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Anomalous localized resonance using a folded geometry in three dimensions

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If a body of dielectric material is coated by a plasmonic structure of negative dielectric material with non-zero loss parameter, then cloaking by anomalous localized resonance (CALR) may occur as the loss parameter tends to zero. If the coated structure is circular (two dimensions) and the dielectric constant of the shell is a negative constant (with loss parameter), then CALR occurs, and if the coated structure is spherical (three dimensions), then CALR does not occur. The aim of this paper is to show that CALR takes place if the spherical coated structure has a specially designed anisotropic dielectric tensor. The anisotropic dielectric tensor is designed by unfolding a folded geometry.

1. Introduction

If a body of dielectric material (core) is coated by a plasmonic structure of negative dielectric constant with non-zero loss parameter (shell), then anomalous localized resonance may occur as the loss parameter tends to zero. To be precise, let Ω be a bounded domain in \mathbb{R}^d , $d = 2, 3$, and D be a domain whose closure is contained in Ω . In other words, D is the core and

$\Omega \setminus \bar{D}$ is the shell. For a given loss parameter $\delta > 0$, the permittivity distribution in \mathbb{R}^d is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \bar{\Omega}, \\ \epsilon_s + i\delta & \text{in } \Omega \setminus \bar{D}, \\ \epsilon_c & \text{in } D. \end{cases} \quad (1.1)$$

Here ϵ_c is a positive constant, but ϵ_s is a negative constant representing the negative dielectric constant of the shell. For a given function f compactly supported in $\mathbb{R}^d \setminus \bar{\Omega}$ satisfying

$$\int_{\mathbb{R}^d} f \, dx = 0 \quad (1.2)$$

(which is required by conservation of charge), we consider the following dielectric problem:

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in } \mathbb{R}^d, \quad (1.3)$$

with the decay condition $V_\delta(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Equation (1.3) is known as the quasi-static equation and the real part of $-\nabla V_\delta(x) e^{-i\omega t}$, where ω is the frequency and t is the time, represents an approximation for the physical electric field in the vicinity of Ω , when the wavelength of the electromagnetic radiation is large compared with Ω .

Let

$$E_\delta := \Im \int_{\mathbb{R}^d} \epsilon_\delta |\nabla V_\delta|^2 \, dx = \int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 \, dx \quad (1.4)$$

(\Im for the imaginary part), which, within a factor proportional to the frequency, approximately represents the time-averaged electromagnetic power produced by the source dissipated into heat. (Note that, for the quasi-static approximation to be valid, it is not necessary for the frequency to be small, only that Ω is sufficiently small compared with the wavelength.) Also for any region \mathcal{Y} , let

$$E_\delta^0(\mathcal{Y}) = \int_{\mathcal{Y}} |\nabla V_\delta|^2 \, dx, \quad (1.5)$$

where, when \mathcal{Y} is outside, Ω approximately represents, within a proportionality constant, the time-averaged electrical energy stored in the region \mathcal{Y} . Anomalous localized resonance is the phenomenon of field blow-up in a localized region. It may (and may not) occur depending upon the structure and the location of the source. Quantitatively, it is characterized by $E_\delta^0(\mathcal{Y}) \rightarrow \infty$, as $\delta \rightarrow 0$ for all regions \mathcal{Y} that overlap the region of anomalous resonance, and this defines that region. Cloaking by anomalous localized resonance (CALR) may occur when the support of the source, or part of it, lies in the anomalously resonant region. Physically the enormous fields in the anomalously resonant region interact with the source to create a sort of optical molasses, against which the source has to do a tremendous amount of work to maintain its amplitude, and this work tends to infinity as $\delta \rightarrow 0$. Quantitatively it is characterized by $E_\delta \rightarrow \infty$ as $\delta \rightarrow 0$.

This phenomenon of anomalous resonance was first discovered by Nicorovici *et al.* [1] and is related to invisibility cloaking [2]: the localized resonant fields created by a source can act back on the source and mask it (assuming the source is normalized to produce fixed power). It is also related to superlenses [3,4] because, as shown by Nicorovici *et al.* [1], the anomalous resonance can create apparent point sources. For these connections and further developments tied to this form of invisibility cloaking, we refer to [5–9] and references therein. Anomalous resonance is also presumably responsible for cloaking owing to complementary media [10–12], although we do not study this here.

The problem of CALR can be formulated as the problem of identifying the sources f such that, first,

$$E_\delta := \int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 \, dx \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \quad (1.6)$$

and, second, $V_\delta/\sqrt{E_\delta}$ goes to zero outside some radius a , as $\delta \rightarrow 0$,

$$|V_\delta(x)/\sqrt{E_\delta}| \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad \text{when } |x| > a. \quad (1.7)$$

Because the quantity E_δ is proportional to the electromagnetic power dissipated into heat by the time-harmonic electric field averaged over time, (1.6) implies an infinite amount of energy dissipated per unit time in the limit $\delta \rightarrow 0$ that is unphysical. If we rescale the source f by a factor of $1/\sqrt{E_\delta}$, then the source will produce the same power independently of δ and the new associated potential $V_\delta/\sqrt{E_\delta}$ will, by (1.7), approach zero outside the radius, a . Hence, CALR occurs. The normalized source is essentially invisible from the outside, yet the fields inside are very large. We also say that the weak CALR occurs if

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty, \quad (1.8)$$

which is weaker than (1.6), and the limit in (1.7) is replaced by \limsup .

In recent papers [5,6], the authors developed a spectral approach to analyse the CALR phenomenon. In particular, they showed that if D and Ω are concentric discs in \mathbb{R}^2 of radii r_i and r_e , respectively, and $\epsilon_s = -1$, then there is a critical radius r_* such that for any source f supported outside r_* CALR does not occur, and for sources f satisfying a mild (gap) condition CALR takes place. The critical radius r_* is given by $r_* = \sqrt{r_e^3/r_i}$, if $\epsilon_c = 1$, and by $r_* = r_e^2/r_i$, if $\epsilon_c \neq 1$. It is also proved that if $\epsilon_s \neq -1$, then CALR does not occur: E_δ is bounded regardless of δ and the location of the source. It is worth mentioning that these results (when $\epsilon_c = -\epsilon_s = 1$) were extended in Kohn *et al.* [13] to the case when the core D is not radial by a different method based on a variational approach. There the source f is assumed to be supported on circles.

The situation in three dimensions is completely different. If D and Ω are concentric balls in \mathbb{R}^3 , CALR does not occur whatever ϵ_s and ϵ_c are, as long as they are constants. We emphasize that this discrepancy comes from the convergence rate of the singular values of the Neumann–Poincaré-type operator associated with the structure. In two dimensions, they converge to 0 exponentially fast, but in three dimensions they converge only at the rate of $1/n$ [6]. The absence of CALR in such coated sphere geometries is also linked with the absence of perfect plasmon waves: see the appendix in Kohn *et al.* [13]. On the other hand, in a slab geometry CALR is known to occur in three dimensions with a single dipolar source [2]. (CALR is also known to occur for the full time-harmonic Maxwell equations with a single dipolar source outside the slab superlens [2,14,15].)

The purpose of this paper is to show that we are able to make CALR occur in three dimensions by using a shell with a specially designed anisotropic dielectric constant. In fact, let D and Ω be concentric balls in \mathbb{R}^3 of radii r_i and r_e , and choose r_0 so that $r_0 > r_e$. For a given loss parameter $\delta > 0$, define the dielectric constant ϵ_δ by

$$\epsilon_\delta(\mathbf{x}) = \begin{cases} \mathbf{I}, & |\mathbf{x}| > r_e, \\ (\epsilon_s + i\delta)a^{-1} \left(\mathbf{I} + \frac{b(b-2|\mathbf{x}|)}{|\mathbf{x}|^2} \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} \right), & r_i < |\mathbf{x}| < r_e, \\ \epsilon_c \frac{r_0}{r_i} \mathbf{I}, & |\mathbf{x}| < r_i, \end{cases} \quad (1.9)$$

where \mathbf{I} is the 3×3 identity matrix, ϵ_s and ϵ_c are constants, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and

$$a := \frac{r_e - r_i}{r_0 - r_e} > 0 \quad \text{and} \quad b := (1+a)r_e. \quad (1.10)$$

Note that ϵ_δ is anisotropic and variable in the shell. This dielectric constant is obtained by push-forwarding (unfolding) that of a folded geometry, as in figure 1. (See the next section for details.) It is worth mentioning that this idea of a folded geometry has been used in Milton *et al.* [16] to prove CALR in the analogous two-dimensional cylinder structure for a finite set of dipolar sources. Folded geometries were first introduced in Leonhardt & Philbin [17] to explain the properties of superlenses, and their unfolding map was generalized in Milton *et al.* [16] to allow for three different fields, rather than a single one, in the overlapping regions. Folded cylinder structures were studied as superlenses in Yan *et al.* [18] and folded geometries using bipolar coordinates were introduced in Chen & Chan [19] to obtain new complementary media cloaking structures. More general folded geometries were rigorously investigated in Nguyen [12].

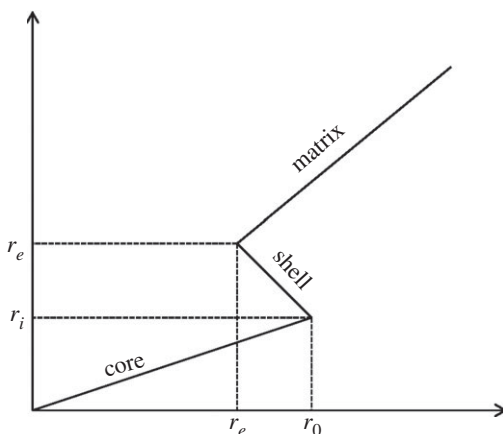


Figure 1. Unfolding map.

For a given source f supported outside $\overline{B_{r_e}}$, let V_δ be the solution to

$$\left. \begin{aligned} \nabla \cdot (\epsilon_\delta \nabla V_\delta) &= f \quad \text{in } \mathbb{R}^3, \\ V_\delta(\mathbf{x}) &\rightarrow 0 \quad \text{as } |\mathbf{x}| \rightarrow \infty, \end{aligned} \right\} \quad (1.11)$$

and define

$$E_\delta = \Im \int_{\mathbb{R}^3} \epsilon_\delta \nabla V_\delta \cdot \nabla \overline{V_\delta} \, d\mathbf{x}, \quad (1.12)$$

where $\overline{V_\delta}$ is the complex conjugate of V_δ . Let F be the Newtonian potential of the source f , i.e.

$$F(\mathbf{x}) := \int_{\mathbb{R}^3} G(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y}, \quad (1.13)$$

with $G(\mathbf{x} - \mathbf{y}) = -1/4\pi |\mathbf{x} - \mathbf{y}|$. Because f is supported in $\mathbb{R}^3 \setminus \overline{B_{r_e}}$, F is harmonic in $|\mathbf{x}| < R$ for some $R > r_e$, and can be expressed there as

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=-n}^n f_n^k |\mathbf{x}|^n Y_n^k(\hat{\mathbf{x}}), \quad (1.14)$$

where $Y_n^k(\hat{\mathbf{x}})$ is the (real) spherical harmonic of degree n and order k . The coefficients f_n^k can be calculated by

$$f_n^k = \frac{1}{4\pi r^{n+2}} \int_{|\mathbf{x}|=r} F(\mathbf{x}) Y_n^k(\hat{\mathbf{x}}) \, d\sigma(\mathbf{x}), \quad (1.15)$$

for any $r < R$. The following is the main result of this paper.

Theorem 1.1. *Let ϵ_δ be the permittivity profile in \mathbb{R}^3 given by (1.9).*

- (i) *If $\epsilon_c = -\epsilon_s = 1$, then weak CALR occurs and the critical radius is $r_* = \sqrt{r_e r_0}$, i.e. if the source function f is supported inside the sphere of radius r_* (and the series in (1.14) does not extend harmonically to $|\mathbf{x}| < r_*$), then the weak CALR occurs, i.e.*

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty, \quad (1.16)$$

and there exists a constant C such that

$$|V_\delta(\mathbf{x})| < C, \quad (1.17)$$

for all \mathbf{x} with $|\mathbf{x}| > r_0^2 r_e^{-1}$. If, in addition, the Fourier coefficients f_n^k of F satisfy the following gap condition:

[GC1]: There exists a sequence $\{n_j\}$ with $n_1 < n_2 < \dots$ such that

$$\lim_{j \rightarrow \infty} \rho^{n_{j+1}-n_j} \sum_{k=-n_j}^{n_j} n_j r_*^{2n_j} |f_{n_j}^k|^2 = \infty,$$

where $\rho := r_e/r_0$, then CALR occurs, i.e.

$$\lim_{\delta \rightarrow 0} E_\delta = \infty, \quad (1.18)$$

and $V_\delta/\sqrt{E_\delta}$ goes to zero outside the radius r_0^2/r_e , as implied by (1.17).

(ii) If $\epsilon_c \neq -\epsilon_s = 1$, then weak CALR occurs and the critical radius is $r_{**} = r_0$. If, in addition, the Fourier coefficients f_n^k of F satisfy

[GC2]: There exists a sequence $\{n_j\}$ with $n_1 < n_2 < \dots$ such that

$$\lim_{j \rightarrow \infty} \rho^{2(n_{j+1}-n_j)} \sum_{k=-n_j}^{n_j} n_j r_0^{2n_j} |f_{n_j}^k|^2 = \infty,$$

then CALR occurs.

(iii) If $-\epsilon_s \neq 1$, then CALR does not occur.

We remark that, even if the source f is located inside in $|\mathbf{x}| < r_*$, the corresponding series (1.14) may be harmonic in $|\mathbf{x}| < r_*$. For example, the Newtonian potential of the form $f = c_1 \chi_{r_1 < |\mathbf{x}| < r_2} - c_2 \chi_{r_3 < |\mathbf{x}| < r_4}$ with $r_e < r_j < r_*$, $1 \leq j \leq 4$, is quadratic in $|\mathbf{x}| < r_e$. We also emphasize that [GC1] and [GC2] are mild conditions on the Fourier coefficients of the Newtonian potential of the source function. For example, if the source function is a dipole in $B_{r_*} \setminus \bar{B}_e$, i.e. $f(\mathbf{x}) = \mathbf{a} \cdot \nabla \delta_{\mathbf{y}}(\mathbf{x})$ for a vector \mathbf{a} and $\mathbf{y} \in B_{r_*} \setminus \bar{B}_e$, where $\delta_{\mathbf{y}}$ is the Dirac delta function at \mathbf{y} , [GC1] and [GC2] hold, and CALR takes place. A proof of this fact is provided in appendix A. Similarly one can show that if f is a quadrupole, $f(\mathbf{x}) = A : \nabla \nabla \delta_{\mathbf{y}}(\mathbf{x}) = \sum_{i,j=1}^2 a_{ij} (\partial^2 / \partial x_i \partial x_j) \delta_{\mathbf{y}}(\mathbf{x})$ for a 3×3 matrix $A = (a_{ij})$ and $\mathbf{y} \in B_{r_*} \setminus \bar{B}_e$, then [GC1] and [GC2] hold.

2. Proof of theorem 1.1

Let r_i , r_e and r_0 be positive constants satisfying $r_i < r_e < r_0$, as before. In terms of spherical coordinates (r, θ, ϕ) , we define a mapping $\Phi = \{\Phi_c, \Phi_s, \Phi_m\}$, called the unfolding map, by

$$\left. \begin{aligned} \Phi_m(r, \theta, \phi) &= (r, \theta, \phi), & r &\geq r_e, \\ \Phi_s(r, \theta, \phi) &= (b - ar, \theta, \phi), & r_e &\leq r \leq r_0 \\ \Phi_c(r, \theta, \phi) &= \left(\frac{r_i}{r_0} r, \theta, \phi \right), & r &\leq r_0, \end{aligned} \right\} \quad (2.1)$$

and

where a and b are constants defined in (1.10). Then, the folding map can be written (with an abuse of notation) as

$$\Phi^{-1}(\mathbf{x}) = \begin{cases} \mathbf{x}, & |\mathbf{x}| > r_e, \\ -a^{-1}\mathbf{x} + a^{-1}b\hat{\mathbf{x}}, & r_i < |\mathbf{x}| < r_e, \\ \frac{r_0}{r_i}\mathbf{x}, & |\mathbf{x}| < r_i. \end{cases} \quad (2.2)$$

Let $\kappa(\mathbf{x})$ be a permittivity profile (in the folded geometry) given by

$$\kappa(\mathbf{x}) = \begin{cases} \kappa_m, & |\mathbf{x}| \geq r_e, \\ \kappa_s, & r_e \leq |\mathbf{x}| \leq r_0, \\ \kappa_c, & |\mathbf{x}| \leq r_0, \end{cases} \quad (2.3)$$

and let ϵ be the push-forward of κ by the unfolding map Φ , namely

$$\epsilon(\mathbf{x}) = \begin{cases} \kappa_m (\det \nabla \Phi_m(\mathbf{y}))^{-1} \nabla \Phi_m(\mathbf{y}) \nabla \Phi_m(\mathbf{y})^t, & |\mathbf{x}| > r_e, \\ \kappa_s (\det \nabla \Phi_s(\mathbf{y}))^{-1} \nabla \Phi_s(\mathbf{y}) \nabla \Phi_s(\mathbf{y})^t, & r_i < |\mathbf{x}| < r_e, \\ \kappa_c (\det \nabla \Phi_c(\mathbf{y}))^{-1} \nabla \Phi_c(\mathbf{y}) \nabla \Phi_c(\mathbf{y})^t, & |\mathbf{x}| < r_i, \end{cases} \quad (2.4)$$

where $\mathbf{x} = \Phi(\mathbf{y})$. By straight-forward computations one can see

$$\epsilon(\mathbf{x}) = \begin{cases} \kappa_m \mathbf{I}, & |\mathbf{x}| > r_e, \\ -\kappa_s a^{-1} \left(\mathbf{I} + \frac{b(b-2|\mathbf{x}|)}{|\mathbf{x}|^2} \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} \right), & r_i < |\mathbf{x}| < r_e, \\ \kappa_c \frac{r_0}{r_i} \mathbf{I}, & |\mathbf{x}| < r_i, \end{cases} \quad (2.5)$$

and $\epsilon = \epsilon_\delta$ in (1.9) if we set

$$\kappa_m = 1, \quad \kappa_s = -(\epsilon_s + i\delta) \quad \text{and} \quad \kappa_c = \epsilon_c. \quad (2.6)$$

For a source f supported outside $\overline{B_{r_e}}$ and the solution V_δ to (1.11), we define

$$\left. \begin{aligned} u_m(\mathbf{x}) &= V_\delta \circ \Phi_m(\mathbf{x}), & \text{if } |\mathbf{x}| > r_e, \\ u_s(\mathbf{x}) &= V_\delta \circ \Phi_s(\mathbf{x}), & \text{if } r_e < |\mathbf{x}| < r_0 \\ u_c(\mathbf{x}) &= V_\delta \circ \Phi_c(\mathbf{x}), & \text{if } |\mathbf{x}| < r_0. \end{aligned} \right\} \quad (2.7)$$

and

Then u_c , u_s and u_m satisfy

$$\left. \begin{aligned} \Delta u_c &= 0 & \text{in } B_{r_0}, \\ \Delta u_s &= 0 & \text{in } B_{r_0} \setminus \overline{B_{r_e}}, \\ \Delta u_m &= f & \text{in } \mathbb{R}^3 \setminus \overline{B_{r_e}}, \\ u_c &= u_s, \quad \kappa_c \frac{\partial u_c}{\partial r} = \kappa_s \frac{\partial u_s}{\partial r} & \text{on } \partial B_{r_0}, \\ u_s &= u_m, \quad \kappa_s \frac{\partial u_s}{\partial r} = \kappa_m \frac{\partial u_m}{\partial r} & \text{on } \partial B_{r_e} \\ u_m(\mathbf{x}) &\rightarrow 0 & \text{as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \right\} \quad (2.8)$$

and

We emphasize that the domains of u_c , u_s and u_m are overlapping on $r_e \leq |\mathbf{x}| \leq r_0$, so that the solutions combined may be considered as the solution of the transmission problem with dielectric constants κ_c , κ_s and κ_m in the folded geometry, as shown in figure 1. We obtain V_δ by unfolding the solution (u_m, u_s, u_c) into one whose domain is not overlapping, following the idea in Milton *et al.* [16].

By the change of variables $\mathbf{x} = \Phi_s(\mathbf{y})$ and (2.4), we have

$$E_\delta = \Im \int_{\mathbb{R}^3} \epsilon(\mathbf{x}) \nabla V_\delta(\mathbf{x}) \cdot \nabla \overline{V_\delta(\mathbf{x})} = \delta \int_{r_e < |\mathbf{y}| < r_0} |\nabla u_s(\mathbf{y})|^2. \quad (2.9)$$

Suppose that the source f is supported in $|\mathbf{x}| > R$ for some $R > r_e$. Then, the solution u to (2.8) can be expressed in $|\mathbf{x}| < R$ as follows:

$$\left. \begin{aligned} u_c(\mathbf{x}) &= \sum_{n=0}^{\infty} \sum_{k=-n}^n a_n^k |\mathbf{x}|^n \gamma_n^k(\hat{\mathbf{x}}), \quad \text{if } |\mathbf{x}| < r_0, \\ u_s(\mathbf{x}) &= \sum_{n=0}^{\infty} \sum_{k=-n}^n (b_n^k |\mathbf{x}|^n + c_n^k |\mathbf{x}|^{-n-1}) \gamma_n^k(\hat{\mathbf{x}}), \quad \text{if } r_e < |\mathbf{x}| < r_0 \\ u_m(\mathbf{x}) &= \sum_{n=0}^{\infty} \sum_{k=-n}^n (e_n^k |\mathbf{x}|^n + d_n^k |\mathbf{x}|^{-n-1}) \gamma_n^k(\hat{\mathbf{x}}), \quad \text{if } r_e < |\mathbf{x}| < R, \end{aligned} \right\} \quad (2.10)$$

and

where the coefficients satisfy the following relations resulting from the interface conditions:

$$\begin{aligned} a_n^k r_0^n &= b_n^k r_0^n + c_n^k r_0^{-n-1}, \\ e_n^k r_e^n + d_n^k r_e^{-n-1} &= b_n^k r_e^n + c_n^k r_e^{-n-1}, \\ \kappa_c a_n^k n r_0^n &= \kappa_s (b_n^k n r_0^n - c_n^k (n+1) r_0^{-n-1}) \end{aligned}$$

and

$$\kappa_s (b_n^k n r_e^n - c_n^k (n+1) r_e^{-n-1}) = \kappa_m (e_n^k n r_e^n - d_n^k (n+1) r_e^{-n-1}).$$

By solving this system of linear equations one can see that

$$a_n^k = a_n e_n^k, \quad b_n^k = b_n e_n^k, \quad c_n^k = c_n e_n^k \quad \text{and} \quad d_n^k = d_n e_n^k,$$

where

$$a_n = \frac{-\rho^{2n+1} (2n+1)^2 \kappa_m \kappa_s}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1} ((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}, \quad (2.11)$$

$$b_n = \frac{-\rho^{2n+1} \kappa_m (2n+1)((n+1)\kappa_s + n\kappa_c)}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1} ((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}, \quad (2.12)$$

$$c_n = \frac{-r_e^{2n+1} \kappa_m n (2n+1)(\kappa_s - \kappa_c)}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1} ((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)} \quad (2.13)$$

$$\text{and} \quad d_n = -\frac{n r_e^{2n+1} [\rho^{2n+1} (\kappa_m - \kappa_s)((n+1)\kappa_s + n\kappa_c) + (\kappa_s - \kappa_c)(n\kappa_m + (n+1)\kappa_s)]}{(n^2+n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1} ((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)}. \quad (2.14)$$

Here ρ is defined to be r_e/r_0 .

Let F be the Newtonian potential of f , as before. Because $u - F$ is harmonic in $|\mathbf{x}| > r_e$ and tends to 0 as $|\mathbf{x}| \rightarrow \infty$, we have

$$e_n^k = f_n^k. \quad (2.15)$$

So u_m (the solution in the matrix) is given by

$$u_m(\mathbf{x}) = F(\mathbf{x}) + \sum_{n=0}^{\infty} \sum_{k=-n}^n f_n^k d_n |\mathbf{x}|^{-n-1} \gamma_n^k(\hat{\mathbf{x}}). \quad (2.16)$$

Because $|d_n| \leq C r_0^{2n}$, we have

$$|u_m(\mathbf{x}) - F(\mathbf{x})| \leq C \sum_{n=0}^{\infty} \sum_{k=-n}^n |f_n^k| r_0^{2n} |\mathbf{x}|^{-n-1} < \infty, \quad (2.17)$$

if $|\mathbf{x}| = r > r_0^2 r_e^{-1}$. This proves (1.17).

The solution in the shell u_s is given by

$$u_s(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{k=-n}^n f_n^k (b_n |\mathbf{y}|^n + c_n |\mathbf{y}|^{-n-1}) Y_n^k(\hat{\mathbf{y}}). \quad (2.18)$$

Using Green's identity and the orthogonality of Y_n^k , we obtain that

$$\begin{aligned} \int_{r_e < |\mathbf{y}| < r_0} |\nabla u_s(\mathbf{y})|^2 &= \int_{|\mathbf{y}|=r_0} u_s \frac{\partial u_s}{\partial r} - \int_{|\mathbf{y}|=r_e} u_s \frac{\partial u_s}{\partial r} \\ &= \sum_{n=0}^{\infty} \sum_{k=-n}^n |f_n^k|^2 [(b_n r_0^n + c_n r_0^{-n-1})(n \bar{b}_n r_0^n - (n+1) \bar{c}_n r_0^{-n-1}) r_0] \\ &\quad - \sum_{n=0}^{\infty} \sum_{k=-n}^n |f_n^k|^2 [(b_n r_e^n + c_n r_e^{-n-1})(n \bar{b}_n r_e^n - (n+1) \bar{c}_n r_e^{-n-1}) r_e] \\ &= \sum_{n=0}^{\infty} \sum_{k=-n}^n |f_n^k|^2 [n |b_n|^2 (r_0^{2n+1} - r_e^{2n+1}) - (n+1) |c_n|^2 (r_0^{-2n-1} - r_e^{-2n-1})] \\ &\approx \sum_{n=0}^{\infty} \sum_{k=-n}^n n |f_n^k|^2 (|b_n|^2 r_0^{2n+1} + |c_n|^2 r_e^{-2n-1}). \end{aligned}$$

The estimate (2.9) yields

$$E_\delta \approx \delta \sum_{n=0}^{\infty} \sum_{k=-n}^n n |f_n^k|^2 (|b_n|^2 r_0^{2n+1} + |c_n|^2 r_e^{-2n-1}). \quad (2.19)$$

Here and afterwards, $a \approx b$ means that there exist constants C_1 and C_2 independent of n and δ such that

$$C_1 a \leq b \leq C_2 a.$$

(i) Suppose that $\epsilon_c = -\epsilon_s = 1$. With the notation in (2.6), we have

$$|(n^2 + n)(\kappa_s - \kappa_c)(\kappa_s - \kappa_m) - \rho^{2n+1}((n+1)\kappa_s + n\kappa_c)((n+1)\kappa_m + n\kappa_s)| \approx n^2(\delta^2 + \rho^{2n+1}),$$

and, hence,

$$|b_n| \approx \frac{\rho^{2n}}{\delta^2 + \rho^{2n}} \quad \text{and} \quad |c_n| \approx \frac{\delta r_e^{2n}}{\delta^2 + \rho^{2n}}. \quad (2.20)$$

It then follows from (2.19) that

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}}. \quad (2.21)$$

Let

$$N_\delta = \frac{\log \delta}{\log \rho}. \quad (2.22)$$

If $n \leq N_\delta$, then we know that $\delta \leq \rho^n$ and $r_e^{2n}/(\delta^2 + \rho^{2n}) \geq \frac{1}{2} r_0^{2n}$. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} &\geq \sum_{n \leq N_\delta} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} \\ &\geq \frac{\delta m r_0^{2m}}{2} \sum_{k=-m}^m |f_m^k|^2 \geq \frac{\delta m}{2(2m+1)} r_0^{2m} \left(\sum_{k=-m}^m |f_m^k| \right)^2 \end{aligned}$$

for any $m \leq N_\delta$. By taking δ to be ρ^n , $n = 1, 2, \dots$, we see that if the following holds

$$\limsup_{n \rightarrow \infty} (r_e r_0)^{n/2} \sum_{k=-n}^n |f_n^k| = \infty, \quad (2.23)$$

then there is a sequence $\{n_k\}$ such that

$$\lim_{k \rightarrow \infty} E_{\rho^{|n_k|}} = \infty, \quad (2.24)$$

i.e. weak CALR occurs.

Suppose that the source function f is supported inside the critical radius $r_* = \sqrt{r_e r_0}$ (and outside r_e) and its Newtonian potential F cannot be extended harmonically in $|x| < r_*$. Then we have

$$\limsup_{n \rightarrow \infty} \left(\sum_{k=-n}^n |f_n^k| \right)^{1/n} > \frac{1}{\sqrt{r_e r_0}}, \quad (2.25)$$

because, otherwise, F given by (1.14) converges in $|x| < r_*$ because $|Y_n^k| \leq \sqrt{2n+1}$. Consequently, (2.23) holds. This proves that if the source function f is supported inside the sphere of critical radius r_* , then weak CALR occurs.

If the source function f is supported outside the sphere of critical radius $r_* = \sqrt{r_e r_0}$, then its Newtonian potential F can be extended harmonically in $|x| < r_* + 2\eta$ for $\eta > 0$ and

$$\sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} \leq \sum_{n=0}^{\infty} \sum_{k=-n}^n n r_*^{2n} |f_n^k|^2 \leq C \|F\|_{L^2(\partial B_{r_*+\eta})}^2 < \infty. \quad (2.26)$$

So E_δ is bounded regardless of δ and CALR does not occur.

Suppose that f is supported inside r_* and [GC1] holds. Let $\{n_j\}$ be the sequence satisfying

$$\lim_{j \rightarrow \infty} \rho^{n_{j+1}-n_j} \sum_{k=-n_j}^{n_j} n_j r_*^{2n_j} |f_{n_j}^k|^2 = \infty.$$

If $\delta = \rho^\alpha$ for some α , let $j(\alpha)$ be the number in the sequence such that

$$n_{j(\alpha)} \leq \alpha < n_{j(\alpha)+1}.$$

Then, we have

$$\begin{aligned} E_\delta &\approx \sum_{n \leq N_\delta} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{2n}} \geq \rho^\alpha \sum_{n \leq N_\delta} \sum_{k=-n}^n \frac{n r_e^{2n} |f_n^k|^2}{\rho^{2n}} \\ &\geq \rho^{|n_{j(\alpha)+1}| - |n_{j(\alpha)}|} \sum_{k=-n_{j(\alpha)}}^{n_{j(\alpha)}} n_{j(\alpha)} r_*^{2n_{j(\alpha)}} |f_{n_{j(\alpha)}}^k|^2 \rightarrow \infty \end{aligned}$$

as $\alpha \rightarrow \infty$. So CALR takes place.

To prove (ii) assume that $\epsilon_c \neq -\epsilon_s = 1$. In this case, we have

$$|b_n| \approx \frac{\rho^{2n}}{\delta + \rho^{2n}} \quad \text{and} \quad |c_n| \approx \frac{r_e^{2n}}{\delta + \rho^{2n}},$$

and

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{\delta n r_e^{2n} |f_n^k|^2}{\delta^2 + \rho^{4n}}.$$

The rest of proof of (ii) is the same as that for (i).

Suppose now that $-\epsilon_s \neq 1$. If $\epsilon_c = 1$, then we have

$$|b_n| \approx \frac{\rho^{2n}}{\delta + \rho^{2n}} \quad \text{and} \quad |c_n| \approx \frac{\delta r_e^{2n}}{\delta + \rho^{2n}},$$

and

$$E_\delta \approx \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{\delta(\delta^2 + \rho^{2n}) n r_e^{2n} |f_n^k|^2}{(\delta + \rho^{2n})^2} \leq \sum_{n=0}^{\infty} \sum_{k=-n}^n n r_e^{2n} |f_n^k|^2 \leq \left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)}^2.$$

Because the source function f is supported outside the radius r_e , we have

$$\left\| \frac{\partial F}{\partial r} \right\|_{L^2(\partial B_e)} \leq C \|f\|_{L^2(\mathbb{R}^3)},$$

and E_δ is bounded independently of δ . The case when $\epsilon_c \neq 1$ can be treated similarly.

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Appendix A. Gap property of dipoles

In this appendix, we show that the Newtonian potentials of dipole source functions satisfy the gap conditions [GC1] and [GC2]. We only prove [GC1], because the other can be proved similarly.

Let f be a dipole in $B_{r_*} \setminus \bar{B}_e$, i.e. $f(\mathbf{x}) = \mathbf{a} \cdot \nabla \delta_{\mathbf{y}}(\mathbf{x})$ for a vector \mathbf{a} and $\mathbf{y} \in B_{r_*} \setminus \bar{B}_e$. Then its Newtonian potential is given by $F(\mathbf{x}) = -\mathbf{a} \cdot \nabla_{\mathbf{y}} G(\mathbf{x} - \mathbf{y})$. It is well known (see [20]) that the fundamental solution $G(\mathbf{x} - \mathbf{y})$ admits the following expansion if $|\mathbf{y}| > |\mathbf{x}|$:

$$G(\mathbf{x} - \mathbf{y}) = - \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} Y_n^k(\hat{\mathbf{x}}) Y_n^k(\hat{\mathbf{y}}) \frac{|\mathbf{x}|^n}{|\mathbf{y}|^{n+1}}.$$

So we have

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} |\mathbf{x}|^n Y_n^k(\hat{\mathbf{x}}) \mathbf{a} \cdot \nabla \left(\frac{1}{|\mathbf{y}|^{n+1}} Y_n^k(\hat{\mathbf{y}}) \right),$$

and, hence,

$$f_n^k = \frac{1}{2n+1} \mathbf{a} \cdot \nabla \left(\frac{1}{|\mathbf{y}|^{n+1}} Y_n^k(\hat{\mathbf{y}}) \right). \quad (\text{A } 1)$$

We show that

$$\sum_{k=-n}^n n r_*^{2n} |f_n^k|^2 \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (\text{A } 2)$$

and hence [GC1] holds. The following lemma is needed.

Lemma A.1. *For any \mathbf{a} and $\hat{\mathbf{y}}$ on S^2 and for any positive integer n there is a homogeneous harmonic polynomial h of degree n such that*

$$\mathbf{a} \cdot \nabla h(\hat{\mathbf{y}}) = 1 \quad (\text{A } 3)$$

and

$$\max_{|\hat{\mathbf{x}}|=1} |h(\hat{\mathbf{x}})| \leq \frac{\sqrt{3}}{n}. \quad (\text{A } 4)$$

Proof. After rotation, if necessary, we may assume that $\hat{\mathbf{y}} = (1, 0, 0)$. We introduce three homogeneous harmonic polynomials of degree n ,

$$h_1(\mathbf{x}) := \frac{1}{2n} [(x_1 + ix_2)^n + (x_1 - ix_2)^n],$$

$$h_2(\mathbf{x}) := \frac{1}{2ni} [(x_1 + ix_2)^n - (x_1 - ix_2)^n]$$

and

$$h_3(\mathbf{x}) := \frac{1}{2ni} [(x_1 + ix_3)^n - (x_1 - ix_3)^n].$$

Then one can easily see that

$$\nabla h_1(\hat{\mathbf{y}}) = (1, 0, 0), \quad \nabla h_2(\hat{\mathbf{y}}) = (0, 1, 0) \quad \text{and} \quad \nabla h_3(\hat{\mathbf{y}}) = (0, 0, 1).$$

So if we define

$$h = a_1 h_1 + a_2 h_2 + a_3 h_3,$$

then (A 3) holds.

Since

$$\max_{|\hat{\mathbf{x}}|=1} |h_j(\hat{\mathbf{x}})| \leq \frac{1}{n} \quad \text{for } j = 1, 2, 3,$$

we obtain (A 4) using the Cauchy–Schwartz inequality. This completes the proof. \blacksquare

Let \mathbf{a} and $\hat{\mathbf{y}}$ be two unit vectors, and let h be a homogeneous harmonic polynomial of degree n satisfying (A 3) and (A 4). Then h can be expressed as

$$h(\mathbf{x}) = \sum_{k=-n}^n \alpha_k |\mathbf{x}|^n Y_n^k(\hat{\mathbf{x}}),$$

where

$$\alpha_k = \frac{1}{4\pi} \int_{S^2} h(\hat{\mathbf{x}}) Y_n^k(\hat{\mathbf{x}}) \, dS. \quad (\text{A } 5)$$

Because of (A 3), we have

$$1 = \mathbf{a} \cdot \nabla h(\hat{\mathbf{y}}) \leq \sum_{k=-n}^n |\alpha_k| |\mathbf{a} \cdot \nabla (|\mathbf{x}|^n Y_n^k(\hat{\mathbf{x}}))|.$$

So there is k , say k_n , between $-n$ and n such that

$$|\alpha_{k_n}| |\mathbf{a} \cdot \nabla (|\mathbf{x}|^n Y_n^{k_n}(\hat{\mathbf{x}}))| \geq \frac{1}{2n+1}. \quad (\text{A } 6)$$

On the other hand, from (A 4) and (A 5), it follows by using Jensen's inequality that

$$|\alpha_{k_n}|^2 \leq \frac{1}{4\pi} \int_{S^2} |h(\hat{\mathbf{x}})|^2 |Y_n^{k_n}(\hat{\mathbf{x}})|^2 \, dS \leq \frac{3}{n^2}.$$

Thus, we have

$$|\mathbf{a} \cdot \nabla (|\mathbf{x}|^n Y_n^{k_n}(\hat{\mathbf{x}}))| \geq \frac{n}{\sqrt{3}(2n+1)} \geq C \quad (\text{A } 7)$$

for some C independent of n .

Now one can see from (A 1) that

$$|f_n^{k_n}| \geq \frac{C}{n|\mathbf{y}|^{n+1}} \quad (\text{A } 8)$$

for some C independent of n . Because $|\mathbf{y}| < r_*$, we obtain that

$$\sum_{k=-n}^n n r_*^{2n} |f_n^k|^2 \geq n r_*^{2n} |f_n^{k_n}|^2 \geq \frac{C}{n} \left(\frac{r_*}{|\mathbf{y}|} \right)^{2n} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

as desired. It is worth mentioning that the constants C in the above may be different at each occurrence, but are independent of n .

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