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Linear independence of *L*-functions

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Abstract. We prove the linear independence of the L-functions, and of their derivatives of any order, in a large class $\mathscr{C}$ defined axiomatically. Such a class contains in particular the Selberg class as well as the Artin and the automorphic L-functions. Moreover, $\mathscr{C}$ is a multiplicative group, and hence our result also proves the linear independence of the inverses of such L-functions.

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1 Introduction

In a previous paper [4], we proved that the L-functions in the Selberg class $\mathscr{S}$ (see the survey papers [3] and [5] for definitions and basic properties) are linearly independent over $\mathbb{C}$ and, more generally, over the ring of the p-finite Dirichlet series (see [4] for definition). Although it is expected that the class $\mathscr{S}$ contains all reasonable global L-functions (see [3], [5]), this is far from being proved at present. As a consequence, there are several examples of classical L-functions, such as the Artin L-functions, which are not yet known to belong to $\mathscr{S}$. It is therefore natural to ask for a more general result, establishing the linear independence inside a suitably larger class of L-functions, unconditionally containing both $\mathscr{S}$ and several classical examples of L-functions.

In this paper we prove a general result on the linear independence of the derivatives of any order of the L-functions, and of their inverses, in a large class $\mathscr{C}$ defined below. Such a class does in fact contain the class $\mathscr{S}$ as well as several important L-functions, in particular the Artin L-functions and the automorphic L-functions. We remark that Nicolae [6] has recently obtained a weaker result of this type in the special case of Artin L-functions. However, his result follows from the arguments in [4]. In fact, the results in this paper are also based on the arguments in [4].

We define the class $\mathscr{C}$ of L-functions by the following axioms. A function $F(s)$ belongs to $\mathscr{C}$ if

i) $F(s)$ is an absolutely convergent Dirichlet series for σ sufficiently large, and has meromorphic continuation to $\mathbb{C}$ as a function of finite order;

ii) $F(s)$ satisfies a functional equation of type

$$\gamma(s)F(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s)$$

where $|\omega| = 1$, $\bar{f}(s) = \overline{f(\bar{s})}$ and the γ-factor $\gamma(s)$ has the form

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

with $Q > 0$, $0 \neq \lambda_j \in \mathbb{R}$ and $\mu_j \in \mathbb{C}$;

iii) for σ sufficiently large

$$F(s) = \prod_p F_p(s)$$

where

$$\log F_p(s) = \sum_{m=1}^{\infty} \frac{b(p^m)}{p^{ms}} \quad \text{with } b(p^m) \ll p^{m\theta} \text{ for some } \theta < \frac{1}{2}.$$

It is easy to check that $\mathscr{C}$ contains the Selberg class $\mathscr{S}$ and also several well known L-functions, such as the Artin L-functions and the GL(n) cuspidal automorphic L-functions (see [2] and [7] for the latter case, ensuring that axioms i)–iii) are satisfied). Moreover, while $\mathscr{S}$ is a multiplicative semigroup, our class $\mathscr{C}$ is a multiplicative group thanks to condition $\lambda_j \neq 0$ in axiom ii). Indeed, if $F \in \mathscr{C}$ satisfies the functional equation in ii), then $F(s)^{-1}$ satisfies

$$\tilde{\gamma}(s)F(s)^{-1} = \frac{1}{\omega}\bar{\tilde{\gamma}}(1-s)\bar{F}(1-s)^{-1}$$

with

$$\tilde{\gamma}(s) = \left(\frac{1}{Q}\right)^s \prod_{j=1}^{r} \Gamma(-\lambda_j s + \lambda_j + \bar{\mu}_j),$$

and it is easy to see that $F(s)^{-1}$ satisfies axioms i) and iii) as well. Therefore, $\mathscr{C}$ contains also the inverse of all the functions in $\mathscr{S}$, of the Artin L-functions and of the GL(n) cuspidal automorphic L-functions.

Denoting as usual by $F^{(k)}(s)$ the k-th derivative of the function $F(s)$ we have

Theorem. *Let $F_1(s), \dots, F_N(s)$ be distinct non-constant functions in $\mathscr{C}$ and K be a non-negative integer. Then the functions*

$$F_1^{(0)}(s), \dots, F_1^{(K)}(s), F_2^{(0)}(s), \dots, F_2^{(K)}(s), \dots, F_N^{(0)}(s), \dots, F_N^{(K)}(s)$$

are linearly independent over $\mathbb{C}$.

As a corollary of the Theorem we obtain for instance that the Artin L-functions, their inverses and the derivatives of any order are linearly independent, and the same applies to most L-functions. Moreover, the same arguments in the proof of the Theorem can be used to prove a more general result, where linear independence is over the ring of p-finite Dirichlet series, derivatives are replaced by suitable convolutions by additive functions, and the axioms of the class $\mathscr{C}$ are suitably relaxed.

The proof of the Theorem is based on two lemmas. Given an arithmetical function $f(n)$ and a non-negative integer k we write $f^{(k)}(n) = (-1)^k f(n) \log^k n$ for its arithmetic k-th derivative. Moreover, two multiplicative arithmetical functions $f(n)$ and $g(n)$ are called *equivalent* if $f(p^m) = g(p^m)$ for all integers $m \geq 1$ and all but finitely many primes p. Further, we denote by $e(n)$ the identity function, defined by $e(1) = 1$ and $e(n) = 0$ for $n \geq 2$. Our first lemma deals with the linear independence of the derivatives of non-equivalent multiplicative functions.

Lemma 1. *Let $f_1(n), \dots, f_N(n)$ be multiplicative functions such that $e(n), f_1(n), \dots, f_N(n)$ are pairwise non-equivalent, and let K be a non-negative integer. Then the functions*

$$f_1^{(0)}(n), \dots, f_1^{(K)}(n), f_2^{(0)}(n), \dots, f_2^{(K)}(n), \dots, f_N^{(0)}(n), \dots, f_N^{(K)}(n)$$

are linearly independent over $\mathbb{C}$.

Our second lemma proves the multiplicity one property for the class $\mathscr{C}$.

Lemma 2. *Let $F, G \in \mathscr{C}$ satisfy $F_p(s) = G_p(s)$ for all but finitely many primes p. Then $F(s) = G(s)$.*

We remark that the bound $b(p^m) \ll p^{m\theta}$ for some $\theta < \frac{1}{2}$ in axiom iii) is crucial for Lemma 2. In fact, Lemma 2 does not hold if condition $\theta < \frac{1}{2}$ is relaxed, as the following example shows. Let $P(s) = (1 - 2^{a-s})(1 - 2^{b-s})$ with $a, b \in \mathbb{R}$ and $a + b = 1$, and hence $b(2^m) \gg 2^{\max(a,b)m}$ in this case. Clearly, $P(s)$ satisfies $2^s P(s) = 2^{1-s}\bar{P}(1-s)$, therefore $P(s)F(s)$ belongs to $\mathscr{C}$ for any $F \in \mathscr{C}$. Thus Lemma 2 does not hold for $\mathscr{C}$ if condition $\theta < \frac{1}{2}$ is relaxed.

Since Lemma 2 shows that the coefficients of functions in $\mathscr{C}$ are pairwise non-equivalent multiplicative functions, the Theorem follows at once from Lemma 1.

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2 Proofs

In the proof of Lemma 1 we may clearly assume that $K \geq 1$, otherwise the result follows from Theorem 2 in [4]. We remark here that there are two misprints in the proof of Theorem 2 of [4]: on page 30, line -8, change "for $j = 2, \ldots, N$" to "for some $j \in \{2, \ldots, N\}$", and on page 31, line 4, change "and the $\tilde{c}_j(n)$ are non-identically vanishing" to "and some $\tilde{c}_j(n)$ is non-identically vanishing".

We prove Lemma 1 by contradiction. Assume that there exists an identically vanishing non-trivial linear combination of derivatives of arithmetical functions satisfying the properties in Lemma 1. That is, suppose there exist $f_1(n), \ldots, f_N(n)$ as in Lemma 1, an integer $K \geq 1$ and complex numbers c_{jk} not all zero such that for every $n \geq 1$

$$(1) \qquad \sum_{j=1}^{N} \sum_{k=0}^{K} c_{jk} f_j^{(k)}(n) = \sum_{j=1}^{N} \sum_{k=0}^{K} (-1)^k c_{jk} f_j(n) \log^k n = 0.$$

We assume that K is minimal over all such linear combinations and, with such a K, also that

$$v = |\{j : c_{jK} \neq 0\}|$$

is minimal.

Suppose that $v \geq 2$, assume without loss of generality that $c_{1K}, c_{2K} \neq 0$, and let $q_0 > 1$ be such that $f_1(q_0) \neq f_2(q_0)$. For every n coprime with q_0 we have

$$(2) \qquad \sum_{j=1}^{N} \sum_{k=0}^{K} c_{jk} f_j^{(k)}(q_0 n) = \sum_{j=1}^{N} \sum_{k=0}^{K} \sum_{l=0}^{k} (-1)^k \binom{k}{l} c_{jk} f_j(q_0) \log^{k-l} q_0 f_j(n) \log^l n = 0.$$

Multiplying (1) by $f_1(q_0)$ and then subtracting from (2) we get

$$(3) \qquad \sum_{j=1}^{N} \sum_{k=0}^{K} (-1)^k \tilde{c}_{jk} \tilde{f}_j(n) \log^k n = 0,$$

where $\tilde{c}_{jk}$ are suitable complex numbers with $\tilde{c}_{jK} = c_{jK}(f_j(q_0) - f_1(q_0))$ and

$$\tilde{f}_j(n) = \begin{cases} f_j(n) & \text{if } (n, q_0) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the functions $\tilde{f}_j(n)$ satisfy the properties required by Lemma 1, the fact that (3) holds with $\tilde{c}_{1K} = 0$ contradicts the minimality of v.

Therefore $v = 1$ and (1) takes the form

$$(4) \qquad cf_1(n)\log^K n + \sum_{j=1}^{N}\sum_{k=0}^{K-1}(-1)^k c_{jk} f_j(n)\log^k n = 0$$

with some $c \neq 0$. Let now $q_1 > 1$ be such that $f_1(q_1) \neq 0$. Arguing as before, using q_1 in place of q_0, from (4) we obtain

$$(5) \qquad \sum_{j=1}^{N}\sum_{k=0}^{K-1}(-1)^k c_{jk}^* f_j^*(n)\log^k n = 0,$$

where c_{jk}^* are suitable complex numbers with $c_{1K-1}^* = cKf_1(q_1)\log q_1$ and the functions $f_j^*(n)$ are defined analogously to the $\tilde{f}_j(n)$'s, with q_0 replaced by q_1. Since $c_{1K-1}^* \neq 0$, equation (5) contradicts the minimality of K, and Lemma 1 follows. □

In order to prove Lemma 2 we write the functional equation of $F \in \mathscr{C}$ in the form

$$\gamma^*(s)F(s) = \omega\overline{\gamma^*}(1-s)\bar{F}(1-s),$$

where the modified γ-factor $\gamma^*(s)$ is defined by

$$\gamma^*(s) = \frac{\prod_{\lambda_j>0}\Gamma(\lambda_j s + \mu_j)}{\prod_{\lambda_j<0}\Gamma(\lambda_j(1-s) + \bar{\mu}_j)}$$

(as usual, an empty product equals 1). Let

$$h(s) = \frac{F(s)}{G(s)} = \prod_{p\in\mathscr{P}_0}\frac{F_p(s)}{G_p(s)} \quad \text{and} \quad H(s) = \frac{\gamma_F^*(s)}{\gamma_G^*(s)}h(s),$$

where $\mathscr{P}_0$ is a finite set of primes and $\gamma_F^*(s), \gamma_G^*(s)$ are modified γ-factors of $F(s)$ and $G(s)$, respectively. By iii), every p-th Euler factor $F_p(s)$ and $G_p(s)$ is holomorphic and non-vanishing for $\sigma \geq \frac{1}{2}$, hence $h(s)$ is holomorphic and non-vanishing for $\sigma \geq \frac{1}{2}$ as well. Moreover, by the properties of the Γ function, the quotient $\gamma_F^*(s)/\gamma_G^*(s)$ is meromorphic with finitely many zeros and poles for $\sigma \geq \frac{1}{2}$, and hence the same property holds for $H(s)$ as well. Therefore $H(s)$ is meromorphic over $\mathbb{C}$ with finitely many zeros and poles by ii), and hence by i) there exists a rational function $R(s)$ such that $R(s)H(s)$ is an entire non-vanishing function of order at most 1. Thus by Hadamard's theory we get

$$(6) \qquad h(s) = e^{as+b}\frac{\gamma_G^*(s)}{R(s)\gamma_F^*(s)}$$

for some $a, b \in \mathbb{C}$.

Now we use the following classical result of Bohr [1]: if $f(t)$ is an almost periodic function satisfying $|f(t)| \geq k > 0$, then $\arg f(t) = \lambda t + \phi(t)$ with $\lambda \in \mathbb{R}$ and $\phi(t)$ almost periodic. Let σ be sufficiently large. Then $h(s)$ is an absolutely convergent Dirichlet series

$$h(\sigma + it) = \sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma+it}}$$

with $c(1) = 1$ and $c(n) \ll n^A$ for some constant A. Therefore $h(s)$ is almost periodic in t and satisfies the hypothesis of Bohr's theorem. Applying Stirling's formula to the right hand side of (6) with a sufficiently large fixed $\sigma = \sigma_0$ we obtain

$$h(\sigma_0 + it) = ce^{\alpha t} t^{\beta} e^{i\gamma t \log t} e^{i\delta t} \left(1 + O\left(\frac{1}{t}\right)\right) \quad t \to +\infty$$

with $c \in \mathbb{C}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By almost periodicity we have that $\alpha = \beta = 0$ and by Bohr's theorem we deduce that $\gamma = 0$ as well, hence

$$e^{-i\delta t} h(\sigma_0 + it) = c + o(1) \quad t \to +\infty. \tag{7}$$

By almost periodicity, the right hand side of (7) must be constant, thus for $s = \sigma_0 + it$ we have

$$h(s) = e^{ds+e}$$

for some $d, e \in \mathbb{C}$. By analytic continuation and by the uniqueness principle for generalized Dirichlet series we deduce that $d = 0$, and hence $h(s) = 1$ since $c(1) = 1$. □

We remark that, once (6) is established, there is a more direct proof of Lemma 2 in the case where the Euler factors of functions in $\mathscr{C}$ are of polynomial type, that is

$$F_p(s) = \prod_{j=1}^{k} \left(1 - \frac{e^{i\theta_j(p)}}{p^s}\right)^{-1} \quad \theta_j(p) \in \mathbb{C}.$$

This is the case, for instance, of automorphic L-functions. In fact, in this case the zeros and poles of $h(s)$ consist of finitely many "vertical progressions" $i\frac{\theta_j(p)}{\log p} + i\frac{2\pi}{\log p}\mathbb{Z}$. The zeros and poles of the RHS of (6) consist of finitely many zeros and poles of $R(s)^{-1}$ as well as finitely many "horizontal semi-progressions", caused by the poles of the Γ function. Therefore, (6) implies that each side is identically one, and Lemma 2 follows. We wish to thank the referee for this remark.

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