Non-uniqueness for a critical heat equation in two dimensions with singular data

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Abstract

Nonlinear heat equations in two dimensions with singular initial data are studied. In recent works nonlinearities with exponential growth of Trudinger-Moser type have been shown to manifest critical behavior: well-posedness in the subcritical case and non-existence for certain supercritical data. In this article we propose a specific model nonlinearity with Trudinger-Moser growth for which we obtain surprisingly complete results: a) for initial data strictly below a certain singular threshold function $\widetilde{u}$ the problem is well-posed, b) for initial data above this threshold function $\widetilde{u}$, there exists no solution, c) for the singular initial datum $\widetilde{u}$ there is non-uniqueness. The function $\widetilde{u}$ is a weak stationary singular solution of the problem, and we show that there exists also a regularizing classical solution with the same initial datum $\widetilde{u}$.

1 Introduction

Consider the following Cauchy problem with Dirichlet boundary condition

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \ \Omega, \ t > 0, \\ u(t,x) = 0 & \text{on } \partial\Omega, \ t > 0, \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where Ω is an open domain in $\mathbb{R}^N$. It is well-known that for bounded initial data u_0 and for C^1-nonlinearities f, this equation has a local-in-time solution $u \in L^\infty_{loc}((0,T]; L^\infty(\Omega))$ for some $T > 0$. In this article we address some questions concerning singular initial data $u_0 \notin L^\infty(\Omega)$. The case of power-type nonlinearity $f(s) = |s|^{p-1}s$ has been widely studied beginning with the seminal works of F. Weissler (see [4, 22, 27, 29, 30, 31, 32] and Section 2 for a description of known results). Let us focus our attention to the so-called critical nonlinearity $f(s) = |s|^{\frac{2}{N-2}}s$, $(N \geq 3)$ and let us consider initial data in the Lebesgue space $L^{\frac{N}{N-2}}(\mathbb{R}^N)$, which is invariant under the scaling of the equation and which has the same integrability as the growth of the nonlinearity. In this case the existence and uniqueness of a local-in-time (classical) solution for any initial data hold. However, some non-uniqueness phenomena of (distributional) solutions appear. Moreover, for small data the solution exists globally in time.

In dimension $N = 2$ this case does not happen and one may expect a critical situation for certain nonlinearities with higher than polynomial growth. In recent works [15, 16, 17, 25, 13] (see also [18] for more general nonlinearities) it was shown that nonlinearities with Trudinger-Moser growth, see [23, 28, 20],

$$f(s) \sim e^{s^2} \quad \text{for} \ \ |s| \ \ \text{large} \tag{1.2}$$

in conjunction with data from the Orlicz space

$$\exp L^2(\Omega) := \left\{ u \in L^1_{loc}(\Omega) : \int_\Omega \left(e^{\alpha u^2} - 1 \right) dx < \infty \ \text{ for some } \alpha > 0 \right\},$$

show some of the critical behavior (see Section 2, Remark 2.1):

– existence of global-in-time solutions for small data u_0 in $\exp L^2(\Omega)$;

– non-existence of solutions for some large initial data $u_0 \in \exp L^2(\Omega)$;

– existence of local-in-time solutions for any initial data $u_0 \in \exp L^2_0(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\exp L^2}}$.

In this paper we set out to complete the picture by proving a non-uniqueness result for a particular equation on a ball $B_\rho(0) \subset \mathbb{R}^2$. Indeed, for a certain nonlinearity $f(t)$ with growth of type (1.2) (more precisely, see (2.1)) we show the existence of a singular solution $\widetilde{u} \in \exp L^2(B_\rho)$ for the corresponding elliptic equation, which gives rise to a singular stationary distributional solution of the parabolic equation. The solution $\widetilde{u}$ has the asymptotic profile $\widetilde{u}(x) \sim \sqrt{-2\log|x|}$, for $|x|$ small, and belongs to $\exp L^2 \backslash \exp L^2_0$. We prove furthermore that the same initial datum $\widetilde{u}$ gives also rise to a regularizing solution, and hence we have non-uniqueness.

Indeed, for this particular initial datum $\widetilde{u}$ and the nonlinearity $f(t)$, we get the following surprisingly complete result:

Theorem A *Let the initial datum u_0 for the problem*

$$\partial_t u - \Delta u = f(u) \quad in \quad B_\rho(0)\ ,\ u = 0 \quad on \quad \partial B_\rho(0), \tag{1.3}$$

be given by $u_0(x) = \mu\, \widetilde{u}(x)$, $\mu > 0$. Then the following hold:

1) (well-posedness) *If $\mu < 1$, then the equation has a unique regular local-in-time solution.*

2) (non-uniqueness) *If $\mu = 1$, then $u_0 = \widetilde{u}$ is a singular (distributional) stationary solution, and there exists a regular solution with the same initial datum $\widetilde{u}$.*

3) (non-existence) *If $\mu > 1$, then the equation has no non-negative solution, in any positive time interval.*

In Section 2 we present more detailed motivations and some background for this problem, and a more precise statement of our results. We point out that the phenomena described in Theorem A are rather subtle, and the function spaces (Orlicz and Lorentz spaces) and related notions of solution have to be chosen very carefully. After introducing these concepts, we formulate a precise statement of Theorem A in Theorem 2.1, see end of Section 2.

In Section 3 we give some preliminary results on the heat kernel in Orlicz spaces and Lorentz spaces which will be needed in the proofs, and the notions of solution (weak, classical) will be introduced.

In Section 4 we construct a singular solution $\widetilde{u}(x)$ of the elliptic equation (1.3): we use that $\sqrt{-2\log|x|}$ is an exact solution of (1.3) for large values of $\widetilde{u}(x)$, and then employ the shooting method to construct a solution with zero boundary values on a suitable ball B_ρ.

In Section 5 we prove the well-posedness of equation (1.3) for initial data below the threshold function $\widetilde{u}$, i.e. statement 1 in Theorem A and Theorem 2.1. This is done with a contraction argument in a suitable function space.

In Section 6 we prove the non-uniqueness result (statement 2 of Theorem A and of Theorem 2.1 below). The stationary singular solution is given by $\widetilde{u}(x)$, as obtained in Section 4. The existence of a regular solution with the same initial datum $\widetilde{u}(x)$ is quite delicate: we first consider an auxiliary equation in a Lorentz space setting with a cubic nonlinearity and with initial datum which belongs to the Lorentz spaces $L^{2,q}$ for all $q > 2$, but not for $q = 2$. From this solution we then produce, by a suitable transformation (inspired by Brezis-Cazenave-Martel-Ramiandrisoa [6] and Fujishima-Ioku [11]), a super-solution of the Cauchy problem (1.3). Finally, applying Perron's monotone method, we then obtain a classical solution of problem (1.3).

In Section 7 we give the proof of the non-existence result (statement 3 in Theorem A and Theorem 2.1). We show that for data above the threshold function $\widetilde{u}(x)$ we encounter instantaneous blow-up, i.e. for no positive time T can a solution exist.

We expect that similar phenomena hold in more general situations, but we note that the growth of the nonlinearity, the behavior of the singular initial data, and the employed function spaces will have to be very carefully calibrated.

2 Origin of the problem and main result

2.1 Polynomial nonlinearities

The study of equation (1.1) with singular data began with the pioneering works of F. Weissler [29], [30]. He considered equation (1.1) on the whole space $\mathbb{R}^N$, with power type nonlinearities $f(s) = |s|^{p-1}s$ and with singular data in certain Lebesgue spaces $L^q(\mathbb{R}^N)$. For power nonlinearities the equation (1.1) enjoys a scale invariance: if u is a solution, then also

$$u_\lambda(t,x) := \lambda^{2/(p-1)}\, u(\lambda^2 t, \lambda x)$$

is a solution. One notes that the initial data space $L^q(\mathbb{R}^N)$ is invariant under this scaling if and only if $q = q_c = \frac{N(p-1)}{2}$. This exponent serves as a limiting or *critical* exponent for the well-posedness of the Cauchy problem (1.1) with $f(s) = |s|^{p-1}s$ and initial data $u_0 \in L^q(\mathbb{R}^N)$. Indeed one has:

- if $q > q_c$, $q \geq 1$ or $q = q_c$, $q > 1$, then the Cauchy problem (1.1) has a unique local-in-time solution in $C([0,T], L^q(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T), L^\infty(\mathbb{R}^N))$ for some $T > 0$, (see [4], [29], [30]). Moreover, in the critical case $q = q_c, q > 1$, for sufficiently small data in $L^{q_c}(\mathbb{R}^N)$ there exist *global-in-time* solutions (see [31]);
- if $1 \leq q < q_c$, then there exist some non-negative initial data in $L^q(\mathbb{R}^N)$ for which there is no non-negative solution for any positive time $T > 0$ (see [4],[30], [32]).

For $q \geq p$, then $C([0,T], L^q(\mathbb{R}^N)) \subset L^p_{loc}((0,T)\times\mathbb{R}^N))$ and for any $u \in C([0,T], L^q(\mathbb{R}^N))$ each term of equation (1.1) is a distribution. Therefore for $q \geq q_c$, $p \geq q$, one may ask whether the solution obtained by Weissler is unique in the larger class $C([0,T], L^q(\mathbb{R}^N))$. The known results are:

- if $q > q_c$, $q \geq p$ or $q = q_c$, $q > p$ uniqueness still holds in the class $C([0,T], L^q(\mathbb{R}^N))$ (see [4], [29]).

In the case $q = q_c$ and $q = p$, then $q = p = \frac{N}{N-2}$ which is referred to as *doubly critical case* in [4, Remark 5], Ni-Sacks [22] proved that (for the unit ball $B_1 \subset \mathbb{R}^N$) there exists a *stationary singular* solution – which is different from the regularizing solution of Weissler. This non-uniqueness result was extended to the whole space $\mathbb{R}^N$ by Terraneo [27].

We remark that if $p > \frac{N}{N-2}$ there exists an explicit singular stationary solution of (1.1) with $f(s) = |s|^{p-1}s$ in $\mathbb{R}^N$. This is another way in which $p = \frac{N}{N-2}$ is critical and so we can say that $q = q_c = \frac{N}{2}(p-1) = p$, i.e. $p = \frac{N}{N-2}$, is doubly critical.

Remark 2.1 *Note that the "doubly critical" case is characterized by the simultaneous appearence of the following two phenomena:*

- *global-in-time existence for small data;*
- *non-uniqueness for some data.*

2.2 The limiting case: the H^s - L^p correspondence

Note that in $\mathbb{R}^2$ the double critical exponent $q = q_c = p = \frac{N}{N-2}$ becomes infinite. If we look for a suitable "critical growth" in two dimensions, we may be guided by recent results for dispersive equations.

Indeed, for the corresponding Nonlinear Schrödinger equation, where one works with energy methods, one has similar phenomena for initial data u_0 in Sobolev spaces $H^s(\mathbb{R}^N)$: again one finds, corresponding to the power nonlinearity $|u|^{p-1}u$, an associated critical space $H^{s_c}(\mathbb{R}^N)$ with $s_c = \frac{N}{2} - \frac{2}{p-1}$. Cazenave-Weissler [8] showed local-in-time existence for all $u_0 \in H^s(\mathbb{R}^N)$ for $s \geq s_c$, and global-in-time existence for small data for $s = s_c$. The critical exponents for the H^s-theory for the Nonlinear Schrödinger and heat equations coincide, while the critical exponents for the H^s-theory and the L^p-theory for the heat equation are related by the Sobolev embedding: $H^{s_c} \subset L^{q_c}$, with $q_c = \frac{2N}{N-2s_c} = \frac{N}{2}(p-1)$.

In the limiting critical case $s_c = \frac{N}{2}$ we have again that $H^{N/2}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$, for all $q \geq 1$, but $H^{N/2}(\mathbb{R}^N) \not\subset L^\infty(\mathbb{R}^N)$. By a result by S. Pohozaev [23] and N. Trudinger [28] we know that for $u \in H^{N/2}$ one has $\int_{\mathbb{R}^N}(e^{u^2} - 1)dx < \infty$, and this is the maximal growth for integrability. Using nonlinearities with this type of growth in the Nonlinear Schrödinger equation (NLS equation)

$$i\partial_t u + \Delta u = f(u) \quad \text{with} \quad f(u) \sim e^{u^2}$$

Nakamura-Ozawa [21] were indeed able to prove a global-in-time existence result for small initial data in $H^{N/2}(\mathbb{R}^N)$, and so in particular in $H^1(\mathbb{R}^2)$ for $N = 2$. For other related results we refer to [9].

2.3 Back to the heat equation

The result of Nakamura-Ozawa was recently transposed to the heat equation by Ibrahim-Jrad-Majdoub-Saanouni [15], showing *local*-in-time existence and uniqueness for the equation (1.1), with $f(u) \sim e^{u^2}$, $x \in \mathbb{R}^2$, and for any initial data $u_0 \in H^1(\mathbb{R}^2)$. Two observations are in order:

– the initial data space $H^1(\mathbb{R}^2)$ is natural for the NLS equation, where one works with energy methods, but less so for the heat equation, where an integrability condition on the initial data ought to be sufficient;

– by Nakamura-Ozawa [21] a *global-in-time* result holds for the NLS equation with $f(u) \sim e^{u^2}$, for *small data* in $H^1(\mathbb{R}^2)$; comparing with the critical case for polynomial nonlinearities, one can say that $f(u) \sim e^{u^2}$ behaves like a critical growth nonlinearity for the NLS equation. However, the uniqueness result in [15] suggests that $f(u) \sim e^{u^2}$ with initial data in $H^1(\mathbb{R}^2)$ is not a double critical case (in the sense of Remark 2.1).

Here we are looking, in dimension $N = 2$, for a data space which has similar "double critical" phenomena as described in Remark 2.1. We propose the Orlicz space determined by the mentioned estimates by Pohozaev and Trudinger, namely $H^1(\mathbb{R}^2) \subset L^\varphi(\mathbb{R}^2)$ with

Young-function $\varphi(t) = e^{t^2} - 1$ (for details, see Section 3.1 below). We will denote this space by $\exp L^2(\mathbb{R}^2) := L^\varphi(\mathbb{R}^2)$. In fact, in [25, 16, 17], small-data global-existence and large-data non-existence result were proved for this space.

In this paper, we focus on the following particular case of an exponential nonlinearity with Trudinger-Moser growth. Consider the nonlinearity $f(s)$ given by

$$f(s) := \begin{cases} \dfrac{1}{|s|^3}\, e^{s^2} & \text{if } |s| > \beta, \\ \alpha s^2 & \text{if } |s| \le \beta \end{cases} \tag{2.1}$$

with $\alpha = \frac{e^{5/2}}{(5/2)^{5/2}}$ and $\beta = \sqrt{\frac{5}{2}}$ such that the function f belongs to $C^1(\mathbb{R})$, it is increasing on $[0, +\infty)$ and convex on $\mathbb{R}$. We will show that the nonlinearity (2.1), together with suitable initial data, shows all the phenomena of a double critical case for the 2-dimensional problem, with respect to existence, non-existence, uniqueness and non-uniqueness.

To this end, we first prove the existence of a radial singular solution for the Dirichlet boundary value problem in $B_\rho \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = f(u) & \text{in } B_\rho, \\ u(x) = 0 & \text{on } \partial B_\rho \end{cases} \tag{2.2}$$

for some $\rho > 0$. By a *singular* solution we mean a solution which belongs to $C^2(B_\rho \setminus \{0\})$, which is unbounded on B_ρ and which satisfies the elliptic equation in the sense of distributions on B_ρ. Moreover this solution $\widetilde{u}$ belongs to the Orlicz space $\exp L^2(B_\rho)$. More precisely, we prove

Proposition 2.1 *There exist a constant $\rho > 0$ and a function $\widetilde{u} \in C^2(B_\rho \setminus \{0\}) \cap C(\overline{B}_\rho \setminus \{0\})$ which is a classical solution on $B_\rho \setminus \{0\}$ for the Dirichlet boundary value problem* (2.2)*. Moreover, the following hold:*

(i) $\widetilde{u}(x) = \sqrt{-2\log(|x|)}$ *in a neighborhood of the origin;*

(ii) $\widetilde{u}$ *is a solution of the elliptic equation* (2.2) *on B_ρ in the sense of distributions.*

Remark 2.2

a) With the change of variable $y = \frac{x}{\rho}$ and the corresponding changes in the nonlinearity $f(u) \rightsquigarrow \rho^2 f(u)$ and initial datum $\widetilde{u}(x) \rightsquigarrow \widetilde{u}(\frac{x}{\rho})$ the equation can be considered on $B_1(0) \subset \mathbb{R}^2$.

b) The nonlinearity $f(s)$ may be generalized to

$$f(s) = \begin{cases} \frac{1}{|s|^3}\, e^{s^2} & , \quad |s| > \beta_p \\ \alpha_p\, s^p & , \quad |s| \le \beta_p \end{cases}$$

for any choice of $p > 1$ and suitable values α_p, β_p (which are uniquely dependent on p since $f(s)$ is required to be of class $C^1(\mathbb{R})$).

The particular form of the nonlinearity (2.1) is due to the existence of the (almost explicit) singular solution given in Proposition 2.1.(i). It would be of interest to prove the existence of singular distributional solutions for equation (2.2) for more general nonlinearities.

2.4 Main result: A heat equation in 2-dimensions with double critical phenomena

Let us now consider the following Cauchy problem with Dirichlet boundary condition on $B_\rho \subset \mathbb{R}^2$

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \ B_\rho, \ t > 0, \\ u(t,x) = 0 & \text{on } \partial B_\rho, \ t > 0, \\ u(0,x) = u_0(x) & \text{in } \ B_\rho, \end{cases} \tag{2.3}$$

where the nonlinear term $f(u)$ is defined in (2.1). We will show that the singular function $\widetilde{u}$ obtained in Proposition 2.1 yields a neat separation into the cases of well-posedness, non-uniqueness and non-existence, and so we may say that we are in a "double critical" situation in the sense of Remark 2.1.

To state the theorem, we denote the Schwarz symmetrization of a measurable function $\varphi : B_\rho \to \mathbb{R}$ by $\varphi^\sharp$ (for details, see Section 3.3). Moreover we introduce the complete metric space for $T, \ \mu^* > 0$,

$$M_{T,\mu^*} = \Big\{ u \in L^\infty(0,T; \exp L^2(B_\rho)) : \sup_{t\in(0,T)} \|u(t)\|_{L^f_\gamma(B_\rho)} \leq \mu^* \Big\}, \tag{2.4}$$

where $\|\cdot\|_{L^f_\gamma}$ is the Luxemburg norm defined by

$$\|u\|_{L^f_\gamma(B)} = \inf\left\{ \lambda > 0 : \ \int_B f\left(\frac{|u(x)|}{\lambda}\right) dx \leq \gamma \right\}$$

with $\gamma = \int_{B_\rho} f(\tilde{u}(x))dx < \infty$. For the definitions of the Orlicz space $\exp L^2(B_\rho)$ with the Luxemburg norm $\|\cdot\|_{L^f_\gamma}$ under specific choice of γ, and of weak and $\exp L^2-$classical solutions, see Sections 3.1 and 3.4.

Theorem 2.1 *Let $\widetilde{u}$ denote the singular solution of the elliptic equation* (2.2) *given by Proposition 2.1.*

1) (well-posedness) If the initial datum u_0 in (2.3) *satisfies*

$$\mu := \sup_{x\in B_\rho} \frac{u_0{}^\sharp(x)}{\widetilde{u}(x)} < 1, \tag{2.5}$$

then problem (2.3) *is well-posed, i.e. for any $\mu < \mu_1 < 1$ there exist a positive time $T = T(\mu_1) > 0$ and a unique function u in the complete metric space M_{T,μ_1} which is a weak solution of the Cauchy problem* (2.3)*. Furthermore, it is an $\exp L^2-$classical solution of* (2.3) *on $(0,T) \times B_\rho$.*

2) (existence and non-uniqueness) If the initial datum u_0 satisfies

$$\mu = \sup_{x\in B_\rho} \frac{u_0{}^\sharp(x)}{\widetilde{u}(x)} \leq 1, \tag{2.6}$$

then (2.3) *admits an $\exp L^2-$classical solution u in some time interval $(0,T)$. If $\mu < 1$ this solution belongs to M_{T,μ_1} for some $\mu < \mu_1 < 1$ (for sufficiently small T), and hence coincides with the solution obtained in 1). If $\mu = 1$ for any $\mu_2 > 1$ the solution belongs to M_{T,μ_2} for some T and may not be unique in this space.*

Indeed, for $u_0 = \widetilde{u}$ the equation (2.3) *has, in addition to this classical solution, the singular stationary (distributional) solution $\widetilde{u}$ which belongs to $M_{T,1} \subset M_{T,\mu_2}$.*

3) (non-existence) Let $u_0 = \mu\,\widetilde{u}$, with $\mu > 1$. Then the problem (2.3) *does not possess non-negative $\exp L^2-$classical solutions on any positive time interval $(0,T)$.*

Remark 2.3
a) The solution in Theorem 2.1.1) can be continued as long as $\mu(u(t)) := \sup_{x\in B_\rho} \frac{u^\sharp(t,x)}{\widetilde{u}(x)} < 1$. *If* $\mu(u(t^*)) = 1$ *for some* $t^* > 0$, *then the local theory fails and non-uniqueness may occur.*

b) Since $\widetilde{u}$ *is a radially symmetric and non-increasing function, the Schwarz symmetrization of* $\widetilde{u}$ *coincides with* $\widetilde{u}$. *Therefore, Theorem A 1) and 2) are particular cases of Theorem 2.1 with* $u_0 = \mu\widetilde{u}$, $0 < \mu < 1$ *and* $u_0 = \widetilde{u}$, *respectively.*

Remark 2.4 *We mention that, with different techniques, Galaktionov-Vazquez [14] and Souplet-Weissler [26] proved similar results for the heat equation with polynomial nonlinearity. Indeed, if* $N > 2$ *and* $p > \frac{N}{N-2}$ *the function* $V(x) = \beta^{1/(p-1)}|x|^{-2/(p-1)}$, *where* $\beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)$ *is an explicit stationary distributional solution for the equation* (1.1) *with* $f(s) = |s|^{p-1}s$. *For* $N > 2$ *and for any* $\frac{N}{N-2} < p < p^*$ *(where* $p^* = +\infty$ *if* $N \leq 10$ *and* $p^* = \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$ *if* $N > 10$*) the equation with initial data* $\mu V(x)$, *with* $\mu \in [1, 1+\varepsilon)$ *for* $\varepsilon > 0$ *small enough, admits at least a nonnegative regular solution* $u(t)$ *that converges to* $\mu V(x)$ *in the sense of distributions as* $t \to 0$. *This implies similar phenomena of non-uniqueness as in part 2) of Theorem 2.1. Moreover, for large values of* μ *the Cauchy problem with initial data* $\mu V(x)$ *has no local nonnegative solution (see [32]).*

3 Preliminary results

Let $B \subset \mathbb{R}^2$ be a ball centered at the origin. In this section we recall some properties of Orlicz and Lorentz spaces on B, and of the heat kernel in these spaces. We also introduce the definition of weak and $\exp L^2-$classical solution of the problem (2.3).

3.1 Orlicz spaces

Let us recall the definition of the Orlicz space $L^\varphi(B)$, where $\varphi(u)$ is a Young function (convex, $\varphi(0) = 0$). First we introduce the *Orlicz class* $K^\varphi(B)$ by

$$K^\varphi(B) = \Big\{ u \in L^1(B) : \int_B \varphi\Big(|u(x)|\Big)dx < +\infty \Big\}.$$

Then the *Orlicz space* $L^\varphi(B)$ is given by the linear hull of the Orlicz class $K^\varphi(B)$ and its norm is given by the Luxemburg type

$$\|u\|_{L^\varphi(B)} := \inf\left\{ \lambda > 0 : \int_B \varphi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

For $\varphi(u) = e^{u^2} - 1$ we define $\exp L^2(B) = L^\varphi(B)$. Let now f be the convex function defined in (2.1). Since for any $0 < b < 1$ there exist $C_1, C_2 > 0$ such that

$$C_1\left(\mathrm{e}^{b\,u^2} - 1\right) \leq f(u) \leq C_2\left(\mathrm{e}^{u^2} - 1\right), \tag{3.1}$$

we have that the Orlicz space $\exp L^2(B)$ coincides with the Orlicz space generated by the convex function f, namely,

$$\exp L^2(B) = L^f(B)$$

and this space can be endowed with the following equivalent norm

$$\|u\|_{L^f_\gamma(B)} = \inf\left\{ \lambda > 0 : \int_B f\left(\frac{|u(x)|}{\lambda}\right) dx \leq \gamma \right\} \tag{3.2}$$

for any fixed positive constant γ. Indeed, we have

Proposition 3.1 *Let $\gamma > 0$. There exist two positive constants c, C such that*

$$c\|u\|_{L^f(B)} \le \|u\|_{L^f_\gamma(B)} \le C\|u\|_{L^f(B)} \tag{3.3}$$

and

$$c\|u\|_{\exp L^2(B)} \le \|u\|_{L^f(B)} \le C\|u\|_{\exp L^2(B)}. \tag{3.4}$$

Furthermore, in (3.3) *one may choose $c = \min(1, \frac{1}{\gamma})$ and $C = \max(1, \frac{1}{\gamma})$.*

Proof. Let us prove the first inequality. Assume $0 < \gamma < 1$. By the definition we get directly $\|u\|_{L^f(B)} \le \|u\|_{L^f_\gamma(B)}$. On the other hand thanks to the convexity of f and the property $f(0) = 0$ we obtain

$$f\Big(\gamma\frac{u}{\lambda}\Big) = f\Big(\gamma\frac{u}{\lambda} + (1-\gamma)0\Big) \le \gamma f\Big(\frac{u}{\lambda}\Big) + (1-\gamma)f(0) = \gamma f\Big(\frac{u}{\lambda}\Big).$$

Therefore it holds

$$\begin{aligned}\|u\|_{L^f(B)} &= \inf\Big\{\lambda > 0 : \int_B \gamma f\Big(\frac{u}{\lambda}\Big)dx \le \gamma\Big\}\\ &\ge \inf\Big\{\lambda > 0 : \int_B f\Big(\frac{\gamma u}{\lambda}\Big)dx \le \gamma\Big\}\\ &= \gamma\|u\|_{L^f_\gamma(B)}.\end{aligned}$$

For $\gamma > 1$ we can apply similar arguments to $0 < \frac{1}{\gamma} < 1$. The second inequality follows from the relation (3.1) and from the definition of Orlicz space (see [1, Section 8.4 and 8.12]). This completes the proof of Proposition 3.1.

In this paper we choose $\gamma := \int_{B_\rho} f(\widetilde{u}(x))dx$. It will be proved in Section 4 that $f(\widetilde{u})$ is integrable, therefore γ is well-defined. This special choice of γ is one of the keys to reach a neat classification as in Theorem 2.1.

3.2 Heat kernel

Now we collect some results concerning the solution of the heat equation on the ball (see Appendix B in [24]). Let us denote by $e^{t\Delta}$ the Dirichlet heat semigroup in B. It is known that for any $\phi \in L^p(B)$, $1 \le p \le +\infty$, the function $u = e^{t\Delta}\phi$ solves the heat equation $u_t - \Delta u = 0$ in $(0, +\infty) \times B$ and $u \in C((0, +\infty) \times \overline{B})$, $u = 0$ on $(0, +\infty) \times \partial B$. Moreover, there exists a positive C^∞ function $G_B : B \times B \times (0, +\infty) \to \mathbb{R}$ (the Dirichlet heat kernel) such that

$$\mathrm{e}^{t\Delta}\phi(x) = \int_B G_B(x, y, t)\phi(y)dy,$$

for any $\phi \in L^p(B)$, $1 \le p \le +\infty$. We prepare several basic lemmas.

Lemma 3.1 *Let $\phi : B \to [0, \infty)$ be a measurable function and $H : \mathbb{R} \to \mathbb{R}$ be a convex function such that $H(0) = 0$. Then*

$$H\left(\mathrm{e}^{t\Delta}\phi\right) \le \mathrm{e}^{t\Delta}H\left(\phi\right).$$

Proof. Let H be a convex function and $\phi \ge 0$ be a measurable function. By Jensen's inequality, denoting $\overline{G} = \overline{G}(x, t) = \int_B G_B(x, y, t)dy$, we obtain

$$H\Big(\frac{1}{\overline{G}(x,t)}\int_B G_B(x, y, t)\phi(y)dy\Big) \le \frac{1}{\overline{G}(x,t)}\int_B G_B(x, y, t)H\big(\phi(y)\big)dy.$$

Therefore

$$H\Big(\frac{\mathrm{e}^{t\Delta}\phi}{\overline{G}}\Big) \le \frac{1}{\overline{G}}\ \mathrm{e}^{t\Delta} H\left(\phi\right). \tag{3.5}$$

Moreover by the convexity of H, the property $H(0) = 0$, and $\overline{G}(x,t) \le 1$ for any $x \in B$ and $t > 0$ we have

$$H(s) = H\Big(\overline{G}\frac{s}{\overline{G}} + (1-\bar{G})0\Big) \le \overline{G}\ H\left(\frac{s}{\overline{G}}\right)$$

and so for $s = \mathrm{e}^{t\Delta}\phi$ we get

$$\frac{H(e^{t\Delta}\phi)}{\overline{G}} \le H\Big(\frac{\mathrm{e}^{t\Delta}\phi}{\overline{G}}\Big). \tag{3.6}$$

Finally, (3.5) and (3.6) imply the desired inequality

$$H\left(\mathrm{e}^{t\Delta}\phi\right) \le \mathrm{e}^{t\Delta} H\left(\phi\right).$$

Lemma 3.2 *There holds*

$$\|e^{t\Delta}\phi\|_{L^f_\gamma} \le \|\phi\|_{L^f_\gamma}$$

for all $t > 0$ *and* $\phi \in L^f_\gamma(B)$.

Proof. Here f is the function in (2.1). Since f is convex on $\mathbb{R}$ and $f(0) = 0$, it follows from the previous Lemma and the property $\overline{G}(x,t) \le 1$ for any $x \in B$ and $t > 0$ that

$$\int_B f\Big(\frac{|e^{t\Delta}\phi|}{\lambda}\Big)dx \le \int_B f\Big(\frac{e^{t\Delta}|\phi|}{\lambda}\Big)dx \le \int_B e^{t\Delta} f\Big(\frac{|\phi|}{\lambda}\Big)dx \le \int_B f\Big(\frac{|\phi|}{\lambda}\Big)dx.$$

This yields the desired estimate.

Lemma 3.3 *Assume* $1 \le p \le 2$. *There exists a positive constant* C *such that*

$$\|e^{t\Delta}\phi\|_{L^f_\gamma(B)} \le C\, t^{-\frac{1}{p}}\Big(\log\left(t^{-1}+1\right)\Big)^{-1/2}\|\phi\|_{L^p(B)}$$

for all $\phi \in L^p(B)$, $t > 0$.

This lemma in the whole space $\mathbb{R}^n$ was proved in [16, Lemma 2.2]. The same method works in B_ρ since we only need the $L^p - L^q$ estimate of the heat kernel which still holds in B_ρ.

3.3 Lorentz spaces and heat kernel

We present some regularizing properties of the heat kernel in Lorentz spaces. We recall the definition of Lorentz spaces $L^{p,q}(B)$ on a ball $B \subset \mathbb{R}^2$. Let ϕ be a measurable function on B, which is finite almost everywhere. We define the distribution function

$$\mu(\lambda,\phi) = |\{x \in B : |\phi(x)| > \lambda\}|, \quad \lambda \ge 0.$$

The decreasing rearrangement of ϕ is the function ϕ^* defined on $[0,\infty)$ by

$$\phi^*(t) = \inf\{\lambda > 0 : \mu(\lambda,\phi) \le t\}, \quad t \ge 0.$$

The Lorentz space $L^{p,q}(B)$, with $1 \le p < \infty$ consists of all Φ measurable on B and finite a.e. for which the quantity

$$\|\Phi\|^*_{L^{p,q}(B)} = \Big(\int_0^\infty (t^{1/p}\Phi^*(t))^q \frac{dt}{t}\Big)^{1/q} \qquad \text{when} \ \ 1 \le q < \infty,$$
$$\|\Phi\|^*_{L^{p,\infty}(B)} = \sup_{t>0} t^{1/p}\Phi^*(t) \qquad \text{when} \ \ q = \infty$$

is finite. In general, $\|\cdot\|^*_{L^{p,q}(B)}$ is a quasi-norm, but when $p>1$ it is possible to replace the quasi-norm with a norm, which makes $L^{p,q}(B)$ a Banach space. In the following we will denote by $\|\cdot\|_{L^{p,q}(B)}$ this norm (see [1, Section 7.25]).

The Lorentz spaces can also be defined using Schwarz symmetrization $\Phi^\sharp$ of Φ, given by $\Phi^\sharp(x) := \Phi^*(\pi|x|^2)$; therefore $\Phi \in L^{p,q}(B)$, $1 \le p < \infty$, if and only if

$$\begin{aligned} &\Big(\int_B \Big(|x|^{\frac{2}{p}}\Phi^\sharp(x)\Big)^q \frac{dx}{|x|^2}\Big)^{\frac{1}{q}} < \infty \qquad && \text{when } \ 1 \le q < \infty, \\ &\sup_{x\in B} |x|^{2/p}\Phi^\sharp(x) < \infty, && \text{when } \ q = \infty. \end{aligned}$$

Lemma 3.4 *Let $1 \le q < \infty$ and $1 < p \le r < \infty$. There exists a positive constant $C > 0$ such that*

$$t^{1/p-1/r}\|\mathrm{e}^{t\Delta}\phi\|_{L^{r,q}(B)} \le C\|\phi\|_{L^{p,q}(B)} \qquad \textit{for all } t > 0.$$

Moreover for $1 < p < r < \infty$ and for all $\phi \in L^{p,q}(B)$ we have

$$\lim_{t\to 0} t^{1/p-1/r}\|\mathrm{e}^{t\Delta}\phi\|_{L^{r,q}(B)} = 0. \tag{3.7}$$

Proof. The first assertion in the lemma is proved by the L^p-L^q estimate of the heat kernel (see [24, Proposition 48.4]) and real interpolation methods (see [2, Theorem 5.3.2]). The second assertion is a consequence of the density of C_0^∞ in $L^{p,q}(B)$ with $1 \le q < \infty$.

3.4 Weak and classical solutions

We now present the notions of weak and classical solution for the Cauchy problem (2.3) with initial data $u_0 \in \exp L^2(B_\rho)$ where B_ρ is the ball centered at the origin and of radius $\rho > 0$. For the sake of simplicity we will omit the underlying space B_ρ.

Definition 3.1 (Weak solution)
Let $u_0 \in \exp L^2$ and $u \in L^\infty(0,T;\exp L^2)$ for some $T \in (0,+\infty]$. We call u a weak solution of the Cauchy problem (2.3) *if u satisfies the differential equation $\partial_t u - \Delta u = f(u)$ in $\mathcal{D}'((0,T)\times B_\rho)$ and $u(t) \to u_0$ in weak* topology as $t \to 0$.*

We recall that $u(t) \to u_0$ in weak* topology as $t \to 0$ if and only if

$$\lim_{t\to 0}\int_{B_\rho} \big(u(t,x) - u_0(x)\big)\,\psi(x)dx = 0$$

for every ψ belonging to the predual space of $\exp L^2$. The predual space of $\exp L^2$ is the Orlicz space defined by the complementary function of $A(t) = e^{t^2} - 1$, denoted by $\widetilde{A}(t)$. This complementary function is a convex function such that $\widetilde{A}(t) \sim t^2$ as $t \to 0$ and $\widetilde{A}(t) \sim t\log^{1/2} t$ as $t \to +\infty$.

Definition 3.2 (Classical solution)
Let $u_0 \in \exp L^2$ and $u \in C((0,T],\exp L^2) \cap L^\infty_{loc}((0,T),L^\infty)$ for some $T \in (0,+\infty]$. We say that the function u is an $\exp L^2$-classical solution of the Cauchy problem (2.3) *in $(0,T]$ if $\big\|u(t) - \mathrm{e}^{t\Delta}u_0\big\|_{\exp L^2} \to 0$ as $t \to 0$, u is C^1 in $t \in (0,T)$, C^2 in $x \in B_\rho$, continuous on $\overline{B}_\rho$ and u is a classical solution* (2.3) *on $(0,T)\times B_\rho$.*

We remark that any $\exp L^2$-classical solution of the Cauchy problem (2.3) is also a weak solution. Indeed we have that $u \in L^\infty(0,\varepsilon;\exp L^2)$ for some $\varepsilon > 0$ and this is a consequence of the inequality

$$\|u(t)\|_{\exp L^2} \le \big\|u(t) - \mathrm{e}^{t\Delta}u_0\big\|_{\exp L^2} + \big\|\mathrm{e}^{t\Delta}u_0\big\|_{\exp L^2}$$

and

$$\left\| u(t) - \mathrm{e}^{t\Delta} u_0 \right\|_{\exp L^2} \to 0, \quad t \to 0.$$

Finally $u(t) \to u_0$ in the weak* topology as $t \to 0$ since $\mathrm{e}^{t\Delta} u_0 \to u_0$ in the weak* topology as $t \to 0$ and $u(t) - \mathrm{e}^{t\Delta} u_0 \to 0$ in $\exp L^2$.

4 Construction of a singular stationary solution

In this section we prove the existence of a radial singular solution for the Dirichlet boundary value problem (2.2) in $B_\rho \subset \mathbb{R}^2$, for a well chosen $\rho > 0$, by using the shooting method (see [7] and [19]); that is, we give the

Proof of Proposition 2.1.
Defining

$$U(r) = \sqrt{-2 \log r},$$

one easily checks that U solves

$$-U'' - \frac{1}{r} U' = \frac{1}{U^3} \, e^{U^2}, \quad 0 < r < 1.$$

The solution U was found by de Figueiredo-Ruf in [10, p. 653].

Let $f(s)$ as in (2.1). We want to continue the solution U to a solution of

$$\begin{cases} -u'' - \dfrac{1}{r} u' = f(u) & \text{in } \ (0, \rho), \\ u(\rho) = 0, & \\ u(r) > 0 & \text{in } \ (0, \rho), \end{cases} \tag{4.1}$$

where ρ will be determined later.
Note that the solution $U(r) = \sqrt{-2 \log r}$ satisfies

$$U(r) \geq \sqrt{\frac{5}{2}} \iff r \leq \frac{1}{e^{5/4}}.$$

Let us consider the following equation

$$\begin{cases} -v'' - \frac{1}{r} \, v' = \alpha \, v^2 \ , \ r \geq \frac{1}{e^{5/4}} \ , \\ v\left(\frac{1}{e^{5/4}} \right) = \sqrt{\frac{5}{2}}, \\ v'\left(\frac{1}{e^{5/4}} \right) = U'\left(\frac{1}{e^{5/4}} \right) = -\frac{e^{5/4}}{\sqrt{5/2}}. \end{cases} \tag{4.2}$$

We now prove that there exists a first zero $\rho > \frac{1}{e^{5/4}}$ of the solution $v(r)$ of the problem (4.2) by using a shooting method and a contradiction argument.

By contradiction, assume that $v(r) > 0$, for all $r > \frac{1}{e^{5/4}}$. Then $v'(r) < 0$, for all $r > \frac{1}{e^{5/4}}$; if not, there would exist r_0 with $v'(r_0) = 0$ and $v''(r_0) \geq 0$, but then $-v''(r_0) = \alpha v^2(r_0) > 0$, which is impossible. It follows from the above argument that $v(r)$ has a limit $L \geq 0$, as $r \to \infty$. We first show that $L = 0$. Indeed, consider the energy

$$E(v, r) := \frac{1}{2} |v'(r)|^2 + \frac{\alpha}{3} v(r)^3.$$

Multiplying the equation of (4.2) by $v'(r)$, we obtain

$$-v''(r)v'(r) - \frac{1}{r}|v'(r)|^2 = \alpha v(r)^2\, v'(r)$$

and so it follows

$$\frac{d}{dr}E(v,r) = v'(r)\, v''(r) + \alpha v(r)^2\, v'(r) = -\frac{1}{r}\,|v'(r)|^2.$$

This yields that $E(v,r)$ is decreasing, and hence

$$|v'(r)|^2 \leq 2\, E\Big(v, \frac{1}{e^{5/4}}\Big).$$

Then, using again the equation of (4.2), we conclude for $r \to \infty$

$$-v''(r) - \frac{1}{r}\, v'(r) = \alpha\, v(r)^2 \to \alpha\, L^2$$

that

$$v''(r) \to -\alpha\, L^2,$$

from which we obtain $L = 0$. We now derive a contradiction by using $L = 0$. Observe that

$$\left(rv'(r) - \frac{1}{e^{5/4}}\, v'\left(\frac{1}{e^{5/4}}\right)\right)' = v'(r) + rv''(r) = -r\alpha\, v(r)^2$$

and hence

$$r\, v'(r) - \frac{1}{e^{5/4}}\, v'\big(\frac{1}{e^{5/4}}\big) = -\int_{1/e^{5/4}}^{r} s\,\alpha\, v(s)^2 ds. \tag{4.3}$$

Therefore

$$\begin{aligned} -r\, v'(r) &= \int_{1/e^{5/4}}^{r} s\,\alpha\, v(s)^2 ds + \sqrt{\frac{2}{5}} \\ &\geq \alpha v(r)^2 \int_{1/e^{5/4}}^{r} sds + \sqrt{\frac{2}{5}} \\ &> \alpha v(r)^2\, \frac{r^2}{2}. \end{aligned}$$

This implies that $\dfrac{1}{v(r)} - \dfrac{\alpha\, r^2}{4}$ is increasing. Thus

$$\frac{1}{v(r)} - \frac{\alpha\, r^2}{4} > \frac{1}{v\left(\frac{1}{e^{5/4}}\right)} - \frac{\alpha}{e^{5/2}\, 4} = \sqrt{\frac{2}{5}} - \frac{1}{4}\left(\frac{2}{5}\right)^{5/2} > 0,$$

which yields

$$\frac{4}{\alpha}\, r^{-2} > v(r)$$

and

$$\int_{1/e^{5/4}}^{\infty} r\, v(r)^2 dr \leq \int_{1/e^{5/4}}^{\infty} r\, \frac{16}{\alpha^2}\, r^{-4} dr < \infty.$$

It follows from (4.3) that there exists $A > 0$ such that

$$r\, v'(r) = \frac{1}{e^{5/4}}\, v'\left(\frac{1}{e^{5/4}}\right) - \alpha \int_{1/e^{5/4}}^{r} s\, v(s)^2 ds \to -A < 0.$$

Hence

$$v(r) = \int_{1/e^{5/4}}^{r} v'(s)ds + v\left(\frac{1}{e^{5/4}}\right) \le C\int_{1/e^{5/4}}^{r} -\frac{A}{s}\,ds \le -AC(\log s)\big|_{1/e^{5/4}}^{r} \to -\infty \quad \text{as } r \to +\infty.$$

This yields a contradiction, and hence there must exist a first zero ρ for $v(r)$.
By the above argument, we see that

$$w(r) := \begin{cases} U(r), & 0 < r < \frac{1}{e^{5/4}}, \\ v(r), & \frac{1}{e^{5/4}} \le r \le \rho, \end{cases}$$

satisfies the equation (4.1). In the following we define

$$\widetilde{u}(x) = w(|x|) = \begin{cases} U(|x|), & 0 < |x| < \frac{1}{e^{5/4}}, \\ v(|x|), & \frac{1}{e^{5/4}} \le |x| \le \rho. \end{cases}$$

We stress that $\widetilde{u}$ belongs to $C^2(B_\rho \setminus \{0\}) \cap C(\overline{B_\rho} \setminus \{0\})$, $\widetilde{u}(x) = 0$ on $|x| = \rho$ and

$$\widetilde{u}(x) = \sqrt{-2\log|x|}, \qquad |x| \le \frac{1}{e^{5/4}}$$

and it is a classical solution of the elliptic equation on $B_\rho \setminus \{0\}$.

It remains to prove that the solution $\widetilde{u}$ satisfies the elliptic equation in the sense of distributions in B_ρ. We use similar arguments as in [5], page 265 and in [22], pages 261-262. Let φ be a C^∞ function with compact support in B_ρ. We prove that

$$\int_{B_\rho} \widetilde{u}\ \Delta\varphi + \ f(\widetilde{u})\ \varphi\ dx = 0.$$

Indeed let $\Phi(r)$ be a $C^\infty(\mathbb{R})$ function, $0 \le \Phi(r) \le 1$ such that

$$\Phi(r) = \begin{cases} 1 & \text{if } \ r < 1/2, \\ 0 & \text{if } \ r \ge 1, \end{cases}$$

and $\Phi_\varepsilon(|x|) = \Phi\left(\frac{\log|x|}{\log\varepsilon}\right)$ for any $x \neq 0$ (these cut-off functions are the same as those used in [5]). By a direct computation for small $\varepsilon > 0$ we get $\Phi_\varepsilon(|x|) = 1$ for $|x| > \sqrt{\varepsilon}$ and $\Phi_\varepsilon(|x|) = 0$ for $|x| \le \varepsilon$ and for $x \neq 0$, we get $\Phi_\varepsilon(|x|) \to 1$ for $\varepsilon \to 0^+$. By the Dominated Convergence Theorem, since $\widetilde{u}$ and

$$f(\widetilde{u}) = \begin{cases} \dfrac{1}{|x|^2(-2\log|x|)^{3/2}} & \text{if } \ 0 < |x| < \dfrac{1}{e^{5/4}}, \\ \alpha\, v^2(|x|) & \text{if } \ \dfrac{1}{e^{5/4}} \le |x| < \rho \end{cases} \tag{4.4}$$

belong to $L^1(B_\rho)$, we have

$$\begin{aligned} &\int_{B_\rho} \widetilde{u}\,\Delta\varphi + f(\widetilde{u})\,\varphi\ dx \\ &= \lim_{\varepsilon\to 0^+} \int_{B_\rho} \Phi_\varepsilon \widetilde{u}\,\Delta\varphi + \Phi_\varepsilon f(\widetilde{u})\ \varphi\ dx \\ &= \lim_{\varepsilon\to 0^+} \int_{B_\rho} \Phi_\varepsilon\,\Delta\widetilde{u}\,\varphi\ dx + 2\int_{B_\rho} \nabla\Phi_\varepsilon\cdot\nabla\widetilde{u}\ \varphi\ dx + \int_{B_\rho} \Delta\Phi_\varepsilon\,\widetilde{u}\,\varphi\ dx + \int_{B_\rho} \Phi_\varepsilon\, f(\widetilde{u})\,\varphi\ dx. \end{aligned}$$

Since $\widetilde{u}$ is a classical solution of the elliptic equation in $B_\rho \setminus \{0\}$ we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_{B_\rho}\Phi_\varepsilon\,\widetilde{u}\,\Delta\varphi+\Phi_\varepsilon f(\widetilde{u})\;\varphi\;dx\\&=\lim_{\varepsilon\to 0^+}2\int_{B_\rho}\nabla\Phi_\varepsilon\cdot\nabla\widetilde{u}\;\varphi\;dx+\int_{B_\rho}\Delta\Phi_\varepsilon\,\widetilde{u}\,\varphi\;dx.\end{aligned}$$

Since

$$\Delta\Phi_\varepsilon=\Phi''\Big(\frac{\log r}{\log\varepsilon}\Big)\frac{1}{r^2(\log\varepsilon)^2}$$

we have

$$\Big|\int_{B_\rho}\widetilde{u}\,\Delta\Phi_\varepsilon\,\varphi\;dx\Big|\le\frac{C}{(\log\varepsilon)^2}\int_\varepsilon^{\sqrt{\varepsilon}}\frac{\sqrt{-2\log(r)}}{r}\;dr$$

and

$$\lim_{\varepsilon\to 0^+}\frac{\int_\varepsilon^{\sqrt{\varepsilon}}\frac{\sqrt{-2\log r}}{r}\;dr}{(\log\varepsilon)^2}=\lim_{\varepsilon\to 0}\frac{2\sqrt{2}-1}{3\sqrt{-\log\varepsilon}}=0.$$

In a similar way

$$\Big|\int_{B_\rho}\nabla\widetilde{u}\cdot\nabla\Phi_\varepsilon\;\varphi\;dx\Big|\le\frac{C}{(-\log\varepsilon)}\int_\varepsilon^{\sqrt{\varepsilon}}\frac{1}{r\sqrt{-2\log r}}\;dr$$

and

$$\lim_{\varepsilon\to 0^+}\frac{\int_\varepsilon^{\sqrt{\varepsilon}}\frac{1}{r\sqrt{-2\log r}}\;dr}{(-\log\varepsilon)}=\lim_{\varepsilon\to 0^+}\frac{\sqrt{2}-1}{\sqrt{-\log\varepsilon}}=0.$$

This proves that the function $\widetilde{u}$ satisfies the equation (2.2) in the sense of distributions.

5 Well-posedness result

In this section we consider the Cauchy problem (2.3) where the initial datum $u_0(x)$ is a measurable function satisfying

$$\mu:=\sup_{x\in B_\rho}\frac{u_0^\sharp(x)}{\widetilde{u}(x)}<1. \tag{5.1}$$

A typical example of such initial data is $u_0=\mu\widetilde{u}(x)$ for $0<\mu<1$.

Recall that $\int_{B_\rho}f(\widetilde{u})\;dx<+\infty$ by (4.4), hence one can choose $\gamma=\int_{B_\rho}f(\widetilde{u})dx$. With this choice of γ, we now prove the well-posedness result 1) in Theorem 2.1. Let $\max\{\mu,\frac{1}{\sqrt{2}}\}<\mu_1<1$ and consider the complete metric space M_{T,μ_1} introduced in (2.4). We prove that there exist a positive time $T=T(\mu_1)$ and a unique function $u\in M_{T,\mu_1}$ which is a weak solution of (2.3).
First, we make the following:

Remark 5.1 *The initial data satisfying* (5.1) *belong to* M_{T,μ_1}. *Indeed, the definition of* γ *and a standard property of the rearrangement yield that*

$$\|\widetilde{u}\|_{L^f_\gamma}=\inf\Big\{\lambda>0:\int_{B_\rho}f\Big(\frac{\widetilde{u}}{\lambda}\Big)dx\le\gamma\Big\}=1\quad\textit{and}\quad\|u_0\|_{L^f_\gamma}=\|u_0^\sharp\|_{L^f_\gamma}\le\mu\|\widetilde{u}\|_{L^f_\gamma}=\mu<\mu_1.$$

In order to prove Theorem 2.1. 1) we first remark that in the space M_{T,μ_1} the differential equation (2.3) admits an equivalent integral formulation as stated in the following proposition.

Proposition 5.1 *Let u_0 be a measurable function such that $\mu = \sup_{x\in B_\rho} \frac{u_0^\sharp(x)}{\tilde{u}(x)} < \mu_1 < 1$, $T \in (0, +\infty]$ and $u \in M_{T,\mu_1}$. The following statements are equivalent:*
i) u is a weak solution of the equation (2.3) in $(0,T)\times B_\rho$;
ii) u satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))ds \quad \text{on } (0,T)\times B_\rho \tag{5.2}$$

in the sense of distributions and $u(t) \to u_0$ as $t \to 0$ in the weak* topology.

The key tool of the proof of Proposition 5.1 is the following lemma:

Lemma 5.1 *Let $0 < \mu_1 < 1$, $T \in (0, +\infty]$ and $u \in M_{T,\mu_1}$. Then*

$$\sup_{t\in(0,T)} \|f(u(t))\|_{L^{\frac{1}{\mu_1{}^2}}} \le (C(\beta,\alpha,\mu_1)\gamma)^{\mu_1{}^2}.$$

Proof of Lemma 5.1. Since $\|u(t)\|_{L^f_\gamma} \le \mu_1$, for any $t \in (0,T)$, we control uniformly with respect to time the L^f_γ-norm of the nonlinearity:

$$\begin{aligned}
\|f(u(t))\|^{\frac{1}{\mu_1{}^2}}_{L^{\frac{1}{\mu_1{}^2}}} &= \int_{B_\rho} f(u(t))^{\frac{1}{\mu_1{}^2}} dx \\
&= \int_{|u|\ge\beta} \frac{\mathrm{e}^{\left(\frac{u}{\mu_1}\right)^2}}{|u|^{\frac{3}{\mu_1{}^2}}} dx + \int_{|u|<\beta} \alpha^{\frac{1}{\mu_1{}^2}} |u|^{\frac{2}{\mu_1{}^2}} dx \\
&\le \int_{|u|\ge\beta} \beta^{3-\frac{3}{\mu_1{}^2}} \frac{\mathrm{e}^{\left(\frac{u}{\mu_1}\right)^2}}{|u|^3} dx + \int_{|u|<\beta} \alpha^{\frac{1}{\mu_1{}^2}} \beta^{\frac{2}{\mu_1{}^2}-2} |u|^2 dx \\
&\le C(\beta,\alpha,\mu_1) \int_{B_\rho} f\Big(\frac{u}{\mu_1}\Big) dx \le C(\beta,\alpha,\mu_1)\gamma
\end{aligned}$$

for all $t \in (0,T)$. This ends the proof of Lemma 5.1.

The proof of Proposition 5.1 relies on the previous lemma and follows the same lines as the proof of Proposition 2.1 in [13].
We are now in position to prove the first part of Theorem 2.1.

Proof of Theorem 2.1. 1)

Let us introduce the integral operator

$$\Phi(u)(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))ds$$

and look for a fixed point of Φ in M_{T,μ_1}.
First we prove that Φ maps the space M_{T,μ_1} into itself for small T. By applying Lemma 3.2 to the linear term and Lemma 3.3 with $p = \frac{1}{\mu_1{}^2}$ $(\frac{1}{2} < \mu_1{}^2 < 1)$ we obtain

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^f_\gamma} &\le \left\|e^{t\Delta}u_0\right\|_{L^f_\gamma} + \int_0^t \left\|e^{(t-s)\Delta} f(u(s))\right\|_{L^f_\gamma} ds \\
&\le \|u_0\|_{L^f_\gamma} + \int_0^t (t-s)^{-\mu_1{}^2} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}} \|f(u(s))\|_{L^{\frac{1}{\mu_1{}^2}}} ds.
\end{aligned}$$

Since $\|u_0\|_{L^f_\gamma} \le \mu$ (Remark 5.1) and the L^f_γ-norm of the nonlinearity is controlled uniformly with respect to time (Lemma 5.1) we get

$$\|\Phi(u)(t)\|_{L^f_\gamma} \le \mu + (C(\alpha,\beta,\mu_1)\gamma)^{{\mu_1}^2} \int_0^t (t-s)^{-{\mu_1}^2} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}}\ ds.$$

Since ${\mu_1}^2 < 1$ and

$$\int_0^t (t-s)^{-{\mu_1}^2} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}}\ ds \to 0 \qquad \text{for} \quad t \to 0,$$

if T is small enough we get for any $0 < t < T$ that

$$(C(\beta,\alpha,\mu_1)\gamma)^{{\mu_1}^2} \int_0^t (t-s)^{-{\mu_1}^2} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}}\ ds \le \mu_1 - \mu$$

and this proves that $\Phi(u)$ belongs to M_{T,μ_1}.

Let us now prove that the integral operator Φ is a contraction from M_{T,μ_1} into itself. Let q be such that $1 < q < \frac{1}{{\mu_1}^2}$. We have

$$\begin{aligned}\|\Phi(u)(t)-\Phi(v)(t)\|_{L^f_\gamma} &\le \int_0^t \left\|e^{(t-s)\Delta}\left(f(u(s))-f(v(s))\right)\right\|_{L^f_\gamma} ds \\ &\le \int_0^t (t-s)^{-\frac{1}{q}} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}} \left\|f(u(s))-f(v(s))\right\|_{L^q} ds.\end{aligned}$$

Since

$$|f(u)-f(v)| \le |u-v|\left(|f'(u)|+|f'(v)|\right)$$

we have

$$\|f(u)-f(v)\|_{L^q} \le \|u-v\|_{L^{\widetilde{r}}}\left(\|f'(u)\|_{L^r}+\|f'(v)\|_{L^r}\right)$$

where $\frac{1}{q} = \frac{1}{\widetilde{r}} + \frac{1}{r}$, for $\widetilde{r}$ large enough such that $q < r < \frac{1}{{\mu_1}^2}$. Since B_ρ is bounded, the Orlicz space is embedded into the Lebesgue space $L^{\widetilde{r}}$ (with $1 < \widetilde{r} < \infty$). Therefore we have $\|u-v\|_{L^{\widetilde{r}}} \le \|u-v\|_{L^f_\gamma}$. Now, since $r < \frac{1}{{\mu_1}^2}$

$$|f'(u)|^r = \begin{cases} \left|2|u| - \dfrac{3}{|u|}\right|^r \left(\dfrac{\mathrm{e}^{u^2}}{|u|^3}\right)^r \le C(\beta,\mu_1,r)\, f\Big(\dfrac{u}{\mu_1}\Big), & |u| \ge \beta, \\ (2\alpha|u|)^r, & |u| < \beta. \end{cases} \tag{5.3}$$

Therefore, thanks to the embedding of the Orlicz space in any Lebesgue space $L^{\widetilde{r}}$, for $1 < \widetilde{r} < \infty$, and since $\sup_{s\in(0,T)} \|u(s)\|_{L^f_\gamma} \le \mu_1$ we have

$$\begin{aligned}\|f'(u)\|_{L^r} &\le C(\beta,\mu_1,r)\left(\int_{|u|\ge\beta} f\left(\frac{u}{\mu_1}\right)dx\right)^{\frac{1}{r}} + \Big(\int_{|u|\le\beta}(2\alpha|u|)^r dx\Big)^{\frac{1}{r}} \\ &\le C(\beta,\mu_1,r)\,(\gamma)^{\frac{1}{r}} + C(\alpha,r)\|u\|_{L^f_\gamma} \\ &\le C(\alpha,\beta,\mu_1,\gamma,r).\end{aligned}$$

Thus it holds

$$\|f(u)-f(v)\|_{L^q} \le C\|u-v\|_{L^f_\gamma},$$

for a constant $C = C(\alpha, \beta, \mu_1, \gamma, r)$. Therefore, for all $0 < t < T$,

$$\|\Phi(u(t)) - \Phi(v(t))\|_{L^f_\gamma} \leq C \sup_{0<t<T} \|u(t) - v(t)\|_{L^f_\gamma} \int_0^t (t-s)^{-\frac{1}{q}} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}} ds$$

and

$$\int_0^t (t-s)^{-\frac{1}{q}} \left(\log\left((t-s)^{-1}+1\right)\right)^{-\frac{1}{2}} ds \to 0, \quad \text{as } t \to 0 \tag{5.4}$$

since $1 < q < \frac{1}{{\mu_1}^2}$. This ends the proof of the contraction argument.
We next prove the convergence to the initial data $\left\|u(t) - \mathrm{e}^{t\Delta}u_0\right\|_{\exp L^2} \to 0$ as $t \to 0$. By the equivalence of L^f_γ and $\exp L^2$ (Proposition 3.1), we prove $\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{\exp L^2} = 0$. Take q so that $1 < q < 1/{\mu_1}^2$. Lemma 3.3 gives us that

$$\|u(t) - e^{t\Delta}u_0\|_{\exp L^2} \leq \int_0^t (t-s)^{-\frac{1}{q}} \big(\log((t-s)^{-1}+1)\big)^{-\frac{1}{2}} \|f(u(s))\|_{L^q} ds.$$

By (3.1), for any $s \in (0,t)$ we have

$$\|f(u(s))\|_{L^q} \leq C\Big(\int_{B_\rho} \left(e^{qu^2} - 1\right) dx\Big)^{\frac{1}{q}} \leq C'\Big(\int_{B_\rho} f\Big(\frac{u}{\mu_1}\Big)\, dx\Big)^{\frac{1}{q}} \leq C'\gamma^{\frac{1}{q}}$$

for some $C, C' > 0$. Thanks to (5.4) this gives $\|u(t) - e^{t\Delta}u_0\|_{\exp L^2} \to 0$ as $t \to 0$.
Moreover u belongs to $L^\infty_{loc}(0,T;L^\infty)$ (and so it is a $\exp L^2-$classical solution of (2.3) on $(0,T) \times B_\rho$). Indeed assume $t > 0$. We know that $e^{t\Delta}u_0$ belongs to L^∞. Moreover, thanks to Lemma 5.1 we get

$$\Big\|\int_0^t e^{(t-s)\Delta} f(u(s))\, ds\Big\|_{L^\infty} \leq \int_0^t (t-s)^{-{\mu_1}^2} \|f(u(s))\|_{L^{\frac{1}{{\mu_1}^2}}} ds \leq C \int_0^t (t-s)^{-{\mu_1}^2} ds < +\infty$$

for fixed $t > 0$. Finally by standard arguments one may check that the solution u belongs to $C((0,T], \exp L^2)$.

6 Existence and Non-uniqueness result

In this section we prove the existence of an $\exp L^2-$classical solution for the Cauchy problem (2.3) for any nonnegative u_0 such that

$$\mu = \sup_{x \in B_\rho} \frac{{u_0}^\sharp(x)}{\widetilde{u}(x)} \leq 1.$$

This will imply the non-uniqueness result.

Non-uniqueness: Since $\widetilde{u}^\sharp(|x|) = \widetilde{u}(x)$, we obtain that for the initial datum $u_0 = \widetilde{u}$ and for any $\mu_2 > 1$ there exist a positive time $T = T(u_0, \mu_2)$ and an $\exp L^2$-classical solution u of the system (2.3) that belongs to M_{T,μ_2}. We recall that $\widetilde{u}$ is a stationary *singular* solution of the system (2.3), it is not bounded and it belongs to the class $M_{T,1}$. Therefore the Cauchy problem (2.3) possesses for $u_0 = \widetilde{u}$ at least two weak solutions in M_{T,μ_2}, even though a weak solution is unique in M_{T,μ_1} for $\mu < \mu_1 < 1$ as in Theorem 2.1 1).

Corollary 6.1 *Assume that* $u_0 = \widetilde{u}$. *For any* $\mu_2 > 1$ *there exist a positive time* $T = T(u_0, \mu_2)$ *and at least two weak solutions on* $(0, T) \times B_\rho$ *of the Cauchy problem* (2.3) *in the space* $M_{T,\,\mu_2}$.

Proof of Theorem 2.1. 2)

The key idea of the proof is to introduce a suitable auxiliary Cauchy problem with a well–chosen polynomial nonlinearity whose solutions can be transformed to *supersolutions* of the Cauchy problem (2.3). Then, applying Perron's monotone method it is possible to prove the existence of a solution of (2.3). To derive the auxiliary equation we apply the generalized Cole-Hopf transformation introduced in [11]. Define

$$F(u) := \int_u^{+\infty} \frac{1}{f(s)} ds, \qquad u > 0,$$

where f is the nonlinearity defined in (2.1). Now let $v_0 = \max\left\{(F(u_0))^{-1/2}, (F(\beta))^{-1/2}\right\}$, where β is as in (2.1). Since $(F(t))^{-1/2}$ is a nondecreasing function we obtain

$$v_0^\sharp(|x|) = \begin{cases} (F(u_0^\sharp(|x|))^{-1/2} & \text{if } \ u_0^\sharp(|x|) > \beta, \\ (F(\beta))^{-1/2} & \text{if } \ u_0^\sharp(|x|) \le \beta, \end{cases}$$

for any $x \in B_\rho$. It follows from the definition of f in (2.1) that

$$F(s) = \int_s^\infty \frac{\eta^3}{e^{\eta^2}} d\eta = \frac{s^2+1}{2e^{s^2}} \qquad \text{for large } s. \tag{6.1}$$

Combining (6.1) to the assumption on u_0, we have

$$v_0^\sharp(|x|) \ \le \begin{cases} \dfrac{\sqrt{2}}{|x|(1-2\log|x|)^{1/2}}, & |x| < \dfrac{1}{\mathrm{e}^{5/4}}, \\ (F(\beta))^{-1/2}, & \dfrac{1}{\mathrm{e}^{5/4}} \le |x| \le \rho. \end{cases}$$

Consider the Cauchy problem

$$\begin{cases} \partial_t v - \Delta v = \dfrac{v^3}{2} & \text{in } \ B_\rho, \ t > 0, \\ v(t,x) = F(\beta)^{-\frac{1}{2}} & \text{on } \partial B_\rho, \ t > 0, \\ v(0,x) = v_0(x). \end{cases} \tag{6.2}$$

If the initial datum of (6.2) belongs to L^2, one can obtain a time-local classical solution by standard contraction mapping arguments developed by Weissler [30] and Brezis-Cazenave [4]. We should remark that the initial datum v_0 belongs to any Lorentz space $L^{2,q}$ with $q > 2$ since

$$v_0 \in L^{2,q} \iff \int_{B_\rho} \left(|x| v_0^\sharp(x)\right)^q \frac{dx}{|x|^2} < \infty$$

and this last inequality is implied by the finiteness of the integral

$$\int_{|x|<e^{-5/4}} \frac{dx}{|x|^2(1-2\log|x|)^{q/2}} < \infty.$$

We remark that v_0 might not belong to L^2, as is the case for $u_0 = \widetilde{u}$. Hence we consider the problem (6.2) in Lorentz space and obtain the following existence result by modifying the arguments in [30, 4].

Proposition 6.1 *Let $2 < q \le 5$. There exists a positive time $T = T(v_0)$ and a unique solution v of the Cauchy problem (6.2) such that $v \in C([0,T], L^{2,q})$, $t^{3/10}v(t) \in C([0,T], L^5)$ and $\lim_{t\to 0} t^{3/10}\|v(t)\|_{L^5} = 0$. Moreover $v \in L^\infty_{loc}((0,T), L^\infty)$ and it is a classical solution of (6.2) on $(0,T)\times B_\rho$.*

We prove this proposition in the Appendix.

We now build a super-solution of the Cauchy problem (2.3) by using the solution of (6.2). Let us define

$$\bar{u} = F^{-1}(v^{-2})$$

where F^{-1} is the inverse function of F and v is the solution constructed in Proposition 6.1. Then $\bar{u}$ belongs to $L^\infty_{loc}((0,T), L^\infty)$ because v belongs to $L^\infty_{loc}((0,T), L^\infty)$ and F^{-1} is a non-increasing function. Moreover, $\bar{u} \ge F^{-1}(F(\beta)) = \beta$, since $v(x,t) \ge (F(\beta))^{-1/2}$. Now by a direct computation we obtain

$$\partial_t \bar{u} - \Delta \bar{u} - f(\bar{u}) = 4 f(\bar{u})\, v^{-4} |\nabla v|^2 \left(\frac{3}{2} - f'(\bar{u})F(\bar{u})\right) \ge 0$$

since $f'(\bar{u})F(\bar{u}) \le 1$ for any $\bar{u} \ge \beta$. Therefore,

$$\partial_t \bar{u} \ge \Delta \bar{u} + f(\bar{u}) \tag{6.3}$$

on $(0,T)\times B_\rho$. Moreover $\bar{u}(0,x) = F^{-1}(v_0(x)^{-2}) \ge u_0(x)$. Therefore, the transformed function $\bar{u}$ is a supersolution of the original problem (2.3). Applying Perron's monotone method, we obtain a classical solution of the problem (2.3) and of the corresponding integral equation (5.2) (for more details, see [11, Proposition 2.1, Lemma 2.3, Remark 6, (1)]).

We prove now the convergence of u to the initial data, as $t \to 0$. We apply the following result.

Lemma 6.1 ([11, Lemma 3.1]) *Let $g(t) = f(F^{-1}(t))$. Assume that there exists some $s_1 > 0$ such that*

$$f'(s)F(s) \le 1 \quad \text{for all } s \ge s_1.$$

Then there exists a constant C such that $g(t) \le Ct^{-1}$ for all $t < t_0 = F(s_1)$.

Since

$$u(x,t) \le \bar{u}(x,t) = F^{-1}(v^{-2}(x,t)) \quad \text{and} \quad v^{-2}(x,t) \le F(\beta),$$

by applying the previous lemma we get

$$\begin{aligned} \left|u(t) - \mathrm{e}^{t\Delta}u_0\right| &= \int_0^t \mathrm{e}^{(t-s)\Delta} f(u(s))ds \\ &\le \int_0^t \mathrm{e}^{(t-s)\Delta} f(F^{-1}(v^{-2}(s))ds \\ &\le C\int_0^t \mathrm{e}^{(t-s)\Delta} v^2(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned} \left\|u(t) - \mathrm{e}^{t\Delta}u_0\right\|_{L^\infty} &\le C\left\| \int_0^t \mathrm{e}^{(t-s)\Delta} v^2(s)ds \right\|_{L^\infty} \\ &\le C\int_0^t \frac{1}{(t-s)^{2/5}s^{3/5}}\, ds \left(\sup_{0<s<t} s^{3/10}\|v(s)\|_{L^5}\right)^2 \\ &\le C\Big(\sup_{0<s<t} s^{3/10}\|v(s)\|_{L^5}\Big)^2 \end{aligned} \tag{6.4}$$

and $\lim_{t\to 0}\sup_{0<s<t} s^{3/10}\|v(s)\|_{L^5} = 0$. This implies that $\|u(t) - \mathrm{e}^{t\Delta}u_0\|_{\exp L^2} \to 0$.

Furthermore, for any $\mu_2 > \mu$, using also Lemma 3.2, we have

$$\begin{aligned}\sup_{0<t<T}\|u(t)\|_{L^f_\gamma} &\leq \sup_{0<t<T}\left\|\mathrm{e}^{t\Delta}u_0\right\|_{L^f_\gamma} + \sup_{0<t<T}\left\|u(t)-\mathrm{e}^{t\Delta}u_0\right\|_{L^f_\gamma}\\ &\leq \|u_0\|_{L^f_\gamma} + \sup_{0<t<T}\left\|u(t)-\mathrm{e}^{t\Delta}u_0\right\|_{L^\infty}\\ &\leq \mu + \sup_{0<t<T}\left\|u(t)-\mathrm{e}^{t\Delta}u_0\right\|_{L^\infty}\\ &\leq \mu_2\end{aligned}$$

for T sufficiently small. Hence $u \in M_{T,\mu_2}$. Note that (6.4) also implies that $u \in L^\infty_{loc}(0,T;L^\infty)$. Finally, by standard arguments one may check that the solution u belongs to $C((0,T],\exp L^2)$.

7 Non-existence result

In this section we prove the non-existence result for $u_0 = \mu\,\widetilde{u}$ with $\mu > 1$, i.e. Theorem 2.1.3). We start by stating the following:

Proposition 7.1 *Let f be a C^2, positive, increasing, convex function in $(0,\infty)$ such that $F(s) := \int_s^\infty \frac{1}{f(\eta)}d\eta < \infty$ for all $s > 0$. Let $u_0 : B_\rho \to [0,\infty]$ and $u : B_\rho \times [0,T] \to [0,\infty]$ be measurable functions satisfying*

$$u(t) \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}f(u(s))ds \qquad a.e.\ in \quad B_\rho \times (0,T). \tag{7.1}$$

Assume that $u(x,t) < \infty$ for a.e. $(x,t) \in B_\rho \times (0,T)$. Then there holds

$$\|e^{t\Delta}u_0\|_{L^\infty} \leq F^{-1}(t) \qquad for\ all\ \ t \in (0,T]. \tag{7.2}$$

Proof. This proposition is essentially proved in [11, Lemma 4.1] by applying the argument developed in Fujita [12, Theorem 2.2] and Weissler [32, Theorem 1]. Here we give a sketch of the proof for the reader's convenience.

Fix $\tau \in (0,T]$ and $t \in (0,\tau)$. Applying $e^{(\tau-t)\Delta}$ to (7.1), we have by Fubini's theorem that

$$e^{(\tau-t)\Delta}u(t) \geq e^{\tau\Delta}u_0 + \int_0^t e^{(\tau-s)\Delta}f(u(s))ds$$

for all $t \in (0,\tau)$. Since f is convex, one can apply Jensen's inequality to obtain

$$e^{(\tau-t)\Delta}u(t) \geq e^{\tau\Delta}u_0 + \int_0^t f\left(e^{(\tau-s)\Delta}u(s)\right)ds. \tag{7.3}$$

Define $H(x,t) := e^{\tau\Delta}u_0 + \int_0^t f\left(e^{(\tau-s)\Delta}u(s)\right)ds$. Then we have

$$-\frac{\partial}{\partial t}\left[F(H(x,t))\right] = \frac{\frac{\partial H}{\partial t}(x,t)}{f(H(x,t))} \geq 1.$$

This yields

$$-F(H(x,t)) + F(H(x,0)) \geq t.$$

Since $F(H(x,t)) \geq 0$ and $H(x,0) = e^{\tau\Delta}u_0$, there holds

$$e^{\tau\Delta}u_0 \leq F^{-1}(t)$$

for all $t \in (0,\tau)$. Taking $t \uparrow \tau$ and the supremum on $x \in B_\rho$, we obtain the desired estimate.

Corollary 7.1 *Let f be the function defined in (2.1). Assume that u_0 and u satisfy the same conditions as in Proposition 7.1. Then there holds*

$$\|e^{t\Delta}u_0\|_{L^\infty} \leq (-\log t)^{\frac{1}{2}} + 1 \quad \textit{for small } t > 0. \tag{7.4}$$

Proof. By (6.1), we have

$$\begin{aligned}\lim_{t\to 0}\left[F^{-1}(t) - (-\log t)^{\frac{1}{2}}\right] &= \lim_{s\to\infty}\left[s - \left(\log\frac{1}{F(s)}\right)^{\frac{1}{2}}\right] \\ &= \lim_{s\to\infty}\left[s - \left(s^2 + \log\frac{2}{s^2+1}\right)^{\frac{1}{2}}\right] = 0.\end{aligned}$$

Hence there holds

$$F^{-1}(t) \leq (-\log t)^{\frac{1}{2}} + 1 \quad \text{for small } t > 0.$$

This and Proposition 7.1 yield the conclusion.

Now we are in the position to prove Theorem 2.1.3).

Proof of Theorem 2.1.3)

Assume that there exists a non-negative $\exp L^2$-classical solution of (2.3) with $u_0 = \mu\widetilde{u},\ \mu > 1$. For any $t > 0,\ s > 0,\ t+s < T$ we have

$$u(t+s) \geq e^{t\Delta}u(s).$$

For $s \to 0$ we get

$$u(t) \geq e^{t\Delta}u_0 \tag{7.5}$$

thanks to the definition of $\exp L^2$-classical solution and the weak* convergence of $u(s) \to u_0$ as $s \to 0$. Since u is an $\exp L^2$-classical solution for any $0 < \tau < t < T$ we have

$$u(t) = e^{(t-\tau)\Delta}u(\tau) + \int_\tau^t e^{(t-s)\Delta}f(u(s))ds. \tag{7.6}$$

Thanks to (7.5) and (7.6) we get

$$u(t) \geq e^{t\Delta}u_0 + \int_\tau^t e^{(t-s)\Delta}f(u(s))ds,$$

and for $\tau \to 0$ by monotone convergence theorem we have:

$$u(t) \geq e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}f(u(s))ds.$$

Therefore applying Corollary 7.1, we get that u satisfies (7.4). We now prove an estimate of $\left\|e^{t\Delta}u_0\right\|_\infty$ from below which is in contradiction with (7.4). Remark that

$$\begin{aligned}\left\|e^{t\Delta}u_0\right\|_\infty &\geq \int_{B_\rho(0)} G(0,y,t)\mu\,\widetilde{u}(y)\ dy \\ &\geq \int_{B_r(0)} G(0,y,t)\mu\sqrt{-2\log|y|}\ dy,\end{aligned}$$

where $r = \frac{1}{e^{5/4}}$. Let us denote by $d = \rho - r$. It is possible to bound on the ball $B_r(0)$ the Dirichlet heat kernel G associated to the ball B_ρ from below by the heat kernel for $\mathbb{R}^2$ (see [3]):

$$G(0, y, t) \geq H(d, t) \frac{e^{-|y|^2/4t}}{4\pi t},$$

where

$$H(d, t) = 1 - e^{-d^2/t} \left(2 + 4\frac{d^2}{t}\right).$$

Therefore

$$\begin{aligned} \left\| e^{t\Delta} u_0 \right\|_\infty &\geq \int_{B_r(0)} G(0, y, t) \mu \sqrt{-2 \log |y|} \, dy \\ &\geq \int_{B_r(0)} H(d, t) \frac{e^{-|y|^2/4t}}{4\pi t} \mu \sqrt{-2 \log |y|} \, dy \\ &\geq H(d, t) \int_{|z| \leq r t^{-1/2}} \frac{e^{-|z|^2/4}}{4\pi} \mu \sqrt{-\log t - 2 \log |z|} \, dz, \end{aligned}$$

where in the last inequality we replace $y = \sqrt{t}\, z$. For $a < 1/2$ and for small values of t we obtain

$$\begin{aligned} \int_{|z| \leq r t^{-1/2}} \frac{e^{-|z|^2/4}}{4\pi} \mu \sqrt{-\log t - 2 \log |z|} dz &\geq \int_{|z| \leq r t^{-a}} \frac{e^{-|z|^2/4}}{4\pi} \mu \sqrt{-\log t - 2 \log |z|} dz \\ &\geq \mu \sqrt{-\log t + 2a \log t - 2 \log r} \int_{|z| \leq r t^{-a}} \frac{e^{-\frac{|z|^2}{4}}}{4\pi} dz \\ &\geq \mu \sqrt{1 - 2a} \sqrt{-\log t} \, (1 - \varepsilon) \end{aligned}$$

for some $\varepsilon > 0$, since $\int_{|z| \leq r t^{-a}} \frac{e^{-|z|^2/4}}{4\pi} dz \to 1$ for $t \to 0^+$. Since also $H(d, t) \to 1$ as $t \to 0^+$, we get

$$\begin{aligned} \left\| e^{t\Delta} u_0 \right\|_\infty &\geq \mu \, H(d, t)(1 - \varepsilon) \sqrt{1 - 2a} \sqrt{-\log t} \\ &\geq \mu \, (1 - \varepsilon)^2 \sqrt{1 - 2a} \, \sqrt{\log \frac{1}{t}} \end{aligned}$$

Thus, for fixed $\mu > 1$ we can choose $\varepsilon > 0$ small and a near 0 such that

$$\mu (1 - \varepsilon)^2 \sqrt{1 - 2a} \geq 1 + \delta$$

for some $\delta > 0$. This contradicts (7.4) in the limit $t \to 0$.

8 Appendix

Proposition 6.1 can be proved by a modification of the standard contraction mapping argument developed by Weissler [30] and Brezis-Cazenave [4] to the framework of Lorentz spaces. We include it for the reader's convenience.

Proof of Proposition 6.1. We look for a solution $v = \bar{v} + F(\beta)^{-\frac{1}{2}}$ where $\bar{v}$ is a solution of the following Cauchy problem with Dirichlet boundary condition:

$$\begin{cases} \partial_t \bar{v} - \Delta \bar{v} = \dfrac{\left(\bar{v} + F(\beta)^{-\frac{1}{2}}\right)^3}{2} & \text{in } \ B_\rho(0), \ t > 0, \\ \bar{v}(t, x) = 0 & \text{on } \partial B_\rho(0), \ t > 0, \\ \bar{v}(0, x) = \bar{v}_0(x) & \text{in } \ B_\rho(0), \end{cases} \tag{8.1}$$

where $\bar{v}_0(x) = v_0(x) - F(\beta)^{-\frac{1}{2}}$. We prove that there exists a solution $\bar{v}$ of the equation (8.1) belonging to the space

$$E_{\delta,M,T} = \left\{ w \in L^\infty(0,T; L^{2,q}) : \begin{array}{c} \sup_{t\in(0,T)} \|w(t)\|_{L^{2,q}} \le M+1, \\ \sup_{t\in(0,T)} t^{3/10}\|w(t)\|_{L^5} \le \delta \end{array} \right\}$$

where $M \ge \sup_{t\in(0,\infty)} \|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^{2,q}}$ and δ and T are well-chosen positive constants.
Let us first remark that the space $E_{\delta,M,T}$ endowed with the metric

$$d(v,w) = \sup_{t\in(0,T)} t^{3/10}\|v(t) - w(t)\|_{L^5}$$

is a nonempty complete metric space. Let us denote $D = F(\beta)^{-\frac{1}{2}}$ and consider the integral operator

$$G(w)(t) = \mathrm{e}^{t\Delta}\bar{v}_0 + \frac{1}{2}\int_0^t \mathrm{e}^{(t-s)\Delta}\left(w(s)+D\right)^3 \, ds.$$

We prove that for some well-chosen positive constants T and δ the operator G maps the space $E_{\delta,M,T}$ into itself and it is a contraction. Indeed let $w \in E_{\delta,M,T}$; by the smoothing effect of the heat semigroup established in Lemma 3.4, $e^{t\Delta}D \le D$ for any positive constant D, and thanks to the inequality $|w+D|^3 \le 4\left(|w|^3 + D^3\right)$, for $t \in (0,T)$, we have

$$\begin{aligned} t^{\frac{3}{10}}\|G(w)(t)\|_{L^5} &\le t^{\frac{3}{10}}\|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^5} + 2t^{\frac{3}{10}}\int_0^t \left\|\mathrm{e}^{(t-s)\Delta}\left(|w(s)|^3 + D^3\right)\right\|_{L^5} ds \\ &\le t^{\frac{3}{10}}\|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^5} + \int_0^t \frac{Ct^{\frac{3}{10}}}{(t-s)^{2/5}s^{9/10}}ds\left(\sup_{0<s<t} s^{\frac{3}{10}}\|w(s)\|_{L^5}\right)^3 + Ct^{\frac{13}{10}} \\ &\le t^{\frac{3}{10}}\|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^5} + C_1\delta^3 + C_2 t^{\frac{13}{10}}. \end{aligned}$$

Therefore

$$\sup_{t\in(0,T)} t^{3/10}\|G(w)(t)\|_{L^5} \le \sup_{t\in(0,T)} t^{3/10}\|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^5} + C_1\delta^3 + C_2 T^{\frac{13}{10}}.$$

Moreover, since $L^2 \subset L^{2,q}$ $(q > 2)$ we obtain

$$\begin{aligned} \|G(w)(t)\|_{L^{2,q}} &\le \|\mathrm{e}^{t\Delta}\bar{v}_0\|_{L^{2,q}} + 2\int_0^t \left\|\mathrm{e}^{(t-s)\Delta}(|w(s)^3 + D^3)\right\|_{L^2} ds \\ &\le M + C\int_0^t \frac{1}{(t-s)^{1/10}s^{9/10}}ds\Big(\sup_{0<t<T} t^{3/10}\|w(t)\|_{L^5}\Big)^3 + Ct \\ &\le M + C_3\delta^3 + C_4 T. \end{aligned} \tag{8.2}$$

Therefore

$$\sup_{t\in(0,T)} \|G(w)(t)\|_{L^{2,q}} \le M + C_4\delta^3 + C_3 T.$$

In a similar way, since $|(w+D)^3 - (v+D)^3| \le C|w-v|(w^2+v^2+D^2)$, for any $v, w \in E_{\delta,M,T}$, we have

$$\begin{aligned} t^{\frac{3}{10}}\|G(v)(t) - G(w)(t)\|_{L^5} &\le Ct^{\frac{3}{10}}\int_0^t \left\|\mathrm{e}^{(t-s)\Delta}|v(s)-w(s)|(v^2(s)+w^2(s)+D^2)\right\|_{L^5} ds \\ &\le \sup_{t\in(0,T)} t^{\frac{3}{10}}\|v(t)-w(t)\|_{L^5}\left(C_5\delta^2 + C_6 T\right). \end{aligned} \tag{8.3}$$

Thus we obtain

$$\sup_{t\in(0,T)} t^{3/10}\|G(v)(t)-G(w)(t)\|_{L^5} \le \sup_{t\in(0,T)} t^{3/10}\|v(t)-w(t)\|_{L^5}\left(C_5\delta^2+C_6T\right).$$

Therefore by choosing δ such that

$$C_1\delta^2\le \frac{1}{2}, \quad C_3\delta^3\le\frac{1}{2}, \quad C_5\delta^2\le\frac{1}{4}$$

and T small enough such that

$$\sup_{t\in(0,T)} t^{3/10}\|e^{t\Delta}\bar v_0\|_{L^5}+C_2T^{\frac{13}{10}}\le\frac{\delta}{2}, \ \ C_4T\le\frac{1}{2}, \ \ C_6T\le\frac{1}{4}$$

we obtain that G maps $E_{\delta,M,T}$ into itself and it is a contraction. We remark that $\sup_{t\in(0,T)} t^{3/10}\|e^{t\Delta}\bar v_0\|_{L^5}\to 0$ as $T\to 0$ since $\bar v_0\in L^{2,q}$, with $2<q\le 5$, thanks to Lemma 3.4. Therefore, the integral equation

$$w(s)=\mathrm{e}^{t\Delta}\bar v_0+\frac{1}{2}\int_0^t \mathrm{e}^{(t-s)\Delta}\left(w(s)+D\right)^3\ ds \tag{8.4}$$

admits a unique solution $\bar v$ in $E_{\delta,T,M}$.

We prove now that the fixed point $\bar v$ belongs to

$$E=E_{\delta,M,T}\cap\Big\{w\in C((0,T],L^5):\lim_{t\to 0}t^{3/10}\|w(t)\|_5=0\Big\}.$$

To this end, it is enough to prove that Φ is a map from E to E, since this implies that the previous contraction mapping argument works in E. It follows from $v_0\in L^{2,q}$ and Lemma 3.4 that $e^{t\Delta}\bar v_0=e^{t\Delta}(v_0-F(\beta)^{-\frac{1}{2}})\in E$. Fix $w\in E$; since $E\cap C([0,T],L^\infty)$ is dense in E with respect to the metric d, there exists a sequence $v_n\in E\cap C([0,T],L^\infty)$ such that $G(v_n)\in E$ and $d(v_n,w)\to 0$ as $n\to\infty$. By (8.3), we have $d(G(v_n),G(w))\to 0$ as $n\to\infty$. This together with the fact that E is a complete metric space with respect to d yields $G(w)\in E$. This proves that the fixed point $\bar v$ belongs to E. Furthermore, (8.2) and $\bar v\in E$ yield

$$\lim_{t\to 0}\|\bar v(t)-e^{t\Delta}\bar v_0\|_{L^{2,q}}=0.$$

Finally we prove that $\bar v$ is a classical solution. Since $\bar v_0$ is nonnegative, the solution $\bar v$ is also nonnegative. Moreover, it belongs to $L^\infty_{loc}(0,T;L^\infty)$ and it is a classical solution on $(0,T)\times B_\rho$. Indeed

$$\begin{aligned}
\|\bar v(t)\|_{L^\infty} &\le \|\mathrm{e}^{t\Delta}\bar v_0\|_{L^\infty}+\frac{1}{2}\Big\|\int_0^t \mathrm{e}^{(t-s)\Delta}(\bar v+D)^3\ ds\Big\|_{L^\infty}\\
&\le \|\mathrm{e}^{t\Delta}\bar v_0\|_{L^\infty}+C\int_0^t\frac{1}{(t-s)^{3/5}}\|\bar v^3(s)\|_{L^{5/3}}ds+tCD^3\\
&\le t^{-1}\|v_0\|_{L^1}+C\int_0^t\frac{1}{(t-s)^{3/5}s^{9/10}}(s^{3/10}\|\bar v(s)\|_{L^5})^3ds+tCD^3\\
&\le t^{-1}\|v_0\|_{L^1}+Ct^{-1/2}\big(\sup_{s\in(0,t)}s^{3/10}\|\bar v(s)\|_{L^5}\big)^3+tCD^3.
\end{aligned} \tag{8.5}$$

Therefore, for any $\epsilon>0$, $v\in L^\infty(\epsilon,T;L^\infty)$ and $\bar v$ is a classical solution on $(0,T)\times B_\rho(0)$.

By denoting $v(x,t)=\bar v(x,t)+D$, $D=F(\beta)^{-1/2}$, we obtain a solution of the differential equation (6.2). The solution v of (6.2) belongs to $C([0,T],L^{2,q})\cap C((0,T],L^5)$ and $\lim_{t\to 0}t^{3/10}\|v(t)\|_{L^5}=0$ and it is bounded on any interval (ε,T), for $\varepsilon>0$. Moreover $v(x,t)\ge F(\beta)^{-1/2}$ for any $(x,t)\in B_\rho\times(0,T)$.

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