

THE STRUCTURE OF NASH EQUILIBRIA IN POISSON GAMES*

CLAUDIA MERONI[†] AND CARLOS PIMIENTA[‡]

ABSTRACT. We show that many results on the structure and stability of equilibria in finite games extend to Poisson games. In particular, the set of Nash equilibria of a Poisson game consists of finitely many connected components and at least one of them contains a stable set (De Sinopoli et al. [1]). In a similar vein, we prove that the number of Nash equilibria in Poisson voting games under plurality, negative plurality, and (when there are at most three candidates) approval rule, as well as in Poisson coordination games, is generically finite. As in finite games, these results are obtained exploiting the geometric structure of the set of Nash equilibria which, in the case of Poisson games, is shown to be semianalytic.

KEY WORDS. Poisson games, voting, stable sets, generic determinacy of equilibria, o-minimal structures.

JEL CLASSIFICATION. C70, C72.

1. INTRODUCTION

Games with population uncertainty and, in particular, Poisson games (Myerson [2]), have been proposed to model economic scenarios where it is more reasonable to assume that agents only have probabilistic information about the number of players. Poisson games have been primarily used to study voting games (for a very incomplete list, see Myerson [3], Bouton and Castanheira [4], Bouton and Gratton [5], Bouton [6], Gratton [7], Huges [8]) but they also have been proven useful in more general economic environments where the number of economic agents is uncertain (see, e.g., Satterthwaite and Shneyerov [9], Makris [10, 11], Ritzberger [12], McLennan [13], Jehiel and Lamy [14]). Similarly to finite games, Poisson games can generate multiple equilibrium outcomes and, moreover, some of them may be only generated by equilibria that are not a plausible description of rational behavior. The economic literature has typically addressed the issue of multiplicity of equilibrium outcomes investigating conditions that guarantee they are at least locally unique (Arrow and Hahn [15,

[†] DEPARTMENT OF ECONOMICS, UNIVERSITY OF VERONA, VERONA, ITALY.

[‡] SCHOOL OF ECONOMICS, THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY, AUSTRALIA.
Email addresses: claudia.meroni@univr.it, c.pimienta@unsw.edu.au.

Corresponding author: Carlos Pimienta.

Date: February 15, 2017.

Chapter 9)). On the other hand, the literature on equilibrium refinements has helped reducing the indeterminacy of game-theoretical models by eliminating implausible equilibria. In this paper, we expand the analysis of the determinacy of equilibria to Poisson games.

Local uniqueness of equilibrium outcomes in a theoretical model is a basic requirement from an applied standpoint. Otherwise, the theory would be too ambiguous in its description of what to expect from the economic agents. Furthermore, without such a property when performing comparative statics, small variations in the environment can lead to abrupt changes in behavior and in the ensuing outcomes. Local uniqueness of equilibrium outcomes is equivalent to finiteness if the strategy space is compact, which is typically the case. Debreu [16] argues that even if it is too demanding a requirement that a model has finitely many equilibrium outcomes for *every* possible description of the environment, in some cases, it is possible to show that “most” of them satisfy such a finiteness result. Harsanyi [17] initiated the analysis of the determinacy of equilibrium in game theory and proved that, for generic assignments of utilities to strategy profiles, every normal form game has finitely many equilibria. This result is of limited significance because in many game-theoretical models (such as voting games) many different strategy profiles lead to the same outcome and, therefore, lead to non-generic games in the space of normal form payoffs. Henceforth, the literature has provided analogous results for extensive-form games (Kreps and Wilson [18]), sender-receiver games (Park [19]), voting games (De Sinopoli [20]), network formation games (Pimienta [21]) and other families of finite games (Govindan and McLennan [22]). We show how many of these results can be extended to Poisson games. In particular, Poisson voting games under plurality, negative plurality, and approval rules (the latter when the number of candidates is no larger than three), as well as Poisson coordination games, have finitely many Nash equilibria for generic utilities over the relevant outcome space.

Generic determinacy results also have a practical implication for strategic stability, which has proved a powerful tool to characterize rational behavior in economic models. In finite games, the set of Nash equilibria consists of finitely many connected components. This is a key result towards a satisfactory theory of equilibrium refinements that conceives a solution to a game as a set-valued object (Kohlberg and Mertens [23], Mertens [24]). If there are finitely many equilibrium outcomes then every point in an equilibrium component necessarily induces the same outcome. In these cases, a set-valued solution concept contained in a connected component of equilibria constitutes a minor departure

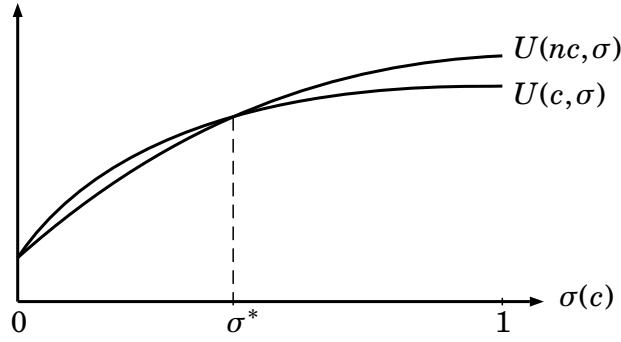


FIGURE 1. The two Nash equilibria of the contribution game.

from a classical single-valued one, as all the points in the same connected component can be considered equivalent. Motivated by this, in Section 3 we prove that every Poisson game has finitely many Nash equilibrium components and that at least one of them contains a stable set.

Stable sets in Poisson games are introduced in De Sinopoli et al. [1] where it is shown that they satisfy existence, admissibility, and robustness against elimination of dominated actions and inferior replies. A stable set in a Poisson game is defined as a set of equilibria that is robust against a suitably chosen family of perturbations. A perturbation in this family is not a strategy tremble as in Kohlberg and Mertens [23], but rather, consists of pushing with vanishing probability the population's average behavior towards a completely mixed measure on the probability simplex over the set of actions in the game. [1] show that such perturbations generate associated utility-perturbed Poisson games. These perturbed games are obtained by adding, for each action available to each type, a constant payoff to the original utility of playing such an action. This constant is equal to a vanishing fraction of the integral of the expected utility function associated with that action with respect to the completely mixed probability measure of the perturbation (see the Appendix for the formal definition.)

To get a feeling for the definition of stability in Poisson games and its implications, consider a simple contribution to a public good game. Suppose that the expected number of players is $n > 0$, that each player is endowed with one dollar, and that players can either commit to contribute (c) or not contribute (nc) to the provision of a public good that costs k dollars, where $k < n$. If less than k players are willing to contribute then the public good is not provided and no player pays anything. If k or more players are willing to contribute then the public good is provided and each of those players contributes equally to the cost. In terms of preferences, let us assume that players derive a utility from the public good that is sufficiently higher than the utility they derive from the dollar.

Figure 1 represents a player's expected utilities under the two actions as functions of the probability $\sigma(c)$ that the average member of the population chooses to contribute (recall that, in a Poisson game, players of the same type play the same randomization over actions and that this game has only one type). Note that no action is dominated. If nobody is willing to contribute then either action leads to the same outcome and the player keeps her dollar anyway. On the other hand, if every player in the population is willing to contribute then the public good is provided with high enough probability so that the player prefers to free ride and not contribute to the public good. This game has two equilibria, an inefficient equilibrium in which no player contributes to the public good and an efficient equilibrium where players commit to contribute with probability σ^* and are indifferent between contributing or not to the public good. The first equilibrium is not stable because it is not robust against every perturbation. In particular, take a perturbation whose probability measure assigns most probability to the segment of the simplex where the utility to contributing is larger than the utility to not contributing. Under this measure, the expected utility function of contributing integrates strictly more than the expected utility function of not doing so. Hence, the associated utility-perturbed game adds a constant to the utility of the first action strictly larger than the constant added to the utility of the second. And this perturbed game has no equilibrium close to the equilibrium of the true game where no player contributes. We conclude that the unique stable set of this game consists of the efficient equilibrium σ^* .

In finite games, most game-theoretical constructions, such as the set of Nash equilibria, generate *semialgebraic sets*, i.e., sets defined by finite systems of polynomial inequalities. This implies that the set of Nash equilibria of every game consists of finitely many connected components and at least one of them contains a stable set (Kohlberg and Mertens [23]). Moreover, powerful results of semialgebraic geometry such as the *Generic Local Triviality Theorem* (Hardt [25], Bochnak et al. [26]) are used to prove most of the generic determinacy results mentioned above.

In Poisson games, the uncertainty about the number of opponents is factored into players' expected utility functions. One consequence is that expected utilities are no longer polynomials and that the tools of semialgebraic geometry are not directly applicable. Nonetheless, utility functions in Poisson games are real analytic functions and, correspondingly, the Nash equilibrium conditions define a bounded *semianalytic set* (that is, a bounded set that can be locally given by the solution of a finite system of analytic inequalities). Similarly to the semialgebraic case, semianalytic sets have a special structure that, e.g., allows us to establish that every Poisson game has finitely many connected components of

equilibria and that each such component is itself a semianalytic set.¹ This also implies that every Poisson game has a stable set contained in one connected component of Nash equilibria. Furthermore, most of the properties of semialgebraic sets that are useful to establish game-theoretical results in finite games can be derived from a set of axioms that define *o-minimal structures* (van den Dries [29]).² Examples of o-minimal structures are the collection of semialgebraic sets and, as showed by van den Dries [33], the collection of *globally subanalytic* sets.³ Every bounded semianalytic set is globally subanalytic so we can use the general version of the Generic Local Triviality Theorem for o-minimal structures to prove the generic finiteness of Nash equilibria in some relevant Poisson games.

In the next section we review the general description of Poisson games. We discuss the geometric structure of the Nash equilibrium set in Section 3 and prove that stable sets in Poisson games satisfy the same version of connectedness as Kohlberg and Mertens [23] stable sets of finite games. We give a quick review of o-minimal structures in Section 4 and use some of their basic properties in Section 5 to establish generic finiteness results.

2. POISSON GAMES

We adopt the same notation used in De Sinopoli et al. [1], where the description of Poisson games closely follows the one introduced by Myerson [2].

A *Poisson game* is a tuple $\Gamma := (n, \mathcal{T}, r, (C_t)_{t \in \mathcal{T}}, \Omega, \theta, v)$. The number of *players* is a Poisson random variable with parameter n . Given n , the probability that there are k players in the game is

$$P(k | n) = \frac{e^{-n} n^k}{k!}.$$

The set $\mathcal{T} = \{1, \dots, T\}$ is the non-empty finite set of possible *types* of players. A player is of type $t \in \mathcal{T}$ with probability r_t . The probabilities that a player is of each type are listed in the vector $r = (r_1, \dots, r_T) \in \Delta(\mathcal{T})$.⁴

We let C_t be the finite set of *actions* that are available to players of type t . The set of all actions is $C := \bigcup_t C_t$. An *action profile* $x \in Z(C)$ specifies for each

¹ Furthermore, using a similar construction to Balkenborg and Vermeulen [27] and combining their results with Łojasiewicz [28], we know that every connected and compact semianalytic set is homeomorphic to a connected component of Nash equilibria of some Poisson game (see the Online Appendix).

² In the economic literature, Blume and Zame [30] apply the properties of o-minimal structures to general equilibrium theory to identify a class of preferences such that, for generic endowments, the corresponding economies have finitely many equilibria (see also Richter and Wong [31]). Bolte et al. [32] use o-minimality to study zero-sum stochastic games.

³ Sometimes also called *finitely subanalytic* sets.

⁴ For any set S , we write $\Delta(S)$ for the set of probability distributions on S with finite support.

action $c \in C$ the number of players $x(c)$ who choose that action. The set of action profiles is $Z(C) := \mathbb{Z}_+^C$.

A player of type $t \in \mathcal{T}$ who chooses action $c \in C_t$ when the action profile of her opponents is $x \in Z(C)$ induces some *outcome* that belongs to the *outcome set* Ω . This information is specified by the *outcome function* $\theta : \mathcal{T} \times C \times Z(C) \rightarrow \Delta(\Omega)$. Note that, e.g., the outcome function can be the identity function as in Myerson [2].

The utility vector $v = (v_1, \dots, v_T)$ summarizes players' preferences over outcomes. Each entry v_t is a bounded function $v_t : \Omega \rightarrow \mathbb{R}$. The utility function that a player of type t has over elements in $C \times Z(C)$ is computed according to $u_t(c, x; v_t) = \sum_{\omega \in \Omega} \theta(t, c, x)(\omega) v_t(\omega)$.

The set $\Delta(C_t)$ is the set of *mixed actions* for type t players. We identify the mixed action that assigns probability one to action c with the pure action $c \in C$. A *strategy function* $\sigma = (\sigma_1, \dots, \sigma_T)$ is a function from \mathcal{T} to $\Delta(C)$ that satisfies $\sigma_t \in \Delta(C_t)$ for all $t \in \mathcal{T}$, i.e. a mapping from the set of types to the set of mixed actions available to each corresponding type. We let Σ denote the set of all strategy functions. We may sometimes refer to strategy functions simply as *strategies*. Let $\tilde{\tau}(\sigma) \in \Delta(C)$ be the population's "average" behavior induced by the strategy σ , which is given by $\tilde{\tau}(\sigma)(c) := \sum_{t \in \mathcal{T}} r(t) \sigma_t(c)$. Moreover, we define the set $\tilde{\tau}(\Sigma) := \{\tau \in \Delta(C) : \tau = \tilde{\tau}(\sigma) \text{ for some } \sigma \in \Sigma\}$. When the population's average behavior is given by $\tau \in \tilde{\tau}(\Sigma)$, the probability that the action profile $x \in Z(C)$ is realized is equal to

$$\mathbf{P}(x \mid \tau) := \prod_{c \in C} \left(e^{-n\tau(c)} \frac{(n\tau(c))^{x(c)}}{x(c)!} \right).$$

The expected payoff to a player of type t who plays action $c \in C_t$ is then

$$U_t(c, \tau; v_t) := \sum_{x \in Z(C)} \mathbf{P}(x \mid \tau) u_t(c, x; v_t). \quad (2.1)$$

From now on we fix $n, \mathcal{T}, r, (C_t)_{t \in \mathcal{T}}, \Omega$, and θ . A Poisson game is then given by a utility vector $v \in \mathbb{R}^{\#\Omega T}$. We denote such a game $\Gamma(v)$.

3. THE SET OF NASH EQUILIBRIA

Definition 1. The strategy function $\sigma \in \Sigma$ is a *Nash equilibrium* of the Poisson game $\Gamma(v)$ if

$$U_t(\sigma_t, \tau(\sigma); v_t) \geq U_t(\sigma'_t, \tau(\sigma); v_t) \quad \text{for all } t \in \mathcal{T}, \sigma'_t \in \Delta(C_t). \quad (3.1)$$

The Nash equilibrium conditions define a system of equalities and inequalities. In finite games, such a system only involves polynomial functions. This defines a semialgebraic set which, in turn, is homeomorphic to a finite simplicial complex (van der Waerden [34]). This fact was used by Kohlberg and

Mertens [23] to show that, for any game, the set of Nash equilibria consists of finitely many connected components. In a Poisson game, the Nash equilibrium conditions (3.1) are clearly not polynomial inequalities. However, Lemma 1 below shows that the expected utility functions are real analytic (i.e., functions that are locally given by a convergent power series) and, therefore, the Nash equilibrium conditions define a *semianalytic set*.

Definition 2. A set $X \subset \mathbb{R}^m$ is *semianalytic* if for each $y \in \mathbb{R}^m$ there is an open neighborhood O such that $O \cap X$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^m : f(x) = 0 \text{ and } g_1(x) > 0, \dots, g_k(x) > 0\}$$

where f, g_1, \dots, g_k are real analytic functions on O .

By definition, the class of semianalytic sets includes also sets defined by weak inequalities. This class is closed under finite union, finite intersection, finite product, and complementation. It is also the case that every compact semianalytic set is homeomorphic to a finite simplicial complex (Łojasiewicz [28]). Therefore, like in the semialgebraic case, compact semianalytic sets also have finitely many connected components.

We now show that expected utility functions in a Poisson game are indeed real analytic.

Lemma 1. *Given a Poisson game $\Gamma(v)$, for every type $t \in \mathcal{T}$ and every action $c \in C_t$, the expected utility function $U_t(c, \cdot; v_t)$ is real analytic.*

Proof. For each $x \in Z(C)$, define the following function with domain \mathbb{R}^C ,

$$U_t^x(c, \tau; v_t) := \mathbf{P}(x \mid \tau) u_t(c, x; v_t).$$

This function is the finite product and composition of real analytic functions with infinite domain of convergence (i.e. polynomials and the exponential functions). Hence, $U_t^x(c, \tau; v_t)$ is also real analytic with infinite domain of convergence. Given a multi-index $\mu \in \mathbb{N}_+^C$ let us write $\tau^\mu := \prod_{c \in C} \tau(c)^{\mu_c}$. The function $U_t^x(c, \tau; v_t)$ can be written as a power series centered at zero $\sum_{\mu \in \mathbb{N}_+^C} a_\mu^x \tau^\mu$. Thus, for every $\tau \in \mathbb{R}^C$,

$$U_t(c, \tau; v_t) = \sum_{x \in Z(C)} U_t^x(c, \tau; v_t) = \sum_{x \in Z(C)} \sum_{\mu \in \mathbb{N}_+^C} a_\mu^x \tau^\mu = \sum_{\mu \in \mathbb{N}_+^C} \sum_{x \in Z(C)} a_\mu^x \tau^\mu = \sum_{\mu \in \mathbb{N}_+^C} A_\mu \tau^\mu.$$

For each $\mu \in \mathbb{N}_+^C$ the value of A_μ is necessarily bounded because the utility function is well-defined. Therefore, $U_t(c, \tau; v_t)$ can be written down as a power series centered at zero with domain of convergence \mathbb{R}^C , so it is a real analytic function. \square

Given our previous discussion, an immediate consequence of this result is the analogue of Proposition 1 in Kohlberg and Mertens [23].

Theorem 1. *The set of Nash equilibria of a Poisson game $\Gamma(v)$ is a compact semianalytic set which, therefore, has finitely many connected components. At least one of those components is such that, for every Poisson game $\Gamma(\hat{v})$ sufficiently close to $\Gamma(v)$, there is a Nash equilibrium of $\Gamma(\hat{v})$ close to it.*

Proof. Lemma 1 implies that the set of Nash equilibria is a semianalytic set. Since it is also compact, it is made of finitely many connected components.

For the second part, we prove a stronger result. Recall that the set of fixed points N of an upper semi-continuous convex-valued correspondence F is an *essential* set of fixed points. That is, for every open $W \supset N$ there is a neighborhood O of the graph of F such that every upper semi-continuous convex-valued correspondence G whose graph is a subset of O has a fixed point in W . Kinoshita [35] proves that if a set of fixed points N is essential and $\{N_1, \dots, N_k\}$ is a partition of N into disjoint compact sets, then some N_j is essential.⁵ Since the best response correspondence is upper semi-continuous and convex valued, and we can choose those compact sets to be connected, every Poisson game has a connected component of Nash equilibria that is essential. \square

A Kohlberg and Mertens [23] *stable set* of a finite game is a minimal (in terms of set inclusion) subset of Nash equilibria such that every close-by game that can be generated by pure-strategy perturbations has a Nash equilibrium close to it. Since robustness is only required against strategy perturbations, every member of a stable set is a perfect (hence undominated) equilibrium. Even if Kohlberg and Mertens [23, p. 1020] list *connectedness* (i.e. every solution should be connected) as one of the main requirements that a set-valued solution concept should satisfy, the form of the definition allows for stable sets that are disconnected. It is also possible that a stable set is made of Nash equilibria that belong to different components.⁶ Nevertheless, stable sets satisfy a weaker version of connectedness. Namely, every game has a stable set which is contained in a single connected component of the set of Nash equilibria.

De Sinopoli et al. [1] propose the analogous definition of stability for Poisson games. As discussed in the Introduction, for any given Poisson game $\Gamma(v)$ we can construct a suitable set of perturbed Poisson games $P(v)$ so that robustness

⁵ See also McLennan [36, Theorem 8.3.2].

⁶ Alternative definitions of stability have been proposed (Hillas [37], Mertens [24, 38]) such that every solution is connected.

against elements in $P(v)$ guarantees that players choose only admissible strategies in $\Gamma(v)$.⁷ Then, a stable set of the Poisson game $\Gamma(v)$ is a minimal (in terms of set inclusion) subset of Nash equilibria such that every close-by game in $P(v)$ has a Nash equilibrium close to the stable set.

Again, we can show that the situation in Poisson games parallels that of finite games. A consequence of Theorem 1 is that, even if not every stable set is connected, every Poisson game has a stable set that is contained in a single connected component of equilibria. (For completeness, the formal definition of stability in Poisson games, as well as an example illustrating the connectedness issue, is contained in the Appendix. We also defer to the Appendix the proof of the following Proposition.)

Proposition 1. *Every Poisson game $\Gamma(v)$ has a stable set contained in a connected component of equilibria. Moreover, every Poisson game has a minimal connected set of Nash equilibria that is robust against every perturbation in $P(v)$.*

From an applied standpoint, a connected component of equilibria (or a stable set contained in it) does not substantially differ from a single-valued equilibrium concept as long as every Nash equilibrium in such a component induces the same probability distribution over outcomes. This is the case in generic finite normal form games (Harsanyi [17]) and in generic finite extensive form games (Kreps and Wilson [18]). Govindan and Wilson [39] show that these results can be easily derived from some basic properties of semialgebraic sets and functions. Such properties are shared by some *o-minimal structures*.

4. O-MINIMAL STRUCTURES ON \mathbb{R}

The objective of this section is to provide the necessary background to show that the graph of the Nash equilibrium correspondence and the set of Nash equilibria of any Poisson game belong to a family of sets with very convenient finiteness properties. In this section we follow van den Dries [33, 29] and Blume and Zame [40]. We simply present the relevant results without their proofs and, where appropriate, we indicate some of their consequences for Poisson games.

Before identifying the family of sets that we are interested in, we need an intermediate step.

Definition 3. A set $X \subset \mathbb{R}^m$ is *subanalytic* if for each $y \in \mathbb{R}^m$ there is an open neighborhood U , a strictly positive integer l , and a bounded semianalytic set $Y \subset \mathbb{R}^{m+l}$ such that $U \cap X$ is the projection of Y onto the first m coordinates.

⁷ Games in $P(v)$ cannot be generated by strategy perturbations in the game $\Gamma(v)$. The definition of the appropriate set of perturbations is given in the Appendix.

It is easy to see that every semianalytic set is subanalytic. Like the collection of semialgebraic sets, the collection of subanalytic sets is closed under finite unions, finite intersections, finite products, and complementation (Gabrielov [41]) but, in contrast with semialgebraic sets, subanalytic sets are not closed under projections. However, the *bounded* subanalytic sets in \mathbb{R}^m are exactly the projections of bounded semianalytic sets $X \subset \mathbb{R}^{m+l}$ on the first m coordinates. Following from this and other basic results, van den Dries [33] shows that the following family of sets is, in addition, closed under projections:

Definition 4. A set $X \subset \mathbb{R}^m$ is *globally subanalytic* (or *finitely subanalytic*) if given the map

$$f(x_1, \dots, x_m) = \left(x_1 / \sqrt{1 + x_1^2}, \dots, x_m / \sqrt{1 + x_m^2} \right)$$

we have that $f(X)$ is a subanalytic subset of \mathbb{R}^m .

The function f is an analytic isomorphism onto the bounded open set $(-1, 1)^m$. Hence, every bounded subanalytic set is globally subanalytic. The collection of globally subanalytic sets forms an o(rder)-minimal structure:

Definition 5. An *o-minimal structure* on \mathbb{R} is a family $\mathcal{S} = \{\mathcal{S}_m\}_{m \in \mathbb{N}_+}$ such that

- (1) For each m , \mathcal{S}_m is a nonempty collection of subsets of \mathbb{R}^m that is closed under formation of finite unions, finite intersections, and complements.
- (2) If $X \in \mathcal{S}_m$ then $\mathbb{R} \times X \in \mathcal{S}_{m+1}$ and $X \times \mathbb{R} \in \mathcal{S}_{m+1}$.
- (3) For each m we have $\{(x_1, \dots, x_m) : x_1 = x_m\} \in \mathcal{S}_m$.
- (4) If $X \in \mathcal{S}_{m+1}$ and $\pi : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m$ is the projection onto the first m coordinates then $\pi(X) \in \mathcal{S}_m$.
- (5) The ordering of the real line $\{(x_1, x_2) : x_1 < x_2\}$ belongs to \mathcal{S}_2 .
- (6) The sets in \mathcal{S}_1 are exactly the finite unions of points and intervals.

Examples of o-minimal structures are the collection of semialgebraic sets, the collection of semilinear sets, and the collection of globally subanalytic sets. Following standard terminology, once we fix an o-minimal structure $\mathcal{S} = \{\mathcal{S}_m\}_{m \in \mathbb{N}_+}$, we say that X is a *definable set* if $X \in \mathcal{S}_m$ for some m . We say that a function or correspondence is *definable* if its graph is a definable set.

To later prove that the graph of the Nash equilibrium correspondence in Poisson games is globally subanalytic we need to use the fact that o-minimal structures are *closed under definability*. This means that if $\Phi(x_1, \dots, x_m)$ is a first order formula, that is, a formula that uses the free variables (x_1, \dots, x_m) , the universal and existential quantifiers, any finite number of quantified variables that range over the definable sets, the logical connectives \wedge (and), \vee (or), \neg (not),

and the definable sets themselves, then

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m : \Phi(x_1, \dots, x_m) \text{ is true}\}$$

is a definable set.⁸

We now present some of the finiteness properties satisfied by o-minimal structures.

Theorem 2. *Each $X \in \mathcal{S}_m$ has only finitely many connected components, and each component also belongs to \mathcal{S}_m . If, furthermore, $f : X \rightarrow \mathbb{R}^l$ is definable then there is a positive integer N_f such that, for each $x \in \mathbb{R}^l$, the set $f^{-1}(x)$ has at most N_f components.*

We already established that the set of Nash equilibria of any Poisson game has finitely many connected components. Lemma 4 below shows that, if the outcome set Ω is finite, the graph of the Nash equilibrium correspondence in Poisson games $\text{graph}(\text{NE})$ is a globally subanalytic set. We can apply the second part of Theorem 2 to the projection $\pi : \text{graph}(\text{NE}) \rightarrow \mathbb{R}^{\#\Omega T}$ to the space of games to conclude that there is a global bound N^* on the number of connected components of equilibria that any Poisson game can have.

Similarly to the family of semialgebraic sets, the family of globally subanalytic sets also contains the graphs of addition and multiplication:

$$(7) \{(x_1, x_2, x_3) : x_3 = x_1 + x_2\} \in \mathcal{S}_3, \text{ and}$$

$$(8) \{(x_1, x_2, x_3) : x_3 = x_1 x_2\} \in \mathcal{S}_3.$$

O-minimal structures satisfying (7) and (8) have more useful properties.

Theorem 3 (Triangulability). *Let \mathcal{S} satisfy (7) and (8). If $X \in \mathcal{S}_m$ then there is a finite simplicial complex \mathcal{K} in \mathbb{R}^m such that X is homeomorphic to a union of (open) simplices of \mathcal{K} .*

If X is a definable set it follows that we can unambiguously define $\dim(X)$, the dimension of X , to be the largest dimension of any such simplex.

Theorem 4 (Generic Triviality). *Let \mathcal{S} satisfy (7) and (8) and let $f : X \rightarrow Y$ be a continuous definable function. There is a lower-dimensional definable set $Z \subset Y$ such that for each of the finitely many connected components C of $Y \setminus Z$ there is a definable set F and a definable homeomorphism $h : C \times F \rightarrow f^{-1}(C)$ with $f(h(c, f)) = c$ for every $(c, f) \in C \times F$.*

⁸ Intuitively, this expression can be replaced by another one involving definable sets and the set theoretic operations allowed by Axioms 1 to 4. We have that \wedge corresponds to the intersection of sets, \vee to the union of sets, and \neg to the complement of a set. Additionally, we can replace the universal and existential quantifiers by suitable projections.

The next two lemmas are used in Section 5 to prove generic determinacy results of Nash equilibria in Poisson games. They follow immediately from the Generic Triviality Theorem. See Govindan and McLennan [22] for their proofs in the context of semialgebraic geometry.

Lemma 2. *Let \mathcal{S} also satisfy (7) and (8) and let $f : X \rightarrow Y$ be a continuous definable function. Then*

$$\dim(X) \leq \dim(Y) + \max_{y \in Y} \dim(f^{-1}(y)).$$

We say that a definable set is generic if its complement is a closed and lower-dimensional definable set. Furthermore, we say that a point satisfying some property is generic if it resides in a generic definable set where every point satisfies such a property.

Lemma 3. *Let \mathcal{S} also satisfy (7) and (8) and let $f : X \rightarrow Y$ be a continuous definable function. If $\dim(X) \leq \dim(Y)$ then, for generic $y \in Y$, $f^{-1}(y)$ is a finite or empty set.*

5. GENERIC DETERMINACY OF EQUILIBRIA IN POISSON GAMES

Consider a Poisson game with expected number of players equal to n where players have to vote on whether or not to implement a new policy. The policy is implemented if at least $n/2$ players vote *yes*. Suppose that there are two types of players t_1 and t_2 , each one with the same prior probability $1/2$. Let t_1 players strictly prefer the new policy to the status quo and let type t_2 players be indifferent between the two. This game has a unique Nash equilibrium component where type t_1 players vote *yes* and type t_2 players randomize between *yes* and *no*. To each point in this continuum corresponds a different probability that the new policy is implemented. However, this indeterminacy of equilibrium disappears if we slightly modify the preferences of t_2 players so that they are no longer indifferent between the new policy and the status quo. Indeed, if type t_2 players are not indifferent then there is a unique Nash equilibrium that therefore induces a unique equilibrium outcome.

This section shows that some relevant classes of Poisson games share this property. That is, games with indeterminacy of equilibria are exceptional within the relevant space of utilities. First, to only consider finite-dimensional spaces we assume:

Assumption 1. The set of outcomes Ω is finite.

This assumption is satisfied in most applications of Poisson games.

We have already seen that given a Poisson game $\Gamma(v)$ the Nash equilibrium conditions define a semianalytic set. Since the set of Nash equilibria is bounded,

they also define a globally subanalytic set. We now show that the Nash equilibrium correspondence is also globally subanalytic.

Lemma 4. *The Nash equilibrium correspondence $\text{NE} : \mathbb{R}^{\# \Omega T} \rightarrow \Sigma$ is globally subanalytic.*

Proof. By definition, we have to show that the graph of NE is a globally subanalytic set. Denote by E the graph of the Nash equilibrium correspondence restricted to the domain where every utility value lies in the bounded interval $(0,1)$. The set E is a bounded semianalytic set, therefore, globally subanalytic. The Nash equilibrium conditions are not altered under an affine transformation of the utility functions $v_t : \Omega \rightarrow \mathbb{R}$. Therefore, the graph of the Nash equilibrium correspondence satisfies

$$\begin{aligned} \text{graph}(\text{NE}) = \Big\{ (\zeta, v) \in \Sigma \times \mathbb{R}^{\# \Omega T} : \exists (\alpha, \beta, \tilde{v}) \in \mathbb{R}_{++}^T \times \mathbb{R}^T \times \mathbb{R}^{\# \Omega T}, \\ \forall t (\tilde{v}_t = \alpha_t \cdot v_t + \beta_t) \wedge ((\zeta, \tilde{v}) \in E) \Big\}. \end{aligned}$$

Since the family of globally subanalytic sets is closed under definability, the correspondence $\text{NE} : \mathbb{R}^{\# \Omega T} \rightarrow \Sigma$ is globally subanalytic. \square

We now derive a result analogous to the generic finiteness of equilibria in normal form games (Harsanyi [17]) and of equilibrium outcomes in finite extensive form games (Kreps and Wilson [18]). Given $\tau \in \tilde{\tau}(\Sigma)$ and an action $c \in C_t$ we can compute a probability distribution $p_t(\cdot \mid c, \tau)$ on Ω where, for every $\omega \in \Omega$, we have $p_t(\omega \mid c, \tau) := \sum_{x \in Z(C)} \mathbf{P}(x \mid \tau) \theta(t, c, x)(\omega)$. Hence, we can write a type t player's expected utility if she chooses action c while the rest of the population plays according to τ as $U_t(c, \tau; v_t) = \sum_{\omega \in \Omega} p_t(\omega \mid c, \tau) v_t(\omega)$.

Definition 6. We say that $\tau \in \tilde{\tau}(\Sigma)$ is a *maximal dimension point* if for every $t \in \mathcal{T}$ the rank of the matrix whose columns are the vectors $(p_t(\cdot \mid c, \tau))_{c \in C_t}$ is $\#C_t$.

The argument below does not substantially differ from the one used by Govindan and McLennan [22] and illustrates how many semialgebraic proofs in finite games can be readily extended to Poisson games. We say that τ is a *Nash equilibrium behavior* if there is a Nash equilibrium $\sigma \in \Sigma$ such that $\tau = \tilde{\tau}(\sigma)$.

Proposition 2. *If every Nash equilibrium behaviour $\tau \in \tilde{\tau}(\Sigma)$ is a maximal dimension point then, for generic utilities, the set of Nash equilibria is finite.*

Proof. It is enough to focus on Nash equilibria where every action of every type of player is used with positive probability. Write CNE for the correspondence that assigns to each utility vector the set of completely mixed Nash equilibria.

For every $t \in \mathcal{T}$ take some action $d \in C_t$. Let $\sigma \in \text{CNE}(v)$ and $\tau = \tilde{\tau}(\sigma)$. The set

$$\left\{ v_t : \sum_{\omega} [p_t(\omega | c, \tau) - p_t(\omega | d, \tau)] v_t(\omega) = 0 \text{ for every } c \in C_t \right\}$$

has dimension $\#\Omega - (\#C_t - 1)$ because τ is a maximal dimension point.

Consider the projection $\pi : \text{graph}(\text{CNE}) \rightarrow \Sigma$. The previous argument implies that $\dim(\pi^{-1}(\sigma)) \leq \#\Omega T - \dim(\Sigma)$. Therefore, by Lemma 2, $\dim(\text{graph}(\text{CNE})) \leq \#\Omega T - \dim(\Sigma) + \dim(\Sigma) = \#\Omega T$. We obtain the desired result applying Lemma 3 to the projection from $\text{graph}(\text{CNE})$ to the space of utilities $\mathbb{R}^{\#\Omega T}$. \square

The next assumption implies that every $\tau \in \tilde{\tau}(\Sigma)$ is a maximal dimension point. It says that, for each type t and each action $c \in C_t$, from the viewpoint of a player of type t , there is an outcome $\omega_{t,c}$ that can be induced with positive probability only if she plays c .

Assumption 2. For every type t and every action $c \in C_t$ there is an outcome $\omega_{t,c} \in \Omega$ such that we can find an $x \in Z(C)$ satisfying $\theta(t, c', x) = \omega_{t,c}$ if and only if $c' = c$.

In a voting model, for instance, this assumption is satisfied if voters care not only about the outcome of the election but also about the ballot they cast. That is, if for every player the situation where candidate A wins and she voted for A is different from the situation where candidate A wins and she voted for B . The reason we discuss Assumption 2 is only to notice that, under such an assumption, the simple semialgebraic proof given by Govindan and Wilson [39] to show the generic finiteness of Nash equilibria in normal form games applies almost verbatim to Poisson games. In fact, using such a proof, we can also conclude that the graph of CNE is a real analytic manifold of dimension $\#\Omega T$.

It makes sense to impose more structure on the outcome function θ so that outcomes are not defined from the viewpoint of a player of a particular type but have a universal description. This is the case in political economy models where the outcome of the game simply corresponds to the winner of the election.

Thus, let us consider a family of Poisson voting games. Let $\Omega = \{\omega_1, \dots, \omega_K\}$ be the finite set of candidates. Every type t has the same set of ballots (actions) C available. A ballot $c \in C$ is a K -dimensional vector that specifies the number of votes c_i given to each candidate ω_i . An electoral system specifies the set of permissible ballots C . For instance, under plurality rule, such a set is the collection of (c_1, \dots, c_K) such that $c_i \in \{0, 1\}$ for every i and $\sum_{i=1}^K c_i \in \{0, 1\}$. Under negative plurality, C is the collection of (c_1, \dots, c_K) such that $c_i \in \{0, 1\}$ for every i and $\sum_{i=1}^K c_i \in \{0, K-1\}$. Under approval voting, C is simply the set of all (c_1, \dots, c_K) such that $c_i \in \{0, 1\}$ for every i . Thus, aggregating the ballots cast by the players,

we can consider the set of ballot (action) profiles to be \mathbb{N}_+^K . The outcome function $\theta : \mathbb{N}_+^K \rightarrow \Delta(\Omega)$ selects for each ballot profile the candidate that has received the most votes. Ties are broken using the uniform probability distribution over the winning candidates. A type t player has preferences over candidates represented by the utility function $v_t \in \mathbb{R}^\Omega$. Thus, a type t player who casts ballot c when the ballot profile is x obtains utility $u_t(c, x; v_t) = \sum_{\omega \in \Omega} \theta(c + x)(\omega) v_t(\omega)$.

De Sinopoli [20] shows that in generic normal form plurality voting games, the set of Nash equilibria such that more than one candidate wins with positive probability is finite. In turn, De Sinopoli et al. [42] show that (1) in generic normal form negative plurality voting games the set of Nash equilibrium outcomes is finite, and (2) in generic normal form approval voting games with three candidates the set of Nash equilibrium outcomes is also finite. Similarly to Proposition 2, we show that analogous results can be easily established in Poisson games. Furthermore, the fact that in a Poisson game every pivotal event receives positive probability allows us to establish such results in a stronger form and with a simpler proof.

Theorem 5. *For generic utility vectors in $\mathbb{R}^{\#\Omega T}$*

- (1) *the corresponding Poisson plurality voting game has finitely many Nash equilibria, and*
- (2) *the corresponding Poisson negative plurality voting game has finitely many Nash equilibria.*
- (3) *Moreover, if $\#\Omega = 3$, the corresponding Poisson approval voting game also has finitely many Nash equilibria.*

Proof. We want to establish a result that holds for generic utilities, so we can restrict the attention to utility vectors in $\mathbb{R}^{\#\Omega T}$ such that every type has a strict ordering over the set of candidates $\Omega = \{\omega_1, \dots, \omega_K\}$.

We start with part (1). With abuse of notation, denote also by ω the action of voting for candidate $\omega \in \Omega$. Take a Nash equilibrium behavior τ of a Poisson plurality voting game and, without loss of generality, assume that $\tau(\omega) > 0$ for every $\omega \in \Omega$, otherwise, consider the reduced game where candidates for which $\tau(\omega) = 0$ are eliminated. Relabeling if necessary, let $\omega_1, \dots, \omega_L$ be the actions that are best response against τ for players of type t . Note that abstention is strictly dominated, so we can also consider the plurality game where $\omega_1, \dots, \omega_L$ are the only actions available to type t players. We have $L < K$ because type t players are not indifferent between any two candidates and $\tau(\omega) > 0$ for every $\omega \in \Omega$.

Consider the square matrix

$$M_t = \begin{pmatrix} p(\omega_1 | \omega_1, \tau) & \dots & p(\omega_L | \omega_1, \tau) \\ \vdots & \ddots & \vdots \\ p(\omega_1 | \omega_L, \tau) & \dots & p(\omega_L | \omega_L, \tau) \end{pmatrix}.$$

Letting \emptyset represent abstention, we construct a new matrix subtracting from each row the row vector $(p(\omega_1 | \emptyset, \tau), \dots, p(\omega_L | \emptyset, \tau))$. For each two candidates ω and ω' let $\pi(\omega | \omega', \tau) = p(\omega | \omega', \tau) - p(\omega | \emptyset, \tau)$. We obtain:

$$M'_t = \begin{pmatrix} \pi(\omega_1 | \omega_1, \tau) & \dots & \pi(\omega_L | \omega_1, \tau) \\ \vdots & \ddots & \vdots \\ \pi(\omega_1 | \omega_L, \tau) & \dots & \pi(\omega_L | \omega_L, \tau) \end{pmatrix}.$$

The expression $\pi(\omega | \omega', \tau)$ represents the increase in the probability that candidate ω wins the election if candidate ω' obtains an additional vote when the population behaves according to τ . Hence, $\pi(\omega | \omega', \tau) > 0$ if $\omega = \omega'$ and $\pi(\omega | \omega', \tau) < 0$ if $\omega \neq \omega'$. Furthermore, for every row r , we have $|\pi(\omega_r | \omega_r, \tau)| > \sum_{\ell=1, \ell \neq r}^L |\pi(\omega_\ell | \omega_r, \tau)|$ because (1) the increase in the probability that candidate ω_r wins has to be equal to the decrease in the probability that the winner of the election is in $\Omega \setminus \{\omega_r\}$, and (2) $L < K$. Thus, M'_t is a strictly diagonally dominant matrix which implies that it has full rank. The same is true for every principal submatrix of M'_t . It follows that M'_t is an M-matrix (Ostrowski [43, p. 95]) since it is a square matrix such that all diagonal elements are strictly positive, all nondiagonal elements weakly (actually, strictly) negative, and all principal minors of all orders strictly positive. Ostrowski [43, p. 97] shows that if we add the same non-negative row vector to each row of an M-matrix then the determinant of the resulting matrix is strictly positive. Thus, the matrix M_t has full rank, τ is a maximal dimension point, and Proposition 2 implies the desired result.

Part (2) can be handled in a similar way, therefore, we skip this part.

For part (3), it is enough to consider a Nash equilibrium behavior τ such that $\tau(\omega) > 0$ for every $\omega \in \Omega = \{a, b, c\}$. Let type t players have preferences $a > b > c$. It is easy to see that action c (approving only candidate c) and abstaining are both strictly dominated, so we can consider the reduced game where a type t player has only action a (approving candidate a) and action ab (approving both candidates a and b) available. Consider the matrix

$$M_t = \begin{pmatrix} p(a | a, \tau) & p(b | a, \tau) & p(c | a, \tau) \\ p(a | ab, \tau) & p(b | ab, \tau) & p(c | ab, \tau) \end{pmatrix}.$$

The second row minus the first row represents the change in probability resulting from the additional approval vote for candidate b . This additional approval vote for b strictly decreases the probability of winning of candidates a and c and

strictly increases the probability of winning of candidate b . Hence, the second row cannot be a multiple of the first row and, therefore, the rank of M_t is two. We conclude that every Nash equilibrium behavior τ is a maximal dimension point. Proposition 2 implies the desired result. \square

We conclude studying the canonical coordination game described by Morris and Shin [44] and used in a Poisson environment in [10]. Every player has the same two actions R and S , where R is a “risky” action (e.g., attacking a currency) and S is a “safe” action (e.g., investing in a safe asset). There are two outcomes ω_0 and ω_1 . The outcome function selects ω_1 when enough agents take the risky action (e.g., the currency attack is successful) and ω_0 otherwise (e.g. the currency attack is not successful). This threshold can be either some number of players strictly larger than 1 or some strictly positive fraction of the actual size of the population. In terms of preferences, we allow for different types. Each $t \in \mathcal{T}$ pays a cost δ_t when taking R . Expected utilities for each type t and each action as functions of the population behavior are

$$\begin{aligned} U_t(R, \tau) &= p(\omega_0 | R, \tau)u_t(\omega_0) + p(\omega_1 | R, \tau)u_t(\omega_1) - \delta_t, \text{ and} \\ U_t(S, \tau) &= u_t(\omega_0). \end{aligned}$$

As far as generic finiteness of equilibria is concerned, it is unimportant whether or not the utility to playing the safe action is the same as the utility to outcome ω_0 . Hence, once we fix the rules of the game, a Poisson coordination game is defined by the players’ preferences as described by a point in $\mathcal{U} := \mathbb{R}^{3T}$.

Let $\text{ME} : \mathcal{U} \rightarrow \Sigma$ be the correspondence that for each $u \in \mathcal{U}$ assigns the set of Nash equilibria such that some type is mixing completely between R and S . Take $\sigma \in \text{ME}(u)$ and let $\tau = \tilde{\tau}(\sigma)$. For each player of type $t \in \mathcal{T}$ who is indifferent between R and S we have:

$$u_t(\omega_0) = \frac{p(\omega_1 | R, \tau)u_t(\omega_1) - \delta_t}{1 - p(\omega_0 | R, \tau)}, \quad (5.1)$$

which is always well-defined because if $\sigma \in \text{ME}(u)$ then the probability of ω_0 is not 1. Denote as ME^* the projection of $\text{graph}(\text{ME})$ on those coordinates that do not correspond to $u_t(\omega_0)$ for types t indifferent between R and S .

Proposition 3. *For generic utility vectors in \mathcal{U} , the corresponding Poisson coordination game has finitely many Nash equilibria.*

Proof. Equation (5.1) defines a continuous function from ME^* to \mathcal{U} . That is, every utility value that we drop when constructing ME^* can be recovered using other utility values and the strategy profile. Of course, $\dim(\mathcal{U}) = 3T$. On the other hand, if T^* is the number of types that are indifferent between R and S (so that the number of types that are mixing completely between R and S

is less than or equal to T^*) then we have $\dim(\text{ME}^*) = \dim(\text{graph}(\text{ME})) - T^* \leq 3T + T^* - T^*$. Lemma 3 implies the result. \square

APPENDIX A. STABLE SETS IN POISSON GAMES

In this section we revise the definition of stable set in Poisson games (De Sinopoli et al. [1]) and prove Proposition 1.

We begin with some preliminary concepts. Let \mathcal{B} be the σ -algebra of Borel sets in $\tilde{\tau}(\Sigma) \subset \Delta(C)$ and let \mathcal{M} be the set of all Borel probability measures over the measurable space $(\tilde{\tau}(\Sigma), \mathcal{B})$. Moreover, let \mathcal{M}° be the subset of measures $\mu \in \mathcal{M}$ that satisfy $\mu(O) > 0$ for every nonempty open set $O \subset \tilde{\tau}(\Sigma)$. For each $\tau \in \tilde{\tau}(\Sigma)$ we let $\delta(\tau)$ be the Dirac measure on $\tilde{\tau}(\Sigma)$ that assigns probability one to $\{\tau\}$. With abuse of notation, if $\sigma \in \Sigma$ we write $\delta(\sigma)$ instead of $\delta(\tilde{\tau}(\sigma))$.

Let us simply write $U_t(c, \tau)$ instead of $U_t(c, \tau; v_t)$. The domain of the utility functions can be extended to \mathcal{M} :

$$\overline{U}_t(c, \mu) := \int_{\tau(\Sigma)} U_t(c, \tau) d\mu.$$

Finally, given a subset $A \subset \tilde{\tau}(\Sigma)$ and a point $\tau \in \tilde{\tau}(\Sigma)$, the distance between τ and A is $d(\tau, A) := \inf\{d(\tau, a) : a \in A\}$.

We are ready to define stable sets in Poisson games. We define a *perturbation* as a pair $(\varepsilon, \mu^\circ) \in (0, 1) \times \mathcal{M}^\circ$. A perturbation acts “moving” the average behavior of the population towards the completely mixed measure μ° with vanishing probability ε . Hence, a Nash equilibrium of such a perturbed game is a strategy function σ that satisfies $\sigma \in \overline{\text{BR}}((1 - \varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$, where the correspondence $\overline{\text{BR}} : \mathcal{M} \rightarrow \Sigma$ is defined in the obvious way.

Given a perturbation $(\varepsilon, \mu^\circ) \in (0, 1) \times \mathcal{M}^\circ$, it can be easily shown that $\sigma \in \overline{\text{BR}}((1 - \varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$ if and only if σ is a Nash equilibrium of a utility perturbed Poisson game with utility functions:⁹

$$u_t(c, x | \varepsilon, \mu^\circ) := (1 - \varepsilon)u_t(c, x) + \varepsilon \int_{\tau(\Sigma)} U_t(c, \tau) d\mu^\circ.$$

Definition 7. A set of equilibria of a Poisson game $\Gamma(v)$ is *stable* if it is minimal with respect to the following property:

Property (S). $S \subset \Sigma$ is a closed set of Nash equilibria of $\Gamma(v)$ satisfying: for any $\varepsilon > 0$ there is a $\bar{\eta} > 0$ such that for any perturbation (η, μ°) with $0 < \eta < \bar{\eta}$ we can find a σ that is ε -close to S and satisfies $\sigma \in \overline{\text{BR}}((1 - \eta)\delta(\sigma) + \eta\mu^\circ)$.

De Sinopoli et al. [1] prove that stable sets in Poisson games satisfy existence, admissibility, and are robust against iterated deletion of dominated strategies and inferior responses. Nonetheless, they also show that a stable set is not

⁹ This defines the set of games $P(v)$ used in page 9.

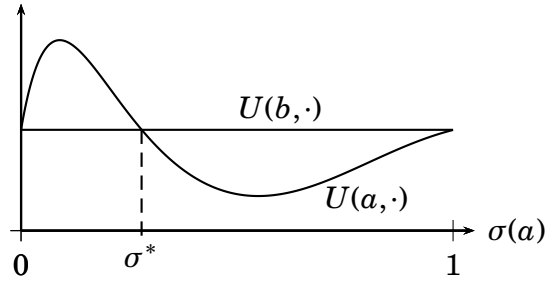


FIGURE 2. Utility functions in a Poisson game with a disconnected stable set.

necessarily connected by means of the example illustrated in Figure 2. It represents the expected utility functions of the unique type of player in the game. The function $U(a, \tau)$ is the expected utility accrued by a player if she chooses action a and the average member of the population plays according to τ . The constant function $U(b, \tau)$ represents the corresponding utility if she chooses action b . The game has two stable sets, $\{\sigma^*\}$ and the disconnected set $\{(a), (b)\}$. The Nash equilibrium strategy a is robust against any perturbation that “lifts” the function $U(a, \tau)$ more than $U(b, \tau)$ while the Nash equilibrium strategy b is robust against any perturbation that “lifts” the function $U(b, \tau)$ more than $U(a, \tau)$. These two strategies belong to different components of the set of Nash equilibria of the game.

Of course, this game has the connected stable set $\{\sigma^*\}$. We can now easily show that any Poisson game has at least one such component.

Proposition 1. *Every Poisson game $\Gamma(v)$ has a stable set contained in a connected component of equilibria. Moreover, every Poisson game has a minimal connected set of Nash equilibria that satisfies Property (S).*

Proof. Stable sets in Poisson games are an example of Q -robust sets of fixed points (McLennan [36, Definition 8.3.5]). In broad terms, a set of fixed points X of a correspondence F is Q -robust if every correspondence close to F that can be obtained through a perturbation in some class Q has a fixed point close to X . The proof of Theorem 1 shows that every Poisson game has a connected set of Nash equilibria that is essential and, therefore, Q -robust. Moreover, McLennan [36, Theorem 8.3.8] shows that, if F is an upper semi-continuous and convex-valued correspondence, every Q -robust set contains a minimal Q -robust set and that every connected Q -robust set contains a minimal connected Q -robust set. The desired result follows. \square

ACKNOWLEDGMENTS

We thank Francesco De Sinopoli for continuous discussions. We also thank Hongyi Li, Francesca Mariani, Adam Solomon, two anonymous referees, and the editor Marciano Siniscalchi for very helpful and constructive comments. Financial support from the Italian Ministry of Education grant PRIN 2011 “New approaches to political economy: positive political theories, empirical evidence and experiments in laboratory” is gratefully acknowledged. Carlos also thanks financial support from the Australian Research Council’s Discovery Projects funding scheme DP140102426.

REFERENCES

- [1] F. De Sinopoli, C. Meroni, C. Pimienta, Strategic stability in Poisson games, *Journal of Economic Theory* 153 (2014) 46–63.
- [2] R. B. Myerson, Population uncertainty and Poisson games, *International Journal of Game Theory* 27 (3) (1998) 375–392.
- [3] R. B. Myerson, Comparison of scoring rules in Poisson voting games, *Journal of Economic Theory* 103 (1) (2002) 219–251.
- [4] L. Bouton, M. Castanheira, One person, many votes: divided majority and information aggregation, *Econometrica* 80 (1) (2012) 43–87.
- [5] L. Bouton, G. Gratton, Majority runoff elections: strategic voting and Duverger’s hypothesis, *Theoretical Economics* 10 (2) (2015) 283–314.
- [6] L. Bouton, A theory of strategic voting in runoff elections, *American Economic Review* 103 (4) (2013) 1248–88.
- [7] G. Gratton, Pandering and electoral competition, *Games and Economic Behavior* 84 (2014) 163–179.
- [8] N. Hugues, Voting in legislative elections under plurality rule, *Journal of Economic Theory* 166 (2016) 51–93.
- [9] M. Satterthwaite, A. Shneyerov, Dynamic matching, two-sided incomplete information, and participation costs: existence and convergence to perfect competition, *Econometrica* 75 (1) (2007) 155–200.
- [10] M. Makris, Complementarities and macroeconomics: Poisson games, *Games and Economic Behavior* 62 (1) (2008) 180–189.
- [11] M. Makris, Private provision of discrete public goods, *Games and Economic Behavior* 67 (1) (2009) 292–299.
- [12] K. Ritzberger, Price competition with population uncertainty, *Mathematical Social Sciences* 58 (2) (2009) 145–157.
- [13] A. McLennan, Manipulation in elections with uncertain preferences, *Journal of Mathematical Economics* 47 (3) (2011) 370–375.
- [14] P. Jehiel, L. Lamy, On discrimination in procurement auctions, mimeo (2014).
- [15] K. J. Arrow, F. H. Hahn, *General Competitive Analysis*, San Francisco: Holden-Day, 1971.

- [16] G. Debreu, Economies with a finite set of equilibria, *Econometrica* 38 (3) (1970) 387–392.
- [17] J. C. Harsanyi, Oddness of the number of equilibrium points: a new proof, *International Journal of Game Theory* 2 (1) (1973) 235–250.
- [18] D. M. Kreps, R. Wilson, Sequential equilibria, *Econometrica* 50 (4) (1982) 863–94.
- [19] I.-U. Park, Generic Finiteness of Equilibrium Outcome Distributions for Sender-Receiver Cheap-Talk Games, *Journal of Economic Theory* 76 (2) (1997) 431–448.
- [20] F. De Sinopoli, On the generic finiteness of equilibrium outcomes in plurality games, *Games and Economic Behavior* 34 (2) (2001) 270–286.
- [21] C. Pimienta, Generic determinacy of Nash equilibrium in network-formation games, *Games and Economic Behavior* 66 (2) (2009) 920–927.
- [22] S. Govindan, A. McLennan, On the generic finiteness of equilibrium outcome distributions in game forms, *Econometrica* 69 (2) (2001) 455–71.
- [23] E. Kohlberg, J.-F. Mertens, On the strategic stability of equilibria, *Econometrica* 54 (5) (1986) 1003–37.
- [24] J.-F. Mertens, Stable equilibria - a reformulation. Part I. Definition and basic properties, *Mathematics of Operations Research* 14 (4) (1989) 575–625.
- [25] R. M. Hardt, Semi-algebraic local triviality in semi-algebraic mappings, *American Journal of Mathematics* 102 (2) (1980) 291–302.
- [26] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, 1998.
- [27] D. Balkenborg, D. Vermeulen, Universality of Nash components, *Games and Economic Behavior* 86 (2014) 67–76.
- [28] S. Łojasiewicz, Triangulation of semi-analytic sets, *Annali della Scuola Normale Superiore di Pisa* 18 (4) (1964) 449–474.
- [29] L. van den Dries, *Tame topology and o-minimal structures*, Vol. 248, Cambridge University Press, 1998.
- [30] L. E. Blume, W. R. Zame, The algebraic geometry of perfect and sequential equilibrium, *Econometrica* 62 (4) (1994) 783–94.
- [31] M. K. Richter, K.-C. Wong, Definable utility in o-minimal structures, *Journal of Mathematical Economics* 34 (2) (2000) 159–172.
- [32] J. Bolte, S. Gaubert, G. Vigeral, Definable Zero-Sum Stochastic Games, *Mathematics of Operations Research* 40 (1) (2015) 171–191.
- [33] L. van den Dries, A generalization of the Tarski-Seidenberg theorem, and some nondefinability results, *Bulletin of the American Mathematical Society* 15 (2) (1986) 189–193.
- [34] B. L. van der Waerden, *Einführung in die Algebraische Geometrie*, Berlin-Heidelberg-New York: Springer Verlag, 1939.
- [35] S. Kinoshita, On essential components of the set of fixed points, *Osaka Mathematical Journal* 4 (1) (1952) 19–22.
- [36] A. McLennan, *Advanced Fixed Point Theory for Economics*, Mimeo, 2012.
- [37] J. Hillas, On the definition of the strategic stability of equilibria, *Econometrica* 58 (6) (1990) 1365–90.

- [38] J.-F. Mertens, Stable equilibria - a reformulation. Part II. Discussion of the definition, and further results, *Mathematics of Operations Research* 16 (4) (1991) 694–753.
- [39] S. Govindan, R. Wilson, Direct proofs of generic finiteness of Nash equilibrium outcomes, *Econometrica* 69 (3) (2001) 765–69.
- [40] L. E. Blume, W. R. Zame, The algebraic geometry of competitive equilibrium, in: *Economic theory and international trade: essays in memoriam J. Trout Rader*, Springer Science & Business Media, 1992, p. 53.
- [41] A. M. Gabriélov, Projections of semi-analytic sets, *Functional Analysis and Its Applications* 2 (4) (1968) 282–291.
- [42] F. De Sinopoli, G. Iannantuoni, C. Pimienta, On stable outcomes of approval, plurality, and negative plurality games, *Social Choice and Welfare* 44 (4) (2015) 889–909.
- [43] A. M. Ostrowski, Note on bounds for some determinants, *Duke Mathematical Journal* 22 (1) (1955) 95–102.
- [44] S. Morris, H. S. Shin, Unique equilibrium in a model of self-fulfilling currency attacks, *American Economic Review* 88 (3) (1998) 587–597.