Stable Determination of Sound-soft Polyhedral Scatterers by a Single Measurement

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ABSTRACT. We prove optimal stability estimates for the determination of a finite number of sound-soft polyhedral scatterers in $\mathbb{R}^3$ by a single far-field measurement. The admissible multiple polyhedral scatterers satisfy minimal a priori assumptions of Lipschitz type and may include at the same time obstacles, screens and even more complicated scatterers. We characterize any multiple polyhedral scatterer by a size parameter h which is related to the minimal size of the cells of its boundary. In a first step we show that, provided the error ε on the far-field measurement is small enough with respect to h, then the corresponding error, in the Hausdorff distance, on the multiple polyhedral scatterer can be controlled by an explicit function of ε which approaches zero, as $\varepsilon \to 0^+$, in an essentially optimal, although logarithmic, way. Then, we show how to improve this stability estimate, provided we restrict our attention to multiple polyhedral obstacles and ε is even smaller with respect to h. In this case we obtain an explicit estimate essentially of Hölder type.

1. INTRODUCTION

Let Σ be a compact subset of $\mathbb{R}^3$ and let us assume that we send an incident time harmonic acoustic plane wave, characterized by its *incident field* $u_i(x) = e^{ik\omega \cdot x}$, $x \in \mathbb{R}^3$. Here $k > 0$ is the *wave number* and $\omega \in \mathbb{S}^2$ is the *direction of propagation*. The incident wave is scattered by the presence of the *scatterer* Σ and the perturbation is denoted by u_s, the *scattered field*. Then, the *total field* u solves

the following exterior boundary value problem

(1.1)
$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } G, \\[2mm] u(x) = u_s(x) + \mathrm{e}^{\mathrm{i}k\omega \cdot x}, & x \in G, \\[2mm] u = 0 & \text{on } \partial G, \\[2mm] \lim_{r \to \infty} r \left(\dfrac{\partial u_s}{\partial r} - \mathrm{i}k u_s \right) = 0, & r = |x|. \end{cases}$$

Here we have assumed that Σ is a so-called *sound-soft* scatterer (to which corresponds the homogeneous Dirichlet condition on ∂G for the total field u) and that $G = \mathbb{R}^3 \setminus \Sigma$. The scattered field $u_s(x)$ satisfies the so-called *Sommerfeld radiation condition*, which is the condition at infinity described in the fourth line of (1.1). We recall that the limit in the Sommerfeld radiation condition has to hold, as $|x|$ goes to ∞, uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. By the Sommerfeld radiation condition, the asymptotic behaviour at infinity of the scattered field u_s is governed by the formula

(1.2)
$$u_s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|} \right) \right\},$$

as $|x|$ goes to ∞, uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. The function u_∞, which is defined on $\mathbb{S}^2$, is called the *far-field pattern* of u_s, see for instance [9].

The inverse acoustic scattering problem consists in the determination of the shape and location of Σ by measuring the far-field pattern u_∞ for one or several different incident waves, in other words for one or several directions of propagation. In fact, throughout the paper, we assume that the wave number $k > 0$ is kept fixed.

This inverse acoustic scattering problem has a long history. The first proof of unique determination for sound-soft *obstacles* (that is scatterers coinciding with the closures of their interiors) is due to Schiffer and it required the use of infinitely many measurements. Then, Colton and Sleeman, in [10], noticed that a simple a priori bound on the location of the unknown obstacles allows us to determine the obstacles with a finite number, which can be explicitly computed, of far-field measurements at a fixed wave number. They also remarked that if the wave number k is small enough, then one measurement would be sufficient. We refer to the book by Colton and Kress, [9], for a much more detailed description of the problem and the related literature. Let us mention that corresponding stability estimates have been developed in [14, 15] and that these estimates, of logarithmic type, are essentially optimal by the analysis developed in [11]. We also recall that, in [22], the result by Colton and Sleeman has been extended to *screens* (that is scatterers

whose interiors are empty) which are simply connected and satisfy minimal a priori regularity assumptions. Although in a less explicit and simple way, also in this case the number of measurements required might be computed.

It is a long standing conjecture that, whatever the wave number k is, one measurement should be enough to determine in a unique way any unknown scatterer, or at least any unknown obstacle. However, such a result has been proved only for scatterers of a special type. Following previous results by Liu and Nachman, [16], and Cheng and Yamamoto, [7, 8], it has been proved in [3] that any *polyhedral scatterer* is determined in a unique way by a single far-field measurement. By a polyhedral scatterer we mean a scatterer whose boundary is the union of a finite number of cells, each *cell* being the closure of a domain contained in a hypersurface. We remark that, by this definition, a polyhedral scatterer may be composed at the same time by obstacles (that is *polyhedra*, in $\mathbb{R}^3$), screens and also quite complicated combinations of the two. This result has been extended to other kinds of boundary conditions, for instance to sound-hard scatterers, in a series of papers [12, 17–19]. Therefore the uniqueness problem for the determination of polyhedral scatterers of different types and nature is by now almost completely solved.

Here we investigate the related problem of stability for the determination of sound-soft polyhedral scatterers by a single far-field measurement. In other words, we consider two different admissible polyhedral scatterers Σ and Σ' and their corresponding far-field patterns (for the same incident field) u_∞ and u'_∞, respectively. If we call $\varepsilon = \|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^2)}$ the error between the measurements, then the stability issue consists in estimating, in a quantitative way, the error between Σ and Σ', which is measured in the Hausdorff distance, with respect to ε.

For any polyhedral scatterer Σ, we introduce a parameter $h > 0$ which is a lower bound on the size of each cell forming $\partial\Sigma$. We assume that the same parameter h is valid for both Σ and Σ'.

The results we obtain are of two kinds. First of all, we consider polyhedral scatterers satisfying minimal regularity assumptions of Lipschitz type, see Section 2 for the precise definitions. Then, in Theorem 4.1, we prove that ε controls the minimum between h and the Hausdorff distance between Σ and Σ' through an explicit logarithmic function. By using the results developed in [11], we show that such an estimate is essentially optimal, Proposition 4.4, and we conclude that, provided the error ε is small enough, in an explicit way, with respect to h, we have a logarithmic stability estimate for our inverse problem.

Actually, if ε is even smaller, again in an explicit way with respect to h, and we limit ourselves to polyhedral obstacles, then the stability estimate may be improved up to an essentially Hölder type estimate, Theorem 4.2. Using again the results described in Proposition 4.4, we conclude that the estimate is essentially optimal. Therefore, asymptotically, which in practice means paying the price of ensuring a very small error on the measurement, the exponential ill-posedness of the problem may be kept under control. This is in accord with several results in which the ill-posedness of an inverse boundary value problem is tamed if the unknown features to be recovered can be described in a discrete way, see for example [2, 4, 5]. With

respect to the unknown discrete boundaries considered in [2, 5], the main novelty here is the fact that we deal with a three-dimensional, instead of two-dimensional, problem and that the number of pieces forming the unknown boundary, namely the number of cells, is not fixed and is not a priori known.

Concerning the proofs, Theorem 4.1 is obtained by repeating the procedure used to prove uniqueness in [3] and by replacing each of its steps by a corresponding quantitative version. About Theorem 4.2, the key step is the following. We make use of the previous general stability estimate to ensure that the two polyhedral obstacles Σ and Σ' are close enough in the Hausdorff distance and we deduce, in Proposition 6.1 and Proposition 6.2, some special geometric conditions which in this case interplay between the two boundaries $\partial\Sigma$ and $\partial\Sigma'$. Then, the proof is concluded by reasonings analogous to the one developed in the proof of Theorem 4.1.

The plan of the paper is as follows. In Section 2, we describe and comment the a priori hypothesis on the scatterers. In Section 3, we formulate the direct acoustic scattering problem (1.1) and recall some properties of its solutions. Then, in Section 4 we state the main stability results, namely a first general stability estimate, Theorem 4.1, for the determination of polyhedral scatterers and a refined stability estimate for the determination of polyhedral obstacles, Theorem 4.2. The section is concluded with a discussion on the optimality of these results, Proposition 4.4. In Section 5 we develop the stability analysis for polyhedral scatterers and prove Theorem 4.1. Finally, in Section 6, we discuss the relationships between two polyhedra which are close in the Hausdorff distance and we prove Theorem 4.2.

2. CLASSES OF ADMISSIBLE SCATTERERS

In order to state our stability results we need first to introduce suitable classes of admissible scatterers. We begin by fixing some notation. Throughout the paper we limit ourselves to the three-dimensional case. For any $x \in \mathbb{R}^3$, and any $r > 0$, with $B_r(x)$ we denote the open ball of center x and radius r. For any $r > 0$, B_r denotes $B_r(0)$. For any subset $A \subset \mathbb{R}^3$, we set $B_r(A) = \bigcup_{x \in A} B_r(x)$. Furthermore, with diam$(A)$ we denote the diameter of A.

Given a point $x \in \mathbb{R}^3$, a direction $\omega_0 \in \mathbb{S}^2$ and two positive constants r, θ, $0 < \theta < \pi/2$, we call $C(x, \omega_0, r, \theta)$ the open cone so defined

$$C(x, \omega_0, r, \theta) = \{y \in \mathbb{R}^3 \mid 0 < |y - x| < r, \ (y - x) \cdot \omega_0 > \cos(\theta)|y - x|\}.$$

Here, x is the *vertex* of the cone, r is its *radius*, the line $\{x + t\omega_0, \ t \geq 0\}$ is its *bisecting line* and θ is its *amplitude angle* or *angle*, for simplicity.

We say that a function $\varphi : A \to B$, A and B being metric spaces, is *bi-Lipschitz* if it is invertible and φ and $\varphi^{-1} : \varphi(A) \to A$ are both Lipschitz functions. If both the Lipschitz constants of φ and φ^{-1} are bounded by $L > 0$, then we say that φ is *bi-Lipschitz* with constant L.

We recall that a *continuum* is a connected set not reduced to a single point. We shall say that σ is a *scatterer* if σ is a compact continuum contained in $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus \sigma$ is connected. If σ is the closure of its interior part, for example it is the closure of a domain, then we call it an *obstacle*, whereas if σ has empty interior, then it is called a *screen*. We shall say that Σ is a *multiple scatterer (obstacle* or *screen*, respectively) if it is the finite union of pairwise disjoint scatterers (obstacles or screens, respectively). We shall denote by G the exterior of a multiple scatterer Σ

$$(2.1) \qquad\qquad G = \mathbb{R}^3 \setminus \Sigma$$

and we observe that it is connected as well. If Σ is a multiple scatterer, for any $x \in \Sigma$, we denote with $\sigma(x)$ the connected component, that is the scatterer, of Σ containing x. We call $r(\Sigma) = \min\{\operatorname{diam}(\sigma(x)) \mid x \in \Sigma\}$.

Let T be the closed equilateral triangle which is contained in the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$ with vertices $V_1 = (0, 1, 0)$, $V_2 = (-\sqrt{3}/2, -\frac{1}{2}, 0)$ and $V_3 = (\sqrt{3}/2, -\frac{1}{2}, 0)$ and $T' \subset \mathbb{R}^2$ be the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, 0) \in T\}$. Fixed a positive constant L, we call an *L-generalized triangle* a set Γ such that, up to a rigid transformation, $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \varphi(T'), \ x_3 = \varphi_1(x_1, x_2)\}$, where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a bi-Lipschitz function with constant L such that $\varphi(0) = 0$ and $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz map with Lipschitz constant bounded by L and such that $\varphi_1(0) = 0$.

The image through φ of any vertex or side of T' will be called a generalized vertex or generalized side of $\varphi(T')$, respectively. The image on the graph of φ_1 of one of the generalized vertices of $\varphi(T')$ will be called a *generalized vertex* of Γ, whereas the image of one of the generalized sides of $\varphi(T')$ will be called a *generalized side* of Γ.

We remark that there exists a constant $L_1 > 0$, depending on L only, such that we can find $\varphi_2 : \mathbb{R}^3 \to \mathbb{R}^3$, a bi-Lipschitz function with constant L_1, such that $\Gamma = \varphi_2(T)$.

Let us fix, throughout the paper, positive constants R, L, δ, c, $0 < c < 1$, θ, $0 < \theta < \pi/2$, α_0, $0 < \alpha_0 < \pi/3$, and k. These constants will be referred to as the *a priori data*.

Definition 2.1. We say that a multiple scatterer Σ belongs to the class $\mathcal{A}$ with constants R, L, δ, c, $0 < c < 1$, and θ, $0 < \theta < \pi/2$, if Σ satisfies the following assumptions

  (i) $\Sigma \subset \bar{B}_R(0)$;

 (ii) $\partial\Sigma = \bigcup_{i=1}^n \Gamma_i$, where n depends on Σ and each Γ_i is an L-generalized triangle;

(iii) for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have that either $\Gamma_i \cap \Gamma_j$ is not empty or $\operatorname{dist}(\Gamma_i, \Gamma_j) \geq \delta$;

(iv) for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, if $\Gamma_i \cap \Gamma_j$ is not empty, then $\Gamma_i \cap \Gamma_j$ is either a common generalized side γ or a common generalized vertex V.

Furthermore, in such a case, for any $x \in \Gamma_i$ we have $\operatorname{dist}(x, \Gamma_j) \geq c \operatorname{dist}(x, y)$ or $\operatorname{dist}(x, \Gamma_j) \geq c|x - V|$, respectively;

(v) for any Γ_i and any x belonging to one of its generalized sides, there exists a direction $\omega_0 \in \mathbb{S}^2$ such that, as $y \in \Gamma_i \cap B_\delta(x)$, the open cones $C(y, \omega_0, \delta, \theta)$ are contained in G and their opposite cones $C(y, -\omega_0, \delta, \theta)$ are all contained either in Σ (for example if $\sigma(x)$ is an obstacle) or in G (for example if $\sigma(x)$ is a screen);

(vi) for any r, $0 < r < \delta/4$, we have that $\mathbb{R}^3 \setminus \bar{B}_r(\Sigma)$ is connected.

We are interested in scatterers of polyhedral type.

Definition 2.2. Let us define a *cell* as the closure of an open domain of a plane in $\mathbb{R}^3$. We shall say that Σ is a *polyhedral multiple scatterer* if Σ is a multiple scatterer such that the boundary of Σ is given by a finite union of cells.

Furthermore, we shall say that a cell C is *triangular* with constants $h > 0$ and α_0, $0 < \alpha_0 < \pi/3$, if C is a triangle of a plane of $\mathbb{R}^3$ whose sides have length greater than or equal to h and whose angles are greater than or equal to α_0.

We say that an L-generalized triangle Γ is *polyhedral* with constants $h > 0$ and α_0, $0 < \alpha_0 < \pi/3$, if Γ is the union of triangular cells, with constants h and α_0, which form a regular triangulation, that is, two different triangular cells may only share a vertex or a side.

Fixed $h > 0$, we say that Σ belongs to the class $\mathcal{A}_p(h)$ if Σ is a polyhedral multiple scatterer belonging to $\mathcal{A}$ and such that each generalized triangle Γ of $\partial\Sigma$ is polyhedral with constants h and α_0. Furthermore, we assume that the triangular cells form a regular triangulation all over $\partial\Sigma$.

Also the following class will be used.

Definition 2.3. We say that a set $S \subset \mathbb{R}^3$ is *uniformly Lipschitz* with constants r and L if for any $x \in S$ there exists a Lipschitz map $\varphi : \mathbb{R}^2 \to \mathbb{R}$, such that $\varphi(0) = 0$ and its Lipschitz constant is bounded by L, such that, up to a rigid transformation, $x = 0$ and

$$S \cap B_r(x) \subset \{y \in \mathbb{R}^3 \mid y_3 = \varphi(y_1, y_2)\}.$$

We say that a multiple scatterer Σ is *uniformly Lipschitz* with constants r and L if $\partial\Sigma$ is a *uniformly Lipschitz* set with constants r and L.

We say that $\Sigma \in \mathcal{A}_o$ if $\Sigma \in \mathcal{A}$ and $\sigma(x)$ is an obstacle for any $x \in \Sigma$ and Σ is a *uniformly Lipschitz* multiple obstacle with constants δ and L. We remark that, for any $x \in \partial\Sigma$, with the previous notation we have

$$\partial\Sigma \cap B_\delta(x) = \{y \in B_\delta(x) \mid y_3 = \varphi(y_1, y_2)\},$$
$$\Sigma \cap B_\delta(x) = \{y \in B_\delta(x) \mid y_3 \leq \varphi(y_1, y_2)\}.$$

Fixed $h > 0$, we define $\mathcal{A}_{p,o}(h)$ as the set of $\Sigma \in \mathcal{A}_p(h)$ such that $\sigma(x)$ is an obstacle for any $x \in \Sigma$ and Σ is a *uniformly Lipschitz* multiple obstacle with constants ch and L.

It is clear that, up to suitable changing the given constants, we can enlarge the class $\mathcal{A}$ and $\mathcal{A}_o$ in such a way that $\mathcal{A}$ and $\mathcal{A}_o$ are not empty and that, for some constant h_0, $0 < h_0 \leq 1$, $\mathcal{A}_p(h)$ and $\mathcal{A}_{p,o}(h) \subset \mathcal{A}_o$ are not empty for any h, $0 < h \leq h_0$. Therefore, in the sequel we shall always assume, without loss of generality, that this is the case. We recall that any element of $\mathcal{A}$ is a nonempty compact set and that $\mathcal{A}$ will be endowed with the Hausdorff distance, which will be denoted by $d_H(\cdot, \cdot)$.

Let us illustrate some of the properties of multiple scatterers belonging to the class $\mathcal{A}$. Assumption (i) is self-explanatory. For what concerns assumptions (ii)–(iv), we may think that the sets Γ_i, $i = 1, \ldots, n$, are a kind of regular triangulation of $\partial\Sigma$. If the Γ_i would be planar, we would have a regular triangulation of the surface $\partial\Sigma$ in the usual sense. Furthermore, we control how the various parts of the triangulation might be close to each other.

About assumption (v), we remark that it implies that G satisfies a uniform cone property. More precisely, there exists $\delta_1 > 0$ and θ_1, $0 < \theta_1 < \pi/2$, depending on the a priori data only, such that for any Γ, generalized triangle of $\partial\Sigma$, and any x belonging to Γ, there exists a direction $\omega_0 \in \mathbb{S}^2$ such that, for any $y \in \Gamma \cap B_{\delta_1}(x)$, the open cones $C(y, \omega_0, \delta_1, \theta_1)$ are contained in G and their opposite cones $C(y, -\omega_0, \delta_1, \theta_1)$ are all contained either in Σ or in G. Furthermore, we notice that any connected component of Σ which is an obstacle satisfies a uniform interior cone property as well. If a connected component of Σ is a *screen*, then it satisfies a uniform exterior cone property on either sides of the screen.

Assumption (vi) is a somewhat stronger version of the following uniform connectedness property of G. In fact, assumption (vi) implies that for any $t > 0$ and for any $x_1 \in G$ so that $\bar{B}_t(x_1)$ is contained in G, we can find a smooth (for instance C^1) curve γ connecting x_1 to $x_0 = (2R, 0, 0)$ so that $\bar{B}_{\eta(t)}(\gamma)$ is contained in G as well. Here $\eta : (0, +\infty) \to (0, +\infty)$ is a strictly increasing function which can be chosen as follows. If t is such that $0 < t \leq \min\{\delta/8, R/2\}$, then we can choose $\eta(t) = t/2$. On $[\min\{\delta/8, R/2\}, \infty)$, we choose as η any continuous strictly increasing function such that $\eta(\min\{\delta/8, R/2\}) = \min\{\delta/8, R/2\}/2$ and $\lim_{t \to \infty} \eta(t) = \min\{\delta/8, R/2\}$. We also observe that, with a completely analogous reasoning, we have that $\mathbb{R}^3 \setminus \bar{B}_r(\Sigma)$ is uniformly connected for any r, $0 < r < \delta/4$, with a, possibly different, function η depending on r and the a priori data only. Actually, we can choose the same function η for any $\Sigma \in \mathcal{A}$ and any $\mathbb{R}^3 \setminus \bar{B}_r(\Sigma)$, $0 < r \leq \delta/8$. Let us finally remark that assumption (vi) is not a particularly strong assumption. In fact, it holds for multiple scatterers whose components are either uniformly Lipschitz obstacles or uniformly Lipschitz screens with some additional assumptions on the boundaries of the surfaces forming the screens, see for instance the reasonings developed in [23].

Other important properties of multiple scatterers Σ belonging to $\mathcal{A}$ can be inferred. We notice that there exists an integer M such that for any $\Sigma \in \mathcal{A}$, such that $\partial\Sigma = \bigcup_{i=1}^n \Gamma_i$, we have $n \leq M$. As a consequence, we have that the number of connected components of Σ (that is the number of scatterers forming Σ) is

bounded by M as well. Furthermore, we have that, for some constant $r_0 > 0$ depending on the a priori data only,

(2.2) $r(\Sigma) = \min\{\operatorname{diam}(\sigma(x)) \mid x \in \Sigma\} \geq r_0,$

(2.3) $\Sigma \cap B_{r_0}(x) \subset \sigma(x), \quad \text{for any } x \in \Sigma,$

that is, we have a lower bound on the size of any scatterer forming Σ, whereas (2.3) is a lower bound on the distance between two different scatterers forming Σ.

   It is not difficult to show that for any $\Sigma \in \mathcal{A}$ and any r, $0 < r \leq \delta/8$, we have that Σ and $\bar{B}_r(\Sigma)$ satisfy Assumptions (a), (b) and (c) of [22] with constants and functions depending on the a priori data only, and not on r. We recall that Assumption (a) of [22] essentially means that Σ is *uniformly thick*. We refer to [13, page 127] for the notion of uniformly thickness of a set and its applications to the regularity of solutions to elliptic equations. Assumption (b) of [22] is the uniform connectedness of the exterior, which we have already discussed, finally Assumption (c) of [22] is a bound on the number of connected components.

   We observe that the following property holds. Let Σ, $\Sigma' \in \mathcal{A}$. There exists $d_0 > 0$, d_0 depending on the a priori data only, such that if $d_H(\Sigma, \Sigma') \leq d_0$, then Σ and Σ' have the same number m of connected component and, up to rearranging their order, we have $\Sigma = \bigcup_{i=1}^m \sigma_i$, $\Sigma' = \bigcup_{i=1}^m \sigma_i'$ and

$$d_H(\sigma_i, \sigma_i') \leq d_H(\Sigma, \Sigma') = \max_{i=1,\dots,m} d_H(\sigma_i, \sigma_i').$$

   Let Σ and Σ' belong to $\mathcal{A}$. Let us define the following modified Hausdorff distance

$$d_H'(\Sigma, \Sigma') = \max\{ \max_{x \in \partial\Sigma\backslash\Sigma'} \operatorname{dist}(x, \partial\Sigma'), \max_{x \in \partial\Sigma'\backslash\Sigma} \operatorname{dist}(x, \partial\Sigma) \},$$

with the convention that if $\partial\Sigma\backslash\Sigma' = \varnothing$, then we pose $\max_{x \in \partial\Sigma\backslash\Sigma'} \operatorname{dist}(x, \partial\Sigma') = 0$. We recall that other versions of modified Hausdorff distance have been previously introduced with similar purposes, see for instance [1]. Let us remark that, obviously, we have

$$d_H'(\Sigma, \Sigma') \leq d_H(\partial\Sigma, \partial\Sigma').$$

The following proposition allows us to control the Hausdorff distance with respect to the modified distance.

   Proposition 2.4. *Let Σ and Σ' belong to $\mathcal{A}$. Then there exist positive constants C_1 and C_2, depending on the a priori data only, such that*

$$C_1 d_H'(\Sigma, \Sigma') \leq C_1 d_H(\partial\Sigma, \partial\Sigma') \leq d_H(\Sigma, \Sigma')$$
$$\leq C_2 d_H'(\Sigma, \Sigma') \leq C_2 d_H(\partial\Sigma, \partial\Sigma').$$

Proof. Let $d = d_H(\Sigma, \Sigma') > 0$ and let us assume, up to swapping Σ and Σ', that there exists $x \in \Sigma \setminus \Sigma'$ such that $\text{dist}(x, \Sigma') = d$. Obviously, $\text{dist}(x, \partial\Sigma') = d$ as well. If $x \in \partial\Sigma$, then we can conclude that $d \leq d'_H(\Sigma, \Sigma')$. Therefore, let us assume that x is an interior point of Σ. Since Σ and Σ' are both contained in $\bar{B}_R$, we a priori know that $d \leq 2R$. For the time being, we assume that $d \leq \min\{\delta/8, R/2\}$. By the uniform connectedness of $G' = \mathbb{R}^3 \setminus \Sigma'$, we can find a simple piecewise smooth curve γ connecting x to $(2R, 0, 0)$ such that $B_{d/4}(\gamma) \subset G'$. Since x belongs to the interior of Σ and $(2R, 0, 0) \in G$, there exists $\bar{x} \in \gamma \cap \partial\Sigma$. We have that $\text{dist}(\bar{x}, \Sigma') = \text{dist}(\bar{x}, \partial\Sigma') \geq d/4$, therefore $d'_H(\Sigma, \Sigma') \geq d/4$ and the second inequality is proved provided $d \leq \min\{\delta/8, R/2\}$. If $d > \min\{\delta/8, R/2\}$, then we can find a simple piecewise smooth curve γ connecting x to $(2R, 0, 0)$ such that $B_{\min\{\delta/8, R/2\}/4}(\gamma) \subset G'$. We conclude that $d'_H(\Sigma, \Sigma') \geq \min\{\delta/8, R/2\}/4 \geq \min\{\delta/8, R/2\}d/(8R)$ and the inequality is proved also in this case. The second inequality is trivial if $d = 0$.

Again up to swapping Σ and Σ', let $x \in \partial\Sigma$ be such that $\text{dist}(x, \partial\Sigma') = d_H(\partial\Sigma, \partial\Sigma') = d'' > 0$. If $d'' = 0$, then $d'_H(\Sigma, \Sigma') = 0$ and, by the second inequality, we have $d = 0$ as well. We have two possibilities. First, if x does not belong to the interior of Σ', then $\text{dist}(x, \partial\Sigma') = \text{dist}(x, \Sigma')$ and therefore $d'' \leq d$. Second, let us assume that x belongs to the interior of Σ', that is $B_{d''}(x) \subset \Sigma'$. Let us further assume, for the time being, that $d'' \leq \delta_1$. By the uniform cone property of $G = \mathbb{R}^3 \setminus \Sigma$, there exists a constant $c_1 > 0$, depending on the a priori data only, such that we can find $x' \in B_{d''/2}(x)$ satisfying $\text{dist}(x', \Sigma) \geq c_1 d''$, hence $c_1 d'' \leq d$. This concludes the proof when $d'' \leq \delta_1$. If $d'' > \delta_1$, then we can find $x' \in B_{\delta_1/2}(x)$ such that $\text{dist}(x', \Sigma) \geq c_1 \delta_1$, that is $d \geq c_1 \delta_1$ and the conclusion is immediate since $d'' \leq 2R$. We have proved that $C_1 d_H(\partial\Sigma, \partial\Sigma') \leq d_H(\Sigma, \Sigma')$ for a positive constant C_1 depending on the a priori data only. Therefore also the first inequality is proved. $\qquad\square$

We conclude this section with the following important compactness result.

Lemma 2.5. *The classes $\mathcal{A}$ and $\mathcal{A}_o$ are compact with respect to the Hausdorff distance. Furthermore, for any h, $0 < h \leq h_0$, $\mathcal{A}_p(h)$ and $\mathcal{A}_{p,o}(h)$ are also compact with respect to the Hausdorff distance.*

Proof. The proof can be obtained by simple modifications of the arguments used to prove Lemma 6.1 in [24]. Some care should be taken in dealing with the uniform Lipschitz property in the case of $\mathcal{A}_o$ and $\mathcal{A}_{p,o}(h)$, however even this case can be treated by standard arguments. $\qquad\square$

3. THE DIRECT SCATTERING PROBLEM

We consider the acoustic scattering problem with a *sound-soft* multiple scatterer Σ.

Throughout the paper we shall keep fixed, beside the wave number $k > 0$, also the direction of propagation $\omega \in \mathbb{S}^2$ of the *incident field* $u_i(x) = e^{ik\omega \cdot x}$. Let u be the complex valued solution to (1.1). It is well-known that a weak solution $u \in W^{1,2}_{\text{loc}}(G)$ to (1.1) exists and is unique, see for instance [21]. We have that u is

analytic in G, but, of course, due to the possible irregularity of the boundary of G, the Dirichlet boundary condition in (1.1) is, in general, satisfied in the weak sense only. We recall that the function $u_s(x) = u(x) - e^{ik\omega \cdot x}$ is called the scattered field and that its asymptotic behaviour at infinity is described by (1.2) and, in particular, by u_∞, a function defined on $\mathbb{S}^2$, which is usually referred to as the far-field pattern of u_s.

Let $\Sigma, \Sigma' \in \mathcal{A}$ and let u be the solution to (1.1) and u' be the solution to the same problem when Σ is replaced by Σ'. Let $u_s(x) = u(x) - e^{ik\omega \cdot x}$ and $u'_s(x) = u'(x) - e^{ik\omega \cdot x}$ and let u_∞ and u'_∞ be the far-field pattern of u_s and u'_s, respectively. In the remaining part of this section, we describe some properties of these solutions and we present auxiliary results which will be useful in the sequel.

By the results proved in [22], the following properties are satisfied by u and $u_s(x)$. We always assume that u (and u' as well) is extended to all of $\mathbb{R}^3$ by setting $u \equiv 0$ outside G (respectively $G' = \mathbb{R} \setminus \Sigma'$). There exists a constant C_1, depending on the a priori data only, such that

$$(3.1) \qquad |u_s(x)| \leq C_1 |x|^{-1} \quad \text{for any } x \in \mathbb{R}^3 : |x| \geq 2R.$$

We immediately infer that there exists a constant R_1, $R_1 \geq 2R$, depending on C_1 only, such that

$$(3.2) \qquad |u(x)| \geq \frac{1}{2} \quad \text{for any } x \in \mathbb{R}^3 : |x| \geq R_1.$$

There exist positive constants C_2 and α, $0 < \alpha < 1$, depending on the a priori data only, such that

$$(3.3) \qquad |u(x) - u(y)| \leq C_2 |x - y|^\alpha \quad \text{for any } x, y \in B_{2R_1},$$

that is u is Hölder continuous with constants depending on the class of admissible multiple scatterers and on k. As a consequence, we have that u is continuous up to ∂G and $u \equiv 0$ on ∂G, that is the Dirichlet boundary condition in (1.1) is satisfied also in a classical sense. Also, we have that there exists a constant C_3, depending on the a priori data only, such that

$$(3.4) \qquad |u(x)| \leq C_3 \quad \text{for any } x \in \mathbb{R}^3.$$

We also have the following stability estimate for the direct problem with respect to the multiple scatterer.

Proposition 3.1. *Let Σ and Σ' belong to $\mathcal{A}$. Then, there exists a constant C_4, depending on the a priori data only, such that*

$$(3.5) \quad |u(x) - u'(x)| \leq C_4 (d_H(\Sigma, \Sigma'))^\alpha \qquad \text{for any } x \in \mathbb{R}^3,$$

$$(3.6) \quad |u(x) - u'(x)| \leq C_4 (d_H(\Sigma, \Sigma'))^\alpha |x|^{-1} \quad \text{for any } x \in \mathbb{R}^3 : |x| \geq 2R.$$

Here α is the same constant appearing in (3.3).

Proof. Let us assume, for the time being, that $d = d_H(\Sigma, \Sigma') \leq \delta/8$. Then $\Sigma \cup \Sigma' \subset \bar{B}_d(\Sigma)$. By (3.3), we have that

$$|u(x)| \leq C_2 d^\alpha \quad \text{for any } x \in \bar{B}_d(\Sigma),$$

whereas, since for any $x \in \bar{B}_d(\Sigma)$ we have $\text{dist}(x, \Sigma') \leq 2d$,

$$|u'(x)| \leq C_2(2d)^\alpha \quad \text{for any } x \in \bar{B}_d(\Sigma).$$

Therefore, we have that

$$|u(x) - u'(x)| = |u_s(x) - u'_s(x)| \leq 3C_2 d^\alpha \quad \text{for any } x \in \bar{B}_d(\Sigma).$$

We have that $u_s - u'_s$ solves the Helmholtz equation on $\mathbb{R}^3 \setminus \bar{B}_d(\Sigma)$, satisfies the Sommerfeld radiation condition and its Dirichlet data on $\partial \bar{B}_d(\Sigma)$ are uniformly bounded by $3C_2 d^\alpha$. By our assumptions on Σ, in particular by Assumption (vi), we can apply the same reasoning used in [22] to conclude the proof provided $d \leq \delta/8$. The fact that $d \leq 2R$, and (3.4) and (3.1), respectively, allow us to conclude the proof of (3.5) and (3.6) also when $d > \delta/8$. $\qquad\square$

As a consequence of Proposition 3.1, we have the following results. Let $r_1 = \frac{3}{2}\max\{R_1, 2/(ek)\}$. Then, there exists a constant C_5, depending on the a priori data only, such that

$$(3.7) \qquad \|u - u'\|_{C^1(\bar{B}_{7er_1/4} \setminus B_{5er_1/4})} \leq C_5(d_H(\Sigma, \Sigma'))^\alpha,$$

and, by the stability of the direct scattering problem on the exterior of a given ball,

$$(3.8) \qquad \|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^2)} \leq C_6 C_5(d_H(\Sigma, \Sigma'))^\alpha,$$

C_6 depending on k and R_1 only.

In this paper we investigate the possibility to reverse the inequalities (3.7) and (3.8), that is to control the Hausdorff distance between Σ and Σ' either by the error on the far-field, $\|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^2)}$, or by the error on the near-field, $\|u - u'\|_{C^1(\bar{B}_{7er_1/4} \setminus B_{5er_1/4})}$. The first step, that is the stability of the determination of the near-field from the far-field, is well-known and is contained in the following lemma, whose proof can be found in [14].

Lemma 3.2. *Let ε be a positive number such that*

$$(3.9) \qquad \|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^2)} \leq \varepsilon.$$

There exists a positive constant $\bar{\varepsilon}_0 < 1/e$, depending on the a priori data only, such that if $0 < \varepsilon \leq \bar{\varepsilon}_0$, then

$$(3.10) \qquad \|u - u'\|_{C^1(\bar{B}_{7er_1/4} \setminus B_{5er_1/4})} \leq \exp\left(-\frac{1}{2}(-\log \varepsilon)^{1/2}\right).$$

We conclude this section with some auxiliary results on classical solutions to the Helmholtz equation. A simple dilation argument and standard interior regularity results for the solutions to the Helmholtz equation lead to the following lemma.

Lemma 3.3. *Let us fix positive constants* ρ_1*,* M*. Let us consider* ρ*,* $0 < \rho \le \rho_1$*, and a function* u *such that*

$$\Delta u + k^2 u = 0 \quad \text{in } B_\rho.$$

If, for some positive $\varepsilon \le M$*,*

(3.11) $$|u(x)| \le \varepsilon \quad \text{for any } x \in B_\rho,$$

then we have that for any constant s*,* $0 < s < 1$*, there exists a constant* C *depending on* k*,* ρ_1*,* M*, and* s *only such that*

(3.12) $$\rho|\nabla u(x)| \le C\varepsilon \quad \text{for any } x \in B_{s\rho}.$$

For any plane Π in $\mathbb{R}^3$, let T_Π be the reflection in Π. If $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$, with respect to a suitably chosen Cartesian coordinate system, then, for any $x = (x_1, x_2, x_3)$, we have $T_\Pi(x) = (x_1, x_2, -x_3)$.

Lemma 3.4. *Let us fix positive constants* ρ_1*,* M*. Let us consider* ρ*,* $0 < \rho \le \rho_1$*, and a function* u *such that*

$$\Delta u + k^2 u = 0 \quad \text{in } B_\rho.$$

Let $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$ *and let* $T = T_\Pi$ *and* $v(x) = -u(T(x))$*. Let us assume that*

$$|u(x)| \le M \quad \text{for any } x \in B_\rho,$$

and, for some positive $\varepsilon \le M$*,*

$$|u(x)| + \rho|\partial_1 u(x)| + \rho|\partial_2 u(x)| \le \varepsilon \quad \text{for any } x \in \Pi \cap B_\rho.$$

Then there exist constants C*,* β*,* $0 < \beta < 1$*, depending on* k *and* ρ_1 *only, such that*

$$|(u - v)(x)| \le CM^{1-\beta}\varepsilon^\beta \quad \text{for any } x \in B_{\rho/2}.$$

Proof. We have that $\Delta v + k^2 v = 0$ in B_ρ and

$$|(u - v)(x)| + \rho|\nabla(u - v)(x)| \le 2\varepsilon \quad \text{for any } x \in \Pi \cap B_\rho.$$

We call $w(x) = (u - v)(\rho x)$ and we have that $\Delta w + k^2 \rho^2 w = 0$ in B_1 and

$$|w(x)| + |\nabla w(x)| \le 2\varepsilon \quad \text{for any } x \in \Pi \cap B_1,$$

whereas $|w(x)| \le 2M$ for any $x \in B_1$. Then, the proof can be easily obtained by applying the estimates derived in [25], see also for similar estimates [20]. ☐

As a consequence of the *three-spheres inequalities* which may be found, for instance, in [6], the following inequality holds true.

Lemma 3.5. *There exist positive constants $\bar{\rho}$, C and c_1, $0 < c_1 < 1$, depending on k only, such that for every $0 < \rho_1 < \rho < \rho_2 \leq \bar{\rho}$ and any function u such that*

$$\Delta u + k^2 u = 0 \quad \text{in } B_{\rho_2},$$

we have, for any s, $\rho < s < \rho_2$,

$$(3.13) \qquad \|u\|_{L^\infty(B_\rho)} \leq C(1 - (\rho/s))^{-3/2} \|u\|_{L^\infty(B_{\rho_2})}^{1-\beta} \|u\|_{L^\infty(B_{\rho_1})}^{\beta},$$

for some β such that

$$(3.14) \qquad \frac{c_1(\log(\rho_2/s))}{\log(\rho_2/\rho_1)} \leq \beta \leq 1 - \frac{c_1(\log(s/\rho_1))}{\log(\rho_2/\rho_1)}.$$

4. The Main Stability Results

The first stability result is the following. Let us call $\eta : (0, 1/e) \to (0, +\infty)$ the following function

$$(4.1) \qquad \eta(s) = \exp(-(\log(-\log s))^{1/2}) \quad \text{for any } s, \ 0 < s < 1/e.$$

Theorem 4.1. *Let Σ, Σ' belong to $\mathcal{A}_p(h)$, with $0 < h \leq h_0$, and let $d = d_H(\Sigma, \Sigma')$.*

There exists a constant $\hat{\varepsilon}_0 > 0$, depending on the a priori data only, such that if

$$(4.2) \qquad \|u_\infty - u'_\infty\|_{L^2(\mathbb{S}^2)} \leq \varepsilon$$

for some $\varepsilon < \hat{\varepsilon}_0$, then for some positive constant C depending on the a priori data only, and not on h, we have

$$(4.3) \qquad \min\{d, h\} \leq 2eR(\eta(\varepsilon))^{C/2}.$$

Therefore,

$$(4.4) \qquad d_H(\Sigma, \Sigma') \leq 2eR(\eta(\varepsilon))^{C/2},$$

provided $\varepsilon < \hat{\varepsilon}_0(h)$ where

$$\hat{\varepsilon}_0(h) = \min\left\{\hat{\varepsilon}_0, \eta^{-1}\left(\left(\frac{h}{2eR}\right)^{2/C}\right)\right\}.$$

This estimate may be improved if we limit ourselves to the determination of multiple obstacles.

Theorem 4.2. *Let* Σ *and* Σ' *belong to* $\mathcal{A}_{p,o}(h)$*, with* $0 < h \leq h_0$.

Then there exists $\hat{\varepsilon}_1(h)$*,* $0 < \hat{\varepsilon}_1(h) \leq \hat{\varepsilon}_0(h)$*, depending on the a priori data and on* h *only, such that if* (4.2) *holds for some* $\varepsilon < \hat{\varepsilon}_1(h)$*, then*

$$(4.5) \qquad d_H(\Sigma, \Sigma') \leq C_1 \exp\left(C_2 \left(\frac{e}{h}\right)^{C_3}\right) \exp\left(-\frac{C_4}{2} h^{C_5} (-\log \varepsilon)^{1/2}\right).$$

Here $C_1, \ldots, C_5$ *are positive constants depending on the a priori data only.*

Remark 4.3. We make the following two observations on Theorem 4.2. First of all, we notice that the estimate is not precisely of Hölder type only because the stability of the determination of the near-field from the far-field is not Hölder, see Lemma 3.2. In fact, with respect to the near-field the stability estimate is Hölder, see Lemma 6.4.

We may also quantify $\hat{\varepsilon}_1(h)$ in an explicit way with respect to h and the a priori data as follows.

The estimate holds provided $d \leq f(h) = c_1 h^2 \exp(-(e/h)^{A_1})$, for some positive constants $c_1 \leq 1$ and A_1 depending on the a priori data only. Therefore, by Theorem 4.1, this is guaranteed provided that

$$\varepsilon < \hat{\varepsilon}_1(h) = \min\left\{\hat{\varepsilon}_0, \eta^{-1}\left(\left(\frac{f(h)}{2eR}\right)^{2/C}\right)\right\}.$$

The proof of the stability theorems is postponed to Section 5 for Theorem 4.1 and to Section 6 for Theorem 4.2. We conclude this section by making some remarks on the optimality of the estimates derived, in particular of estimate (4.3).

In order to prove the optimality of our estimates, let $\Sigma = [-1, 1] \times [-1, 1] \times [-1, 1]$. Let us take a part of the upper face of the cube, namely, $[-\ell h, \ell h] \times [-\ell h, \ell h] \times \{1\}$, where ℓ is an integer such that $\frac{1}{2} \leq \ell h \leq \frac{3}{4}$ (we may assume $h \leq \frac{1}{8}$). Therefore, $1/(2h) \leq \ell \leq 3/(4h)$. We divide such a part into $4\ell^2$ squares of side h. Let us consider a pyramid with base given by a square of side h, height $d \leq h$ and vertex exactly above the center of the square.

If we modify Σ by replacing one or several of the squares with the corresponding pyramid, we obtain $2^{4\ell^2}$ different obstacles. It is easy to verify that we may choose constants R, L, δ, c, $0 < c < 1$, θ, $0 < \theta < \pi/2$, α_0, $0 < \alpha_0 < \pi/3$, and c_1, $0 < c_1 < 1$, none of them depending on h or d, such that $\Sigma_j \in \mathcal{A}_{p,o}(\bar{h})$ for any $j = 1, \ldots, 2^{4\ell^2}$, where $\bar{h} = c_1 h$. Let us remark that there exists $j \in \{1, \ldots, 2^{4\ell^2}\}$ such that $\Sigma = \Sigma_j$. Finally we remark that $d_H(\Sigma_j, \Sigma) \leq d$ for any j and $d_H(\Sigma_i, \Sigma_j) = d$ for any $i \neq j$. We also fix k. By slight modifications of the computations developed in [11], we obtain that there exists a positive constant $h_1 \leq \frac{1}{8}$, depending on the a priori data only, such that, if $0 < h \leq h_1$, then we can find $i \neq j$ satisfying

$$\|u_\infty^i(\omega, \hat{x}) - u_\infty^j(\omega, \hat{x})\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)} \leq 2 \exp(-h^{-(1/3)}) d^\alpha,$$

with α as in (3.3).

Obviously, there exists ω_0 such that

$$\varepsilon = \|u_\infty^i(\omega_0, \cdot) - u_\infty^j(\omega_0, \cdot)\|_{L^2(\mathbb{S}^2)}$$

$$\leq (4\pi)^{-1/2}\|u_\infty^i(\omega, \hat{x}) - u_\infty^j(\omega, \hat{x})\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^2)}.$$

We observe that, for any $0 < d \leq h$, we have $\min\{d, \bar{h}\} \geq c_1 d$ and that, if we pick $d = h$, then $\min\{d, \bar{h}\} = \bar{h} = c_1 h$. Then the following result holds.

Proposition 4.4. *Under the previous assumptions, we infer that for any h, $0 < h \leq h_1$, and if $d = h$, then*

$$\min\{d, \bar{h}\} = \bar{h} = c_1 h = c_1 d \geq c_1(-\log(\sqrt{\pi}\varepsilon))^{-3}.$$

More precisely,

$$\bar{h} \geq c_1(-\log(\sqrt{\pi}\varepsilon))^{-3}$$

and

$$d = h \geq (-\log(\sqrt{\pi}\varepsilon))^{-3}.$$

Furthermore, for any h, $0 < h \leq h_1$, and any d, $0 < d \leq h$, we obtain

$$d \geq (\exp(h^{-1/3})\sqrt{\pi}\varepsilon)^{1/\alpha},$$

and, consequently,

$$\min\{d, \bar{h}\} \geq c_1 d \geq c_1(\exp(h^{-1/3})\sqrt{\pi}\varepsilon)^{1/\alpha}.$$

This proposition shows that in general logarithmic estimates are optimal and that we may improve them at most with Hölder type estimates whose multiplicative constant, however, blows up in an exponential way with respect to h.

5. THE STABILITY ANALYSIS FOR POLYHEDRAL MULTIPLE SCATTERERS

Throughout this section, we also fix h, $0 < h \leq h_0$, and we let $\mathcal{A}_p = \mathcal{A}_p(h)$.

Let Σ, Σ' belong to $\mathcal{A}_p$ and let $d = d_H(\Sigma, \Sigma')$. We observe that $d \leq 2R$. Let ε be a positive number such that $\varepsilon \leq \bar{\varepsilon}_0$, $\bar{\varepsilon}_0$ as in Lemma 3.2, and

(5.1) $$\|u_\infty - u_\infty'\|_{L^2(\mathbb{S}^2)} \leq \varepsilon.$$

Let us call ε_1 the error on the near-field, namely

(5.2) $$\|u - u'\|_{C^1(\bar{B}_{7er_1/4} \setminus B_{5er_1/4})} = \varepsilon_1.$$

We observe that, since $\varepsilon \le \bar{\varepsilon}_0 < 1/e$, then, by Lemma 3.2, we have that

$$\varepsilon_1 \le \exp\left(-\frac{1}{2}(-\log \varepsilon)^{1/2}\right).$$

There exists $\bar{\varepsilon}_1$, $0 < \bar{\varepsilon}_1 \le \bar{\varepsilon}_0$, depending on the a priori data only, such that $\exp(-\frac{1}{2}(-\log \bar{\varepsilon}_1)^{1/2}) \le 1/(2e)$. Therefore, we assume, without loss of generality, that either $\varepsilon < \bar{\varepsilon}_1$ or $\varepsilon_1 < 1/(2e)$. We observe that we also have

$$\log \frac{1}{\varepsilon_1} \ge \frac{1}{2}\left(\log \frac{1}{\varepsilon}\right)^{1/2}.$$

Furthermore, by (3.4), we have that

(5.3) $$|u(x)| + |u'(x)| \le E \quad \text{for any } x \in \mathbb{R}^3,$$

where E depends on the a priori data only and it may be assumed to be greater than or equal to 1.

Let us now proceed with the proof of Theorem 4.1. We need to introduce the following notation. Let H be the connected component of $G \cap G'$, where $G' = \mathbb{R}^3 \setminus \Sigma'$, such that $\mathbb{R}^3 \setminus \bar{B}_R$ is contained in H.

Definition 5.1. We say that a sequence of balls $B_{\rho_i}(z_i)$, $i = 0, \ldots, n$, forms a *regular chain* with respect to an open set G if the following properties are satisfied

(i) for any $i = 0, 1, \ldots, n$, $B_{8\rho_i}(z_i) \subset G$;
(ii) for any $i = 1, \ldots, n$, we have $\rho_i \le \rho_{i-1}$ and $B_{\rho_i/8}(z_i) \subset B_{3\rho_{i-1}/8}(z_{i-1})$ and, for any $i = 0, \ldots, n-1$, we have $B_{\rho_i/8}(z_i) \subset B_{7\rho_{i+1}/8}(z_{i+1})$.

Proof of Theorem 4.1. We proceed into several steps, alongside with the geometric construction and its corresponding estimate. Without loss of generality, up to swapping Σ with Σ', we can find $x_1 \in \Sigma'$ such that $d = \text{dist}(x_1, \Sigma)$. We also fix $x_0 = ((3/2)er_1, 0, 0)$.

Step I: From x_0 to x_1. We construct a sequence of balls $B_{\rho_i}(z_i)$, $i = 0, \ldots, n$, such that they are a regular chain with respect to G and the following conditions are satisfied. First, $z_0 = x_0$ and ρ_0 is a positive constant, depending on the a priori data only, such that $8\rho_0 \le \min\{R/4, \bar{\rho}\}$, and $z_n = x_1$.

The sequence is constructed as follows. Let y_1 be a point of Σ such that $|x_1 - y_1| = d$. By the properties of Σ, there exists a direction $\omega_0 \in \mathbb{S}^2$ such that the open cone $C(y_1, \omega_0, \delta_1, \theta_1)$ is contained in G and its opposite cone $C(y_1, -\omega_0, \delta_1, \theta_1)$ is contained either in Σ or in G. We have that $B_d(x_1)$ has a

nonempty intersection with one of these two cones, say $C(y_1, \omega_0, \delta_1, \theta_1)$. There-
fore, there exist positive constants c_1 and c_2, depending on the a priori data only,
such that we may find $y_2 = y_1 + s\, d\omega_0$ with the following properties. First,
$c_1 \leq s \leq 1$ and, second, for any x belonging to the segment connecting x_1
to y_2, we have $B_{c_2 d}(x) \subset G$. We connect y_2 to x_1 with a regular chain made
of balls with fixed radius, $s_1 d$, centred on the segment connecting y_2 to x_1.
The positive constant $s_1 \leq c_2/8$ depends on the a priori data only and will be
chosen later. The number of balls needed to reach x_1 from y_2 is controlled
by a constant depending on the a priori data only. Then, let us call y_3 the
point $y_1 + \bar{\delta}\omega_0$, where $\bar{\delta} = (1/(1 + \sin(\theta_1))) \min\{\delta_1, R/4, \bar{\rho}\}$. We have that
$B_{\sin(\theta_1)\bar{\delta}}(y_3) \subset C(y_1, \omega_0, \delta_1, \theta_1) \subset G$. We connect y_3 to y_2 with a regular chain
made of balls centred on the bisecting line of the cone. Let $B_\rho(z)$ and $B_{\rho_1}(z_1)$
be two consecutive balls. First of all we require that $8\rho = \bar{c}\sin(\theta_1)|z - y_1|$
and $8\rho_1 = \bar{c}\sin(\theta_1)|z_1 - y_1|$, for some constant $\bar{c}$, $0 < \bar{c} \leq 1$. We can find
a constant $\bar{c}$, depending on the a priori data only, such that both the conditions
$|z - y_1| \geq |z_1 - y_1| \geq |z - y_1| - \rho/8 - \rho_1/8$ and $\rho/3 \leq \rho_1 \leq \rho$ are satis-
fied. The first ball is $B_{(1/8)\bar{c}\sin(\theta_1)\bar{\delta}}(y_3)$ and the last one is $B_{(1/8)\bar{c}\sin(\theta_1)sd}(y_2)$. We
can choose $\bar{c}$ in such a way that we also have $\bar{c}\sin(\theta_1)s \leq c_2$ and we pick s_1 as
$\frac{1}{8}\bar{c}\sin(\theta_1)s$. The number of balls needed to reach y_2 from y_3 can be bounded
by $\bar{C}\log(2eR/d)$, with $\bar{C}$ depending on the a priori data only. Then, we connect,
using the connection properties of the exterior of neighbourhoods of Σ, z_0 to y_3
with a regular chain of balls with fixed radius $\frac{1}{8}\bar{c}\sin(\theta_1)\bar{\delta}$, whose number can
be bounded again by a constant depending on the a priori data only. We may
therefore conclude that the sequence exists and that

$$n \leq \bar{C}\log(2eR/d)$$

with $\bar{C}$ depending on the a priori data only.

Starting from $z_0 = x_0$, we take $j \in \{1, \ldots, n\}$ such that, for any $i =
0, 1, \ldots, j-1$, $B_{\rho_i/2}(z_i) \subset H$ and $B_{\rho_j/2}(z_j) \cap \Sigma' \neq \emptyset$. We apply the three-spheres
inequality of Lemma 3.5 as follows. For any $i = 0, 1, \ldots, j-1$,

$$\|u - u'\|_{L^\infty(B_{\rho_{i+1}/8}(z_{i+1}))} \leq \|u - u'\|_{L^\infty(B_{3\rho_i/8}(z_i))}$$

$$\leq C\|u - u'\|_{L^\infty(B_{\rho_i/2}(z_i))}^{1-\beta_i} \|u - u'\|_{L^\infty(B_{\rho_i/8}(z_i))}^{\beta_i}.$$

If β_i, $i = 0, 1, 2, \ldots$, are positive constants, we shall use the following notation for
any $j = 0, 1, 2, \ldots$

$$\mathcal{B}_j = \sum_{r=0}^{j} \prod_{i=r}^{j} \beta_i, \quad \Gamma_j = \prod_{i=0}^{j} \beta_i.$$

By iterating the estimate, and recalling that $\|u - u'\|_{L^\infty(\mathbb{R}^3)} \leq C_1 d^\alpha$ and (5.2), we obtain

$$(5.4) \qquad \|u - u'\|_{L^\infty(B_{\rho_j/8}(z_j))} = \varepsilon_2 \leq C^{1+\mathcal{B}_{j-1}}(C_1 d^\alpha)^{1-\Gamma_{j-1}}\varepsilon_1^{\Gamma_{j-1}}.$$

We recall that any β_i, $i = 0, \ldots, j - 1$, satisfies

$$0 < a \leq \beta_i \leq b < 1$$

where a and b depend on k only.

We are now ready to proceed with the second step.

Step II: Towards the face. We call $\hat{d} = \min\{d, h\}$. Let us describe our starting point for this step. Let $z = z_j$ and $\rho = \rho_j$. Then, $B_{\rho/8}(z) \subset H$, $B_{8\rho}(z) \subset G$ and there exists $w \in \Sigma'$ such that $|z - w| < \rho/2$. Let C' be one of the cells of $\partial\Sigma'$ to which w belongs. We call Π the plane containing C' and, up to a rigid change of coordinates, without loss of generality, we assume $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$. We know that for a direction $\omega_0 \in \mathbb{S}^2$, the open cone $C(y, \omega_0, \delta_1, \theta_1)$ is contained in G' for any $y \in C' \cap B_{\delta_1}(w)$ and its opposite cone $C(y, -\omega_0, \delta_1, \theta_1)$ is contained either in Σ' or in G'. We have that $B_{|z-w|}(z)$ has a nonempty intersection with one of these two cones, say $C(w, \omega_0, \delta_1, \theta_1)$. First of all, as in the previous construction from y_2 to x_1, we call $w_1 = w + s|z - w|\omega_0$, with s with the same properties as in the previous step. Then, we construct a regular chain with respect to H with balls of constant radius $s_2 d$ and centered on the segment connecting z to w_1 such that the first one is centered at z, the last one is centered at w_1 and s_2 satisfies $s_2 d \leq (c_2/8)|z - w|$ and $s_2 d \leq \rho/8$. Since $|z - w|$ and ρ are greater than a positive constant times d, we may take s_2 as a positive constant depending on the a priori data only, to be chosen later. In any case, the number of these balls is bounded by a constant depending on the a priori data only. Having reached w_1, we observe that, by the properties of C', there exists $\omega_1 \in \mathbb{S}^2 \cap \Pi$ and constants c_3, $0 < c_3 \leq 1$, and θ_2, $0 < \theta_2 < \pi/2$, depending on α_0 only, such that $C(w, \omega_1, c_3\hat{d}, \theta_2) \cap \Pi \subset C'$. By looking at the points on the bisecting line of $C(w, \omega_1, c_3\hat{d}, \theta_2)$, we may find w_2 on this line such that $B_{s_3\hat{d}}(w_2) \cap \Pi \subset C'$, $B^+_{s_4\hat{d}}(w_2) \subset \bigcup_{y \in B_{s_3\hat{d}}(w_2) \cap \Pi} C(y, \omega_0, (1 + \sin(\theta_3))s|z - w|, \theta_3) \subset H$, where $B^+ = B \cap \{x_3 > 0\}$. Furthermore, we may ensure that

$$\bigcup_{y \in B_{s_3\hat{d}}(w_2) \cap \Pi} C\big(y, \omega_0, (1 + \sin(\theta_3))s|z - w|, \theta_3\big),$$

$$\bigcup_{y = w + t\omega_1, 0 \leq t \leq |w_2 - w|} C\big(y, \omega_0, (1 + \sin(\theta_3))s|z - w|, \theta_3\big)$$

are both contained in H and in $B_\rho(z)$; that we may choose s_2 so that $B_{8s_2d}(w_1)$ is contained in $C(w, \omega_0, (1 + \sin(\theta_3))s|z - w|, \theta_3)$; and that all the positive constants s_2, s_3, s_4, and θ_3, $0 < \theta_3 < \pi/2$, depend on the a priori data only. With a further regular chain of balls with constant radius s_2d, we proceed from $w_1 = w + s|z - w|\omega_0$ towards $w_2' = w_2 + s|z - w|\omega_0$. If we take the balls centered on the segment connecting w_1 to w_2', it takes only a finite number, bounded by a constant depending on the a priori data only, of balls to reach w_2'. Then we proceed inside $C(w_2, \omega_0, (1 + \sin(\theta_3))s|z - w|, \theta_3)$, with a reasoning analogous to the one used to connect y_3 to y_2, and we obtain that we may continue our regular chain, with respect to $H \cup B_{s_4\hat{d}}(w_2)$, from $B_{s_2d}(w_2')$ up to $B_{s_5\hat{d}}(w_2)$, $s_5 > 0$ depending on the a priori data only. The number of balls forming the regular chain, with respect to $H \cup B_{s_4\hat{d}}(w_2)$, used to reach, along this path, $B_{s_5\hat{d}}(w_2)$ from $B_{s_2d}(z) \subset B_{\rho/8}(z)$, can be bounded by $\bar{C}\log(ed/\hat{d})$, with $\bar{C}$ depending on the a priori data only.

First of all, we notice that we may extend, by a reflection argument, u' on $B_{s_4\hat{d}}(w_2)$. Namely, if T_Π is the reflection in the plane Π, that is $T_\Pi(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, we set, for any $x \in B_{s_4\hat{d}}^-(w_2)$, $u'(x) = -u'(T_\Pi(x))$. In this way, u' satisfies the Helmholtz equation all over $B_{s_4\hat{d}}(w_2)$. Therefore, arguing as in the first step, we conclude that

$$(5.5) \qquad \|u - u'\|_{L^\infty(B_{s_5\hat{d}/8}(w_2))} = \varepsilon_3 \le C^{1+\mathcal{B}_{m-1}}(C_1 d^\alpha)^{1-\Gamma_{m-1}} \varepsilon_2^{\Gamma_{m-1}}$$

where

$$m \le \bar{C}\log\frac{ed}{\hat{d}}$$

and any β_i, $i = 0, \ldots, m-1$, satisfies $0 < a \le \beta_i \le b < 1$, with a and b depending on k only.

Actually, coupling (5.5) with (5.4), we also have

$$(5.6) \qquad \|u - u'\|_{L^\infty(B_{s_5\hat{d}/8}(w_2))} \le C^{1+\mathcal{B}_{m-1}}(C_1 d^\alpha)^{1-\Gamma_{m-1}} \varepsilon_1^{\Gamma_{m-1}}$$

where

$$m \le \bar{C}\left(\log\frac{ed}{\hat{d}} + \log\frac{2eR}{d}\right)$$

and any β_i, $i = 0, \ldots, m-1$, satisfies $0 < a \le \beta_i \le b < 1$.

Step III: Reflection. We recall that Π is the plane containing the face C' and T_Π is the reflection in Π. We define Σ_1 as the reflection of Σ with respect to the plane Π, $G_1 = \mathbb{R}^3 \setminus \Sigma_1$, and u_1 as the reflection of u with respect to the same plane Π, defined as follows. For any $x \in \mathbb{R}^3$, we pose $u_1(x) = -u(T_\Pi(x))$. Let us observe that $B_{s_5\hat{d}}(w_2) \subset B_\rho(z) \subset B_{8\rho}(z) \subset G$. Therefore, $B_\rho(z) \subset B_{2\rho}(w_2) \subset B_{3\rho}(w_2) \subset G \cap G_1$. Both u and u_1 satisfy the Helmholtz equation on $B_{3\rho}(w_2)$.

We apply Lemma 3.3 to $u - u'$. We recall that $u' = 0$ on $\Pi \cap B_{s_5\hat{d}}(w_2)$ and we apply Lemma 3.4 to $u - u_1$ and we find that, if $s_6 = s_5/32$, then

$$\|u - u_1\|_{L^\infty(B_{s_6\hat{d}}(w_2))} = \varepsilon_4 \leq C(2E)^{1-\beta}\varepsilon_3^\beta, \tag{5.7}$$

where C and β, $0 < \beta < 1$, again depend on the a priori data only.

Step IV: From the face back to $B_{\rho_j}(z_j)$. We apply the three-spheres inequality to $u - u_1$ with balls centered at w_2, radius $s_6\hat{d}$, $2\rho_j$ and $3\rho_j$, and we obtain that

$$\|u - u_1\|_{L^\infty(B_{\rho_j}(z_j))} = \varepsilon_5 \leq \|u - u_1\|_{L^\infty(B_{2\rho_j}(w_2))} \leq C(2E)^{1-\bar{\beta}}\varepsilon_4^{\bar{\beta}} \tag{5.8}$$

where

$$c_1\frac{\log(6/5)}{\log(3\rho/s_6\hat{d})} \leq \bar{\beta} \leq 1 - c_1\frac{\log(5\rho/2s_6\hat{d})}{\log(3\rho/s_6\hat{d})}.$$

By coupling (5.7) and (5.8), we obtain

$$\|u - u_1\|_{L^\infty(B_{\rho_j}(z_j))} = \varepsilon_5 \leq C^{1+\bar{\beta}}(2E)^{1-\beta\bar{\beta}}\varepsilon_3^{\beta\bar{\beta}}. \tag{5.9}$$

By coupling (5.9) with (5.6), we conclude that, since $C_1 d^\alpha \leq 2E$,

$$\|u - u_1\|_{L^\infty(B_{\rho_j/8}(z_j))} \leq C^{1+\mathcal{B}_n}(2E)^{1-\Gamma_n}\varepsilon^{\Gamma_n} \tag{5.10}$$

where $C \geq 1$ and $2E \geq 1$ are constants depending on the a priori data only and for any β_i, $i = 0, \ldots, n-1$, we have $0 < a \leq \beta_i \leq b < 1$, whereas β_n satisfies

$$c_1\frac{\log(6/5)}{\log(c_2\rho_0/\hat{d})} \leq \beta_n \leq 1 - c_1 + c_1\frac{\log(6/5)}{\log(c_2\rho_j/\hat{d})}$$

and, finally,

$$n \leq \bar{C}\left(\log\frac{ed}{\hat{d}} + \log\frac{2eR}{d}\right).$$

Here a, b, c_1, c_2, $\bar{C}$ depend on the a priori data only.

Step V: Returning back towards x_0. Let us now consider the regular chain of balls $B_{\rho_i}(z_i)$, $i = 0, \ldots, j$, we have constructed in Step I. We have that $B_{\rho_j}(z_j)$ is contained in G_1. We proceed backwards along the chain, until we find j_1, $0 \leq j_1 < j$, such that, for any $i = j_1 + 1, \ldots, j$, we have $B_{\rho_i}(z_i) \cap G_1 = \emptyset$, whereas $B_{\rho_{j_1}}(z_{j_1}) \cap G_1 \neq \emptyset$. Then, we apply Step II, III and IV to u, u_1, Σ and Σ_1. By reflection in a suitable plane Π_1, from Σ we obtain Σ_2 and from u we obtain u_2. And we estimate, in an analogous way as (5.10), $\|u - u_2\|_{L^\infty(B_{\rho_{j_1}/8}(z_{j_1}))}$.

We repeat this procedure as many times as needed (actually at most j times), until we are able to estimate from above $\|u - u_N\|_{L^\infty(B_{\rho_0/8}(z_0))}$, where $N \leq j$, and u_N is the reflection of u with respect to a suitable plane Π_{N-1}. The last step is the following.

Step VI: Along ∂B_{3er_1}. Since, without loss of generality, we assume that $|z_i| \leq \frac{3}{2}er_1$, for any $i = 0, \ldots, j$, then there exists a point $\tilde{z}$ belonging to the intersection of ∂B_{3er_1} with Π_{N-1}. Again with a regular chain of balls with constant radius ρ_0, we proceed from z_0 towards $\tilde{z}$. Two cases may occur. Either, for any of the balls $B_\rho(z)$ of this sequence, we have $B_{\rho/2}(z) \cap \Sigma_N = \varnothing$, or not. In the first case, with the usual iteration of the three-spheres inequality, we may estimate $|(u - u_N)(\tilde{z})|$ from above with respect to $\|u - u_N\|_{L^\infty(B_{\rho_0/8}(z_0))}$. We then observe that on $\tilde{z}$, by construction, $|(u - u_N)(\tilde{z})| = 2|u(\tilde{z})|$. Otherwise, if some $B_{\rho/2}(z) \cap \Sigma_N$ is not empty, we apply Step II, we find a corresponding point $\tilde{w}_2 \in \partial \Sigma_N$ and we estimate from above $|(u - u_N)(\tilde{w}_2)|$. Since $u_N(\tilde{w}_2) = 0$, we infer that $|(u - u_N)(\tilde{w}_2)| = |u(\tilde{w}_2)|$. In either cases, if we set $\tilde{y}$ as $\tilde{z}$ in the first case or $\tilde{w}_2$ in the second, then we observe that $|\tilde{y}| > er_1 \geq R_1$ and we have an estimate from above of $|u(\tilde{y})|$.

We may then conclude our proof in the following way. We have found a point $\tilde{y}$ such that $|\tilde{y}| > R_1$ and, for some β_i, $i = 0, \ldots, n$,

(5.11)
$$\frac{1}{2} \leq |u(\tilde{y})| \leq C^{1+\mathcal{B}_n}(2E)^{1-\Gamma_n}\varepsilon_1^{\Gamma_n}$$

where $C \geq 1$ and $2E \geq 1$ are constants depending on the a priori data only and

$$n \leq \tilde{C}\log\frac{2eR}{d}\left(\log\frac{2eR}{d} + \log\frac{ed}{\hat{d}}\right).$$

Furthermore, there are at most $j \leq \tilde{C}_1\log(2eR/d)$ of these β such that

$$c_1\frac{\log(6/5)}{\log(c_2\rho_0/\hat{d})} \leq \beta \leq 1 - c_1 + c_1\frac{\log(6/5)}{\log(c_2\rho_j/\hat{d})}$$

and they are never consecutive ones, and all the others satisfy $0 < a \leq \beta \leq b < 1$. Here a, b, c_1, c_2, $\tilde{C}$, and $\tilde{C}_1$ depend on the a priori data only.

We observe that, first of all, $(2E)^{1-\Gamma_n} \leq 2E$. Second, let us observe that $1 + \mathcal{B}_n \leq 2\sum_{i=0}^{n}b^i$, thus

$$C^{1+\mathcal{B}_n} \leq C^{2/(1-b)} = \tilde{C}_2.$$

Let us call $C = 2E\tilde{C}_2$ and let us assume, without loss of generality, that $C > e$.

Let ℓ be a positive integer such that $a^\ell \leq c_1(\log\frac{6}{5})/(\log(c_2\rho_0/\hat{d}))$. Let us observe that we may choose ℓ such that

$$\ell \leq \tilde{C}_3\left(\log\left(\log\frac{2eR}{\hat{d}}\right) + 1\right),$$

for some constant $\bar{C}_3$ depending on the a priori data only. Then

$$\varepsilon_1^{\Gamma_n} \leq \varepsilon_1^{a^{\bar{n}}}$$

where

$$(5.12) \qquad \bar{n} \leq \bar{C}_4 \log \frac{2eR}{d} \left(\log \frac{2eR}{d} + \log \frac{ed}{\hat{d}} + \log \left(\log \frac{2eR}{\hat{d}} \right) \right),$$

$\bar{C}_4$ still depending on the a priori data only. Thus, we obtain the crucial estimate

$$(5.13) \qquad \frac{1}{2} \leq C \varepsilon_1^{a^{\bar{n}}}.$$

Therefore, by straightforward computations, if we set $C_1 = \log(\log(2C))$ and $C_2 = \log(1/a)$, we obtain

$$\log \left(\log \frac{1}{\varepsilon_1} \right) \leq C_1 + C_2 \bar{n}.$$

Using (5.12), we have, for a constant C_3 depending on the a priori data only,

$$\log \left(\log \frac{1}{\varepsilon_1} \right) \leq C_3 \log \frac{2eR}{d} \left(\log \frac{2eR}{d} + \log \frac{ed}{\hat{d}} + \log \left(\log \frac{2eR}{\hat{d}} \right) \right).$$

We may conclude that

$$\log \left(\log \frac{1}{\varepsilon_1} \right) \leq C_4 \left(\log \frac{2eR}{\hat{d}} \right)^2.$$

Therefore, by straightforward computations, (4.3) and, in turn, (4.4), follow. $\qquad \square$

6. THE CASE OF POLYHEDRAL MULTIPLE OBSTACLES

We begin by investigating the relationships between two multiple polyhedral obstacles which are close in the Hausdorff distance.

Proposition 6.1. *Let Σ, $\Sigma' \in \mathcal{A}_{p,o}(h)$, $0 < h \leq h_0$. Then there exist positive constants $c_1 \leq 1$ and $\bar{c}_1 \leq c$, depending on the a priori data only, such that if $d_H(\Sigma, \Sigma') \leq c_1 h$, then the following holds.*
Let $x \in \partial\Sigma$ and, up to a rigid transformation, let

$$\partial\Sigma \cap B_{ch}(x) = \{y = (y_1, y_2, y_3) \in B_{ch}(x) \mid y_3 = \phi(y_1, y_2)\},$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant bounded by L and such that $\phi(x_1, x_2) = x_3$. Then there exists a Lipschitz function $\phi' : \mathbb{R}^2 \to \mathbb{R}$ with

Lipschitz constant bounded by L_1, *where* L_1 *depends on the a priori data and on* h_0 *only, such that, with respect to the same coordinate system, we have*

$$\partial \Sigma' \cap B_{\bar{c}_1 h}(x) = \{ y = (y_1, y_2, y_3) \in B_{\bar{c}_1 h}(x) \mid y_3 = \phi'(y_1, y_2) \}.$$

Proof. Recalling Proposition 2.4, let us assume that $c_1 = sC_1$, for some $0 < s \le c/4$. Therefore, we have $d_H(\partial\Sigma, \partial\Sigma') \le sh$.

Let us take any $x' \in \partial\Sigma'$ and any cell C' of $\partial\Sigma'$ such that $x' \in C' \subset \partial\Sigma'$ and $|x' - x| \le ch/4$. Let n' be the unit vector which is normal to the plane Π' containing the cell C' and such that $n' \cdot e_3 \ge 0$, where $e_3 = (0, 0, 1)$ in the coordinate system used for defining ϕ. By the properties of the cells of $\partial\Sigma'$, we can find x_1' satisfying, for some constants c_2, $0 < c_2 \le c/8$, and θ_2, $0 < \theta_2 < \pi/2$, depending on the a priori data only, $|x' - x_1'| = c_2 h$ and $B_{\sin(\theta_2)c_2 h}(x_1') \cap \Pi' \subset C'$.

Let θ', $0 \le \theta' \le \pi/2$, be the angle between n' and e_3. We can find two points in $\bar{B}_{\sin(\theta_2)c_2 h}(x_1') \cap \Pi'$ whose third coordinates differ by $2\sin(\theta')\sin(\theta_2)c_2 h$. On the other hand, on the projection of $\bar{B}_{\sin(\theta_2)c_2 h}(x_1') \cap \Pi'$ on $\{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_3 = 0\}$, we have that the oscillation of ϕ is bounded by $2L\cos(\theta')\sin(\theta_2)c_2 h$. Therefore, provided $\tan(\theta') \ge L_2$, $L_2 \ge L$ depending on the a priori data only, we can find $x_2' \in \bar{B}_{\sin(\theta_2)c_2 h}(x_1') \cap \Pi'$ such that $\{x_2' + tc_3 h e_3 \mid -1 \le t \le 1\} \cap \partial\Sigma$ is empty. Here c_3 is a positive constant depending on the a priori data only such that $c_3 \le c_2$. By the Lipschitz property of $\partial\Sigma$, we infer that for some constant $c_4 > 0$, depending on c_3 and L only, we have $\mathrm{dist}(x_2', \partial\Sigma) \ge c_4 h$. This is a contradiction if $s = \min\{c_4/2, c/4\}$.

Therefore, provided $c_1 = C_1 \min\{c_4/2, c/4\}$, we obtain that $\tan(\theta') \le L_2$ for any $x' \in C' \subset \partial\Sigma'$ and $|x' - x| \le ch/4$. Similar reasonings lead to the fact that, possibly taking a smaller c_1 and carefully choosing $\bar{c}_1$, for any $(y_1, y_2) \in \mathbb{R}^2$ such that $(y_1 - x_1)^2 + (y_2 - x_2)^2 \le (\bar{c}_1 h)^2$ there exists at most one y_3 such that $(y_1, y_2, y_3) \in \partial\Sigma' \cap B_{\bar{c}_1 h}(x)$. Otherwise, we would contradict the fact that, with respect to some coordinate system, $\partial\Sigma'$ is locally the graph of a Lipschitz function with constant L. Therefore $\partial\Sigma'$ is, in $B_{\bar{c}_1 h}(x)$, the graph of a Lipschitz function with Lipschitz constant bounded by L_2 with respect to the same coordinate system. We then extend this Lipschitz function all over $\mathbb{R}^2$ and the proof is concluded. $\square$

Proposition 6.2. *Let* Σ, Σ' *belong to* $\mathcal{A}_{p,o}(h)$, *for some* h, $0 < h \le h_0$, *and let* $d = d_H(\Sigma, \Sigma')$. *Then, there exist positive constants* $\kappa \le 1$, $\bar{\kappa} \le 1$, $\kappa_1 \le \bar{\kappa}$, K, K_1, *and* L_1, *depending on the a priori data only, such that if* $d \le \kappa h$, *then there exist* $x \in \partial\Sigma$ *and* $x' \in \partial\Sigma'$ *such that the following conditions are satisfied. Up to a rigid transformation,* $x = (0, 0, 0)$, $x' = (0, 0, a')$ *and*

$$\partial\Sigma \cap B_{\bar{\kappa}h} = \{ y = (y_1, y_2, y_3) \in B_{\bar{\kappa}h} \mid y_3 = \phi(y_1, y_2) \},$$

$$\partial\Sigma' \cap B_{\bar{\kappa}h} = \{ y = (y_1, y_2, y_3) \in B_{\bar{\kappa}h} \mid y_3 = \phi'(y_1, y_2) \},$$

where ϕ, $\phi' : \mathbb{R}^2 \to \mathbb{R}$ *are Lipschitz functions with Lipschitz constants bounded by* L_1 *and such that* $\phi(0) = 0$ *and* $\phi'(0) = a'$.

Furthermore, on $C = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq (\kappa_1 h)^2\}$ we have

$$\phi(y_1, y_2) = \ell_1 y_1 + \ell_2 y_2,$$
$$\phi'(y_1, y_2) = \ell'_1 y_1 + \ell'_2 y_2 + a', \quad \text{for any } (y_1, y_2) \in C,$$
$$S = \{(y_1, y_2, \phi(y_1, y_2)) \mid (y_1, y_2) \in C\} \subset \partial\Sigma,$$
$$S' = \{(y_1, y_2, \phi'(y_1, y_2)) \mid (y_1, y_2) \in C\} \subset \partial\Sigma',$$
$$(\ell_1 - \ell'_1)^2 + (\ell_2 - \ell'_2)^2 \leq (Kd/h)^2 \quad \text{and} \quad |a'| \leq Kd,$$
$$|\phi(y_1, y_2) - \phi'(y_1, y_2)| \geq K_1 d^3, \quad \text{for any } (y_1, y_2) \in C.$$

We postpone the rather technical proof of this proposition to the end of the section and we state the following immediate corollary.

Corollary 6.3. *Let Σ, Σ' belong to $\mathcal{A}_{p,o}(h)$, for some h, $0 < h \leq h_0$, and let $d = d_H(\Sigma, \Sigma')$. Then, there exist positive constants $\hat{\kappa} \leq \kappa$, $\hat{\kappa}_1$, $\bar{K}_1$, and $\bar{K}_2$ depending on the a priori data only, such that if $d \leq \hat{\kappa}h$, then up to a rigid transformation and up to swapping Σ with Σ', we have the following properties. First, $\hat{S} = \bar{B}_{\hat{\kappa}_1 h} \cap \{y_3 = 0\} = \partial\Sigma \cap (\hat{S} \times [\hat{\kappa}_1 h, \hat{\kappa}_1 h])$ and $\bar{B}_{\hat{\kappa}_1 h} \cap \{y_3 \leq 0\} = \Sigma \cap (\hat{S} \times [\hat{\kappa}_1 h, \hat{\kappa}_1 h])$. Then, we call $\hat{S}' = \{(y_1, y_2, \phi'(y_1, y_2)) \mid (y_1, y_2) \in \hat{S}\}$, where ϕ' is an affine function such that $\hat{S}' = \partial\Sigma' \cap (\hat{S} \times [\hat{\kappa}_1 h, \hat{\kappa}_1 h])$, $\hat{S}' \subset G$ and*

$$\bar{K}_1 d^3 \leq \phi'(y_1, y_2) \leq \bar{K}_2 d \leq \frac{\hat{\kappa}_1 h}{2}, \quad \text{for any } (y_1, y_2) \in \hat{S}.$$

Let us state the corresponding stability result for the near-field measurement.

Lemma 6.4. *Let Σ and Σ' belong to $\mathcal{A}_{p,o}(h)$, $0 < h \leq h_0$.*
Then there exists $\check{\varepsilon}_1(h)$, $0 < \check{\varepsilon}_1(h) \leq 1/(2e)$, depending on the a priori data and on h only, such that if (5.2) holds for some $\varepsilon_1 < \check{\varepsilon}_1(h)$, then

$$(6.1) \qquad d_H(\Sigma, \Sigma') \leq C_1 \exp(C_2(e/h)^{C_3}) \varepsilon_1^{C_4 h^{C_5}}.$$

Here $C_1, \ldots, C_5$ are positive constants depending on the a priori data only.

We observe that Theorem 4.2 is an immediate consequence of Lemma 6.4 and Lemma 3.2. Therefore in this section we concentrate our attention on proving Lemma 6.4. The first step in this proof is the following property of the solutions to (1.1) when Σ is a polyhedral multiple scatterer.

Proposition 6.5. *Let $\Sigma \in \mathcal{A}_p(h)$ for some h, $0 < h \leq h_0$. Let $x \in C \subset \partial\Sigma$ where the cell C is contained in the plane Π. Let us assume that for some constant s, $0 < s \leq 1$, we have that $B_{sh}(x) \cap \Pi \subset C$. Then, we obtain that*

$$(6.2) \qquad \|\nabla u \cdot v\|_{L^\infty(B_{sh}(x) \cap \Pi)} \geq \exp(-(e/h)^{A_1}),$$

where $A_1 > 0$ depends on the a priori data and on s only.

Proof. We argue as follows. Let $p = \|\nabla u \cdot \nu\|_{L^\infty(B_{sh}(x) \cap \Pi)}$. Then by the estimates derived in [25], already used in the proof of Lemma 3.4, and by the iterated use of the three-spheres inequality, using a construction completely analogous to the one developed in Step I and Step II of the previous section, we infer that, assuming without loss of generality that $p < 1$ and recalling that $h \leq h_0 \leq 1$,

$$\frac{1}{2} \leq |u(x_0)| \leq C p^{a^n}$$

where

$$n \leq \bar{C} \log \frac{\mathrm{e}}{h},$$

and $C \geq 1$, $\bar{C}$ and a, $0 < a < 1$, depend on the a priori data and on s only. The conclusion follows by straightforward computations. □

Proof of Lemma 3.2. By the proof of Theorem 4.1, we infer that, if either

$$\varepsilon_1 < \min \left\{ \frac{1}{2e}, \eta^{-1} \left(\left(\frac{\hat{\kappa} h}{2eR} \right)^{1/C} \right) \right\},$$

or

$$\varepsilon < \min \left\{ \hat{\varepsilon}_0, \eta^{-1} \left(\left(\frac{\hat{\kappa} h}{2eR} \right)^{2/C} \right) \right\},$$

then the conclusions of Corollary 6.3 hold true.

We observe that, by a reflection argument and by applying twice Lemma 3.3, we obtain that for suitable positive constants s, $0 < s \leq \hat{\kappa}_1/4$, and C, depending on the a priori data only,

(6.3) $$\|D^2 u\|_{L^\infty(\tilde{S}_1 \times [0, sh])} \leq \frac{C}{h^2},$$

where $S_1 = B_{sh} \cap \{x_3 = 0\}$.

Let us apply to S_1 and u the Proposition 6.5. Then we obtain that

(6.4) $$\|\nabla u \cdot e_3\|_{L^\infty(S_1)} \geq \exp(-(\mathrm{e}/h)^{A_1}).$$

Let $x = (x_1, x_2, 0) \in S_1$ and $x' = (x_1, x_2, \phi'(x_1, x_2))$ and let us compute $|u(x')|$. Let us assume, for the time being, that we also have $d \leq (s/\bar{K}_2)h$, thus, again by Corollary 6.3, $0 \leq \phi'(x_1, x_2) \leq sh$. Then, we have that there exists $x \in S_1$ such that $|\nabla u(x) \cdot e_3| \geq \exp(-(\mathrm{e}/h)^{A_1})$. We recall that $|x' - x| = \phi'(x_1, x_2)$ and by a Taylor expansion

$$|u(x')| \geq \exp(-(\mathrm{e}/h)^{A_1})|x' - x| - \frac{C}{2h^2}|x' - x|^2.$$

Provided $(C/h^2)\bar{K}_2 d \leq \exp(-(e/h)^{A_1})$ and recalling that $\bar{K}_1 d^3 \leq \phi'(y_1, y_2)$, we conclude that

$$|u(x')| \geq \frac{\bar{K}_1}{2} \exp(-(e/h)^{A_1}) d^3.$$

By reasonings completely analogous to the ones described in Step I and Step II of the previous section, we conclude that there exist positive constants C_1, $\tilde{C}_1$, α, $0 < \alpha < 1$, and a, $0 < a < 1$, depending on the a priori data only, such that either $d^\alpha \leq \varepsilon_1$ or, if $d^\alpha \geq \varepsilon_1$, then

$$|u'(x)| \leq C_1 d^\alpha \left(\frac{\varepsilon_1}{d^\alpha}\right)^{a^n}$$

where n can be bounded by

$$n \leq \tilde{C}_1 \log \frac{e}{h}.$$

Coupling these two last equations, we conclude that, provided $\varepsilon_1^{1/\alpha} \leq d \leq c_1 h^2 \exp(-(e/h)^{A_1}) \leq \hat{\kappa} h$, where $c_1 \leq 1/(C\bar{K}_2)$ depends on the a priori data only,

(6.5) $$\frac{\bar{K}_1}{2} \exp(-(e/h)^{A_1}) d^3 \leq C_1 d^\alpha \left(\frac{\varepsilon_1}{d^\alpha}\right)^{a^n}.$$

If we further assume that $\varepsilon_1 < \varepsilon_1^{1/2} \leq d^\alpha$, then (6.5) turns into

(6.6) $$d^{3-\alpha} \leq A_2 \exp((e/h)^{A_1}) \varepsilon_1^{(1/2)a^n}.$$

Then the proof may be concluded by easy computations. $\square$

Proof of Proposition 6.2. Up to swapping Σ with Σ', let us assume that there exists $x_1 \in \partial\Sigma$ such that $\text{dist}(x_1, \partial\Sigma') = d_H(\partial\Sigma, \partial\Sigma') = d''$. By Proposition 2.4, we know that $C_1 d'' \leq d \leq C_2 d''$.

For the time being, let us assume that $d \leq c_1 h$, c_1 as in Proposition 6.1, and by using Proposition 6.1, that $x_1 = (0, 0, 0)$ and that

$$\partial\Sigma \cap B_{\tilde{c}_1 h} = \{y = (y_1, y_2, y_3) \in B_{\tilde{c}_1 h} \mid y_3 = \phi(y_1, y_2)\},$$

$$\partial\Sigma' \cap B_{\tilde{c}_1 h} = \{y = (y_1, y_2, y_3) \in B_{\tilde{c}_1 h} \mid y_3 = \phi'(y_1, y_2)\},$$

where $\phi, \phi' : \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq (\tilde{c}_1 h)^2\} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants bounded by L_2 and such that $x_1 = (0, 0, 0)$ belongs to the graph of ϕ. Let $x_1' = (0, 0, \phi'(0, 0)) = (0, 0, b')$. We remark that

$$\frac{d}{C_2} \leq d'' \leq |x_1' - x_1| = |b'| \leq C_3 d'' \leq C_3 d/C_1,$$

where C_3 depends on the a priori data only.

For any $r > 0$ we set $B'_r = \{(y_1, y_2, y_3) \in B_r \mid y_3 = 0\}$. Let us now choose a positive constant $\bar{c}_2$, depending on the a priori data only, such that $\bar{c}_2 \leq \bar{c}_1$ and such that for any $(y_1, y_2, 0) \in B'_{\bar{c}_2 h}$, we have that both $(y_1, y_2, \phi(y_1, y_2))$ and $(y_1, y_2, \phi'(y_1, y_2))$ belong to $B_{\bar{c}_1 h}$. Furthermore, for a positive constant C_4, depending on the a priori data only, for any $(y_1, y_2, 0) \in B'_{\bar{c}_2 h}$ we have

$$|\phi(y_1, y_2) - \phi'(y_1, y_2)| \leq C_4 d.$$

Let us consider the projections of the sides of $\partial \Sigma \cap B_{\bar{c}_1 h}$ and $\partial \Sigma' \cap B_{\bar{c}_1 h}$ onto the planar region $B'_{\bar{c}_2 h}$. Let T and T' be the union of the sides and $\mathcal{V}$ and $\mathcal{V}'$ the set of vertices of these planar triangulations, respectively. We observe that two sides belonging to the same triangulation meet in a vertex with an angle greater than or equal to θ_3, $0 < \theta_3 < \pi/3$ depending on the a priori data only. We observe that there exists θ_4, $0 < \theta_4 \leq \theta_3$, and $\bar{c}_3$, $0 < \bar{c}_3 \leq \bar{c}_2/10$, depending on the a priori data only, such that we can find a two-dimensional cone in $B'_{\bar{c}_2 h}$ with vertex at the origin, radius $\bar{c}_3 h$ and angle θ_4 whose intersection with T is empty. Another cone with the same vertex, radius and angle but maybe with another bisecting line, has empty intersection with T'. Moreover we can choose $\bar{c}_2$, still depending on the a priori data only, such that both $\mathcal{V}$ and $\mathcal{V}'$ have at most one point in $B'_{\bar{c}_2 h}$.

Let us assume, for the time being, that for some constant t, $0 < t \leq 1$, there exists a two-dimensional cone in $B'_{\bar{c}_2 h}$ with vertex at the origin, radius $t\bar{c}_3 h$ and angle $\theta_4/4$ whose intersection with $T \cup T'$ is empty. Then we argue in the following way. On such a cone $\phi(y_1, y_2) = \ell_1 y_1 + \ell_2 y_2$, $\phi'(y_1, y_2) = \ell'_1 y_1 + \ell'_2 y_2 + b'$ and, with a reasoning which is analogous to the one used to prove Proposition 6.1, we have $(\ell_1 - \ell'_1)^2 + (\ell_2 - \ell'_2)^2 \leq (C_5 d/(th))^2$, C_5 depending on the a priori data only. Therefore, since $|b'| \geq d/C_2$, we may find t_1, $0 < t_1 \leq t$ depending on the a priori data and on t only, such that

$$|\phi(y_1, y_2) - \phi'(y_1, y_2)| \geq d/(2C_2)$$

for any (y_1, y_2) belonging to the cone with the same vertex, same bisecting line, same angle and radius $t_1 \bar{c}_3 h$. Therefore, if this is the case, the proof easily follows.

Let us investigate when we can guarantee that this two-dimensional cone exists for a suitable t depending on the a priori data only. We exploit the properties of the triangulations and we infer that the following cases allow us to construct the cone with the required properties in a rather straightforward way.

First, if either $\text{dist}(0, T) \geq t\bar{c}_3 h$ or $\text{dist}(0, T') \geq t\bar{c}_3 h$. Second, if either $0 \in T$ and $\text{dist}(0, \mathcal{V}) \geq t\bar{c}_3 h$ or $0 \in T'$ and $\text{dist}(0, \mathcal{V}') \geq t\bar{c}_3 h$. Third, if either $0 \in \mathcal{V}$ or $0 \in \mathcal{V}'$.

Let us concentrate our attention on the other cases. If $0 \notin T \cup T'$, then we proceed along the half plane on which $|\phi - \phi'|$ increases. If we are able to proceed until we are at a distance of $t\bar{c}_3 h$ from the origin without meeting any point either of T or of T', then we have a cone of amplitude angle $\pi/2$ which

does not intersect $T \cup T'$ and we can use the previous argument to conclude the proof. If we meet a point of T or of T', then we have found a point y in $B'_{t\bar{c}_3 h}$ such that $y \in T \cup T'$ and $|\phi(y) - \phi'(y)| \geq |b'|$.

Then, let us assume that we have $y \in B'_{t\bar{c}_3 h}$ such that $y \in T \cup T'$ and $|\phi(y) - \phi'(y)| \geq |b'|$. Clearly, y might be the origin as well. Let us assume, for the time being, that $y \notin T \cap T'$. Without loss of generality, we assume that $y \in T$ and $0 < \text{dist}(y, \mathcal{V}) \leq t\bar{c}_3 h$. We proceed in the direction of the side of T containing y along which $|\phi - \phi'|$ increases. Three situations may occur. First, we might reach the vertex V of $\mathcal{V}$, and if this is the case we use the previous argument to conclude the proof, recalling that $|V| \leq 2t\bar{c}_3 h$. Second, we might reach a point y_1 whose distance from the vertex V is greater than $t\bar{c}_3 h$ and again, since the distance of y_1 from 0 is at most $2t\bar{c}_3 h$, we use the previous argument to conclude the proof. Third, we might reach a point of T'. Therefore, without loss of generality, the only case which remains to be treated is the following. We have a point $y_1 \in T \cap T'$ such that $|y_1| \leq 2t\bar{c}_3 h$, $|y_1 - V| \leq t\bar{c}_3 h$, $|y_1 - V'| \leq t\bar{c}_3 h$ and $|\phi(y_1) - \phi'(y_1)| \geq |b'| \geq d/C_2$. Here V and V' are the vertices in $B'_{\bar{c}_2 h}$ of T and T' respectively.

Since we have assumed $\bar{c}_3 \leq \bar{c}_2/10$, then we have that $B'_{8t\bar{c}_3 h}(y_1) \subset B'_{\bar{c}_2 h}$. Hence for any w such that $3t\bar{c}_3 h \leq |w - y_1| \leq 7t\bar{c}_3 h$, we have $B'_{t\bar{c}_3 h}(w) \subset B'_{8t\bar{c}_3 h}(y_1) \subset B'_{\bar{c}_2 h}$ and $\text{dist}(V, B'_{t\bar{c}_3 h}(w)) \geq t\bar{c}_3 h$ and $\text{dist}(V', B'_{t\bar{c}_3 h}(w)) \geq t\bar{c}_3 h$.

Let us call $\bar{d} = \max\{|\phi(w) - \phi'(w)|, 3t\bar{c}_3 h \leq |w - y_1| \leq 7t\bar{c}_3 h\}$. Let us recall that $\bar{d} \leq C_4 d \leq C_4$. Without loss of generality, we may also assume that $C_4 d \leq 1$. Let us observe that in this case the proof of the proposition may be concluded by using similar arguments with respect to those already developed, provided we replace, in the last line of the proposition, d^3 with $\bar{d}$. Therefore, our aim is to prove that there exists a positive constant C_6, depending on the a priori data only, such that $\bar{d} \geq C_6 d^3$.

By contradiction, let us therefore assume that $\bar{d} \leq d^3$. We observe that, in $B'_{\bar{c}_2 h}$, T is formed by n half-lines t_i, $i = 1, \ldots, n$, with vertex V, and T' is formed by n' half-lines t'_j, $j = 1, \ldots, n'$, with vertex V'. We observe that n and n' are bounded by a constant N depending on θ_3 only. We order the lines in the counterclockwise sense and identify $n + 1$ and $n' + 1$, respectively, with 1. Therefore T divides $B'_{\bar{c}_2 h}$ into n cones D_i, $i = 1, \ldots, n$, and T' divides the same region into n' cones D'_j, $i = j, \ldots, n'$, where D_i is contained between t_i and t_{i+1} and D'_j is contained between t'_j and t'_{j+1}, respectively. On D_i the affine function ϕ is characterized by the vector ℓ_i and the constant b_i, whereas, respectively, on D'_j the affine function ϕ' is characterized by the vector ℓ'_j and the constant b'_j.

We construct the triangulation T_1, contained in T, in the following way. For a positive constant s, to be chosen later, we erase from T all the lines t_i such that the adjacent cones have corresponding vectors whose difference in norm is less than or equal to $sd^{3/2}/h$.

We observe that we can find a cone D' with vertex in y_1, radius $\bar{c}_3 h$ and angle θ_4 which has empty intersection with T'. If $t \leq \frac{1}{4}$, then such a cone intersects one of the cones D_i, let us say D_1, on a ball of radius $s_1 h$, s_1 depending on the a priori data only. We have the following cases.

First, the new triangulation T_1 is empty. We construct an affine function φ on $B'_{\bar{c}_2 h}$ in the following way. We set φ in such a way that $\varphi(w) = \phi(w)$ for any $w \in D_1$. We observe that, by construction, $|\varphi(w) - \phi(w)| \leq 2N s \bar{c}_2 d^{3/2}$ for any $w \in B'_{\bar{c}_2 h}$. Therefore, $|\varphi(w) - \phi'(w)| \leq 2N s \bar{c}_2 d^{3/2} + C_4 d$ for any $w \in B'_{\bar{c}_2 h}$ and $|\varphi(y_1) - \phi'(y_1)| \geq d/C_2 - 2N s \bar{c}_2 d^{3/2}$. Then we apply the previous reasonings to φ and ϕ' and the proof may be easily concluded provided s is small enough, namely for any s, $0 < s \leq s_0$, s_0 depending on the a priori data only.

The second case is the following. Let us assume that T_1 is composed exactly by two lines and the angle θ between these two lines, which we relabel as t_1 and t_2, is such that $|\theta - \pi| \leq \theta_4/4$. Let us call K_1 and K_2 the two cones determined by T_1. If $y_1 \in T_1$, then we can find a cone with vertex in y_1, angle $\theta_4/4$ and radius $\bar{c}_3 h$ whose intersection with T_1 and T' is empty. Such a cone is obtained as a subset of D' and we assume, without loss of generality, that K_1 contains it and D_1, as well. We construct an affine function φ on K_1 in the following way. We set φ in such a way that $\varphi(w) = \phi(w)$ for any $w \in D_1$. We observe that, by construction, $|\varphi(w) - \phi(w)| \leq 2N s \bar{c}_2 d^{3/2}$ for any $w \in K_1$. Therefore, $|\varphi(w) - \phi'(w)| \leq 2N s \bar{c}_2 d^{3/2} + C_4 d$ for any $w \in K_1$ and $|\varphi(y_1) - \phi'(y_1)| \geq d/C_2 - 2N s \bar{c}_2 d^{3/2}$. Then we apply the previous reasonings to φ and ϕ' and the proof may be easily concluded provided s is small enough, namely for any s, $0 < s \leq s_0$, s_0 depending on the a priori data only.

In the case when y_1 does not belong to T_1, we may assume, without loss of generality, that $y_1 \in K_1$. If D' is contained in K_1, then we proceed exactly as when $y_1 \in T_1$, with the same construction of φ. Otherwise, we consider the line passing through y_1 and V' and we consider its intersection with T_1. If all the intersection points (which may be none, one or two) lie, with respect to y_1, on the same side of the line of V', then we may reduce ourselves to the previous case in a rather straightforward way. Also, if there are two intersection points, then the two lines of T_1 intersect the line passing through y_1 and V' at such a small angle that the previous construction may be analogously performed. Therefore, without loss of generality, we may assume that the intersection consists of a single point y_2, with y_1 between y_2 and V'. We may also assume that y_2 belongs to the boundary of D' and to T_1. We construct an affine function φ on K_1 such that $\varphi(w) = \phi(w)$ on the halfline of T_1 containing y_2 (if y_2 is exactly equal to V, we may choose one of the two halflines arbitrarily) and on the cone D_i which is adjacent to it and contained in K_1. We have that $|\varphi(w) - \phi'(w)| \leq 2N s \bar{c}_2 d^{3/2} + C_4 d$ for any $w \in K_1$ and $|\varphi(y_1) - \phi'(y_1)| \geq d/C_2 - 2N s \bar{c}_2 d^{3/2}$. We proceed along the line of T' containing y_1 in the direction where $|\varphi - \phi'|$ increases. We have two possibilities. First, we reach V' and we apply previous reasonings. If, otherwise, we reach y_2, we notice that $|\phi(y_2) - \phi'(y_2)| \geq d/C_2 - 2N s \bar{c}_2 d^{3/2}$, thus we conclude as before, replacing y_1 with y_2.

The third case is the most difficult to treat. In the third case we have that T_1 contains at least two segments and two of them intersect with an angle θ such that $\theta_5 < \theta < \pi - \theta_5$, where θ_5 depends on the a priori data only and $0 < \theta_5 \leq \min\{\theta_3, \theta_4\}$. Let us take one of the segments of T_1 which divides, for simplicity, the cone D_1 from D_2 and let us take as x its intersection with either $\partial B_{4t\bar{c}_3 h}(y_1)$ or $\partial B_{6t\bar{c}_3 h}(y_1)$. If $B_r(x) \cap T'$ is empty, for some $0 < r \leq t\bar{c}_3 h$, then $B_r(x)$ is contained in D'_j, for some j and we have $r\|\ell_i - \ell'_j\| \leq C_7 \bar{d}$ for any $i = 1, 2$ and for some constant C_7 depending on the a priori data only. Therefore, $r\|\ell_1 - \ell_2\| \leq 2C_7 \bar{d}$ and, since by our assumptions we have $\|\ell_1 - \ell_2\| > sd^{3/2}/h$ and $\bar{d} \leq d^3$, we conclude that $r \leq 2C_7 d^{3/2} h/s$. Thus, there exists $x' \in T'$ such that $|x - x'| \leq 3C_7 d^{3/2} h/s$. We deduce that, provided $d \leq \bar{c}_4 h$, with $\bar{c}_4$ a positive constant depending on the a priori data only, we obtain that, to each segment in T_1, there corresponds a segment in T' such that the angle between the lines in which they are contained is bounded by $C_8 d^{3/2}/(ts)$. We may infer that $|V - V'| \leq C_9 d^{3/2} h/s$, where C_9 depends on the a priori data only, in particular it depends on θ_5.

Let us take $x \in \bar{B}_{7t\bar{c}_3 h}(y_1) \setminus B_{3t\bar{c}_3 h}(y_1)$. The segment connecting x to V, which we denote by $\overline{xV}$, belongs to D_i for some i. We can find x' satisfying the following conditions with the constant $r_0 = C_{10} d^{3/2} h/s$, C_{10} depending on the a priori data only. First, $|x - x'| \leq r_0$ and there exists $w \in \partial B_{4t\bar{c}_3 h}(y_1)$ such that $B_{r_0/2}(w) \subset D_i \cap D'_j$ where D'_j is the cone containing x'. We may infer that the angle between the lines containing, respectively, the segments $\overline{xV}$ and $\overline{x'V'}$ is bounded by $C_{11} d^{3/2}/(ts)$, C_{11} depending on the a priori data only. For any $y \in \overline{xV}$, we can find a corresponding $y' \in \overline{x'V'}$ such that $|y - y'| \leq C_{12} d^{3/2} h/s$. Since $B_{r_0/2}(w) \subset D_i \cap D'_j$ and $w \in \partial B_{4t\bar{c}_3 h}(y_1)$, with the usual reasoning, we conclude that $\|\ell_i - \ell'_j\| \leq sC_{13} d^{3/2}/h$. Using the Lipschitz properties of ϕ and ϕ', we may conclude that the following estimates hold

$$|\phi'(V) - \phi'(V')| + |\phi'(y) - \phi'(y')| \leq C_{14} L_1 d^{3/2} h/s,$$

$$|(\phi(y) - \phi(V)) - (\phi'(y') - \phi'(V'))| \leq C_{14}(L_1 d^{3/2} h/s + std^{3/2}).$$

Therefore,

$$|(\phi(y) - \phi'(y)) - (\phi(V) - \phi'(V))| \leq C_{14}(3L_1 d^{3/2} h/s + std^{3/2}) = Ad^{3/2}.$$

Here, as usual, C_{12}, C_{13} and C_{14} are constants depending on the a priori data only.

By taking $y = x$, we infer that

$$|\phi(V) - \phi'(V)| \leq Ad^{3/2} + \bar{d} \leq Ad^{3/2} + d^3.$$

Finally, for any $y \in B_{5t\bar{c}_3 h}(y_1)$ we may conclude that

$$|\phi(y) - \phi'(y)| \leq |(\phi(y) - \phi'(y)) \pm (\phi(V) - \phi'(V))| \leq 2Ad^{3/2} + d^3.$$

If we apply the previous estimate to $y = 0$, we conclude that

$$\frac{d}{C_2} \le 2Ad^{3/2} + d^3.$$

If d is small enough, this leads to a contradiction, thus the proof is concluded. $\quad\square$

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