

# PLAYING WITH THE HADAMARD PRODUCT IN DECOMPOSING GINI, CONCENTRATION, REDISTRIBUTION AND RE-RANKING INDEXES

ACHILLE VERNIZZI<sup>1</sup>

## Abstract

Gini and concentration indexes are well known useful tools in analysing redistribution and re-ranking effects of taxes with respect to a population of income earners.

Aronson, Johnson and Lambert (1994), Urban and Lambert (2008) decompose Gini and re-ranking indices to analyse potential redistribution effects and the unfairness of a tax systems. Urban and Lambert (2008) consider contiguous income groups which are created by dividing the pre-tax income parade according to the same bandwidth. However, earners may be very often split into groups characterized by social and demographic aspects or by other characteristics: in these circumstances groups can easily overlap. In this paper we consider a more general situation that takes into account overlapping among groups; we obtain matrix compact forms for Gini and concentration indexes, and consequently, for redistribution and re-ranking indexes. In deriving formulae the so called matrix Hadamard product is extensively used. Matrix algebra allows to write indexes aligning incomes in a non decreasing order either with respect to post-tax income or to pre-tax incomes. Moreover, matrix compact formulae allow an original discussion for the signs of the *within group*, *across group*, *between* and *transvariation* components into which the Atkinson-Plotnick-Kakwany (Plotnick, 1981) re-ranking index can split.

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## Introduction

It is known that, dealing with a transferable phenomenon where units are classifiable into groups, Gini index fails to decompose additively into a between and a within component if the group ranges overlap. Following Bahattacharya and Mahalanobis (1967), a number of Gini decompositions was proposed ( Rao (1969), Pyatt (1976), ), Mookherjee and Shorrocks (1982), Silber (1989), Yitzhaki and Lerner (1991), Lambert and Aronson (1993), Ytzhaki (1994), Dagum (1997)) and after Lambert and Aronson (1993), the third component of the conventional Gini index decomposition is denoted by overlapping term.

Monti (2007) shows that the conventional and the Dagum (1997) decomposition are identical, so that an alternative way to calculate the overlapping term can be derived from the decomposition suggested by this author.

Aronson, Johnson and Lambert (1994), Urban and Lambert (2008), use Gini and concentration index decomposition to identify and evaluate potential distributive effects and unfairness in a tax system. These authors consider contiguous income groups created by dividing

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the pre-tax income parade according to an identical bandwidth, so that the pre-tax income parade excludes overlapping by construction.

In the present paper we consider incomes gathered into groups characterized by social, demographic or income sources characteristics, so that overlapping among groups need not to be excluded. Our results are obtained using the Gini index decomposition derived from Dagum decomposition (Monti and Santoro 2007, Monti 2008).

Making use of the Hadamard product, in the first section we present Gini and concentration indexes in compact matrix forms. In the second section we introduce groups, present Gini and concentration indexes and show how within groups, across, between groups and transvariation components can be written in matrix compact forms. Links from matrix compact forms and scalar forms are reported: some scalar expressions are well known in literature, while others appears as modifications of already well known forms.

Section 3 presents matrix forms for redistribution and re-ranking indexes, together with their within, across, between groups and transvariation components.

In the fourth section we show how the signs of Atkinson-Plotnick-Kakwani (Plotnick 1981) re-ranking index components can be analysed, thanks to the algebraic tools presented in the paper.

## 1 Matrix forms for concentration and Gini indexes

Let be  $X$  and  $Y$  two real non negative statistical variables that describe a transferable phenomenon for a population of  $K$  units,  $K \in \mathbb{N}$ . In this paper we suppose that  $X$  represents income before taxation and  $Y$  after-tax income; not infrequently the pair  $(x_i, y_i)$  has associated a weight  $p_i$  ( $i=1, \dots, K$ ),  $\sum_{i=1}^K p_i = N$ . Furthermore in measuring concentration we generally need to rank either  $x_i$  or  $y_i$  in a non-decreasing order: when the  $X$  elements are ranked in a non-decreasing order, the sequence of  $(x_i, y_i, p_i)$  triplets will be indicated as  $\{(x_i, y_i, p_i)\}_X$ ; analogously  $\{(x_i, y_i, p_i)\}_Y$  will denote the sequence of  $(x_i, y_i, p_i)$ , when the  $Y$  elements are ranked in a non-decreasing order.

The concentration index <sup>2</sup> for  $Y$ , in the ordering  $\{(x_i, y_i, p_i)\}_X$ , is defined as <sup>3</sup>

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<sup>2</sup> The author is in debt with Maria Monti for the suggestion to express the concentration index by differences between incomes: this suggestion is at the basis of this paper. In the appendix, §A1, it is shown that in expressions (1) the first formula is equal to the second one.

In the right hand side of (1), the first component calculates the normalized concentration. In the case where the  $y$ 's are in a non decreasing order, the second one is the normalized mean absolute difference, that is  $G_Y = (1/2\mu_Y N^2) \sum_{i=1}^K \sum_{j=1}^K |y_i - y_j| p_i p_j = \Delta/2\mu_Y$ .

<sup>3</sup> The indicator function  $I_{i-j}$  is a particular case of generalized functions considered in Faliva (2000): this article can be consulted for  $I_{i-j}$  properties.

$$\begin{aligned}
C_{Y|X} &= 1 - \sum_{i=1}^K \left( \sum_{j=1}^i \frac{y_j p_j}{\mu_Y N} + \sum_{j=1}^{i-1} \frac{y_j p_j}{\mu_Y N} \right) \frac{p_i}{N} = \frac{1}{\mu_Y N^2} \sum_{i=1}^K \sum_{j=1}^{i-1} (y_i - y_j) p_i p_j = \\
&= \frac{1}{2\mu_Y N^2} \sum_{i=1}^K \sum_{j=1}^K (y_i - y_j) p_i p_j I_{i-j} \\
I_{i-j} &= \begin{cases} 1: & i - j > 0 \\ 0: & i - j = 0 \\ -1: & i - j < 0 \end{cases}
\end{aligned} \tag{1}$$

where  $\mu_Y$  is the weighed mean of the observations on  $Y$ . Obviously in the ordering  $\{(x_i, y_i, p_i)\}_Y$ , the concentration index  $C_{Y|Y}$  coincides with the Gini index  $G_Y$  and, analogously in the ordering  $\{(x_i, y_i, p_i)\}_X$ ,  $C_{X|X} \equiv G_X$ <sup>4</sup>. Generally, when tax effects are analyzed, one considers the Gini index for the pre-tax distribution  $G_X$ , the Gini index for the post-tax distribution  $G_Y$ , and the concentration index for the post tax distributions,  $C_{Y|X}$ , with incomes ranked according to the  $\{(x_i, y_i, p_i)\}_X$  ordering.

In order to pass to a matrix representation, we stack the  $K$  observations on  $X$ ,  $Y$  and the weights  $P$  into  $K \times 1$  vectors: when referring to the ordering  $\{(x_i, y_i, p_i)\}_X$ , the vectors will be indicated as  $\mathbf{x}$ ,  $\mathbf{y}_X$  and  $\mathbf{p}_X$ , while, referring to the ordering  $\{(x_i, y_i, p_i)\}_Y$ , the vectors will be labelled as  $\mathbf{x}_Y$ ,  $\mathbf{y}$  and  $\mathbf{p}_Y$ , that is, when elements in a vector are ranked in a non-decreasing order no label will be added, conversely, when they are ordered according to a non-decreasing order for another variable, this variable will be explicitly indicated.

We also introduce the following definitions:

$\mathbf{S} = [s_{i,j}]$  will denote a  $K \times K$  emi-symmetric matrix with diagonal elements equal to zero, super-diagonal elements equal to 1 and sub-diagonal elements equal to  $-1$ ;

$\mathbf{j}$  for a  $K \times 1$  vector that has entries equal to 1;

$\mathbf{D}_X$  and  $\mathbf{D}_Y$  will denote the  $K \times K$  matrices  $\mathbf{D}_X = (\mathbf{j}\mathbf{x}' - \mathbf{x}\mathbf{j}')$ ,  $\mathbf{D}_Y = (\mathbf{j}\mathbf{y}' - \mathbf{y}\mathbf{j}')$ .

Then, by making use of the Hadamard product  $\odot$ , we can express the indexes  $G_Y$  and  $G_X$  as follows<sup>5</sup>:

$$G_Y = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{S} \odot \mathbf{D}_Y) \mathbf{p}_Y \qquad G_X = \frac{1}{2\mu_X N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{p}_X \tag{2}$$

<sup>4</sup> For definitions concerning concentration indexes and their relations with Gini indexes, see e.g. Kakwani (1980), in particular Ch. 5 and 8.

<sup>5</sup> The Hadamard product for two matrices  $\mathbf{A}$  and  $\mathbf{B}$  is defined if both of them have the same number of rows and the same number of columns:  $[a_{i,j}] \odot [b_{i,j}] = [a_{i,j} \cdot b_{i,j}]$ . For the definition and properties of the Hadamard product see, e.g., Faliva (1983, Appendix) and (1987, Ch. 3), Schott (2005, Ch. 5).

where  $\mu_Y$  and  $\mu_X$  are the weighed mean of the observations on  $Y$  and on  $X$ , respectively.

In addition, by introducing the  $K \times K$  matrix  $\mathbf{D}_{Y|X} = (\mathbf{j}\mathbf{y}'_X - \mathbf{y}_X\mathbf{j}')$ , we can write the concentration index in compact form as

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \quad (3)$$

The transformation from vectors  $\mathbf{y}$  and  $\mathbf{p}_Y$  to vectors  $\mathbf{y}_X$  and  $\mathbf{p}_X$  can be performed by a proper  $K \times K$  permutation matrix  $\mathbf{E}$ . The reverse transformation from  $\mathbf{y}_X$  and  $\mathbf{p}_X$  to  $\mathbf{y}$  and  $\mathbf{p}_Y$  can be obtained through the matrix  $\mathbf{E}^{-1}$  which is equal to  $\mathbf{E}'$ . Formally

$$\begin{cases} \mathbf{y}_X = \mathbf{E}\mathbf{y}, & \mathbf{y} = \mathbf{E}'\mathbf{y}_X \\ \mathbf{x} = \mathbf{E}\mathbf{x}_Y, & \mathbf{x}_Y = \mathbf{E}'\mathbf{x} \\ \mathbf{p}_X = \mathbf{E}\mathbf{p}_Y, & \mathbf{p}_Y = \mathbf{E}'\mathbf{p}_X \end{cases} \quad (4)$$

We shall show that, with some suitable algebraic permutations of the elements of  $\mathbf{S}$ , it is possible to reformulate both the matrices  $\mathbf{D}$  and the vectors  $\mathbf{p}$  in (2) and (3) according either to the  $\{(x_i, y_i, p_i)\}_X$  or to the  $\{(x_i, y_i, p_i)\}_Y$  ordering, maintaining both Gini and concentration indexes unchanged. This leads to rewrite the expressions of formula (2) as

$$G_Y = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{E}\mathbf{S}\mathbf{E}' \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \quad \text{and} \quad G_X = \frac{1}{2\mu_X N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{p}_X \quad (5)$$

or as

$$G_Y = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{S} \odot \mathbf{D}_Y) \mathbf{p}_Y \quad \text{and} \quad G_X = \frac{1}{2\mu_X N^2} \mathbf{p}_Y' (\mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_{X|Y}) \mathbf{p}_Y \quad (6)$$

where  $\mathbf{D}_{Y|X} = (\mathbf{j}\mathbf{y}'_X - \mathbf{y}_X\mathbf{j}')$  and  $\mathbf{D}_{X|Y} = (\mathbf{j}\mathbf{x}'_Y - \mathbf{x}_Y\mathbf{j}')$ , respectively.

Moreover,  $C_{Y|X}$  can be given in the following alternative form:

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_Y) \mathbf{p}_Y \quad (7)$$

### Proof

Consider  $G_X$  as specified in (2) and (6). As  $\mathbf{E}\mathbf{E}' = \mathbf{E}'\mathbf{E} = \mathbf{I}$ , the following holds:

$$\mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{p}_X = \mathbf{p}_X' \mathbf{E}\mathbf{E}' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{E}\mathbf{E}' \mathbf{p}_X = \mathbf{p}_Y' (\mathbf{E}'\mathbf{S}\mathbf{E} - \mathbf{E}'\mathbf{D}_X\mathbf{E}) \mathbf{p}_Y$$

by keeping in mind the noteworthy property of the Hadamard product,

$$\mathbf{E}' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{E} = (\mathbf{E}'\mathbf{S}\mathbf{E}) \odot (\mathbf{E}'\mathbf{D}_X\mathbf{E}) \quad (\text{Faliva 1996, property } vii, \text{ page. 157}).$$

Noticing that

$$\mathbf{E}'\mathbf{D}_X\mathbf{E} = \mathbf{E}'(\mathbf{j}\mathbf{x}' - \mathbf{x}\mathbf{j}')\mathbf{E} = (\mathbf{j}\mathbf{x}'\mathbf{E} - \mathbf{E}'\mathbf{x}\mathbf{j}') = (\mathbf{j}\mathbf{x}'_Y - \mathbf{x}_Y\mathbf{j}') = \mathbf{D}_{X|Y}, \quad \text{as } \mathbf{E}'\mathbf{j} = \mathbf{j} \text{ and } \mathbf{j}'\mathbf{E} = \mathbf{j}',$$

the equivalence of expression (2) and expression (6) for  $G_X$  is proved.

The equivalence of expressions (2) and (5) for  $G_Y$  can be likewise proved.

Indeed the following holds:

$$\mathbf{p}_Y'(\mathbf{S} \odot \mathbf{D}_Y)\mathbf{p}_Y = \mathbf{p}_Y'\mathbf{E}'\mathbf{E}(\mathbf{S} \odot \mathbf{D}_Y)\mathbf{E}'\mathbf{E}\mathbf{p}_Y = \mathbf{p}_X'(\mathbf{E}\mathbf{S}\mathbf{E}' - \mathbf{D}_{Y|X})\mathbf{p}_X$$

upon noticing that  $\mathbf{E}\mathbf{D}_Y\mathbf{E}' = \mathbf{j}\mathbf{y}'\mathbf{E}' - \mathbf{E}\mathbf{y}\mathbf{j}' = \mathbf{j}\mathbf{y}_X' - \mathbf{y}_X\mathbf{j}' = \mathbf{D}_{Y|X}$ .

As far as  $C_{Y|X}$  is concerned, expression (3) turns out to be equivalent to expression (7), upon noticing that

$$\mathbf{E}'\mathbf{D}_{Y|X}\mathbf{E} = \mathbf{j}\mathbf{y}_X'\mathbf{E} - \mathbf{E}'\mathbf{y}_X\mathbf{j}' = \mathbf{j}\mathbf{y}' - \mathbf{y}\mathbf{j}' = \mathbf{D}_Y.$$

□

## 2 Introducing groups

A population of income earners can be partitioned into  $H$  groups,  $H \in \mathbb{N}$ , which can be characterized by income sources or by social and demographic aspects: typical group characterizations are family composition, dependent/not-dependent worker, men/women, geographic area and so on.

Dagum (1997) decomposes the Gini coefficient into *within groups* (henceforth  $W$ ) and an *across groups* (henceforth  $AG$ ) component. Dagum calls this latter component *gross between*.

Hence  $G_Y = G_Y^W + G_Y^{AG}$ . In addition Dagum splits the  $AG$  component into a *between* and a *transvariation* component:  $G_Y^{AG} = G_Y^B + G_Y^T$ . The between component  $G_Y^B$  is the Gini (weighed) index which results when all values within a same group are replaced by their (weighed) average; the transvariation component  $G_Y^T$  measures the overlapping among groups: it is zero when no overlapping exists and it is equal to  $G_Y^{AG}$  when all group averages are equal<sup>6</sup>. Extending Dagum's decompositions to concentration indexes, we can split  $C_{Y|X}$  into the two components  $W$  and  $AG$ , and write  $C_{Y|X} = C_{Y|X}^W + C_{Y|X}^{AG}$ , accordingly with

$$C_{Y|X}^W = \frac{1}{2\mu_Y N^2} \sum_{i=1}^K \sum_{j=1}^K (y_i - y_j) p_i p_j \cdot I_{i,j \in h} \cdot I_{i-j} \quad (8)$$

$$C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2} \sum_{i=1}^K \sum_{j=1}^K (y_i - y_j) p_i p_j \cdot (1 - I_{i,j \in h}) \cdot I_{i-j} \quad (9)$$

<sup>6</sup> For more details on the expression of the Gini components in the Dagum decomposition, see e.g. Monti (2008).

In (8) and (9)  $I_{i-j}$  is as defined in (1) above, and  $I_{i,j \in h}$  is an indicator function:  $I_{i,j \in h} = 1$  if both  $y_i$  and  $y_j$  belong to a same group  $h$  ( $h=1,2,\dots,H$ ),  $I_{i,j \in h} = 0$  if  $y_i$  and  $y_j$  do not.

Similar expressions hold for  $C_{Y|Y}^W = G_Y^W$ ,  $C_{Y|Y}^{AG} = G_Y^{AG}$  and  $C_{X|X}^W = G_X^W$ ,  $C_{X|X}^{AG} = G_X^{AG}$ . In particular, for what concerns  $G^W$  and  $G^{AG}$ , the product  $(y_i - y_j) \cdot I_{i-j}$  can be replaced by the absolute difference  $|y_i - y_j|$ .

In order to formalize compact matrix forms for  $C_{Y|X}^W$  and  $C_{Y|X}^{AG}$ , it is worth to introduce a proper notation. More precisely,  $\mathbf{J}$  will denote a  $K \times K$  matrix with all elements equal to one,  $\mathbf{W}_X = \sum_{h=1}^H \mathbf{w}_{X,h} \mathbf{w}_{X,h}'$  a  $K \times K$  matrix in the  $\{(x_i, y_i, p_i)\}_X$  ordering, where  $\mathbf{w}_{X,h}$  stands for a  $K \times 1$  vector with the  $i$ -th entry equal to one if the income in the  $i$ -th position belongs to group  $h$  ( $h=1,2,\dots,H$ ), whereas it is zero otherwise. The matrix  $\mathbf{W}_X$ , when applied to  $\mathbf{S} \odot \mathbf{D}_{Y|X}$  in expression (3) allows to detect the  $\sum_{h=1}^H K_h^2$  differences belonging to the same group from the whole  $K^2$   $[s_{i,j} \cdot (y_i - y_j)]$  income differences. Conversely the matrix  $(\mathbf{J} - \mathbf{W}_X)$ , when applied to  $\mathbf{S} \odot \mathbf{D}_{Y|X}$ , allows to detect the  $\left(K^2 - \sum_{h=1}^H K_h^2\right)$  differences between incomes belonging to different groups.

Consider now the following expressions for the  $W$  and  $AG$  components of  $C_{Y|X}$ :

$$C_{Y|X}^W = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{W}_X \odot \mathbf{S} \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \quad (10)$$

$$C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot \mathbf{S} \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \quad (11)$$

It is immediate to verify that  $C_{Y|X} = C_{Y|X}^W + C_{Y|X}^{AG}$ . Similar expressions for  $G_Y^W = C_{Y|Y}^W$  and for  $G_Y^{AG} = C_{Y|Y}^{AG}$  can be obtained by substituting  $\mathbf{p}_X$  with  $\mathbf{p}_Y$ ,  $\mathbf{W}_X$  with  $\mathbf{W}_Y$  and  $\mathbf{D}_{Y|X}$  with  $\mathbf{D}_Y$ . Likewise, the corresponding expressions for  $G_X^W = C_{X|X}^W$  and  $G_X^{AG} = C_{X|X}^{AG}$  are obtained by replacing  $\mu_Y$  with  $\mu_X$ , and  $\mathbf{D}_X$  with  $\mathbf{D}_{Y|X}$ . Observe also <sup>7</sup> that  $\mathbf{W}_Y = \mathbf{E}' \mathbf{W}_X \mathbf{E}$  and  $\mathbf{W}_X = \mathbf{E} \mathbf{W}_Y \mathbf{E}'$ .

Moreover, Dagum (1997) splits  $G_Y^{AG}$  into the components  $G_Y^B$  and  $G_Y^T$ , bringing subdivision to the fore. Let's now label each subject triplet of observations on  $X$ ,  $Y$  and  $P$  by a pair of indexes  $(h,i)$ , instead of one as before:  $h$  refers to the group ( $h=1,2,\dots,H$ ), whereas  $i$  ( $i=1,2,\dots,K_h$ ) refers to

<sup>7</sup>  $\mathbf{w}_{X,h} = \mathbf{E} \mathbf{w}_{Y,h}$  and  $\mathbf{w}_{Y,h} = \mathbf{E}' \mathbf{w}_{X,h}$ .

the position that the subject occupies within the  $h$ -th group; note that  $\sum_{i=1}^{K_h} p_{h,i} = N_h$  and

$$\sum_{h=1}^H \sum_{i=1}^{K_h} p_{h,i} = \sum_{h=1}^H N_h = N.$$

Dagum's representations are:

$$G_Y = \frac{1}{2\mu_Y N^2} \sum_{h=1}^H \sum_{g=1}^H \left( \sum_{i=1}^{K_h} \sum_{j=1}^{K_g} |y_{h,i} - y_{g,j}| p_{h,i} p_{g,j} \right) \quad (12)$$

$$G_Y^W = \frac{1}{2\mu N^2} \sum_{h=1}^H \sum_{i=1}^{K_h} \sum_{j=1}^{K_h} |y_{h,i} - y_{h,j}| p_{h,i} p_{h,j} \quad (13)$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2} \sum_{h=1}^H \sum_{g \neq h}^H \left( \sum_{i=1}^{K_h} \sum_{j=1}^{K_g} |y_{h,i} - y_{g,j}| p_{h,i} p_{g,j} \right) \quad (14)$$

$$G_Y^B = \frac{1}{2\mu_Y N^2} \sum_{h=1}^H \sum_{g=1}^H \left( \sum_{i=1}^{K_h} \sum_{j=1}^{K_g} |\mu_{Yh} - \mu_{Yg}| p_{h,i} p_{g,j} \right) = \frac{1}{2\mu_Y N^2} \sum_{h=1}^H \sum_{g=1}^H |\mu_{Yh} - \mu_{Yg}| \bar{p}_h \bar{p}_g \quad (15a)$$

where  $\mu_{Yh}$  represents the income average of the  $h$ -th group ( $h=1, 2, \dots, H$ ).

$$G_Y^B = \frac{1}{\mu_Y N^2} \sum_{h=2}^H \sum_{g=1}^{h-1} \left( \sum_{i=1}^{K_h} \sum_{j=1}^{K_g} (y_{h,i} - y_{g,j}) p_{h,i} p_{g,j} \right) \quad (15b)$$

$$G_Y^T = \frac{2}{\mu_Y N^2} \sum_{h=2}^H \sum_{g=1}^{h-1} \left( \sum_i^{K_g} \sum_{j \in \{y_{h,i} < y_{g,j}\}}^{K_h} |y_{h,i} - y_{g,j}| p_{h,i} p_{g,j} \right) \quad (16)$$

where  $\bar{p}_h = \sum_{i=1}^{K_h} p_{h,i}$  and  $\bar{p}_g = \sum_{j=1}^{K_g} p_{g,j}$ .

We refer to Monti and Santoro (2007), formula (6) in particular, for the derivation of expression (15b). Expressions (12) (13), (14) and the first term on the right hand side in (15a) do not need ranking  $Y$  values; whereas (15b) and (16) need groups to be ranked according to their averages.

Let's now order the  $Y$  values (and the related  $P$  and, possibly,  $X$  values) so that

- (i) within each group they are ranked in a non-decreasing order;
- (ii) groups are aligned in a non-decreasing order with respect to the their averages.

Then the  $Y$  values parade becomes

$$\mathbf{y}_A' = \left[ (y_{1,1}, y_{1,2}, \dots, y_{1,K_1}), \dots, (y_{h,1}, y_{h,2}, \dots, y_{h,K_h}), \dots, (y_{H,1}, y_{H,2}, \dots, y_{H,K_H}) \right] \quad (17)$$

$$y_{h,i} \leq y_{h,i+1} \quad (i=1, 2, \dots, K_h) \quad \text{and} \quad \mu_{Yh} \leq \mu_{Yh+1} \quad (h=1, 2, \dots, H)^8.$$

<sup>8</sup> It is not excluded that  $y_{h,i} > y_{g,j}$ ,  $g > h$ .

We shall denote the ordering given by (17) as the  $\{(x_i, y_i, p_i)\}_{AY}$  ordering.

The  $\{(x_i, y_i, p_i)\}_{AX}$  ordering can be introduced likewise: according to this ordering the  $X$  values, together with the related  $Y$  and  $P$  values, are distributed into the  $H$  groups such that

- (i) within each group the  $x$ 's are ranked in a non-decreasing order;
- (ii) groups are in a non-decreasing order with respect to their  $X$  averages.

Thus, for what concerns the  $X$  values, the  $\{(x_i, y_i, p_i)\}_{AX}$  ordering will appear as

$$\mathbf{x}_A' = \left[ (x_{1,1}, x_{1,2}, \dots, x_{1,K_1}), \dots, (x_{h,1}, x_{h,2}, \dots, x_{h,K_h}), \dots, (x_{H,1}, x_{H,2}, \dots, x_{H,K_H}) \right] \quad (18)$$

$$x_{h,i} \leq x_{h,i+1} \quad (i=1,2,\dots,K_h) \quad \text{and} \quad \mu_{Xh} \leq \mu_{Xh+1} \quad (h=1,2,\dots,H)^9.$$

The vectors  $\mathbf{y}_A$  in (17) and  $\mathbf{x}_A$  in (18) can be expressed as functions of  $\mathbf{y}$  and  $\mathbf{x}$  respectively, by introducing proper  $K \times K$  permutation matrices  $\mathbf{A}_Y$  and  $\mathbf{A}_X$ , such that  $\mathbf{y}_A = \mathbf{A}_Y \mathbf{y}$  and  $\mathbf{x}_A = \mathbf{A}_X \mathbf{x}$ .

Since  $\mathbf{A}_Y$  and  $\mathbf{A}_X$  are permutation matrices, the following holds:  $\mathbf{A}_Y^{-1} = \mathbf{A}_Y'$  and  $\mathbf{A}_X^{-1} = \mathbf{A}_X'$ .

The  $Y$  vector corresponding to the  $\{(x_i, y_i, p_i)\}_{AX}$  ordering can be obtained as  $\mathbf{y}_{AX} = \mathbf{A}_X \mathbf{y}_X$ , and likewise  $\mathbf{p}_{AX} = \mathbf{A}_X \mathbf{p}_X$ .

Also  $\mathbf{x}_{AY} = \mathbf{A}_Y \mathbf{x}_X$  and  $\mathbf{p}_{AY} = \mathbf{A}_Y \mathbf{p}_X$  contain the  $Y$  and the  $P$  elements, respectively, aligned according to the  $\{(x_i, y_i, p_i)\}_{AY}$  ordering.

If we work out (3), (10) and (11), by making use of the property  $\mathbf{A}_X' \mathbf{A}_X = \mathbf{I}$ , we get

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} \quad (19)$$

$$C_{Y|X}^W = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' (\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' (\mathbf{W}_{AX} \odot \mathbf{S} \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} \quad (20)$$

$$C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (21)$$

where  $\mathbf{W}_{AX} = \mathbf{A}_X \mathbf{W}_X \mathbf{A}_X'$  and  $\mathbf{D}_{Y|AX} = (\mathbf{j} \mathbf{y}_X' \mathbf{A}_X' - \mathbf{A}_X \mathbf{y}_X \mathbf{j}') = (\mathbf{j} \mathbf{y}_{AX}' - \mathbf{y}_{AX} \mathbf{j}')$ .

For what concerns  $C_{Y|X}^W$  in (21), it is shown in Appendix A2 that

$$\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' = \mathbf{W}_{AX} \odot \mathbf{S}.$$

Focusing on  $C_{Y|X}^{AG}$  decomposition, notice that:

<sup>9</sup> Here also it is not excluded that  $x_{h,i} > x_{g,j}$ ,  $g > h$ .

$$C_{Y|X}^B = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot \mathbf{S} \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (22)$$

$$C_{Y|X}^T = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' - \mathbf{S}) \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (23)$$

Summing (22) and (23) yields (21).

Should  $C_{Y|Y} \equiv G_Y$ ,  $C_{Y|Y}^W \equiv G_Y^W$ ,  $C_{Y|Y}^{AG} \equiv G_Y^{AG}$ ,  $C_{Y|Y}^B \equiv G_Y^B$  and  $C_{Y|Y}^T \equiv G_Y^T$ , then (19), (20), (21), (22) and (23) would take the following forms:

$$G_Y = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} \quad (24)$$

$$G_Y^W = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' (\mathbf{W}_{AY} \odot \mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' (\mathbf{W}_{AY} \odot \mathbf{S} \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} \quad (25)$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot \mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (26)$$

$$G_Y^B = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot \mathbf{S} \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (27)$$

$$G_Y^T = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' - \mathbf{S}) \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (28)$$

where  $\mathbf{D}_{AY} = (\mathbf{j}_Y' \mathbf{A}_Y' - \mathbf{A}_Y \mathbf{j}_Y') = (\mathbf{j}_Y' \mathbf{A}_Y' - \mathbf{y}_Y \mathbf{j}_Y')$  and  $\mathbf{W}_{AY} = \mathbf{A}_Y \mathbf{W}_Y \mathbf{A}_Y'$ .

The matrix compact forms (24), (25), (26) (27) and (28) correspond to the scalar expressions (19), (20), (21), (22) and (23), respectively.

We conclude this section by providing closed-form expressions for  $C_{Y|X}^B$  and  $C_{Y|X}^T$ , by bearing in mind  $C_{Y|X}^{AG}$ , as specified in (11), under the  $\{(x_i, y_i, p_i)\}_X$  ordering:

$$C_{Y|X}^B = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \quad (29)$$

$$C_{Y|X}^T = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \quad (30)$$

### 3 Redistribution and re-ranking indexes

The redistributive effect of a tax system can be measured by the difference between the Gini index for the pre-tax income distribution  $X$  and the Gini index for the post-tax income distribution  $Y$ <sup>10</sup>: following e.g. Urban and Lambert (2008), we shall denote difference by the acronym  $RE$ .

The Atkinson-Plotnick-Kakwani index is generally applied to measure the re-ranking effect generated by a tax system; it is defined as the difference between the Gini index for the post-tax income distribution and the concentration index for net incomes  $Y$  in the  $\{(x_i, y_i, p_i)\}_X$  ordering<sup>11</sup>. The Atkinson, Plotnick; Kakwani index is usually denoted by the acronym  $R$ .

In considering the effects of a tax, it may be interesting to evaluate how  $RE$  and  $R$  act within and across groups and, eventually, also how they modify both group average positions and group intersections. This can be attained by splitting either  $RE$  or  $R$  into the within groups, across groups, between groups and transvariation components, introduced in the previous section.

One of the advantages of the compact expressions introduced in the previous sections is that all indexes can be calculated either aligning incomes according to the pre-tax or according to the post-tax ranking.

#### 3.1 The $RE$ index

From the definition of  $RE$  we can write

$$RE = G_X - G_Y = (G_X^W + G_X^{AG}) - (G_Y^W + G_Y^{AG}) = (G_X^W + G_X^B + G_X^T) - (G_Y^W + G_Y^B + G_Y^T)$$

Rearranging terms we get

$$RE = (G_X^W - G_Y^W) + (G_X^{AG} - G_Y^{AG}) = RE^W + RE^{AG} \quad (32)$$

Here, in what concerns  $RE^{AG}$ , bearing in mind that  $G^{AG} = G^B + G^T$ , we get

$$RE^{AG} = (G_X^B - G_Y^B) + (G_X^T - G_Y^T) = RE^B + RE^T \quad (32')$$

We will present the  $RE$  index and its decompositions by writing  $\mathbf{D}$  matrices and  $\mathbf{p}$  vectors either according to the  $\{(x_i, y_i, p_i)\}_X$  or the  $\{(x_i, y_i, p_i)\}_Y$  orderings, when individual income units are considered as well, either according to the  $\{(x_i, y_i, p_i)\}_{AX}$  or the  $\{(x_i, y_i, p_i)\}_{AY}$  orderings, when group subdivision is made explicit.

<sup>10</sup> See e.g. Lambert (2001, Ch. 2, Section 2.5).

<sup>11</sup> Plotnick (1981), Lambert (2001, Ch. 2, Section 2.5).

Representing RE by individual units

From (5) and (6) it follows that

$$\begin{aligned} RE &= G_X - G_Y = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_X' \left[ \mu_Y (\mathbf{S} \odot \mathbf{D}_X) - \mu_X (\mathbf{E}\mathbf{S}\mathbf{E}' \odot \mathbf{D}_{Y|X}) \right] \mathbf{p}_X \end{aligned} \quad (33a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_Y' \left[ \mu_Y (\mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_{X|Y}) - \mu_X (\mathbf{S} \odot \mathbf{D}_Y) \right] \mathbf{p}_Y \quad (33b)$$

The  $RE^W$  components can be written, according to (32) and bearing in mind (10) as

$$\begin{aligned} RE^W &= G_X^W - G_Y^W = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_X' \left\{ \mathbf{W}_X \odot \left[ \mu_Y (\mathbf{S} \odot \mathbf{D}_X) - \mu_X (\mathbf{E}\mathbf{S}\mathbf{E}' \odot \mathbf{D}_{Y|X}) \right] \right\} \mathbf{p}_X \end{aligned} \quad (34a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_Y' \left\{ \mathbf{W}_Y \odot \left[ \mu_Y (\mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_{X|Y}) - \mu_X (\mathbf{S} \odot \mathbf{D}_Y) \right] \right\} \mathbf{p}_Y \quad (34b)$$

Likewise the  $RE^{AG}$  components can be written as

$$\begin{aligned} RE^{AG} &= G_X^{AG} - G_Y^{AG} = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_X' \left\{ (\mathbf{J} - \mathbf{W}_X) \odot \left[ \mu_Y (\mathbf{S} \odot \mathbf{D}_X) - \mu_X (\mathbf{E}\mathbf{S}\mathbf{E}' \odot \mathbf{D}_{Y|X}) \right] \right\} \mathbf{p}_X \end{aligned} \quad (35a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_Y' \left\{ (\mathbf{J} - \mathbf{W}_Y) \odot \left[ \mu_Y (\mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_{X|Y}) - \mu_X (\mathbf{S} \odot \mathbf{D}_Y) \right] \right\} \mathbf{p}_Y \quad (35b)$$

Resorting to (29) and (30),  $RE^B$  and  $RE^T$  can be rewritten as

$$\begin{aligned} RE^B &= G_X^B - G_Y^B = \\ &= \frac{1}{2\mu_X N^2} \mathbf{p}_X' \left[ (\mathbf{J} - \mathbf{W}_X) \odot \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \odot \mathbf{D}_X \right] \mathbf{p}_X - \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' \left[ (\mathbf{J} - \mathbf{W}_Y) \odot \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \odot \mathbf{D}_Y \right] \mathbf{p}_Y \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_X' \left\{ (\mathbf{J} - \mathbf{W}_X) \odot \left[ \mu_Y (\mathbf{A}_X' \mathbf{S} \mathbf{A}_X \odot \mathbf{D}_X) - \mu_X (\mathbf{E} \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \mathbf{E}' \odot \mathbf{D}_{Y|X}) \right] \right\} \mathbf{p}_X \end{aligned} \quad (36a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_Y' \left\{ (\mathbf{J} - \mathbf{W}_Y) \odot \left[ \mu_Y (\mathbf{E}' \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \mathbf{E} \odot \mathbf{D}_{X|Y}) - \mu_X (\mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \odot \mathbf{D}_Y) \right] \right\} \mathbf{p}_Y \quad (36b)$$

$$\begin{aligned} RE^T &= G_X^T - G_Y^T = \\ &= \frac{1}{2\mu_X N^2} \mathbf{p}_X' \left[ (\mathbf{J} - \mathbf{W}_X) \odot (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \odot \mathbf{D}_X \right] \mathbf{p}_X - \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' \left[ (\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \odot \mathbf{D}_Y \right] \mathbf{p}_Y \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_X' \left\{ (\mathbf{J} - \mathbf{W}_X) \odot \left[ \mu_Y (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \odot \mathbf{D}_X - \mu_X \mathbf{E} (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \mathbf{E}' \odot \mathbf{D}_{Y|X} \right] \right\} \mathbf{p}_X \end{aligned} \quad (37a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_Y' \left\{ (\mathbf{J} - \mathbf{W}_Y) \odot \left[ \mu_Y \mathbf{E}' (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \mathbf{E} \odot \mathbf{D}_{X|Y} - \mu_X (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \odot \mathbf{D}_Y \right] \right\} \mathbf{p}_Y \quad (37b)$$

Representing RE by units gathered into groups

If we wish to explicit group subdivision, we can work out (33a) and (33b). By making use of the equalities  $\mathbf{A}_X' \mathbf{A}_X = \mathbf{I}$  and  $\mathbf{A}_Y' \mathbf{A}_Y = \mathbf{I}$ , simple computations yield

$$\begin{aligned} RE^B &= G_X^B - G_Y^B = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AX}' \left[ \mu_Y (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{AX}) - \mu_X (\mathbf{A}_X \mathbf{E} \mathbf{S} \mathbf{E}' \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \right] \mathbf{p}_{AX} \end{aligned} \quad (38a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AY}' \left[ \mu_Y (\mathbf{A}_Y \mathbf{E}' \mathbf{S} \mathbf{E} \mathbf{A}_Y' \odot \mathbf{D}_{X|AY}) - \mu_X (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \right] \mathbf{p}_{AY} \quad (38b)$$

From (34a) and (34b) one gets

$$\begin{aligned} RE^W &= (G_X^W - G_Y^W) = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AX}' \left\{ \mathbf{W}_{AX} \odot \left[ \mu_Y (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{AX}) - \mu_X (\mathbf{A}_X \mathbf{E} \mathbf{S} \mathbf{E}' \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \right] \right\} \mathbf{p}_{AX} \end{aligned} \quad (39a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AY}' \left\{ \mathbf{W}_{AY} \odot \left[ \mu_Y (\mathbf{A}_Y \mathbf{E}' \mathbf{S} \mathbf{E} \mathbf{A}_Y' \odot \mathbf{D}_{X|AY}) - \mu_X (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \right] \right\} \mathbf{p}_{AY} \quad (39b)$$

Likewise, from (35a) and (35b), the AG component can be worked out as follows

$$\begin{aligned} RE^{AG} &= G_X^{AG} - G_Y^{AG} = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AX}' \left\{ (\mathbf{J} - \mathbf{W}_{AX}) \odot \left[ \mu_Y (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{AX}) - \mu_X (\mathbf{A}_X \mathbf{E} \mathbf{S} \mathbf{E}' \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \right] \right\} \mathbf{p}_{AX} \end{aligned} \quad (40a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AY}' \left\{ (\mathbf{J} - \mathbf{W}_{AY}) \odot \left[ \mu_Y (\mathbf{A}_Y \mathbf{E}' \mathbf{S} \mathbf{E} \mathbf{A}_Y' \odot \mathbf{D}_{X|AY}) - \mu_X (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \right] \right\} \mathbf{p}_{AY} \quad (40b)$$

For what concerns the between groups component, from (36a) and (36b), we obtain

$$\begin{aligned} RE^B &= G_X^B - G_Y^B = \\ &= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AX}' \left\{ (\mathbf{J} - \mathbf{W}_{AX}) \odot \left[ \mu_Y (\mathbf{S} \odot \mathbf{D}_{AX}) - \mu_X (\mathbf{A}_X \mathbf{E} \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \mathbf{E}' \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \right] \right\} \mathbf{p}_{AX} \end{aligned} \quad (41a)$$

$$= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AY}' \left\{ (\mathbf{J} - \mathbf{W}_{AY}) \odot \left[ \mu_Y (\mathbf{A}_Y \mathbf{E}' \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \mathbf{E} \mathbf{A}_Y' \odot \mathbf{D}_{X|AY}) - \mu_X (\mathbf{S} \odot \mathbf{D}_{AY}) \right] \right\} \mathbf{p}_{AY} \quad (41b)$$

Finally, for what concerns the transvariation component, from (37a) and (37b) it follows that

$$\begin{aligned}
RE^T &= G_X^T - G_Y^T = \\
&= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AX}' \left\{ (\mathbf{J} - \mathbf{W}_{AX}) \odot \left[ \mu_Y (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' - \mathbf{S}) \odot \mathbf{D}_{AX} - \mu_X \mathbf{A}_X \mathbf{E} (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \mathbf{E}' \mathbf{A}_X' \odot \mathbf{D}_{Y|AX} \right] \right\} \mathbf{p}_{AX}
\end{aligned} \tag{42a}$$

$$\begin{aligned}
&= \frac{1}{2\mu_X\mu_Y N^2} \mathbf{p}_{AY}' \left\{ (\mathbf{J} - \mathbf{W}_{AY}) \odot \left[ \mu_Y \mathbf{A}_Y \mathbf{E}' (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \mathbf{E} \mathbf{A}_Y' \odot \mathbf{D}_{X|AY} - \mu_X (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' - \mathbf{S}) \odot \mathbf{D}_{AY} \right] \right\} \mathbf{p}_{AY}
\end{aligned} \tag{42b}$$

### 3.2 The $R$ (Atkinson-Plotnick-Kakwani) index

From the definition of  $R$  we can write

$$R = G_Y - C_{Y|X} = (G_Y^W + G_Y^{AG}) - (C_{Y|X}^W + C_{Y|X}^{AG}) = (G_Y^W + G_Y^B + G_Y^T) - (C_{Y|X}^W + C_{Y|X}^B + C_{Y|X}^T)$$

Rearranging the terms we get

$$R = (G_Y^W - C_{Y|X}^W) + (G_Y^{AG} - C_{Y|X}^{AG}) = R^W + R^{AG} \tag{43}$$

and in particular, for what concerns  $R^{AG}$ , we have

$$R^{AG} = (G_Y^B - C_{Y|X}^B) + (G_Y^T - C_{Y|X}^T) = R^B + R^T \tag{43'}$$

#### Representing $R$ by individual units

When considering income units individually, from (2), (3), (5) and (7) the index  $R$ , and its components, can be written as follows

$$\begin{aligned}
R = G_Y - C_{Y|X} &= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{S} \odot \mathbf{D}_Y) \mathbf{p}_Y - \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{S} - \mathbf{E}' \mathbf{S} \mathbf{E}) \odot \mathbf{D}_Y] \mathbf{p}_Y
\end{aligned} \tag{44a}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{E} \mathbf{S} \mathbf{E}' - \mathbf{S}) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \tag{44b}$$

From (10) and (44) it follows that

$$\begin{aligned}
R^W &= G_Y^W - C_{Y|X}^W = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{W}_Y \odot \mathbf{S} \odot \mathbf{D}_Y) \mathbf{p}_Y - \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{W}_X \odot \mathbf{S} \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [\mathbf{W}_Y \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \odot \mathbf{D}_Y] \mathbf{p}_Y \tag{45a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [\mathbf{W}_X \odot (\mathbf{E}\mathbf{S}\mathbf{E}' - \mathbf{S}) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \tag{45b}$$

From (11) and (44) it follows that

$$\begin{aligned}
R^{AG} &= G_Y^{AG} - C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot \mathbf{S} \odot \mathbf{D}_Y] \mathbf{p}_Y - \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot \mathbf{S} \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \odot \mathbf{D}_Y] \mathbf{p}_Y \tag{46a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot (\mathbf{E}\mathbf{S}\mathbf{E}' - \mathbf{S}) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \tag{46b}$$

From (29) the component  $R^B$  of  $R$  can be expressed as

$$\begin{aligned}
R^B &= G_Y^B - C_{Y|X}^B = \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \odot \mathbf{D}_Y] \mathbf{p}_Y - \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y - \mathbf{E}' \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \mathbf{E}) \odot \mathbf{D}_Y] \mathbf{p}_Y \tag{47a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot (\mathbf{E} \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \mathbf{E}' - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \tag{47b}$$

From (30) the component  $R^T$  can be expressed as

$$\begin{aligned}
R^T &= G_Y^T - C_{Y|X}^T = \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \odot \mathbf{D}_Y] \mathbf{p}_Y - \frac{1}{2\mu_Y N^2} \mathbf{p}_X' [(\mathbf{J} - \mathbf{W}_X) \odot (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \odot \mathbf{D}_{Y|X}] \mathbf{p}_X \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' \{ (\mathbf{J} - \mathbf{W}_Y) \odot [(\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) - \mathbf{E}' (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \mathbf{E}] \odot \mathbf{D}_Y \} \mathbf{p}_Y \tag{48a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_X' \{ (\mathbf{J} - \mathbf{W}_X) \odot [\mathbf{E} (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \mathbf{E}' - (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X)] \odot \mathbf{D}_{Y|X} \} \mathbf{p}_X \tag{48b}$$

Either from the definitions of  $R^{AG}$  and  $R^B$  or by rearranging the terms in (48a) and (48b),  $R^T$  can be given the following representations:

$$\begin{aligned}
R^T &= (R^{AG} - R^B) = \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_Y' \{ (\mathbf{J} - \mathbf{W}_Y) \odot [(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) - (\mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y - \mathbf{E}' \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \mathbf{E})] \odot \mathbf{D}_Y \} \mathbf{p}_Y \quad (48c)
\end{aligned}$$

$$= \frac{1}{2\mu_y N^2} \mathbf{p}_X' \{ (\mathbf{J} - \mathbf{W}_X) \odot [(\mathbf{E}\mathbf{S}\mathbf{E}' - \mathbf{S}) - (\mathbf{E} \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \mathbf{E}' - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X)] \odot \mathbf{D}_{Y|X} \} \mathbf{p}_X \quad (48d)$$

Representing R by units gathered into groups

Bearing in mind (19) and (44), the representation by groups for R is given by

$$\begin{aligned}
R &= G_Y - C_{Y|X} = \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} - \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' [\mathbf{A}_Y (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \mathbf{A}_Y' \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (49a)
\end{aligned}$$

$$= \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' [\mathbf{A}_X (\mathbf{E}\mathbf{S}\mathbf{E}' - \mathbf{S}) \mathbf{A}_Y' \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (49b)$$

From (20), (25), (45) and (49) it follows that

$$\begin{aligned}
R^W &= G_Y^W - C_{Y|X}^W \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' (\mathbf{W}_{AY} \odot \mathbf{S} \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} - \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' (\mathbf{W}_{AX} \odot \mathbf{S} \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' (\mathbf{W}_{AY} \odot \mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}) \mathbf{p}_{AY} - \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' (\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}) \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' [\mathbf{W}_{AY} \odot (\mathbf{S} - \mathbf{A}_Y \mathbf{E}' \mathbf{S} \mathbf{E} \mathbf{A}_Y') \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (50a)
\end{aligned}$$

$$= \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' [\mathbf{W}_{AX} \odot (\mathbf{A}_X \mathbf{E} \mathbf{S} \mathbf{E}' \mathbf{A}_X' - \mathbf{S}) \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (50b)$$

Likewise from (21), (26), (46) and (49) it follows that

$$\begin{aligned}
R^{AG} &= (G_Y^{AG} - C_{Y|X}^{AG}) = \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot \mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} - \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' - \mathbf{A}_Y \mathbf{E}' \mathbf{S} \mathbf{E} \mathbf{A}_Y') \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \quad (51a)
\end{aligned}$$

$$= \frac{1}{2\mu_y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot (\mathbf{A}_X \mathbf{E} \mathbf{S} \mathbf{E}' \mathbf{A}_X' - \mathbf{A}_Y \mathbf{S} \mathbf{A}_Y') \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \quad (51b)$$

The quantity  $R^{AG}$  can then be split into  $R^B$  and  $R^T$ . By resorting to (22), (27), (47) and (51) we obtain

$$\begin{aligned}
R^B &= G_Y^B - C_{Y|X}^B = \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot \mathbf{S} \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} - \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot \mathbf{S} \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot (\mathbf{S} - \mathbf{A}_Y \mathbf{E}' \mathbf{A}_X' \mathbf{S} \mathbf{A}_X \mathbf{E} \mathbf{A}_Y') \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} \tag{52a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot (\mathbf{A}_X \mathbf{E} \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y \mathbf{E}' \mathbf{A}_X' - \mathbf{S}) \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \tag{52b}$$

Besides, by keeping in mind (23), (28), (48), and (51), we get

$$\begin{aligned}
R^T &= G_Y^T - C_{Y|X}^T = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' [(\mathbf{J} - \mathbf{W}_{AY}) \odot (\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' - \mathbf{S}) \odot \mathbf{D}_{AY}] \mathbf{p}_{AY} + \\
&\quad - \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' [(\mathbf{J} - \mathbf{W}_{AX}) \odot (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' - \mathbf{S}) \odot \mathbf{D}_{Y|AX}] \mathbf{p}_{AX} \\
&= \frac{1}{2\mu_Y N^2} \mathbf{p}_{AY}' \{ (\mathbf{J} - \mathbf{W}_{AY}) \odot [(\mathbf{A}_Y \mathbf{S} \mathbf{A}_Y' - \mathbf{S}) - \mathbf{A}_Y \mathbf{E}' (\mathbf{S} - \mathbf{A}_X' \mathbf{S} \mathbf{A}_X) \mathbf{E} \mathbf{A}_Y'] \odot \mathbf{D}_{AY} \} \mathbf{p}_{AY} \tag{53a}
\end{aligned}$$

$$= \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}' \{ (\mathbf{J} - \mathbf{W}_{AX}) \odot [\mathbf{A}_X \mathbf{E} (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \mathbf{E}' \mathbf{A}_X' - (\mathbf{A}_X \mathbf{S} \mathbf{A}_X' - \mathbf{S})] \odot \mathbf{D}_{Y|AX} \} \mathbf{p}_{AX} \tag{53b}$$

#### 4 The issue of the signs of $R$ and its components

We will now analyse the signs of  $R$  and of its decompositions, by making use of the matrix tools introduced in the previous sections. Although most of the results presented in this section are available in the specialized literature<sup>12</sup>, we think that our reappraisal of the issue through a tailor-made matrix toolkit provides some additional insights on the matter. Demonstrations will be carried out by inspecting the quadratic form which the  $R$  index and its decompositions are proportional to.

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<sup>12</sup> Mussini (2008, Ch. 6, § 6.1, page 92) discusses the signs of  $R$  and its components  $R^W$ ,  $R^B$  and  $R^T$ . The author observes also that  $R^T$  can be positive, null or negative in the framework of non contiguous pre-tax income groups: the proofs reported here complete the author's statements, especially in what concerns  $R^T$ . See also Vernizzi (2007) for considerations on  $G$  and  $C$  components especially for pre-tax non overlapping groups.

## **R**

It is well known that for the concentration  $C$  index the property  $-G \leq C \leq +G$  holds<sup>13</sup>, from which it follows that  $R = G_Y - C_{Y|X} \geq 0$ . This result will be proved considering expression (44a).

### Statement 1

The quadratic form  $\mathbf{p}_Y'[(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \odot \mathbf{D}_Y]\mathbf{p}$  is non-negative definite.

### Proof

Recall that (i) matrix  $\mathbf{S} = [s_{i,j}]$  has all super-diagonal elements equal to +1 and sub-diagonal ones equal to -1; (ii) the elements of  $\mathbf{E}'\mathbf{S}\mathbf{E} = [s_{i,j}^e]$  may not necessarily respect the same repartition as in  $\mathbf{S}$ , due to permutations performed by  $\mathbf{E}$ . So, for all entries of  $\mathbf{S}$  and  $\mathbf{E}'\mathbf{S}\mathbf{E}$  which present the same values,  $s_{i,j} - s_{i,j}^e = 0$ , otherwise for  $i < j$  we would have  $s_{i,j} - s_{i,j}^e = 2$  and, for  $i > j$ ,  $s_{i,j} - s_{i,j}^e = -2$ . Bearing in mind that for  $i < j$ , the matrix  $\mathbf{D}_Y = [d_{i,j}^Y]$  has super-diagonal elements non-negative and sub-diagonal ones non-positive, the product  $(s_{i,j} - s_{i,j}^e) \cdot d_{i,j}^Y$  will in any case result to be non-negative, which proves the Statement. □

## **R<sup>W</sup> and R<sup>AG</sup>**

We will prove that  $R^W = G_Y^W - C_{Y|X}^W \geq 0$  and  $R^{AG} = G_Y^{AG} - C_{Y|X}^{AG} \geq 0$ , by considering expressions (45a) and (46a) respectively.

### Statement 2

The quadratic forms

$\mathbf{p}_Y'[\mathbf{W}_Y \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \odot \mathbf{D}_Y]\mathbf{p}_Y$  and  $\mathbf{p}_Y'[(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \odot \mathbf{D}_Y]\mathbf{p}_Y$  are non-negative definite.

### Proof

Statement 2 is a corollary of Statement 1. Bearing in mind the considerations reported to prove Statement 1, the entries that are in the super-diagonal part of  $[\mathbf{W}_Y \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E})]$  and  $[(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E})]$  are either 0 or 2, while in the sub-diagonal part of both matrices they are

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<sup>13</sup> Kakwani (1980, Corollary 8.7, page 175).

either 0 or -2; since in  $[\mathbf{W}_Y \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E})] \odot \mathbf{D}_Y$  and in  $[(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E})] \odot \mathbf{D}_Y$ , super-diagonal elements of  $\mathbf{D}_Y = [d_{i,j}^Y]$ , which are non-negative, are multiplied either by 0 or by 2, while sub-diagonal elements of  $\mathbf{D}_Y = [d_{i,j}^Y]$ , which are non-positive, are multiplied either by 0 or by -2, the Statement is shown to hold true. □

## $G^T$

In the present context  $G_Y^T$  is not a re-ranking index: however,  $R^T$  coincides with  $G_Y^T$  in the particular framework considered by Aronson, Johnson and Lambert and Urban <sup>14</sup>, because pre-tax groups do not overlap by construction and, consequently,  $G_X^T = 0$  and  $C_{Y|X}^T = 0$ .

That  $G_Y^T \geq 0$  will be shown by inspection of expression (30).

### Statement 3

The quadratic form

$$\mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \odot \mathbf{D}_Y] \mathbf{p}_Y \text{ is non-negative definite.}$$

### Proof

From expression (30)  $G_Y^T \propto \mathbf{p}_Y' [(\mathbf{J} - \mathbf{W}_Y) \odot (\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) \odot \mathbf{D}_Y] \mathbf{p}_Y$ , which is quite analogous to  $R^{AG}$  expression, except for the term  $\mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y$ , where  $\mathbf{A}_Y$  substitutes  $\mathbf{E}$ . When applied to  $\mathbf{S}$ , the effect of  $\mathbf{A}_Y$  is quite analogous to the effect created by  $\mathbf{E}$  even if it involves elements which belong to different groups. When permutations are applied, they still consist in permuting sub-diagonal entries (that are -1) with their symmetric super-diagonal ones (that are +1), so that  $\mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y = [s_{i,j}^a]$  may present some super-diagonal elements equal to -1 and some sub-diagonal entries equal to +1: so when permutations are applied the  $(\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y) = [s_{i,j} - s_{i,j}^a]$  matrix presents  $s_{i,j} - s_{i,j}^a = 2$  if  $i < j$ , and  $s_{i,j} - s_{i,j}^a = -2$  if  $i > j$ . As elements  $s_{i,j}^a$  that do not permute from lower diagonal entries to upper ones and vice-versa cancel out the corresponding  $s_{i,j}$  in  $\mathbf{S}$ , it follows that the non-zero entries of matrix  $(\mathbf{S} - \mathbf{A}_Y' \mathbf{S} \mathbf{A}_Y)$  can be but +2, if super-diagonal, -2 if sub-diagonal.

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<sup>14</sup> Aronson and Lambert (1993, 1994), Aronson, Johnson, and Lambert (1994), Urban and Lambert (2008); Urban and Lambert define  $G_Y^T = R^T$  as  $R^{AJL}$ .

Bearing in mind the considerations put forward for previous statements, as super-diagonal  $(s_{i,j} - s_{i,j}^a) = 2$  multiplies super-diagonal entries of  $\mathbf{D}_Y = [d_{i,j}^Y]$  which are non-negative, and sub-diagonal  $(s_{i,j} - s_{i,j}^a) = -2$  multiplies sub-diagonal entries of  $\mathbf{D}_Y = [d_{i,j}^Y]$  which are non-positive, the Statement is proved. □

## $R^B$

We now prove that  $R^B = G_Y^B - C_{Y|X}^B \geq 0$ . In order to carry out the proof as for the previous Statements, it is convenient to consider a matrix compact form that corresponds in a straightforward manner to the second term in the right hand side of (15a). Let's define the  $H \times 1$  vector  $\boldsymbol{\mu}_Y = [\mu_{Y1}, \mu_{Y2}, \dots, \mu_{YH}]'$  of group averages,  $\mu_{Yh} \leq \mu_{Yh+1}$  ( $h=1,2,\dots,H$ ), the  $H \times 1$  vector  $\bar{\mathbf{p}}_Y = [\bar{p}_1, \bar{p}_2, \dots, \bar{p}_H]'$  of group weights  $\bar{p}_h = \sum_{i=1}^{K_h} p_{h,i}$  and the  $H \times H$  matrix  $\bar{\mathbf{D}}_Y = (\mathbf{1}\boldsymbol{\mu}_Y' - \boldsymbol{\mu}_Y\mathbf{1}')$  of group average differences. Then

$$G_Y^B = \frac{1}{2\mu_Y N^2} \sum_{h=1}^H \sum_{g=1}^H |\mu_{Y,h} - \mu_{Y,g}| \bar{p}_h \bar{p}_g = \frac{1}{2\mu_Y N^2} \bar{\mathbf{p}}_Y' [\mathbf{S} \odot \bar{\mathbf{D}}_Y] \bar{\mathbf{p}}_Y \quad (54)$$

where  $\mathbf{S}$  is now an  $H \times H$  matrix.

After having defined  $\boldsymbol{\mu}_{Y|X}$  and  $\bar{\mathbf{p}}_X$ , respectively, as the  $H \times 1$  vector of  $\mu_{Yh}$  and the  $H \times 1$  vector of  $\bar{p}_h$ , aligned according to the  $\{(x_i, y_i, p_i)\}_{AX}$  order, and the  $H \times H$  matrix  $\bar{\mathbf{D}}_{Y|X} = (\mathbf{1}\boldsymbol{\mu}_{Y|X}' - \boldsymbol{\mu}_{Y|X}\mathbf{1}')$ , (22) can be rewritten in this way:

$$C_Y^B = \frac{1}{2\mu_Y N^2} \bar{\mathbf{p}}_X' [\mathbf{S} \odot \bar{\mathbf{D}}_{Y|X}] \bar{\mathbf{p}}_X \quad (55)$$

Finally, by denoting by  $\bar{\mathbf{E}}$  the  $H \times H$  full rank permutation matrix such that  $\boldsymbol{\mu}_{Y|X} = \bar{\mathbf{E}}\boldsymbol{\mu}_Y$ ,  $\boldsymbol{\mu}_Y = \bar{\mathbf{E}}'\boldsymbol{\mu}_{Y|X}$ ,  $\bar{\mathbf{p}}_X = \bar{\mathbf{E}}\bar{\mathbf{p}}_Y$  and  $\bar{\mathbf{p}}_Y = \bar{\mathbf{E}}'\bar{\mathbf{p}}_X$ , (52) can be rewritten as

$$R^B = G_Y^B - C_Y^B = \frac{1}{2\mu_Y N^2} \bar{\mathbf{p}}_Y' [(\mathbf{S} - \bar{\mathbf{E}}'\mathbf{S}\bar{\mathbf{E}}) \odot \bar{\mathbf{D}}_Y] \bar{\mathbf{p}}_Y \quad (56)$$

### Statement 4

The quadratic form

$$\bar{\mathbf{p}}_Y' [(\mathbf{S} - \bar{\mathbf{E}}'\mathbf{S}\bar{\mathbf{E}}) \odot \bar{\mathbf{D}}_Y] \bar{\mathbf{p}}_Y \text{ is n. n. definite.}$$

Proof

Considerations analogous to those reported above hold for  $(\mathbf{S} - \bar{\mathbf{E}}'\mathbf{S}\bar{\mathbf{E}}) \odot \bar{\mathbf{D}}_Y$ . In  $\bar{\mathbf{D}}_Y$  the super-diagonal entries are non-negative, the sub-diagonal entries are non-positive: while the former are multiplied either by 0 or by +2 entries which are in the super-diagonal part of  $(\mathbf{S} - \bar{\mathbf{E}}'\mathbf{S}\bar{\mathbf{E}})$ , the latter by 0 or by -2 entries which are in the sub-diagonal part of  $(\mathbf{S} - \bar{\mathbf{E}}'\mathbf{S}\bar{\mathbf{E}})$ , and hence it is proved that  $R^B \geq 0$ .

□

$R^T$

Differently from  $R$ ,  $R^W$ ,  $G^T$  and  $R^B$ , that are all non negative,  $R^T$  can be either positive or negative, and, obviously, equal to zero.

Statement 5

In expression (48a) the quadratic form

$\mathbf{p}_Y' \{ (\mathbf{J} - \mathbf{W}_Y) \odot [(\mathbf{S} - \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y) - \mathbf{E}'(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}] \odot \mathbf{D}_Y \} \mathbf{p}_Y$  can be zero, positive or negative.

Proof

Both in matrix  $(\mathbf{S} - \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y) = [{}_Y\omega_{i,j}]$  and in matrix  $(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X) = [{}_X\omega_{i,j}]$  non zero super-diagonal entries are +2, non zero sub-diagonal are -2. Due to permutation performed by  $\mathbf{E}'$  and  $\mathbf{E}$ ,  $\mathbf{E}'(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E} = [{}_X\omega_{i,j}^e]$  can present some -2 as super-diagonal entries and, symmetrically, some +2 as sub-diagonal entries: hence, not considering the cases when both  ${}_Y\omega_{i,j}$  and  ${}_X\omega_{i,j}^e$  are zero, the super-diagonal differences in  $[(\mathbf{S} - \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y) - \mathbf{E}'(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}] = \{ [{}_Y\omega_{i,j}] - [{}_X\omega_{i,j}^e] \}$  may assume values  $[2] - [2] = 0$ ,  $[2] - [0] = 2$ ,  $[2] - [-2] = 4$ ,  $[0] - [-2] = 2$ ,  $[0] - [2] = -2$ . It follows that non-negative super-diagonal entries of  $\mathbf{D}_Y$  can be multiplied by a negative value. Symmetrically, sub-diagonal entries of  $[(\mathbf{S} - \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y) - \mathbf{E}'(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}]$  can now be equal non only to  $[-2] - [0] = -2$ ,  $[-2] - [2] = -4$ , and to  $[0] - [2] = -2$ , but also to  $[0] - [-2] = 2$ , so that non-positive sub-diagonal entries of  $\mathbf{D}_Y$  can be multiplied by a positive value, which proves the Statement.

□

## Conclusions

By use of the Hadamard product, an elegant compact representation in matrix notation has been obtained not only for Gini, concentration indexes and for their decompositions, but for redistribution and re-ranking indexes and their decompositions as well. The matrix toolkit introduced in this paper paves the way to obtain informative expressions for both the said indexes and their components, with incomes aligned either according to the pre-tax non-decreasing order or to the post-tax non-decreasing order.

Moreover, the compact representation introduced in this paper leads to establish in a straightforward manner the signs of the Atkinson-Plotnick-Kakwani index and of its components. We prove that  $R$ ,  $R^W$ ,  $R^{AG}$  and  $R^B$  are non-negative quantities, both when pre-tax income groups do overlap and when do not. In the latter case  $R^T \equiv G^T$  ( $R^T \equiv R^{AJL}$ , following Urban and Lambert, 2008, notation) is non negative, whereas in the former case we show  $R^T$  can be either positive or negative. Even if it is well known that  $R$  and  $G^T \equiv R^{AJL}$  are non-negative, the proofs presented in this paper are new.

## References

- Bhattacharya, N., B. Mahalanobis (1967). "Regional disparities in household consumption in India", *Journal of the American Statistical Association*, 62, pp. 143-161.
- Dagum C. (1997), "A new approach to the decomposition of Gini income inequality ratio", *Empirical Economics*, 22, pp. 515-531.
- Aronson R. J., P. J. Lambert (1993), "Inequality decomposition analysis and the Gini coefficient revisited", *The Economic Journal*, 103, pp. 1221-1227.
- Aronson R. J., P. J. Lambert (1994), "Decomposing the Gini coefficient to reveal the vertical, horizontal and re-ranking effects of income taxation", *National Tax Journal*, 47, pp. 273-294.
- Aronson R. J., P. J. Johnson, P. J. Lambert (1994), "Redistributive effect and unequal income tax treatment", *The Economic Journal*, 104, pp. 262-270.
- Faliva M. (1983), *Identificazione e Stima nel Modello Lineare ad Equazioni Simultanee*, Vita e Pensiero, Milano.
- Faliva M. (1987), *Econometria, Principi e Metodi*, UTET, Torino.

- Faliva M. (1996), "Hadamard matrix product, graph and system theories: motivations and role in Econometrics", *Matrices and Graphs, Theory and Applications to Economics*, S. Camiz, S. Stefani eds., World Scientific, London.
- Faliva M. (2000), "Su alcune funzioni generalizzate utilizzate nell'analisi dei processi armonizzabili", *Statistica*, 60, 4, pp.655-667.
- Kakwani N. C. (1980), *Income Inequality and Poverty: Methods of Estimation and Policy Applications*, Oxford University Press.
- Lambert P. (2001), *The Distribution and Redistribution of Income*, Manchester University Press.
- Landenna G. (1994), *Fondamenti di Statistica Descrittiva*, il Mulino, Bologna.
- Monti M., A. Santoro (2007), "The Gini decomposition: an alternative formulation with an application to tax reform", *ECINEQ*, 2<sup>nd</sup> Conference, Berlin, available at [http://www.diw.de/documents/dokumentenarchiv/17/60106/papermonti\\_santoro.pdf](http://www.diw.de/documents/dokumentenarchiv/17/60106/papermonti_santoro.pdf).
- Monti, M. (2007), "On the Dagum decomposition of the Gini inequality index", *DEAS Università degli studi di Milano*, W.P. 2007-16.
- Monti M. (2008), "A note on the residual term R in the decomposition of the Gini Index", *Argumenta Oeconomica*, 20, 1, pp. 107-138.
- Mookherjee, D., A. Shorrocks (1982). "A Decomposition Analysis of the Trend in UK Income Inequality", *The Economic Journal*, 92, pp. 886-902.
- Mussini M. (2008), *La misurazione del riordinamento: aspetti metodologici ed un'applicazione con riguardo ai redditi delle famiglie milanesi*, PhD thesis, Università degli Studi di Milano-Bicocca.
- Plotnick R. (1981), "A measure of horizontal inequity", *Review of economics and Statistics*, 63, 283-288.
- Pyatt, G. (1976). "On the Interpretation and desegregations of Gini Coefficient", *Economic Journal*, v86, 243-255.
- Rao, V. (1969). "The Decomposition of the Concentration ratio", *Journal of the Royal Statistical Society*, 132, 418-25.
- Schott J. F. (2005), *Matrix Analysis for Statistics*, 2<sup>nd</sup> edition, Wiley, Hoboken, New Jersey.
- Silber, J.(1989). "Factor Components, Population Subgroups and the Computation of the Gini Index of Inequality", *The Review of Economics and Statistics*, 71, 107-115.
- Urban I., P. J. Lambert, (2008), "Redistribution, horizontal inequity and re-ranking: how to measure them properly", *Public Finance Review*, 20, n. 10, pp.1-24.
- Vernizzi A. (2007), "Una precisazione sulla scomposizione dell'indice di redistribuzione RE di Aronson-Johnson-Lambert e una proposta di estensione dell'indice di Plotnick", *Economia Pubblica*, 37, n. 1-2, pp. 145-153, and *DEAS, Università degli Studi di Milano*, WP 2006-28.
- Yitzhaki, S., R. Lerman, (1991). "Income stratification and income inequality", *Review of income and wealth*, 37, 313-329.
- Yitzhaki, S. (1994). "Economic distance and overlapping of distributions", *Journal of Econometrics*,. 61, pp. 147-159.

## APPENDIX

In his Appendix we will establish two results that proved useful in Section 1 and in Section 2, respectively.

### A1 The concentration index by differences

In formula (1) we expressed the concentration index both as the normalized concentration area and as differences between incomes: we shall now proof that the latter approach leads to the same result as the former.

Given the sequence of incomes  $X$  associated to weights  $P$

$$\left[ (x_1, p_1), (x_2, p_2), \dots, (x_K, p_K) \right], \quad \sum_{i=1}^K p_i = N, \quad \sum_{i=1}^K x_i p_i = N\mu \quad (\text{A1})$$

with the  $x_i$  ( $i = 1, 2, \dots, K$ ) not necessarily in a non decreasing order, for the concentration index  $C$  the following relation holds:

$$C = 1 - \sum_{i=1}^K \left( \sum_{j=1}^i \frac{x_j p_j}{\mu N} + \sum_{j=1}^{i-1} \frac{x_j p_j}{\mu N} \right) \frac{p_i}{N} = \frac{1}{\mu N^2} \sum_{i=1}^K \sum_{j=1}^i (x_i - x_j) p_i p_j \quad (\text{A2})$$

The former term in the right hand side of expression (A1) is the normalized concentration area which is delimited by the equidistribution line and the concentration line, the latter having coordinates

$$\left( \frac{1}{N} \sum_{j=1}^i p_j, \frac{1}{N\mu} \sum_{j=1}^i x_j p_j \right) \quad (i = 1, 2, \dots, K)$$

while the latter would be the average absolute normalized differences, in the case the sequence (A1) presented the  $x_i$ 's in a non decreasing order.

#### Proof

The proof follows the demonstration that Landenna (1994, Ch. 4, § 4.4.), gives for the Gini index, namely

$$\begin{aligned} C &= 1 - \sum_{i=1}^K \left( \sum_{j=1}^i \frac{x_j p_j}{\mu N} + \sum_{j=1}^{i-1} \frac{x_j p_j}{\mu N} \right) \frac{p_i}{N} = 1 - \sum_{i=1}^K \left( \sum_{j=1}^i \frac{x_j p_j}{\mu N} + \sum_{j=1}^{i-1} \frac{x_j p_j}{\mu N} \right) \left( \sum_{j=1}^i \frac{p_j}{N} - \sum_{j=1}^{i-1} \frac{p_j}{N} \right) = \\ &= 1 - \left[ \frac{x_1 p_1}{\mu N} \cdot \frac{p_1}{N} + \sum_{i=2}^K \left( \sum_{j=1}^i \frac{x_j p_j}{\mu N} + \sum_{j=1}^{i-1} \frac{x_j p_j}{\mu N} \right) \frac{p_i}{N} \right] = \end{aligned}$$

$$\begin{aligned}
&= 1 - \frac{1}{\mu N^2} \left[ p_1 x_1 p_1 + p_2 (x_1 p_1 + x_1 p_1 + x_2 p_2) + p_3 (x_1 p_1 + x_2 p_2 + x_1 p_1 + x_2 p_2 + x_3 p_3) + \right. \\
&\quad \left. + \dots + p_K (x_1 p_1 + \dots + x_{K-1} p_{K-1} + x_1 p_1 + \dots + x_K p_K) \right] = \\
&= 1 - \frac{1}{\mu N^2} \left[ \sum_{i=1}^K p_i \sum_{j=1}^i x_j p_j + \sum_{i=1}^{K-1} x_i p_i \left( N - \sum_{j=1}^i p_j \right) \right] = \\
&= 1 - \frac{1}{\mu N^2} \left[ \sum_{i=1}^K p_i \sum_{j=1}^i x_j p_j + \sum_{i=1}^K x_i p_i \left( N - \sum_{j=1}^i p_j \right) - x_K p_K \left( N - \sum_{j=1}^K p_j \right) \right] =
\end{aligned}$$

Observe now that  $\left( N - \sum_{j=1}^K p_j \right) = 0$

$$\begin{aligned}
&= 1 - \frac{1}{\mu N^2} \left[ \sum_{i=1}^K p_i \sum_{j=1}^i x_j p_j + N \sum_{i=1}^K x_i p_i - \sum_{i=1}^K x_i p_i \sum_{j=1}^i p_j \right] = \\
&= 1 - \frac{1}{\mu N^2} \left[ \sum_{i=1}^K p_i \sum_{j=1}^i x_j p_j + N^2 \mu - \sum_{i=1}^K x_i p_i \sum_{j=1}^i p_j \right] =
\end{aligned}$$

After having erased the 1 outside the parenthesis with the  $-1$  generated inside after division by  $N^2 \mu$ , by multiplying by  $-1$  and then reordering what remains,  $C$  can be written as

$$C = \frac{1}{\mu N^2} \left[ \sum_{i=1}^K x_i p_i \sum_{j=1}^i p_j - \sum_{i=1}^K p_i \sum_{j=1}^i x_j p_j \right] = \frac{1}{\mu N^2} \sum_{i=1}^K p_i \left[ x_i \sum_{j=1}^i p_j - \sum_{j=1}^i x_j p_j \right]$$

Observing that

$$C = \frac{1}{\mu N^2} \sum_{i=1}^K \sum_{j=1}^i (x_i - x_j) p_i p_j = \frac{1}{\mu N^2} \sum_{i=1}^K p_i \left( x_i \sum_{j=1}^i p_j - \sum_{j=1}^i x_j p_j \right)$$

it can be easily seen that expression (A1) and (A2) coincide.

□

## A2 On simplifying $C_W$

We will prove the simplification used in formula (20), that is

$$\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' = \mathbf{W}_{AX} \odot \mathbf{S} \quad (\text{A3})$$

### Proof

The elements  $w_{i,j}$  of matrix  $\mathbf{W}_X$  and the elements  $w_{l,m}^a = \mathbf{a}_l' \mathbf{W}_X \mathbf{a}_m$  of matrix  $\mathbf{W}_{AX}$  are equal to 1 if the associated pair of incomes,  $x_i$  and  $x_j$ , belong to a same group, they are zero otherwise. As all super-diagonal elements in matrix  $\mathbf{S}$  are plus 1 and sub-diagonal elements are  $-1$ , we have to prove that all super-diagonal elements of matrix  $\mathbf{A}_X \mathbf{S} \mathbf{A}_X'$ , that are selected by  $\mathbf{W}_{AX}$ , are 1, and all sub-diagonal elements of  $\mathbf{A}_X \mathbf{S} \mathbf{A}_X'$  selected by  $\mathbf{W}_{AX}$  are  $-1$ .

Observe that incomes belonging to a same group remain ranked in a non decreasing order within each group, also according to the  $\{(x_i, y_i, p_i)\}_{AX}$  ordering: therefore

- (i) if in the  $\{(x_i, y_i, p_i)\}_X$  ordering  $x_i$  occupies the  $i$ -th position and  $x_j$  the  $j$ -th one, with  $i < j$ , in the  $\{(x_i, y_i, p_i)\}_{AX}$  ordering,  $x_i$  will occupy the  $l$ -th position and  $x_j$  the  $m$ -th one with  $l < m$ ;
- (ii) symmetrically, in the  $\{(x_i, y_i, p_i)\}_X$  ordering, all pairs of incomes  $x_i > x_j$ , belonging to a same group, will respectively be in positions  $i$  and  $j$ ,  $i > j$ , and in the  $\{(x_i, y_i, p_i)\}_{AX}$  ordering, in positions  $l$  and  $m$ ,  $l > m$ , respectively.

This implies that the entry  $s_{i,j}$  of  $\mathbf{S}$  will be shifted to the entry  $s_{l,m}^a$  of  $\mathbf{A}_X \mathbf{S} \mathbf{A}_X'$ , with  $l < m$  if  $i < j$ , and  $l > m$  if  $i > j$ , so that in the super-diagonal part of  $\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X'$  all elements will be equal to 1, and in the sub-diagonal part, all elements will be equal to  $-1$ , which proves (A3). □