Limits of integrals involving almost periodic functions

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Abstract

Let $\mathrm{Sp} \subset \mathbb{R}^+$ be a discrete countable set, let $\{a_\lambda\}_{\lambda\in\mathrm{Sp}}$ be a sequence in $l^1(\mathrm{Sp})$ and $f(x) := \sum_{\lambda\in\mathrm{Sp}} a_\lambda \sin(\lambda x)$. f is an almost periodic odd function with $\{\lambda : \pm\lambda \in \mathrm{Sp}\}$ as spectrum. We give some conditions about the set S so that $\int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} \to 0$ whenever $R \to +\infty$, $R \in S$.

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Motivations and results

The Banach algebra AP of Bohr's almost periodic functions is obtained completing with respect to the uniform norm the complex vector space generated by the functions $e^{i\lambda x}$, with $\lambda \in \mathbb{R}$ (see [1]). Over AP it is possible to define a continuous functional $\mathcal{M}$ such that $\mathcal{M}(e^{i\lambda x}) = \delta_{\lambda,0}$, where $\delta_{\lambda,0} = 1$ if $\lambda = 0$ and 0 otherwise. An important feature of $\mathcal{M}$ is that for every $f \in \mathrm{AP}$, $\mathcal{M}(f(x)e^{i\lambda x}) = 0$ for all but a countable set of values for λ which constitutes the *spectrum* Sp of f. Usually $\mathcal{M}$ is defined as

$$\mathcal{M}(f) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(x)dx \ ,$$

but there are other possibilities. In fact, for every $\alpha \in [0,1]$ we can consider

$$\mathcal{M}_\alpha(f) := \lim_{T\to+\infty} \frac{1}{\mu_\alpha([1,T])}\int_1^T f(x)\frac{dx}{x^\alpha} \ , \quad \text{where} \quad \mu_\alpha([1,T]) := \int_1^T \frac{dx}{x^\alpha} \ .$$

The existence and the continuity of $\mathcal{M}_\alpha$ as a functional over AP can be proved following the same argument proving existence and continuity of $\mathcal{M}(f)$ (see [1]). Since a direct check shows that $\mathcal{M}_\alpha(e^{i\lambda x}) = \delta_{\lambda,0}$ for every $\alpha \in [0,1]$, we conclude that $\mathcal{M}_\alpha$ is only a different definition of $\mathcal{M}$; in other words, we have

$$\int_1^T f(x)e^{-iRx}\frac{dx}{x^\alpha} = (a_R + o(1))\mu_\alpha([1,T]) \tag{1}$$

where a_R is independent of α and is zero if $R \notin \mathrm{Sp}_f$. The behavior of the integral in (1) for $\alpha = 0$ and $\alpha \neq 0$ is different when a more exact asymptotic behavior is looked for. In fact, suppose $R \notin \mathrm{Sp}_f$ so that $a_R = 0$, and consider the case $\alpha = 0$, i.e., the function

$$F(T) := \int_1^T f(x)e^{-iRx}dx \ .$$

A classical result (Theorem 4.1 of [1]) states that if F is bounded then it is almost periodic, therefore when $\alpha = 0$ the limit

$$\lim_{T\to+\infty}\int_1^T f(x)e^{-iRx}\frac{dx}{x^\alpha} \tag{2}$$

does not exist if $f \not\equiv 0$. On the contrary, when $\alpha > 0$ an integration by parts

$$\int_1^T f(x)e^{-iRx}\frac{dx}{x^\alpha} = \frac{F(x)}{x^\alpha}\Big|_1^T + \alpha\int_1^T F(x)\frac{dx}{x^{\alpha+1}}$$

is sufficient to realize that (2) exists, at least when F is bounded. When $\alpha \neq 0$, therefore, it is quite natural to enquire the behavior of (2) as a function of R, in particular we are interested in finding the behavior of

$$\lim_{\substack{R\to+\infty\\ R\notin \mathrm{Sp}}}\int_1^{+\infty} f(x)e^{-iRx}\frac{dx}{x^\alpha}\ . \tag{3}$$

When $f(x)x^{-\alpha} \in L^1(\mathbb{R})$ the Riemann-Lebesgue theorem implies that the limit in (3) is zero. This fact is not useful when f is almost periodic and not identically zero, but we would like to know the conditions we have to assume in order to prove that the limit is again zero. The function $g(x) := f(x)e^{-iRx}$ is almost periodic with $\mathrm{Sp}_g = \mathrm{Sp}_f - R$, and it is known that the primitive of an almost periodic function is bounded when 0 is not a limit point for its spectrum (see [1], Chapter IV), hence we conjecture that (3) exists and is zero if and only if R runs over a set of points whose distance from Sp_f is large enough, in some sense. Our principal result, the theorem below, shows that this conjecture is true for a large class of almost periodic functions.

We have inquired both the case $0 < \alpha < 1$ and the case $\alpha = 1$. We have found similar (but not identical) conclusions but the second case is complicated by the non-integrability at $x = 0$ hence we have chosen to present our result only for $\alpha = 1$. Moreover, for our applications it is useful to know the behavior of

$$\lim_{\substack{R\to+\infty\\ R\notin \mathrm{Sp}}}\int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x}$$

($\sin(Rx)/x$ is the Fourier transform of the characteristic function $\chi_{[-R,R]}(x)$) so that we state our result directly for such object. The proofs are based on explicit formulas, hence only almost periodic functions which are associated with l^1 sequences are considered. Summarizing, our setting is the following: $\mathrm{Sp} \subset \mathbb{R}^+$ is a discrete set, $\{a_\lambda\}_{\lambda\in\mathrm{Sp}}$ is a sequence in $l^1(\mathrm{Sp})$ so that

$$f(x) := \sum_{\lambda\in\mathrm{Sp}} a_\lambda \sin(\lambda x)$$

is an almost periodic odd function whose spectrum is a subset of $\{\pm\lambda : \lambda \in \mathrm{Sp}\}$.

Remark 1. The referee pointed to our attention that in the context of the signal processing theory $\sin(Rx)/x$ represents the impulse response of a reconstruction filter to the unit step rectangular pulse on the time interval $[-R, R]$, that the limit $R \to +\infty$ implies an increase of bandwidth and that $\mathcal{M}$ defines the spectral average detection performed by a signal responding instrument. In this context the spectral weights adopt a novel and interesting character. Since we are not expert of this subject, we prefer to demand to the specialized literature (for example [2] and [3]) the interested reader.

The following lemma gives an explicit and alternative formula for the integral we are studying.

Lemma 1. *Let $R > 0$ be fixed, then the limit*

$$\lim_{M\to+\infty}\int_1^M f(x)\sin(Rx)\frac{dx}{x}$$

exists if and only if $a_R = 0$ and in this case

$$\int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} = \sum_{\lambda\in\mathrm{Sp}}\frac{a_\lambda}{2}\int_{|\lambda-R|}^{\lambda+R}\cos x\frac{dx}{x}\ . \tag{4}$$

Proof. The series defining f converges uniformly on $\mathbb{R}$, therefore

$$\begin{aligned}\int_1^M f(x)\sin(Rx)\frac{dx}{x} &= \sum_{\lambda\in\mathrm{Sp}} a_\lambda\int_1^M \sin(\lambda x)\sin(Rx)\frac{dx}{x}\\ &= -\sum_{\lambda\in\mathrm{Sp}}\frac{a_\lambda}{2}\int_1^M [\cos(\lambda+R)x-\cos(\lambda-R)x]\frac{dx}{x}\\ &= -\sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}}\frac{a_\lambda}{2}\Big[\int_{\lambda+R}^{M(\lambda+R)}\cos x\frac{dx}{x}-\int_{|\lambda-R|}^{M|\lambda-R|}\cos x\frac{dx}{x}\Big]-\frac{a_R}{2}\Big[\int_1^M\cos(2Rx)\frac{dx}{x}-\int_1^M\frac{dx}{x}\Big]\ .\end{aligned}$$

When $R\neq\lambda$ and $M\gg R$ we have $|\lambda-R|\leq\lambda+R\leq M|\lambda-R|\leq M(\lambda+R)$, so that

$$= -\sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}}\frac{a_\lambda}{2}\Big[\int_{M|\lambda-R|}^{M(\lambda+R)}\cos x\frac{dx}{x}-\int_{|\lambda-R|}^{\lambda+R}\cos x\frac{dx}{x}\Big]-\frac{a_R}{2}\int_{2R}^{2RM}\cos x\frac{dx}{x}+\frac{a_R}{2}\ln M.$$

The series depending on M can be uniformly estimated since

$$\int_{M|\lambda-R|}^{M(\lambda+R)}\cos x\frac{dx}{x} = \frac{\sin x}{x}\Big|_{M|\lambda-R|}^{M(\lambda+R)}+\int_{M|\lambda-R|}^{M(\lambda+R)}\sin x\ \frac{dx}{x^2}\ll\frac{1}{M|\lambda-R|}\ , \tag{5}$$

so that

$$\sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}}\frac{|a_\lambda|}{2}\Big|\int_{M|\lambda-R|}^{M(\lambda+R)}\cos x\frac{dx}{x}\Big|\ll\sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}}|a_\lambda|\frac{1}{M|\lambda-R|}\ll\frac{1}{M}\ .$$

A similar upper bound, this time with $M = 1$, proves that also the second series converges, therefore as $M\to+\infty$ we have

$$\int_1^M f(x)\sin(Rx)\frac{dx}{x} = \sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}}\frac{a_\lambda}{2}\int_{|\lambda-R|}^{\lambda+R}\cos x\frac{dx}{x}-\frac{a_R}{2}\int_{2R}^{+\infty}\cos x\frac{dx}{x}+\frac{a_R}{2}\ln M+O(M^{-1})\ ,$$

and the claim follows. □

Now we approximate Identity (4) in such a way that only the elements of Sp which are near to R appear explicitly.

Lemma 2. *Let* Sp *and* a_λ *as before, with* $a_R = 0$. *Let* c *be an arbitrary positive constant, then*

$$\int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} = -\sum_{\substack{\lambda\in\mathrm{Sp}\\|\lambda-R|<c}} \frac{a_\lambda}{2}\ln|\lambda-R| + O_c(\sum_{\substack{\lambda\in\mathrm{Sp}\\R/2<\lambda<2R}} |a_\lambda|) + O(R^{-1}) \ .$$

Proof. In fact, from (5) we have the upper bound

$$\sum_{\substack{\lambda\in\mathrm{Sp}\\\lambda\le R/2}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} \ll \sum_{\substack{\lambda\in\mathrm{Sp}\\\lambda\le R/2}} \frac{|a_\lambda|}{|\lambda-R|} \ll R^{-1} \ ,$$

the same argument holds in the range $\lambda \ge 2R$, therefore

$$\sum_{\lambda\in\mathrm{Sp}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} = \sum_{\substack{\lambda\in\mathrm{Sp}\\R/2<\lambda<2R}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} + O(R^{-1}) \ .$$

Using (5) again but in ranges $R/2 < \lambda < R-c$ and $R+c < \lambda < 2R$, we obtain

$$\sum_{\lambda\in\mathrm{Sp}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} = \sum_{\substack{\lambda\in\mathrm{Sp}\\|\lambda-R|<c}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} + O_c(\sum_{\substack{\lambda\in\mathrm{Sp}\\R/2<\lambda<2R\\|\lambda-R|\ge c}} |a_\lambda|) + O(R^{-1}) \ . \tag{6}$$

Since

$$\int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} = \int_{|\lambda-R|}^{1} \cos x \frac{dx}{x} + O(1) = -\ln|\lambda-R| + \int_{|\lambda-R|}^{1} \frac{\cos x - 1}{x}\, dx + O(1)$$
$$= -\ln|\lambda-R| + O(1)$$

uniformly on $\lambda \in \mathrm{Sp}$ and $R \in \mathbb{R}^+$, from (6) we get

$$\sum_{\lambda\in\mathrm{Sp}} a_\lambda \int_{|\lambda-R|}^{\lambda+R} \cos x \frac{dx}{x} = -\sum_{\substack{\lambda\in\mathrm{Sp}\\|\lambda-R|<c}} a_\lambda \ln|\lambda-R| + O(\sum_{\substack{\lambda\in\mathrm{Sp}\\|\lambda-R|<c}} |a_\lambda|) + O_c(\sum_{\substack{\lambda\in\mathrm{Sp}\\R/2<\lambda<2R\\|\lambda-R|\ge c}} |a_\lambda|) + O(R^{-1})$$

which is the claim. □

Lemma 2 immediately implies the following theorem.

Theorem. *Given* $\phi : \mathbb{R} \to \mathbb{R}^+$, *suppose that* $S_\phi := \{x \in \mathbb{R}^+ : |x-\lambda| \ge 1/\phi(\lambda),\ \forall \lambda \in \mathrm{Sp}\}$ *is unbounded and that*

$$\lim_{R\to+\infty} \sum_{\substack{\lambda\in\mathrm{Sp}\\|\lambda-R|<1}} |a_\lambda \ln\phi(\lambda)| = 0 \ ,$$

then

$$\lim_{\substack{R\to+\infty\\R\in S_\phi}} \int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} = 0 \ . \tag{7}$$

We note that S_ϕ is unbounded if and only if $\Delta_\lambda := \inf\{|\eta-\lambda|,\ \eta\in \mathrm{Sp}, \eta\neq\lambda\} > 1/\phi(\lambda)$ for infinitely many $\lambda\in\mathrm{Sp}$, so that a function ϕ as in Theorem exists if and only if

$$\lim_{R\to+\infty} \sum_{\substack{\lambda\in\mathrm{Sp}\\ |R-\lambda|<1}} |a_\lambda \ln(\Delta_\lambda)| = 0 \ . \tag{8}$$

This condition is always satisfied when Sp is well spaced, i.e., $\Delta_\lambda \gg 1$, but can fail if $\inf_\lambda \Delta_\lambda = 0$.

Remark 2. The restriction $R\in S_\phi$ in (7) is necessary, in fact the limit can be non-zero when R runs on a set sufficiently near to Sp. For example, suppose $\mathrm{Sp} = \mathbb{N}\backslash\{0\}$ and let $a_n = n^{-2}$. Then, from Lemma 2 (with $c=1/2$) we have

$$\int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} = -\frac{\ln\|R\|}{2\lfloor R\rceil^2} + o(1)$$

where $\lfloor R\rceil$ is the integer which is nearest to R and $\|R\| := |R-\lfloor R\rceil|$: obviously the limit can be zero, positive or infinite for suitable choices of R.

Remark 3. The similar problem for even functions is easier. In fact, let $g(x) := \sum_{\lambda\in\mathrm{Sp}} b_\lambda \cos(\lambda x)$, where Sp is a discrete countable set and $\{b_\lambda\}_{\lambda\in\mathrm{Sp}}$ is a sequence in $l^1(\mathrm{Sp})$. Then, an argument similar to that one proving Lemma 1 shows that

$$\int_1^{+\infty} g(x)\sin(Rx)\frac{dx}{x} = \pi\sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda<R}} b_\lambda - \sum_{\substack{\lambda\in\mathrm{Sp}\\ \lambda\neq R}} \frac{b_\lambda}{2}\int_{\lambda-R}^{\lambda+R}\sin x\frac{dx}{x} + \frac{b_R}{2}\int_{2R}^{+\infty}\sin x\frac{dx}{x} \qquad \forall R,$$

so that by the dominated convergence theorem we conclude that $\int_1^{+\infty} g(x)\sin(Rx)\frac{dx}{x}$ tends to zero as R tends to infinity, without any restriction about the set containing R.

An application

Let $\mathrm{Sp} := \{\lambda : \lambda = \ln n, n\in\mathbb{N}, n>1\}$ so that $\Delta_\lambda \sim e^{-\lambda}$, and take $a_\lambda = \lambda^{-2}e^{-\lambda}$ so that

$$f(x) = \sum_{\lambda\in\mathrm{Sp}} a_\lambda \sin(\lambda x) = \sum_{n=2}^{\infty}\frac{\sin(x\ln n)}{n\ln^2 n} \ .$$

Since $\sum_{|R-\lambda|<1}|a_\lambda \ln(\Delta_\lambda)| \asymp \lambda^{-1}$, by (8) and our theorem we know that there exists a function ϕ (for example, $\phi(\lambda)\asymp\Delta_\lambda^{-1} = e^\lambda$) such that $\int_1^{+\infty} f(x)\ \sin(Rx)dx/x$ tends to 0 as $R\to\infty$ in S_ϕ. It is interesting to check this claim when R runs over some particular sequence, for example, what happens if we take $R\in\mathbb{N}$? An answer to this question follows from known results about the transcendence measure of logarithms of algebraic numbers; in particular, we use the following fact: there exists $c>0$ such that

$$\forall p,q,n\in\mathbb{N}\backslash\{0\}, \qquad |\tfrac{p}{q} - \ln n| > e^{-c(\ln n)\ln(q\ln n)} \tag{9}$$

(for $q=1$ this claim is due to Mahler, the generalization we consider here is an immediate consequence of Theorem 9.1 of [5]). Let $r:\mathbb{R}^+\to\mathbb{R}^+$ be an arbitrary function monotonously decreasing to 0 and satisfying $r(x) \geq x^{-1}\ln x$. Let

$$\phi(\lambda) := e^{c\lambda^2 r(\lambda)},$$
$$S_\phi := \{x : |x-\lambda| > \phi^{-1}(\lambda), \forall\lambda \in \mathrm{Sp}\},$$
$$S_{\phi,r} := \{p/q \in \mathbb{Q} : 1 \leq q \leq \lambda^{-1}e^{\lambda r(\lambda)},\ \text{where } \lambda = \lambda(p/q) \in \mathrm{Sp} \text{ and } |\tfrac{p}{q}-\lambda| = \min_{\eta\in\mathrm{Sp}}|\tfrac{p}{q}-\eta|\}.$$

The inclusion $S_{\phi,r} \subset S_\phi$ follows by (9) and since

$$\sum_{|\lambda-R|<1} |a_\lambda \ln\phi(\lambda)| \ll \sum_{|\lambda-R|<1} \frac{r(\lambda)}{e^\lambda} \ll r(R-1) \sum_{|\lambda-R|<1} \frac{1}{e^\lambda} \asymp r(R-1) \to 0\ ,$$

the theorem gives

$$\lim_{\substack{R\to+\infty\\ R\in S_{\phi,r}}} \int_1^{+\infty} f(x)\sin(Rx)\frac{dx}{x} = 0\ . \tag{10}$$

When $r(x) = x^{-1}\ln x$ we have $S_{\phi,r} = \mathbb{N}$, but for other choices of r, $S_{\phi,r}$ can be significantly larger than $\mathbb{N}$.

We note that $f(x) = -\Im F(1+ix)$ where $F(s) := \sum_{n=2}^\infty n^{-s}/\ln^2 n$, so that by (10) and Remark 3 we conclude that

$$\lim_{\substack{R\to+\infty\\ R\in S_{\phi,r}}} \int_1^{+\infty} F(\sigma+ix)\sin(Rx)\frac{dx}{x} = 0 \tag{11}$$

when $\sigma = 1$. By similar arguments it is possibile to prove the validity of (11) for every $\sigma \geq 1$. The function $F(s)$ has an analytical continuation to $\mathbb{C}\backslash(-\infty,1]$ coming from the equality $F''(s)+1 = \zeta(s)$ where $\zeta(s)$ is the Riemann zeta function. It is probable that (11) holds whenever $F(\sigma+ix) = o(x)$, in particular, we conjecture that (11) holds whenever $\mu_F(\sigma) < 1$, where $\mu_F(\sigma) := \inf\{a > 0 : F(\sigma+ix) \ll_a x^a,\ x > 1\}$ is the Lindelöf function of F. It is known that the the Lindelöf function is a convex function and it is simple to prove that $\mu_F(1) = 0$, therefore (11) should be true also in some range $\sigma \in (c,1)$. In particular, assuming LH (i.e., the Lindelöf hypothesis for $\zeta(s)$ stating $\mu_\zeta(\sigma) = 0$ when $\sigma \in [1/2,1]$; see [4]) we get $\mu_F(\sigma) \leq 4(1-\sigma)$ when $\sigma \in [1/2,1]$, hence (11) should be correct at least when $\sigma \in (3/4,1)$. Our inquires in this direction have been fruitless.

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