Growth of Sobolev norms for time dependent periodic Schrödinger equations with sublinear dispersion

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Abstract: In this paper we consider Schrödinger equations with sublinear dispersion relation on the one-dimensional torus $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. More precisely, we deal with equations of the form $\partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u]$ where $\mathcal{V}(\omega t)$ is a quasi-periodic in time, self-adjoint pseudo-differential operator of the form $\mathcal{V}(\omega t) = V(\omega t, x)|D|^M + \mathcal{W}(\omega t)$, $0 < M \leq 1$, $|D| := \sqrt{-\partial_{xx}}$, V is a smooth, quasi-periodic in time function and $\mathcal{W}$ is a quasi-periodic time-dependent pseudo-differential operator of order strictly smaller than M. Under suitable assumptions on V and $\mathcal{W}$, we prove that if ω satisfies some non-resonance conditions, the solutions of the Schrödinger equation $\partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u]$ grow at most as t^η, $t \to +\infty$ for any $\eta > 0$. The proof is based on a reduction to constant coefficients up to smoothing remainders of the vector field $\mathrm{i}\mathcal{V}(\omega t)$ which uses Egorov type theorems and pseudo-differential calculus. The *homological equations* arising in the reduction procedure involve both time and space derivatives, since the dispersion relation is sublinear. Such equations can be solved by imposing some Melnikov non-resonance conditions on the frequency vector ω.

Keywords: Growth of Sobolev norms, linear Schrödinger equations, pseudo-differential operators.

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Contents

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1 Introduction and main result

In this paper we consider linear quasi-periodic in time Schrödinger-type equations with sublinear dispersion relation of the form

$$\partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u] \,, \quad x \in \mathbb{T} \tag{1.1}$$

where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ is the one-dimensional torus, $\omega \in \Omega \subset \mathbb{R}^\nu$, $\mathcal{V}(\varphi)$, $\varphi \in \mathbb{T}^\nu$ is a L^2 self-adjoint, pseudo-differential Schrödinger operator of the form

$$\mathcal{V}(\varphi) := V(\varphi, x)|D|^M + \mathcal{W}(\varphi) \,, \quad |D| := \sqrt{-\partial_{xx}} \,, \quad 0 < M \le 1 \,. \tag{1.2}$$

We assume that the set of parameters $\Omega \subseteq \mathbb{R}^\nu$ is a bounded domain of $\mathbb{R}^\nu$. Moreover V is a real valued $\mathcal{C}^\infty$ function defined on $\mathbb{T}^\nu \times \mathbb{T}$ satisfying $\inf_{(\varphi,x)\in\mathbb{T}^\nu\times\mathbb{T}} V(\varphi, x) > 0$, in the case $0 < M < 1$ and *close* to 1 in the case $M = 1$. The operator $\mathcal{W}(\varphi)$ is a time-dependent pseudo differential operator of order strictly smaller than M. Our main goal is to show that given $t_0 \in \mathbb{R}$, $s \ge 0$, $u_0 \in H^s(\mathbb{T})$, for *most values* of the parameters $\omega \in \Omega$, the Cauchy problem

$$\begin{cases} \partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u] \\ u(t_0, x) = u_0(x) \end{cases} \tag{1.3}$$

admits a unique solution $u(t)$ satisfying, for any $\eta > 0$, the bound

$$\|u(t)\|_{H^s} \le C(s, \eta)(1 + |t - t_0|)^\eta \|u_0\|_{H^s} \,, \quad \forall t \in \mathbb{R} \tag{1.4}$$

for some constant $C(s, \eta) > 0$. Here, $H^s(\mathbb{T})$ denotes the standard Sobolev space on the 1-dimensional torus $\mathbb{T}$ equipped with the norm $\| \cdot \|_{H^s}$.
The problem of estimating the high Sobolev norms for solutions of linear Schrödinger-type equations of the form $\partial_t u = \mathrm{i}(H + V(t))u$, in the case when H satisfies the so called *spectral gap condition*, has been extensively investigated. Such a condition states that the spectrum of the operator H can be enclosed in disjont clusters $(\sigma_j)_{j\ge0}$ such that the distance between σ_j and σ_{j+1} tends to $+\infty$ for $j \to +\infty$.
For the Schrödinger operator $\mathcal{V}(t) = -\Delta + V(t, x)$ on the d-dimensional torus $\mathbb{T}^d$, the growth $\sim t^\eta$ of the $\| \cdot \|_{H^s}$ norm of the solutions of $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$ has been proved by Bourgain in [10] for smooth quasi-periodic in time potentials and in [11] for smooth and bounded time dependent potentials. In the case where the potential V is analytic and quasi-periodic in time, Bourgain [10] proved also that $\|u(t)\|_{H^s}$ grows like a power of $\log(t)$. Moreover, this bound is optimal, in the sense that he constructed an example for which $\|u(t)\|_{H^s}$ is bounded from below by a power of $\log(t)$. The result obtained in [11] has been extended by Delort [12] for Schrödinger operators on Zoll manifolds. Furthermore, the logarithmic growth of $\|u(t)\|_{H^s}$ proved in [10] has been extended by Wang [23] in dimension 1, for any real analytic and bounded potential.
All the aforementioned results concern Schrödinger equations with bounded perturbations. The first result in which the growth of $\|u(t)\|_{H^s}$ is established in the case of unbounded perturbations is due to Maspero-Robert [19]. More precisely, they prove the growth $\sim t^\eta$ of $\|u(t)\|_{H^s}$, for Schrödinger equations of the form $\mathrm{i}\partial_t u = L(t)[u]$ where $L(t) = H + P(t)$, H is a time-independent operator of order $\mu + 1$ satisfying the *spectral gap condition* and $P(t)$ is an operator of order $\nu \le \mu/(\mu + 1)$ (see Theorem 1.8 in [19]). This last paper has been generalized, independently and at the same time, by Bambusi-Grebert-Maspero-Robert [7] in the case in which the order of $P(t)$ is strictly smaller than the one of H and in [21] for periodic 1-dimensional Schrödinger equations where the order of $P(t)$ is the same as the order of H.

In [7], the authors deal with some Schrödinger equations in which the spectral gap condition is violated. In particular, they study the non-resonant harmonic oscillator on $\mathbb{R}^d$, in which the gaps are dense and the relativistic Schrödinger equation on Zoll manifolds, in which the distance between the gaps are constants. In all the papers mentioned above, the dispersion relation is at least linear. The purpose of the present paper is to provide some results concerning the growth of Sobolev norms of the solutions of some Schrödinger equations with sublinear dispersion, in the case in which the order of H is the same as the order of $P(t)$.
We also mention that in the case of quasi-periodic systems with *small* perturbations, i.e. $\mathrm{i}\partial_t u = L(\omega t)[u]$, $L(\omega t) = H + \varepsilon P(\omega t)$ it is often possible to prove that $\|u(t)\|_{H^s}$ is uniformly bounded in time for ε small enough and for a *large* set of frequencies ω. The general strategy to deal with these quasi-periodic systems is called *reducibility*. It consists in costructing, for most values of the frequencies ω and for ε small enough, a bounded quasi-periodic change of variables $\Phi(\omega t)$ which transforms the equation $\mathrm{i}\partial_t u = L(\omega t)u$ into a time independent system $\mathrm{i}\partial_t v = \mathcal{D}v$ whose solution preserves the Sobolev norms $\|v(t)\|_{H^s}$. We mention the results of Eliasson-Kuksin [13] which proved the reducibility of the Schrödinger equation on $\mathbb{T}^d$ with a small, quasi-periodic in time analytic potential and Grebert-Paturel [18] which proved the reducibility of the quantum harmonic oscillator on $\mathbb{R}^d$. Concerning KAM-reducibility with unbounded perturbations, we mention Bambusi [4], [5] for the reducibility of the quantum harmonic oscillator with unbounded perturbations (see also [6] in any dimension), [1], [2], [17] for fully non-linear KdV-type equations, [14], [15] for fully-nonlinear Schrödinger equations, [8], [9] for the water waves system and [20] for the Kirchhoff equation. Note that in [1], [2], [17], [8], [9], [20] the reducibility of the linearized equations is obtained as a consequence of the KAM theorems proved for the corresponding nonlinear equations.
In the case of sublinear growth of the eigenvalues, the first KAM-reducibility result is proved in [3] for the pure gravity water waves equations and the technique has been extended in [22] to deal with a class of linear wave equations on $\mathbb{T}^d$ with smoothing quasi-periodic in time perturbations.
We now state in a precise way the main results of this paper. First, we introduce some notations. For any function $u \in L^2(\mathbb{T})$, we introduce its Fourier coefficients

$$\widehat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{T}} u(x) e^{-\mathrm{i}x\xi}\, dx\,, \qquad \forall \xi \in \mathbb{Z}\,. \tag{1.5}$$

For any $s \geq 0$, we introduce the Sobolev space of complex valued functions $H^s \equiv H^s(\mathbb{T})$, as

$$H^s := \Big\{ u \in L^2(\mathbb{T}) : \|u\|_{H^s}^2 := \sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 < +\infty \Big\}\,, \quad \langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}\,. \tag{1.6}$$

Given two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, we denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators from X to Y equipped with the usual operator norm $\|\cdot\|_{\mathcal{B}(X,Y)}$. If $X = Y$, we simply write $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$.
Given a linear operator $\mathcal{R} \in \mathcal{B}(L^2(\mathbb{T}))$, we denote by $\mathcal{R}^*$ the adjoint operator of $\mathcal{R}$ with respect to the standard L^2 inner product

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{T}} u(x) \overline{v(x)}\, dx\,, \quad \forall u, v \in L^2(\mathbb{T})\,. \tag{1.7}$$

We say that the operator $\mathcal{R}$ is self-adjoint if $\mathcal{R} = \mathcal{R}^*$. For any domain $\Omega \subset \mathbb{R}^d$, we also denote by $\mathcal{C}_b^\infty(\Omega)$ the space of the $\mathcal{C}^\infty$ functions on Ω with all the derivatives bounded. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}^\nu$ we define its lenght by $|\alpha| := \alpha_1 + \ldots \alpha_\nu$ and $\partial_\varphi^\alpha = \partial_{\varphi_1}^{\alpha_1} \ldots \partial_{\varphi_\nu}^{\alpha_\nu}$.
Since the equation we deal with is a Hamiltonian PDE, we briefly describe the Hamiltonian formalism. We define the symplectic form $\Omega : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \mathbb{R}$ by

$$\Omega[u_1, u_2] := \mathrm{i} \int_{\mathbb{T}} (u_1 \bar{u}_2 - \bar{u}_1 u_2)\, dx\,, \quad \forall u_1, u_2 \in L^2(\mathbb{T})\,. \tag{1.8}$$

Given a family of linear operators $\mathcal{R} : \mathbb{T}^\nu \to \mathcal{B}(L^2)$ such that $\mathcal{R}(\varphi) = \mathcal{R}(\varphi)^*$ for any $\varphi \in \mathbb{T}^\nu$, we define the φ-dependent quadratic Hamiltonian associated to $\mathcal{R}$ as

$$\mathcal{H}(\varphi, u) := \langle \mathcal{R}(\varphi)[u]\,,\, u \rangle_{L^2_x} = \int_{\mathbb{T}} \mathcal{R}(\varphi)[u]\, \overline{u}\, dx\,, \qquad \forall u \in L^2(\mathbb{T})\,.$$

The Hamiltonian vector field associated to the Hamiltonian $\mathcal{H}$ is defined by

$$X_{\mathcal{H}}(\varphi, u) := \mathrm{i}\nabla_{\overline{u}}\mathcal{H}(\varphi, u) = \mathrm{i}\mathcal{R}(\varphi)[u] \tag{1.9}$$

where the gradient $\nabla_{\overline{u}}$ stands for

$$\nabla_{\overline{u}} := \frac{1}{\sqrt{2}}(\nabla_v + \mathrm{i}\nabla_\psi)\,, \quad v = \mathrm{Re}(u)\,, \quad \psi := \mathrm{Im}(u)\,.$$

We say that $\Phi : \mathbb{T}^\nu \to \mathcal{B}(L^2(\mathbb{T}))$ is symplectic if and only if

$$\Omega\Big[\Phi(\varphi)[u_1], \Phi(\varphi)[u_2]\Big] = \Omega[u_1, u_2]\,, \qquad \forall u_1, u_2 \in L^2(\mathbb{T})\,, \quad \forall \varphi \in \mathbb{T}^\nu\,.$$

We recall that if $X_{\mathcal{H}}$ is a Hamiltonian vector field, then the flow map generated by $X_{\mathcal{H}}$ is symplectic.

Let us consider a vector field $X : \mathbb{T}^\nu \to \mathcal{B}(L^2(\mathbb{T}))$ and a differentiable family of invertible maps $\Phi : \mathbb{T}^\nu \to \mathcal{B}(L^2(\mathbb{T}))$. Given $\omega \in \mathbb{R}^\nu$, under the change of variables $u = \Phi(\omega t)[v]$, the equation $\partial_t u = X(\omega t)[u]$ transforms into the equation $\partial_t v = X_+(\omega t)[v]$ where the *quasi-periodic push-forward* $X_+(\varphi)$ of the vector field $X(\varphi)$ is defined by

$$X_+(\varphi) := \Phi_{\omega *}X(\varphi) := \Phi(\varphi)^{-1}\Big(X(\varphi)\Phi(\varphi) - \omega\cdot\partial_\varphi\Phi(\varphi)\Big)\,, \quad \varphi \in \mathbb{T}^\nu\,. \tag{1.10}$$

If Φ is symplectic and X is a Hamiltonian vector field, then the push-forward $X_+ = \Phi_{\omega *}X$ is still a Hamiltonian vector field.
In the next two definitions, we also define the class of pseudo differential operators on $\mathbb{T}$ that we shall use along the paper.

Definition 1.1 (The symbol class S^m). *Let $m \in \mathbb{R}$. We say that a $\mathcal{C}^\infty$ function $a : \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ belongs to the symbol class S^m if and only if for any multiindex $\alpha \in \mathbb{N}^\nu$, for any $k, n \in \mathbb{N}$, there exists a constant $C_{\alpha,n,k} > 0$ such that*

$$|\partial_\varphi^\alpha \partial_x^k \partial_\xi^n a(\varphi, x, \xi)| \le C_{\alpha,n,k}\langle\xi\rangle^{m-n}\,, \quad \forall(\varphi, x, \xi) \in \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R}\,. \tag{1.11}$$

We define the class of smoothing symbols $S^{-\infty} := \cap_{m\in\mathbb{R}} S^m$.

Definition 1.2. (the class of operators OPS^m) *Let $m \in \mathbb{R}$ and $a \in S^m$. We define the φ-dependent linear operator $A(\varphi) = \mathrm{Op}\big(a(\varphi, x, \xi)\big) = a(\varphi, x, D)$ as*

$$A(\varphi)[u](x) := \sum_{\xi\in\mathbb{Z}} a(\varphi, x, \xi)\widehat{u}(\xi)e^{\mathrm{i}x\xi}\,, \qquad \forall u \in \mathcal{C}^\infty(\mathbb{T})\,.$$

We say that the operator A is in the class OPS^m.
We define the class of smoothing operators $OPS^{-\infty} := \cap_{m\in\mathbb{R}} OPS^m$.

Now we are ready to state the main results of this paper.

- **(H1)** The operator $\mathcal{V}(\varphi) = V(\varphi, x)|D|^M + \mathcal{W}(\varphi)$ in (1.2) is L^2 self-adjoint for any $\varphi \in \mathbb{T}^\nu$.
- **(H2)** The operator $\mathcal{W}(\varphi)$ is a φ-dependent pseudo-differential operator $\mathcal{W}(\varphi) = \mathrm{Op}(w(\varphi, x, \xi))$, with symbol $w \in S^{M-\mathfrak{e}}$ for some $\mathfrak{e} > 0$.

We also need a third hypothesis. For this last assumption, we need to distinguish between the cases $0 < M < 1$ and $M = 1$. In the case $0 < M < 1$ we assume

$(\mathbf{H3})_{\mathbf{M<1}}$ The function $V(\varphi, x)$ in (1.2) is in $\mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$, strictly positive and bounded from below, i.e. $\delta := \inf_{(t,x)\in\mathbb{T}^\nu\times\mathbb{T}} V(\varphi, x) > 0$.

In the case $M = 1$ we assume

(H3)$_{\mathbf{M=1}}$ The function $V(\varphi, x)$ in (1.2) has the form $V(\varphi, x) = 1 + \varepsilon P(\varphi, x)$ where $\varepsilon \in (0,1)$ and $P \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$.

Given $\tau > \nu - 1$ and $\gamma \in (0,1)$, we also introduce the set of diophantine frequencies

$$DC(\gamma, \tau) := \Big\{ \omega \in \Omega : |\omega \cdot \ell| \geq \frac{\gamma}{|\ell|^\tau}, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\} \Big\}. \tag{1.12}$$

It is well known that the Lebesgue measure of $\Omega \setminus DC(\gamma, \tau)$ is of order $O(\gamma)$. The main results of this paper are the following

Theorem 1.3 (**Growth of Sobolev norms, the case** $0 < M < 1$)**.** *Assume the hypotheses* (**H1**), (**H2**), (**H3**)$_{\mathbf{M<1}}$. *Let* $s > 0$, $\gamma \in (0,1)$, $\tau > \nu$, $t_0 \in \mathbb{R}$, $u_0 \in H^s(\mathbb{T})$, $\omega \in DC(\gamma, \tau)$. *Then there exists a unique global solution* $u \in \mathcal{C}^0(\mathbb{R}, H^s(\mathbb{T}))$ *of the Cauchy problem* (1.3) *which satisfies the bound* (1.4).

Theorem 1.4 (**Growth of Sobolev norms, the case** $M = 1$)**.** *Assume the hypotheses* (**H1**), (**H2**), (**H3**)$_{\mathbf{M=1}}$. *Let* $s > 0$, $u_0 \in H^s(\mathbb{T})$, $t_0 \in \mathbb{R}$, $\gamma \in (0,1)$, $\tau > \nu$. *Then there exists a constant* $\delta_0 \in (0,1)$ *such that if* $\varepsilon\gamma^{-1} \leq \delta_0$ *the following holds: there exists a Borel set* $\Omega_{\gamma,\tau} \subseteq \Omega$ *such that for any* $\omega \in \Omega_{\gamma,\tau}$ *there exists a unique global solution* $u \in \mathcal{C}^0(\mathbb{R}, H^s(\mathbb{T}))$ *of the Cauchy problem* (1.3) *which satisfies the bound* (1.4).

Remark 1.5. *The explicit expression of the set* $\Omega_{\gamma,\tau}$ *in Theorem 1.4 is provided in* (4.3), (4.28). *It is obtained by imposing some first order Melnikov conditions on the frequency* $\omega \in \Omega \subset \mathbb{R}^\nu$. *Arguing as in the proof of Corollary 4.2 in [16], it can be proved that the Lebesgue measure of* $\Omega \setminus \Omega_{\gamma,\tau}$ *is of order* $O(\gamma)$.

The two theorems stated above will be deduced by the following two *normal form* theorems stated below.

Theorem 1.6 (**Normal-form theorem, the case** $0 < M < 1$)**.** *Assume the hypotheses* (**H1**), (**H2**), (**H3**)$_{\mathbf{M<1}}$ *and let* $\omega \in DC(\gamma, \tau)$. *For any* $K > 0$ *there exists a* φ*-dependent family of invertible maps* $\mathcal{T}_K(\varphi) \in \mathcal{B}(H^s)$, $s \geq 0$ *such that the following holds: the vector field* $\mathrm{i}\mathcal{V}(\varphi)$ *is transformed, by the map* $\mathcal{T}_K(\varphi)$, *into the vector field*

$$\mathrm{i}\mathcal{V}_K(\varphi) := (\mathcal{T}_K)_{\omega *}(\mathrm{i}\mathcal{V})(\varphi) = \mathrm{i}\Big(\lambda_K(D) + \mathcal{R}_K(\varphi)\Big) \tag{1.13}$$

where $\lambda_K(D) := \mathrm{Op}(\lambda_K(\xi)) \in OPS^M$ *is a time-independent Fourier multiplier with real symbol and* $\mathcal{R}_K \in OPS^{-K}$.

Theorem 1.7 (**Normal-form theorem, the case** $M = 1$)**.** *Assume* (**H1**), (**H2**), (**H3**)$_{\mathbf{M=1}}$ *and let* $\gamma \in (0,1)$. *Then if* $\varepsilon\gamma^{-1} \leq \delta$, *for any* $\omega \in \Omega_{\gamma,\tau}$ *(where the constant* δ *and the set* $\Omega_{\gamma,\tau}$ *are given in Theorem 1.4), the following holds. For any* $K > 0$ *there exists a* φ*-dependent family of invertible maps* $\mathcal{T}_K(\varphi) \in \mathcal{B}(H^s)$, $s \geq 0$ *such that the push-forward of the Hamiltonian vector field* $\mathrm{i}\mathcal{V}(\varphi)$ *by means of the map* $\mathcal{T}_K(\varphi)$ *has the form*

$$\mathrm{i}\mathcal{V}_K(\varphi) := (\mathcal{T}_K)_{\omega *}(\mathrm{i}\mathcal{V})(\varphi) = \mathrm{i}\Big(\lambda_K(D) + \mathcal{R}_K(\varphi)\Big) \tag{1.14}$$

where $\lambda_K(D) := \mathrm{Op}(\lambda_K(\xi)) \in OPS^1$ *is a time-independent Fourier multiplier with real symbol and* $\mathcal{R}_K \in OPS^{-K}$.

In the remaining part of the introduction, we shall explain the main ideas needed to prove the theorems stated above.

1.1 Ideas of the proof

The proof of Theorems 1.3, 1.4 is deduced in a straightforward way by Theorems 1.6, 1.7. Indeed, once we transform the equation $\partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u]$ into another one which is an arbitrarily regularizing perturbation of a constant coefficients equation, fixing the number of regularization steps $K \simeq s$, we get immediately that the Sobolev norm $\|u(t)\|_{H^s}$ grows linearly in t. Then in order to get the desired estimate (1.4), it is enough to apply the classical Riesz-Thorin interpolation Theorem.

The proof of Theorems 1.6, 1.7 is based on a *normal form* procedure, which transforms the vector field $\mathrm{i}\mathcal{V}(\varphi)$ into another one which is an arbitrarily regularizing perturbation of a *space diagonal* vector field. Such a normal form method is based on symbolic calculus and Egorov type Theorems (see Propositions 2.13-2.14) and it is performed by constructing iteratively changes of variables which reduce to constant coefficients order by order the vector field $\mathrm{i}\mathcal{V}(\varphi)$. The procedure is different in the case $0 < M < 1$ and $M = 1$. The intuitive reason is the following: In the equation $\partial_t u = \mathrm{i}\mathcal{V}(t)[u]$ we have the interaction between the time differential operator ∂_t and the space pseudo-differential operator $|D|^M$. If $M < 1$, the effect of the operator ∂_t is stronger than the one of $|D|^M$, hence in the *homological equations* which arise in our reduction procedure at any order, we remove first the dependence on the time variable, by requiring that the frequency ω is diophantine and then the dependence on the space variable. In the case $M = 1$, we deal with the operators ∂_t and $|D|$ which are both of order 1. Hence the homological equations arising along the reduction procedure involve both time and space variable and they are solved by imposing *first order Melnikov conditions*, see (4.3), (4.28). We also remark that in [21], it is studied the case where $M > 1$, in which the effect of the operator $|D|^M$ is much stronger than the effect of the time differential operator ∂_t and we do not need any parameter to perform the reduction procedure.
In the following we shall explain more in detail these two reduction procedures. It is convenient for the sequel to introduce the following notation. Given $A = \mathrm{Op}(a), B = \mathrm{Op}(b) \in S^m$ and $m' < m$, we write

$$\begin{aligned} & A = B + OPS^{m'} \quad \text{if} \quad A - B \in OPS^{m'}, \\ & a = b + S^{m'} \quad \text{if} \quad a - b \in S^{m'}. \end{aligned} \tag{1.15}$$

In particular if $A - B$ is a smoothing operator, we write

$$A = B + OPS^{-\infty} \quad \text{and} \quad a = b + S^{-\infty}. \tag{1.16}$$

1.1.1 Normal form in the case $M < 1$

The normal form procedure for the vector field $\mathrm{i}\mathcal{V}(\omega t)$ in the case $M < 1$ is developed in Section 3.

1. *Reduction of the highest order term.* The first step is to reduce to constant coefficient the highest order term $\mathrm{i}V(\omega t, x)|D|^M$ of the vector field $\mathrm{i}\mathcal{V}(\omega t)$. This is done in Section 3.1. First, in Section 3.1.1, we remove the time-dependence from the operator $V(\omega t, x)|D|^M$ by conjugating the vector field $\mathrm{i}\mathcal{V}(\omega t)$ by means of the flow map generated by a Hamiltonian vector field $\mathrm{i}\mathcal{G}_0^{(1)}(\varphi)$ of the form $\mathcal{G}_0^{(1)}(\varphi) := \alpha(\varphi, x)|D|^M + |D|^M \alpha(\varphi, x) = 2\alpha(\varphi, x)|D|^M + OPS^{M-1}$. The expansion of the symbol of the transformed vector field $\mathrm{i}\mathcal{V}_0^{(1)}(\varphi)$ is given in (3.12) and it has the form

$$\mathcal{V}_0^{(1)}(\varphi) = \Big(2\omega \cdot \partial_\varphi \alpha(\varphi, x) + V(\varphi, x)\Big)|D|^M + OPS^{M-\bar{\mathfrak{e}}}$$

where the constant $\bar{\mathfrak{e}} > 0$ is defined in (3.1). By choosing $\omega \in \mathbb{R}^\nu$ diophantine (see (1.12)), we can determine the function α so that

$$2\omega \cdot \partial_\varphi \alpha(\varphi, x) + V(\varphi, x) = \langle V \rangle_\varphi(x), \quad \langle V \rangle_\varphi(x) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} V(\varphi, x)\, d\varphi$$

so that

$$\mathcal{V}_0^{(1)}(\varphi) = \langle V \rangle_\varphi(x)|D|^M + OPS^{M-\bar{\mathfrak{e}}}$$

has the highest order $\langle V \rangle_\varphi(x)|D|^M$ independent of φ. Then, in Section 3.1.2 we remove the x dependence from the highest order term $\langle V \rangle_\varphi(x)|D|^M$. In order to achieve this purpose, we conjugate the vector field $\mathrm{i}\mathcal{V}_0^{(1)}$ by means of the time-1 flow map $\Phi_0^{(2)}$ of the transport equation

$$\partial_\tau u = \Big(b(\tau; x)\partial_x + \frac{b_x(\tau; x)}{2}\Big)u, \quad b(\tau; x) := -\frac{\beta(x)}{1 + \tau\beta_x(x)}.$$

Note that the function b is φ-independent, implying that also the map $\Phi_0^{(2)}$ is φ-independent, therefore the transformed vector field is given by $\mathrm{i}\mathcal{V}_1(\varphi)$, $\mathcal{V}_1(\varphi) := \Phi_0^{(2)} \mathcal{V}_0^{(1)} (\Phi_0^{(2)})^{-1}$. The expansion of the operator $\mathcal{V}_1(\varphi) = \mathrm{Op}\big(v_1(\varphi, x, \xi)\big)$ is given in Lemma 3.2 and its highest order term is given by

$$\Big[\langle V\rangle_\varphi(y)\big(1 + \partial_y \widetilde{\beta}(y)\big)^M\Big]_{y = x + \beta(x)} |D|^M$$

where $y \mapsto y + \widetilde{\beta}(y)$ is the inverse diffeomorphism of $x \mapsto x + \beta(x)$. Then we determine the function $\widetilde{\beta}$ and a constant $\lambda > 0$ so that

$$\langle V\rangle_\varphi(y)\big(1 + \partial_y \widetilde{\beta}(y)\big)^M = \lambda\,.$$

In order to solve this equation, we use the hypothesis $(\mathbf{H3})_{\mathbf{M<1}}$, i.e. $\inf_{(\varphi,x) \in \mathbb{T}^{\nu+1}} V(\varphi, x) > 0$, then one gets that $\mathcal{V}_1(\varphi) = \lambda |D|^M + OPS^{M - \bar{\mathfrak{e}}}$. Thanks to the Hamiltonian structure one also gets that $\lambda \in \mathbb{R}$.

2. *Reduction of the lower order terms.* Given $N > 0$, the next step is to transform the vector field $\mathrm{i}\mathcal{V}_1(\varphi)$, $\mathcal{V}_1(\varphi) = \lambda|D|^M + \mathcal{W}_1(\varphi)$, $\mathcal{W}_1 \in OPS^{M - \bar{\mathfrak{e}}}$ into another one of the form $\mathrm{i}\mathcal{V}_N(\varphi)$, $\mathcal{V}_N(\varphi) = \lambda|D|^M + \mu_N(D) + OPS^{M - N\bar{\mathfrak{e}}}$ where $\mu_N(D)$ is a time independent Fourier multiplier of order $M - \bar{\mathfrak{e}}$. This is achieved in Section 3.2, by means of an iterative procedure. At the n-th step of such a procedure, we deal with $\mathrm{i}\mathcal{V}_n(\varphi)$, $\mathcal{V}_n(\varphi) = \lambda|D|^M + \mu_n(D) + \mathcal{W}_n(\varphi)$ where $\mu_n(D)$ is a time independent Fourier multiplier of order $M - \bar{\mathfrak{e}}$ and $\mathcal{W}_n(\varphi) = \mathrm{Op}\big(w_n(\varphi, x, \xi)\big) \in OPS^{M - n\bar{\mathfrak{e}}}$. First, in Section 3.2.1, we remove the φ-dependence from the symbol $w_n(\varphi, x, \xi)$ by conjugating $\mathrm{i}\mathcal{V}_n(\varphi)$ with the time one flow map $\Phi_n^{(1)}(\varphi)$ of a Hamiltonian vector field $\mathrm{i}\mathcal{G}_n^{(1)}(\varphi)$, $\mathcal{G}_n^{(1)}(\varphi) = \mathrm{Op}\big(g_n^{(1)}(\varphi, x, \xi)\big) \in OPS^{M - n\bar{\mathfrak{e}}}$. The transformed vector field is given by

$$\mathcal{V}_n^{(1)}(\varphi) = \lambda|D|^M + \mu_n(D) + \mathrm{Op}\Big(w_n(\varphi, x, \xi) + \omega \cdot \partial_\varphi g_n^{(1)}(\varphi, x, \xi)\Big) + OPS^{M - (n+1)\bar{\mathfrak{e}}}$$

see (3.42). Since $\omega \in \mathbb{R}^\nu$ is a diophantine frequency, we choose the symbol $g_n^{(1)}$ so that

$$w_n(\varphi, x, \xi) + \omega \cdot \partial_\varphi g_n^{(1)}(\varphi, x, \xi) = \langle w_n\rangle_\varphi(x, \xi), \quad \langle w_n\rangle_\varphi(x, \xi) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} w_n(\varphi, x, \xi)\, d\varphi\,.$$

therefore we have removed the φ-dependence at the order $M - n\bar{\mathfrak{e}}$ and

$$\mathcal{V}_n^{(1)}(\varphi) = \lambda|D|^M + \mu_n(D) + \langle w_n\rangle_\varphi(x, D) + OPS^{M - (n+1)\bar{\mathfrak{e}}}\,.$$

Then, in Section 3.2.2, we remove the x-dependence from the symbol $\langle w_n\rangle_\varphi$, by conjugating the vector field $\mathrm{i}\mathcal{V}_n^{(1)}(\varphi)$ with the time one flow map $\Phi_n^{(2)}$ of a Hamiltonian vector field $\mathrm{i}\mathcal{G}_n^{(2)}$, $\mathcal{G}_n^{(2)} = \mathrm{Op}(g_n^{(2)}(x, \xi)) \in OPS^{1 - n\bar{\mathfrak{e}}}$. The expansion of the transformed vector field $\mathrm{i}\mathcal{V}_{n+1}(\varphi)$ is given by

$$\mathcal{V}_{n+1}(\varphi) = \Phi_n^{(2)} \mathcal{V}_n^{(1)}(\varphi) (\Phi_n^{(2)})^{-1} = \lambda|D|^M + \mu_n(D) + \mathrm{Op}\Big(\langle w_n\rangle_\varphi + \{g_n^{(2)}\,,\, \lambda|\xi|^M\}\Big) + OPS^{M - (n+1)\bar{\mathfrak{e}}}\,.$$

Then in Lemma 3.5, we determine the symbol $g_n^{(2)}$ so that

$$\langle w_n\rangle_\varphi + \{g_n^{(2)}\,,\, \lambda|\xi|^M\} = \langle w_n\rangle_{\varphi,x} + OPS^{M - (n+1)\bar{\mathfrak{e}}}, \quad \langle w_n\rangle_{\varphi,x} := \frac{1}{2\pi} \int_{\mathbb{T}} \langle w_n\rangle_\varphi(x, \xi)\, dx$$

implying that

$$\mathcal{V}_{n+1}(\varphi) = \lambda|D|^M + \mu_{n+1}(D) + OPS^{M - (n+1)\bar{\mathfrak{e}}}, \quad \mu_{n+1}(D) := \mu_n(D) + \mathrm{Op}\big(\langle w_n\rangle_{\varphi,x}(\xi)\big)\,.$$

1.1.2 Normal form in the case $M = 1$

The regularization procedure for the vector field $\mathrm{i}\mathcal{V}(\varphi)$, $\mathcal{V}(\varphi) = V(\varphi, x)|D| + \mathcal{W}(\varphi)$, $\mathcal{W} \in OPS^{1 - \mathfrak{e}}$, $\mathfrak{e} > 0$, $V(\varphi, x) = 1 + \varepsilon P(\varphi, x)$ is developed in Section 4. In the following we will give a sketch of the proof of such a procedure.

1. *Reduction of the highest order term.* In order to reduce to constant coefficients the highest order term $\mathrm{i}V(\varphi,x)|D|$, we conjugate the vector filed $\mathrm{i}\mathcal{V}(\varphi)$ with the map

$$\Phi(\varphi) := \Phi_+(\varphi)^{-1}\Pi_+ + \Phi_-(\varphi)^{-1}\Pi_-$$

where Π_+ (resp. Π_-) are the projection operators on the positive (resp. negative) Fourier modes and $\Phi_\pm(\varphi)$ are the operators given by

$$\Phi_\pm(\varphi) : u(x) \mapsto \sqrt{1 + (\partial_x \alpha_\pm)(\varphi,x)}\; u(x + \alpha_\pm(\varphi,x))$$

with α_+, α_- being $\mathcal{C}^\infty$ functions to be determined. The transformed vector field is $\mathrm{i}\mathcal{V}_1(\varphi)$ with

$$\begin{aligned}\mathcal{V}_0(\varphi) = \Pi_+\Big(\Big(\big(1+\varepsilon P\big)\big(1+(\partial_y\widetilde{\alpha}_+)\big) - \omega\cdot\partial_\varphi\widetilde{\alpha}_+\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D|\Big)\Pi_+ \\ + \Pi_-\Big(\Big(\big(1+\varepsilon P\big)\big(1+(\partial_y\widetilde{\alpha}_-)\big) + \omega\cdot\partial_\varphi\widetilde{\alpha}_-\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D|\Big)\Pi_- \\ + OPS^{M-\bar{\mathfrak{e}}}\end{aligned} \tag{1.17}$$

where the constant $\bar{\mathfrak{e}} > 0$ is defined in (4.2) and the map $y \mapsto y + \widetilde{\alpha}_\pm(\varphi,y)$ is the inverse diffeomorphism of $x \mapsto x + \alpha(\varphi,x)$. The reason for which we introduce the projectors Π_+ and Π_- is the following: there are two terms which give a contribution to the highest order in the transformed vector field $\mathrm{i}\mathcal{V}_0(\varphi)$. The ones coming from the conjugation $\Phi_\pm(\varphi)\mathrm{i}\mathcal{V}(\varphi)\Phi_\pm(\varphi)^{-1}$ are given by $\mathrm{i}\Big(\big(1+\varepsilon P\big)\big(1+(\partial_y\widetilde{\alpha}_\pm)\big)\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D|$ and the other ones coming from $\Phi_\pm(\varphi)\omega\cdot\partial_\varphi\big(\Phi_\pm(\varphi)^{-1}\big)$ are $\mp\Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\Big)_{|y=x+\alpha_\pm(\varphi,x)}\partial_x$. Then, in order to reduce to constant coefficients the term of order one, we have to compare the operators $\mathrm{i}|D|$ and ∂_x. These two operators are the same (up to a sign) if we project them to positive and negative Fourier modes, since they satisfy the elementary properties $\mathrm{i}|D|\Pi_+ = \partial_x\Pi_+$, $\mathrm{i}|D|\Pi_- = -\partial_x\Pi_-$.

   In order to reduce to constant coefficients the highest order term in (1.17), we look for *small* functions $\widetilde{\alpha}_\pm \in \mathcal{C}^\infty$ and constants $\lambda_\pm \in \mathbb{R}$ close to 1, so that

$$\big(1+\varepsilon P\big)\big(1+(\partial_y\widetilde{\alpha}_\pm)\big) \mp \omega\cdot\partial_\varphi\widetilde{\alpha}_\pm = \lambda_\pm\,.$$

   This is a quasi-periodic transport equation, which is solved by applying Proposition 6.1 in the appendix. Note that this is the only point in which we require a smallness condition on the parameter ε.

2. *Reduction of the lower order terms* : The expansion of the vector field $\mathrm{i}\mathcal{V}_0(\varphi)$ is given in (4.27) and it has the form $\mathcal{V}_0(\varphi) = \Pi_+\big(\lambda_+|D| + \mathcal{W}_{0,+}(\varphi)\big)\Pi_+ + \Pi_-\big(\lambda_-|D| + \mathcal{W}_{0,-}(\varphi)\big)\Pi_- + OPS^{-\infty}$ (see (4.27)) where $\mathcal{W}_{0,\pm} \in OPS^{1-\bar{\mathfrak{e}}}$. Given $N > 0$, our next goal is to transform the vector field $\mathrm{i}\mathcal{V}_0(\varphi)$ into another one $\mathrm{i}\mathcal{V}_N(\varphi)$, which has the form $\mathcal{V}_N(\varphi) = \Pi_+\big(\lambda_+|D| + \mu_{N,+}(D)\big)\Pi_+ + \Pi_-\big(\lambda_-|D| + \mu_{N,-}(D)\big)\Pi_- + OPS^{1-N\bar{\mathfrak{e}}}$ where $\mu_{N,\pm}(D) \in OPS^{1-\bar{\mathfrak{e}}}$ is a φ-independent Fourier multiplier with real symbol. This is achieved by means of an iterative procedure which is developed in Section 4.2. At the n-th step of such a procedure, we deal with a vector field $\mathrm{i}\mathcal{V}_n(\varphi)$, $\mathcal{V}_n(\varphi) = \Pi_+\mathcal{V}_{n,+}(\varphi)\Pi_+ + \Pi_-\mathcal{V}_{n,-}(\varphi)\Pi_- + OPS^{-\infty}$, $\mathcal{V}_{n,\pm}(\varphi) = \lambda_\pm|D| + \mu_{n,\pm}(D) + \mathcal{W}_{n,\pm}(\varphi)$ where $\mu_{n,\pm}(D)$ are Fourier multipliers with real symbols of order $1-\bar{\mathfrak{e}}$ and $\mathcal{W}_{n,\pm}(\varphi) \in OPS^{1-n\bar{\mathfrak{e}}}$. The vector fields $\mathrm{i}\mathcal{V}_{n,\pm}$ are Hamiltonian, i.e. $\mathcal{V}_{n,\pm}$ are self-adjoint operators. In order to reduce to constant coefficients the terms of order $1-n\bar{\mathfrak{e}}$, we conjugate the vector field $\mathrm{i}\mathcal{V}_n(\varphi)$ by means of the map

$$\Phi_n(\varphi) = \Phi_{n,+}(\varphi)^{-1}\Pi_+ + \Phi_{n,-}(\varphi)^{-1}\Pi_-$$

   where $\Phi_{n,\pm}(\varphi)$ are the time one flow maps of Hamiltonian vector fields $\mathrm{i}\mathcal{G}_{n,\pm}(\varphi)$ with $\mathcal{G}_{n,\pm}(\varphi) = \mathrm{Op}\big(g_{n,\pm}(\varphi,x,\xi)\big) \in OPS^{1-n\mathfrak{e}}$. Note that, even if $\Phi_{n,\pm}(\varphi)$ is symplectic, the map Φ_n is not symplectic. By applying Lemma 2.15, one gets that the transformed vector field has the form $\mathrm{i}\mathcal{V}_{n+1}(\varphi)$

$$\mathcal{V}_{n+1}(\varphi) = \Pi_+\mathcal{V}_{n+1,+}(\varphi)\Pi_+ + \Pi_-\mathcal{V}_{n+1,-}(\varphi)\Pi_- + OPS^{-\infty}$$

where $\mathrm{i}\mathcal{V}_{n+1,\pm} = (\Phi_{n,\pm}^{-1})_{\omega*}(\mathrm{i}\mathcal{V}_{n,\pm})$ are Hamiltonian vector fields. The symbols $v_{n+1,\pm}$ of $\mathcal{V}_{n+1,\pm}$ have the expansion

$$v_{n+1,\pm} = \lambda_\pm|\xi| + \mu_{n,\pm} + w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\mathrm{sign}(\xi) + S^{1-(n+1)\bar{\mathfrak{e}}}\,.$$

The order $1 - n\bar{\mathfrak{e}}$ is reduced to constant coefficients in Lemma 4.3, by choosing the symbol $g_{n,\pm}$ so that

$$w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\mathrm{sign}(\xi) = \langle w_n \rangle_{\varphi,x} + S^{1-(n+1)\bar{\mathfrak{e}}}\,,$$
$$\langle w_n \rangle_{\varphi,x}(\xi) := \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} w_n(\varphi, x, \xi)\, d\varphi\, dx\,.$$

In order to make this choice, we impose some first order Melnikov conditions on ω, i.e. we require that

$$|\omega \cdot \ell + \lambda_\pm j| \geq \frac{\gamma}{\langle \ell \rangle^\tau}\,, \quad \forall (\ell, j) \in \mathbb{Z}^{\nu+1} \setminus \{(0,0)\}\,.$$

Then $v_{n+1,\pm} = \lambda_\pm|\xi| + \mu_{n+1,\pm} + S^{1-(n+1)\bar{\mathfrak{e}}}$ with $\mu_{n+1,\pm} = \mu_{n,\pm} + \langle w_{n,\pm}\rangle_{\varphi,x}$. Thanks to the Hamiltonian structure of the vector fields $\mathrm{i}\mathcal{V}_{n,\pm}$ one gets that $\mu_{n+1,\pm}(D)$ is a Fourier multiplier with real symbol.

The paper is organized as follows: in Section 2 we provide some technical tools which are needed for the proof of Theorems 1.6 and 1.7. In Sections 3, 4, we develop the normal form procedures described in Sections 1.1.1, 1.1.2 and we prove Theorems 1.6, 1.7. Finally, in Section 5 we prove Theorems 1.3, 1.4.

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2 Pseudo differential operators and flows of pseudo-PDEs

In this section, we recall some well-known definitions and results concerning pseudo differential operators on the torus $\mathbb{T}$. We always consider φ-dependent symbols $a(\varphi, x, \xi)$, $(\varphi, x, \xi) \in \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R}$, depending in a $\mathcal{C}^\infty$ way on the whole variables. Since φ plays the role of a parameter, all the well known results apply to these symbols without any modification (we refer for instance to [24], [25]). For the symbol class S^m given in the definition 1.1 and the operator class OPS^m given in the definition 1.2, the following standard inclusions hold:

$$S^m \subseteq S^{m'}\,, \quad OPS^m \subseteq OPS^{m'}\,, \quad \forall m \leq m'\,. \tag{2.1}$$

Theorem 2.1 (Calderon-Vallancourt). *Let $m \in \mathbb{R}$ and $A = a(\varphi, x, D) \in OPS^m$. Then for any $\alpha \in \mathbb{N}^\nu$ the operator $\partial_\varphi^\alpha A(\varphi) \in \mathcal{B}(H^{s+m}(\mathbb{T}), H^s(\mathbb{T}))$ with $\sup_{\varphi \in \mathbb{T}^\nu} \|\partial_\varphi^\alpha A(\varphi)\|_{\mathcal{B}(H^{s+m}, H^s)} < +\infty$.*

Definition 2.2 (Asymptotic expansion). *Let $(m_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers converging to $-\infty$ and $a_k \in S^{m_k}$ for any $k \in \mathbb{N}$. We say that $a \in S^{m_0}$ has the asymptotic expansion $\sum_{k\geq 0} a_k$, i.e. $a \sim \sum_{k\geq 0} a_k$ if $a - \sum_{k=0}^N a_k \in S^{m_{N+1}}$ for any $N \in \mathbb{N}$.*

Theorem 2.3 (Composition). *Let $m, m' \in \mathbb{R}$ and $A(\varphi) = a(\varphi, x, D) \in OPS^m$, $B(\varphi) = b(\varphi, x, D) \in OPS^{m'}$. Then the composition operator $A(\varphi)B(\varphi) := A(\varphi) \circ B(\varphi) = \mathrm{Op}(\sigma_{AB})$ is a pseudo-differential operator in $OPS^{m+m'}$. The symbol σ_{AB} has the following asymptotic expansion*

$$\sigma_{AB} \sim \sum_{\alpha \geq 0} \frac{1}{\mathrm{i}^\alpha \alpha!} \partial_\xi^\alpha a \partial_x^\alpha b\,, \tag{2.2}$$

meaning that for any $N \in \mathbb{N}$,

$$\sigma_{AB} = \sum_{\beta=0}^{N-1} \frac{1}{\mathrm{i}^\alpha \alpha!} \partial_\xi^\alpha a\, \partial_x^\alpha b + S^{m+m'-N}\,. \tag{2.3}$$

Corollary 2.4. *Let $m, m' \in \mathbb{R}$ and let $A = \mathrm{Op}(a)$, $B = \mathrm{Op}(b)$. Then the commutator $[A, B] = \mathrm{Op}(a \star b)$, with $a \star b \in S^{m+m'-1}$ having the following asymptotic expansion:*

$$a \star b = -\mathrm{i}\{a, b\} + S^{m+m'-2}\,, \quad \{a, b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b \in S^{m+m'-1}\,.$$

Theorem 2.5 (**Adjoint of a pseudo-differential operator).** *If $A(\varphi) = a(\varphi, x, D) \in OPS^m$ is a pseudo-differential operator with symbol $a \in S^m$, then its L^2-adjoint is a pseudo-differential operator $A^* = \mathrm{Op}(a^*) \in OPS^m$. The symbol $a^* \in S^m$ admits the asymptotic expansion*

$$a^* \sim \sum_{\alpha \in \mathbb{N}} \frac{1}{\mathrm{i}^\alpha \alpha!} \overline{\partial_x^\alpha \partial_\xi^\alpha a} \tag{2.4}$$

meaning that for any $N \in \mathbb{N}$,

$$a^* = \sum_{\alpha=0}^{N-1} \frac{1}{\mathrm{i}^\alpha \alpha!} \overline{\partial_x^\alpha \partial_\xi^\alpha a} + S^{m-N}\,.$$

Lemma 2.6. *Let $A = \mathrm{Op}(a) \in OPS^m$ be self-adjoint, i.e. $a = a^*$ and let $\varphi(\xi)$ be a real Fourier multiplier of order m'. We define the symbol $b(\varphi, x, \xi) := \varphi(\xi) a(\varphi, x, \xi) \in S^{m+m'}$. Then $b^* = b + S^{m+m'-1}$.*

Proof. See Lemma 2.6 in [21]. □

Lemma 2.7. *Let $a \in S^m$, $m \in \mathbb{R}$ and let $g(\xi)$ be a Fourier multiplier in S^0 satisfying the following property: There exists $\delta > 0$ such that $\partial_\xi g(\xi) = 0$ for any $|\xi| \geq \delta$. Then the commutator $[\mathrm{Op}(g), \mathrm{Op}(a)] \in OPS^{-\infty}$.*

Proof. By applying Theorem 2.3 the symbol $\sigma(\varphi, x, \xi)$ of the commutator $[\mathrm{Op}(g), \mathrm{Op}(a)] \in OPS^{-\infty}$ has the asymptotic expansion $\sigma \sim \sum_{\alpha \geq 1} \frac{1}{\mathrm{i}^\alpha \alpha!} (\partial_\xi^\alpha g)(\partial_x^\alpha a)$. Since for any $\alpha \geq 1$, $\partial_\xi^\alpha g = 0$ for $|\xi| \geq \delta$, all the symbols $(\partial_\xi^\alpha g)(\partial_x^\alpha a) \in S^{-\infty}$, implying that $\sigma \in S^{-\infty}$. □

We define the operator ∂_x^{-1} by setting

$$\partial_x^{-1}[1] := 0\,, \quad \partial_x^{-1}[e^{\mathrm{i}xk}] := \frac{e^{\mathrm{i}xk}}{\mathrm{i}k}\,, \quad \forall k \in \mathbb{Z} \setminus \{0\} \tag{2.5}$$

and for any diophantine frequency vector $\omega \in DC(\gamma, \tau)$, we define the operator $(\omega \cdot \partial_\varphi)^{-1}$ by setting

$$(\omega \cdot \partial_\varphi)^{-1}[1] = 0\,, \quad (\omega \cdot \partial_\varphi)^{-1}[e^{\mathrm{i}\ell \cdot \varphi}] = \frac{e^{\mathrm{i}\ell \cdot \varphi}}{\mathrm{i}\omega \cdot \ell}\,, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}\,. \tag{2.6}$$

Given $\omega \in \mathbb{R}^\nu$ and $\lambda \in \mathbb{R}$ satisfying the non-resonance condition

$$|\omega \cdot \ell + \lambda\, j| \geq \frac{\gamma}{\langle \ell \rangle^\tau}\,, \quad \forall (\ell, j) \in \mathbb{Z}^{\nu+1} \setminus \{(0,0)\}\,, \tag{2.7}$$

we define the operator $(\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1}$ by setting

$$(\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1}[1] = 0\,, \quad (\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1}[e^{\mathrm{i}\ell \cdot \varphi} e^{\mathrm{i}jx}] := \frac{e^{\mathrm{i}\ell \cdot \varphi} e^{\mathrm{i}jx}}{\mathrm{i}(\omega \cdot \ell + \lambda\, j)}\,, \quad \forall (\ell, j) \neq (0,0)\,. \tag{2.8}$$

Furthermore, given a symbol $a \in S^m$, we define the averaged symbols $\langle a \rangle_\varphi, \langle a \rangle_{\varphi,x}$ by

$$\langle a \rangle_\varphi(x, \xi) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} a(\varphi, x, \xi)\, d\varphi\,, \quad \langle a \rangle_{\varphi,x}(\xi) \quad := \frac{1}{2\pi} \int_{\mathbb{T}} \langle a \rangle_\varphi(x, \xi)\, dx\,. \tag{2.9}$$

The following elementary properties hold:

$$a \in S^m \Longrightarrow \partial_x^{-1} a\,, \ (\omega \cdot \partial_\varphi)^{-1} a\,, \ (\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1} a\,, \langle a \rangle_\varphi\,, \ \langle a \rangle_{\varphi,x} \in S^m\,. \tag{2.10}$$

Lemma 2.8. *Given a symbol $a \in S^m$, the following holds.*
(i) $\langle a^* \rangle_\varphi = \big(\langle a \rangle_\varphi\big)^*$, $\langle a^* \rangle_{\varphi,x} = \overline{\langle a \rangle_{\varphi,x}} = \big(\langle a \rangle_{\varphi,x}\big)^*$.
(ii) $\partial_x^{-1}(a^*) = (\partial_x^{-1} a)^*$.
(iii) If $\omega \in DC(\gamma, \tau)$ then $(\omega \cdot \partial_\varphi)^{-1} a^* = \Big((\omega \cdot \partial_\varphi)^{-1} a\Big)^*$.
(iv) If ω satisfies the condition (2.7) *then* $(\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1} a^* = \Big((\omega \cdot \partial_\varphi + \lambda \partial_x)^{-1} a\Big)^*$.

Proof. We prove item (iv). The proof of items $(i)-(iii)$ can be done arguing similarly. The symbol a^* is given by

$$a^*(\varphi,x,\xi) := \overline{\sum_{\eta\in\mathbb{Z}} \widehat{a}(\varphi,\eta,\xi-\eta)e^{\mathrm{i}\eta x}}, \quad \widehat{a}(\varphi,\eta,\xi-\eta) := \frac{1}{2\pi}\int_{\mathbb{T}} a(\varphi,x,\xi-\eta)e^{-\mathrm{i}x\eta}\,dx\,.$$

Writing also the Fourier expansion w.r. to $\varphi\in\mathbb{T}^\nu$ one gets that $\widehat{a}(\varphi,\eta,\xi-\eta)=\sum_{\ell\in\mathbb{Z}^\nu}\widehat{a}_\ell(\eta,\xi-\eta)e^{\mathrm{i}\ell\cdot\varphi}$ and by recalling (2.8), one has

$$(\omega\cdot\partial_\varphi+\lambda\partial_x)^{-1}a^* = \overline{\sum_{(\ell,\eta)\neq(0,0)} \frac{\widehat{a}_\ell(\eta,\xi-\eta)}{\mathrm{i}(\omega\cdot\ell+\lambda\eta)}e^{\mathrm{i}(\ell\cdot\varphi+\eta x)}}\,. \tag{2.11}$$

Now let $\sigma=(\omega\cdot\partial_\varphi+\lambda\partial_x)^{-1}a$. One has that

$$\sigma^* = \overline{\sum_{\eta\in\mathbb{Z}}\widehat{\sigma}(\varphi,\eta,\xi-\eta)e^{\mathrm{i}\eta x}} = \overline{\sum_{(\ell,\eta)\in\mathbb{Z}^{\nu+1}}\widehat{\sigma}_\ell(\eta,\xi-\eta)e^{\mathrm{i}(\ell\cdot\varphi+\eta x)}} \tag{2.12}$$

and using again (2.8), one has that $\widehat{\sigma}_0(0,\xi-\eta)=0$ and $\widehat{\sigma}_\ell(\eta,\xi-\eta)=\frac{\widehat{a}_\ell(\eta,\xi-\eta)}{\mathrm{i}(\omega\cdot\ell+\lambda\eta)}$ for any $(\ell,\eta)\in\mathbb{Z}^{\nu+1}\setminus\{(0,0)\}$, implying that (2.11) coincides with (2.12). ☐

For any $\alpha\in\mathbb{R}$, we define the operator $|D|^\alpha$ as follows. Let $\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function satisfying

$$\chi(\xi) := \begin{cases} 1 & \text{if } |\xi|\geq 1 \\ 0 & \text{if } |\xi|\leq\frac{1}{2}\,. \end{cases} \tag{2.13}$$

We then define for any $\alpha\in\mathbb{R}$

$$|D|^\alpha := \mathrm{Op}\big(|\xi|^\alpha\chi(\xi)\big)\,. \tag{2.14}$$

Clearly $|D|^\alpha\in OPS^\alpha$ and the action on any 2π-periodic function $u\in L^2(\mathbb{T})$ is given by

$$|D|^\alpha u(x) = \sum_{\xi\in\mathbb{Z}\setminus\{0\}}|\xi|^\alpha\widehat{u}(\xi)e^{\mathrm{i}x\xi}\,.$$

2.1 Well posedness of some linear PDEs

In this section we state some classical properties of the flow of some linear pseudo-PDEs. Before to give the statement of the following Lemma, we introduce the following notation. For any $s\in\mathbb{R}$, we write $A\lesssim_s B$ if there exists a constant $C(s)>0$ depending on s such that $A\leq C(s)B$.

Lemma 2.9. *Let $\mathcal{A}(\tau;\varphi):=\mathrm{Op}\Big(a(\tau;\varphi,x,\xi)\Big)$, $\tau\in[0,1]$ be a smooth τ-dependent family of pseudo differential operators in OPS^1. Assume that $\mathcal{A}(\tau;\varphi)+\mathcal{A}(\tau;\varphi)^*\in OPS^0$. Then the following holds.*
(i) Let $s\geq 0$, $u_0\in H^s(\mathbb{T})$, $\tau_0\in[0,1]$. Then there exists a unique solution $u\in\mathcal{C}^0_b\Big([0,1],H^s(\mathbb{T})\Big)$ of the Cauchy problem

$$\begin{cases}\partial_\tau u = \mathcal{A}(\tau;\varphi)[u] \\ u(\tau_0;x)=u_0(x)\end{cases} \tag{2.15}$$

satisfying the estimate

$$\|u\|_{\mathcal{C}^0([0,1],H^s)}\lesssim_s\|u_0\|_{H^s}\,.$$

As a consequence, for any $\tau_0,\tau\in[0,1]$, the flow map $\Phi(\tau_0,\tau;\varphi)$, which maps the initial datum $u(\tau_0)=u_0$ into the solution $u(\tau)$ of (2.15) at the time τ, is in $\mathcal{B}(H^s)$ with $\sup_{\substack{\tau_0,\tau\in[0,1]\\ \varphi\in\mathbb{T}^\nu}}\|\Phi(\tau_0,\tau;\varphi)\|_{\mathcal{B}(H^s)}<+\infty$ for any $s\geq 0$. Moreover, the operator $\Phi(\tau_0,\tau;\varphi)$ is invertible with inverse $\Phi(\tau_0,\tau;\varphi)^{-1}=\Phi(\tau,\tau_0;\varphi)$.
(ii) For any $\tau_0,\tau\in[0,1]$, the flow map $\varphi\mapsto\Phi(\tau_0,\tau;\varphi)$ is differentiable and

$$\sup_{\substack{\tau_0,\tau\in[0,1]\\ \varphi\in\mathbb{T}^\nu}}\|\partial_\varphi^\alpha\Phi(\tau_0,\tau;\varphi)\|_{\mathcal{B}(H^{s+|\alpha|},H^s)}<+\infty\,,\quad\forall\alpha\in\mathbb{N}^\nu,\quad s\geq 0\,. \tag{2.16}$$

Proof. The proof of item (i) is classical. We refer for instance to [25], Section 0.8. The proof of item (ii) can be obtained arguing as in Lemma 2.9 in [21]. $\square$

Lemma 2.10. *Let $g(\xi)$ be a Fourier multiplier in OPS^0 which satisfies the following property: there exists $\delta > 0$ such that $\partial_\xi g(\xi) = 0$ for any $|\xi| \geq \delta$. Let $\mathcal{A}(\tau;\varphi) := \mathrm{Op}\Big(a(\tau;\varphi,x,\xi)\Big)$, $\varphi \in \mathbb{T}^\nu$, $\tau \in [0,1]$ be a smooth τ-dependent family of periodic pseudo differential operators in OPS^η with $\eta \leq 1$. Assume that $\mathcal{A}(\tau;\varphi) + \mathcal{A}(\tau;\varphi)^* \in OPS^{\eta-1}$. Let $\Phi(\tau;\varphi)$ be the flow of the pseudo PDE $\partial_\tau u = \mathcal{A}(\tau;\varphi)[u]$, i.e.*

$$\begin{cases} \partial_\tau \Phi(\tau;\varphi) = \mathcal{A}(\tau;\varphi)\Phi(\tau;\varphi) \\ \Phi(0;\varphi) = \mathrm{Id}\,. \end{cases}$$

Then the commutator $[\Phi(\tau;\varphi), \mathrm{Op}(g)] \in OPS^{-\infty}$.

Proof. Let $\Phi_g(\tau;\varphi) := [\Phi(\tau;\varphi), \mathrm{Op}(g)]$. A direct calculation shows that $\Phi_g(\tau;\varphi)$ solves

$$\begin{cases} \partial_\tau \Phi_g(\tau;\varphi) = \mathcal{A}(\tau;\varphi)\Phi_g(\tau;\varphi) + \mathcal{R}_g(\tau;\varphi) \\ \Phi_g(0;\varphi) = 0\,, \end{cases} \qquad \mathcal{R}_g(\tau;\varphi) := [\mathcal{A}(\tau;\varphi), \mathrm{Op}(g)]\Phi(\tau;\varphi)\,.$$

By Lemma 2.7, $[\mathcal{A}(\tau;\varphi), \mathrm{Op}(g)] \in OPS^{-\infty}$ and since $\mathcal{A}(\tau;\varphi)$ satisfies the hypothesis of Lemma 2.9, the flow $\Phi(\tau;\varphi)$ satisfies

$$\Phi(\tau;\varphi)^{\pm 1} \in \mathcal{B}(H^s(\mathbb{T}))\,, \quad \forall s \geq 0\,, \tag{2.17}$$

implying that the operator $\mathcal{R}_g(\tau;\varphi) = [\mathcal{A}(\tau;\varphi), \mathrm{Op}(g)]\Phi(\tau;\varphi) \in OPS^{-\infty}$. Using the Duhamel principle one gets that

$$\Phi_g(\tau;\varphi) = \int_0^\tau \Phi(\tau;\varphi)^{-1}\Phi(\zeta;\varphi)\mathcal{R}_g(\zeta;\varphi)\,d\zeta \in OPS^{-\infty}$$

and the proof of the lemma is then concluded. $\square$

2.2 Some Egorov-type theorems

In this section we collect some abstract egorov type theorems, namely we study how a pseudo differential operator transforms under the action of the flow of a first order hyperbolic PDE. Let $\alpha : \mathbb{T}^\nu \times \mathbb{T} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying

$$\alpha \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})\,, \qquad \inf_{(\varphi,x)\in\mathbb{T}^{\nu+1}} \big(1 + \alpha_x(\varphi,x)\big) > 0\,. \tag{2.18}$$

We then consider the non-autonomous transport equation

$$\partial_\tau u = \mathcal{A}(\tau;\varphi)u\,, \qquad \mathcal{A}(\tau;\varphi) := b_\alpha(\tau;\varphi,x)\partial_x + \frac{(\partial_x b_\alpha)(\tau;\varphi,x)}{2}\,, \tag{2.19}$$

$$b_\alpha(\tau;\varphi,x) := -\frac{\alpha(\varphi,x)}{1+\tau\alpha_x(\varphi,x)}\,, \qquad \tau \in [0,1]\,. \tag{2.20}$$

Note that the condition (2.18) implies that $\inf_{\substack{\tau\in[0,1] \\ (\varphi,x)\in\mathbb{T}^\nu\times\mathbb{T}}} \big(1 + \tau\alpha_x(\varphi,x)\big) > 0$, hence the function $b \in \mathcal{C}^\infty([0,1] \times \mathbb{T}^\nu \times \mathbb{T})$ and $\mathcal{A}(\tau;\cdot) \in OPS^1$, $\tau \in [0,1]$ is a smooth family of pseudo-differential operators. It is straightforward to verify that $\mathcal{A}(\tau;\varphi) + \mathcal{A}(\tau;\varphi)^* = 0$, therefore, the hypotheses of Lemma 2.9 are verified, implying that, for any $\tau \in [0,1]$, the flow $\Phi(\tau;\varphi) \equiv \Phi(0,\tau;\varphi)$, $\tau \in [0,1]$ of the equation (2.19), i.e.

$$\begin{cases} \partial_\tau \Phi(\tau;\varphi) = \mathcal{A}(\tau;\varphi)\Phi(\tau;\varphi) \\ \Phi(0;t) = \mathrm{Id} \end{cases} \tag{2.21}$$

is a well defined map and satisfies all the properties stated in the items (i), (ii) of Lemma 2.9. Furthermore, arguing as in Section 2.2 of [21], the map $\Phi(\tau;\varphi)$ is symplectic. We then have the following

Lemma 2.11. *The flow* $\Phi(\tau;\varphi)$ *given by* (2.21) *is a symplectic, invertible map satisfying*

$$\sup_{\substack{\tau\in[0,1]\\ \varphi\in\mathbb{T}^\nu}} \|\partial_\varphi^\alpha \Phi(\tau;\varphi)^{\pm 1}\|_{\mathcal{B}(H^{s+|\alpha|},H^s)} < +\infty\,, \quad \forall \alpha\in\mathbb{N}^\nu\,, \quad s\geq 0\,.$$

In order to state Proposition 2.13 of this section, we need some preliminary results.

Lemma 2.12. *Let* $\alpha\in\mathcal{C}^\infty(\mathbb{T}^\nu\times\mathbb{T},\mathbb{R})$ *satisfy the condition* (2.18)*. Then for any* $\varphi\in\mathbb{T}^\nu$*, the map*

$$\psi_\varphi:\mathbb{T}\to\mathbb{T}\,, \quad x\mapsto x+\alpha(\varphi,x)$$

is a diffeomorphism of the torus whose inverse has the form

$$\psi_\varphi^{-1}:\mathbb{T}\to\mathbb{T}, \quad y\mapsto y+\widetilde{\alpha}(\varphi,y)\,, \tag{2.22}$$

with $\widetilde{\alpha}:\mathbb{T}^\nu\times\mathbb{T}\to\mathbb{R}$ *satisfying*

$$\widetilde{\alpha}\in\mathcal{C}^\infty(\mathbb{T}^\nu\times\mathbb{T},\mathbb{R})\,, \qquad \inf_{(\varphi,y)\in\mathbb{T}^\nu\times\mathbb{T}}\big(1+\widetilde{\alpha}_y(\varphi,y)\big)>0\,. \tag{2.23}$$

Furthermore, the following identities hold:

$$1+\alpha_x(\varphi,x)=\frac{1}{1+\widetilde{\alpha}_y\big(\varphi,x+\alpha(\varphi,x)\big)}\,, \quad 1+\widetilde{\alpha}_y(\varphi,y)=\frac{1}{1+\alpha_x\big(\varphi,y+\widetilde{\alpha}(\varphi,y)\big)} \tag{2.24}$$

Proof. The proof is the same as the one of Lemma 2.12 in [21]. □

Now, we are ready to state the following Proposition.

Proposition 2.13. *Let* $m\in\mathbb{R}$*,* $\mathcal{V}(\varphi)=\mathrm{Op}\big(v(\varphi,x,\xi)\big)$ *be in the class* S^m *and* $\Phi(\tau;\varphi)$*,* $\tau\in[0,1]$ *be the flow map of the PDE* (2.21)*. Then* $\mathcal{P}(\tau;\varphi):=\Phi(\tau;\varphi)\mathcal{V}(\varphi)\Phi(\tau;\varphi)^{-1}$ *is a pseudo differential operator in the class* OPS^m*, i.e.* $\mathcal{P}(\tau;\varphi)=\mathrm{Op}\big(p(\tau;\varphi,x,\xi)\big)$ *with* $p(\tau,\cdot,\cdot,\cdot)\in S^m$*,* $\tau\in[0,1]$*. Furthermore* $p(\tau;\varphi,x,\xi)$ *admits the expansion*

$$p(\tau;\varphi,x,\xi)=p_0(\tau;\varphi,x,\xi)+p_{\geq 1}(\tau;\varphi,x,\xi)\,, \qquad p_0(\tau,\cdot,\cdot,\cdot)\in S^m\,, \quad p_{\geq1}(\tau;\cdot,\cdot,\cdot)\in S^{m-1}$$

and the principal symbol p_0 *has the form*

$$p_0(\tau;\varphi,x,\xi):=v\Big(\varphi,x+\tau\alpha(\varphi,x),(1+\tau\alpha_x(\varphi,x))^{-1}\xi\Big)\,,$$
$$\forall(\varphi,x,\xi)\in\mathbb{T}^\nu\times\mathbb{T}\times\mathbb{R}\,, \quad \forall\tau\in[0,1]\,.$$

Proof. The proof is the same as the one of Theorem 2.14 in [21]. □

We also state another *semplified* version of the Egorov theorem in which we conjugate a symbol by means of the flow of a vector field which is a pseudo differential operator of order strictly smaller than one. We consider a pseudo differential operator $\mathcal{G}(\varphi)=\mathrm{Op}(g(\varphi,x,\xi))$, with $g\in S^\eta$, $\mathcal{G}(\varphi)=\mathcal{G}(\varphi)^*$, $\eta<1$ and for any $\tau\in[0,1]$, let $\Phi_{\mathcal{G}}(\tau;\varphi)$ be the flow of the pseudo-PDE $\partial_\tau u=\mathrm{i}\mathcal{G}(\varphi)[u]$, i.e.

$$\begin{cases}\partial_\tau\Phi_{\mathcal{G}}(\tau;\varphi)=\mathrm{i}\mathcal{G}(\varphi)\Phi_{\mathcal{G}}(\tau;\varphi)\\ \Phi_{\mathcal{G}}(0;\varphi)=\mathrm{Id}\,.\end{cases} \tag{2.25}$$

which is a well-defined invertible map by Lemma 2.9. The following Proposition holds.

Proposition 2.14. *Let* $m\in\mathbb{R}$*,* $\mathcal{V}(\varphi)=\mathrm{Op}\big(v(\varphi,x,\xi)\big)\in OPS^m$ *and* $\mathcal{G}(\varphi)=\mathrm{Op}(g(\varphi,x,\xi))$*, with* $g\in S^\eta$*,* $\eta<1$*. Then for any* $\tau\in[0,1]$*, the operator* $\mathcal{P}(\tau;\varphi):=\Phi_{\mathcal{G}}(\tau;\varphi)\mathcal{V}(\varphi)\Phi_{\mathcal{G}}(\tau;\varphi)^{-1}$ *is a pseudo differential operator of order* m *with symbol* $p(\tau;\cdot,\cdot,\cdot)\in S^m$*. The symbol* $p(\tau;\varphi,x,\xi)$ *admits the expansion*

$$p(\tau;\varphi,x,\xi)=v(\varphi,x,\xi)+\tau\{g,v\}(\varphi,x,\xi)+p_{\geq2}(\tau;\varphi,x,\xi)\,, \quad p_{\geq2}(\tau;\varphi,x,\xi)\in S^{m-2(1-\eta)}\,. \tag{2.26}$$

Proof. The proof is the same as the one of Theorem 2.16 in [21]. □

We now consider the projection operators $\Pi_+, \Pi_- : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by

$$\Pi_+ u(x) := \sum_{\xi \geq 0} \widehat{u}(\xi) e^{\mathrm{i}x\xi} , \quad \Pi_- u(x) := \sum_{\xi < 0} \widehat{u}(\xi) e^{\mathrm{i}x\xi} , \quad u \in L^2(\mathbb{T}) . \tag{2.27}$$

The following elementary properties hold:

$$\Pi_+ + \Pi_- = \mathrm{Id} , \quad \Pi_\pm^2 = \Pi_\pm , \quad \Pi_\pm \Pi_\mp = 0 . \tag{2.28}$$

Given two cut-off functions $\chi_\pm \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\chi_+(\xi) = \begin{cases} 1 , & \forall \xi \geq 0 \\ 0 , & \forall \xi \leq -\frac{1}{2}, \end{cases} \qquad \chi_-(\xi) = \begin{cases} 0 , & \forall \xi \geq -\frac{2}{3} \\ 1 , & \forall \xi \leq -1, \end{cases} \tag{2.29}$$

we often identify the operators $\Pi_\pm$ with $\mathrm{Op}(\chi_\pm(\xi))$, i.e.

$$\Pi_\pm \equiv \mathrm{Op}(\chi_\pm(\xi)) . \tag{2.30}$$

Lemmata 2.15-2.18 below will be used in order to develop the reduction procedure of Section 4.

Lemma 2.15. *Let $\mathcal{V}(\varphi) = \mathrm{Op}\big(v(\varphi, x, \xi)\big) \in OPS^m$. Let $\mathcal{G}_\pm(\tau; \varphi) = \mathrm{Op}\big(g_\pm(\tau; \varphi, x, \xi)\big) \in OPS^\eta$, $\eta \leq 1$ satisfy $\mathcal{G}_\pm(\tau; \varphi) + \mathcal{G}_\pm(\tau; \varphi)^* \in OPS^{\eta - 1}$, $\tau \in [0, 1]$. Let $\Phi_\pm(\tau; \varphi)$ by the flow associated to the vector field $\mathcal{G}_\pm(\tau; \varphi)$, i.e.*

$$\begin{cases} \partial_\tau \Phi_\pm(\tau; \varphi) = \mathcal{G}_\pm(\tau; \varphi) \Phi_\pm(\tau; \varphi) \\ \Phi_\pm(0; \varphi) = \mathrm{Id} . \end{cases}$$

Then the following holds.
(i) The map

$$\Phi(\tau; \varphi) := \Phi_+(\tau; \varphi)^{-1} \Pi_+ + \Phi_-(\tau; \varphi)^{-1} \Pi_- \tag{2.31}$$

is invertible, with inverse given by

$$\Phi(\tau; \varphi)^{-1} = \Pi_+ \Phi_+(\tau; \varphi) + \Pi_- \Phi_-(\tau; \varphi) . \tag{2.32}$$

Furthermore the quasi-periodic push forward of the vector field $\mathcal{V}(\varphi)$ by means of the map Φ has the form

$$\mathcal{V}_1(\tau; \varphi) := \Phi_{\omega *} \mathcal{V}(\tau; \varphi) = \Pi_+ \mathcal{V}_{1,+}(\tau; \varphi) \Pi_+ + \Pi_- \mathcal{V}_{1,-}(\tau; \varphi) \Pi_- + OPS^{-\infty} \tag{2.33}$$

where

$$\mathcal{V}_{1,\pm}(\tau; \varphi) := (\Phi_\pm^{-1})_{\omega *} \mathcal{V}(\varphi) = \Phi_\pm(\tau; \varphi) \mathcal{V}(\varphi) \Phi_\pm(\tau; \varphi)^{-1} - \Phi_\pm(\tau; \varphi) \omega \cdot \partial_\varphi \Phi_\pm(\tau; \varphi)^{-1} . \tag{2.34}$$

(ii) Assume that $\mathcal{V}(\varphi) = \Pi_+ \mathcal{V}_+(\varphi) \Pi_+ + \Pi_- \mathcal{V}_-(\varphi) \Pi_-$ where $\mathcal{V}_+(\varphi), \mathcal{V}_-(\varphi) \in OPS^m$. Then $\mathcal{V}_1(\tau; \varphi) = \Phi_{\omega *} \mathcal{V}(\tau; \varphi)$ has the form

$$\begin{aligned} &\mathcal{V}_1(\tau; \varphi) = \Pi_+ \mathcal{V}_{1,+}(\tau; \varphi) \Pi_+ + \Pi_- \mathcal{V}_{1,-}(\tau; \varphi) \Pi_- + OPS^{-\infty} , \\ &\mathcal{V}_{1,\pm}(\tau; \varphi) := \Phi_\pm(\tau; \varphi) \mathcal{V}_\pm(\varphi) \Phi_\pm(\tau; \varphi)^{-1} - \Phi_\pm(\tau; \varphi) \omega \cdot \partial_\varphi \Phi_\pm(\tau; \varphi)^{-1} . \end{aligned} \tag{2.35}$$

Proof. Proof of (i) By the hypotheses on $\mathcal{G}_\pm(\tau; \varphi)$, we can apply Lemma 2.9 obtaining the the flow $\Phi_\pm(\tau; \varphi)$ is invertible and for any $s \geq 0$

$$\Phi_\pm(\tau; \varphi), \Phi_\pm(\tau; \varphi)^1 \in \mathcal{B}(H^s) , \quad \sup_{\substack{\tau \in [0,1] \\ \varphi \in \mathbb{T}^\nu}} \|\Phi_\pm(\tau; \varphi)\|_{\mathcal{B}(H^s)}, \|\Phi_\pm(\tau; \varphi)^{-1}\|_{\mathcal{B}(H^s)} < +\infty . \tag{2.36}$$

By the properties (2.28), one can verify that the map $\Phi(\tau;\varphi)$ is invertible and the inverse is given by the formula (2.31). We now compute the push forward

$$\mathcal{V}_1(\tau;\varphi) = \Phi(\tau;\varphi)^{-1}\mathcal{V}(\varphi)\Phi(\tau;\varphi) - \Phi(\tau;\varphi)^{-1}\omega\cdot\partial_\varphi\Phi(\tau;\varphi)\,. \tag{2.37}$$

By (2.31), (2.32), one has

$$\begin{aligned}\Phi(\tau;\varphi)^{-1}\mathcal{V}(\varphi)\Phi(\tau;\varphi) &= \Big(\Pi_+\Phi_+(\tau;\varphi) + \Pi_-\Phi_-(\tau;\varphi)\Big)\mathcal{V}(\varphi)\Big(\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \Phi_-(\tau;\varphi)^{-1}\Pi_-\Big)\\ &= \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_-\\ &\quad + \Pi_-\Phi_-(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \Pi_-\Phi_-(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_-\\ &= \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \Pi_-\Phi_-(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_- + \mathcal{R}_\infty^{(1)}(\tau;\varphi)\end{aligned} \tag{2.38}$$

where, using the properties (2.28), one has that

$$\mathcal{R}_\infty^{(1)}(\tau;\varphi) := [\Pi_+, \Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}]\Pi_- + [\Pi_-, \Phi_-(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}]\Pi_+\,. \tag{2.39}$$

Moreover, using again (2.28), one gets

$$\begin{aligned}\Phi(\tau;\varphi)^{-1}\omega\cdot\partial_\varphi\Phi(\tau;\varphi) &= \Big(\Pi_+\Phi_+(\tau;\varphi) + \Pi_-\Phi_-(\tau;\varphi)\Big)\Big(\omega\cdot\partial_\varphi\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \omega\cdot\partial_\varphi\Phi_-(\tau;\varphi)^{-1}\Pi_-\Big)\\ &= \Pi_+\Phi_+(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \Pi_-\Phi_-(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_-(\tau;\varphi)^{-1}\Pi_-\\ &\quad + \mathcal{R}_\infty^{(2)}(\tau;\varphi)\end{aligned} \tag{2.40}$$

where

$$\mathcal{R}_\infty^{(2)}(\tau;\varphi) := [\Pi_+, \Phi_+(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_-(\tau;\varphi)^{-1}]\Pi_- + [\Pi_-, \Phi_-(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_+(\tau;\varphi)^{-1}]\Pi_+\,. \tag{2.41}$$

Thus (4.12)-(2.41) imply that

$$\mathcal{V}_1(\tau;\varphi) = \Pi_+\mathcal{V}_+(\tau;\varphi)\Pi_+ + \Pi_-\mathcal{V}_-(\tau;\varphi)\Pi_- + \mathcal{R}_\infty(\tau;\varphi) \tag{2.42}$$

where

$$\begin{aligned}&\mathcal{V}_\pm(\tau;\varphi) := \Phi_\pm(\tau;\varphi)\mathcal{V}(\varphi)\Phi_\pm(\tau;\varphi)^{-1} - \Phi_\pm(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_\pm(\tau;\varphi)^{-1}\,,\\ &\mathcal{R}_\infty(\tau;\varphi) := \mathcal{R}_\infty^{(1)}(\tau;\varphi) - \mathcal{R}_\infty^{(2)}(\tau;\varphi)\,.\end{aligned} \tag{2.43}$$

It remains only to prove that the operator $\mathcal{R}_\infty(\tau;\varphi)$ satisfies the property (2.34). All the terms in (2.39), (2.41) can be analyzed in the same way, then we consider only the operator

$$\mathcal{R}(\tau;\varphi) := [\Pi_+, \Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}]\Pi_- \overset{(2.28)}{=} \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_-\,. \tag{2.44}$$

Uisng the property (2.28), we write

$$\begin{aligned}\mathcal{R}(\tau;\varphi) &= [\Pi_+, \Phi_+(\tau;\varphi)]\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1} + \Phi_+(\tau;\varphi)[\Pi_+, \mathcal{V}(\varphi)]\Phi_-(\tau;\varphi)^{-1}\Pi_-\\ &\quad + \Phi_+(\tau;\varphi)\mathcal{V}(\varphi)[\Pi_+, \Phi_-(\tau;\varphi)^{-1}]\Pi_-\,.\end{aligned} \tag{2.45}$$

By (2.30), $\Pi_\pm \equiv \mathrm{Op}(\chi_\pm(\xi))$. Furthermore (2.29) implies that $\partial_\xi\chi_\pm(\xi) = 0$ for any $|\xi| \geq 1$. Therefore we can apply Lemma 2.10 to the operators $[\Pi_+, \Phi_+(\tau;\varphi)]$, $[\Pi_+, \Phi_-(\tau;\varphi)^{-1}]$ and Lemma 2.7 to the operator $[\Pi_+, \mathcal{V}(\varphi)]$, which together with (2.36) imply that $\mathcal{R}(\tau;\varphi) \in OPS^{-\infty}$.
Proof of (ii). Applying item (i) to the operator $\mathcal{V}(\varphi) = \Pi_+\mathcal{V}_+(\varphi)\Pi_+ + \Pi_-\mathcal{V}_-(\varphi)\Pi_-$, one gets that

$$\begin{aligned}&\mathcal{V}_1(\tau;\varphi) = \Phi_{\omega *}\mathcal{V}(\tau;\varphi) = \Pi_+\mathcal{P}_+(\tau;\varphi)\Pi_+ + \Pi_-\mathcal{P}_-(\tau;\varphi)\Pi_- + OPS^{-\infty}\,,\\ &\mathcal{P}_\pm(\tau;\varphi) := \Phi_\pm(\tau;\varphi)\mathcal{V}(\varphi)\Phi_\pm(\tau;\varphi)^{-1} - \Phi_\pm(\tau;\varphi)\omega\cdot\partial_\varphi\Phi_\pm(\tau;\varphi)^{-1}\,.\end{aligned} \tag{2.46}$$

Using that $\mathcal{V} = \Pi_+ \mathcal{V}_+ \Pi_+ + \Pi_- \mathcal{V}_- \Pi_-$ one gets

$$\begin{aligned}\Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ &= \Pi_+\Phi_+(\tau;\varphi)\Pi_+\mathcal{V}_+(\varphi)\Pi_+\Phi_+(\tau;\varphi)^{-1}\Pi_+ \\ &\quad + \Pi_+\Phi_+(\tau;\varphi)\Pi_-\mathcal{V}_-(\varphi)\Pi_-\Phi_+(\tau;\varphi)^{-1}\Pi_+ \\ &= \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}_+(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ + \mathcal{R}_{\infty,+}(\tau;\varphi) \end{aligned} \tag{2.47}$$

where, using the properties (2.28)

$$\begin{aligned}\mathcal{R}_{\infty,+}(\tau;\varphi) := &\ \Pi_+[\Pi_+, \Phi_+(\tau;\varphi)]\mathcal{V}_+(\varphi)\Pi_+\Phi_+(\tau;\varphi)^{-1}\Pi_+ \\ &+ \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}_+(\varphi)[\Pi_+, \Phi_+(\tau;\varphi)^{-1}]\Pi_+ \\ &+ [\Pi_+, \Phi_+(\tau;\varphi)]\Pi_-\mathcal{V}_-(\varphi)\Pi_-\Phi_+(\tau;\varphi)^{-1}\Pi_+ \, . \end{aligned} \tag{2.48}$$

Using the same arguments used to analyze the remainder $\mathcal{R}(\tau;\varphi)$ in (2.45), one can show that the remainder $\mathcal{R}_{\infty,+}(\tau;\varphi) \in OPS^{-\infty}$ implying that

$$\Pi_+\Phi_+(\tau;\varphi)\mathcal{V}(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ = \Pi_+\Phi_+(\tau;\varphi)\mathcal{V}_+(\varphi)\Phi_+(\tau;\varphi)^{-1}\Pi_+ + OPS^{-\infty} \, . \tag{2.49}$$

In a similar way, one gets that

$$\Pi_-\Phi_-(\tau;\varphi)\mathcal{V}(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_- = \Pi_-\Phi_-(\tau;\varphi)\mathcal{V}_-(\varphi)\Phi_-(\tau;\varphi)^{-1}\Pi_- + OPS^{-\infty} \, . \tag{2.50}$$

Therefore the expansion (2.35) follows by (2.46), (2.47), (2.49), (2.50) and the lemma is proved. □

We now recall some well known properties of the Hilbert transform. The Hilbert transform is defined by

$$\mathcal{H}(1) := 0 \, , \quad \mathcal{H}(e^{\mathrm{i}jx}) := -\mathrm{i}\,\mathrm{sign}(j) e^{\mathrm{i}jx} \, , \quad \forall j \in \mathbb{Z} \setminus \{0\} \, . \tag{2.51}$$

We often identify the operator $\mathcal{H}$ with the operator associated to the symbol $-\mathrm{i}\,\mathrm{sign}(\xi)\chi(\xi)$ (recall the definition (2.13)), since the action on the 2π-periodic functions of $\mathcal{H}$ and $\mathrm{Op}\big(-\mathrm{i}\,\mathrm{sign}(\xi)\chi(\xi)\big)$ is the same, i.e.

$$\mathcal{H} \equiv \mathrm{Op}\Big(-\mathrm{i}\,\mathrm{sign}(\xi)\chi(\xi)\Big) \, . \tag{2.52}$$

We now state some classical results concerning the Hilbert transform.

Lemma 2.16. *Let $a \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1})$. Then the commutator $[a, \mathcal{H}] \in OPS^{-\infty}$.*

Let $\alpha \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ satisfy the condition (2.18). Then by Lemma 2.12, the map $x \mapsto x + \alpha(\varphi, x)$ is a diffeomorphism of the torus, with inverse given by $y \mapsto y + \widetilde{\alpha}(\varphi, y)$ and $\widetilde{\alpha} \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ satisfy the properties stated in Lemma 2.12. The following lemma holds:

Lemma 2.17. *Let $\alpha \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ satisfy the condition (2.18).*
(i) The operator $\Psi(\varphi)[u] := u(x + \alpha(\varphi, x))$ is a bounded linear operator $H^s \to H^s$, for any $s \geq 0$, whose inverse is given by $\Psi(\varphi)^{-1}[u] = u(y + \widetilde{\alpha}(\varphi, y))$ where $y \mapsto y + \widetilde{\alpha}(\varphi, y)$, $\widetilde{\alpha} \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ is the inverse diffeomorphism of $x \mapsto x + \alpha(\varphi, x)$. The map $\Phi(\varphi) := \sqrt{1 + \alpha_x(\varphi, x)}\Psi(\varphi)$ is a symplectic bounded invertible operator $H^s \to H^s$ for any $s \geq 0$ and the inverse is given by $\Phi(\varphi)^{-1} = \sqrt{1 + \widetilde{\alpha}(\varphi, y)}\Psi(\varphi)^{-1}$. Furthermore, the following holds:

$$\sup_{\varphi \in \mathbb{T}^\nu} \{ \|\partial_\varphi^\alpha \Psi(\varphi)^{\pm 1}\|_{\mathcal{B}(H^{s+|\alpha|}, H^s)}, \|\partial_\varphi^\alpha \Phi(\varphi)^{\pm 1}\|_{\mathcal{B}(H^{s+|\alpha|}, H^s)} \} < +\infty \, , \quad \forall s \geq 0 \, , \, \alpha \in \mathbb{N}^\nu \, . \tag{2.53}$$

(ii) Let $A(\varphi) \in OPS^m$. Then $\Phi(\varphi)A(\varphi)\Phi(\varphi)^{-1} \in OPS^m$.

Proof. We prove the lemma for the map $\Phi(\varphi)$. the claimed statements for $\Psi(\varphi)$ follow by similar arguments. Define

$$\Phi(\tau;\varphi)[u] := \sqrt{1 + \tau(\partial_x \alpha)(\varphi, x)}\; u(x + \tau\alpha(\varphi, x)) \, , \quad u \in L^2(\mathbb{T}) \, , \quad \tau \in [0, 1] \, . \tag{2.54}$$

A direct verification shows that $\Phi(\tau;\varphi)$ is an invertible symplectic operator, whose inverse is given by

$$\Phi(\tau;\varphi)^{-1}[u] := \sqrt{1+(\partial_y \widetilde{\alpha})(\tau;\varphi,y)}\; u(y+\widetilde{\alpha}(\tau;\varphi,y))\,, \quad u \in L^2(\mathbb{T}) \tag{2.55}$$

where $y \mapsto y + \widetilde{\alpha}(\tau;\varphi,y)$ is the inverse diffeomorphism of $x \mapsto x + \tau\alpha(\varphi,x)$. A direct calculation shows that the map $\Phi(\tau;\varphi)$ is the flow map of the transport PDE

$$\begin{aligned} &\partial_\tau u = \mathcal{A}(\tau;\varphi)[u]\,, \quad \mathcal{A}(\tau;\varphi) := b_1(\tau;\varphi,x)\partial_x + b_0(\tau;\varphi,x)\,, \\ &b_1 := \frac{\alpha}{\sqrt{1+\tau(\partial_x\alpha)}}\,, \quad b_0 := \frac{\partial_\tau\big(\sqrt{1+\tau(\partial_x\alpha)}\big)}{\sqrt{1+\tau(\partial_x\alpha)}}\,. \end{aligned} \tag{2.56}$$

The estimate (2.53) for $\Phi(\varphi) = \Phi(1;\varphi)$ follows by applying Lemma 2.9. Moreover, by the classical Egorov Theorem (see Theorem in A.0.9 in [25]), one gets that if $A(\varphi) \in OPS^m$, $\Phi(\varphi)A(\varphi)\Phi(\varphi)^{-1} \in OPS^m$. $\square$

Lemma 2.18. *The following property holds:* $\Psi(\varphi)\mathcal{H}\Psi(\varphi)^{-1} - \mathcal{H} \in OPS^{-\infty}$ *and* $\Phi(\varphi)\mathcal{H}\Phi(\varphi)^{-1} - \mathcal{H} \in OPS^{-\infty}$*.*

Proof. The fact that $\Psi(\varphi)\mathcal{H}\Psi(\varphi)^{-1} - \mathcal{H} \in OPS^{-\infty}$ is a classical result. For a detailed proof see for instance Lemmata 2.32, 2.36 in [9]. Let us prove that $\Phi(\varphi)\mathcal{H}\Phi(\varphi)^{-1} - \mathcal{H} \in OPS^{-\infty}$. To shorten notations we neglect the dependence on φ. By Lemma 2.17, we have

$$\begin{aligned} \Phi\mathcal{H}\Phi^{-1} - \mathcal{H} &= \sqrt{1+\alpha_x}\Psi\mathcal{H}\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} - \mathcal{H} \\ &= \sqrt{1+\alpha_x} \circ (\Psi\mathcal{H}\Psi^{-1}) \circ \Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} - \mathcal{H} \\ &= \sqrt{1+\alpha_x} \circ (\Psi\mathcal{H}\Psi^{-1} - \mathcal{H}) \circ \Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} \\ &\quad + \sqrt{1+\alpha_x} \circ \mathcal{H} \circ \Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} - \mathcal{H}\,. \end{aligned} \tag{2.57}$$

Note that $\Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1}$ is a multiplication operator given by

$$\Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} = \sqrt{1+\widetilde{\alpha}_y(\varphi, x+\alpha_x(\varphi,x))} \overset{(2.24)}{=} \frac{1}{\sqrt{1+\alpha_x(\varphi,x)}}\,. \tag{2.58}$$

Hence

$$\begin{aligned} \sqrt{1+\alpha_x} \circ \mathcal{H} \circ \Psi\sqrt{1+\widetilde{\alpha}_y}\Psi^{-1} - \mathcal{H} &= \sqrt{1+\alpha_x} \circ \mathcal{H} \circ \frac{1}{\sqrt{1+\alpha_x}} - \mathcal{H} \\ &= \sqrt{1+\alpha_x}\Big[\mathcal{H}, \frac{1}{\sqrt{1+\alpha_x}}\Big]\,. \end{aligned} \tag{2.59}$$

Finally (2.57)-(2.59), Lemma 2.16 and using that $\Psi\mathcal{H}\Psi^{-1} - \mathcal{H} \in OPS^{-\infty}$ one obtains that

$$\Phi\mathcal{H}\Phi^{-1} - \mathcal{H} = \sqrt{1+\alpha_x} \circ (\Psi\mathcal{H}\Psi^{-1} - \mathcal{H}) \circ \frac{1}{\sqrt{1+\alpha_x}} + \sqrt{1+\alpha_x}\Big[\mathcal{H}, \frac{1}{\sqrt{1+\alpha_x}}\Big] \in OPS^{-\infty}\,.$$

$\square$

We conclude this section by stating an interpolation theorem, which is an immediate consequence of the classical Riesz-Thorin interpolation theorem in Sobolev spaces.

Theorem 2.19. *Let* $0 \le s_0 < s_1$ *and let* $A \in \mathcal{B}(H^{s_0}) \cap \mathcal{B}(H^{s_1})$*. Then for any* $s_0 \le s \le s_1$ *the operator* $A \in \mathcal{B}(H^s)$ *and*

$$\|A\|_{\mathcal{B}(H^s)} \le \|A\|^{\lambda}_{\mathcal{B}(H^{s_0})}\|A\|^{1-\lambda}_{\mathcal{B}(H^{s_1})}\,, \quad \lambda := \frac{s_1 - s}{s_1 - s_0}\,.$$

3 Regularization of the vector field $\mathrm{i}\mathcal{V}(\varphi)$: the case $0 < M < 1$

In this section we develop the regularization procedure on the vector field $\mathrm{i}\mathcal{V}(\varphi) = \mathrm{i}\big(V(\varphi,x)|D|^M + \mathcal{W}(\varphi)\big)$, $0 < M < 1$, see (1.2), which is needed to prove Theorem 1.6. We assume the hypotheses **(H1)**, **(H2)**, $\mathbf{(H3)_{M<1}}$, i.e. $\mathcal{V}(\varphi)$ is self-adjoint, $\mathcal{W} \in OPS^{M-\mathfrak{e}}$, $\mathfrak{e} > 0$ and $V \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ satisfies $\inf_{(\varphi,x)\in\mathbb{T}^{\nu+1}} V(\varphi,x) > 0$.
In Section 3.1 we reduce to constant coefficients the highest order term $V(\varphi,x)|D|^M$, see Proposition 3.1. Then, in Section 3.2, we perform the reduction of the lower order terms up to arbitrarily regularizing remainders, see Proposition 3.3. At each step of the regularization procedure, the reduction to constant coefficients is split in two parts: first we remove the dependence on φ and in a second step, we remove the dependence on x.

3.1 Reduction of the highest order

In order to state precisely the main result of this section, we define

$$\bar{\mathfrak{e}} := \min\{1 - M, \mathfrak{e}\} \tag{3.1}$$

so that

$$M - \bar{\mathfrak{e}} \geq M - \mathfrak{e}\,,\ M - (1 - M) \tag{3.2}$$

(recall that $0 < M < 1$). We prove the following

Proposition 3.1. *Let $\gamma \in (0,1), \tau > \nu - 1$ and $\omega \in DC(\gamma,\tau)$ (recall (1.12)). There exist a symplectic family of invertible maps $\Phi_0(\varphi)$, $\varphi \in \mathbb{T}^\nu$ satisfying*

$$\sup_{\varphi\in\mathbb{T}^\nu} \|\Phi_0(\varphi)^{\pm 1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s \geq 0\,, \tag{3.3}$$

a constant $\lambda > 0$ and a self-adjoint operator $\mathcal{W}_1(\varphi) = \mathrm{Op}\Big(w_1(\varphi,x,\xi)\Big) \in OPS^{M-\bar{\mathfrak{e}}}$ such that

$$(\Phi_0^{-1})_{\omega*}(\mathrm{i}\mathcal{V})(\varphi) = \mathrm{i}\mathcal{V}_1(\varphi) \quad \textit{with} \quad \mathcal{V}_1(\varphi) := \lambda|D|^M + \mathcal{W}_1(\varphi)\,. \tag{3.4}$$

The rest of this subsection is devoted to the proof of Proposition 3.1. In Section 3.1.1 we show how to remove the dependence on φ from the highest order. Then, in Section 3.1.2, we remove the dependence on x.

3.1.1 Time reduction

We consider a function $\alpha \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$ (to be determined) and an operator of the form

$$\mathcal{G}_0^{(1)}(\varphi) := \alpha(\varphi,x)|D|^M + |D|^M\alpha(\varphi,x)\,. \tag{3.5}$$

Note that $\mathcal{G}_0^{(1)}(\varphi) = \mathcal{G}_0^{(1)}(\varphi)^*$ for any $\varphi \in \mathbb{T}^\nu$ and by applying Theorem 2.3,

$$\mathcal{G}_0^{(1)}(\varphi) = \mathrm{Op}\Big(g_0^{(1)}(\varphi,x,\xi)\Big)\,, \quad g_0^{(1)}(\varphi,x,\xi) := 2\alpha(\varphi,x)|\xi|^M\chi(\xi) + S^{M-1} \tag{3.6}$$

(recall the notation (1.15)). Since $\mathcal{G}_0^{(1)}$ fullfills the hypotheses of Lemma 2.9, one has that the flow $\Phi_0^{(1)}(\tau;\varphi)$, $\tau \in [0,1]$, $\varphi \in \mathbb{T}^\nu$ of the autonomous PDE $\partial_\tau u = \mathrm{i}\mathcal{G}_0^{(1)}(\varphi)[u]$, i.e.

$$\begin{cases} \partial_\tau \Phi_0^{(1)}(\tau;\varphi) = \mathrm{i}\mathcal{G}_0^{(1)}(\varphi)\Phi_0^{(1)}(\tau;\varphi) \\ \Phi_0^{(1)}(0;\varphi) = \mathrm{Id} \end{cases} \tag{3.7}$$

is an invertible map $H^s \to H^s$ and satisfies

$$\sup_{\substack{\tau\in[0,1]\\ \varphi\in\mathbb{T}^\nu}} \|\Phi_0^{(1)}(\tau;\varphi)\|_{\mathcal{B}(H^s)}\,, \quad \forall s \geq 0\,. \tag{3.8}$$

Since $\mathcal{G}_0^{(1)}(\varphi)$ is self-adjoint, $\Phi_0^{(1)}(\tau;\varphi)$ is a symplectic map. We set $\Phi_0^{(1)}(\varphi) := \Phi_0^{(1)}(1;\varphi)$ and we have

$$\big((\Phi_0^{(1)})^{-1}\big)_{\omega*}\mathcal{V}(\varphi) = \mathrm{i}\mathcal{V}_0^{(1)}(\varphi)\,, \quad \mathcal{V}_0^{(1)}(\varphi) := \Phi_0^{(1)}(\varphi)\mathcal{V}(\varphi)\Phi_0^{(1)}(\varphi)^{-1} + \mathrm{i}\Phi_0^{(1)}(\varphi)\omega\cdot\partial_\varphi\Phi_0^{(1)}(\varphi)^{-1}\,.$$

By applying Proposition 2.14 with $m = \eta = M$, since

$$v(\varphi,x,\xi) = V(\varphi,x)|\xi|^M\chi(\xi) + w(\varphi,x,\xi) \quad \text{with} \quad w \in S^{M-\mathfrak{e}}\,,$$

by recalling (3.1), (3.2), using that by (2.1), $S^{M-\mathfrak{e}}, S^{M-(1-M)} \subseteq S^{M-\bar{\mathfrak{e}}}$ one gets that

$$\Phi_0^{(1)}(\varphi)\mathcal{V}(\varphi)\Phi_0^{(1)}(\varphi)^{-1} = \mathrm{Op}\Big(V(\varphi,x)|\xi|^M\chi(\xi)\Big) + OPS^{M-\bar{\mathfrak{e}}}\,. \tag{3.9}$$

Moreover, defining $\Psi(\tau;\varphi) := \mathrm{i}\Phi_0^{(1)}(\tau;\varphi)\omega\cdot\partial_\varphi\big(\Phi_0^{(1)}(\tau;\varphi)^{-1}\big)$, a direct calculation shows that

$$\Psi(\tau;\varphi) = \mathrm{i}\int_0^\tau \mathcal{S}(\zeta;\varphi)\,d\zeta\,, \quad \mathcal{S}(\zeta;\varphi) := \Phi_0^{(1)}(\zeta;\varphi)\omega\cdot\partial_\varphi\mathcal{G}_0^{(1)}(\varphi)\Phi_0^{(1)}(\zeta;\varphi)^{-1}, \quad \zeta\in[0,1]\,.$$

Since $\omega\cdot\partial_\varphi\mathcal{G}_0^{(1)}(\varphi) \in OPS^M$, by Proposition 2.14 (applied with $m = \eta = M$) and using (3.6), one has

$$\Psi(\varphi) = \Psi(1;\varphi) = \mathrm{i}\Phi_0^{(1)}(1;\varphi)\omega\cdot\partial_\varphi\big(\Phi_0^{(1)}(1;\varphi)^{-1}\big) = \mathrm{Op}\Big(\psi(\varphi,x,\xi)\Big) \in OPS^M \tag{3.10}$$

with

$$\psi(\varphi,x,\xi) := 2\omega\cdot\partial_\varphi\alpha(\varphi,x)|\xi|^M\chi(\xi) + r_1(\varphi,x,\xi)\,, \quad r_1 \in S^{M-(1-M)} \overset{(3.2),(2.1)}{\subseteq} S^{M-\bar{\mathfrak{e}}}\,. \tag{3.11}$$

Therefore, (3.9)-(3.11) imply that $\mathcal{V}_0^{(1)}(\varphi) = \mathrm{Op}\Big(v_0^{(1)}(\varphi,x,\xi)\Big)$ with

$$v_0^{(1)}(\varphi,x,\xi) = \Big(V(\varphi,x) + 2\omega\cdot\partial_\varphi\alpha(\varphi,x)\Big)|\xi|^M\chi(\xi) + S^{M-\bar{\mathfrak{e}}}\,. \tag{3.12}$$

Defining

$$\langle V\rangle_\varphi(x) := \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu} V(\varphi,x)\,d\varphi\,, \tag{3.13}$$

we want to choose the function α_0 so that

$$V(\varphi,x) + 2\omega\cdot\partial_\varphi\alpha(\varphi,x) = \langle V\rangle_\varphi(x)\,. \tag{3.14}$$

Since $\langle V\rangle_\varphi - V$ has zero average w.r. to $\varphi\in\mathbb{T}^\nu$ and $\omega\in DC(\gamma,\tau)$, one has that

$$\alpha := (\omega\cdot\partial_\varphi)^{-1}\Big[\frac{\langle V\rangle_\varphi - V}{2}\Big] \tag{3.15}$$

solves the equation (3.14). Note that $\alpha\in\mathcal{C}^\infty(\mathbb{T}^\nu\times\mathbb{T},\mathbb{R})$, since $V\in\mathcal{C}^\infty(\mathbb{T}^\nu\times\mathbb{T},\mathbb{R})$ and since V satisfies the hypothesis $(\mathbf{H3})_{\mathbf{M<1}}$, by (3.13), one sees that

$$\inf_{x\in\mathbb{T}}\langle V\rangle_\varphi(x) > 0\,. \tag{3.16}$$

Finally (3.12), (3.14) imply that

$$\mathcal{V}_0^{(1)}(\varphi) = \mathrm{Op}\Big(v_0^{(1)}(\varphi,x,\xi)\Big)\,, \quad v_0^{(1)}(\varphi,x,\xi) = \langle V\rangle_\varphi(x)|\xi|^M\chi(\xi) + S^{M-\bar{\mathfrak{e}}}\,. \tag{3.17}$$

Since $\mathcal{V}(\varphi)$ is self-adjoint and $\Phi_0^{(1)}(\varphi)$ is symplectic, then also $\mathcal{V}_0^{(1)}(\varphi)$ is self-adjoint.

3.1.2 Space reduction

In this section, our purpose is to remove the dependence on x from the highest order term $\mathrm{i}\langle V\rangle_\varphi(x)|D|^M$ of the vector field $\mathrm{i}\mathcal{V}_0^{(1)}(\varphi)$ given in (3.17). To this aim, we consider a function $\beta \in \mathcal{C}^\infty(\mathbb{T},\mathbb{R})$ (that will be fixed later) satisfying the ansatz

$$\inf_{x\in\mathbb{T}}(1+\partial_x\beta(x)) > 0\,. \tag{3.18}$$

We then define

$$b(\tau;x) := -\frac{\beta(x)}{1+\tau\partial_x\beta(x)}\,, \quad (\tau,x)\in[0,1]\times\mathbb{T} \tag{3.19}$$

and we consider the τ-dependent vector field

$$\mathcal{G}_0^{(2)}(\tau) := b(\tau;x)\partial_x + \frac{\partial_x b}{2}\,. \tag{3.20}$$

As explained in Section 2.2, the flow of the PDE $\partial_\tau u = \mathcal{G}_0^{(2)}(\tau)[u]$, i.e. the map $\Phi_0^{(2)}(\tau)$, $\tau\in[0,1]$ which solves

$$\begin{cases}\partial_\tau\Phi_0^{(2)}(\tau) = \mathcal{G}_0^{(2)}(\tau)\Phi_0^{(2)}(\tau)\\ \Phi_0^{(2)}(0) = \mathrm{Id}\,.\end{cases} \tag{3.21}$$

is a bounded invertible symplectic map satisfying

$$\sup_{\tau\in[0,1]}\|\Phi_0^{(2)}(\tau)^{\pm1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s\geq 0\,. \tag{3.22}$$

Note that $\mathcal{G}_0^{(2)}(\tau)$ and $\Phi_0^{(2)}(\tau)$ are independent of $\varphi\in\mathbb{T}^\nu$. We set $\Phi_0^{(2)} := \Phi_0^{(2)}(1)$ and the transformed vector field is given by

$$\big((\Phi_0^{(2)})^{-1}\big)_{\omega*}\mathrm{i}\mathcal{V}_0^{(1)}(\varphi) = \mathrm{i}\mathcal{V}_1(\varphi)\,, \quad \mathcal{V}_1(\varphi) := \Phi_0^{(2)}\mathcal{V}_0^{(1)}(\varphi)(\Phi_0^{(2)})^{-1}\,. \tag{3.23}$$

By applying Proposition 2.13, the operator $\mathcal{V}_1(\varphi) = \mathrm{Op}\Big(v_1(\varphi,x,\xi)\Big)$ admits the expansion

$$v_1(\varphi,x,\xi) = v_0^{(1)}\Big(\varphi, x+\beta(x), (1+\partial_x\beta(x))^{-1}\xi\Big) + S^{M-1}\,. \tag{3.24}$$

Lemma 3.2. *The symbol $v_1(\varphi,x,\xi)$ has the form*

$$v_1(\varphi,x,\xi) = \Big[\langle V\rangle_\varphi(y)\big(1+\partial_y\widetilde{\beta}(y)\big)^M\Big]_{y=x+\beta(x)}|\xi|^M\chi(\xi) + S^{M-\bar{\mathfrak{e}}} \tag{3.25}$$

where we recall the definitions (2.13), (2.14) *and $y\mapsto y+\widetilde{\beta}(y)$ is the inverse diffeomorphism of $x\mapsto x+\beta(x)$.*

Proof. The Lemma follows by using the same aguments used in the proof of Lemma 3.2 in [21], hence the proof is omitted. □

We now determine the function $\widetilde{\beta}$ and a constant $\lambda>0$ so that

$$\langle V\rangle_\varphi(y)\big(1+\partial_y\widetilde{\beta}(y)\big)^M = \lambda\,. \tag{3.26}$$

The equation (3.26) is equivalent to the equation

$$\partial_y\widetilde{\beta}(y) = \frac{\lambda^{\frac{1}{M}}}{\langle V\rangle_\varphi(y)^{\frac{1}{M}}} - 1\,. \tag{3.27}$$

Note that by (3.16), the function $\langle V\rangle_\varphi$ does not vanish. Then we choose λ so that the right hand side of the equation (3.27) has zero average, i.e.

$$\lambda := \Big(\frac{1}{2\pi}\int_{\mathbb{T}}\langle V\rangle_\varphi(y)^{-\frac{1}{M}}\,dy\Big)^{-M} \tag{3.28}$$

and hence we define $\widetilde{\beta}$ as

$$\widetilde{\beta} := \partial_y^{-1}\Big[\frac{\lambda^{\frac{1}{M}}}{\langle V\rangle_\varphi(y)^{\frac{1}{M}}} - 1\Big]\,. \tag{3.29}$$

By the definition (3.28) and recalling the property (3.16), one has that

$$\lambda > 0\,. \tag{3.30}$$

By (3.27), (3.16) one has that $\widetilde{\beta} \in \mathcal{C}^\infty(\mathbb{T})$ and satisfies

$$\inf_{y\in\mathbb{T}}(1 + \partial_y\widetilde{\beta}(y)) > 0\,,$$

hence, by applying Lemma 2.12, the inverse diffeomorphism $x \mapsto x + \beta(x)$ satisfies the ansatz (3.18). Finally, (3.25), (3.26) imply that

$$\mathcal{V}_1(\varphi) = \mathrm{Op}\Big(v_1(\varphi, x, \xi)\Big)\,, \quad v_1(\varphi, x, \xi) = \lambda|\xi|^M\chi(\xi) + w_1(\varphi, x, \xi)\,, \quad w_1 \in S^{M-\bar{\mathfrak{e}}}\,. \tag{3.31}$$

We then define $\Phi_0(\varphi) := \Phi_0^{(1)}(\varphi) \circ \Phi_0^{(2)}$. By (3.8), (3.22), the symplectic map $\Phi_0(\varphi)$ satisfies the property (3.3). Since Φ_0 is symplectic the vector field $\mathrm{i}\mathcal{V}_1$ is Hamiltonian, i.e. $\mathcal{V}_1(\varphi)$ is self-adjoint. Since $\lambda \in \mathbb{R}$, and hence $\lambda|D|^M$ is self-adjoint, then also $\mathcal{W}_1(\varphi) = \mathcal{V}_1(\varphi) - \lambda|D|^M$ is self-adjoint and then the proof of Proposition 3.1 is concluded.

3.2 Reduction of the lower order terms

We now prove the following

Proposition 3.3. *Let $\gamma \in (01), \tau > \nu - 1$, $\omega \in DC(\gamma, \tau)$ and $N \in \mathbb{N}$. For any $n = 1, \dots, N$ there exists a linear Hamiltonian vector field $\mathrm{i}\mathcal{V}_n(\varphi)$ of the form*

$$\mathcal{V}_n(\varphi) := \lambda|D|^M + \mu_n(D) + \mathcal{W}_n(\varphi) \tag{3.32}$$

where

$$\mu_n(D) := \mathrm{Op}\Big(\mu_n(\xi)\Big)\,, \qquad \mu_n \in S^{M-\bar{\mathfrak{e}}}\,, \tag{3.33}$$

$$\mathcal{W}_n(\varphi) := \mathrm{Op}\Big(w_n(\varphi, x, \xi)\Big)\,, \qquad w_n \in S^{M-n\bar{\mathfrak{e}}}\,, \tag{3.34}$$

with $\mu_n(\xi)$ real and $\mathcal{W}_n(\varphi)$ self-adjoint, i.e. $w_n = w_n^*$ (see Theorem 2.5).
For any $n \in \{1, \dots, N-1\}$, there exists a symplectic invertible map $\Phi_n(\varphi)$ satisfying

$$\sup_{\varphi\in\mathbb{T}^\nu} \|\Phi_n(\varphi)^{\pm1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s \geq 0 \tag{3.35}$$

and

$$\mathrm{i}\mathcal{V}_{n+1}(\varphi) = (\Phi_n^{-1})_{\omega*}(\mathrm{i}\mathcal{V}_n)(\varphi)\,, \qquad \forall n \in \{1, \dots, N-1\}\,. \tag{3.36}$$

The remaining part of this section is devoted to the proof of this Proposition. We describe the inductive step. Given $n \in \{1, \dots, N\}$, we assume that the vector field $\mathrm{i}\mathcal{V}_n(\varphi)$ satisfies the properties (3.32)-(3.34). The reduction to constant coefficients of the order $M - n\bar{\mathfrak{e}}$ is split in two parts: in Section 3.2.1 we eliminate the dependence on φ from the symbol w_n and in Section 3.2.2, we eliminate the dependence on x.

3.2.1 Time reduction

We consider an operator

$$\begin{aligned} & \mathcal{G}_n^{(1)}(\varphi) = \mathrm{Op}(g_n^{(1)}(\varphi, x, \xi)) \,, \qquad g_n^{(1)} \in S^{M - n\bar{\mathfrak{e}}} \,, \\ & \mathcal{G}_n^{(1)}(\varphi) = \mathcal{G}_n^{(1)}(\varphi)^* \,. \quad \forall \varphi \in \mathbb{T}^\nu \end{aligned} \tag{3.37}$$

Since the operator $\mathcal{G}_n^{(1)}$ satisfies the hypotheses of Lemma 2.9, the flow $\Phi_n^{(1)}(\tau; \varphi)$ of the autonomous PDE $\partial_\tau u = \mathrm{i}\mathcal{G}_n^{(1)}(\varphi)u$, i.e.

$$\begin{cases} \partial_\tau \Phi_n^{(1)}(\tau; \varphi) = \mathrm{i}\mathcal{G}_n^{(1)}(\varphi)\Phi_n^{(1)}(\tau; \varphi) \\ \Phi_n^{(1)}(0; \varphi) = \mathrm{Id} \,. \end{cases} \tag{3.38}$$

is a well defined, invertible map satisfying

$$\sup_{\substack{\tau \in [0,1] \\ \varphi \in \mathbb{T}^\nu}} \|\Phi_n^{(1)}(\tau; \varphi)\|_{\mathcal{B}(H^s)} < +\infty \,, \quad \forall s \geq 0 \,. \tag{3.39}$$

Furthermore, since $\mathcal{G}_n^{(1)}$ is self-adjoint, the map $\Phi_n^{(1)}(\tau; \varphi)$ is symplectic. We define $\Phi_n^{(1)}(\varphi) := \Phi_n^{(1)}(1; \varphi)$. We then compute

$$\begin{aligned} & (\Phi_n^{(1)})_{\omega *}^{-1} \mathrm{i}\mathcal{V}_n(\varphi) = \mathrm{i}\mathcal{V}_n^{(1)}(\varphi) \,, \\ & \mathcal{V}_n^{(1)}(\varphi) := \Phi_n^{(1)}(\varphi)\mathcal{V}_n(\varphi)\Phi_n^{(1)}(\varphi)^{-1} + \mathrm{i}\Phi_n^{(1)}(\varphi)\omega \cdot \partial_\varphi \Phi_n^{(1)}(\varphi)^{-1} \,. \end{aligned} \tag{3.40}$$

A direct calculation shows that

$$\begin{aligned} & \mathrm{i}\Phi_n^{(1)}(\varphi)\omega \cdot \partial_\varphi \Phi_n^{(1)}(\varphi)^{-1} = \int_0^1 \mathcal{S}_n(\tau; \varphi) \, d\tau \,, \\ & \mathcal{S}_n(\tau; \varphi) := \Phi_n^{(1)}(\tau; \varphi)\omega \cdot \partial_\varphi \mathcal{G}_n^{(1)}(\varphi)\Phi_n^{(1)}(\tau; \varphi)^{-1} \,. \end{aligned} \tag{3.41}$$

By (3.40)-(3.41), by applying Proposition 2.14 (with $m = M$ and $\eta = M - n\bar{\mathfrak{e}}$ for the term $\Phi_n^{(1)}(\varphi)\mathcal{V}_n(\varphi)\Phi_n^{(1)}(\varphi)^{-1}$ and $m = \eta = M - n\bar{\mathfrak{e}}$ for the term given in (3.41)), one gets that

$$\begin{aligned} & \mathcal{V}_n^{(1)}(\varphi) = \mathrm{Op}\big(v_n^{(1)}(\varphi, x, \xi)\big) \in OPS^M \,, \\ & v_n^{(1)}(\varphi, x, \xi) = \lambda |\xi|^M \chi(\xi) + \mu_n(\xi) + w_n(\varphi, x, \xi) + \omega \cdot \partial_\varphi g_n^{(1)}(\varphi, x, \xi) + S^{M - (n+1)\bar{\mathfrak{e}}} \,. \end{aligned} \tag{3.42}$$

In order to eliminate the φ-dependence from the symbol $w_n(\varphi, x, \xi) + \omega \cdot \partial_\varphi g_n^{(1)}(\varphi, x, \xi)$ (of order $M - n\bar{\mathfrak{e}}$), we choose the symbol $g_n^{(1)}(\varphi, x, \xi)$ so that

$$w_n(\varphi, x, \xi) + \omega \cdot \partial_\varphi g_n^{(1)}(\varphi, x, \xi) = \langle w_n \rangle_\varphi(x, \xi) \tag{3.43}$$

where we recall the definition (2.9). Then, since $\omega \in DC(\gamma, \tau)$, the equation (3.43) is solved by defining

$$g_n^{(1)}(\varphi, x, \xi) := (\omega \cdot \partial_\varphi)^{-1} \Big[\langle w_n \rangle_\varphi(x, \xi) - w_n(\varphi, x, \xi) \Big] \,. \tag{3.44}$$

Note that, since $\mathcal{W}_n = \mathrm{Op}(w_n)$ is self-adjoint, i.e. $w_n = w_n^*$, by applying Lemma 2.8, one gets that $g_n^{(1)} = (g_n^{(1)})^*$ implying that $\mathcal{G}_n^{(1)} = \mathrm{Op}(g_n^{(1)})$ is self-adjoint. By (3.42), (3.43) one then has

$$\begin{aligned} & \mathcal{V}_n^{(1)} = \mathrm{Op}(v_n^{(1)}) \,, \quad v_n^{(1)}(\varphi, x, \xi) = \lambda |\xi|^M \chi(\xi) + \mu_n(\xi) + \langle w_n \rangle_\varphi(x, \xi) + S^{M - (n+1)\bar{\mathfrak{e}}} \,, \\ & \mu_n \in S^{M - \bar{\mathfrak{e}}} \,, \quad \langle w_n \rangle_\varphi \in S^{M - n\bar{\mathfrak{e}}} \,. \end{aligned} \tag{3.45}$$

Since $\Phi_n^{(1)}(\varphi)$ is symplectic one has that $\mathcal{V}_n^{(1)}(\varphi)$ is self-adjoint.

3.2.2 Space reduction

In order to remove the x-dependence from the symbol $\langle w_n\rangle_\varphi$ in (3.45), we consider a φ-independent pseudo-differential operator

$$\mathcal{G}_n^{(2)} = \mathrm{Op}\Big(g_n^{(2)}(x,\xi)\Big)\,, \quad g_n^{(2)} \in S^{1-n\bar{\mathfrak{e}}}\,, \quad \mathcal{G}_n^{(2)} = (\mathcal{G}_n^{(2)})^*\,. \tag{3.46}$$

Since the operator $\mathcal{G}_n^{(2)}$ satisfies the hypotheses of Lemma 2.9, the flow $\Phi_n^{(2)}(\tau)$, $\tau\in[0,1]$ of the PDE $\partial_\tau u = \mathrm{i}\mathcal{G}_n^{(2)}[u]$, i.e.

$$\begin{cases}\partial_\tau \Phi_n^{(2)}(\tau) = \mathrm{i}\mathcal{G}_n^{(2)}\Phi_n^{(2)}(\tau)\\ \Phi_n^{(2)}(0) = \mathrm{Id}\end{cases} \tag{3.47}$$

is a well defined invertible map satisfying

$$\sup_{\tau\in[0,1]} \|\Phi_n^{(2)}(\tau)^{\pm 1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s\geq 0\,,. \tag{3.48}$$

Note that $\mathcal{G}_n^{(2)}$ as well as $\Phi_n^{(2)}$ is φ-independent. Since $\mathcal{G}_n^{(2)}$ is self-adjoint, the flow $\Phi_n^{(2)}(\tau)$ is symplectic. We set $\Phi_n^{(2)} := \Phi_n^{(2)}(1)$. The transformed vector field is then given by

$$\mathrm{i}\mathcal{V}_{n+1} := ((\Phi_n^{(2)})^{-1})_{\omega*}(\mathrm{i}\mathcal{V}_n^{(1)})(\varphi)\,, \quad \mathcal{V}_{n+1}(\varphi) = \Phi_n^{(2)}\mathcal{V}_n^{(1)}(\varphi)(\Phi_n^{(2)})^{-1}\,. \tag{3.49}$$

We prove the following

Lemma 3.4. *The operator $\mathcal{V}_{n+1}(\varphi) = \mathrm{Op}\big(v_{n+1}(\varphi,x,\xi)\big)$, $v_{n+1}\in S^M$ has the expansion*

$$v_{n+1} = \lambda|\xi|^M\chi(\xi) + \mu_n + \langle w_n\rangle_\varphi - M\lambda|\xi|^{M-2}\xi\chi(\xi)\partial_x g_n^{(2)} + S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.50}$$

Proof. By applying Proposition 2.14 (with $m=M$ and $\eta = 1-n\bar{\mathfrak{e}}$), one gets that

$$\mathcal{V}_{n+1}(\varphi) = \mathrm{Op}\Big(v_{n+1}(\varphi,x,\xi)\Big)\,, \quad v_{n+1}\in S^M \tag{3.51}$$

and the symbol v_{n+1} admits the expansion

$$v_{n+1} = v_n^{(1)} + \{g_{n+1}^{(2)}, v_n^{(1)}\} + p_{n,\geq 2}\,, \quad p_{n,\geq 2}\in S^{M-2n\bar{\mathfrak{e}}} \overset{(2.1)}{\subseteq} S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.52}$$

Recalling (3.45), one gets

$$\begin{aligned}\{g_n^{(2)}, v_n^{(1)}\} &= \{g_n^{(2)}, \lambda|\xi|^M\chi(\xi) + \mu_n + \langle w_n\rangle_\varphi + w_n^{(1)}\}\\ &= \{g_n^{(2)}, \lambda|\xi|^M\chi(\xi)\} + \{g_n^{(2)}, \mu_n + \langle w_n\rangle_\varphi + w_n^{(1)}\}\\ &= -\lambda(\partial_x g_n^{(2)})\Big(\partial_\xi(|\xi|^M\chi(\xi))\Big) + \{g_n^{(2)}, \mu_n + \langle w_n\rangle_\varphi + w_n^{(1)}\}\\ &= -M\lambda|\xi|^{M-2}\xi\chi(\xi)\partial_x g_n^{(2)} - \lambda|\xi|^M(\partial_\xi\chi(\xi))(\partial_x g_n^{(2)}) + \{g_n^{(2)}, \mu_n + \langle w_n\rangle_\varphi + w_n^{(1)}\}\,.\end{aligned} \tag{3.53}$$

Since $\partial_\xi\chi(\xi) = 0$ for $|\xi|\geq 1$ (see (2.13)), by Corollary 2.4 and by (3.45), (3.46), one gets that

$$M\lambda|\xi|^{M-2}\xi\chi(\xi)\partial_x g_n^{(2)} \in S^{M-n\bar{\mathfrak{e}}}\,, \quad \lambda|\xi|^M(\partial_\xi\chi(\xi))(\partial_x g_n^{(2)}) \in S^{-\infty}\,, \quad \{g_n^{(2)}, \mu_n\}\in S^{M-(n+1)\bar{\mathfrak{e}}}\,, \tag{3.54}$$

$$\{g_n^{(2)}, \langle w_n\rangle_\varphi\} \in S^{M-2n\bar{\mathfrak{e}}} \subseteq S^{M-(n+1)\bar{\mathfrak{e}}}\,, \quad \{g_n^{(2)}, w_n^{(1)} \in S^{M-(2n+1)\bar{\mathfrak{e}}} \subseteq S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.55}$$

Thus, the expansion (3.50) follows by (3.45), (3.52)-(3.55). ☐

Lemma 3.5. *There exists a symbol $g_n^{(2)}\in S^{1-n\bar{\mathfrak{e}}}$ with $g_n^{(2)} = (g_n^{(2)})^*$ and satisfying*

$$\langle w_n\rangle_\varphi - \langle w_n\rangle_{\varphi,x} - M\lambda|\xi|^{M-2}\xi\chi(\xi)\partial_x g_n^{(2)} \in S^{M-(n+1)\bar{\mathfrak{e}}} \tag{3.56}$$

where

$$\langle w_n\rangle_{\varphi,x}(\xi) := \frac{1}{2\pi}\int_{\mathbb{T}}\langle w_n\rangle_\varphi(x,\xi)\,dx = \frac{1}{(2\pi)^{\nu+1}}\int_{\mathbb{T}^{\nu+1}} w_n(\varphi,x,\xi)\,d\varphi\,dx\,. \tag{3.57}$$

Proof. Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function satisfying

$$\begin{aligned} &\chi_0(\xi) = 1\,, \quad \forall |\xi| \geq 2\,, \\ &\chi_0(\xi) = 0\,, \quad \forall |\xi| \leq \frac{3}{2}\,. \end{aligned} \tag{3.58}$$

Writing $1 = \chi_0 + 1 - \chi_0$, one gets that

$$\begin{aligned} &-\lambda M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n^{(2)})(x,\xi) + \langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi) \\ &= -\lambda M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n^{(2)})(x,\xi) + \chi_0(\xi)\big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\big) \\ &\quad + \big(1-\chi_0(\xi)\big)\big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\big)\,. \end{aligned} \tag{3.59}$$

By the definition of χ_0 given in (3.58), one easily gets that

$$\big(1-\chi_0(\xi)\big)\big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\big) \in S^{-\infty}\,, \tag{3.60}$$

therefore we look for a solution $g_n^{(2)}$ of the equation

$$-\lambda M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n^{(2)})(x,\xi) + \chi_0(\xi)\big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\big) \in S^{M-(n+1)\bar{\mathfrak{e}}}\,. \tag{3.61}$$

Since we require that $\mathcal{G}_n^{(2)} = \mathrm{Op}(g_n^{(2)})$ is self-adjoint, we look for a symbol of the form

$$g_n^{(2)}(x,\xi) = \sigma_n(x,\xi) + \sigma_n^*(x,\xi) \in S^{1-n\bar{\mathfrak{e}}} \tag{3.62}$$

with the property that

$$\sigma_n^*(x,\xi) = \sigma_n(x,\xi) + r_n(x,\xi), \qquad r_n \in S^{-n\bar{\mathfrak{e}}}. \tag{3.63}$$

Plugging the ansatz (3.62) into the equation (3.61), using (3.63) and since

$$-\lambda M|\xi|^{M-2}\xi\chi(\xi)(\partial_x r_n)(x,\xi) \in S^{M-1-n\bar{\mathfrak{e}}} \subseteq S^{M-(n+1)\bar{\mathfrak{e}}}, \tag{3.64}$$

we are led to solve the equation

$$-2\lambda M|\xi|^{M-2}\xi\chi(\xi)(\partial_x\sigma_n)(x,\xi) + \chi_0(\xi)\Big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\Big) = 0 \tag{3.65}$$

whose solution is given by

$$\sigma_n(x,\xi) := \frac{\chi_0(\xi)\partial_x^{-1}\Big[\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\Big]}{2\lambda M|\xi|^{M-2}\xi}\,. \tag{3.66}$$

Since $w_n, \langle w_n\rangle_x \in S^{M-n\bar{\mathfrak{e}}}$, $\frac{\chi_0(\xi)}{|\xi|^{M-2}\xi} \in S^{1-M}$, one gets that $\sigma_n \in S^{1-n\bar{\mathfrak{e}}}$ and hence also $g_n = \sigma_n + \sigma_n^* \in S^{1-n\bar{\mathfrak{e}}}$. We now use Lemma 2.6 with $\varphi(\xi) = \frac{\chi_0(\xi)}{2\lambda M|\xi|^{M-2}\xi}$, $a(x,\xi) = \partial_x^{-1}\Big[\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\Big]$. Recalling that $w_n = w_n^*$, by Lemma 2.8 one has that $a = a^*$. Hence we can apply Lemma 2.6, obtaining that the ansatz (3.63) is satisfied. By (3.59), (3.62), (3.63), (3.65) one then gets

$$\begin{aligned} &-\lambda M\,|\xi|^{M-2}\xi\chi(\xi)(\partial_x g_n^{(2)})(x,\xi) + \langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi) \\ &= \big(1-\chi_0(\xi)\big)\big(\langle w_n\rangle_\varphi(x,\xi) - \langle w_n\rangle_{\varphi,x}(\xi)\big) - \lambda M|\xi|^{M-2}\xi\chi(\xi)(\partial_x r_n)(x,\xi) \end{aligned}$$

and recalling (3.60), (3.64) one then gets (3.56). □

By applying Lemmata 3.4, 3.5, one obtains that

$$\begin{aligned} &\mathcal{V}_{n+1}(\varphi) = \mathrm{Op}\Big(v_{n+1}(\varphi,x,\xi)\Big) \\ &v_{n+1}(\varphi,x,\xi) = \lambda|\xi|^M\chi(\xi) + \mu_{n+1}(\xi) + w_{n+1}(\varphi,x,\xi) \\ &\mu_{n+1}(\xi) := \mu_n + \langle w_n\rangle_{\varphi,x} \in S^{M-\bar{\mathfrak{e}}}\,, \quad w_{n+1} \in S^{M-(n+1)\bar{\mathfrak{e}}} \end{aligned} \tag{3.67}$$

Since $\Phi^{(2)}_{n+1}$ is symplectic, then $\mathcal{V}_{n+1}(\varphi)$ is self-adjoint. Since $w_n = w_n^*$ and $\mu_n(\xi)$ is real by the induction hypothesis, Lemma 2.8 implies that $\langle w_n \rangle_{\varphi,x}(\xi)$ is real and therefore $\mu_{n+1}(\xi) = \mu_n(\xi) + \langle w_n \rangle_{\varphi,x}(\xi)$ is real too. This implies that

$$\mathcal{W}_{n+1}(\varphi) = \mathcal{V}_{n+1}(\varphi) - \mathrm{Op}\Big(\lambda|\xi|^M \chi(\xi) + \mu_{n+1}(\xi)\Big)$$

is self-adjoint. The proof of Proposition 3.3 is then concluded by setting $\Phi_n(\varphi) := \Phi_n^{(1)}(\varphi) \circ \Phi_n^{(2)}$ and by applying (3.39), (3.48) to obtain (3.35).

3.3 Proof of Theorem 1.6.

Let $K > 0$ and let us fix the integer $N_K \in \mathbb{N}$ so that $M - N_K \bar{\mathfrak{e}} < -K$, i.e.

$$N_K := \Big[\frac{M+K}{\bar{\mathfrak{e}}}\Big] + 1 \tag{3.68}$$

(recall the definition of $\bar{\mathfrak{e}}$ given in (3.1)) where for any $a \in \mathbb{R}$, we denote by $[a]$ its integer part. Then we define

$$\begin{aligned} &\mathcal{T}_K(\varphi) := \Phi_0(\varphi)^{-1} \circ \Phi_1(\varphi)^{-1} \circ \ldots \circ \Phi_{N_K-1}(\varphi)^{-1}\,, \quad \varphi \in \mathbb{T}^\nu \\ &\mathcal{R}_K(\varphi) := \mathcal{W}_{N_K}(\varphi)\,, \quad \varphi \in \mathbb{T}^\nu \\ &\lambda_K(D) := \lambda|D|^M + \mu_{N_K}(D) \end{aligned} \tag{3.69}$$

where the map $\Phi_0(\varphi)$ is given in Proposition 3.1, λ is defined in (3.28) and for any $n \in \{1, \ldots, N_K - 1\}$, $\Phi_n(\varphi)$, $\mathcal{W}_n(\varphi)$, $\mu_n(D)$ are given in Proposition 3.3. By (3.3), (3.35), one gets that $\mathcal{T}_K^{\pm 1} \in \mathcal{B}(H^s)$, $s \geq 0$ with $\sup_{\varphi \in \mathbb{T}^\nu} \|\mathcal{T}_K(\varphi)^{\pm 1}\|_{\mathcal{B}(H^s)} < +\infty$. Finally, by (3.4), (3.32), (3.36) one obtains (1.13), with $\lambda_K(D)$, $\mathcal{R}_K(\varphi)$ defined in (3.69).

4 Regularization of the vector field $\mathrm{i}\mathcal{V}(\varphi)$: the case $M = 1$

In this section we develop the regularization procedure on the vector field $\mathrm{i}\mathcal{V}(\varphi)$, $\mathcal{V}(\varphi) = V(\varphi, x)|D| + \mathcal{W}(\varphi)$ needed to prove Theorem 1.7. Recall that in order to prove such a theorem, we assume the hypotheses (**H1**), (**H2**), $(\mathbf{H3})_{\mathbf{M=1}}$, hence $\mathcal{V}(\varphi)$ is self-adjoint, $\mathcal{W} \in OPS^{1-\mathfrak{e}}$ for some $\mathfrak{e} > 0$ and $V(\varphi, x) = 1 + \varepsilon P(\varphi, x)$ with $P \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$. In Section 4.1 we reduce to constant coefficients the highest order term $\mathrm{i}V(\varphi, x)|D|$, see Proposition 4.1. In order to perform such a reduction, we need a smallness condition on the parameter ε, since we shall apply Proposition 6.1. Then, in Section 4.2, we perform the reduction of the lower order terms up to arbitrarily regularizing remainders, see Proposition 4.2.

4.1 Reduction of the highest order

In this section we prove the following proposition. We recall that for a function $\mu : \Omega_o \to \mathbb{R}$, $\Omega_o \subseteq \mathbb{R}^\nu$, given $\gamma > 0$, we define

$$|\mu|^{\mathrm{Lip}(\gamma)} := |\mu|^{\sup} + \gamma |\mu|^{\mathrm{lip}}\,, \quad |\mu|^{\sup} := \sup_{\omega \in \Omega_o} |\mu(\omega)|\,, \quad |\mu|^{\mathrm{lip}} := \sup_{\substack{\omega_1, \omega_2 \in \Omega_o \\ \omega_1 \neq \omega_2}} \frac{|\mu(\omega_1) - \mu(\omega_2)|}{|\omega_1 - \omega_2|}\,. \tag{4.1}$$

In order to state precisely the main result of this section we define the constant $\bar{\mathfrak{e}} > 0$ as

$$\bar{\mathfrak{e}} := \min\{\mathfrak{e}, 1\} \quad \text{so that} \quad 1 - \bar{\mathfrak{e}} \geq \max\{1 - \mathfrak{e}\,,\, 0\}\,. \tag{4.2}$$

Proposition 4.1. *Let $\gamma \in (0,1)$ and $\tau > \nu$. Then there exists $\delta \in (0,1)$ such that if $\varepsilon\gamma^{-1} \leq \delta$ the following holds. There exist two Lipschitz functions $\mu_\pm : \Omega \to \mathbb{R}$ satisfying $|\mu_\pm - 1|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon$ such that for any $\omega \in \Omega^{\mu_+}_{\gamma,\tau} \cap \Omega^{\mu_-}_{\gamma,\tau}$, where*

$$\Omega^{\mu_\pm}_{\gamma,\tau} := \Big\{\omega \in \Omega : |\omega \cdot \ell + \mu_\pm(\omega)\, j| \geq \frac{\gamma}{\langle \ell \rangle^\tau}\,, \quad \forall (\ell, j) \in \mathbb{Z}^{\nu+1} \setminus \{(0,0)\}\Big\} \tag{4.3}$$

there exists an invertible map $\Phi_0(\varphi)$ *satisfying*

$$\sup_{\varphi \in \mathbb{T}^\nu} \|\Phi_0(\varphi)^{\pm 1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s \geq 0 \tag{4.4}$$

such that the push forward of the vector field $\mathrm{i}\mathcal{V}(\varphi)$ *has the form*

$$\begin{aligned} &(\Phi_0)_{\omega *}(\mathrm{i}\mathcal{V})(\varphi) = \mathrm{i}\mathcal{V}_1(\varphi)\,, \\ &\mathcal{V}_1(\varphi) = \Pi_+ \mathcal{V}_{1,+}(\varphi)\Pi_+ + \Pi_- \mathcal{V}_{1,-}(\varphi)\Pi_- + OPS^{-\infty}\,, \\ &\mathcal{V}_{1,\pm}(\varphi) = \lambda_\pm |D| + \mathcal{W}_{1,\pm}(\varphi) \end{aligned} \tag{4.5}$$

where the projection operators Π_+, Π_- *are defined in* (2.27)*, the functions* $\lambda_\pm : \Omega^\gamma_{\mu_+} \cap \Omega^\gamma_{\mu_-} \to \mathbb{R}$ *are Lipschitz and satisfy* $|\lambda_\pm - 1|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon$*,* $\mathcal{W}_{1,\pm}(\varphi) \in OPS^{1-\bar{\mathfrak{e}}}$ *are self-adjoint operators.*

The rest of this section is devoted to the proof of Proposition 4.1. Let $\alpha_+, \alpha_- \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$ satisfy the ansatz

$$\inf_{(\varphi,x) \in \mathbb{T}^{\nu+1}} \big(1 + (\partial_x \alpha_\pm)(\varphi, x)\big) > 0\,. \tag{4.6}$$

Then, by applying Lemma 2.17, the operators

$$\Phi_\pm(\varphi) := \sqrt{1 + (\partial_x \alpha_\pm)(\varphi, x)}\,\Psi_\pm(\varphi)\,, \quad \Psi_\pm(\varphi)[u] := u(x + \alpha_\pm(\varphi, x)) \tag{4.7}$$

are symplectic bounded linear operators $H^s \to H^s$, $s \geq 0$ with inverse given by

$$\Phi_\pm(\varphi)^{-1}[u] := \sqrt{1 + (\partial_x \widetilde{\alpha}_\pm)(\varphi, x)}\,\Psi_\pm(\varphi)^{-1}[u]\,, \quad \Psi(\varphi)^{-1}[u](y) = u(y + \widetilde{\alpha}_\pm(\varphi, y))\,, \tag{4.8}$$

where $y \mapsto y + \widetilde{\alpha}(\varphi, y)$ is the inverse diffeomorphism of $x \mapsto x + \alpha_\pm(\varphi, x)$ and. By Lemma 2.12, $\widetilde{\alpha}_\pm$ satisfies

$$\inf_{(\varphi,y) \in \mathbb{T}^{\nu+1}} \big(1 + (\partial_y \widetilde{\alpha}_\pm)(\varphi, y)\big) > 0\,. \tag{4.9}$$

We then consider the operator

$$\Phi_0(\varphi) := \Phi_+(\varphi)^{-1}\Pi_+ + \Phi_-(\varphi)^{-1}\Pi_-\,, \quad \tau \in [0,1]\,, \tag{4.10}$$

whose inverse is given by

$$\Phi_0(\varphi)^{-1} := \Pi_+ \Phi_+(\varphi) + \Pi_- \Phi_-(\varphi)\,, \tag{4.11}$$

see Lemma 2.15-(i). The property (4.4) for $\Phi_0(\varphi)^{\pm 1}$ holds by applying Lemma 2.17-(i). By Lemma 2.15, the push forward $\mathrm{i}\mathcal{V}_1(\varphi) := (\Phi_0)_{\omega *}(\mathrm{i}\mathcal{V})(\varphi)$ is given by

$$(\Phi_0)_{\omega *}(\mathrm{i}\mathcal{V})(\varphi) = \mathrm{i}\mathcal{V}_1(\varphi)\,, \quad \mathcal{V}_1(\varphi) := \Phi_0(\varphi)^{-1}\mathcal{V}(\varphi)\Phi_0(\varphi) + \mathrm{i}\Phi_0(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi_0(\varphi)\,, \tag{4.12}$$

$$\mathcal{V}_1(\varphi) = \Pi_+ \mathcal{V}_{1,+}(\varphi)\Pi_+ + \Pi_- \mathcal{V}_{1,-}(\varphi)\Pi_- + OPS^{-\infty} \tag{4.13}$$

where

$$\mathcal{V}_{1,\pm}(\varphi) := \Phi_\pm(\varphi)\mathcal{V}(\varphi)\Phi_\pm(\varphi)^{-1} + \mathrm{i}\Phi_\pm(\varphi)\omega \cdot \partial_\varphi \Phi_\pm(\varphi)^{-1}\,. \tag{4.14}$$

Note that, since $\Phi_\pm(\varphi)$ are symplectic maps, then $\mathcal{V}_{1,\pm}(\varphi)$ are self-adjoint operators. This implies that even if the transformation $\Phi(\varphi)$ (see (4.10)) is not symplectic, the transformed vector field $\mathrm{i}\mathcal{V}_1(\varphi)$ is Hamiltonian *up to smoothing operators.*

4.1.1 Expansion of $\mathcal{V}_{1,\pm}(\varphi)$.

In this section we provide an expansion of the operators $\mathcal{V}_{1,\pm}(\varphi)$ given in (4.14). One has

$$\Phi_\pm(\varphi)\mathcal{V}(\varphi)\Phi_\pm(\varphi)^{-1} = \Phi_\pm(\varphi)V(\varphi, x)|D|\Phi_\pm(\varphi)^{-1} + \Phi_\pm(\varphi)\mathcal{W}(\varphi)\Phi_\pm(\varphi)^{-1}\,. \tag{4.15}$$

By applying Lemma 2.17-(ii), using that $V|D| \in OPS^1$, $\mathcal{W} \in OPS^{1-\mathfrak{e}}$, one gets that

$$\Phi_\pm(\varphi)V(\varphi,x)|D|\Phi_\pm(\varphi)^{-1} \in OPS^1\,, \quad \Phi_\pm(\varphi)\mathcal{W}(\varphi)\Phi_\pm(\varphi)^{-1} \in OPS^{1-\mathfrak{e}}\,. \tag{4.16}$$

In order to compute the highest order term in (4.15) (the term of order 1), we need to expand the pseudo-differential operator $\Phi_\pm(\varphi)V(\varphi,x)|D|\Phi_\pm(\varphi)^{-1} \in OPS^1$. We write $|D| = \partial_x\mathcal{H}$ where $\mathcal{H}$ is the Hilbert transform, see (2.51), (2.52). Hence,

$$\begin{aligned}\Phi_\pm(\varphi)V(\varphi,x)|D|\Phi_\pm(\varphi)^{-1} &= \Phi_\pm(\varphi)V(\varphi,x)\partial_x\Phi_\pm(\varphi)^{-1}\Phi_\pm(\varphi)\mathcal{H}\Phi_\pm(\varphi)^{-1}\\ &= \Phi_\pm(\varphi)V(\varphi,x)\partial_x\Phi_\pm(\varphi)^{-1}\mathcal{H}\\ &\quad + \Phi_\pm(\varphi)V(\varphi,x)\partial_x\Phi_\pm(\varphi)^{-1}\Big(\Phi_\pm(\varphi)\mathcal{H}\Phi_\pm(\varphi)^{-1} - \mathcal{H}\Big)\\ &= \sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(V(\varphi,x)(1+(\partial_y\widetilde{\alpha}_\pm))^{\frac{3}{2}}\Big)\partial_x\mathcal{H} + \mathcal{R}_{\pm,V}(\varphi)\,,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\mathcal{R}_{\pm,V}(\varphi) := &\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(V(\varphi,x)\partial_y\sqrt{1+(\partial_y\widetilde{\alpha}_\pm)}\Big)\\ &+ \Phi_\pm(\varphi)V(\varphi,x)\partial_x\Phi_\pm(\varphi)^{-1}\Big(\Phi_\pm(\varphi)\mathcal{H}\Phi_\pm(\varphi)^{-1} - \mathcal{H}\Big)\,.\end{aligned} \tag{4.18}$$

The property (2.24) applied to $\alpha_\pm$, $\widetilde{\alpha}_\pm$ implies that

$$\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(V(1+(\partial_y\widetilde{\alpha}_\pm))^{\frac{3}{2}}\Big) = \Big(V\big(1+(\partial_y\widetilde{\alpha}_\pm)\big)\Big)_{|y=x+\alpha_\pm(\varphi,x)}\,. \tag{4.19}$$

and therefore, $\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(V(\varphi,x)\partial_y\sqrt{1+(\partial_y\widetilde{\alpha}_\pm)}\Big)$ is the multiplication operator by the function $\Big(V\big(1+(\partial_y\widetilde{\alpha}_\pm)\big)\Big)_{|y=x+\alpha_\pm(\varphi,x)}$. Moreover, recalling Lemma 2.18, the operator $\Phi_\pm(\varphi)V(\varphi,x)\partial_x\Phi_\pm(\varphi)^{-1}\Big(\Phi_\pm(\varphi)\mathcal{H}\Phi_\pm(\varphi)^{-1} - \mathcal{H}\Big) \in OPS^{-\infty}$, hence $\mathcal{R}_{\pm,V} \in OPS^0$. Then, summarizing (4.15), (4.16), (4.17), (4.19), recalling that $1-\overline{\mathfrak{e}} \geq 1-\mathfrak{e}\,, 0$ (see (4.2)), so that $OPS^{1-\mathfrak{e}}\,, OPS^0 \subseteq OPS^{1-\overline{\mathfrak{e}}}$, one gets that

$$\Phi_\pm(\varphi)\mathcal{V}(\varphi)\Phi_\pm(\varphi)^{-1} = \Big(V\big(1+(\partial_y\widetilde{\alpha}_\pm)\big)\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D| + OPS^{1-\overline{\mathfrak{e}}}\,. \tag{4.20}$$

Now we compute the term $\mathrm{i}\Phi_\pm(\varphi)\omega\cdot\partial_\varphi\Phi_\pm(\varphi)^{-1}$ appearing in the definitions of $\mathcal{V}_{\pm,1}(\varphi)$ given in (4.14). A direct calculation shows that

$$\begin{aligned}\mathrm{i}\Phi_\pm(\varphi)\omega\cdot\partial_\varphi\Phi_\pm(\varphi)^{-1} = &\,\mathrm{i}\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\sqrt{1+(\partial_y\widetilde{\alpha}_\pm)}\Big)\partial_x\\ &+ \mathrm{i}\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(\omega\cdot\partial_\varphi\sqrt{1+(\partial_y\widetilde{\alpha}_\pm)}\Big)\,.\end{aligned} \tag{4.21}$$

By applying the equality (2.24) to $\alpha_\pm$, $\widetilde{\alpha}_\pm$, one has

$$\sqrt{1+(\partial_x\alpha_\pm)}\Psi_\pm\Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\sqrt{1+(\partial_y\widetilde{\alpha}_\pm)}\Big) = \Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\Big)_{y=x+\alpha_\pm(\varphi,x)}\,, \tag{4.22}$$

thus (4.21) becomes

$$\mathrm{i}\Phi_\pm(\varphi)\omega\cdot\partial_\varphi\Phi_\pm(\varphi)^{-1} = \mathrm{i}\Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\Big)_{y=x+\alpha_\pm(\varphi,x)}\partial_x + OPS^0\,. \tag{4.23}$$

Therefore (4.14), (4.20), (4.23) and $OPS^0 \subseteq OPS^{1-\overline{\mathfrak{e}}}$ imply that

$$\mathcal{V}_{\pm,1}(\varphi) = \Big(V\big(1+(\partial_y\widetilde{\alpha}_\pm)\big)\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D| + \mathrm{i}\Big(\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm\Big)_{y=x+\alpha_\pm(\varphi,x)}\partial_x + OPS^{1-\overline{\mathfrak{e}}}\,. \tag{4.24}$$

We now provide the final expansion of the operator $\mathcal{V}_1(\varphi)$ defined in (4.13). Using the elementary properties $\mathrm{i}\partial_x \Pi_+ = -|D|\Pi_+$, $\mathrm{i}\partial_x \Pi_- = |D|\Pi_-$, by (4.24) one gets

$$
\begin{aligned}
\mathcal{V}_1(\varphi) = {} & \Pi_+\Big(\Big(V(1+(\partial_y\widetilde{\alpha}_+)) - \omega\cdot\partial_\varphi\widetilde{\alpha}_+\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D| + \mathcal{W}_{1,+}(\varphi)\Big)\Pi_+ \\
& + \Pi_-\Big(\Big(V\big(1+(\partial_y\widetilde{\alpha}_-)\big) + \omega\cdot\partial_\varphi\widetilde{\alpha}_-\Big)_{|y=x+\alpha_\pm(\varphi,x)}|D| + \mathcal{W}_{1,-}(\varphi)\Big)\Pi_- \\
& + OPS^{-\infty}\,.
\end{aligned}
\tag{4.25}
$$

with $\mathcal{W}_{1,\pm} \in OPS^{1-\overline{\mathfrak{e}}}$. By applying Proposition 6.1, given $\gamma \in (0,1)$, for $\varepsilon\gamma^{-1} \le \delta$, for some $\delta \in (0,1)$ small enough, there exist Lipschitz functions $\mathtt{c}_\pm : \Omega \to \mathbb{R}$ such that for any $\omega \in \Omega^{\mu_+}_{\gamma,\tau} \cap \Omega^{\mu_-}_{\gamma,\tau}$ (see (4.3)) there exist two $\mathcal{C}^\infty$ functions $\widetilde{\alpha}_\pm$ (depending on ω) such that

$$
\begin{aligned}
& \mp\,\omega\cdot\partial_\varphi\widetilde{\alpha}_\pm + (1+\varepsilon P)\partial_y\widetilde{\alpha}_\pm + \varepsilon P = \mathtt{c}_\pm \\
& \|\widetilde{\alpha}_\pm\|_s \lesssim_s \varepsilon\gamma^{-1}\,, \quad \forall s \ge 0\,, \quad |\mathtt{c}_\pm|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon\,.
\end{aligned}
\tag{4.26}
$$

For $\varepsilon\gamma^{-1}$ small enough, $\widetilde{\alpha}_\pm$ satisfies the property (4.9), therefore by applying Lemma 2.12, the functions $\alpha_\pm$ satisfy the ansatz (4.6). The operator $\mathcal{V}_0(\varphi)$ defined in (4.25) takes the form

$$
\begin{aligned}
& (\Phi_0)_{\omega*}\mathrm{i}\mathcal{V}(\varphi) = \mathrm{i}\mathcal{V}_1(\varphi)\,, \\
& \mathcal{V}_1(\varphi) = \Pi_+\Big(\lambda_+|D| + \mathcal{W}_{1,+}(\varphi)\Big)\Pi_+ + \Pi_-\Big(\lambda_-|D| + \mathcal{W}_{1,-}(\varphi)\Big)\Pi_- + OPS^{-\infty}\,, \\
& \lambda_\pm := 1 + \mathtt{c}_\pm\,, \quad |\mathtt{c}_\pm|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon\,, \qquad \mathcal{W}_{1,\pm} \in OPS^{1-\overline{\mathfrak{e}}}\,,
\end{aligned}
\tag{4.27}
$$

where $\mathcal{W}_{1,+}(\varphi), \mathcal{W}_{1,-}(\varphi)$ are self-adjoint. This concludes the proof of Proposition 4.1.

4.2 Reduction of the lower order terms.

In order to state the main result of this section, for $\gamma \in (0,1)$ and $\tau > \nu$, we introduce the set

$$
\Omega_{\gamma,\tau} := \Big\{\omega \in \Omega^{\mu_+}_{\gamma,\tau} \cap \Omega^{\mu_-}_{\gamma,\tau} : |\omega\cdot\ell + \lambda_\pm(\omega)\,j| \ge \frac{\gamma}{\langle\ell\rangle^\tau}\,, \quad \forall(\ell,j) \in \mathbb{Z}^{\nu+1}\setminus\{(0,0)\}\Big\}\,.
\tag{4.28}
$$

Proposition 4.2. *Let $\gamma \in (0,1)$, $\tau > \nu$, $N \in \mathbb{N}$. Then for any $\omega \in \Omega_{\gamma,\tau}$, for any $n = 1,\dots,N$ there exists a linear vector field $\mathrm{i}\mathcal{V}_n(\varphi)$, $\varphi \in \mathbb{T}^\nu$ of the form*

$$
\mathcal{V}_n(\varphi) = \Pi_+\mathcal{V}_{n,+}(\varphi)\Pi_+ + \Pi_-\mathcal{V}_{n,-}(\varphi)\Pi_- + OPS^{-\infty}
\tag{4.29}
$$

(recall (2.27)) where the vector fields $\mathrm{i}\mathcal{V}_{n,\pm}(\varphi)$ are Hamiltonian and have the form

$$
\mathcal{V}_{n,\pm}(\varphi) := \lambda_\pm|D| + \mu_{n,\pm}(D) + \mathcal{W}_{n,\pm}(\varphi)\,,
\tag{4.30}
$$

where

$$
\mu_{n,\pm}(D) := \mathrm{Op}\Big(\mu_{n,\pm}(\xi)\Big)\,, \qquad \mu_{n,\pm} \in S^{1-\overline{\mathfrak{e}}}\,,
\tag{4.31}
$$

$$
\mathcal{W}_{n,\pm}(\varphi) := \mathrm{Op}\Big(w_{n,\pm}(\varphi,x,\xi)\Big)\,, \qquad w_{n,\pm} \in S^{1-n\overline{\mathfrak{e}}}\,,
\tag{4.32}
$$

$\mu_{n,\pm}(\xi)$ is real and $\mathcal{W}_{n,\pm}(\varphi)$ is self-adjoint.
For any $n \in \{1,\dots,N-1\}$, there exist an invertible map $\Phi_n(\varphi)$ satisfying

$$
\sup_{\varphi\in\mathbb{T}^\nu}\|\Phi_n(\varphi)^{\pm1}\|_{\mathcal{B}(H^s)} < +\infty\,, \quad \forall s \ge 0
\tag{4.33}
$$

such that

$$
\mathrm{i}\mathcal{V}_{n+1}(\varphi) = (\Phi_n)_{\omega*}\mathrm{i}\mathcal{V}_n(\varphi)\,, \qquad \forall n \in \{1,\dots,N-1\}\,.
\tag{4.34}
$$

The rest of this section is devoted to the proof of the above proposition, which is proved by induction. We describe the induction step of the proof. Assume that for $n \in \{1, \ldots, N-1\}$ there exists an operator $\mathcal{V}_n(\varphi)$ satisfying the properties (4.2)-(4.32). We consider an operator

$$\mathcal{G}_{n,\pm}(\varphi) = \mathrm{Op}\big(g_{n,\pm}(\varphi, x, \xi)\big)\,, \quad g_{n,\pm} \in S^{1-n\bar{\mathtt{e}}}\,, \quad g_{n,\pm} = g_{n,\pm}^*\,. \tag{4.35}$$

Let $\Phi_{n,\pm}(\tau;\varphi)$, $\tau \in [0,1]$ be the symplectic flow generated by the Hamiltonian vector field $\mathrm{i}\mathcal{G}_{n,\pm}(\varphi)$, namely

$$\begin{cases} \partial_\tau \Phi_{n,\pm}(\tau;\varphi) = \mathrm{i}\mathcal{G}_{n,\pm}(\varphi)\Phi_{n,\pm}(\tau;\varphi) \\ \Phi_{n,\pm}(0;\varphi) = \mathrm{Id}\,. \end{cases} \tag{4.36}$$

By applying Lemma 2.9, the maps $\Phi_{n,\pm}(\tau;\varphi)$ satisfy the property (4.33). We then define for any $\tau \in [0,1]$ the map

$$\Phi_n(\tau;\varphi) := \Phi_{n,+}(\tau;\varphi)^{-1}\Pi_+ + \Phi_{n,-}(\tau;\varphi)^{-1}\Pi_-\,, \tag{4.37}$$

whose inverse is given by

$$\Phi_n(\tau;\varphi)^{-1} = \Pi_+\Phi_{n,+}(\tau;\varphi) + \Pi_-\Phi_{n,-}(\tau;\varphi)\,, \tag{4.38}$$

see Lemma 2.15-(i). We set $\Phi_n(\varphi) := \Phi_n(1;\varphi)$, $\Phi_{n,\pm}(\varphi) := \Phi_{n,\pm}(1;\varphi)$. By applying Lemma 2.15-(ii) one gets that $\mathrm{i}\mathcal{V}_{n+1} = (\Phi_n)_{\omega *}\mathrm{i}\mathcal{V}_n$ with

$$\begin{aligned} &\mathcal{V}_{n+1}(\varphi) = \Pi_+\mathcal{V}_{n+1,+}(\varphi)\Pi_+ + \Pi_-\mathcal{V}_{n+1,-}(\varphi)\Pi_- + OPS^{-\infty}\,, \\ &\mathcal{V}_{n+1,\pm}(\varphi) := \Phi_{n,\pm}(\varphi)\mathcal{V}_{n,\pm}(\varphi)\Phi_{n,\pm}(\varphi)^{-1} + \mathrm{i}\Phi_{n,\pm}(\varphi)\omega\cdot\partial_\varphi\Phi_{n,\pm}(\varphi)^{-1}\,. \end{aligned} \tag{4.39}$$

Since $\Phi_{n,\pm}(\varphi)$ are symplectic maps, the vector fields $\mathrm{i}\mathcal{V}_{n+1,\pm}(\varphi)$ are Hamiltonian, i.e. $\mathcal{V}_{n+1,\pm}(\varphi)$ are self-adjoint operators. In the following, we provide an expansion of the operators $\mathcal{V}_{n+1,\pm}(\varphi)$.
ANALYSIS OF $\Phi_{n,\pm}(\varphi)\mathcal{V}_{n,\pm}(\varphi)\Phi_{n,\pm}(\varphi)^{-1}$. The symbol $v_{n,\pm} \in S^1$ of the operator $\mathcal{V}_{n,\pm}$ is given by

$$v_{n,\pm}(\varphi, x, \xi) = \lambda_\pm|\xi|\chi(\xi) + \mu_{n,\pm}(\xi) + w_{n,\pm}(\varphi, x, \xi)\,. \tag{4.40}$$

Applying Proposition 2.14 one gets that

$$\Phi_{n,\pm}(\varphi)\mathcal{V}_{n,\pm}(\varphi)\Phi_{n,\pm}(\varphi)^{-1} = \mathrm{Op}\big(v_{n+1,\pm}^{(1)}\big)\,, \tag{4.41}$$

where

$$v_{n+1,\pm}^{(1)} := v_{n,\pm} + \{g_{n,\pm}, v_{n,\pm}\} + r_{n,\pm}^{(\geq 2)}\,, \quad r_{n,\pm}^{(\geq 2)} \in S^{1-2n\bar{\mathtt{e}}} \subseteq S^{1-(n+1)\bar{\mathtt{e}}}\,. \tag{4.42}$$

We write

$$\begin{aligned} \{g_{n,\pm}, v_{n,\pm}\} &= \lambda_\pm\{g_{n,\pm}, |\xi|\chi(\xi)\} + \{g_{n,\pm}, \mu_{n,\pm}\} + \{g_{n,\pm}, w_{n,\pm}\} \\ &= -\lambda_\pm\partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) - \lambda_\pm\partial_x g_{n,\pm}|\xi|\partial_\xi\chi(\xi) + \{g_{n,\pm}, \mu_{n,\pm}\} + \{g_{n,\pm}, w_{n,\pm}\} \end{aligned} \tag{4.43}$$

By Corollary 2.4 and since $\partial_\xi\chi(\xi) = 0$ for $|\xi| \geq 1$ (see (2.13)), $\mu_{n,\pm} \in S^{1-\bar{\mathtt{e}}}$ and $w_{n,\pm}, g_{n,\pm} \in S^{1-n\bar{\mathtt{e}}}$, one gets that

$$\begin{aligned} &-\lambda_\pm\partial_x g_{n,\pm}|\xi|\partial_\xi\chi(\xi) \in S^{-\infty} \subseteq S^{1-(n+1)\bar{\mathtt{e}}}\,, \quad \{g_{n,\pm}, \mu_{n,\pm}\} \in S^{1-(n+1)\bar{\mathtt{e}}}\,, \\ &\{g_{n,\pm}, w_{n,\pm}\} \in S^{1-2n\bar{\mathtt{e}}} \subseteq S^{1-(n+1)\bar{\mathtt{e}}}\,. \end{aligned} \tag{4.44}$$

By (4.40), (4.42)–(4.44) one gets

$$v_{n+1,\pm}^{(1)} = \lambda_\pm|\xi|\chi(\xi) + \mu_{n,\pm} + w_{n,\pm} - \lambda_\pm\partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) + S^{1-(n+1)\bar{\mathtt{e}}}\,. \tag{4.45}$$

ANALYSIS OF THE TERM $\mathrm{i}\Phi_{n,\pm}(\varphi)\omega\cdot\partial_\varphi\Phi_{n,\pm}(\varphi)^{-1}$. We define $\mathcal{V}_{n+1,\pm}^{(2)}(\tau;\varphi) := \mathrm{i}\Phi_{n,\pm}(\tau;\varphi)\omega\cdot\partial_\varphi\big(\Phi_{n,\pm}(\tau;\varphi)^{-1}\big)$. A direct calculation shows that

$$\mathcal{V}_{n+1,\pm}^{(2)}(\tau;\varphi) = \int_0^\tau \mathcal{S}_{n+1,\pm}(\zeta;\varphi)\,d\zeta\,, \quad \mathcal{S}_{n+1,\pm}(\zeta;\varphi) := \Phi_{n,\pm}(\zeta;\varphi)\circ\omega\cdot\partial_\varphi\mathcal{G}_{n,\pm}(\varphi)\circ\Phi_{n,\pm}(\zeta;\varphi)^{-1}, \quad \zeta \in [0,1]\,.$$

Since $\omega \cdot \partial_\varphi \mathcal{G}_{n,\pm}(\varphi) \in OPS^{1-n\bar{\mathfrak{e}}}$, by Proposition 2.14

$$\mathcal{V}^{(2)}_{n+1,\pm}(\varphi) = \mathcal{V}^{(2)}_{n+1,\pm}(1;\varphi) = \mathrm{Op}\Big(v^{(2)}_{n+1,\pm}(\varphi,x,\xi)\Big) \in OPS^{1-n\bar{\mathfrak{e}}} \tag{4.46}$$

with

$$v^{(2)}_{n+1,\pm}(\varphi,x,\xi) := \omega \cdot \partial_\varphi g_{n,\pm}(\varphi,x,\xi) + S^{1-(n+1)\bar{\mathfrak{e}}}\,. \tag{4.47}$$

Collecting (4.39), (4.41), (4.42), (4.47) one then gets

$$v_{n+1,\pm} = \lambda_\pm |\xi|\chi(\xi) + \mu_{n,\pm} + w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) + S^{1-(n+1)\bar{\mathfrak{e}}}\,. \tag{4.48}$$

In the next Lemma we show how to choose $\mathcal{G}_{m,\pm}(\varphi) = \mathrm{Op}\big(g_{n,\pm}(\varphi,x,\xi)\big)$ in order to reduce to constant coefficients the term $w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi)$ of order $1-n\bar{\mathfrak{e}}$ in (4.48).

Lemma 4.3. *For any $\omega \in \Omega_{\gamma,\tau}$ (see (4.28)) there exist symbols $g_{n+1,\pm} \in S^{1-n\bar{\mathfrak{e}}}$ satisfying $g_{n+1,\pm} = g^*_{n+1,\pm}$ and*

$$w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) - \langle w_{n,\pm}\rangle_{\varphi,x} \in S^{1-(n+1)\bar{\mathfrak{e}}}$$

(recall the definition (2.9)).

Proof. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function satisfying:

$$\begin{aligned} &\chi_1(\xi) = 0\,, \quad \forall |\xi| \le \frac{3}{2}\,,\\ &\chi_1(\xi) = 1\,, \quad \forall |\xi| \ge 2\,. \end{aligned} \tag{4.49}$$

Writing $1 = \chi_1 + (1-\chi_1)$, one has

$$\begin{aligned} &w_{n,\pm} + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) - \langle w_{n,\pm}\rangle_{\varphi,x}\\ &= \chi_1(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x}) + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) + (1-\chi_1)(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x})\,. \end{aligned} \tag{4.50}$$

By the definition of χ_1 given in (4.49), one has that

$$(1-\chi_1)(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x}) \in S^{-\infty}, \tag{4.51}$$

therefore we look for a symbol $g_{n,\pm}$ satisfying

$$\chi_1(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x}) + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi) \in S^{1-(n+1)\bar{\mathfrak{e}}}\,. \tag{4.52}$$

Since we require that $g_{n,\pm} = g^*_{n,\pm}$, we look for a symbol of the form $g_{n,\pm} = \frac{q_{n,\pm}+q^*_{n,\pm}}{2}$ and we make the ansatz

$$q^*_{n,\pm} = q_{n,\pm} + S^{-n\bar{\mathfrak{e}}}\,. \tag{4.53}$$

By (4.52), (4.53), we write

$$\begin{aligned} &\chi_1(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x}) + \omega \cdot \partial_\varphi g_{n,\pm} - \lambda_\pm \partial_x g_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi)\\ &= \chi_1(w_{n,\pm} - \langle w_{n,\pm}\rangle_{\varphi,x}) + \omega \cdot \partial_\varphi q_{n,\pm} - \lambda_\pm \partial_x q_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi)\\ &\quad + \Big(\omega \cdot \partial_\varphi - \lambda_\pm \frac{\xi}{|\xi|}\chi(\xi)\partial_x\Big)\Big[\frac{q^*_n - q_n}{2}\Big]\,. \end{aligned} \tag{4.54}$$

By (4.53) we have that

$$\Big(\omega\cdot\partial_\varphi-\lambda_\pm\frac{\xi}{|\xi|}\chi(\xi)\partial_x\Big)\Big[\frac{q_n^*-q_n}{2}\Big]\in S^{-n\bar{\mathfrak{e}}}\overset{\bar{\mathfrak{e}}\le 1}{\subseteq}S^{1-(n+1)\bar{\mathfrak{e}}}\,, \tag{4.55}$$

hence it is enough to determine the symbol $q_{n,\pm}$ so that

$$\chi_1(w_{n,\pm}-\langle w_{n,\pm}\rangle_{\varphi,x})+\omega\cdot\partial_\varphi q_{n,\pm}-\lambda_\pm\partial_x q_{n,\pm}\frac{\xi}{|\xi|}\chi(\xi)=0\,. \tag{4.56}$$

The equation above, can be solved for any $\omega\in\Omega_{\gamma,\tau}$, by defining $q_{n,\pm}$ as

$$\begin{aligned} q_{n,\pm}(\varphi,x,\xi)&:=(\omega\cdot\partial_\varphi-\lambda_\pm\partial_x)^{-1}\big[\langle w_{n,\pm}\rangle_{\varphi,x}(\xi)-w_{n,\pm}(\varphi,x,\xi)\big]\chi_1^+(\xi)\\ &\quad+(\omega\cdot\partial_\varphi+\lambda_\pm\partial_x)^{-1}\big[\langle w_{n,\pm}\rangle_{\varphi,x}(\xi)-w_{n,\pm}(\varphi,x,\xi)\big]\chi_1^-(\xi) \end{aligned} \tag{4.57}$$

where $\chi_1^+(\xi):=\chi_1(\xi)\mathbb{I}_{\{\xi>0\}}$, $\chi_1^-(\xi):=\chi_1(\xi)\mathbb{I}_{\{\xi\le 0\}}$ where $\mathbb{I}_{\{\xi>0\}}$ (resp. $\mathbb{I}_{\{\xi\le 0\}}$) is the characteristic function of the set $\{\xi\in\mathbb{R}:\xi>0\}$ (resp. $\{\xi\in\mathbb{R}:\xi\le 0\}$). Note that, by (4.49), the functions χ_1^+,χ_1^- are C^∞. Using that $w_{n,\pm}\in S^{1-n\bar{\mathfrak{e}}}$ and recalling the property (2.10), one has that $q_{n,\pm}\in S^{1-n\bar{\mathfrak{e}}}$ and therefore $g_{n,\pm}\in S^{1-n\bar{\mathfrak{e}}}$. Furthermore, using that $w_{n,\pm}=w_{n,\pm}^*$, by applying Lemma 2.6 (with $\varphi(\xi)=\chi_1^+(\xi)$, $a=(\omega\cdot\partial_\varphi-\lambda_\pm\partial_x)^{-1}\big[\langle w_{n,\pm}\rangle_{\varphi,x}-w_{n,\pm}\big]$ and $\varphi(\xi)=\chi_1^-(\xi)$, $a=(\omega\cdot\partial_\varphi+\lambda_\pm\partial_x)^{-1}\big[\langle w_{n,\pm}\rangle_{\varphi,x}-w_{n,\pm}\big]$) and Lemma 2.8, one gets that the symbols $q_{n,\pm}$ verify the ansatz (4.53). Finally collecting (4.50), (4.51), (4.54), (4.55), (4.56), the claimed statement follows. □

By (4.39), (4.48) and Lemma 4.3, one gets that

$$\mathcal{V}_{n+1}(\varphi)=\Pi_+\mathcal{V}_{n+1,+}(\varphi)\Pi_++\Pi_-\mathcal{V}_{n+1,-}(\varphi)\Pi_-+OPS^{-\infty} \tag{4.58}$$

where

$$\begin{aligned} &\mathcal{V}_{n+1,\pm}(\varphi)=\lambda_\pm|D|+\mu_{n+1,\pm}(D)+\mathcal{W}_{n+1,\pm}(\varphi)\,,\\ &\mu_{n+1,\pm}(D):=\mu_{n,\pm}(D)+\mathrm{Op}\big(\langle w_{n,\pm}\rangle_{\varphi,x}(\xi)\big)\,,\quad \mathcal{W}_{n+1,\pm}(\varphi)\in OPS^{1-(n+1)\bar{\mathfrak{e}}}\,. \end{aligned} \tag{4.59}$$

By the induction hypothesis $w_{n,\pm}=w_{n,\pm}^*$ and $\mu_{n,\pm}(\xi)$ is real, hence by Lemma 2.8 one has that $\langle w_{n,\pm}\rangle_{\varphi,x}(\xi)$ is real and therefore $\mu_{n+1,\pm}(\xi)=\mu_{n,\pm}(\xi)+\langle w_{n,\pm}\rangle_{\varphi,x}(\xi)$ is real too. Since $\Phi_{n,\pm}$ are symplectic maps, $\mathrm{i}\mathcal{V}_{n\pm}$ are Hamiltonian vector fields, then also $\mathrm{i}\mathcal{V}_{n+1,\pm}=(\Phi_{n,\pm}^{-1})_{\omega*}\mathrm{i}\mathcal{V}_{n,\pm}$ are Hamiltonian vector fields, implying that

$$\mathcal{W}_{n+1,\pm}(\varphi)=\mathcal{V}_{n+1,\pm}(\varphi)-\lambda_\pm|D|-\mu_{n+1,\pm}(D)$$

are self-adjoint operators. The proof of Proposition 4.2 is then concluded.

4.3 Proof of Theorem 1.7.

Let $K>1$ and let fix an integer N_K so that $1-N_K\bar{\mathfrak{e}}\le -K$

$$N_K:=\Big[\frac{K+1}{\bar{\mathfrak{e}}}\Big]+1\,. \tag{4.60}$$

We define

$$\mathcal{T}_K(\varphi):=\Phi_0(\varphi)\circ\Phi_1(\varphi)\circ\ldots\circ\Phi_{N_K-1}(\varphi) \tag{4.61}$$

where $\Phi_0(\varphi)$ is defined in (4.10) and for any $n=1,\ldots,N_K-1$, the maps $\Phi_n(\varphi)$ are given in Lemma 4.2. By (4.11), (4.4), (4.33), the map $\mathcal{T}_K(\varphi)$ is invertible with inverse given by $\mathcal{T}_K(\varphi)^{-1}=\Phi_{N_K}(\varphi)^{-1}\circ\ldots\circ\Phi_1(\varphi)^{-1}\circ\Phi_0(\varphi)^{-1}$ and $\mathcal{T}_K(\varphi)^{\pm1}$ satisfy

$$\sup_{\varphi\in\mathbb{T}^\nu}\|\mathcal{T}_K(\varphi)^{\pm1}\|_{\mathcal{B}(H^s)}<+\infty\,,\quad\forall s\ge 0\,. \tag{4.62}$$

By (4.27) and by Lemma 4.2, one gets that $(\mathcal{T}_K)_{\omega*}\mathrm{i}\mathcal{V}(\varphi)=\mathrm{i}\mathcal{V}_{N_K}(\varphi)$ where $\mathcal{V}_{N_K}(\varphi)$ is given by formula (4.29) for $n=N_K$. Then we can write

$$\mathcal{V}_{N_K}(\varphi)=\lambda_K(D)+\mathcal{R}_K(\varphi)$$

where, recalling that $|D| = \mathrm{Op}(|\xi|\chi(\xi))$ (see (2.14)) and $\Pi_\pm = \mathrm{Op}(\chi_\pm(\xi))$ (see (2.30))

$$\lambda_K(D) = \mathrm{Op}(\lambda_K(\xi)), \quad \lambda_K(\xi) := \big(\lambda_+|\xi|\chi(\xi) + \mu_{N_K,+}(\xi)\big)\chi_+(\xi) + \big(\lambda_-|\xi|\chi(\xi) + \mu_{N_K,-}(\xi)\big)\chi_-(\xi),$$
$$\mathcal{R}_K(\varphi) := \Pi_+\mathcal{W}_{N_K,+}(\varphi)\Pi_+ + \Pi_-\mathcal{W}_{N_K,-}(\varphi)\Pi_- + OPS^{-\infty}.$$

Note that $\lambda_K(\xi)$ is real. By (4.32) (applied with $n = N_K$), using that $1 - N_K\bar{\mathfrak{e}} \le -K$ and recalling Theorem 2.1, one gets that $\mathcal{R}_K \in OPS^{-K}$. The proof of Theorem 1.7 is then concluded.

5 Proof of Theorems 1.3, 1.4.

Let $s > 0$, $t_0 \in \mathbb{R}$, $u_0 \in H^s(\mathbb{T})$. We fix the constant $K \in \mathbb{N}$ appearing in Theorem 1.6 so that $K > s$,

$$K = K_s := [s] + 1. \tag{5.1}$$

By applying Theorems 1.6, 1.7, taking $\omega \in DC(\gamma,\tau)$ in the case $M < 1$ and $\omega \in \Omega_{\gamma,\tau}$, $\varepsilon\gamma^{-1}$ small enough, in the case $M = 1$, one has that $u(t)$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u] \\ u(t_0) = u_0 \end{cases} \tag{5.2}$$

if and only if $v(t) := \mathcal{T}_{K_s}^{-1}(\omega t)[u(t)]$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t v = \mathrm{i}\lambda_{K_s}(D)v + \mathrm{i}\mathcal{R}_{K_s}(\omega t)[v] \\ v(t_0) = v_0\,, \end{cases} \qquad v_0 := \mathcal{T}_{K_s}^{-1}(\omega t_0)[u_0] \tag{5.3}$$

with $\lambda_{K_s}(D) = \mathrm{Op}(\lambda_{K_s}(\xi)) \in OPS^M$ with $\lambda_{K_s}(\xi) = \overline{\lambda_{K_s}(\xi)}$. Moreover, since $K_s > s > 0$, one has that $\mathcal{R}_{K_s} \in OPS^{-K_s} \subset OPS^{-s}$ implying that

$$\sup_{\varphi\in\mathbb{T}^\nu} \|\mathcal{R}_{K_s}(\varphi)\|_{\mathcal{B}(L^2,H^s)} < +\infty. \tag{5.4}$$

Moreover, since $\mathcal{T}_{K_s}(\varphi)$ is bounded and invertible, we have that

$$\|u(t)\|_{H^s} \simeq_s \|v(t)\|_{H^s} \quad \text{and} \quad \|u(t)\|_{L^2} \simeq \|v(t)\|_{L^2}, \quad \forall t \in \mathbb{R}. \tag{5.5}$$

Writing the Duhamel formula for the Cauchy problem (5.3), one obtains

$$v(t) = e^{\mathrm{i}\lambda_{K_s}(D)t}v_0 + \int_{t_0}^t e^{\mathrm{i}\left(t-\tau\right)\lambda_{K_s}(D)}\mathcal{R}_{K_s}(\omega\tau)[v(\tau)]\,d\tau. \tag{5.6}$$

Using that $\lambda_{K_s}(\xi)$ is real which implies that the propagator $e^{\mathrm{i}t\lambda_{K_s}(D)}$ is unitary on $H^s(\mathbb{T})$ for any $t \in \mathbb{R}$ and by (5.4), (5.5) one gets the estimate $\|v(t)\|_{H^s} \lesssim_s \|v_0\|_{H^s} + |t-t_0|\|v_0\|_{L^2}$. Applying again (5.5) we then get

$$\|u(t)\|_{H^s} \lesssim_s \|u_0\|_{H^s} + |t-t_0|\|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}. \tag{5.7}$$

The above argument implies that the propagator $\mathcal{U}(t_0,t)$ of the PDE $\partial_t u = \mathrm{i}\mathcal{V}(\omega t)[u]$, i.e.

$$\begin{cases} \partial_t \mathcal{U}(t_0,t) = \mathrm{i}\mathcal{V}(\omega t)\mathcal{U}(t_0,t) \\ \mathcal{U}(t_0,t_0) = \mathrm{Id} \end{cases}$$

satisfies

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^s)} \lesssim_s 1 + |t-t_0|, \quad \forall s > 0, \quad \forall t, t_0 \in \mathbb{R}. \tag{5.8}$$

Furthermore, since $\mathcal{V}(\varphi)$ is self-adjoint, the L^2 norm of the solutions is constant, namely

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(L^2)} = 1, \quad \forall t, t_0 \in \mathbb{R}. \tag{5.9}$$

Hence, for any $0 < s < S$, by applying Theorem 2.19, one gets that

$$\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^s)} \le \|\mathcal{U}(t_0,t)\|_{\mathcal{B}(L^2)}^{\frac{S-s}{S}}\|\mathcal{U}(t_0,t)\|_{\mathcal{B}(H^S)}^{\frac{s}{S}} \overset{(5.8),(5.9)}{\lesssim_S} (1 + |t-t_0|)^{\frac{s}{S}}. \tag{5.10}$$

Then, for any $\eta > 0$, choosing S large enough so that $s/S \le \eta$, the estimate (1.4) follows.

6 Appendix: a quasi-periodic transport equation

In this appendix we state some results concerning quasi-periodic transport equations. The following statement is a direct consequence of Corollary 4.3 in [16].

Proposition 6.1. *Let $\gamma \in (0,1)$, $\tau > \nu$, $P \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$, $\mathtt{m} \in \mathbb{R}$, $\frac{1}{2} < |\mathtt{m}| < 2$. There exists a constant $\delta_* = \delta_*(\tau, \nu) > 0$, such that if $\varepsilon\gamma^{-1} \le \delta_*$, then the following holds: there exists a Lipschitz function $\mu : \Omega \to \mathbb{R}$ satisfying $|\mu - \mathtt{m}|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon$ (recall the definition (4.1)) such that for any ω in the set*

$$\Omega^\mu_{\gamma,\tau} := \Big\{ \omega \in \Omega : |\omega \cdot \ell + \mu(\omega)\, j| \ge \frac{\gamma}{\langle \ell \rangle^\tau}\,, \quad \forall (\ell, j) \in \mathbb{Z}^{\nu+1} \setminus \{(0,0)\} \Big\} \tag{6.1}$$

there exists a $\mathcal{C}^\infty$ function $\alpha(\varphi, x; \omega)$ satisfying $\|\alpha\|_s \lesssim_s \varepsilon\gamma^{-1}$, $\forall s \ge 0$ and a Lipschitz family of constants $\mathtt{c}(\omega)$ satisfying $|\mathtt{c}|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon$, such that

$$\omega \cdot \partial_\varphi \alpha(\varphi, x) + \Big(\mathtt{m} + \varepsilon P(\varphi, x)\Big)\partial_x \alpha(\varphi, x) + \varepsilon P(\varphi, x) = \mathtt{c}\,, \quad \forall (\varphi, x) \in \mathbb{T}^\nu \times \mathbb{T}\,. \tag{6.2}$$

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