38, Theorem 3.7] for the control theorem in our setting and [11, Corollary B5.7.1] and [20, Corollary 11.4] for these kind of applications of the control theorem). These lifts to eigenfamilies in $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\leq h}$ provide such a kind of examples.

Next, we define the $\mathcal{O}_{\underline{\mathbf{k}}}$-valued $\mathcal{O}_{\underline{\mathbf{k}}}$-linear functionals

$$\mathcal{L}_{p,i}^{\underline{\alpha}} : M_p^{\underline{\alpha}} := M_p^\diamond(\mathcal{D}_{\mathbf{k}_1}(W), \omega_{0,p}^{\mathbf{k}_1})^{\alpha_1} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_2}(W), \omega_{0,p}^{\mathbf{k}_2})^{\alpha_2} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_3}(W), \omega_{0,p}^{\mathbf{k}_3})^{\alpha_3} \to \mathcal{O}_{\underline{\mathbf{k}}}$$

via the formula

$$\mathcal{L}_{p,1}^{\underline{\alpha}}\left(\varphi_1 \otimes \varphi_2 \otimes \varphi_3\right) := \frac{p+1}{\alpha_1} t_{1,\underline{\mathbf{k}}}^{\circ}\left(\varphi_1 \mid W_3 \otimes \varphi_2 \otimes \varphi_3\right),$$

and $\mathcal{L}_{p,2}^{\underline{\alpha}}$ and $\mathcal{L}_{p,3}^{\underline{\alpha}}$ are defined in a similar way. We will use the notation $\underline{\varphi}$ to denote an element of $M_p^{\underline{\alpha}}$, that we may and will assume to be a pure tensor product.

Let us assume, from now on, that $\mathbf{k}_i$ corresponds to $U_i \subset \mathcal{X}$ and, if $\phi_i : \mathbf{k}_i \to k_i \in \mathbb{N} \cap U_i$ is obtained from $k_i \in U_i \subset \mathcal{X}$, $F \in \mathcal{O}_{\underline{\mathbf{k}}} = \mathcal{O}\left(U_1 \times U_2 \times U_3\right)$ and $\underline{k} = (k_1,k_2,k_3)$, let $F(k) := \left(\phi_1 \otimes \phi_2 \otimes \phi_3\right)(F)$ be its evaluation at $\underline{k}$. Also, we write $\varphi_{i,k_i} := \phi_{i,*}^{\mathrm{alg}}\left(\varphi_i\right)$. Because we assume that α_i is invertible in $\mathcal{O}_i$ and, hence, it has finite slope, except for a finite number of points φ_{i,k_i} is old at p and, more precisely, there is a unique $\varphi^{\#}_{i,k_i} \in M^{\diamond}(\mathbf{V}_{k_i,F}, \omega_0^{k_i})^{\mathbf{GL}_2(\mathbb{Z}_p)}$ such that $\varphi_{i,k_i} = \varphi^{\#,(\alpha_i)}_{i,k_i}$. Let us write $\underline{U} := U_1 \times U_2 \times U_3$.

Definition 7.2. We say that $\underline{k} = (k_1, k_2, k_3) \in \mathbb{N}^3 \cap \underline{U} \subset \mathcal{X}^3$ (resp. $k_i \in \mathbb{N} \cap U_i \subset \mathcal{X}$) is a generic integer point for $\underline{\varphi}$ (resp. φ_i) if φ_{i,k_i} is old at p for $i = 1, 2, 3$ (resp. φ_{i,k_i} is old at p).

If $\underline{k} \in \mathbb{N}^3 \cap \underline{U}$, we write $\varphi_{\underline{k}} := \varphi_{1,k_1} \otimes \varphi_{2,k_2} \otimes \varphi_{3,k_3}$ and, when $\underline{k}$ is a generic integer point, we also write $\varphi^{\#}_{\underline{k}} := \varphi^{\#}_{1,k_1} \otimes \varphi^{\#}_{2,k_2} \otimes \varphi^{\#}_{3,k_3}$. When it happens that $\varphi_{\underline{k}}$ belongs to an irreducible representation, we denote it by $\Pi\left(\varphi_{\underline{k}}\right)$ and let $\Pi'\left(\varphi_{\underline{k}}\right)$ be its Jacquet-Langlands lifts to $\mathbf{GL}_2$, so that $\Pi\left(\varphi_{\underline{k}}\right) = \Pi\left(\varphi^{\#}_{\underline{k}}\right)$ and $\Pi'\left(\varphi_{\underline{k}}\right) = \Pi'\left(\varphi^{\#}_{\underline{k}}\right)$ when $\underline{k}$ is generic. Finally, we write $M_p^{\diamond}\left(\underline{\varphi}\right) \subset M_p^{\underline{\alpha}}$ for the $\mathbf{B}^{\times 3}\left(\mathbb{A}_{\mathrm{f}}^p\right)$-representation generated by $\underline{\varphi}$ over $\mathcal{O}_{\mathbf{k}}$: note that, if $\underline{\varphi}' \in M_p^{\diamond}\left(\underline{\varphi}\right)$, then $\Pi\left(\varphi'_{\underline{k}}\right) = \Pi\left(\varphi_{\underline{k}}\right)$ for every integer point $\underline{k}$. Finally, we choose vectors $\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}} \in \Pi\left(\varphi^{\#}_{\underline{k}}\right)$ such that $\left(\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}\right)_{\underline{k}} \neq 0$ (see Lemma 3.2 for their existence and (21) for a specific choice); we further assume that they satisfy the property that the local components at p equals the local component at p of $\varphi^{\#}_{\underline{k}}$ (indeed, because $\varphi^{\#}_{\underline{k}}$ is new at p, $\varphi^{\#}_{\underline{k}}$ is the tensor product of its local component at p, which is defined, and a prime to p-component). Having setup our notations, we can state our main result, which is a combination of Theorem 3.4, (46), Corollary 5.6 and Proposition 6.5.

Theorem 7.3. *There is a unique $\mathcal{O}_{\underline{\mathbf{k}}}$-valued $\mathcal{O}_{\underline{\mathbf{k}}}$-linear functional $\mathcal{L}_p^{\underline{\alpha}} : M_p^{\underline{\alpha}} \to \mathcal{O}_{\underline{\mathbf{k}}}$ such that, for every $\underline{\varphi} \in M_p^{\underline{\alpha}}$ and every balanced generic integer point $\underline{k} \in \underline{U}$ for $\underline{\varphi}$,*

$$\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}\right)\left(\underline{k}\right) := \mathcal{E}_p\left(\underline{\alpha}, \underline{k}\right) t_{\underline{k}}\left(\varphi^{\#}_{\underline{k}}\right). \tag{71}$$

We have, indeed, $\mathcal{L}_p^{\underline{\alpha}} = \mathcal{L}_{p,i}^{\underline{\alpha}}$ for $i = 1, 2, 3$ and, furthermore, if $\underline{\varphi} \in M_p^{\underline{\alpha}}$ is a tensor product of three families and $\varphi_{\underline{k}}$ belongs to the irreducible representation $\Pi\left(\varphi_{\underline{k}}\right)$, then

$$\begin{aligned}\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}\right)\left(\underline{k}\right)^2 &= \mathcal{E}_p\left(\underline{\alpha}, \underline{k}\right)^2 \frac{C_{\underline{k}}}{2^9 3^2} \frac{\zeta_{\mathbb{Q}}^2(2)\, L\left(1/2, \Pi'\left(\varphi^{\#}_{\underline{k}}\right)\right)}{L\left(1, \Pi'\left(\varphi^{\#}_{\underline{k}}\right), \mathrm{Ad}\right)} \prod_v I_v(\varphi^{\#}_{\underline{k}}) \\ &= \mathcal{E}_p\left(\underline{\alpha}, \underline{k}\right)^2 \frac{\left(\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}\right)_{\underline{k}}}{2L\left(1, \Pi'\left(\varphi^{\#}_{\underline{k}}\right), \mathrm{Ad}\right)} L\left(1/2, \Pi'\left(\varphi^{\#}_{\underline{k}}\right)\right) \prod_{v \neq \infty, p} C_v^{\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#}_{\underline{k}}\right),\end{aligned} \tag{72}$$

where $C_{\underline{k}} \neq 0$ is defined in (22) and $I_v(\varphi^{\#}_{\underline{k}})$ and $C_v^{\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#}_{\underline{k}}\right)$ are defined in (25).

Also, suppose that there is a balanced generic integer point $\underline{k}^0$ for $\underline{\varphi}$ such that $B = B_{\Pi'\left(\varphi^{\#}_{\underline{k}^0}\right)}$ is the quaternion algebra predicted by [34] *and $L\left(1/2, \Pi'\left(\varphi^{\#}_{\underline{k}^0}\right)\right) \neq 0$. Then, up to shrinking $\underline{U}$ in a neighbourhood of $\underline{k}^0$, there exists $\underline{\varphi}' \in M_p^{\diamond}\left(\underline{\varphi}\right)$ such that, for every balanced generic integer point $\underline{k} \in \underline{U}$, we know that $B = B_{\Pi'\left(\varphi'^{\#}_{\underline{k}}\right)} = B_{\Pi'\left(\varphi^{\#}_{\underline{k}}\right)}$ is the quaternion algebra predicted by* [34] *and we have satisfied the equivalence*

$$\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}'\right)\left(\underline{k}\right) \neq 0 \Leftrightarrow L\left(1/2, \Pi'\left(\varphi'^{\#}_{\underline{k}}\right)\right) = L\left(1/2, \Pi'\left(\varphi^{\#}_{\underline{k}}\right)\right) \neq 0.$$

Proof. Applying Corollary 5.6 and Proposition 6.5 to any one of the $\mathcal{L}_{p,i}^{\underline{\alpha}}$'s gives (71) with $\mathcal{L}_p^{\underline{\alpha}} := \mathcal{L}_{p,i}^{\underline{\alpha}}$ and the uniqueness follows from the Zariski density of the balanced generic integer points in the open affinoid subdomain $U_1 \times U_2 \times U_3$ of $\mathcal{X}^3$. Then everything is clear from Theorem 3.4, except we have to explain why we have excluded $C_p^{\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#}_{\underline{k}}\right)$ from our local constants. This is because $\varphi^{\#}_{\underline{k}}$ is the tensor product of p-new vectors which gives rise, in the local representation at p attached to $\Pi\left(\varphi^{\#}_{\underline{k}}\right)$, to a vector ψ_p which is the tensor product of the (unique up to a scalar factor) p-new vectors in the local representations attached to the $\varphi^{\#,(\alpha_i)}_{i,k_i}$'s. Then, because the local components of $\varphi^{\flat\#}_{\underline{k}}$ and $\varphi^{\flat\flat\#}_{\underline{k}}$ at p equals the local component at p

of $\varphi^{\#}_{\underline{k}}$, the equality $C_p^{\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#}_{\underline{k}}\right)=1$ follows from [31, Lemma 2.2] (see §8.2 below for more details). The last assertion follows from (72), Theorem 3.4 (2) and Remark 1.1. $\square$

Remark 7.4. The equality (72) should be understood as an equality of quadratic forms although, with an eye to its applications (see §8), we have suggestively stated it as if $\varphi^{\#}_{\underline{k}}$ were a pure tensor. In general, even in case $\underline{\varphi}$ is a tensor product of three families, it may happen that $\varphi_{\underline{k}}$ belongs to a sum of irreducible automorphic representations: then the scalar factor relating the two sides of (72) depend of these irreducible components via the above L-values. Let us also remark that, in the applications, one usually start with a tensor product of three eigenfamilies $\underline{\varphi}$ and then take linear combinations of them: in this case $\varphi_{\underline{k}}$ belongs to a single irreducible representation $\Pi\left(\varphi_{\underline{k}}\right)$ for every balanced integer point.

Let us now discuss the value of $\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}\right)$ at some balanced integer point $\underline{k}\in\underline{U}$ where one, two or three of the Galois representations attached to φ_{i,k_i} are semistable: more precisely, we suppose that, in this case, the p-new form φ_{i,k_i} has even weight and trivial central character, thus forcing the corresponding $\omega_{\mathrm{f},i}$ of the family φ_i to be 1. The proof is essentially the same of Theorem 7.3: one replaces 6.5 by either 6.7, 6.8 or 6.9.

Proposition 7.5. *Suppose that $\underline{k}\in\underline{U}$ is a balanced integer point such that (i) φ_{3,k_3}, resp. (ii) φ_{2,k_2} and φ_{3,k_3} or resp. (iii) φ_{1,k_2}, φ_{2,k_2} and φ_{3,k_3} are p-new with even weight and trivial central character. Then (72) holds with the following modified Euler factor E replacing $\mathcal{E}_p\left(\underline{\alpha},\underline{k}\right)^2$, where we write $\xi_{k_i}:=(p+1)\,\omega_{\mathrm{f},i}(\mathbf{p}')^{-1}p^{k_i}$:*

- *(i) $E=\xi_{k_1}\frac{\mathcal{E}^2_{p,1}(\underline{\alpha},\underline{k})\mathcal{E}^2_{p,3}(\underline{\alpha},\underline{k})}{\alpha_1^2}=\xi_{k_2}\frac{\mathcal{E}^2_{p,2}(\underline{\alpha},\underline{k})\mathcal{E}^2_{p,3}(\underline{\alpha},\underline{k})}{\alpha_2^2}$;*
- *(ii) $E=\frac{1}{p}\xi^2_{k_1}\frac{\mathcal{E}_{p,1}(\underline{\alpha},\underline{k})^2}{\alpha_1^2}=\frac{1}{p}\left(p+1\right)^2\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)^2=\frac{1}{p}\left(p+1\right)^2\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)^2$;*
- *(iii) $E=\frac{2}{p}\left(1+\frac{1}{p}\right)\left(1+w_{1,p}w_{2,p}w_{3,p}\right)^2$.*

Proof. Case (i). Applying Corollary 5.6 and Proposition 6.7 gives (71) with $t_{\underline{k}}\left(\varphi^{\#}_{\underline{k}}\right)$ replaced by $t_{\underline{k}}\left(\varphi_1^{\#(p)},\varphi_2^{\#},\varphi_3^{\#}\right)$ (resp. $t_{\underline{k}}\left(\varphi_1^{\#},\varphi_2^{\#(p)},\varphi_3^{\#}\right)$) and $\mathcal{E}_p\left(\underline{\alpha},\underline{k}\right)$ replaced by the modified Euler factor $E':=\frac{p+1}{\alpha_1}\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)$, $\frac{p+1}{\alpha_2}w_{p,3}\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)^{-1}p^{(k_2-k_1)/2}\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)$ or $\frac{p+1}{\alpha_1}\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right)$, which can be checked to agree (resp. $\frac{p+1}{\alpha_1}w_{p,3}\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)^{-1}p^{(k_1-k_2)/2}\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)$, $\frac{p+1}{\alpha_2}\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)$ or $\frac{p+1}{\alpha_2}\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)$, which again can be checked to agree). Next, as above one applies Theorem 3.4 now with $\varphi^{\#}_{\underline{k}}$ replaced by $\varphi^{\#(p)}_{\underline{k}}:=\varphi_1^{\#(p)}\otimes\varphi_2^{\#}\otimes\varphi_3^{\#}$ (resp. $\varphi^{\#(p)}_{\underline{k}}:=\varphi_1^{\#}\otimes\varphi_2^{\#(p)}\otimes\varphi_3^{\#}$) and again with $\varphi^{\flat\#}_{\underline{k}}$ and $\varphi^{\flat\flat\#}_{\underline{k}}$, which fixes $\left(\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}\right)_{\underline{k}}$ and the local constants $C_p^{\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)$. On the other hand, with the notations introduced in the proof of Theorem 3.4, set $\psi^{\flat\#(p)}:=f_{\Lambda_{\underline{k}/E}}\left(\varphi^{\flat\#(p)}_{\underline{k}}\right)$, $\psi^{\flat\flat\#(p)}:=f_{\Lambda_{\underline{k}/E}}\left(\varphi^{\flat\flat\#(p)}_{\underline{k}}\right)$ and $\psi^{\flat\#(p)\vee}:=(\check{\psi}^{\flat\flat\#(p)})$ where $\varphi^{\flat\#(p)}_{\underline{k}}:=\varphi^{\flat\#(p)}_{k_1}\otimes\varphi^{\#}_{k_2}\otimes\varphi^{\#}_{k_3}$ (resp. $\varphi^{\flat\#(p)}_{\underline{k}}:=\varphi^{\#}_{k_1}\otimes\varphi^{\#(p)}_{k_2}\otimes\varphi^{\#}_{k_3}$) and the same for $\varphi^{\flat\flat\#(p)}_{\underline{k}}$. Similarly as in the global calculation (63), one checks that $\left\langle\psi_p^{\flat\#(p)},\psi_p^{\flat\#(p)\vee}\right\rangle_p=\omega^{-1}_{\mathrm{f},1}(\mathbf{p}')p^{k_1}\left\langle\psi_p^{\flat\#},\psi_p^{\flat\#\vee}\right\rangle_p$ (resp. $\left\langle\psi_p^{\flat\#(p)},\psi_p^{\flat\#(p)\vee}\right\rangle_p=\omega^{-1}_{\mathrm{f},2}(\mathbf{p}')p^{k_2}\left\langle\psi_p^{\flat\#},\psi_p^{\flat\#\vee}\right\rangle_p$) and, consequently, $C_p^{\varphi^{\flat\#(p)}_{\underline{k}},\varphi^{\flat\flat\#(p)}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)=\omega_{\mathrm{f},1}(\mathbf{p}')p^{-k_1}C_p^{\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)$ (resp. $C_p^{\varphi^{\flat\#(p)}_{\underline{k}},\varphi^{\flat\flat\#(p)}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)=\omega_{\mathrm{f},2}(\mathbf{p}')p^{-k_2}C_p^{\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)$). Because the local components of $\varphi^{\flat\#}_{\underline{k}}$ and $\varphi^{\flat\flat\#}_{\underline{k}}$ at p equals the local component at p of $\varphi^{\#}_{\underline{k}}$, the local components at p of $\varphi^{\flat\#(p)}_{\underline{k}}$ and $\varphi^{\flat\flat\#(p)}_{\underline{k}}$ equals the local component at p of $\varphi^{\#(p)}_{\underline{k}}$ and [42, Corollary 4.2] gives $C_p^{\varphi^{\flat\#(p)}_{\underline{k}},\varphi^{\flat\flat\#(p)}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)=\frac{1}{p}\left(1+\frac{1}{p}\right)^{-1}=\frac{1}{p+1}$ (see §8.2 below for more details). Hence

$$C_p^{\varphi^{\flat\#}_{\underline{k}},\varphi^{\flat\flat\#}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)=\omega^{-1}_{\mathrm{f},1}(\mathbf{p}')p^{k_1}C_p^{\varphi^{\flat\#(p)}_{\underline{k}},\varphi^{\flat\flat\#(p)}_{\underline{k}}}\left(\varphi^{\#(p)}_{\underline{k}}\right)=\frac{\omega^{-1}_{\mathrm{f},1}(\mathbf{p}')p^{k_1}}{p+1}$$

(resp. $C_p^{\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^{\#(p)}\right)=\frac{\omega_{\mathrm{f},2}^{-1}(\mathbf{p}')p^{k_2}}{p+1}$) and we see that $C_p^{\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^{\#(p)}\right)E'^2=E$, as claimed. The proof of the cases (ii) and (iii) is similar, noticing that we already have everything expressed in term of $\varphi_{\underline{k}}^{\#}$ and $C_p^{\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^{\#}\right)=p^{-1}$ (resp. $C_p^{\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#}}\left(\varphi_{\underline{k}}^{\#}\right)=2p^{-1}\left(1+p^{-1}\right)$) in case (ii) (resp. (iii)) thanks to [42, Proposition 4.3] (resp. [42, Proposition 4.4]). $\square$

Remark 7.6. It follows from Deligne's proof of the generalized Ramanujan conjecture that, in the setting of the above Proposition 7.5, we may have the vanishing of the Euler factor E only in case (i) or (iii). In the first case, we have indeed $\mathcal{E}_{p,3}\left(\underline{\alpha},\underline{k}\right)\neq 0$ and $\mathcal{E}_{p,1}\left(\underline{\alpha},\underline{k}\right)=\mathcal{E}_{p,2}\left(\underline{\alpha},\underline{k}\right)=0$ if and only if the equivalence

$$\frac{\alpha_1}{\alpha_2}=-w_{p,3}p^{(k_1-k_2)/2}\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)^{-1}\Leftrightarrow\frac{\alpha_2}{\alpha_1}=-w_{p,3}p^{(k_2-k_1)/2}\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)^{-1}$$

is satisfied (recall $\omega_{\mathrm{f},1}\left(\mathbf{p}'\right)\omega_{\mathrm{f},2}\left(\mathbf{p}'\right)=1$). In the second case, we have $E=0$ if and only if $w_{1,p}w_{2,p}w_{3,p}=-1$ and, in this case, we see from Proposition 6.9 that there is an extra vanishing due to the complex L-function.

Finally, let us discuss improved p-adic L-functions. Suppose that $c\in\mathbb{N}$ and consider the plane

$$H_i^c:=\left\{\underline{\kappa}\in\underline{U}:\underline{\kappa}_i^*=c\right\}\subset\underline{U}.$$

Let us remark that the Euler factor $\mathcal{E}_{p,i}\left(\underline{\alpha},\underline{k}\right)$ extends to a rigid analytic function on H_i^c. Suppose that $\underline{\mathbf{k}}'$ is such that $\underline{\mathbf{k}}_1'^*=c$; geometrically, this means that $\underline{\mathbf{k}}':\mathbb{Z}_p^{\times 3}\to\mathcal{O}_{\underline{\mathbf{k}}}$ factors through the morphism $\mathcal{O}_{\underline{\mathbf{k}}}=\mathcal{O}\left(\underline{U}\right)\twoheadrightarrow\mathcal{O}\left(H_1^c\right)$ which corresponds to $H_1^c\subset\underline{U}$. Then we can consider the $\mathcal{O}_{\underline{\mathbf{k}}'}$-valued $\mathcal{O}_{\underline{\mathbf{k}}'}$-linear functional $\mathcal{L}_{p,1}^{\underline{\alpha},c}:M_p^{\underline{\alpha}}\to\mathcal{O}_{\underline{\mathbf{k}}'}$ defined via the formula

$$\mathcal{L}_{p,1}^{\underline{\alpha},c}\left(\varphi_1\otimes\varphi_2\otimes\varphi_3\right):=\frac{p+1}{\alpha_1}t_{1,\underline{\mathbf{k}}}\left(\varphi_1\mid W_p\otimes\varphi_2\otimes\varphi_3\right),$$

and $\mathcal{L}_{p,2}^{\underline{\alpha},c}$ and $\mathcal{L}_{p,3}^{\underline{\alpha},c}$ are defined in a similar way when $\underline{\mathbf{k}}_2'^*=c$ or, respectively, $\underline{\mathbf{k}}_3'^*=c$ (see (44) for the definition of $t_{i,\underline{\mathbf{k}}}$). Taking $\underline{\mathbf{k}}'$ to be the morphism which corresponds to $H_i^c\subset\underline{U}$ and applying Corollary 5.6 yields the following result.

Proposition 7.7. *The above $\mathcal{O}\left(H_i^c\right)$-linear functional $\mathcal{L}_{p,i}^{\underline{\alpha},c}$ is uniquely characterized by the property that, for every $\underline{\varphi}\in M_p^{\underline{\alpha}}$,*

$$\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}\right)_{|H_i^c}=\mathcal{E}_{p,i}\left(\underline{\alpha},-\right)\mathcal{L}_{p,i}^{\underline{\alpha},c}\left(\underline{\varphi}\right)$$

as rigid analytic functions on H_i^c.

7.2. **Variants.** Let us explain how one can rewrite the term $\frac{\Omega(\varphi_{\underline{k}}^{\#})}{L\left(1,\Pi'\left(\varphi_{\underline{k}}^{\#}\right),\mathrm{Ad}\right)}$ that appears in the interpolation formula (72) when $\Omega(\varphi_{\underline{k}}^{\#})\neq 0$. Suppose that $f\in S_k\left(N,\varepsilon\right)$ is a normalized newform with nebetype ε having conductor N_ε and write $\pi\left(f\right)=\bigotimes_v\pi_v\left(f\right)$ for the corresponding automorphic representation: we recall that a formula of Shimura and Hida relates $L\left(\mathrm{ad}\left(\pi\left(f\right)\right),1\right)$ and the Petersson inner product $\left(f,f\right)_k$ that we normalized as in (1). Let us define $L\left(\mathrm{ad}\left(\pi\left(f\right)\right),s\right)=\prod_v L_v\left(\mathrm{ad}\left(\pi\left(f\right)\right),s\right)$ and $L^H\left(\mathrm{ad}\left(f\right),s\right)=\prod_{v\neq\infty}L_v^H\left(\mathrm{ad}\left(f\right),s\right)$, where

$$L_\infty\left(\mathrm{ad}\left(\pi\left(f\right)\right),s\right)=2\left(2\pi\right)^{-(s+k-1)}\Gamma\left(s+k-1\right)\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)$$

and the Euler factors are defined as follows:

$$L_l\left(\mathrm{ad}\left(\pi\right),s\right)^{-1}=\begin{cases}\left(1-l^{-s}\right)\left(1-\chi_1\overline{\chi_2}\left(l\right)l^{-s}\right)\left(1-\overline{\chi_1}\chi_2\left(l\right)l^{-s}\right), & \pi_l=\pi\left(\chi_1,\chi_2\right)\text{ is principal}\\ 1-l^{-1-s}, & \pi_l\text{ is special},\\ 1+l^{-s}, & \pi_l\text{ is supercuspidal and }\pi_l\simeq\pi_l\otimes\eta_l,\\ 1, & \pi_l\text{ is supercuspidal and }\pi_l\not\cong\pi_l\otimes\eta_l.\end{cases}$$

$$L_l^H\left(\mathrm{ad}\left(f\right),s\right)^{-1}=\begin{cases}\left(1-l^{k-1-s}\right)\left(1-\varepsilon\left(q\right)^{-1}\alpha_l^2l^{k-1-s}\right)\left(1-\varepsilon\left(q\right)^{-1}\beta_l^2l^{k-1-s}\right), & \text{if }l\nmid N,\\ \left(1-l^{-1-s}\right)\left(1+l^{-s}\right), & \text{if }l\parallel N\text{ and }l\nmid N_\varepsilon,\\ 1-l^{-1-s}, & \text{if }l\mid N\text{ and }l\nmid N/N_\varepsilon,\\ 1+l^{-s}, & \text{otherwise};\end{cases}$$

Then (see [26, Theorem 5.1] and [14, Theorem 2.2.3 and Corollary 2.2.4] for the notations employed here):

$$L\left(\mathrm{ad}\left(\pi\left(f\right)\right),1\right)=\frac{2^{k}\pi}{3}\left(f,f\right)_{k}\prod_{l|N}\frac{L_{l}\left(\mathrm{ad}\left(\pi\left(f\right)\right),1\right)}{L_{l}^{H}\left(\mathrm{ad}\left(f\right),1\right)}. \tag{73}$$

Let us now go back to our family $\underline{\varphi}\in M_{p}^{\underline{\alpha}}$ specializing to $\varphi_{\underline{k}}=\varphi_{k_1}^{\#(\alpha_1)}\otimes\varphi_{k_1}^{\#(\alpha_2)}\otimes\varphi_{k_1}^{\#(\alpha_3)}$ and write the $\mathbf{GL}_2$ representation $\pi'\left(\varphi_{k_i}\right)$ attached to φ_{k_i} as $\pi'\left(\varphi_{k_i}\right)=\pi\left(\mathbf{f}_{k_i}^{\#}\right)$ where $\mathbf{f}_{k_i}^{\#}\in S_{k_i}\left(N_i,\varepsilon_i\right)$ is a normalized newform (in particular, $\varepsilon_i\left(z\right)=\omega_{\mathrm{f},i}^{-1}\left(z\right)$ viewing $z\in\mathbb{Z}$ diagonally embedded in $\mathbb{A}_{\mathrm{f}}$) and set $\mathbf{f}_{\underline{k}}^{\#}:=\left(\mathbf{f}_{k_1}^{\#},\mathbf{f}_{k_2}^{\#},\mathbf{f}_{k_3}^{\#}\right)$ and $\Omega(\mathbf{f}_{\underline{k}}^{\#}):=\left(f_{k_1}^{\#},f_{k_1}^{\#}\right)_{k_1}\left(f_{k_2}^{\#},f_{k_2}^{\#}\right)_{k_2}\left(f_{k_1}^{\#},f_{k_1}^{\#}\right)_{k_3}$. Then we have $L_v\left(s,\Pi'\left(\varphi_{\underline{k}}^{\#}\right),\mathrm{Ad}\right)=\prod_{i=1,2,3}L_v\left(\mathrm{ad}\left(\pi\left(\mathbf{f}_{k_i}^{\#}\right)\right),s\right)$ and we define $L_l^{H}\left(s,\mathbf{f}_{\underline{k}}^{\#},\mathrm{Ad}\right):=\prod_{i=1,2,3}L_l^{H}\left(\mathrm{ad}\left(\mathbf{f}_{k_i}^{\#}\right),s\right)$ and $M:=lcm(N_1,N_2,N_3)$). The following result is a direct consequence of Lemmas 5.7 and 6.6.

Proposition 7.8. *Suppose that $\mathbf{k}:\mathbb{Z}_p^{\times}\to\mathcal{O}$ corresponds to an open affinoid subdomain and that $\varphi^{\flat},\varphi^{\flat\flat}\in M_p^{\diamond}\left(\mathcal{D}_{\mathbf{k}}\left(W\right),\omega_{0,p}^{\mathbf{k}}\right)^{\alpha}$ for the same $\alpha\in\mathcal{O}^{\times}$. Then $\frac{p+1}{\alpha}\left(\varphi^{\flat},\varphi^{\flat\flat}\right)\in\mathcal{O}$ is the unique rigid analytic function such that, for every generic integer $k\in\mathbb{N}\cap U$ (for $\varphi^{\flat}$ and $\varphi^{\flat\flat}$),*

$$\frac{p+1}{\alpha}\left(\varphi^{\flat},\varphi^{\flat\flat}\right)(k)=\left(1-\omega_{\mathrm{f}}\left(\mathbf{p}'\right)^{-1}\alpha^{-2}p^{k}\right)\left(1-\omega_{\mathrm{f}}\left(\mathbf{p}'\right)^{-1}\alpha^{-2}p^{k+1}\right)\left(\varphi_k^{\flat\#},\varphi_k^{\flat\flat\#}\right)_k,$$

where $\varphi_k^{\flat\#}$ (resp. $\varphi_k^{\flat\flat\#}$) is the unique vector in $M^{\diamond}(\mathbf{V}_{k,F},\omega_0^{k})^{\mathbf{GL}_2(\mathbb{Z}_p)}$ such that $\varphi_k^{\flat\#(\alpha)}=\varphi_k^{\flat}$ (resp. $\varphi_k^{\flat\flat\#(\alpha)}=\varphi_k^{\flat\flat}$) and $\varphi_k^{\flat}$ (resp. $\varphi_k^{\flat\flat}$) is the specialization of $\varphi^{\flat}$ (resp. $\varphi^{\flat\flat}$) at k.

In particular, if $\varphi_i^{\flat},\varphi_i^{\flat\flat}\in M_p^{\diamond}(\mathcal{D}_{\mathbf{k}_i}(W),\omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$, $\underline{\varphi}^{\flat}:=\varphi_1^{\flat}\otimes\varphi_2^{\flat}\otimes\varphi_3^{\flat}\in M_p^{\underline{\alpha}}$ and $\underline{\varphi}^{\flat\flat}:=\varphi_1^{\flat\flat}\otimes\varphi_2^{\flat\flat}\otimes\varphi_3^{\flat\flat}\in M_p^{\underline{\alpha}}$, we can define

$$\left(\underline{\varphi}^{\flat},\underline{\varphi}^{\flat\flat}\right)_p:=(p+1)^3\,\frac{\left(\varphi_1^{\flat},\varphi_1^{\flat\flat}\right)}{\alpha_1}\widehat{\otimes}\frac{\left(\varphi_2^{\flat},\varphi_2^{\flat\flat}\right)}{\alpha_2}\widehat{\otimes}\frac{\left(\varphi_3^{\flat},\varphi_3^{\flat\flat}\right)}{\alpha_3}\in\mathcal{O}_{\underline{\mathbf{k}}},$$

which satisfies, at every generic integer point $\underline{k}\in\mathbb{N}^3\cap\underline{U}$ (for $\underline{\varphi}^{\flat}$ and $\underline{\varphi}^{\flat\flat}$), the interpolation property

$$\left(\underline{\varphi}^{\flat},\underline{\varphi}^{\flat\flat}\right)_p(\underline{k})=\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)(\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#})_{\underline{k}},$$

where

$$\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right):=\prod_{i=1,2,3}\left(1-\omega_{\mathrm{f},i}\left(\mathbf{p}'\right)^{-1}\alpha_i^{-2}p^{k_i}\right)\left(1-\omega_{\mathrm{f},i}\left(\mathbf{p}'\right)^{-1}\alpha_i^{-2}p^{k_i+1}\right)$$

and $\varphi_{\underline{k}}^{\flat\#}:=\varphi_{k_1}^{\flat\#}\otimes\varphi_{k_2}^{\flat\#}\otimes\varphi_{k_3}^{\flat\#}$ (resp. $\varphi_{\underline{k}}^{\flat\flat\#}:=\varphi_{k_1}^{\flat\flat\#}\otimes\varphi_{k_2}^{\flat\flat\#}\otimes\varphi_{k_3}^{\flat\flat\#}$) if $\varphi_{k_i}^{\flat\#}$ (resp. $\varphi_{k_i}^{\flat\flat\#}$) is the unique vector in $M^{\diamond}(\mathbf{V}_{k_i,F},\omega_0^{k_i})^{\mathbf{GL}_2(\mathbb{Z}_p)}$ such that $\varphi_{k_i}^{\flat\#(\alpha_i)}=\varphi_{k_i}^{\flat}$ (resp. $\varphi_{k_i}^{\flat\flat\#(\alpha_i)}=\varphi_{k_i}^{\flat\flat}$) and $\varphi_{k_i}^{\flat}$ (resp. $\varphi_{k_i}^{\flat\flat}$) is the specialization of $\varphi_i^{\flat}$ (resp. $\varphi_i^{\flat\flat}$) at k_i. We may choose $\varphi_i^{\flat}$ and $\varphi_i^{\flat\flat}$ in such a way that, for every generic integer point $\underline{k}\in\mathbb{N}^3\cap\underline{U}$ for both $\underline{\varphi}^{\flat}$ and $\underline{\varphi}^{\flat\flat}$, we have $(\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#})_{\underline{k}}\neq0$ and so that the local components at p of the specialization of $\varphi_i^{\flat}$ and $\varphi_i^{\flat\flat}$ equals the local components at p of the specializations of φ_i: indeed, for example, we may take $\varphi_i^{\flat}$ a newvector of tame level N_i and $\varphi_i^{\flat\flat}:=\varphi_i^{\flat}\mid W_{N_i}$ (see (21).

Then, thanks to (73) and Proposition 7.8, we find for every generic integer point $\underline{k}\in\mathbb{N}^3\cap\underline{U}$ for both $\underline{\varphi}$, $\underline{\varphi}^{\flat}$ and $\underline{\varphi}^{\flat\flat}$ (the notation $\overset{(\Omega NE)}{=}$ means that the equality holds when $\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\neq0$, see Remark 1.2):

$$\begin{aligned}\frac{(\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#})_{\underline{k}}}{L\left(1,\Pi'\left(\varphi_{\underline{k}}^{\#}\right),\mathrm{Ad}\right)}&=\frac{27}{4^{\underline{k}^*}\pi^3}\frac{(\varphi_{\underline{k}}^{\flat\#},\varphi_{\underline{k}}^{\flat\flat\#})_{\underline{k}}}{\Omega(\mathbf{f}_{\underline{k}}^{\#})}\prod_{l|M}\frac{L_l^{H}\left(s,\mathbf{f}_{\underline{k}}^{\#},\mathrm{Ad}\right)}{L_l\left(s,\Pi'\left(\varphi_{\underline{k}}^{\#}\right),\mathrm{Ad}\right)}\\&\overset{(\Omega NE)}{=}\frac{27}{4^{\underline{k}^*}\pi^3}\prod_{l|M}\frac{L_l^{H}\left(s,\mathbf{f}_{\underline{k}}^{\#},\mathrm{Ad}\right)}{L_l\left(s,\Pi'\left(\varphi_{\underline{k}}^{\#}\right),\mathrm{Ad}\right)}\frac{\left(\underline{\varphi}^{\flat},\underline{\varphi}^{\flat\flat}\right)_p(\underline{k})}{\mathcal{E}_p^{\Omega}\left(\underline{\alpha},\underline{k}\right)\Omega(\mathbf{f}_{\underline{k}}^{\#})}.\end{aligned} \tag{74}$$

8. An explicit example

Recall the cuspidal finite slope h_i Coleman eigenfamilies $\mathbf{f}_i$ of tame level N_i, trivial nebentype $\varepsilon_i = 1$ and U_p-eigenvalue $\alpha_i \in \mathcal{O}$ defined in the connected affinoid subdomain U_i of the weight space $\mathcal{X}$ that was considered in the introduction (we are going to relax the assumption on the level that we did there, which is no longer in force).

Remark 8.1. Let us fix a Dirichlet character $\varepsilon : \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^{\times} \to \mathbb{C}$ of level prime to p. It follows from the work of Coleman that prime to p newforms vary in families: there is a sheaf of families of overconvergent modular forms $\mathbf{f}$ of finite slope on the weight space whose specializations at k belongs to $S_{k+2}\left(\Gamma_0\left(pN\right), \varepsilon\right)^{N-\mathrm{new}}$ for almost every k's (see [11, discussion after Corollary B5.7.1]). Working over connected affinoids $U \subset \mathcal{X}$ there are plenty of sections of this sheaf. More precisely, let us fix a real number $h \geq 0$ and let us assume that $f \in S_{k_0+2}\left(\Gamma_0\left(pN\right), \varepsilon\right)^{N-\mathrm{new}}$ is such that $k_0 > h + 1$ and has slope h. Under a mild further assumption, it is shown in [11, Corollary B5.7.1] the existence of a family $\mathbf{f}$ such that $\mathbf{f}_{k_0} = f$ and, for every integer $k > h + 1$ (an integer points in our notations), $\mathbf{f}_k \in S_{k+2}\left(\Gamma_0\left(pN\right), \varepsilon\right)^{N-\mathrm{new}}$ and has slope h. Let us also remark that, if $\mathbf{f}$ is a finite slope overconvergent modular forms as above defined on a connected affinoid, then its slope is constant, say h, and $\mathbf{f}_k \in S_{k+2}\left(\Gamma_0\left(pN\right), \varepsilon\right)^{N-\mathrm{new}}$ for every integer $k > h+1$, as it follows from the Coleman's classicality result.

One can give the following geometric interpretation of this result. As explained in [20, §12.1 and Corollary 10.7], the work on Ash-Stevens on slope decompositions combined with standard techniques due to Coleman-Mazur and Buzzard yields the construction of a curve $w : \mathcal{C}_{N,\varepsilon}^{\leq h} \to \mathcal{X}$ parametrizing cuspidal eigenforms of level $\Gamma_0\left(pN\right)$ and nebetype ε which are N-new and of slope $\leq h$. The Coleman-Mazur eigencurve $w : \mathcal{C}_{N,\varepsilon} \to \mathcal{X}$ parametrizing these objects with the relaxed finite slope condition admits $\mathcal{C}_{N,\varepsilon}^{\leq h}$ as a closed curve[7] and it is the union of them. The families $\mathbf{f}$ obtained from [11, Corollary B5.7.1] corresponds to sections $s : U \to \mathcal{C}_{N,\varepsilon}^{\leq h} \subset \mathcal{C}_{N,\varepsilon}$ of w (see [20, Corollary 10.7 (*iii*)]). Moreover, if $s : U \to \mathcal{C}_{N,\varepsilon}$ is a section of $w : \mathcal{C}_{N,\varepsilon} \to \mathcal{X}$, because U is a connected affinoid, $s\left(U\right) \subset \mathcal{C}_{N,\varepsilon}^{\leq h}$ for some h.

Recall that, for every generic integer point k, the specialization $\mathbf{f}_{i,k} = f_{i,k}$ is the p-stabilization of a level N_i newform $\mathbf{f}_{i,k}^{\#} = f_{i,k}^{\#}$ and we let $\pi_{i,k} = \bigotimes_v \pi_{i,k,v}$ be the associated automorphic representation. We write

$$D := gcd(N_1, N_2, N_3) \text{ (resp. } M := lcm(N_1, N_2, N_3))$$

for the greatest common multiple (resp. the least common multiple).

Lemma 8.2. *Suppose that* $\mathbf{f}$ *is a finite slope cuspidal p-adic Coleman eigenfamily new of tame level N and nebetype ε defined on some open affinoid subdomain U as in Remark 8.1 above. Let us write $\pi_k = \bigotimes_v \pi_{k,v}$ for the automorphic representation attached to an integer point $k \in U$. If $\pi_{k_0,v}$ is a principal series, special or supercuspidal representation of conductor $c_{k_0,v}$ at an integer point $k_0 \in U$ and $v \neq p, \infty$, then $\pi_{k,v}$ has the same property for every integer point $k \in U$.*

Proof. According to the work of Coleman-Mazur [13] (see also [2, Theorem 5.1]), there is a pseudocharacter $T : G_{\mathbb{Q}} \to \mathcal{O}_{\mathcal{C}_{N,\varepsilon}}$ such that the specialization T_y of T at a classical point $y \in \mathcal{C}_{N,\varepsilon}$ is (the pseudocharacter of) the representation ρ_y attached to the eigenform y, characterized by the fact that the trace of the geometric Frobenius at l is the eigenvalue of T_l acting on y for almost every l. Let us write $s^{\#} : \mathcal{O}_{\mathcal{C}_{N,\varepsilon}} \to \mathcal{O} := \mathcal{O}_{\mathcal{X}}\left(U\right)$ for the morphism induced by $s : U \to \mathcal{C}_{N,\varepsilon}$, where s corresponds to $\mathbf{f}$, as explained in Remark 8.1. Then $T_s := s^{\#} \circ T$ has the property that its specialization at an integer point $k \in U$ is (the pseudocharacter of) the representation $\rho_k := \rho_{\mathbf{f}_k}$ attached to the eigenform $\mathbf{f}_k$. Because $\mathcal{O}$ is a PID, it follows from [13, Theorem 5.1.2 and Remark after it] that $T = \rho$ is indeed (the pseudocharacter of) a representation $\rho : G_{\mathbb{Q}} \to \mathbf{GL}_2\left(\mathcal{O}\right)$ which interpolates the representations ρ_k's. We also refer the reader to [2, §6.2] for an alternative construction of this representation.

According to [5, Lemma 7.8.14], writing W_v for the Weil-Deligne group of $\mathbb{Q}$ at v, we have that $\rho_{|W_v}$ is monodromic in the sense of [36, Definition 2.2]. This means that we can attach to it a Weil-Deligne

[7] Via the morphism $\mathcal{O}_{\mathcal{C}_{N,\varepsilon}} \to \mathcal{O}_{\mathcal{C}_{N,\varepsilon}^{\leq h}}$ which sends a Hecke operator acting on finite slope overconvergent modular forms to the Hecke operator acting on overconvergent modular forms of slope $\leq h$.

representation $WD_v(\rho)$ with coefficients in $\mathcal{O}$ via [5, Definition 7.8.13]: we remark that, writing $WD_v(\rho)_k$ for its specialization at k, we have $WD_v(\rho)_k = WD_v(\rho_k)$ by the uniqueness assertion of [5, Lemma 7.8.12] characterizing the monodromy and the definition of r in loc.cit. Let us remark that the automorphic type of $\pi_{k,v}$ and its conductor are encoded in the Frobenius semisimplification $WD_v(\rho_k)^{\mathrm{Fr-ss}}$ of $WD_v(\rho_k)$ thanks to the local-global compatibility, which is well known in this case and widely covered by the Hilbert case handled in [8]. We can now apply [36, Theorem 3.1 (1) and (4)] and [37, Theorem 3.1] to deduce that the automorphic type of $\pi_{k,v}$ and its conductor are constant if $WD_v(\rho_k)^{\mathrm{Fr-ss}}$ is pure, i.e. it satisfies the monodromy conjecture (see [36, Definition 2.10]). Since the conjecture is well known in our setting and, again, widely covered by [8], the lemma is proved.

Let us also remark that the assertion about the conductor also follows from the fact that $\mathbf{f}_k$ is new outside p and the costancy of automorphic types can also be proved along the lines of [19, Lemma 2.14]. Indeed, the proof of [19, Lemma 2.14] needs as an input a representation interpolating the ρ_k's as the ρ above and then the arguments of loc.cit. apply. More precisely, case (a) of loc.cit. does not need any further explanation, while case (b) of loc.cit. take advantage of a base change argument and, hence, needs a theory of Coleman families attached to Hilbert modular forms, which has been developed in [2, Theorem 5.1], and also a base change theorem for Coleman families. This latter ingredient can be proved as in the case of Hida families, but we have not been able to provide a reference in our broader setting. Alternatively, we remark that case (b) of loc.cit. also follows from [18, Lemma 2.6.2] without the need to make a base change, but again this result is formulated in the setting of Hida families. Let us explain another approach based on [36, Theorem 3.1 (2)] which avoids the base change argument and applies to Coleman families. The main point of (b) is proving that, if $\pi_{k_0,v}$ is special, then the same is true for $\pi_{k,v}$. By the local Langlands correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$, this means that if the monodromy of $WD_v\left(\rho_{k_0}\right)^{\mathrm{Fr-ss}}$ is non-trivial, then the same is true for $WD_v(\rho_k)^{\mathrm{Fr-ss}}$: but this follows from [36, Theorem 3.1 (2)]. More precisely, writing $\overline{\mathcal{L}}$ for an algebraic closure of the fraction field of $\mathcal{O}$, let us write t (resp. t_k) for the smallest integer s such that $N^s = 0$ on $\left(\overline{\mathcal{L}} \otimes_{\mathcal{O}} WD_v(\rho)\right)^{\mathrm{Fr-ss}}$ (resp. $WD_v(\rho_k)^{\mathrm{Fr-ss}}$). Then a priori $t_k \leq t$ and [36, Theorem 3.1 (2)] gives $t_k = t$ whenever $WD_v(\rho_k)^{\mathrm{Fr-ss}}$ is pure, from which the required $t_k = t = t_{k_0} = 2$ follows. We note that [36, Theorem 3.1 (2)] is already proved in the first paragraph of pag. 890 of loc.cit. taking into account that the integer t_k can be characterized by the fact that $2(t_k - 1)$ equals the difference between the larger and the smaller weight of $WD_v(\rho_k)^{\mathrm{Fr-ss}}$ when $WD_v(\rho_k)^{\mathrm{Fr-ss}}$ is pure (or when it is indecomposable). $\square$

According to Lemma 8.2, the conductor of $\pi_{i,k,l}$ at a finite prime $l \neq p$ is a well defined quantity which does not depend on the choice of the point $k \in \mathbb{N} \cap U_i$: we denote it by $c_l(\mathbf{f}_i)$ and let $c_l := \max_{i=1,2,3}\{c_l(\mathbf{f}_i)\}$. We recall that, if l is a finite prime, an irreducible admissible representation of $\mathbf{GL}_2(\mathbb{Q}_l)$ admits a Jacquet-Langlands lift if and only if it is special or supercuspidal. Hence, in view of Lemma 8.2, it makes sense to consider

$S_{JL} := \{l \neq p \text{ finite primes} : \pi_{i,k,l} \text{ admits a JL-lift}, \forall k \in \mathbb{N} \cap U_i \text{ and } i = 1,2,3\}$ and $D_{JL} := \prod_{l \in S_{JL}} l^{c_l} \mid D$.

Next, we suppose that D_{JL} is squarefree and define

$$S_{JL}^{-} := \left\{ l \mid D_{JL} : -a_l(f_{1,k_1})\, a_l(f_{2,k_2})\, a_l(f_{3,k_3})\, l^{-\frac{k_1+k_2+k_3}{2}} = -1 \right\} \text{ and } D_{JL}^{-} := \textstyle\prod_{l \in S_{JL}^{-}} l,$$

where we remark that D_{JL}^{-} is indeed independent of $\underline{k}$: because $l \neq p$, the function of $\underline{k}$ that appears in the definition of D_{JL}^{-} is rigid analytic and, being $\{\pm 1\}$-valued on the Zariski dense subset of integer points of the connected affinoid subdomain $U_1 \times U_2 \times U_3$, is indeed constant. Because we assume that D_{JL} is squarefree, we have the equality

$$\varepsilon_l\left(\mathbf{f}_{\underline{k}}^{\#}\right) = -a_l(f_{1,k_1})\, a_l(f_{2,k_2})\, a_l(f_{3,k_3})\, l^{-\frac{k_1+k_2+k_3}{2}},$$

and we see that there is a well defined finite "generic sign" $\varepsilon_{\mathrm{fin}}(\underline{\mathbf{f}})$. Next, we make the following further assumptions that will be in force until the end of this section.

(Bal) $\varepsilon_{\mathrm{fin}}(\underline{\mathbf{f}}) = -1$, i.e. D_{JL}^{-} is the product of an *odd* number of primes;

(Con) For every $l \mid M$, we have $c_l(\mathbf{f}_i) \leq 1$ for $i = 1,2,3$ or there is an index i_l such that $c_l(\mathbf{f}_{i_l}) \geq 2\max_{i \neq i_l}\{c_l(\mathbf{f}_i), 1\}$.

Let us write α_i for the U_p-eigenvalues attached to $\mathbf{f}_i$, so that $\alpha_i \in \mathcal{O}$. Write B for the definite quaternion algebra of discriminant D_{JL}^-: then $\mathbf{f}_i$ lifts to an element of $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$ (by the p-adic Jacquet-Langlands correspondence, see [10]) and it follows from (Bal) that B is the quaternion algebra predicted by [34] at every integer and balanced point $\underline{k}$.

Under these assumptions, we can prove the following result.

Theorem 8.3. *There exists*

$$\underline{\varphi} \in M_p^\diamond(\mathcal{D}_{\mathbf{k}_1}(W), \omega_{0,p}^{\mathbf{k}_1})^{\alpha_1} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_2}(W), \omega_{0,p}^{\mathbf{k}_2})^{\alpha_2} \otimes M_p^\diamond(\mathcal{D}_{\mathbf{k}_3}(W), \omega_{0,p}^{\mathbf{k}_3})^{\alpha_3}$$

such that, for every balanced generic integer point $\underline{k} \in \underline{U}$ for $\underline{\varphi}$,

$$\mathcal{L}_p^{\underline{\alpha}}\left(\underline{\varphi}\right)\left(\underline{k}\right)^2 = \mathcal{E}_p\left(\underline{\alpha}, \underline{k}\right)^2 \frac{\left(\varphi_{\underline{k}}^{\#}, \varphi_{\underline{k}}^{\#}\right)_{\underline{k}}}{2L\left(1, \Pi_{\underline{k}}, \mathrm{Ad}\right)} L\left(1/2, \Pi_{\underline{k}}\right) \prod_{l \mid M} C_l,$$

where the constants C_l are defined as follows.

- *If $l \mid D_{JL}^-$, then $C_l = 2\frac{1}{l}\left(1 - \frac{1}{l}\right)$;*
- *If $l \mid M/D_{JL}^-$ and $c_l(\mathbf{f}_i) \leq 1$ for $i = 1, 2, 3$, then*[8]

$$C_l = \begin{cases} \frac{1}{l}\left(1 + \frac{1}{l}\right)^{-1}, & \textit{if one of the representations is special unramified.} \\ \frac{1}{l}, & \textit{if two of the representations are special unramified,} \\ \frac{2}{l}\left(1 + \frac{1}{l}\right), & \textit{if three of the representations are special unramified;} \end{cases}$$

- *If $l \mid M/D_{JL}^-$ and $c := c_l(\mathbf{f}_{i_l}) \geq 2\max_{i \neq i_l}\{c_l(\mathbf{f}_i), 1\}$, then $C_l = \frac{L_v\left(1, \Pi_v', \mathrm{Ad}\right)}{\zeta_{\mathbb{Q}_v}^2(2) L_v(1/2, \Pi_v')} C_l'$, where $C_l' = \frac{\prod_{i \neq i_l}(1 - A(\pi_i))}{l^c}\left(1 + \frac{1}{l}\right)$ and*

$$A(\pi_i) := \begin{cases} \frac{1}{1-l}, & \textit{if } \pi_i \textit{ is supercuspidal or } \pi_i = \pi\left(\chi_1, \chi_2\right) \textit{ with } \chi_k \textit{ ramified for } k = 1, 2, \\ \frac{1}{1+l}\left(\frac{a_l(\mathbf{f}_i)^2}{\varepsilon(l) l^{k_i+1}} - \left(1 + \frac{1}{l}\right)\right), & \textit{if } \pi_i = \pi\left(\chi_1, \chi_2\right) \textit{ is principal unramified,} \\ -\frac{1}{l}, & \textit{if } \pi_i \textit{ is special unramified,} \\ 0, & \textit{if } \pi_i = \pi\left(\chi_1, \chi_2\right) \textit{ with one } \chi_k \textit{ ramified and the other unramified.} \end{cases}$$

Remark 8.4. The contribution of the local L-factors in the third case depends on the nature of the representations. For example, we have $C_l = C_l'$ when the two representations having the smaller conductor are unramified.

8.1. **The p-adic Jacquet-Langlands correspondence and the choice of the test vector.** By Chenevier's p-adic Jacquet-Langlands correspondence (see [10]), the eigenvector $\mathbf{f}_i$ corresponds to an eigenvector $\varphi_{\mathbf{f}_i}$ in $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$ with the property that its specializations are new at all the primes l, the only possible exception being at $l = p$. If $l \nmid D_{JL}^- p$ is a finite prime, we write $\widehat{\delta}_{l^c}$ for the element of $\mathbf{GL}_2\left(\mathbb{A}_{\mathrm{f}}\right)$ defined by the following local conditions:

$$\left(\widehat{\delta}_{l^c}\right)_v = \begin{cases} 1 & \text{if } v \neq l, \\ \delta_{l^c} = \begin{pmatrix} l^c & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v = l. \end{cases}$$

Set $\underline{\varphi}_{\underline{\mathbf{f}}} := \varphi_{\mathbf{f}_1} \otimes \varphi_{\mathbf{f}_2} \otimes \varphi_{\mathbf{f}_3}$ and define $\underline{\varphi}$ in $M_p^\diamond(\mathcal{D}_{\mathbf{k}_i}(W), \omega_{0,p}^{\mathbf{k}_i})^{\alpha_i}$ as follows. For every finite prime l, let i_l be the index which realizes the largest conductor $c_l(\mathbf{f}_{i_l})$ and let $j_l \neq i_l$ be the index which realizes the second largest conductor $c_l(\mathbf{f}_{j_l})$. Suppose we have, for example, $i_l = 3$ and $j_l = 2$: then we define $\widetilde{\delta}_l := 1 \otimes \widehat{\delta}_{l^{c_l(\mathbf{f}_3) - c_l(\mathbf{f}_2)}} \otimes 1$. Finally, we set $\widetilde{\delta}_M := \prod_{l \mid M} \widetilde{\delta}_l$ and define $\underline{\varphi} := \underline{\varphi}_{\underline{\mathbf{f}}} \widetilde{\delta}_M$. Let us remark that $\underline{\varphi}$ is designed so that its specialization at an arithmetic point $\underline{k}$ is a pure tensor whose component at the finite primes different from p are either a tensor product of new vectors when $l \nmid M$ or, assuming as above that $i_l = 3$ and $j_l = 2$ and taking into account the twist (15):

$$\psi_l = l^{k_2/2} \psi_{1,l} \otimes \psi_{2,l} \delta_{l^{c_l(\mathbf{f}_3) - c_l(\mathbf{f}_2)}} \otimes \psi_{3,l} =: l^{k_2/2} \psi_l^0, \tag{75}$$

[8]By a special unramified representation, we mean the twist by an unramified character of the special representation. Of course, this is a ramified representation of conductor 1

where $\psi_{i,l}$ is a new vector in $\pi_{i,k_i,l}$. In particular, the same property is enjoyed by $\underline{\varphi}^{\#}_{\underline{k}}$, whose local component at the primes $l \neq p$ agrees with the local component of $\underline{\varphi}$.

8.2. **Proof of Theorem 8.3.** We have to make explicit the local constants $C_v^{\psi_v,\check{\psi}_v}(\psi_v) := \frac{I_v(\psi_v\otimes\check{\psi}_v)}{(\psi_v,\check{\psi}_v)_v}$ associated to the local components of $\varphi^{\#}_{\underline{k}}$ whose p-stabilization $\varphi_{\underline{k}}$ is the specialization of $\underline{\varphi}$ at the finite primes that appear in Theorem 7.3 where we choose $\varphi^{\flat\#}_{\underline{k}} = \varphi^{\flat\flat\#}_{\underline{k}} = \varphi^{\#}_{\underline{k}}$ (at the same time we will see that $\left(\varphi^{\flat\#}_{\underline{k}}, \varphi^{\flat\flat\#}_{\underline{k}}\right)_{\underline{k}} = \left(\varphi^{\#}_{\underline{k}}, \varphi^{\#}_{\underline{k}}\right)_{\underline{k}} \neq 0$ from the local calculation and (22)). As noticed in Remark 3.5, $C_v^{\psi_v,\psi_v^\vee}(\psi_v,\psi_v^\vee)$ does not depend on the non-zero vector in the lines spanned by either ψ_v or $\psi_v^\vee$: thanks to our assumption on the central characters, we see that $\check{\psi}_{i,l} = \omega_{i,l}^{-1}\psi_{i,l}$ (resp. $(\psi_{i,l}\check{\delta}_{l^c}) = \omega_{i,l}(l)^c\check{\psi}_{i,l}\delta_{l^c} = \omega_{i,l}(l)^c\omega_{i,l}^{-1}\psi_{i,l}\delta_{l^c}$) is a new vector (resp. the translate by δ_{l^c} of a new vector) as it is $\overline{\psi}_{i,v}$ (resp. $\overline{\psi}_{i,v}\delta_{l^c}$); it follows that $\langle \psi_l, \check{\psi}_l\rangle_l \neq 0$ and, from (75), that we have $C_v^{\psi_v,\check{\psi}_v}(\psi_v) = C_v^{\psi_v,\overline{\psi}_v}(\psi_v,\overline{\psi}_v) = C_v^{\psi_v^0,\overline{\psi}_v^0}(\psi_v^0,\overline{\psi}_v^0)$. Hence our claim follows from the following computations of the local constants $C_v(\psi_v^0,\overline{\psi}_v^0)$, which always uses (75) as a test vector. When l is prime to M, $C_l(\psi_l) = 1$ by [31, Lemma 2.2]. When $l \mid D_{JL}^-$ this is done in [42, Proposition 4.5]. When $l \mid M/D_{JL}^-$ and $c_l(\mathbf{f}_i\ ) \leq 1$, then $\pi_{1,k_1,l}\otimes\pi_{2,k_2,l}\otimes\pi_{3,k_3,l}$ is the tensor product of unramified principal series representations with either one, two or three special unramified representations: then we may apply [42, Corollary 4.2], [42, Proposition 4.3] or, respectively, [42, Proposition 4.4][9]. Finally, the computation of $\frac{\zeta^2_{\mathbb{Q}_v}(2)L_v\left(1/2,\Pi'_v\right)}{L_v(1,\Pi'_v,\mathrm{Ad})}C_v(\psi_v)$ in the last case follows from [30, Theorem 4.1], once we remark that the roots α_l and β_l of the Hecke polynomial of a weight $k+2$ supercuspidal eigenform f at l satisfy $\chi_1(l) = \frac{\alpha_l}{p^{(k+1)/2}}$ and $\chi_2(l) = \frac{\beta_l}{p^{(k+1)/2}}$, when the associated automorphic representation at l is principal unramified of the form $\pi(\chi_1,\chi_2)$, implying that we have

$$\frac{\chi_1(l)}{\chi_2(l)} + \frac{\chi_2(l)}{\chi_1(l)} = \frac{\chi_1^2(l)+\chi_2^2(l)}{\chi_1(l)\chi_2(l)} = \frac{\alpha_l^2+\beta_l^2}{\alpha_l\beta_l} = \frac{(\alpha_l+\beta_l)^2 - 2\alpha_l\beta_l}{\alpha_l\beta_l}$$

in loc.cit.

8.3. **Variants.** Let us remark that we have $(\check{\psi g}) = \omega(\underline{\mathrm{nrd}}(g))\check{\psi}g = \omega_{\mathrm{f}}(\underline{\mathrm{nrd}}(g))\check{\psi}g$ and then we see from the invariance of the left hand side of (20) (or directly from the definition of $(-,-)_{\underline{k}}$) that we have

$$(\varphi_1 g, \varphi_2 g)_{\underline{k}} = \omega_{\mathrm{f}}(\underline{\mathrm{nrd}}(g))^{-1}(\varphi_1,\varphi_2)_{\underline{k}}.$$

Write $\varphi^{\#}_{\underline{\mathbf{f}},\underline{k}} := \varphi^{\#}_{\mathbf{f}_1,k_1}\otimes\varphi^{\#}_{\mathbf{f}_2,k_2}\otimes\varphi^{\#}_{\mathbf{f}_3,k_3}$, where $\varphi^{\#}_{\mathbf{f}_i,k_i}$ is the unique vector whose p-stabilization $\varphi_{\mathbf{f}_i,k_i}$ is the specialization of $\underline{\varphi}_{\underline{\mathbf{f}}}$. Then it follows from (75) that we have an equality $\left(\varphi^{\#}_{\underline{k}}, \varphi^{\#}_{\underline{k}}\right)_{\underline{k}} = \mu\left(\varphi^{\#}_{\underline{\mathbf{f}},\underline{k}}, \varphi^{\#}_{\underline{\mathbf{f}},\underline{k}}\right)_{\underline{k}}$ for a non-zero constant μ which does not depend on $\underline{k}$: a product of values of the kind $\omega_{l,i}^{-1}(\mathrm{nrd}(\delta_{l^c}))$, indeed, which is therefore 1. Thus we can substitute

$$\left(\varphi^{\#}_{\underline{k}}, \varphi^{\#}_{\underline{k}}\right)_{\underline{k}} = \left(\varphi^{\#}_{\underline{\mathbf{f}},\underline{k}}, \varphi^{\#}_{\underline{\mathbf{f}},\underline{k}}\right)_{\underline{k}} \tag{76}$$

in Theorem 8.3 and also apply (74).

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