

STRONG KAC'S CHAOS IN THE MEAN-FIELD BOSE-EINSTEIN CONDENSATION

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ABSTRACT. A stochastic approach to the (generic) mean-field limit in Bose-Einstein Condensation is described and the convergence of the ground state energy as well as of its components are established. For the one-particle process on the path space a total variation convergence result is proved. A strong form of Kac's chaos on path-space for the k -particles probability measures are derived from the previous energy convergence by purely probabilistic techniques notably using a simple chain-rule of the relative entropy. The Fisher's information chaos of the fixed-time marginal probability density under the generic mean-field scaling limit and the related entropy chaos result are also deduced.

1. Introduction

We consider the problem of justifying the general mean-field approach to Bose-Einstein Condensation in the ground state framework, starting from the N body Hamiltonian for N Bose particles and by performing a suitable limit of infinitely many particles. We say general because we include the cases corresponding to $0 \leq \beta < 1$, where β is the parameter that allows to model the N -dependence of the range of the interacting potential. The case $\beta = 0$ corresponds properly to the mean-field approximation where the potential range is fixed and the intensity of the interacting potential decreases as $1/N$. Differently the regime corresponding to $0 < \beta < 1$ is more difficult since the interaction potential goes to a delta function in the sense of the measures convergence. This case is not very well studied and it is usually denoted as the non linear Schroedinger limit (see [33]). There are many results and some quantitative estimates of the convergence rate for small values of β (see [33, 50] and references therein). We face the general mean-field convergence problem by using the hard results for the case $\beta = 1$, known as Gross-Pitaevskii scaling limit, obtained in [34, 35] and, recently, in [43]. We prove the convergence of the one-particle ground-state energy to the ground-state energy of the non-linear Schroedinger functional for the case of purely repulsive interacting potential (Theorem 2.2). After this, we are also able to discriminate how the single

2000 *Mathematics Subject Classification.* Primary 60J60, 60K35, 81S20, 94A17; Secondary 26D15, 81S20, 60G10, 60G40.

Key words and phrases. Bose-Einstein Condensation, Mean-field scaling limit, Stochastic Mechanics, interacting Nelson diffusions, strong Kac's chaos, Fisher's and entropy chaos, convergence of probability measures on path space.

terms of the energy converge (Theorem 2.4).

Nelson's Stochastic Mechanics ([44, 45, 10]) allows to rigorously associate a system of N interacting diffusions to the N body Hamiltonian ([7]) and consequently to consider the convergence problem of the one-particle probability measure on the path-space to the limit measure of McKean-Vlasov type. In this paper we prove that the convergence holds in total variation (Theorem 4.6), which is stronger with respect to the weak convergence recently obtained in [4] for the Gross-Pitaevskii scaling limit. Successively we establish both the usual Kac's chaos and a strong form of Kac's chaos for the law of the N interacting Nelson diffusions (Theorem 5.4). It is well-known indeed that Kac's chaos is usually expressed as the weak convergence of the law of any k -components diffusions to the asymptotic k -product measure. In our paper we state that the cited convergence on path space is in total variation sense (Theorem 5.1 and Theorem 5.4). The proof, based on Girsanov Theorem and on the relative entropy between the two involved probability measures, is essentially a probabilistic one. The relevant analytic result used in both the proofs is the L^2 -convergence of the difference of the drifts which can be deduced from Theorem 2.4 without the usual assumptions of bounded or Lipschitz drifts (that are not satisfied in our quantum mathematical setting).

The plan of the present paper is the following. In Section 2 we introduce the quantum framework of the derivation of the non-linear Schroedinger model from the initial N body Hamiltonian describing the N Bose particles through a suitable scaling limit of general mean-field type and we prove the weak- L^1 convergence of the ground-state energies to the asymptotic non linear energy as well as the convergence of the single terms of the associated energy functional.

In Section 3 we briefly introduce the Nelson-Carlen scheme of Stochastic Mechanics. The total variation convergence of the one-particle measure on path-space is proved in Section 4. In Section 5 both the usual and the strong Kac's chaos is established and in the Section 6 we derive the entropy chaos from Fisher chaos for the fixed time symmetric probability law on the product space \mathbb{R}^{3N} .

2. Convergence of the mean-field quantum energy functional

We want to study a thermodynamic limit of a system of Bosons in \mathbb{R}^3 . For this reason we consider the following N -body quantum Hamiltonian

$$H_N = \sum_{i=1}^N \left(-\frac{\hbar^2}{2m} \Delta_i + V(\mathbf{r}_i) \right) + \sum_{1 \leq i < j \leq N} v_N(\mathbf{r}_i - \mathbf{r}_j) \quad (2.1)$$

where V is a confining potential, v_N a pair-wise repulsive interaction potential (depending on the number of particles for obtaining a meaningful thermodynamic limit) and $\mathbf{r}_i \in \mathbb{R}^3$, $i = 1, \dots, N$. It operates on symmetric wave functions Ψ in the complex $L^2(\mathbb{R}^{3N})$ -space in order to satisfy the symmetry permutation prescription for Bose particles.

We consider the mean quantum mechanical energy

$$\mathcal{E}_N(\Psi) = \langle \Psi, H_N \Psi \rangle, \quad \Psi \in H^1(\mathbb{R}^{3N}) \quad (2.2)$$

If there exists a minimizing function Ψ_N of \mathcal{E}_N it is called a *ground state* and the corresponding energy $E_N[\Psi_N]$ given by

$$E_N[\Psi_N] := \inf \left\{ \mathcal{E}(\Psi) : \int_{\mathbb{R}^{3N}} |\Psi|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N = 1 \right\}$$

is known as *ground state energy*.

Under suitable assumptions on the potentials V and v_N one can prove the existence of the ground state Ψ_N for (2.1). Uniqueness of the ground state is to be understood as uniqueness apart from an *overall phase*.

The Bose-Einstein condensate is obtained as the thermodynamic limit of the previous N -body problem and it is usually described by a wave $\phi \in H^1(\mathbb{R}^3)$, also called wave function of the condensate, which is the minimizer of the Hartree or of the non-linear Schroedinger (energy) functional (depending on the dependence of v_N from N)

$$\mathcal{E}^H[\phi] = \int \left(\frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + V(r) |\phi(\mathbf{r})|^2 \right) d\mathbf{r} + \int \int |\phi(\mathbf{r})|^2 v_0(\mathbf{r} - \mathbf{r}_1) |\phi(\mathbf{r}_1)|^2 d\mathbf{r} d\mathbf{r}_1, \quad (2.3)$$

$$\mathcal{E}^{nls}[\phi] = \int \left(\frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + V(r) |\phi(\mathbf{r})|^2 + g |\phi(\mathbf{r})|^4 \right) d\mathbf{r}, \quad (2.4)$$

under the L^2 -normalization condition

$$\int_{\mathbb{R}^3} |\phi(\mathbf{r})|^2 d\mathbf{r} = 1$$

and where

$$g = \int v_0(x) dx.$$

Here v_0 denotes the potential of the interaction between two particles before the thermodynamic limit is taken and its relation with v_N is given by the following equation

$$v_N(\mathbf{r}) = \frac{N^{3\beta}}{N-1} v_0(N^\beta \mathbf{r}), \quad 0 \leq \beta < 1 \quad (2.5)$$

From now on we consider the more difficult case given by \mathcal{E}^{nls} which implies $\beta > 0$.

We denote by E_{nls} the minimum of the energy (2.4) and by ϕ_{nls} the minimizer which solves the stationary cubic non-linear equation (called non-linear Schroedinger equation, nlS, or Gross-Pitaevskii equation, GP) ([33], or [26, 47])

$$-\frac{\hbar^2}{2m} \Delta \phi + V\phi + 2g|\phi|^2\phi = \lambda\phi \quad (2.6)$$

λ , the real-valued Lagrange multiplier of the normalization constraint, is usually called chemical potential. For the GP case one can prove that ϕ is continuously differentiable and strictly positive ([34]).

A stochastic quantization approach for the system of N interacting Bose particles has been faced for the first time in [38].

In [41] it has been proved that to the N -body problem associated to H_N there correspond a well defined diffusion process describing the motion of the single particle in the condensate, under the Gross-Pitaevskii scaling limit as introduced in [34], which allows to prove the existence of an exact Bose-Einstein condensation for the ground state of H^N (see [34, 35]). For the time-dependent derivation of the Gross-Pitaevskii equation see [1] and [21]. For the non linear Schroedinger case see [22].

For simplicity of notations, let us put $\hbar = 2m = 1$.

We consider the mean energy (2.2)

$$\mathcal{E}_N[\Psi_N] = \int \sum_{i=1}^N (|\nabla_i \Psi_N|^2 + V(\mathbf{r}_i)) |\Psi_N|^2 + \sum_{1 \leq i < j \leq N} v_N(\mathbf{r}_i - \mathbf{r}_j) |\Psi_N|^2 d\mathbf{r}_1 \cdots \mathbf{r}_N$$

We assume

h1) $V(|\mathbf{r}_i|)$ is locally bounded, continuous, strictly positive and going to infinity when $|\mathbf{r}_i|$ goes to infinity.

h2) v_0 is smooth, compactly supported, non negative, spherically symmetric.

Remark 2.1. The scaling case with $\beta = 1$ does not belong to the mean-field regime. It is known as Gross-Pitaevskii scaling limit and it involves the scattering length of the interaction potential. The convergence of the ground state energy in this setting is difficult and has been provided by [34, 35, 43].

In this paper we propose a proof of the above convergence result in the thermodynamic limit for a generic mean-field case (i.e. with $0 \leq \beta < 1$), but only under the assumption h2) corresponding to a positive-definite interaction, by taking advantage of the hard results of the GP regime.

Theorem 2.2. *Under the previous hypothesis h1), h2) we have that*

$$\lim_{N \uparrow \infty} \frac{E_N[\Psi_N]}{N} = E_{nls}[\phi_{nls}] \quad (2.7)$$

and

$$\lim_{N \uparrow \infty} \int |\Psi_N|^2 d\mathbf{r}_2 \cdots \mathbf{r}_N = |\phi_{nls}|^2 \quad (2.8)$$

where ϕ_{nls} is the minimizer of the non-linear Schroedinger functional (2.4) and the convergence is in the weak $L^1(\mathbb{R}^3)$ sense.

Remark 2.3. The one-particle marginal density $\rho_N^{(1)}$ converges weakly to ρ_{nls} in the sense that the probability measures $\rho_N^{(1)} d\mathbf{r}$ weakly converge as $N \rightarrow \infty$ towards the probability measure $\rho_{nls} d\mathbf{r}$ on \mathbb{R}^3 .

Proof. First of all we note that the following estimate

$$E_{nls} + CN^{-\beta} \geq \frac{E_N}{N}$$

is proved in ([33], Proposition 2.3 or Theorem 2.4). This means that $\lim_{N \uparrow \infty} \frac{E_N}{N} \leq E_{nls}$.

In order to prove that $\lim_{N \uparrow \infty} \frac{E_N}{N} \geq E_{nlS}$ let us introduce the functions u_m^N such that

$$u_m^N(\mathbf{r}) = \frac{m^3}{m-1} u_0^N(m\mathbf{r}) \quad (2.9)$$

and

$$u_0^N(\mathbf{r}) = N^{3\beta-3} v_0(N^{\beta-1}(\mathbf{r})) \quad (2.10)$$

By using the previous equations we obtain

$$u_N^N(\mathbf{r}) = v_N(\mathbf{r}) = \frac{N^{3\beta}}{N-1} v_0(N^\beta(\mathbf{r})), \quad 0 \leq \beta < 1 \quad (2.11)$$

By relevant results on the convergence of ground-state energies in the GP scaling limit (see [34, 32] which corresponds to considering v_N as in equation (2.5)) if we denote by E_m^N the ground-state energy associated to the potential u_m^N we have that, for all fixed N ,

$$\lim_{m \uparrow \infty} \frac{1}{m} E_m^N = E_{GP}^N \quad (2.12)$$

where E_{GP}^N is the minimum energy of

$$\mathcal{E}_{GP}^N[\phi] = \int (|\nabla \phi(\mathbf{r})|^2 + V(r)|\phi(\mathbf{r})|^2 + 4\pi a_N |\phi(\mathbf{r})|^4) d\mathbf{r} \quad (2.13)$$

where a_N is the scattering length of the potential u_0^N (see [34, 36] for the definition of scattering length).

First of all we prove that $\lim_{N \uparrow \infty} 4\pi a_N = g$. We note that by (2.10) and by changing integration variables

$$\int u_0^N(\mathbf{r}) d\mathbf{r} = \int v_0(\mathbf{r}) d\mathbf{r} = g \quad (2.14)$$

By the upper bound for the scattering length (see [36] Appendix B) we have

$$g = \int u_0^N(\mathbf{r}) d\mathbf{r} \geq 4\pi a_N$$

We look for an estimate from below for $4\pi a_N$ converging to g . Let us recall how the scattering length is defined. Denoting by R_0 is the maximum radius of the support of v_0 , let

$$R_N = N^{1-\beta} R_0$$

be the maximum radius of the support of u_0^N . Setting

$$\mathcal{E}_R^N[\phi] = \int_{B_R} (|\nabla \phi(\mathbf{r})|^2 + u_0^N(\mathbf{r})|\phi(\mathbf{r})|^2) d\mathbf{r} \quad (2.15)$$

and denoting by \tilde{E}_R^N the minimum energy with respect to ϕ in $L^2(B_R)$ (with B_R the ball of radius R) subject to the constrain $\|\phi\|_{B_R} = 1$, we have for all $R \geq R_N$:

$$\tilde{E}_R^N = \frac{4\pi a_N R}{(R - a_N)} \quad (2.16)$$

We observe that for $R > N^{1-\beta} R_0$ we can rewrite (2.15) as

$$\mathcal{E}_R^N[\phi] = \int_{\frac{B_R}{N^{1-\beta}}} (N^{1-\beta} |\nabla \phi(\mathbf{r})|^2 + v_0(\mathbf{r})|\phi(\mathbf{r})|^2) d\mathbf{r} \quad (2.17)$$

Denoting by E_R^N the minimum energy with respect to ϕ in $L^2(\frac{B_R}{R})$ subject to the constrain $\|\phi\|_{\frac{B_R}{R}} = 1$, we want to give a lower bound for E_R^N .

If ϕ_R^N denotes the minimizer of the previous functional we have that

$$E_R^N \geq \int_{\frac{B_R}{N}} v_0(\mathbf{r}) |\phi_R^N(\mathbf{r})|^2 d\mathbf{r}$$

So it is sufficient to give a lower bound for $|\phi_R^N|$. To this aim let us introduce the potential:

$$v_0^k(\mathbf{r}) = \min(v_0(\mathbf{r}), k), \quad k > 0$$

If $E_R^{N,k}$ denotes the minimum energy when the interaction potential is $v_0^k(\mathbf{r})$, then we note that

$$E_R^N \geq E_R^{N,k} \geq \int_{\frac{B_R}{N^{1-\beta}}} v_0^k(\mathbf{r}) |\phi_R^{N,k}(\mathbf{r})|^2 d\mathbf{r} \quad (2.18)$$

where the function $\phi_R^{N,k}$ satisfies the equation $L_N^k(\phi_R^{N,k}) = 0$ with L_N^k given by

$$L_N^k(\phi) = N^{1-\beta} \Delta \phi + v_0^k \phi$$

By the maximum principle if

$$L_N^k(\phi) \leq 0, \quad \phi \geq 0, \quad \|\phi\|_{\frac{B_R}{N^{1-\beta}}} = 1$$

then $\phi \leq \phi_R^{N,k}$.

We choose $\phi_R^{N,k,\epsilon} = C + \epsilon \|x^2\|$ where ϵ is such that:

$$-3N^{1-\beta}\epsilon + (C + \epsilon R_0^2)k \leq 0$$

When $N \uparrow \infty$ we can choose $\epsilon \downarrow 0$ and C such that

$$\left(C + \epsilon \frac{R_0^2}{N^{1-\beta}} \right) = 1$$

In particular we have:

$$\phi_R^{N,k} \geq \phi_R^{N,k,\epsilon} \geq 1 - \epsilon \frac{R^2}{N^{1-\beta}}$$

Therefore from (2.18) we get

$$E_R^N \geq E_R^{N,k} \geq \left(\int v_0^k(\mathbf{r}) d\mathbf{r} \right) \left(1 - \epsilon \frac{R^2}{N^{1-\beta}} \right)$$

Sending $\epsilon \downarrow 0$, $N \uparrow \infty$, $k \uparrow \infty$ and $R \uparrow \infty$ we obtain

$$\begin{aligned} \lim_{N \uparrow \infty} 4\pi a_N &\geq \lim_{N \uparrow \infty R \uparrow \infty} \left(\frac{4\pi a_N R}{R - a_N} \right) \geq \\ &\geq \lim_{k \uparrow \infty} \int v_0^k(\mathbf{r}) d\mathbf{r} = \int v_0(\mathbf{r}) d\mathbf{r}. \end{aligned}$$

Therefore

$$\lim_{N \uparrow \infty} 4\pi a_N = \int v_0(\mathbf{r}) d\mathbf{r}.$$

Let us now consider the following functional

$$\mathcal{E}_{GP}^{N,R}[\phi] = \int (|\nabla\phi(\mathbf{r})|^2 + V(r)|\phi(\mathbf{r})|^2 + 4\pi a_N |\phi(\mathbf{r})|^4) d\mathbf{r} \quad (2.19)$$

and let $E_{GP}^{N,R} = \mathcal{E}_{GP}^{N,R}[\phi]$ where ϕ is in $H^1(B_R)$ subject to the constrain $\|\phi\| = 1$ and with Neumann boundary conditions.

In [34] it has be proven that

$$E_m^N \geq E_{GP}^{N,R} (1 - C(R, \phi_{GP}^{N,R}) m^{-\frac{1}{10}}) \quad (2.20)$$

where $C(R, \phi_{GP}^{N,R})$ is a continuous, locally bounded function of R and of the minimizer $\phi_{GP}^{N,R}$ with respect to the norm $\|\phi_{GP}\|_{L^\infty} + \|\nabla\phi_{GP}\|_{L^\infty}$. Indeed in the proof of Theorem 4.1 in [34] we can see that $\phi_{GP}^{N,R}$ depends from both

$$\min_{\Lambda_L} |\phi_{GP}^{N,R}|^2, \quad \max_{\Lambda_L} |\phi_{GP}^{N,R}|^2 - \min_{\Lambda_L} |\phi_{GP}^{N,R}|^2$$

where Λ_L is the box of length L and that the two previous quantities are continuous when $\phi_{GP}^{N,R}$ varies in a continuous way with respect to the norm $\|\phi_{GP}\|_{L^\infty} + \|\nabla\phi_{GP}\|_{L^\infty}$. Consequently if we are able to prove that $C(R, \phi_{GP}^{N,R})$ is bounded with respect to N then we obtain

$$\lim_{N \uparrow \infty} \frac{E_N}{N} = \lim_{N \uparrow \infty} \frac{E_N^N}{N} \geq \lim_{N \uparrow \infty} \left[E_{GP}^{N,R} (1 - C(R, \phi_{GP}^{N,R}) N^{-\frac{1}{10}}) \right] = \lim_{N \uparrow \infty} E_{GP}^{N,R}. \quad (2.21)$$

On the other hand, since $4\pi a_N \rightarrow g$, we have that $\lim_{N \uparrow \infty} E_{GP}^{N,R} = E_{nlS}^R$, where E_{nlS}^R is the minimum of the functional

$$\mathcal{E}_{nlS}^R[\phi] = \int (|\nabla\phi(\mathbf{r})|^2 + V(r)|\phi(\mathbf{r})|^2 + g|\phi(\mathbf{r})|^4) d\mathbf{r} \quad (2.22)$$

with respect to ϕ in $H^1(B_R)$ subject to the constraint $\|\phi\| = 1$ with Neumann boundary conditions.

Since E_{nlS}^R converges to E_{nlS} when $N \uparrow \infty$ we finally have

$$\lim_{N \uparrow \infty} \frac{E_N}{N} \geq E_{nlS} \quad (2.23)$$

It remains to prove that $C(R, \phi_{GP}^{N,R})$ is bounded with respect to N . We provide this by showing that

$$\lim_{N \uparrow \infty} \phi_{GP}^{N,R} = \phi_{nlS}^R \quad (2.24)$$

with respect to the norm $\|\cdot\|_{L^\infty} + \|\nabla \cdot\|_{L^\infty}$.

Let us first note that $\phi_{GP}^{N,R}$ satisfies the equation

$$-\Delta\phi_{GP}^{N,R} + V\phi_{GP}^{N,R} + 8\pi a_N |\phi_{GP}^{N,R}|^2 \phi_{GP}^{N,R} = \mu_{GP}^R(a_N) \phi_{GP}^{N,R} \quad (2.25)$$

from which we obtain that $\Delta\phi_{GP}^{N,R} \in L^{3/2+\epsilon}(B_R)$ with the bounds

$$\|\Delta\phi_{GP}^{N,R}\|_{L^{3/2+\epsilon}(B_R)} \leq C_1(a_N^{3/2} + 1) \int_{B_R} (\phi_{GP}^{N,R}(\mathbf{r}))^4 d\mathbf{r} \leq C_1 \frac{a_N^{3/2} + 1}{4\pi a_N} E_{GP}^{N,R},$$

where C_1 is positive constant independent of N . Using a bootstrap argument we obtain that $\|\phi_{GP}^{N,R}\|_{C^{2-\epsilon}} \leq F(E_{GP}^{N,R})$ where F is a continuous increasing function from \mathbb{R}_+ into itself and $C^{2-\epsilon}$ is the space of $2-\epsilon$ Hölder functions with $0 < \epsilon < 1$. Since $a_N \rightarrow g$ and $E_{GP}^{N,R}$ depends continuously on a_N , we have that $\sup_N \|\phi_{GP}^{N,R}\|_{C^{2-\epsilon}} < +\infty$, and so $\phi_{GP}^{N,R}$ stays in a compact set of C^1 with respect to $\|\cdot\|_{L^\infty} + \|\nabla \cdot\|_{L^\infty}$ norm. Since equation (2.25) has a unique solution and by Berge Maximum Theorem (see [13] Theorem 17.31) the map $a_N \mapsto \phi_{GP}^{N,R}$ is continuous with respect to $\|\cdot\|_{L^\infty} + \|\nabla \cdot\|_{L^\infty}$ norm, we have that $\lim_{N \uparrow \infty} \phi_{GP}^{N,R} = \phi_{nlS}^R$ in C^1 and so $C(R, \phi_{GP}^{N,R})$ is bounded with respect to N . \square

We aim at characterizing the limit of the single components of the ground state energy $E_N[\Psi_N]$.

Let us introduce the following energy functionals, for any $\lambda > 0$:

$$\mathcal{E}^1[\Psi_N, \lambda] = \int \sum_{i=1}^N (|\nabla_i \Psi_N|^2 + \lambda V(\mathbf{r}_i)) |\Psi_N|^2 + \sum_{1 \leq i < j \leq N} v(\mathbf{r}_i - \mathbf{r}_j) |\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N, \quad (2.26)$$

$$\mathcal{E}^2[\Psi_N, \lambda] = \int \sum_{i=1}^N (|\nabla_i \Psi_N|^2 + V(\mathbf{r}_i)) |\Psi_N|^2 + \sum_{1 \leq i < j \leq N} \lambda v(\mathbf{r}_i - \mathbf{r}_j) |\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N, \quad (2.27)$$

$$\mathcal{E}_{nlS}^1[\phi, \lambda] = \int \left(\frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + \lambda V(r) |\phi(\mathbf{r})|^2 + g |\phi(\mathbf{r})|^4 \right) d\mathbf{r}, \quad \beta > 0, \quad (2.28)$$

$$\mathcal{E}_{nlS}^2[\phi, \lambda] = \int \left(\frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + V(r) |\phi(\mathbf{r})|^2 + \lambda g |\phi(\mathbf{r})|^4 \right) d\mathbf{r}, \quad \beta > 0 \quad (2.29)$$

We denote by $E_N^1(\lambda)$, $E_N^2(\lambda)$, $E_{nlS}^1(\lambda)$, $E_{nlS}^2(\lambda)$ the minimum of the four above energy functionals respectively.

We introduce the following hypothesis.

Hypothesis A. For λ in an neighbour of 1

$$\lim_{N \rightarrow \infty} \frac{E_N^1(\lambda)}{N} = E_{nlS}^1(\lambda) \quad (2.30)$$

$$\lim_{N \rightarrow \infty} \frac{E_N^2(\lambda)}{N} = E_{nlS}^2(\lambda) \quad (2.31)$$

If Theorem 2.2 holds, then Hypothesis A is true when $v(\mathbf{r}) \in L^1(\mathbb{R}^3)$, positive and with compact support. Indeed $g_\lambda = \int_{\mathbb{R}^3} \lambda v(\mathbf{r}) d\mathbf{r} = \lambda g$.

Theorem 2.4 (Energy Components). *Under the same hypothesis h1), h2) as in Theorem 2.2, and the Hypothesis A, let us suppose that $\mathcal{E}_{nlS}^1[\phi, \lambda]$ and $\mathcal{E}_{nlS}^2[\phi, \lambda]$ admit a unique minimizer in a neighbour of $\lambda = 1$. Then*

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} |\nabla_1 \Psi_N(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^3} |\nabla \phi_{nlS}(\mathbf{r})|^2 d\mathbf{r} \quad (2.32)$$

and, moreover,

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} V(\mathbf{r}_1) |\Psi_N(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^3} V(\mathbf{r}) |\phi_{nlS}(\mathbf{r})|^2 d\mathbf{r} \quad (2.33)$$

$$\lim_{N \uparrow \infty} \frac{1}{2} \sum_{j=2}^N \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} v(|\mathbf{r}_1 - \mathbf{r}_j|) |\Psi_N(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = g \int_{\mathbb{R}^3} |\phi_{nlS}(\mathbf{r})|^4 d\mathbf{r} \quad (2.34)$$

where Ψ_N and ϕ_{nlS} are the unique minimizers of \mathcal{E}_N and \mathcal{E}_{nlS} respectively.

Proof of Theorem 2.4. Since $\mathcal{E}^1[\Psi_N, \lambda]$, $\mathcal{E}^2[\Psi_N, \lambda]$, $\mathcal{E}_{nlS}^1[\phi, \lambda]$, $\mathcal{E}_{nlS}^2[\phi, \lambda]$ depend linearly by λ we have that $E_N^1(\lambda)$, $E_N^2(\lambda)$, $E_{nlS}^1(\lambda)$, $E_{nlS}^2(\lambda)$ are concave in λ (as minimizers of functionals which are linear in λ).

Thanks to Hypothesis A and using the fact that $E_N^1(\lambda)$, $E_N^2(\lambda)$, $E_{nlS}^1(\lambda)$, $E_{nlS}^2(\lambda)$ are concave we have

$$\lim_{N \uparrow \infty} \frac{1}{N} \partial_\lambda E_N^1(\lambda) = \partial_\lambda E_{nlS}^1(\lambda) \quad (2.35)$$

$$\lim_{N \uparrow \infty} \frac{1}{N} \partial_\lambda E_N^2(\lambda) = \partial_\lambda E_{nlS}^2(\lambda) \quad (2.36)$$

whenever $\partial_\lambda E_{nlS}^1(\lambda)$ and $\partial_\lambda E_{nlS}^2(\lambda)$ are well defined. By using the Hellman-Feynman Principle we obtain

$$\frac{\partial_\lambda E_N^1(\lambda)}{N} = \int_{\mathbb{R}^3} \sum_{i=1}^N V(\mathbf{r}_i) |\Psi_N^{1,\lambda}(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N \quad (2.37)$$

$$\frac{\partial_\lambda E_N^2(\lambda)}{N} = \int_{\mathbb{R}^3} \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) |\Psi_N^{2,\lambda}(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N \quad (2.38)$$

where $\Psi_N^{i,\lambda}$ is the minimizer of $\mathcal{E}^i[\Psi_N, \lambda]$ with $i = 1, 2$.

For calculating the derivatives of (for example) $E_{nlS}^2(\lambda)$ we recall that

$$-\Delta \phi_{nlS}^{2,\lambda} + V \phi_{nlS}^{2,\lambda} + 2\lambda g |\phi_{nlS}^{2,\lambda}|^2 \phi_{nlS}^{2,\lambda} = \mu_{nlS}(\lambda) \phi_{nlS}^{2,\lambda}, \quad (2.39)$$

where $\mu_{nlS}(\lambda) = E_{nlS}^2(\lambda) + \lambda g \int_{\mathbb{R}^3} |\phi_{nlS}^{2,\lambda}|^4 d\mathbf{r}$. It is simple to see that the function $\mu_{nlS}(\lambda)$ is continuous with respect to λ and it is differentiable whenever $E_{nlS}^2(\lambda)$ is differentiable.

Since $\mathcal{E}_{nlS}^2[\lambda]$ has only one minimum, using a reasoning similar to the one of the proof of Theorem 2.2 and by Berge Maximum Theorem (see [13]) the map $\lambda \mapsto \phi_{nlS}^{2,\lambda}$ is a continuous function in λ and also differentiable whenever $\partial_\lambda E_{nlS}^2(\lambda)$ is well defined.

By differentiating (2.39) with respect to λ , when $E_{nlS}^2(\lambda)$ is differentiable, and by multiplying for $\phi_{nlS}^{2,\lambda}$ and finally by integrating we obtain

$$\begin{aligned}
& - \int \phi_{nlS}^{2,\lambda} \triangle \phi_{nlS}^{2,\lambda} + \int V \phi_{nlS}^{2,\lambda} \partial_\lambda \phi_{nlS}^{2,\lambda} + 2\lambda g \int |\phi_{nlS}^{2,\lambda}|^3 \partial_\lambda \phi_{nlS}^{2,\lambda} = \\
& = \mu_{nlS}(\lambda) \int \phi_{nlS}^{2,\lambda} \partial_\lambda \phi_{nlS}^{2,\lambda} + \partial_\lambda E_{nlS}^2[\lambda] - g \int |\phi_{nlS}^{2,\lambda}|^4 \quad (2.40)
\end{aligned}$$

Since equation (2.39) holds, we deduce that

$$\partial_\lambda E_{nlS}^2(\lambda) = g \int |\phi_{nlS}^{2,\lambda}|^4 \quad (2.41)$$

Now, $\phi_{nlS}^{2,\lambda}$ is continuous with respect to λ and since $E_{nlS}^2(\lambda)$ is concave it is almost everywhere differentiable. Then $E_{nlS}^2(\lambda)$ is continuously differentiable $\forall \lambda$. By the concavity of the previous functions we finally obtain

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^{3N}} \sum_{i=1}^N V(\mathbf{r}_1) |\Psi_N^{1,\lambda}(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int V(\mathbf{r}_1) |\phi_{nlS}^{1,\lambda}(\mathbf{r})|^2 d\mathbf{r} \quad (2.42)$$

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^{3N}} \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) |\Psi_N^{2,\lambda}(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int g |\phi_{nlS}^{2,\lambda}(\mathbf{r})|^4 d\mathbf{r} \quad (2.43)$$

□

3. Stochastic mechanics and Bose-Einstein condensation

Nelson's Stochastic Mechanics allows to study quantum phenomena using a well determined class of diffusion processes ([44, 45, 7, 8]). See [10] for a more recent review on Stochastic Mechanics.

We will briefly introduce the class of *Nelson* diffusions which are associated to a solution of a Schrödinger equation.

Let the complex-valued function (*wave function*) $\psi(x, t)$ be a solution of the equation:

$$i\partial_t \psi(x, t) = H\psi(x, t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad (3.1)$$

with $\psi(x, 0) = \psi_0(x)$, corresponding to the Hamiltonian operator:

$$H = -\frac{\hbar^2}{2m} \triangle + V(x),$$

where \hbar denotes the reduced Planck constant, m denotes the mass of a particle, and V is some scalar potential such that H is realized as a self-adjoint lower semibounded operator on a dense domain $\mathcal{D}(H) \subset L^2(\mathbb{R}^d)$.

Let us set:

$$u(x, t) := \Re \left[\frac{\nabla \psi(x, t)}{\psi(x, t)} \right] \quad (3.2)$$

$$v(x, t) := \Im \left[\frac{\nabla \psi(x, t)}{\psi(x, t)} \right] \quad (3.3)$$

when $\psi(x, t) \neq 0$ and, otherwise, set both $u(x, t)$ and $v(x, t)$ to be equal to zero. Let us put

$$b(x, t) := u(x, t) + v(x, t) \quad (3.4)$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t)$, $t \geq 0$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, be the evaluation stochastic process $X_t(\omega) = \omega(t)$, with $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the natural filtration.

Carlen ([7, 8, 9]) proved that if $\|\nabla \Psi\|_2^2 < \infty$ then there exists a unique Borel probability measure \mathbb{P} on Ω such that

- i) $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P})$ is a Markov process;
- ii) the image of \mathbb{P} under X_t has a density $\rho(t, x) := \rho_t(x)$, for every $t > 0$;
- iii) $W_t := X_t - X_0 - \int_0^t b(X_s, s) ds$ is a $(\mathbb{P}, \mathcal{F}_t)$ -Brownian Motion.

The continuity problem for the above Nelson-Carlen map (from solutions of Schrödinger equations to probability measures on the path space given by the laws of the corresponding Nelson-Carlen diffusions X_t) is investigated in [17]. For a generalization to the case of Hamiltonian operators with magnetic potential see [48].

From now on we will mainly consider the case where $d = 3$.

We adopt the following notations: capital letters for stochastic processes or, otherwise, we will explicitly specify them, $\hat{Y} = (Y_1, \dots, Y_N)$ to denote arrays in \mathbb{R}^{3N} , $N \in \mathbb{N}$, and bold letters for vectors in \mathbb{R}^3 .

Here we precisely identify the interacting diffusions system rigorously associated to the ground state solution Ψ_N^0 of the Hamiltonian (2.1).

Introducing the probability space $(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t)$, with $\hat{Y}_t(\omega) = \omega(t)$ the evaluation stochastic process, with $\mathcal{F}_t^N = \sigma(Y_s, s \leq t)$ the natural filtration, since $\|\nabla \Psi_N\|^2 \leq \infty$, then by Carlen's Theorem there exists a unique Borel probability measure \mathbb{P}_N such that

- i) $(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t, \mathbb{P}_N)$ is a Markov process;
- ii) the image of \mathbb{P}_N under \hat{Y}_t has density $\rho_N(\mathbf{r})$;
- iii) $\hat{W}_t := \hat{Y}_t - \hat{Y}_0 - \int_0^t b_N(\hat{Y}_s) ds$ where

$$b_N(\hat{Y}_t) := \frac{\nabla^{(N)} \Psi_N^0}{\Psi_N^0} = \frac{1}{2} \frac{\nabla^{(N)} \rho_N}{\rho_N}.$$

The stationary probability measure \mathbb{P}_N with density ρ_N can be alternatively defined as the one of the Markov diffusion process (properly) associated to the Dirichlet form ([3, 24, 25, 39]):

$$\epsilon_{\rho_N}(f, g) := \frac{1}{2} \int_{\mathbb{R}^{3N}} \nabla f(r) \cdot \nabla g(r) \rho_N dr^{3N} \quad f, g \in C_c^\infty(\mathbb{R}^{3N}) \quad (3.5)$$

4. A total variation convergence of the one-particle measure on the path space

In the present section we focus on a convergence problem for the probability measure on the path space corresponding to the one-particle process.

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω is $C(\mathbb{R}^+ \rightarrow \mathbb{R}^3)$, $N \in \mathbb{N}$ and \mathcal{F} is its Borel sigma-algebra as introduced in Section 3. We denote by $\hat{Y} := (Y_1, \dots, Y_N)$ the coordinate process and by \mathcal{F}_t^N the natural filtration.

Let us introduce a process X^{nlS} with invariant density ρ_{nlS} , that is we assume that X^{nlS} is a weak solution of the SDE

$$dX_t^{nlS} := u_{nlS}(X_t^{nlS})dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW_t \quad (4.1)$$

where,

$$u_{nlS} := \frac{1}{2} \frac{\nabla \rho_{nlS}}{\rho_{nlS}}$$

The vector field u_{nlS} is well defined since ρ_{nlS} is continuously differentiable and strictly positive under hypothesis h1) and h2) (see [34, 36]). We denote again by \mathbb{P}_N the measure corresponding to the weak solution of the $3N$ - dimensional stochastic differential equation

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t \quad (4.2)$$

where \hat{Y}_0 is a random variable with probability density equal to ρ_N , while \hat{W}_t is a $3N$ -dimensional \mathbb{P}_N standard Brownian motion.

In this section we use the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$.

We denote by \mathbb{P}_{nlS}^N the measure corresponding to the weak solution of the $3N$ -dimensional stochastic differential equation

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{u}_{nlS}(\hat{Y}_s) ds + \hat{W}_t', \quad (4.3)$$

where

$$\hat{u}_{nlS}(\mathbf{r}_1, \dots, \mathbf{r}_N) = (u_{nlS}(\mathbf{r}_1), \dots, u_{nlS}(\mathbf{r}_N)),$$

\hat{Y}_0 is a random variable with probability density equal to ρ_N and \hat{W}_t' is a $3N$ -dimensional \mathbb{P}_{nlS}^N standard Brownian motion.

Following [41], the next lemma computes the one-particle relative entropy between the three-dimensional *one-particle* non markovian diffusions Y_1 and X^{nlS} .

Lemma 4.1. *Under hypothesis h1) and h2) we have*

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_N} [\|b_1^N(\hat{Y}_s) - u^{nlS}(Y_1(s))\|^2] &= \int \frac{|\nabla \Psi_N|^2}{2} d\mathbf{r}_1 \dots d\mathbf{r}_N - \mu_{nlS} + \\ &+ \int V(\mathbf{r}_1) |\Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N + 2g \int \phi_{nlS}^2(\mathbf{r}_1) |\Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N. \end{aligned} \quad (4.4)$$

Proof. By simple computation and recalling that ϕ_{nlS} is strictly positive and C^2 and hypothesis h1) we have

$$\mathbb{E}_{\mathbb{P}_N} [\|b_1^N(\hat{Y}_s) - u^{nlS}(Y_1(s))\|^2] = \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla_1 \left(\frac{\Psi_N}{\phi_{nlS}} \right) \right|^2 \phi_{nlS}^2 d\mathbf{r}_1 \dots d\mathbf{r}_N$$

where Ψ_N is the ground state of the N body Hamiltonian (2.1).

We now want to prove that $\int_{\mathbb{R}^n} \left| \nabla_1 \left(\frac{\Psi_N}{\phi_{nlS}} \right) \right|^2 \phi_{nlS}^2 d\mathbf{r}_1 \dots d\mathbf{r}_N$ is finite and equal to the right hand side of equation (4.4). In order to prove this, let $\Psi_{R,N}$ be the ground state of Hamiltonian (2.1) restricted on the ball B_R , with radius R and

centre in 0, with Dirichlet boundary condition. Using integration by parts, the fact that $\Psi_{N,R}|_{\partial B_R} = 0$ and the nLS equation (2.6) we obtain

$$\begin{aligned} & \frac{1}{2} \int_{B_R} \left| \nabla_1 \left(\frac{\Psi_{N,R}}{\phi_{nLS}} \right) \right|^2 \phi_{nLS}^2 d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &= \int_{B_R} \left(\frac{|\nabla_1 \Psi_{N,R}|^2}{2} - \nabla_1 \left(\frac{|\Psi_{N,R}|^2}{\phi_{nLS}} \right) \cdot \nabla_1 \phi_{nLS} \right) d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &= \int_{B_R} \frac{|\nabla \Psi_{N,R}|^2}{2} d\mathbf{r}_1 \dots d\mathbf{r}_N + \int_{B_R} V(\mathbf{r}_1) |\Psi_{N,R}|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N + \\ & \quad - \mu_{nLS} + 2g \int \phi_{nLS}^2(\mathbf{r}_1) |\Psi_{N,R}|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N. \end{aligned} \quad (4.5)$$

Using the fact that $|\Psi_{N,R}|^2 \rightarrow |\Psi_N|^2$ weakly and that $\int_{B_R} \frac{|\nabla \Psi_{N,R}|^2}{2} d\mathbf{r}_1 \dots d\mathbf{r}_N \rightarrow \int_{\mathbb{R}^{3N}} \frac{|\nabla \Psi_N|^2}{2} d\mathbf{r}_1 \dots d\mathbf{r}_N$ as $R \rightarrow +\infty$ the lemma is proved. \square

Remark 4.2. Lemma 4.1 has two important consequences. First of all, since Ψ_N^0 is the minimizer of $E^N[\Psi]$, the following finite energy conditions hold:

$$\mathbb{E}_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty \quad (4.6)$$

$$\mathbb{E}_{\mathbb{P}_N} \int_0^t \|\hat{u}_s^{nLS}\|^2 ds < \infty. \quad (4.7)$$

Furthermore, by Theorem 2.4, we obtain

$$\mathbb{E}_{\mathbb{P}_N} [\|b_1^N(\hat{Y}_s) - u^{nLS}(Y_1(s))\|^2] \rightarrow 0$$

as $N \rightarrow \infty$.

Lemma 4.3. *The one-particle (or normalized) relative entropy is given by*

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nLS}^N)|_{\mathcal{F}_t} = \frac{1}{2} \mathbb{E}_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{nLS}(Y_1(s))\|^2 ds \quad (4.8)$$

Proof. The proof is similar to the one proposed for GP limit in [41]. For this reason we report here only a sketch.

The inequalities (4.6) and (4.7) are *finite entropy conditions* (see, e.g. [23]) which imply that $\forall t > 0$

$$\mathbb{P}_N|_{\mathcal{F}_t} \ll \hat{W}|_{\mathcal{F}_t}, \quad \mathbb{P}_{nLS}^N|_{\mathcal{F}_t} \ll \hat{W}'|_{\mathcal{F}_t}$$

(where \ll stands for absolute continuity) By Girsanov's theorem, we have, for all $t > 0$,

$$\left. \frac{d\mathbb{P}_N}{d\mathbb{P}_{nLS}^N} \right|_{\mathcal{F}_t} = \exp \left\{ - \int_0^t (\hat{b}_s^N - \hat{u}_s^{nLS}) \cdot d\hat{W}_s + \frac{1}{2} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{nLS}\|^2 ds \right\}, \quad (4.9)$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^{3N} . The relative entropy restricted to \mathcal{F}_t reads

$$\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{nls}^N)|_{\mathcal{F}_t} =: \mathbb{E}_{\mathbb{P}_N} \left[\log \frac{d\mathbb{P}_N}{d\mathbb{P}_{nls}^N} \Big|_{\mathcal{F}_t} \right] = \frac{1}{2} \mathbb{E}_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{nls}\|^2 ds \quad (4.10)$$

Since under \mathbb{P}_N the $3N$ -dimensional process \hat{Y} is a solution of (4.2) with invariant probability density ρ_N , we can write, recalling also (4.6) and (4.7), and by using the symmetry of \hat{b}^N and ρ_N

$$\begin{aligned} \mathcal{H}(\mathbb{P}_N, \mathbb{P}_{nls}^N)|_{\mathcal{F}_t} &= \\ &= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \sum_{i=1}^N \|b_i^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{nls}(\mathbf{r}_i)\|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N = \\ &= \frac{1}{2} N t \int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{nls}(\mathbf{r}_1)\|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N = \\ &= \frac{1}{2} N t \mathbb{E}_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u_{nls}(Y_1(s))\|^2 ds, \quad (4.11) \end{aligned}$$

By defining the one-particle relative entropy as the *normalized relative entropy* we obtain

$$\begin{aligned} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nls}^N)|_{\mathcal{F}_t} &=: \frac{1}{N} \mathcal{H}(\mathbb{P}_N, \mathbb{P}_{nls}^N)|_{\mathcal{F}_t} = \\ &= \frac{1}{2} \mathbb{E}_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u_{nls}(Y_1(s))\|^2 ds \quad (4.12) \end{aligned}$$

□

By Theorem 2.4 we deduce that for any $t > 0$ the one particle relative entropy converges to zero in the scaling limit.

Remark 4.4. In the more complicated GP scaling limit the one-particle relative entropy does not converge to zero but to a finite constant. This fact has important consequences for the convergence of the one-particle probability measure. See [19] for the proof of the existence of the limit probability measure, [4] for the proof of weak convergence of the one-particle process and [42] for the localization phenomenon of the relative entropy.

The proof the theorem takes advantage of the following lemma, which represents a useful chain-rule of the relative entropy when the reference measure is a *product one*.

Lemma 4.5. *We consider $M = X \times Y$, where X and Y are Polish spaces. Let \mathbb{P} be a measure on M and \mathbb{Q}_1 and \mathbb{Q}_2 probability measures on X and Y respectively. We denote by $\mathbb{Q} = \mathbb{Q}_1 \otimes \mathbb{Q}_2$ the product measure on M of the measures \mathbb{Q}_1 and \mathbb{Q}_2 and we suppose that $\mathbb{P} \ll \mathbb{Q}$. Then we have*

$$\mathcal{H}(\mathbb{P}|\mathbb{Q}) \geq \mathcal{H}(\mathbb{P}_1|\mathbb{Q}_1) + \mathcal{H}(\mathbb{P}_2|\mathbb{Q}_2), \quad (4.13)$$

where \mathbb{P}_1 and \mathbb{P}_2 are the marginal probabilities of \mathbb{P} .

Proof. The proof can be found in Lemma 5.1 of [19]. \square

Theorem 4.6 (Total variation convergence on the path space). *Under the same hypothesis h1),h2) as in Theorem 2.2, the one-particle measure \mathbb{P}_N^1 converges in total variation to \mathbb{P}_{nlS} , the latter being uniquely associate to the non linear Schroedinger functional.*

Proof. By Lemma 4.3 the one-particle relative entropy reads

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nlS}^N)|_{\mathcal{F}_t} = \frac{1}{2} \mathbb{E}_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{nlS}(Y_1(s))\|^2 ds \quad (4.14)$$

and by Remark 4.2 we obtain

$$\lim_{N \uparrow \infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nlS}^N)|_{\mathcal{F}_t} = \frac{t}{2} \lim_{N \uparrow \infty} \mathbb{E}_{\mathbb{P}_N} [\|b_1^N(\hat{Y}_s) - u^{nlS}(Y_1(s))\|^2] = 0 \quad (4.15)$$

Let us introduce the total variation distance between the one-particle measure \mathbb{P}_N^1 and \mathbb{P}_{nlS} :

$$d_{TV}(\mathbb{P}_N^1, \mathbb{P}_{nlS})|_{\mathcal{F}_t} = \sup_{A \in \mathcal{F}_t} |\mathbb{P}_N^1(A) - \mathbb{P}_{nlS}(A)| = \sup_{A \in \mathcal{F}_t} \left| \int_A \left(\frac{d\mathbb{P}_N^1}{d\mathbb{P}_{nlS}} - 1 \right) d\mathbb{P}_{nlS} \right| \quad (4.16)$$

By the well-known Csiszar-Kullback inequality ([16],[30]), which is valid in arbitrary Polish spaces,

$$d_{TV}(\mathbb{P}_N^1, \mathbb{P}_{nlS})|_{\mathcal{F}_t} \leq \sqrt{2\mathcal{H}(\mathbb{P}_N^1, \mathbb{P}_{nlS})|_{\mathcal{F}_t}}. \quad (4.17)$$

By using Lemma 4.5 we can write

$$\mathcal{H}(\mathbb{P}^N | \mathbb{P}_{nlS}^N) \geq \mathcal{H}(\mathbb{P}_N^1 | \mathbb{P}_{nlS}^1) + \mathcal{H}(\mathbb{P}_N^{N-1} | \mathbb{P}_{nlS}^{N-1}), \quad (4.18)$$

and by repeating this procedure we obtain

$$\mathcal{H}(\mathbb{P}_N^1, \mathbb{P}_{nlS})|_{\mathcal{F}_t} \leq \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nlS}^N)|_{\mathcal{F}_t}. \quad (4.19)$$

From equation (4.15) and inequalities (4.19), (5.12) we deduce that the sequence \mathbb{P}_N^1 converges in total variation to \mathbb{P}_{nlS} . \square

5. Strong Kac's chaos on path space

It is well known that the (usual) Kac's chaos is implied by statements of the kind given in the following theorem:

Theorem 5.1 (Kac's chaos on path space). *$\forall T > 0$ fixed, $\forall i = 1, 2, \dots, N$ we have*

$$\lim_{N \uparrow +\infty} E_{\mathbb{P}_N} [\sup_{t \leq T} |Y_t^1 - Y_t|^2] = 0 \quad (5.1)$$

Proof. Let T be a fixed time. For all $t \leq T$, by exploiting (4.2) and (4.3), with Y_t^1 and Y_t the first components of \hat{Y}_t in (4.2) and (4.3) respectively, we have

$$\sup_{t \leq T} |Y_t^1 - Y_t|^2 \leq \left| \int_0^T (b_1^N(\hat{Y}_s) - u_{nlS}(Y_s^1)) ds \right|^2 \quad (5.2)$$

and by taking the expectation with respect to the measure \mathbb{P}_N we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_N}[\sup_{t \leq T} |Y_t^1 - Y_t|^2] &\leq \mathbb{E}_{\mathbb{P}_N} \left[\left| \int_0^T (b_1^N(\hat{Y}_s) - u_{nlS}(Y_s^1)) ds \right|^2 \right] = \\
&= \mathbb{E}_{\mathbb{P}_N} \left[T^2 \left(\frac{1}{T} \int_0^T (b_1^N(\hat{Y}_s) - u_{nlS}(Y_s^1)) ds \right)^2 \right] \leq \\
&\leq \mathbb{E}_{\mathbb{P}_N} \left[T^2 \frac{1}{T} \int_0^T (b_1^N(\hat{Y}_s) - u_{nlS}(Y_s^1))^2 ds \right] = \\
&= T^2 \mathbb{E}_{\mathbb{P}_N} [(b_1^N(\hat{Y}_s) - u_{nlS}(Y_s^1))^2] \quad (5.3)
\end{aligned}$$

where we have applied Jensen's inequality and we have taken into account the stationarity of both our systems. By Remark 4.2 we deduce the statement of the theorem. \square

Remark 5.2. The Kac's chaos on path space implies the Kac's chaos for the fixed time marginal probability densities, that is for any fixed $k \in \mathbb{N}$, the measures $\rho_N^k(\mathbf{r}_1, \dots, \mathbf{r}_N) d\mathbf{r}_1 \dots d\mathbf{r}_N$ converges weakly to $\rho^{\otimes k}(\mathbf{r}) d\mathbf{r}$. The latter can be derived by the complete Bose-Einstein Condensation (see [53]).

Let us denote by \mathbb{P}_N^k the measure \mathbb{P}_N restricted with respect to the σ -algebra generated by (any) k particles (due to the symmetry).

We prove a result which states that when the normalized relative entropy goes to zero, also the relative entropy between any k marginal probability measures goes to zero.

Lemma 5.3. *If for all fixed $t > 0$, when $N \uparrow +\infty$*

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nlS}^N)|_{\mathcal{F}_t} \rightarrow 0 \quad (5.4)$$

then, for every fixed $k \in \mathbb{N}$,

$$\lim_{N \uparrow +\infty} \mathcal{H}(\mathbb{P}_N^k, \mathbb{P}_{nlS}^k)|_{\mathcal{F}_t} = 0 \quad (5.5)$$

Proof. We prove the statement by induction on k . By simplicity of notations we do not write the restriction to \mathcal{F}_t . The assertion is true for $k = 1$ as we have shown in (4.19).

Let us write $N = kN_k + r_k$ and suppose that the statement is true for any $r_k < k$. Applying Lemma 4.5 we have

$$\mathcal{H}(\mathbb{P}_N | \mathbb{P}_{nlS}^N) \geq N_k \mathcal{H}(\mathbb{P}_N^k | \mathbb{P}_{nlS}^k) + \mathcal{H}(\mathbb{P}_N^{r_k} | \mathbb{P}_{nlS}^{r_k}), \quad (5.6)$$

which implies:

$$\begin{aligned}
\mathcal{H}(\mathbb{P}_N^k | \mathbb{P}_{nlS}^k) &\leq \frac{(k+1)}{N_k} \left\{ \mathcal{H}(\mathbb{P}_N | \mathbb{P}_{nlS}^N) + \sum_{r=1}^{k-1} \mathcal{H}(\mathbb{P}_N^r | \mathbb{P}_{nlS}^r) \right\} \\
&= (k+1) \frac{N}{N_k} \bar{\mathcal{H}}(\mathbb{P}_N | \mathbb{P}_{nlS}^N) + \frac{(k+1)}{N_k} \sum_{r=1}^{k-1} \mathcal{H}(\mathbb{P}_N^r | \mathbb{P}_{nlS}^r) \quad (5.7)
\end{aligned}$$

Since $N \uparrow +\infty$ if and only if $N_k \uparrow +\infty$, by induction hypothesis we obtain the result. \square

Theorem 5.4 (A strong form of Kac's chaos on path space). *Under the same hypothesis h_1, h_2) as in Theorem 2.2, for all fixed $t > 0, \forall k \in \mathbb{N}$ we have*

$$\lim_{N \uparrow +\infty} d_{TV}(\mathbb{P}_N^k, \mathbb{P}_{nIS}^k)|_{\mathcal{F}_t} = 0 \quad (5.8)$$

Proof. Since by Lemma 4.3

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{nIS}^N)|_{\mathcal{F}_t} = \frac{1}{2} \mathbb{E}_{\mathbb{P}_N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{nIS}(Y_1(s))\|^2 ds \quad (5.9)$$

by Remark 4.2 we obtain that

$$\lim_{N \uparrow +\infty} \bar{\mathcal{H}}(\mathbb{P}_N^k, \mathbb{P}_{nIS}^k) = 0 \quad (5.10)$$

By Lemma 5.3 it follows that for every fixed k ,

$$\lim_{N \uparrow +\infty} \mathcal{H}(\mathbb{P}_N^k, \mathbb{P}_{nIS}^k) = 0 \quad (5.11)$$

Since by the Csiszar-Kullback inequality we have:

$$d_{TV}(\mathbb{P}_N^k, \mathbb{P}_{nIS}^k)|_{\mathcal{F}_t} \leq \sqrt{2\mathcal{H}(\mathbb{P}_N^k, \mathbb{P}_{nIS}^k)|_{\mathcal{F}_t}}, \quad (5.12)$$

we obtain the result. \square

6. Fisher information chaos in Bose-Einstein Condensation

In this section we consider the symmetric probability law G^N of our N interacting diffusions on the product space \mathbb{R}^{3N} , which are absolutely continuous with density $\rho_N := |\Psi_N^0|^2$.

First we define the Fisher information associated to a probability measure G on a space \mathbb{R}^{3n} .

Definition 6.1. For $G \in W^{1,1}(S^n)$ we put

$$I_n(G) := \int_{\mathbb{R}^{3n}} \frac{|\nabla G|^2}{G}$$

otherwise equal to $+\infty$. We consider the normalized Fisher information $I := \frac{1}{n}I_n$

The Fisher information has the following important properties:

- (1) I is proper, convex, l.s.c. (in the sense of the weak convergence of measures) on $\mathcal{P}(\mathbb{R}^{3n})$ (the space of probability measures on S^n).
- (2) (a) For $1 \leq l \leq n$, $I_l(G_l) \leq I(G)$ where $G \in \mathcal{P}(\mathbb{R}^{3n})$
 (b) The (non normalized) Fisher information is super-additive, i.e.

$$I_n(G) \geq I_l(G_l) + I_{n-l}(G_{n-l})$$

with (in the case $I_l(G_l) + I_{n-l}(G_{n-l}) < +\infty$) equality if and only if $G = G_l G_{n-l}$

- (c) If $I(G_1) < +\infty$, the equality $I(G_1) = I(G)$ holds if and only if $G = (G_1)^{\otimes j}$

Proof. For 1) see, e.g., [27], Lemma 3.5. For 2) see [12] or [27], Lemma 3.7. \square

Proposition 6.2 (Fisher's chaos in mean-field BEC). *There is Fisher's chaos in the generic mean-field Bose-Einstein Condensation.*

Proof. The Fisher information associated to the symmetric law \mathbb{P}_N is:

$$\begin{aligned} I(G_N) &= \frac{1}{N} \int_{\mathbb{R}^{3N}} \left| \frac{\nabla \rho_N}{\rho_N} \right|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N = \frac{4}{N} \int_{\mathbb{R}^{3N}} \left| \frac{\nabla \rho_N}{2\rho_N} \right|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &= \frac{1}{N} \int_{\mathbb{R}^{3N}} |\nabla \Psi_N|^2 d\mathbf{r}_1 \dots d\mathbf{r}_N \end{aligned}$$

By Theorem 2.4, in particular by (2.32), we get

$$\lim_{N \uparrow \infty} I(G_N) = I(G_{nls})$$

Moreover, by Theorem 2.4, we also have that $\rho_N^{(1)}$ converges to ρ_{nls} weakly in $\mathcal{P}(S)$ as $N \rightarrow +\infty$ and so we have that Fisher's chaos holds. \square

Proposition 6.3 (Entropy chaos in mean-field BEC). *Entropy chaos holds in the generic mean-field Bose-Einstein Condensation.*

Proof. The proof of the fact that the Fisher's information chaos implies entropy chaos and Kac's chaos can be found in [27]. \square

Acknowledgements

The first and second authors would like to thank the Department of Mathematics, Università degli Studi di Milano for the warm hospitality. The second author is supported by the German Research Foundation (DFG) via SFB 1060.

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