UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Dipartimento di Matematica Federigo Enriques

Corso di Dottorato di Ricerca in Matematica

XXXI ciclo

Tesi di Dottorato di Ricerca

GEOMETRY OF FOURFOLDS WITH AN ADMISSIBLE K3 SUBCATEGORY

MAT03

Relatore:
Prof. Paolo Stellari

Coordinatore del dottorato:
Prof. Vieri Mastropietro

Candidata:
Laura Pertusi

A. A. 2017-2018

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Introduction

It is classically known that many geometric properties of a smooth projective scheme Z can be recovered from the study of its derived category of bounded complexes of coherent sheaves $\mathrm{D^b}(Z)$. For example, a famous result by Bondal and Orlov states that smooth projective varieties with ample (anti)canonical bundle and equivalent bounded derived categories are isomorphic. The category $\mathrm{D^b}(Z)$ is in general a difficult object to directly investigate; however, in some situations, it is possible to divide it into subcategories which are easier to describe. More precisely, a semiorthogonal decomposition for $\mathrm{D^b}(Z)$ is a collection of full admissible subcategories generating the bounded derived category and satisfying certain orthogonality conditions.

In this thesis, we focus on the case of cubic fourfolds and Gushel-Mukai fourfolds, which are two classes of smooth Fano varieties of dimension four (defined over the complex numbers). By the mentioned result of Bondal and Orlov, we know that the isomophism class of such a fourfold is determined by its derived category. On the other hand, by the work of Kuznetsov for cubic fourfolds, and Kuznetsov-Perry for Gushel-Mukai fourfolds, there is an admissible subcategory of K3 type arising as a non trivial component of a semiorthogonal decomposition of their derived categories.

The aim of this work is to investigate certain aspects of the geometry of these fourfolds which are encoded by their K3 subcategory.

Historical motivations. A cubic fourfold Y is a smooth cubic hypersurface in $\mathbb{P}^5$, while a Gushel-Mukai (GM) fourfold X is a smooth four-dimensional quadric section of a linear section of the cone over the Grassmannian $\mathrm{Gr}(2,5)$. Even if these fourfolds have been deeply studied in the last twenty years, the problem of understanding their irrationality/rationality remains one of the most challenging in this area. A folklore conjecture connects the rationality of a cubic fourfold Y to the property of having a Hodge-associated K3 surface, in the sense of Hassett (see [39]). Similar definitions have been stated in [27] by Debarre, Iliev and Manivel for a GM fourfold X.

In 2008, Kuznetsov proposed a new categorical approach in order to deal with this question. In particular, he proved in [57] that the derived category of a cubic fourfold Y has a semiorthogonal decomposition of the form

$$\mathrm{D^b}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle.$$

Here $\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)$ are line bundles on Y and $\mathrm{Ku}(Y)$ is the right orthogonal to this exceptional collection. It turns out that $\mathrm{Ku}(Y)$ is a K3 category, in the sense that it has the same Serre functor and the same Hochschild homology of that of the derived category of a K3 surface. He conjectured that Y is rational if and only if $\mathrm{Ku}(Y)$ is equivalent to the derived category of a K3 surface.

More recently, Kuznetsov and Perry found a semiorthogonal decomposition for the derived category of a GM fourfold X, of the form

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1) \rangle.$$

Again $\mathrm{Ku}(X) := \langle \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1) \rangle^{\perp}$ is an admissible K3 subcategory of $\mathrm{D^b}(X)$ (see [61]).

The noncommutative K3 surfaces $\mathrm{Ku}(Y)$ and $\mathrm{Ku}(X)$ are known as the *Kuznetsov components* of Y and X, respectively. Notice that the known examples of rational cubic fourfolds and GM fourfolds are consistent with Kuznetsov's conjecture, but in general this remains an open problem. A better understanding of the Kuznetsov component and its relation with the geometry of the fourfold could be a step toward a possible strategy to address the rationality question.

Cubic fourfolds. The first problem we deal with in this thesis is whether the Kuznetsov component determines the isomorphism class of a cubic fourfold Y. As expected in relation to what happens in the case of K3 surfaces, we provide a negative answer to this question. This is the content of [87].

More precisely, a cubic fourfold Y' is a *Fourier-Mukai partner* of Y if there is an exact equivalence $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y')$ which is of Fourier-Mukai type, i.e. such that the composition with the inclusion of $\mathrm{Ku}(Y)$ in $\mathrm{D^b}(Y)$ and the left adjoint of the inclusion is a Fourier-Mukai functor. In [47], Huybrechts proved that the number of isomorphism classes of Fourier-Mukai partners of Y is finite; moreover, if Y is very general (i.e. the rank of $H^{2,2}(Y, \mathbb{Z})$ is one), then every Fourier-Mukai partner is isomorphic to Y.

It is natural to ask whether a special cubic fourfold Y, i.e. such that $\mathrm{rk}(H^{2,2}(Y, \mathbb{Z})) \geqslant 2$, admits Fourier-Mukai partners which are not isomorphic to Y. The first result of this work is a counting formula for the number of isomorphism classes of Fourier-Mukai partners for very general special cubic fourfolds admitting an associated K3 surface. In particular, we deduce that there exist cubic fourfolds with discriminant d, admitting an arbitrary number of Fourier-Mukai partners, depending on the number of distinct odd primes in the prime factorization of d.

Secondly, we obtain a similar formula for very general cubic fourfolds Y of discriminant d admitting an associated twisted K3 surface (X, α), if 9 does not divide the discriminant d. Indeed, we find a lower bound for the number of Fourier-Mukai partners of Y, which is controlled by the number of distinct primes in the prime factorization of $d/2$ divided by the square of the order of the Brauer class α and by the Euler function evaluated in $\mathrm{ord}(\alpha)$.

These results complete the expected analogy between cubic fourfolds and K3 surfaces, stated in [47]. They also represent a first step in order to understand whether cubic fourfolds with equivalent Kuznetsov components are birational.

The second problem we discuss is the study of moduli spaces of rational curves of low degree on a cubic fourfold Y. In particular, we give a description of the Fano variety of lines and of the hyperkähler eightfold associated to twisted cubic curves in Y as moduli spaces of Bridgeland stable objects in $\mathrm{Ku}(Y)$. This is the content of [65], which is a joint work with Chunyi Li and Xiaolei Zhao.

The general feeling is that, in order to have a better understanding of the category $\mathrm{Ku}(Y)$ and its relation with the geometry of the cubic fourfold, we should know more about moduli spaces parametrizing stable objects in the Kuznetsov component. This is now possible by one of the main results of [7], where they construct Bridgeland stability conditions for $\mathrm{Ku}(Y)$. Since $\mathrm{Ku}(Y)$ is a K3 category, these moduli spaces come naturally equipped with a holomorphic nondegenerate two-form; thus, they are good candidates to provide examples of complete families of hyperkähler manifolds.

On the other hand, there are two well-known hyperkähler varieties naturally associated to Y. In 1982 Beauville and Donagi proved that the Fano variety parametrizing lines in Y is a smooth irreducible homolorphic symplectic variety of dimension four, deformation equivalent to the Hilbert square on a K3 surface. More recently, Lehn, Lehn, Sorger and van Straten constructed a smooth hyperkähler eightfold M_Y of $\mathrm{K3}^{[4]}$-type, from the irreducible component of the Hilbert scheme parametrizing twisted cubic curves in Y.

Our strategy is to consider the projection in $\mathrm{Ku}(Y)$ of a certain twist of the ideal sheaf of a line and of a twisted cubic curve in Y, and to prove that it is stable with respect to the Bridgeland stability conditions constructed in [7]. In the case of twisted cubic curves, this approach simplifies a lot the description of the construction of LLSvS eightfold, because, involving only homological computations, it does not require a detailed analysis of the singularities of the cubic surface containing the curve. For example, we interpret the contraction of the divisor of non CM twisted curves, performed in order to get M_Y, via wall-crossing in stability. As a consequence, we get that all birational models of M_Y are obtained by crossing a wall in Bridgeland stability.

As an application of these results, we give an alternative proof of the categorical Torelli Theorem for cubic fourfolds and we obtain the identification of the period point of LLSvS eightfold with that of the Fano variety. Finally, we suggest a possible stategy to treat the derived Torelli Theorem for cubic fourfolds and we prove it in a simple case.

Gushel-Mukai fourfolds. In the second part of this thesis, we study the formulation of some results proved in [4], [2] and [47] for cubic fourfolds, in the case of GM fourfolds. In particular, we discuss the conditions under which the double cover $\tilde{Y}_A$ of the EPW sextic hypersurface associated to a GM fourfold X is birationally equivalent to a moduli space of (twisted) stable sheaves on a K3 surface. Then, we characterize when $\tilde{Y}_A$ is birational to the Hilbert square on a K3 surface. This is the content of [88].

As already observed, cubic fourfolds and GM fourfolds share a lot of similarities from the Hodge theoretical and the derived categorical point of view. The interest in understanding these analogies has increased after the recent work [29], where Debarre and Kuznetsov prove that the period point of X is identified to that of a double EPW sextic $\tilde{Y}_A$, which is a hyperkähler fourfold deformation equivalent to the Hilbert square on a K3 surface constructed from X. This is analogous to Beauville and Donagi's result for the Fano variety of a cubic fourfold.

In order to study $\tilde{Y}_A$, we define the *Mukai lattice* $K(\mathrm{Ku}(X))_{\mathrm{top}}$ for $\mathrm{Ku}(X)$, as done by Addington and Thomas in [4] for the Kuznetsov component of a cubic fourfold. In particular, we find two classes in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ such that the orthogonal complement to the lattice they generate is Hodge isometric to the degree four vanishing cohomology of X. As a first consequence, we prove that X has a twisted associated K3 surface if and only if the discriminant of the GM fourfold satisfies a certain numerical condition (this was done in [47] for cubic fourfolds).

Using [2], we prove that the property of having $\tilde{Y}_A$ birational to the Hilbert square on a K3 surface is a divisorial condition; however, this is not true if we require that $\tilde{Y}_A$ is birational to a moduli space of stable sheaves on a K3 surface. In particular, we construct examples of GM fourfolds having a hyperbolic plane embedded in the algebraic part of $K(\mathrm{Ku}(X))_{\mathrm{top}}$, but without a Hodge-associated K3 surface. This shows a different behavior with respect to the case of cubic fourfolds.

As a byproduct, we get that if a very general GM fourfold has the Kuznetsov component equivalent to the derived category of a K3 surface, then it has a Hodge-associated K3 surface. In contrast to what proved in [4] for cubic fourfolds, it is not guaranteed whether this holds for every GM fourfold.

In the last part of this work, we describe a conic fibration associated to an ordinary GM fourfold. This construction has been used in the work [84] joint with Mattia Ornaghi, to prove Voevodsky's conjecture for general GM fourfolds. In a joint work in progress with Alex Perry and Xiaolei Zhao, we are trying to use this geometric picture in order to induce Bridgeland stability conditions on $\mathrm{Ku}(X)$, in the same spirit of [7]. This technique would allow to study moduli spaces of stable objects in $\mathrm{Ku}(X)$, as in the case of cubic fourfolds, leading to many interesting applications.

Acknowledgements

These pages are the concrete product of a journey started three years ago, during which I had the opportunity to scratch the surface of a fascinating world to discover. One of the most important aspect is represented by all the people I have met in this period and I wish to thank all of them. Here I remember who deeper shared the travel with me.

A special thanks is for my advisor Paolo Stellari, for having made possible this adventure introducing me in his beautiful research domain, for his constant support with advices, smart ideas and interesting activities, and for having encouraged me in a moment I needed. His way of doing and sharing mathematics will be a model for me to follow.

I am deeply indebted to Xiaolei Zhao, who played a key role in my intellectual growth. I thank him for having shared with me his deep knowledge and for making so enjoyable the time we work together.

It is a pleasure to thank Emanuele Macrì, for plenty of inspiring conversations and for his involving enthusiasm in talking about mathematics.

I would like to thank the anonymous referee of one of my papers, who have taught me a lot on writing mathematics and for being patience in pointing out many mistakes. I am also grateful to the referees for their careful reading of the thesis and useful suggestions.

I also thank Alex Perry, Arend Bayer, Beren Sanders, Enrico Arbarello, Manfred Lehn, Nicolas Addington and Richard Thomas, for their interest in my work and many useful comments, Alexander Kuznetsov, Daniel Huybrechts and Pietro Pirola, for answering a question, and Chunyi Li, Martí Lahoz and Riccardo Moschetti for interesting discussions.

I wish to acknowledge the people of the Mathematics Department in Milano, in particular Alice and Chiara, for helpful conversations, and Alfonso, Andreas, Bert, Diego, Elisabetta, Gilberto, prof. Lanteri and Valerio, for the nice atmosphere they have created.

I thank Alberto Albano and Cinzia Casagrande, for making so pleasant my visits in Torino.

I take the opportunity to thank my friends Enrica, Fabio and Mirko in London, for many happy moments spent together and whose support started before, in a situation I would have given up without them.

My life as a Ph.D. student would have been much harder without Mattia, whom I thank for being my favourite mate of crazy summer schools with his sense of humor and for many chats. I also thank the other students in Milano, in particular Alberto, Alessandra, Claudia, Davide, Francesco, Francesco, Giulio, Jacopo, Luca, Martino, Mascotto, Matteo, Tommaso, for many chats, funny dinners, Bang games and aperitivi. I thank Eleonora, Gioia, Irene, Ivan, Paola, Sara, Vincenzo, and the other mathematicians I talked with in conferences.

I cannot thank enough my family, in particular my parents and my sister for their constant support, Giulia for playing with me and Martina who is arriving, my boyfriend's parents for making me feel at home, my friends of a life Alessia, Francesca and Marta, the other friends Massimiliano and Romina, and my friends in Torino I have no more space to list.

Finally, I thank my boyfriend Stefano, for always knowing how to make me smile, for staying next to me in every situation and for all the rest the words cannot explain.

My hope is to have the chance to continue the journey in this direction...

Chapter 0

General preliminaries

In this chapter we introduce the basic material in order to understand Part I and Part II of this thesis. In particular, we discuss semiorthogonal decompositions of derived categories, lattice theory, hyperkähler varieties and stability conditions.

0.1 Categorical setting in the geometric context

In this section, we firstly recall the abstract notions of abelian category, triangulated category, derived category of an abelian category and derived functor. Then, passing to the geometric setting, we deal with the bounded derived category of a smooth projective scheme and the notion of semiorthogonal decomposition. Finally, we provide some known examples of semiorthogonal decompositions, dealing with the case of cubic fourfolds and GM fourfolds, which are required in the next chapters. Our main reference for the first three sections is [44], while we follow [59] for the last two sections.

0.1.1 Triangulated categories

The aim of this part is to encode the framework in which we work from the categorical point of view. Since we will consider smooth projective schemes over a field k, our categories will be k-linear. Let us firstly recall this notion.

Definition 0.1.1. A category $\mathcal{A}$ is **additive** if for every couple of objects $A, B \in \mathcal{A}$ the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is endowed with the structure of an abelian group and the following conditions hold:

1. The compositions $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \times \mathrm{Hom}_{\mathcal{A}}(A_2, A_3) \to \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$ sending (f, g) to $g \circ f$ are bilinear.

2. There exists a zero object $0 \in \mathcal{A}$.

3. For any two objects A_1, $A_2 \in \mathcal{A}$ there exists an object $B \in \mathcal{A}$ with morphisms $j_i : A_i \to B$ and $p_i : B \to A_i$ for $i = 1, 2$, which make B the direct sum and the direct product of A_1 and A_2.

A functor $F : \mathcal{A} \to \mathcal{A}'$ between additive categories is additive if the induced maps

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}'}(F(A), F(B))$$

are group homomorphisms.

Definition 0.1.2. A k**-linear category** is an additive category $\mathcal{A}$ such that the groups $\mathrm{Hom}_{\mathcal{A}}(A, B)$ are k-vector spaces and all compositions are k-bilinear.
An additive functor F between k-linear categories is k**-linear** if for every pair of objects A and B in $\mathcal{A}$ the natural map $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}'}(F(A), F(B))$ is k-linear.

We also recall the definition of a functor which takes a key role in the geometric context.

Definition 0.1.3. Let $\mathcal{A}$ be a k-linear category. A **Serre functor** is a k-linear equivalence $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ such that for every pair of objects A and B in $\mathcal{A}$ there is a functorial isomorphism of k-vector spaces

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(B, S_{\mathcal{A}}(A))^{\vee}.$$

We end this section explaining the notions of abelian categories and triangulated categories.

Definition 0.1.4. An **abelian category** is an additive category such that every morphism admits kernel and cokernel, and the natural map between coimage and image is an isomorphism.

Example 0.1.5. In the geometric context, there are three abelian categories naturally associated to a scheme X over k: the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules, the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves, and the category $\mathrm{Coh}(X)$ of coherent sheaves on X. We recall that they are related by the following inclusions:

$$\mathcal{O}_X\text{-Mod} \supset \mathrm{QCoh}(X) \supset \mathrm{Coh}(X).$$

Definition 0.1.6. Let $\mathcal{T}$ be an additive category. The structure of **triangulated category** for $\mathcal{T}$ is the data of an additive autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ called *shift functor*, and a set of sequences of the form

$$A \to B \to C \to A[1]$$

called *distinguished triangles*, satisfying the following axioms.

TR1
- $A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ is a distinguished triangle.
- Any triangle isomorphic to a distinguished triangle is distinguished.
- Any morphism $f : A \to B$ sits inside a distinguished triangle $A \to B \to C \to A[1]$.

TR2 A triangle $A \to B \to C \to A[1]$ is distinguished if and only if $B \to C \to A[1] \to B[1]$ is.

TR3 Suppose we have a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & A_1[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & A_2[1].
\end{array}$$

Then there is a (non-unique) morphism $h : C_1 \to C_2$ completing the diagram.

TR4 Given two morphisms $f : A \to B$ and $g : B \to C$, there is a distinguished triangle $D_1 \to D_3 \to D_2 \to D_1[1]$, where D_1, D_2 and D_3 are given by $A \xrightarrow{f} B \to D_1 \to A[1]$, $B \xrightarrow{g} C \to D_2 \to B[1]$ and $A \xrightarrow{g\circ f} C \to D_3 \to A[1]$, which sits in the following commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \longrightarrow & D_1 & \longrightarrow & A[1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\
A & \xrightarrow{g\circ f} & C & \longrightarrow & D_3 & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{g} & C & \longrightarrow & D_2 & \longrightarrow & B[1].
\end{array}$$

A functor between triangulated categories is **exact** if it commutes with the shift functors and it respects distinguished triangles.

We point out that the left (resp. right) adjoint of an exact functor is exact (see [44], Proposition 1.41).

Definition 0.1.7. A subcategory $\mathcal{T}'$ of a triangulated category $\mathcal{T}$ is a **triangulated subcategory** of $\mathcal{T}$ if $\mathcal{T}'$ has a structure of triangulated category such that the (faithful) inclusion functor $i : \mathcal{T}' \hookrightarrow \mathcal{T}$ is exact.

A triangulated subcategory $\mathcal{T}'$ is **full** if i is a full functor. We say that a full triangulated subcategory $\mathcal{T}'$ is **left admissible** (resp. **right admissible**) if i has a left (resp. right) adjoint functor $i^* : \mathcal{T} \to \mathcal{T}'$ (resp. $i^! : \mathcal{T} \to \mathcal{T}'$). Moreover, $\mathcal{T}'$ is **admissible** if it is left and right admissible.

In the next section, we will describe two important examples of triangulated categories: the homotopy category and the derived category of an abelian category.

0.1.2 Derived categories

Given an abelian category $\mathcal{A}$, we can associate two triangulated categories to the category of complexes of objects in $\mathcal{A}$ in the following way. We recall that a complex in $\mathcal{A}$ is the data of a sequence

$$A^\bullet : \cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

where A^i is an object of $\mathcal{A}$ and d^i is a morphism in $\mathcal{A}$ such that $d^{i+1} \circ d^i = 0$ for every $i \in \mathbb{Z}$. Moreover, a morphism f between two complexes $A^\bullet$ and $B^\bullet$ is the data of a collection of morphisms $f^i : A^i \to B^i$ for $i \in \mathbb{Z}$ such that $f^{i+1} \circ d^i_A = d^i_B \circ f^i$. We denote by $\mathrm{C}(\mathcal{A})$ the category of complexes in $\mathcal{A}$, whose objects are complexes in $\mathcal{A}$ and whose morphisms are morphisms of complexes. We recall that the cohomology of a complex $A^\bullet$ in $\mathcal{A}$ is

$$H^i(A^\bullet) := \frac{\ker d^i_A}{\mathrm{Im} d^{i-1}_A} \in \mathcal{A}$$

and given f in $\mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ we have an induced map $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ for every $i \in \mathbb{Z}$.

Definition 0.1.8. Two morphisms f and g in $\mathrm{C}(\mathcal{A})$ are **homotopy equivalent** if there is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$. In this case, we write $f \sim g$. We denote by $\mathrm{K}(\mathcal{A})$ the **homotopy category** of $\mathrm{C}(\mathcal{A})$ whose objects are complexes in $\mathcal{A}$ and whose set of morphisms for every pair $A^\bullet$, $B^\bullet$ in $\mathrm{K}(\mathcal{A})$ is given by

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim .$$

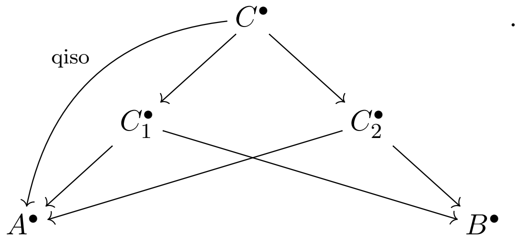
Definition 0.1.9. A morphism $f : A^\bullet \to B^\bullet$ in $\mathrm{C}(\mathcal{A})$ is a **quasi-isomorphism** if $H^i(f)$ is an isomorphism for every $i \in \mathbb{Z}$.

Notice that if f is homotopy equivalent to g, then $H^i(f) = H^i(g)$ for every index i. Thus, the condition of being a quasi-isomorphism is well-defined in $\mathrm{K}(\mathcal{A})$.

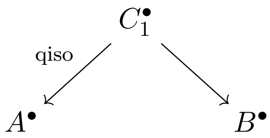
Definition 0.1.10. Given an abelian category $\mathcal{A}$, the **derived category** $\mathrm{D}(\mathcal{A})$ is the category whose objects are complexes in $\mathcal{A}$ and whose sets of morphisms $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ are formed by equivalence classes of diagrams of the form

$$A^\bullet \xleftarrow{\text{qiso}} C^\bullet \longrightarrow B^\bullet$$

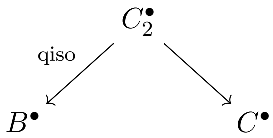
where $C^\bullet \to A^\bullet$ is a quasi-isomorphism. Two such diagrams are equivalent if there exists a commutative diagram in $\mathrm{K}(\mathcal{A})$ of the form



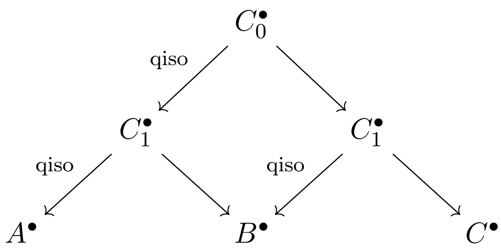
Given two morphisms



and



in $\mathrm{D}(\mathcal{A})$, the composition is defined by the equivalence class of a commutative diagram



in $\mathrm{K}(\mathcal{A})$. In [44], Proposition 2.17 and Corollary 2.18, it is proved that this diagram exists and it is unique up to equivalence (the proof relies on the definition of the cone of a morphism which is explained below).

The idea behind the definition is that quasi-isomorphisms are isomorphisms in the derived category. More precisely, the derived category of $\mathcal{A}$ is obtained by *localization* of $\mathrm{K}(\mathcal{A})$ with respect to the class of quasi-isomorphisms (see [34] for a precise definition). In particular, we have a natural functor $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$.

The homotopy category and the derived category of $\mathcal{A}$ carry a triangulated structure. In particular, the shift functor is the equivalence which acts on complexes by shifting by one the degree, i.e. given a complex $A^\bullet$ in $\mathcal{A}$, then $A^\bullet[1]$ is the complex with $A[1]^i := A^{i+1}$ and $d^i_{A[1]} := d^{i+1}_A$. In order to define distiguished triangles, we need to introduce the mapping cone construction. Given a morphism $f : A^\bullet \to B^\bullet$, the **mapping cone** of f is the complex $C(f)^\bullet$ defined by $C(f)^i := A^{i+1} \oplus B^i$ and

$$d^i_{C(f)^\bullet} = \begin{pmatrix} -d^{i+1}_A & 0 \\ -f^{i+1} & d^i_B \end{pmatrix}.$$

Notice that $A^\bullet$, $B^\bullet$ and $C(f)^\bullet$ sit in the sequence

$$A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A^\bullet[1],$$

where the second map is the inclusion $B^i \hookrightarrow A^{i+1} \oplus B^i$ and the third map is the projection $A^{i+1} \oplus B^i \twoheadrightarrow A^{i+1}$. The set of isomorphism classes of sequences of this form defines the set of distinguished triangles. Moreover, the localization functor $Q_{\mathcal{A}}$ is exact. We refer to [34], Chapter IV.2 for the proof.

We point out that we can consider the categories $\mathrm{C}^+(\mathcal{A})$, $\mathrm{C}^-(\mathcal{A})$ and $\mathrm{C}^{\mathrm{b}}(\mathcal{A})$ defined respectively as the categories of bounded below complexes, bounded above complexes and bounded complexes. We can then construct the homotopy categories $\mathrm{K}^+(\mathcal{A})$, $\mathrm{K}^-(\mathcal{A})$, $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ and the derived categories $\mathrm{D}^+(\mathcal{A})$, $\mathrm{D}^-(\mathcal{A})$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ with the same procedure used before.

Remark 0.1.11. Notice that there is a fully faithful functor $i : \mathcal{A} \hookrightarrow \mathrm{D^b}(\mathcal{A})$ sending an object $A \in \mathcal{A}$ to the bounded complex $\dots \to 0 \to A \to 0 \to \dots$, where the only non zero factor is A sitted in degree 0.

Definition 0.1.12. (**Example**) Let X be a noetherian scheme over k. Then the **derived category of** X is the derived category of boundend complexes of coherent sheaves $\mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X))$ constructed from the abelian category $\mathrm{Coh}(X)$.

Remark 0.1.13. As seen in Example 0.1.5, we have other abelian categories associated to X. In particular, we can consider also the derived categories $\mathrm{D^b}(\mathrm{QCoh}(X))$ and $\mathrm{D^b}(\mathcal{O}_X\text{-Mod})$. If X is a noetherian scheme, we have that $\mathrm{D^b}(\mathrm{QCoh}(X))$ is identified with the full triangulated subcategory of $\mathrm{D^b}(\mathcal{O}_X\text{-Mod})$ of bounded complexes with coherent cohomology (see [44], Proposition 3.3). Moreover, the derived category $\mathrm{D^b}(\mathrm{Coh}(X))$ is equivalent via the natural inclusion in $\mathrm{D^b}(\mathrm{QCoh}(X))$ to the full triaungulated subcategory of bounded complexes of quasi-coherent sheaves with coherent cohomology (see [44], Proposition 3.5).

0.1.3 Derived functors

We are now interested to study functors between abelian categories which induce a functor between the derived categories. It is not difficult to check that an exact functor between abelian categories defines a functor between the associated derived categories which preserves the triangulated structure. However, there are many non exact functors which are interesting in the geometrical setting, for example pullback, pushforward, tensor product and many others. In any case, there is a general construction which allows to associate an exact functor between the derived categories under weaker hypotheses than exactness. A functor obtained in this way is a **derived functor**. Firstly, we need to assume that the additive functor $F : \mathcal{A} \to \mathcal{B}$ is left or right exact. In particular, if F is left exact (resp. right exact), we will obtain the right derived functor $\mathrm{R}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ (resp. the left derived functor $\mathrm{L}F : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$).

Let us recall the construction of $\mathrm{R}F$ in the case that F is left exact. Assume that the category $\mathcal{A}$ has *enough injectives*. Explicitely, this means that for every object $A \in \mathcal{A}$ there is an injective morphism $A \to I$, where I is injective. Notice that if $\mathcal{A}$ has enough injective objects, then every $A \in \mathcal{A}$ has an injective resolution, i.e. an exact sequence of the form

$$0 \to A \to I^0 \to I^1 \to \dots,$$

where I^i is an injective object in $\mathcal{A}$ for every index i. We denote by $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ of injective objects. The key result for the construction of derived functor is the following property.

Proposition 0.1.14 ([44], Proposition 2.40). *Let $\mathcal{A}$ be an abelian category with enough injectives. Then the exact functor $i : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$ given by the composition of the functor induced by the natural inclusion $\mathcal{I} \subset \mathcal{A}$ with the localization functor $\mathrm{K}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$ is an equivalence.*

We are now ready to define the right derived functor $\mathrm{R}F$.

Definition 0.1.15. The **right derived functor** of $F : \mathcal{A} \to \mathcal{B}$ is given by the composition

$$\mathrm{R}F := Q_{\mathcal{B}} \circ \mathrm{K}F \circ i^{-1} : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B}).$$

Here $\mathrm{K}F : \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$ is the (well-defined) natural functor sending a complex $\dots \to A^{i-1} \to A^i \to A^{i+1} \to \dots$ in $\mathrm{K}^+(\mathcal{A})$ to $\dots \to F(A^{i-1}) \to F(A^i) \to F(A^{i+1}) \to \dots$, and a morphism $\{f^i : A^i \to B^i\}$ in $\mathrm{K}^+(\mathcal{A})$ to $\{Ff^i : F(A^i) \to F(B^i)\}$.

In the following, we will often write F instead of $\mathrm{R}F$ or $\mathrm{L}F$ to denote the derived functor between the derived categories to simplify the notation.

Remark 0.1.16. Assume that $\mathcal{A}$ is an abelian category with enough injective objects. Then for every $A, B \in \mathcal{A} \hookrightarrow \mathrm{D^b}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[k]) \cong \mathrm{Ext}^k_{\mathcal{A}}(A, B)$$

(see [44], Proposition 2.56). This property is very useful in order to compute Hom groups.

In the geometric setting, an important class of functors between the derived categories of smooth projective varieties is given by Fourier-Mukai functors.

Definition 0.1.17. An exact functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(X')$ is of **Fourier-Mukai type** if there exists an object K in the derived category $\mathrm{D^b}(X \times X')$ of the product and an isomorphism of exact functors

$$F(-) \cong \Phi_K(-) := \mathrm{R}p_{X'*}(K \overset{\mathrm{L}}{\otimes} \mathrm{L}p_X^*(-)),$$

where $p_X : X \times X' \to X$ and $p_{X'} : X \times X' \to X'$ are the natural projections. The object K is called *Fourier-Mukai kernel.*

By definition, a Fourier-Mukai functor Φ_K is exact and it is possible to check that the composition of Fourier-Mukai functors is again of Fourier-Mukai type (see [44], Proposition 5.10).

Remark 0.1.18. A famous result by Orlov states that every fully and faithful exact functor between the derived category of two smooth projective varieties is of Fourier-Mukai type and the kernel is unique up to isomorphism (see [44], Theorem 5.14). However, in [90], they construct an example of an exact functor between the derived categories of two smooth projective schemes which is not of Fourier-Mukai type, relaxing the hypothesis of fully faithfulness. For a complete survey about related questions, we recommend [22].

Example 0.1.19. Let X be a smooth projective variety with canonical sheaf ω_X. By Serre duality for coherent sheaves, it follows that the exact autoequivalence of $\mathrm{D^b}(X)$ defined by

$$S_X(-) := (-) \otimes \omega_X[\dim(X)],$$

is a Serre functor for $\mathrm{D^b}(X)$. Notice also that S_X is a Fourier-Mukai functor with kernel $(\Delta_X)_*\omega_X[\dim(X)]$, where $\Delta_X : X \to X \times X$ denotes the diagonal embedding. Indeed, for every E in $\mathrm{D^b}(X)$, by projection formula we have that

$$\begin{aligned}\Phi_{(\Delta_X)_*\omega_X[\dim(X)]}(E) &= p_{2*}(p_1^*E \otimes (\Delta_X)_*\omega_X[\dim(X)]) \\ &= p_{2*}(\Delta_X)_*((\Delta_X)^*p_1^*E \otimes \omega_X))[\dim(X)] \\ &= (p_2 \circ (\Delta_X))_*((p_1 \circ (\Delta_X))^*E \otimes \omega_X)[\dim(X)] \cong E \otimes \omega_X[\dim(X)],\end{aligned}$$

where for the last identification we have used that $p_i \circ (\Delta_X) = \mathrm{id}$.

0.1.4 Semiorthogonal decompositions of $\mathbf{D^b}(X)$

The derived category $\mathrm{D^b}(X)$ of a smooth projective scheme X over a field k is in general difficult to describe. The following definition provides a tool in order to decompose $\mathrm{D^b}(X)$ into "easier" subcategories.

Definition 0.1.20. A **semiorthogonal decomposition** of $\mathrm{D^b}(X)$ is the data of a sequence $\mathcal{T}_1, \dots, \mathcal{T}_n$ of full triangulated subcategories of $\mathrm{D^b}(X)$ satisfying the folllowing conditions:

1. $\mathrm{Hom}_{\mathrm{D^b}(X)}(A_i, A_j) = 0$ for every $A_i \in \mathcal{T}_i$, $A_j \in T_j$ and $n \geqslant i > j \geqslant 0$;

2. For any non trivial object E in $\mathrm{D^b}(X)$, there is a chain of morphisms

$$0 = E_n \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} E_{n-2} \dots E_1 \xrightarrow{f_1} E_0 = E$$

such that the cone $C(f_i)$ is in $\mathcal{T}_i$ for every $1 \leqslant i \leqslant n$.

We use the notation $\mathrm{D^b}(X) = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle$ for such a semiorthogonal decomposition.

Notice that the semiorthogonality condition of item 1 implies that the filtration of item 2 is unique up to isomorphism. The second requirement means that $\mathrm{D^b}(X)$ is generated by the subcategories $\mathcal{T}_1, \dots, \mathcal{T}_n$.

Remark 0.1.21. 1) If $\mathrm{D^b}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$, then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible. Indeed, notice that by Definition 0.1.20, for every $C \in \mathrm{D^b}(X)$, there is a unique (up to isomorphism) distinguished triangle

$$C_{\mathcal{B}} \to C \to C_{\mathcal{A}} \to C_{\mathcal{B}}[1] \tag{1}$$

with $C_{\mathcal{B}} \in \mathcal{B}$ and $C_{\mathcal{A}} \in \mathcal{A}$. Furthermore, let $f : C \to C'$ be a morphism in $\mathrm{D^b}(X)$ and consider the triangle

$$C'_{\mathcal{B}} \to C' \to C'_{\mathcal{A}} \to C'_{\mathcal{B}}[1]. \tag{2}$$

Applying $\mathrm{Hom}_{\mathrm{D^b}(X)}(C_{\mathcal{B}}, -)$ to (2) and using the semiorthogonality condition, we get

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(C_{\mathcal{B}}, C') \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(C_{\mathcal{B}}, C'_{\mathcal{B}}).$$

Thus, there is a morphism $f_{\mathcal{B}}$ sitting in the commutative diagram

$$\begin{array}{ccccc} C_{\mathcal{B}} & \longrightarrow & C & \longrightarrow & C_{\mathcal{A}} \\ \downarrow{\scriptstyle f_{\mathcal{B}}} & & \downarrow{\scriptstyle f} & & \\ C'_{\mathcal{B}} & \longrightarrow & C' & \longrightarrow & C'_{\mathcal{A}}. \end{array}$$

The uniqueness of the filtration allows to define a functor $i^!_{\mathcal{B}} : \mathrm{D^b}(X) \to \mathcal{B}$ such that $i^!_{\mathcal{B}}(C) = C_{\mathcal{B}}$ and $i^!_{\mathcal{B}}(f) = f_{\mathcal{B}}$. We prove that $i^!_{\mathcal{B}}$ is right adjoint to the inclusion $i_{\mathcal{B}} : \mathcal{B} \to \mathrm{D^b}(X)$. Indeed, for every $B \in \mathcal{B}$, applying $\mathrm{Hom}_{\mathrm{D^b}(X)}(i_{\mathcal{B}}(B), -)$ to (1) and using the semiorthogonality condition, we get $\mathrm{Hom}_{\mathrm{D^b}(X)}(i_{\mathcal{B}}(B), C) \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(i_{\mathcal{B}}(B), C_{\mathcal{B}}) \cong \mathrm{Hom}_{\mathcal{B}}(B, C_{\mathcal{B}})$. This proves that $\mathcal{B}$ is right admissible. Analogously, we deduce the left admissibility of $\mathcal{A}$.
2) Since $\mathrm{D^b}(X)$ has Serre functor, we have also the semiorthogonal decompositions

$$\mathrm{D^b}(X) = \langle S_X(\mathcal{B}), \mathcal{A} \rangle = \langle \mathcal{B}, S_X^{-1}(\mathcal{A}) \rangle.$$

In particular, by item 1), it follows that $\mathcal{A}$ and $\mathcal{B}$ are admissible subcategories.

It may happen that some factors of a semiorthogonal decomposition are subcategories generated by a single object. This special situation is explained by the following definition.

Definition 0.1.22. An **exceptional object** is an element $E \in \mathrm{D^b}(X)$ such that

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(E, E[l]) = \begin{cases} k & \text{if } l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

An **exceptional collection** is a sequence $E_1, \dots E_n$ of exceptional objects such that

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(E_i, E_j[l]) = 0 \text{ if } i > j \text{ and for all } l.$$

An exceptional collection $E_1, \dots, E_n$ is **full** if any full triangulated subcategory containing $E_1, \dots, E_n$ is equivalent to $\mathrm{D^b}(X)$.

Given an exceptional object E, we denote by $\langle E\rangle$ the full triangulated subcategory of $\mathrm{D^b}(X)$, whose objects are elements of the form $\oplus E[i]^{\oplus j_i}$. Notice that we can identify the subcategory $\langle E\rangle$ with the derived category of k-vector spaces. More precisely, the functor $i : \mathrm{D^b}(k) \to \mathrm{D^b}(X)$, defined over objects by $i(V^\bullet) = V^\bullet \otimes_k E$ and in the natural way over morphisms, is full and faithful.

We claim that the subcategory $\langle E\rangle$ is admissible. Indeed, an easy computation shows that the functors $i^*, i^! : \mathrm{D^b}(X) \to \mathrm{D^b}(k)$ defined over objects by

$$i^*(F) = \mathrm{Hom}_{\mathrm{D^b}(X)}(F, E)^\vee \quad \text{and} \quad i^!(F) = \mathrm{Hom}_{\mathrm{D^b}(X)}(E, F)$$

are, respectively, left and right adjoint to i.

We consider the left and right orthogonal categories

$$E^\perp := \{F \in \mathrm{D^b}(X) : \mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[l]) = 0 \text{ for every } l\}$$

and

$${}^\perp E := \{F \in \mathrm{D^b}(X) : \mathrm{Hom}_{\mathrm{D^b}(X)}(F, E[l]) = 0 \text{ for every } l\}.$$

Then there are semiorthogonal decompositions of the form

$$\mathrm{D^b}(X) = \langle E^\perp, E\rangle \quad \text{and} \quad \mathrm{D^b}(X) = \langle E, {}^\perp E\rangle,$$

where E stands for the subcategory $\langle E\rangle$. Let us verify this claim for the first decomposition (the proof in the other case is analogous). It is enough to verify the second condition of Definition 0.1.20, since the first one holds by definition. Let F be in $\mathrm{D^b}(X)$ and we set $G := i^!(F) \in \langle E\rangle$. Recall that by adjunction we have

$$\mathrm{Hom}_{\langle E\rangle}(G, G) \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(G, F).$$

Thus the identity morphism in $\mathrm{Hom}_{\langle E\rangle}(G, G)$ corresponds to a morphism $G \to F$. We denote by G' the cone of this morphism and we consider the exact triangle

$$G \to F \to G' \to G[1].$$

Now notice that

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F) \cong \mathrm{Hom}_{\langle E\rangle}(E, i^!(F)) \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(E, G).$$

Thus, applying the functor $\mathrm{Hom}_{\mathrm{D^b}(X)}(E, -)$ to the above triangle, we deduce that $\mathrm{Hom}_{\mathrm{D^b}(X)}(E, G') = 0$. We conclude that G' is an object in $\langle E\rangle^\perp$ as desired.

The argument above generalizes to any exceptional collection. In particular, we have the following well-known fact.

Proposition 0.1.23. *If $E_1, \ldots, E_n$ is an exceptional collection in $\mathrm{D^b}(X)$, then there are semiorthogonal decompositions of the form*

$$\mathrm{D^b}(X) = \langle \mathcal{C}, E_1, \ldots, E_n\rangle \quad \textit{and} \quad \mathrm{D^b}(X) = \langle E_1, \ldots, E_n, \mathcal{D}\rangle$$

where $\mathcal{C} := \langle E_1, \ldots, E_n\rangle^\perp = E_1^\perp \cap \cdots \cap E_n^\perp$ and $\mathcal{D} := {}^\perp\langle E_1, \ldots, E_n\rangle = {}^\perp E_1 \cap \cdots \cap {}^\perp E_n$.

Remark 0.1.24. We point out that $\mathcal{C}$ and $\mathcal{D}$ are admissible subcategories by Remark 0.1.21.

Notice that semiorthogonal decompositions are not unique, as seen for example in item 2 of Remark 0.1.21. In fact, given an exceptional object $E \in \mathrm{D^b}(X)$, we define the **left and right mutation functors** $\mathbb{L}_E, \mathbb{R}_E : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ by

$$\mathbb{L}_E(G) = \mathrm{cone}(\oplus_p \mathrm{Hom}_{\mathrm{D^b}(X)}(E, G[p])[-p] \otimes E \to G)$$

and

$$\mathbb{R}_E(G) = \operatorname{cone}(G \to \oplus_p \operatorname{Hom}_{\mathrm{D^b}(X)}(G, E[p])[-p]^\vee \otimes E)[-1].$$

Thus, if we have a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathcal{T}_1, \dots, \mathcal{T}_k, E, \mathcal{T}_{k+1}, \dots, \mathcal{T}_m \rangle,$$

then we also have

$$\mathrm{D^b}(X) = \langle \mathcal{T}_1, \dots, \mathcal{T}_k, \mathbb{L}_E(\mathcal{T}_{k+1}), E, \mathcal{T}_{k+2}, \dots, \mathcal{T}_m \rangle \quad \text{and} \quad \mathrm{D^b}(X) = \langle \mathcal{T}_1, \dots, \mathcal{T}_{k-1}, E, \mathbb{R}_E(\mathcal{T}_k), \mathcal{T}_{k+1}, \dots, \mathcal{T}_m \rangle.$$

0.1.5 Examples of semiorthogonal decompositions

Interesting examples of semiorthogonal decompositions come from Fano varieties, as we recall in the following.

Example 0.1.25 (Projective space)**.** The n-dimensional projective space $\mathbb{P}^n$ over a field k admits a semiorthogonal decomposition of the form

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}(a), \mathcal{O}(1+a), \dots, \mathcal{O}(n+a) \rangle$$

for every $a \in \mathbb{Z}$. Indeed, by Remark 0.1.16, we have that

$$\operatorname{Hom}_{\mathrm{D^b}(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(j)[l]) \cong \operatorname{Ext}^l(\mathcal{O}(i), \mathcal{O}(j)) \cong H^l(\mathbb{P}^n, \mathcal{O}(j-i)) = \begin{cases} 0 & \text{if } i > j, l \in \mathbb{Z} \text{ or } i = j, l \in \mathbb{Z} - \{0\} \\ k & \text{if } i = j, l = 0. \end{cases}$$

This proves that the sequence of line bundles we considered forms an exceptional collection. The fact that these objects generate $\mathrm{D^b}(\mathbb{P}^n)$ is a consequence of the Beilinson spectral sequence, as explained in [44], Corollary 8.29.

Example 0.1.26 (Quadric hypersurfaces)**.** Assume that k is a field with $\operatorname{char}(k) \neq 2$. Let Q be a quadric hypersurface in $\mathbb{P}^{n+1}$. By [53], Section 4, there is a semiorthogonal decomposition of the form

$$\mathrm{D^b}(Q) = \begin{cases} \langle \Sigma_-(-n), \Sigma_+(-n) \mathcal{O}_Q(-n+1), \dots, \mathcal{O}_Q \rangle & \text{if } n \text{ is even} \\ \langle \Sigma(-n), \mathcal{O}_Q(-n+1), \dots, \mathcal{O}_Q \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Here $\Sigma_+, \Sigma_-, \Sigma$ are *spinor bundles*, which are vector bundles over Q constructed from the associated Clifford algebra (see also [1]).

Example 0.1.27 (Projective hypersurfaces)**.** Let Y be a smooth complex hypersurface of degree d in $\mathbb{P}^{n+1}$ such that $d < n+2$. By adjunction formula, we have that $\omega_Y = \mathcal{O}_Y(d-n-2)$. Then there is a semiorthogonal decomposition

$$\mathrm{D^b}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n-d+1) \rangle,$$

where $\mathcal{O}_Y(i)$ is the restriction of the line bundle $\mathcal{O}(i)$ on $\mathbb{P}^{n+1}$ and

$$\mathrm{Ku}(Y) := \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n+1-d) \rangle^\perp$$

is known as the **Kuznetsov component** of Y. Indeed, for $n+1-d \geqslant i > j \geqslant 0$, we have $0 < i-j < n+2-d$. We set $h := n+2-d-(i-j) > 0$; by Kodaira vanishing Theorem, for $l \neq 0$ we have that

$$\operatorname{Hom}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y(i), \mathcal{O}_Y(j)[l]) \cong \operatorname{Ext}^l(\mathcal{O}_Y(i), \mathcal{O}_Y(j)) \cong H^l(Y, \mathcal{O}_Y(j-i)) = H^l(Y, \mathcal{O}_Y(h) \otimes \omega_Y) = 0.$$

Since $\mathcal{O}_Y(1)$ is an ample line bundle on Y, we deduce that $H^0(Y, \mathcal{O}_Y(j-i)) = 0$; using the sequence

$$0 \to \mathcal{O}(-d) \to \mathcal{O} \to \mathcal{O}_Y \to 0$$

we get

$$\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y(i), \mathcal{O}_Y(i)[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\{\mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n-d+1)\}$ is an exceptional collection. By Proposition 0.1.23, we deduce the desired semiorthogonal decomposition. As seen in the previous example, if $d = 2$, there is an explicit description of $\mathrm{Ku}(Y)$ in terms of the spinor bundles. For a more intrinsic discussion working over an algebraic closed field k of $\mathrm{char}(k) \neq 2, 3$, we suggest [69], Section 3.1.

Example 0.1.28 (Prime Fano varieties)**.** The examples considered above belong to the class of prime Fano varieties. We recall that a prime Fano variety X is a smooth complex projective variety with ample anticanonical bundle and Picard number one. The index of X is the integer r such that $\omega_X = \mathcal{O}_X(-r)$, where $\mathcal{O}_X(1)$ is an ample line bundle. By the same argument explained in the case of projective hypersurfaces, it is possible to prove that there is a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \dots, \mathcal{O}_X(r-1) \rangle,$$

where $\mathrm{Ku}(X) := \langle \mathcal{O}_X, \dots, \mathcal{O}_X(r-1) \rangle^{\perp}$. The three examples above are prime Fano varieties of index $n+1$, n and $n+2-d$ respectively.

Example 0.1.29 (Grassmannians)**.** Assume that k is a field of characteristic zero. Let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-dimensional subvector spaces in a n-dimensional k-vector space. It is known that $\mathrm{Gr}(k, n)$ provides an other example of prime Fano variety with $\omega_{\mathrm{Gr}(k,n)} = \mathcal{O}_{\mathrm{Gr}(k,n)}(-n)$. However, it is proved in [53] that there is an other full exceptional collection involving the tautological subbundle $\mathcal{U}$ on $\mathrm{Gr}(k, n)$. More precisely, there is a semiorthogonal decomposition of the form

$$\mathrm{D^b}(\mathrm{Gr}(k, n)) = \langle \Sigma^{\alpha} \mathcal{U}^{\vee} \rangle_{\alpha \in \mathsf{Y}_{k,n-k}},$$

where $\mathsf{Y}_{k,n-k}$ is the set of Young diagrams inscribed in a $k \times (n-k)$-rectangle and Σ^{α} is the Schur functor associated to the Young diagram α. We point out that in [33], examples of *minimal Lefschetz decompositions* for $\mathrm{D^b}(\mathrm{Gr}(k, n))$, which are semiorthogonal decompositions of a particular form, are provided.

Example 0.1.30 (GM varieties)**.** Gushel-Mukai varieties are an interesting class of prime Fano varieties obtained as quadric sections of linear sections of the cone over Grassmannian $\mathrm{Gr}(2, 5)$ (see Definition 0.1.20). In [61], using the result recalled in Example 0.1.29, they prove that a Gushel-Mukai variety of dimension $3 \leqslant n \leqslant 6$ has a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^{\vee}, \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^{\vee}(n-3) \rangle,$$

where $\mathcal{U}_X$ is the pullback of the tautological bundle over the Grassmannian via the projection $X \to \mathrm{Gr}(2, 5)$ from the vertex of the cone. In this case, we refer to $\mathrm{Ku}(X)$ as the **Kuznetsov component** of X. We will give more details on this example in the second part of this thesis.

Finally, we consider the more general situations of the blow-up of a smooth projective subvariety and of a flat quadric fibration, respectively, recalling the semiorthogonal decompositions arising in this context.

Example 0.1.31 (Blow-up)**.** Let X be a smooth projective scheme and let Y be a locally complete intersection subscheme of X of codimension $r \geqslant 2$. We denote by $\tilde{X} := \mathrm{Bl}_Y(X) \xrightarrow{\sigma} X$ the blow up of X in Y. Let $i : E \hookrightarrow \tilde{X}$ be the exceptional divisor. A classical result by Orlov (see [82]) states that there is a semiorthogonal decomposition of the form

$$\mathrm{D^b}(\tilde{X}) = \langle \Phi_{\mathcal{O}_E(1-r)}(\mathrm{D^b}(Y)), \ldots, \Phi_{\mathcal{O}_E(-1)}(\mathrm{D^b}(Y)), \sigma^*(\mathrm{D^b}(X))\rangle.$$

Here $\mathcal{O}_E(E) \cong \mathcal{O}_{E/Y}(-1)$ is the Grothendieck line bundle on the projectivization $E \cong \mathbb{P}(\mathcal{N}_{Y|X})$, and for every $k \in \mathbb{Z}$, the Fourier-Mukai functor $\Phi_{\mathcal{O}_E(k)} : \mathrm{D^b}(Y) \to \mathrm{D^b}(\tilde{X})$ defined by

$$\Phi_{\mathcal{O}_E(k)}(-) = i_*((\sigma|_E)^*(-) \otimes \mathcal{O}_{E/Y}(k))$$

is fully faithful.

Example 0.1.32 (Quadric fibrations)**.** In [55], Kuznetsov constructed a semiorthogonal decomposition for the derived category of a flat quadric fibration, which we recall in the following.

Given a smooth algebraic variety S, let $\mathcal{F}$ be a rank n vector bundle on S. We denote by $\pi : \mathbb{P}_S(\mathcal{F}) \to S$ the projectivization of $\mathcal{F}$ in S and let $\mathcal{O}_{\mathbb{P}_S(\mathcal{F})}(1)$ be the Grothendieck line bundle on it. Notice that $H^0(S, S^2\mathcal{F} \otimes \mathcal{L}^*) \cong H^0(\mathbb{P}_S(\mathcal{F}), \mathcal{O}_{\mathbb{P}_S(\mathcal{F})}(2) \otimes \mathcal{L}^*)$, where $\mathcal{L}$ is a line bundle on S. We denote by $X \subset \mathbb{P}_S(\mathcal{F})$ the zero locus of a non trivial section σ of $\mathcal{O}_{\mathbb{P}_S(\mathcal{F})}(2) \otimes \mathcal{L}^*$. Then the restriction $p : X \to S$ of π defines a flat quadric fibration, whose fibers have dimension $n-2$.

As every quadratic form carries a natural Clifford algebra, we consider the sheaf of Clifford algebras associated to the quadric fibration p and let $\mathcal{B}_0$ be its even part. We denote by $\mathrm{Coh}(S, \mathcal{B}_0)$ the abelian category of coherent sheaves with the structure of $\mathcal{B}_0$-modules over S and we set

$$\mathrm{D^b}(S, \mathcal{B}_0) := \mathrm{D^b}(\mathrm{Coh}(S, \mathcal{B}_0)).$$

Then by [55], Theorem 4.2, there is a fullly faithful functor $\Phi : \mathrm{D^b}(S, \mathcal{B}_0) \to \mathrm{D^b}(X)$ and a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \Phi(\mathrm{D^b}(S, \mathcal{B}_0)), p^*\,\mathrm{D^b}(S) \otimes \mathcal{O}_{X/S}(1), \ldots, p^*\,\mathrm{D^b}(S) \otimes \mathcal{O}_{X/S}(n-2)\rangle.$$

We do not recall the precise definition of Φ as it is not important for the following, but in Chapter 4 we will describe its left adjoint in the particular case of the conic fibration obtained by projecting from a line in a cubic fourfold.

0.2 Basics on lattice theory

The aim of this section is to give a summary of some well-known definitions and properties involving lattices, and to fix the notation we will use in the following. Our main reference is [76].

Definition 0.2.1. A **lattice** is a free abelian group L of finite rank with a nondegenerate symmetric bilinear form $(\ ,\ ) : L \times L \to \mathbb{Z}$.

Let us recall some terminology related to a lattice $(L, (\ ,\ ))$.

- An **isometry** of L is an automorphism of L preserving $(\ ,\ )$. We denote by $\mathrm{O}(L)$ the group of isometries of L.
- We say that a lattice L is **even** if (l, l) is even for every $l \in L$, and L is **odd** otherwise.
- We consider the matrix representing the bilinear form $(\ ,\ )$ in a fixed basis for L. The **discriminant** of L is the determinant of this matrix and we denote it by $\mathrm{discr}(L)$. We say that L is **unimodular** if $\mathrm{discr}(L) = \pm 1$.

- A lattice L is **positive definite** (resp. **negative definite**) is $l^2 := (l, l) > 0$ (resp. < 0) for every non zero $l \in L$. Otherwise, we say that L is **indefinite**.
- The **dual** of a lattice L is $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{l' \in L \otimes \mathbb{Q} : (l', l) \in \mathbb{Z} \text{ for every } l \in L\}$. Notice that there is a natural inclusion of L in its dual sending $l \in L$ to $(l, -) \in L^\vee$, and the bilinear form $(\, , \,)$ extends to a symmetric bilinear form over $L^\vee$ with values in $\mathbb{Q}$. Given a basis $l_1, \dots, l_r$ for L, the dual basis $l_1^\vee, \dots, l_r^\vee$ of $L^\vee$ such that $l_i^\vee(l_j) = \delta_{ij}$ is a $\mathbb{Z}$-basis for $L^\vee$. In particular, the dual of a lattice is a lattice.
- The **discriminant group** of L is the finite group $d(L) := L^\vee/L$. The bilinear form on $L^\vee$ induces a symmetric bilinear form on $d(L)$ and we denote by $q_L : d(L) \to \mathbb{Q}/\mathbb{Z}$ the associated quadratic form. The form q_L is known as the **discriminant form**.

  Notice that if L is even, then q_L takes values in $\mathbb{Q}/2\mathbb{Z}$. Finally, we observe that the order of $d(L)$ is equal to $|\mathrm{discr}(L)|$. Indeed, assume that these lattices have rank r, fix a basis for L and its dual on $L^\vee$. We denote by A the representative matrix of the inclusion $j : L \hookrightarrow L^\vee$ in these bases. By definition, we have that $\mathrm{discr}(L) = \det A$. Passing to the Smith normal form, we write $A = PDQ$, where P and Q are invertible matrices with entries in $\mathbb{Z}$, and D is diagonal with entries $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ such that $\lambda_i \mid \lambda_{i+1}$ for $1 \leqslant i < r$. Then the image of j is generated by vectors $\lambda_i v_i$ where the v_i's form a basis for $L^\vee$. It follows that $L^\vee/L \cong \mathbb{Z}/\lambda_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\lambda_r\mathbb{Z}$, whose order is equal to $|\det A|$, as we wanted. We denote by $l(d(L))$ the minimal number of generators of $d(L)$.
- The **genus** of L is the set $\mathcal{G}(L)$ of all isometry classes of lattices L' with the same signature of L and $d(L') \cong d(L)$.
- Let $\mathrm{O}(d(L))$ be the group of automorphisms of $d(L)$ preserving q_L. Using the inclusion $L \hookrightarrow L^\vee$, we obtain a homomorphism $r_L : \mathrm{O}(L) \to \mathrm{O}(d(L))$. We use the notation $\bar{f} := r_L(f)$ for $f \in \mathrm{O}(L)$.

Example 0.2.2. (1) The **hyperbolic lattice**, denoted by U, is the free group $\mathbb{Z}^{\oplus 2}$ with the bilinear form represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is an even unimodular lattice of signature $(1, 1)$.

(2) We denote by E_8 the unique even, unimodular lattice of signature $(8, 0)$. More explicitely, E_8 is the abelian group $\mathbb{Z}^{\oplus 8}$ with the bilinear form represented by the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

(3) Given a lattice $(L, (\, , \,))$ and a non-zero integer m, we denote by $L(m)$ the lattice $(L, m(\, , \,))$.

(4) We set $I_{r,s} := I_1^r \oplus I_1(-1)^s$, where I_1 is the lattice $\mathbb{Z}$ with bilinear form (1).

(5) We denote by A_2 the free group $\mathbb{Z}^{\oplus 2}$ with the bilinear form represented by the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. It is an even lattice of discriminant 3 and signature $(2, 0)$. The lattice A_2 has a key role in the definition of the Mukai lattice of a cubic fourfold as recalled in Part I of this thesis.

The first four examples we recalled above actually describe all possible lattices in the unimodular case, as explained in the following result.

Theorem 0.2.3 ([72], Theorem 1.3)**.** *Let L be an indefinite unimodular lattice. If L is odd, then $L \cong I_{m,n}$ for some m and n. If L is even, then $L \cong U^{\oplus m} \oplus E_8(\pm 1)^{\oplus n}$ for some m and n. In particular, the signature and the parity of L determine the lattice up to isometry.*

Example 0.2.4. In the geometric context, unimodular lattices arise in the studying of torsion free higher degree cohomology groups of Kähler varieties.

For example, let Y be a *cubic fourfold*, i.e. a smooth hypersurface of degree three in $\mathbb{P}^5_{\mathbb{C}}$. We recall that the degree four integral cohomology group $L := H^4(Y, \mathbb{Z})$ is a torsion free abelian group; the intersection form $(\, , \,)$ on L is a symmetric nondegenerate bilinear form, whose signature is determined by the Riemann bilinear relations. Moreover, by Poincaré duality we have an isomorphism of L with its dual. We conclude that $(L, (\, , \,))$ is a unimodular lattice of signature $(21, 2)$. We denote by H^2 the class in L of a cubic surface. Since $(H^2, H^2) = H^4 = \deg(Y) = 3$, we deduce that L is an odd lattice. By Theorem 0.2.3, we conclude that $L \cong I_{21,2}$ as a lattice.

Relaxing the unimodularity condition, we have the following classification result for even lattices.

Theorem 0.2.5 ([76], Corollary 1.14.2)**.** *Let L be an even indefinite lattice satisfying $rk(L) \geqslant l(d(L)) + 2$. Then the genus of L contains only one class and the map $r_L : \mathrm{O}(L) \to \mathrm{O}(d(L))$ is surjective.*

We are now interested in studying embeddings of even lattices. Firstly, we deal with non primitive embeddings; then, we state some results for primitive sublattices of unimodular lattices which will be useful in the next.

An **overlattice** of a given lattice L is the data of a lattice L' and an embedding $L \hookrightarrow L'$ such that L'/L is a finite abelian group. We denote by $H_{L'}$ this quotient. Since $L \hookrightarrow L' \hookrightarrow L'^{\vee} \hookrightarrow L^{\vee}$, we have that $H_{L'} \subset L'^{\vee}/L \subset d(L)$. In particular, we point out that

$$\mathrm{disc}(L) = i^2 \mathrm{discr}(L'),$$

where $i := [L' : L]$ is the index of L in L' (by the same argument used above to compute the order of the discriminant group). Two overlattices $L \hookrightarrow L'$ and $L \hookrightarrow L''$ are **isomorphic** if there is an isometry of L extending to an isomorphism between L' and L''.

Proposition 0.2.6 ([76], Proposition 1.4.2)**.** *Two even overlattices $L \hookrightarrow L'$ and $L \hookrightarrow L''$ are isomorphic if and only if the isotropy subgroups $H_{L'}$ and $H_{L''}$ are conjugate under some isometry of L.*

On the other hand, an embedding $i : M \hookrightarrow L$ of a lattice M in L is **primitive** if $L/i(M)$ is a free abelian group. The **orthogonal complement** of M in L is

$$M^{\perp} := \{l \in L : (l, m) = 0 \text{ for every } m \in M\}.$$

We recall the following results for primitive embeddings in unimodular lattices.

Proposition 0.2.7 ([76], Proposition 1.6.1)**.** *A primitive embedding of an even lattice M into an even unimodular lattice L determines an isometry $\gamma : (d(M), q_M) \cong (d(M^{\perp}), -q_{M^{\perp}})$.*

Notice that the isometry γ defines an isomorphism $\psi_M : \mathrm{O}(d(M)) \cong \mathrm{O}(d(M^{\perp}))$. Recall the notation $\bar{f} := r_L(f)$ introduced in the list at the beginning of this section.

Proposition 0.2.8 ([76], Proposition 1.6.1, Corollary 1.5.2)**.** *Let M be an even primitive sublattice M of an even unimodular lattice L. An isometry f of M lifts to an isometry of L if and only if there is an isometry g of $M^{\perp}$ such that $\psi_M(\bar{f}) = \bar{g}$.*

Theorem 0.2.9 ([76], Proposition 1.14.4)**.** *Let M be an even lattice with invariants (t_+, t_-, q_M) and L be an even unimodular lattice of signature (s_+, s_-). Suppose that*

$$t_+ < s_+, \quad t_- < s_-, \quad \text{and} \quad \mathrm{rk}(L) - \mathrm{rk}(M) \geqslant l(d(M)) + 2.$$

Then there exists a unique primitive embedding of M in L.

0.3 Irreducible holomorphic symplectic manifolds

In Section 0.3.1 we briefly recall the definition and some basic properties of hyperkähler manifolds. Our main references are [13] and [35]. In Section 0.3.2 we introduce the irreducible holomorphic symplectic manifolds which will play an important role in the results presented in Part I and Part II: the Fano variety of lines in a cubic fourfold, the LLSvS eightfold and double EPW sextics.

0.3.1 Introductory definitions and properties

Let X be a complex variety and we denote by T_X and Ω_X the holomorphic tangent and cotangent bundle over X, respectively. We recall that a holomorphic symplectic structure on X is a holomorphic closed two form on X which is non degenerate in every point of X.

Definition 0.3.1. An **irreducible holomophic symplectic manifold** (or a **hyperkähler manifold**) X is a compact complex simply connected Kähler manifold such that

$$H^0(X, \Omega_X^2) := H^0(X, \bigwedge^2 \Omega_X) = \mathbb{C}\,\eta,$$

where η is a symplectic structure on X.

Let us list some immediate consequences of the definition.

1. The existence of a symplectic structure implies that the dimension of the complex manifold X is even; we denote it by $2n$.

2. Notice that the canonical bundle ω_X is trivial, because the $(2n, 0)$-form η^n is a generator for it.

3. Since η is nondegenerate, it follows that the antisymmetric morphism $\eta : T_X \to \Omega_X$, induced by η, is bijective, i.e. $T_X \cong \Omega_X$.

4. Since X is a compact Kähler manifold, its cohomology carries a Hodge structure: for every $0 \leqslant k \leqslant 4n$, we have
$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$
where $H^{p,q}(X) = H^q(X, \Omega_X^p)$ and $H^{q,p}(X) = \overline{H^{p,q}(X)}$. Moreover, since X is simply connected, we deduce that
$$H^0(X, \Omega_X) = H^1(X, \mathcal{O}_X) = 0.$$
On the other hand, in degree 2 we have
$$\begin{aligned} H^2(X, \mathbb{C}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \\ &= \mathbb{C}\,\eta \oplus H^{1,1}(X) \oplus \mathbb{C}\,\bar{\eta}. \end{aligned}$$

A key property in the hyperkähler setting is that the group $H^2(X, \mathbb{Z})$ can be equipped with a primitive quadratic form q, known as the **Beauville-Bogomolov-Fujiki form**, which gives to $H^2(X, \mathbb{Z})$ the structure of a lattice. Explicitely, assuming the normalization $\int_X \eta^n \bar{\eta}^n = 1$, we consider $q : H^2(X, \mathbb{C}) \times H^2(X, \mathbb{C}) \to \mathbb{C}$ defined by

$$q(\alpha) := ab + \frac{n}{2} \int_X \beta^2 \eta^{n-1} \bar{\eta}^{n-1}$$

for every $\alpha = a\eta + \beta + b\bar{\eta} \in H^2(X, \mathbb{C})$ with $a, b \in \mathbb{C}$ and $\beta \in H^{1,1}(X)$. Then, the following properties hold.

1. By [13], Théorème 5, we have that q comes from an integral quadratic form defined over $H^2(X,\mathbb{Z})$ (which we denote again q) with signature $(3, b_2(X) - 3)$, where $b_2(X) := \dim H^2(X,\mathbb{C})$. In particular, we have that q is positive over the Kähler class ω and over $(H^{2,0} \oplus H^{0,2})(X)$, while q is negative over the primitive $(1,1)$-part of the cohomology. Up to a scalar multiplication, the form q is primitive.

2. (Local Torelli Theorem) We set

$$\Omega := \left\{[\alpha] \in \mathbb{P}(H^2(X,\mathbb{C})) : q(\alpha) = 0, q(\alpha + \bar{\alpha}) > 0\right\}.$$

   The local period map $p : \mathrm{Def}(X) \to \mathbb{P}(H^2(X,\mathbb{Z}))$ defined by $X \mapsto [\mathbb{C}\,\eta]$ has image in Ω and p is a local isomorphism (see [13], Théoréme 5 for more details).

Let us recall some examples of irreducible holomorphic symplectic manifolds which will appear in the next. We start with the two dimensional case.

Example 0.3.2 (K3 surfaces)**.** Hyperkähler manifolds of dimension two are K3 surfaces. In fact, a **K3 surface** S is a compact complex surface with $H^1(S,\mathcal{O}_S) = 0$ and $\omega_S \cong \mathcal{O}_S$. Notice that a section η trivializing the canonical bundle defines a nondegenerate holomorphic 2-form over S, which is unique up to scalar multiplication. Moreover, Siu proved that every K3 surface is a Kähler manifold (see [92], exposé XII), and it admits a deformation into a simply connected K3 surface as explained in [92], exposé VI. Thus, K3 surfaces are irreducible holomorphic symplectic.

In this case, the form q is given by the intersection form and $H^2(S,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ as a lattice. Moreover, up to isomorphism, a K3 surface can be recovered from the lattice structure and the Hodge structure on $H^2(S,\mathbb{Z})$, as explained below.

Theorem 0.3.3 (Global Torelli Theorem, [86], [21])**.** *Two K3 surfaces S_1 and S_2 are isomorphic if and only of there is an isometry of Hodge structure $H^2(S_1,\mathbb{Z}) \cong H^2(S_2,\mathbb{Z})$.*

For a detailed survey on results about K3 surfaces and for references we recommend [46].

Remark 0.3.4. It is natural to ask whether Torelli Theorem holds in higher dimension. The answer is no if we keep the same formulation used for K3 surfaces, but Verbitsky proved a weaker form stating the generically injectivity of the period map restricted to a connected component of the moduli space (see [93] for the original proof or [45], for details and references).

Examples in dimension greater than two are difficult to construct. In the following, we recall the known examples, which are not deformation equivalent to each others.

Example 0.3.5 (Hilbert schemes on a K3 surface)**.** Let S be a K3 surface and we fix a positive integer $r > 1$. Assume for simplicity that S is projective (for the aims of this thesis, it is enough to consider this case, anyway the same construction works in the analytic case). We denote by η the symplectic form on S. Notice that the product $S^r := S \times \cdots \times S$ carries many symplectic forms obtained by pulling back η via the projections $p_i : S^r \to S$ over the ith-factor. In order to get unicity, we consider the quotient $S^{(r)} := S^r/\mathfrak{S}_r$ with respect to the symmetric group $\mathfrak{S}_r$. Since the form $\sum_{i=1}^r p_i^*\eta$ is invariant with respect to the action of $\mathfrak{S}_r$, it comes from a form over $S^{(r)}$. We observe that $S^{(r)}$ is singular along the preimage of the diagonal in S^r via the quotient map. A resolution of $S^{(r)}$ is given by the map $\varepsilon : S^{[r]} \to S^{(r)}$, where $S^{[r]}$ is the Hilbert scheme of zero dimensional, length r subschemes in S, which sends $[Z] \in S^{[r]}$ to the 0-cycle

$$\sum_{p \in \mathrm{Supp}(Z)} \mathrm{length}(\mathcal{O}_{Z,p})\,[p]\,.$$

Notice that the Hilbert scheme $S^{[r]}$ is projective. By [13], Théoréme 3, we have that $S^{[r]}$ is an irreducible holomorphic symplectic manifold of dimension $2r$. Moreover, there is an injective morphism of Hodge structures $i : H^2(S,\mathbb{C}) \to H^2(S^{[r]},\mathbb{C})$, such that

$$H^2(S^{[r]},\mathbb{Z}) = i(H^2(S,\mathbb{C})) \oplus \mathbb{Z}\,\delta,$$

where δ is a class in $H^2(S^{[r]},\mathbb{Z})$ of square $2-2r$ associated to the exceptional divisor of ε (see [13], Proposition 6).

Remark 0.3.6. Consider $X = S^{[r]}$. Via the morphism i defined above, we have that

$$\mathrm{NS}(X) := H^{1,1}(X) \cap H^2(X,\mathbb{Z}) = i(\mathrm{NS}(S)) \oplus \mathbb{Z}\,\delta.$$

In particular, the rank of the Néron-Severi lattice of X is greater than 1. As a consequence, we deduce that a generic projective deformation of X cannot be of the form $S'^{[r]}$ for a K3 surface S'. In particular, Hilbert schemes over a K3 surface form a 19-dimensional family in the 20-dimensional moduli space of polarized projective hyperkähler manifolds (see [13], Proposition 11).

Remark 0.3.7. By [13], Proposition 6, if X is deformation equivalent to $S^{[r]}$ for a K3 surface S, then $H^2(X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus I_1(2-2r)$, as a lattice.

Example 0.3.8 (Moduli spaces of stable sheaves on a K3 surface)**.** In [74], Mukai considered more generally moduli spaces of stable sheaves on a K3 surface and he proved that they provide examples of smooth hyperkähler manifolds deformation equivalent to those in Example 0.3.5. The key point is that if F is a *simple* sheaf on a K3 surface S (i.e. $\mathrm{Hom}(F,F) \cong \mathbb{C}$), then the tangent space at the point $[F]$ to the moduli space of simple sheaves on S, is isomorphic to $\mathrm{Ext}^1(F,F)$ and by Serre duality the natural pairing

$$\mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^2(F,F) \cong \mathrm{Hom}(F,F) \cong \mathbb{C}$$

is non degenerate. As before, this construction describe codimension one loci in the polarized moduli space. See also [9], Theorem 1.3 for the generalization to Bridgeland stability.

In a similar fashion, starting from a complex torus of dimension two, it is possible to construct a hyperkähler manifold known as generalized Kummer variety of dimension $2r$ for every $r > 1$ (see [13], Section 7). Finally, O'Grady constructed two examples of dimension 10 and 6 respectively, as desingularizations of moduli spaces of semistable sheaves on a K3 surface and on an abelian surface. These four classes of examples are not equivalent by deformation, because they have different Betti numbers (see [35], Section 21.2 for a list). On the other hand, all the other constructions of hyperkähler manifolds known till now are deformation equivalent to one of those we described.

0.3.2 Examples of hyperkähler manifolds of K3 type

An irreducible holomorphic symplectic manifold X is of **K3 type** if it is deformation equivalent to $S^{[r]}$ for a K3 surface S and an integer $r > 1$. In the last part of this section, we describe three examples of hyperkähler varieties, which are of K3 type. We remark that the first two are associated to a cubic fourfold, while the third one is related to a Gushel-Mukai fourfold, as recalled in the second part of this thesis.

Example 0.3.9 (Fano variety of lines on a cubic fourfold)**.** Let Y be a cubic fourfold. We denote by F_Y the Fano variety parametrizing lines ℓ contained in Y. In [14], Beauville and Donagi proved that F_Y is a smooth projective hyperkähler fourfold of K3 type. The idea of their strategy is to consider a special class of cubic fourfolds, called *Pfaffian* cubic fourfolds. For such a general Y, there is a degree 14 associated K3 surface S and they prove that $F_Y \cong S^{[2]}$. Then, they use a deformation argument in order to extend the result to every Y. We point out that in [60], Section 5, they provide a way to construct directly a symplectic form over F_Y. It is important to observe that in [14], Proposition 6, they show that there is a Hodge isometry between the primitive cohomologies $H^4(Y,\mathbb{Z})_{\text{prim}}$ and $H^2(F_Y,\mathbb{Z})_{\text{prim}}$. In particular, the period point of Y and F_Y are identified.

Example 0.3.10 (LLSvS eightfold)**.** In [64], Lehn, Lehn, Sorger and van Straten construct a hyperkäher eightfold M_Y from the irreducible component of the Hilbert scheme parametrizing twisted cubic curves on a cubic fourfold Y non containing a plane. Here we summarize their results.

We recall that a smooth rational curve C of degree 3 is projectively equivalent to the image of $\mathbb{P}^1$ via the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ defined by $[s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$. The space H_0 parametrizing these curves is then identified with automorphisms of $\mathbb{P}^3$ modulo automorphisms of $\mathbb{P}^1$; thus, it is smooth and irreducible of dimension 12. On the other hand, by Riemann-Roch, we have that the Hilbert polynomial of C is $\chi(\mathcal{O}_C(m)) = 3m+1$. Let $\text{Hilb}^{3m+1}(\mathbb{P}^3)$ be the Hilbert scheme parametrizing curves in $\mathbb{P}^3$ with Hilbert polynomial $3m+1$. Since H_0 is contained in $\text{Hilb}^{3m+1}(\mathbb{P}^3)$, we denote by $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$ the closure of H_0 in $\text{Hilb}^{3m+1}(\mathbb{P}^3)$. We refer to the objects in $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$ as (generalized) **twisted cubic curves**.

In [85], they proved that $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$ is a smooth 12-dimensional irreducible component of the Hilbert scheme $\text{Hilb}^{3m+1}(\mathbb{P}^3)$. Moreover, the Hilbert scheme $\text{Hilb}^{3m+1}(\mathbb{P}^3)$ is the union of $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$ and an other component, which intersect $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$ transversely in a smooth divisor J of $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$. According to their result, we can distinguish two classes of curves in $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)$.

1. A curve C giving a point in $\text{Hilb}^{\text{gtc}}(\mathbb{P}^3)\backslash J$ is **arithmetically Cohen-Macaulay (aCM)**, i.e. its affine cone in $\mathbb{C}^4$ is Cohen-Macaulay at the origin. The homogeneous ideal of C is generated by a net of quadrics (q_0, q_1, q_2) given by the minors of a 3×2-matrix with linear entries.

2. A curve C in J is not **Cohen-Macaulay (non CM)**. In appropriate coordinates on $\mathbb{P}^3$, the homogeneous ideal of C is $(x_0^2, x_0x_1, x_0x_2, q(x_1,x_2,x_3))$, where q is a cubic polynomial, defining a cubic curve in the plane $\{x_0 = 0\}$ which is singular at the point $[0:0:0:1]$. In particular, we have that C is a plane singular cubic curve with an embedded point giving a direction emerging from the plane.

Now, consider the Hilbert scheme $\text{Hilb}^{3m+1}(\mathbb{P}^5)$ and the irreducible component $\text{Hilb}^{\text{gtc}}(\mathbb{P}^5)$, defined analogously. If Y is a cubic fourfold in $\mathbb{P}^5$, then we consider the Hilbert scheme $\text{Hilb}^{3m+1}(Y)$ and we set

$$M_3 := \text{Hilb}^{\text{gtc}}(\mathbb{P}^5) \cap \text{Hilb}^{3m+1}(Y),$$

which is the irreducible component of the Hilbert scheme parametrizing twisted cubic curves on Y.

There is a natural map

$$M_3 \to \text{Gr}(\mathbb{P}^3, \mathbb{P}^5), \quad [C] \to \langle C \rangle \cong \mathbb{P}^3,$$

sending a twisted cubic curve C to its linear span. Furthermore, if we fix a 3-plane $\mathbb{P}(W)$ in $\mathbb{P}^5$, then the fiber over the point $[\mathbb{P}(W)]$ is the Hilbert scheme of twisted cubic curves in the cubic surface $S := Y \cap \mathbb{P}(W)$. Notice that if Y does not contain a plane, then S is reduced and irreducible. Under the assumption that Y does not contain a plane, we have the following results.

1. The component M_3 is a smooth and irreducible projective variety of dimension 10 (see [64], Theorem A).

2. There exist a holomorphic symplectic projective variety M_Y of dimension 8 and a morphism $u : M_3 \to M_Y$, such that the following diagram

$$\begin{array}{ccc} M_3 & \xrightarrow{\quad u \quad} & M_Y \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle \sigma} & \\ & M'_Y & \end{array}$$

where $a : M_3 \to M'_Y$ is a $\mathbb{P}^2$-fiber bundle and $\sigma : M'_Y \to M_Y$ is the contraction of the Cartier divisor of non CM twisted curves, commutes (see [64], Theorem B). Moreover, the cubic fourfold Y embeds in M_Y as a Lagrangian submanifold (with respect to the symplectic structure) and σ is the blow-up of M_Y in Y.

3. By [3], the variety M_Y is deformation equivalent to the Hilbert scheme of points of length 4 on a K3 surface.

Remark 0.3.11. We point out that these examples provide complete families of hyperkähler manifolds. Indeed, cubic fourfolds are defined as the zero locus of a degree 3 homogeneous polynomial in 6 variables; thus, their moduli space is described as an open subset of the (GIT) quotient

$$|\mathcal{O}(3)|//\mathrm{PGL}(V_6),$$

which has dimension $56 - 1 - (36 - 1) = 20$.

Example 0.3.12 (Double EPW sextic)**.** In [79], O'Grady proved that the smooth double cover of a sextic hypersurface in $\mathbb{P}^5$ is a hyperkähler fourfold of K3 type. Let us briefly recall the construction.

Let V_6 be a six dimensional $\mathbb{C}$-vector space. We observe that the wedge product $\wedge : \bigwedge^3 V_6 \times \bigwedge^3 V_6 \to \bigwedge^6 V_6 \cong \mathbb{C}$ defines a symplectic form on $\bigwedge^3 V_6$ which we denote by η. We recall that a subspace A of $\bigwedge^3 V_6$ is **Lagrangian** with respect to η if $\eta|_{A \times A} = 0$ and A has dimension 10. We define the Lagrangian subbundle $F \subset \bigwedge^3 V_6 \otimes \mathcal{O}_{\mathbb{P}(V_6)}$, whose fiber over $v \in V_6$ is the Lagrangian subspace $F_v := v \wedge \bigwedge^2 V_6$ of $\bigwedge^3 V_6$.

Fix a Lagrangian subspace A of $\bigwedge^3 V_6$ and consider the composition

$$\lambda_A : F \hookrightarrow \bigwedge^3 V_6 \otimes \mathcal{O}_{\mathbb{P}(V_6)} \to \frac{\bigwedge^3 V_6}{A} \otimes \mathcal{O}_{\mathbb{P}(V_6)}.$$

Then the hypersurface Y_A is the zero locus in $\mathbb{P}(V_6)$ of $\det \lambda_A$ (when it is not $\mathbb{P}(V_6)$). Explicitely, we have that points of Y_A are classes of vectors v in $\mathbb{P}(V_6)$ where λ_A has rank < 10, i.e.

$$Y_A = \{[U_1] \in \mathbb{P}(V_6) : A \cap (U_1 \wedge \bigwedge^2 V_6) \neq 0\}.$$

Notice that Y_A is a sextic hypersurface. Indeed, since λ_A is a section of $\mathcal{H}om(F, \bigwedge^3 V_6/A \otimes \mathcal{O}_{\mathbb{P}(V_6)}) \cong F^\vee \otimes \bigwedge^3 V_6/A \otimes \mathcal{O}_{\mathbb{P}(V_6)}$, we have that $\det \lambda_A$ is a section of $\det(F^\vee) \cong \mathcal{O}_{\mathbb{P}(V_6)}(6)$, where the last isomorphism follows from the fact that $c_1(F) = -6h$ (see [32], Section 2.1.2).

Moreover, we can consider the closed subschemes

$$Y_A^{\geqslant l} := \{[U_1] \in \mathbb{P}(V_6) : \dim(A \cap (U_1 \wedge \bigwedge^2 V_6)) \geqslant l\} \quad \text{for } l \geqslant 0,$$

giving a stratification of $\mathbb{P}(V_6)$ such that $Y_A^{\geqslant l} \subset Y_A^{\geqslant l-1}$ for $l > 0$ and $Y_A = Y_A^{\geqslant 1}$. The results are the following.

- Assume that A has not decomposable vectors, i.e. $A \cap \mathrm{Gr}(3, V_6) = 0$. Then, we have that Y_A is a normal sextic hypersurface, known as *Eisenbud-Popescu-Walter (EPW) sextic*, which is singular along the integral surface $Y_A^{\geqslant 2}$ ([79], Proposition 2.8).
- Let $\tilde{Y}_A$ be the double cover of the EPW sextic Y_A branched over $Y_A^{\geqslant 2}$. If $Y_A^{\geqslant 3}$ is empty, (e.g. for generic A), then the **double EPW sextic** $\tilde{Y}_A$ is a smooth hyperkähler fourfold of K3 type (see [79], Theorem 1.1).

In the second part of this thesis, we will explain how a double EPW sextic is associated to a GM fourfold, as observed by Debarre and Kuznetsov. This construction provides a complete family of 4-dimensional hyperkähler manifolds of K3 type.

0.4 (Weak) Stability conditions on triangulated categories

In this section we recall the definition of (weak) stability condition for a $\mathbb{C}$-linear triangulated category $\mathcal{T}$, following the summary in [7], Section 2 (see also [69]). In particular, we review the tilting procedure and we explain how it is used to produce examples of weak stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$.

0.4.1 Definition and examples

Essentially, a (weak) stability condition is the data of the heart of a bounded t-structure and of a (weak) stability function, satisfying certain conditions.

Definition 0.4.1. The **heart of a bounded t-structure** is a full subcategory $\mathcal{A}$ of $\mathcal{T}$ such that

1. for E, F in $\mathcal{A}$ and $n < 0$, we have $\mathrm{Hom}(E, F[n]) = 0$;
2. for every E in $\mathcal{T}$, there exists a sequence of morphisms

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{m-1}} E_{m-1} \xrightarrow{\phi_m} E_m = E$$

such that the cone of ϕ_i is of the form $A_i[k_i]$, for some sequence $k_1 > k_2 > \dots > k_m$ of integers and A_i in $\mathcal{A}$. The object A_i is the i-th **cohomology object** of E with respect to $\mathcal{A}$ and it is denoted by $A_i := \mathcal{H}^{-k_i}_{\mathcal{A}}(E)$.

Remark 0.4.2. Recall that the heart of a bounded t-structure is an abelian category by [15].

Remark 0.4.3. We do not recall the definition of a bounded t-structure, which can be found in [15]. The reason is that by [18], Lemma 3.2, the heart uniquely determines the bounded t-structure.

Example 0.4.4. If $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(X)$ for a smooth projective variety X, then $\mathrm{Coh}(X)$ is the heart of a bounded t-structure. Indeed, condition 1 of the definition follows from the fact that coherent sheaves have no Ext groups in negative degree. In order to prove item 2, we take a complex $E \in \mathcal{T}$ and we consider its cohomology sheaves $A_i := \mathcal{H}^{-k_i}(E) \in \mathrm{Coh}(X)$. Then the desired filtration is constructed by iterative projections over the first non trivial cohomology with an appropriate shift.

Definition 0.4.5. Let $\mathcal{A}$ be an abelian category. A group homomorphism $Z : K(A) \to \mathbb{C}$ is a **weak stability function** (resp. a **stability function**) on $\mathcal{A}$ if, for $E \in \mathcal{A}$, we have $\Im Z(E) \geqslant 0$, and in the case that $\Im Z(E) = 0$, we have $\Re Z(E) \leqslant 0$ (resp. $\Re Z(E) < 0$ when $E \neq 0$).

Recall that the **Grothendieck group** $K(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is the free abelian group generated by isomorphism classes $[F]$ of objects $F \in \mathcal{T}$ with respect to the relation $[E] - [F] + [G] = 0$ if there is a triangle $E \to F \to G \to E[1]$ in $\mathcal{T}$. Let Λ be a finite rank lattice with a surjective homomorphism $v : K(\mathcal{T}) \twoheadrightarrow \Lambda$.

Definition 0.4.6. A **weak stability condition** on $\mathcal{T}$ is the data of a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$ and $Z : \Lambda \to \mathbb{C}$ is a group morphism called **central charge**, satisfying the following properties:

1. The composition $K(\mathcal{A}) = K(\mathcal{T}) \xrightarrow{v} \Lambda \xrightarrow{Z} \mathbb{C}$ is a weak stability function on $\mathcal{A}$. We will write $Z(-)$ instead of $Z(v(-))$ for simplicity.

   For any $E \in \mathcal{A}$, the **slope** with respect to Z is given by

$$\mu_\sigma(E) = \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

A non zero object $E \in \mathcal{A}$ is **σ-semistable** (resp. **σ-stable**) if for every proper subobject F of E, we have $\mu_\sigma(F) \leqslant \mu_\sigma(E)$ (resp. $\mu_\sigma(F) < \mu_\sigma(E)$). We say that E is strictly semistable with respect to σ if E is σ-semistable, but not stable. An object $F \in \mathcal{T}$ is σ-semistable if $F = E[n]$, where E is σ-semistable in $\mathcal{A}$ and $n \in \mathbb{Z}$.

2. (HN-filtration) Any object of $\mathcal{A}$ has a Harder-Narasimhan filtration with σ-semistable factors. Explicitly, for every $E \in \mathcal{A}$, there is a filtration

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{m-1} \hookrightarrow E_m = E$$

where $A_i := E_i/E_{i-1}$ is σ-semistable for $i = 1, \dots, m$ and

$$\mu_\sigma(A_1) > \cdots > \mu_\sigma(A_m).$$

The object A_i is a **Harder-Narasimhan (HN) factor** of E. We set $\phi^+(E) := \phi(A_1)$ and $\phi^-(E) := \phi(A_m)$.

3. (Support property) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that the restriction of Q to $\ker Z$ is negative definite and $Q(E) \geqslant 0$ for all σ-semistable objects E in $\mathcal{A}$.

In addition, if $Z \circ v$ is a stability function, then σ is a **Bridgeland stability condition**.

The main difference between weak stability and Bridgeland stability is that in the first case there could be non-zero objects $E \in \mathcal{A}$ with $Z(E) = 0$, whose slope is $+\infty$ by definition.

Remark 0.4.7. Using item 1 in Definition 0.4.1, it is possible to prove that the HN-filtration is unique. Moreover, item 2 of Definition 0.4.1 and of Definition 0.4.6 imply that every object in $\mathcal{T}$ has a HN-filtration.

Remark 0.4.8. The original formulation of the support property is due to Kontsevich and Soibelman and it is different from that we gave. In [11], Appendix A, it is proved that these definitions are in fact equivalent.

We need to introduce some terminology we will use in the following. Let σ be a (weak) stability condition for $\mathcal{T}$.

Definition 0.4.9. The **phase** of a σ-semistable object $E \in \mathcal{A}$ is

$$\phi(E) := \frac{1}{\pi}\arg(Z(E)) \in (0, 1]$$

and for $F = E[n]$, we set

$$\phi(E[n]) := \phi(E) + n.$$

A **slicing** $\mathcal{P}$ of $\mathcal{T}$ is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for $\phi \in \mathbb{R}$, such that:

- for $\phi \in (0, 1]$, the subcategory $\mathcal{P}(\phi)$ is given by the zero object and all σ-semistable objects with phase ϕ;
- for $\phi + n$ with $\phi \in (0, 1]$ and $n \in \mathbb{Z}$, we set $\mathcal{P}(\phi + n) := \mathcal{P}(\phi)[n]$.

Remark 0.4.10. There is an other definition of Bridgeland stability condition involving a general notion of slicing and phase, which is equivalent to ours in Definition 0.4.6. As this is not relevant in the thesis, we do not recall it here, and we suggest [68], Section 5 for a detailed comparison between the two approaches. Essentially, in that case a slicing gives a way to list all the semistable objects in $\mathcal{T}$, while Definition 0.4.6 provides a function detecting them.

Let us recall the following properties of a (weak) stability condition $\sigma = (\mathcal{A}, Z)$.

1. (See-saw principle) Let $0 \to E \to F \to G \to 0$ be an exact sequence in the heart $\mathcal{A}$. Then we have
$$\mu_\sigma(E) \leqslant \mu_\sigma(F) \text{ if and only if } \mu_\sigma(F) \leqslant \mu_\sigma(G)$$
and
$$\mu_\sigma(E) \geqslant \mu_\sigma(F) \text{ if and only if } \mu_\sigma(F) \geqslant \mu_\sigma(G).$$
Indeed, notice that $Z(F) = Z(E) + Z(G)$ and these are complex numbers with non negative imaginary part or non positive real numbers (it can be helpful to draw a picture as suggested by the name of the property).

2. Let E and F be two σ-semistable objects in $\mathcal{A}$. If $\mu_\sigma(E) > \mu_\sigma(F)$, then
$$\mathrm{Hom}(E, F) = 0. \tag{3}$$
Indeed, consider a morphism $f : E \to F$. Notice that f cannot be injective, because otherwise E would be a subobject of F with greater slope, contradicting the semistability of F. Analogously, f is not surjective.

   Since the kernel of f is a subobject of E which is semistable, we have $\mu_\sigma(\ker f) \leqslant \mu_\sigma(E)$. By the see-saw property, it follows that $\mu_\sigma(E) \leqslant \mu_\sigma(\mathrm{Im} f)$. Again, since the image of f is a subobject of F which is semistable, we get $\mu_\sigma(\mathrm{Im} f) \leqslant \mu_\sigma(F)$. We deduce that $\mu_\sigma(E) \leqslant \mu_\sigma(F)$ in contradiction with our assumption. We conclude that $f = 0$, as claimed.

3. (Jordan-Hölder filtration) Assume that σ is a stability condition. Every σ-semistable object $E \in \mathcal{P}(\phi)$ admits a (non unique) finite filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$, where the quotients E_i/E_{i-1} are stable with the same phase ϕ. This follows from the fact that $\mathcal{P}(\phi)$ is an abelian category of finite length.

   Proof. Firstly, we show that $\mathcal{P}(\phi)$ is abelian. Notice that it is enough to consider the case $\phi \in (0, 1]$, because every $\mathcal{P}(\phi)$ is obtained by shift of the subcategories of semistable objects with such a phase. So given a morphism $f : E \to F$ in $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$, we prove that $K := \ker f$ and $I := \mathrm{Im} f$ are in $\mathcal{P}(\phi)$. Assume for simplicity that K and I are semistable. Since E and F are semistable, we get $\phi(K) \leqslant \phi(E)$ and $\phi(I) \leqslant \phi(F)$. By the see-saw principle, we deduce that $\phi(E) \leqslant \phi(I) \leqslant \phi(F)$, i.e. $\phi(I) = \phi$. As $Z(E) = Z(K) + Z(I)$, we also get $\phi(K) = \phi$. In the case that K and I are not semistable, we argue in the same way considering their HN-factors. In particular, we have $\phi^+(I) \leqslant \phi(F)$ and $\phi(E) \leqslant \phi^-(I)$, because the HN-factors of I with phase $\phi^+(I)$ and $\phi^-(I)$ are a subobject and a quotient of F and E, respectively. Thus, they must have the same slope ϕ, which implies that I is semistable with phase ϕ. Similarly, we have $\phi^+(K) \leqslant \phi$; thus, every HN-factor of K has phase $\leqslant \phi$. As $Z(K) = \sum Z(A_i)$, where the A_i's are its HN-factors, the only possibility is that there is only one factor, i.e. K is in $\mathcal{P}(\phi)$.

   Secondly, we prove that for every descending sequence $\cdots \subset E_i \subset E_{i-1} \subset \cdots \subset E_1$ in $\mathcal{P}(\phi)$, there is an index j such that $E_i = E_j$ for every $i \geqslant j$. Indeed, as $Z(E_i) = Z(E_{i+1}) + Z(E_i/E_{i+1})$, we have that the sequence of the $Z(E_i/E_{i+1})$ converges to 0. On the other hand, the support property implies that Z has discrete image in $\mathbb{C}$ over the set of semistable objects. Thus there is an index j such that $Z(E_i/E_{i+1}) = 0$ for $i \geqslant j$. As Z is a stability function, we deduce the statement. □

4. If E is σ-stable, then $\mathrm{Hom}(E, E) \cong \mathbb{C}$, i.e. stable objects with phase ϕ are simple objects in $\mathcal{P}(\phi)$. Indeed, by the see-saw principle, every morphism $f \in \mathrm{Hom}(E, E)$ is an isomorphism.

We denote by $\mathrm{Stab}^{\mathrm{w}}(\mathcal{T})$ (resp. $\mathrm{Stab}(\mathcal{T})$) the set of weak stability conditions (resp. of stability conditions) on $\mathcal{T}$. These sets comes with a natural topology which is the coarsest topology such that the maps $(\mathcal{A}, Z) \mapsto Z$, $(\mathcal{A}, Z) \mapsto \phi^+(E)$, $(\mathcal{A}, Z) \mapsto \phi^-(E)$ are continuous for every $E \in \mathcal{T}$. In particular, the sets $\mathrm{Stab}^{\mathrm{w}}(\mathcal{T})$ and $\mathrm{Stab}(\mathcal{T})$ are topological spaces. A very deep result of Bridgeland is that $\mathrm{Stab}(\mathcal{T})$ is actually a complex manifold, as stated below.

Theorem 0.4.11 (Bridgeland Deformation Theorem, [18]). *The continuous map $\mathcal{Z} : \mathrm{Stab}(\mathcal{T}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ defined by $(\mathcal{A}, Z) \mapsto Z$, is a local homeomorphism. In particular, the topological space $\mathrm{Stab}(\mathcal{T})$ has the structure of a complex manifold of dimension $\mathrm{rk}(\Lambda)$.*

Example 0.4.12 (Slope stability). Let X be a smooth projective variety of dimension n and let H be a hyperplane class on X. We consider the triangulated category $\mathrm{D^b}(X)$ with Grothendieck group $K(X) := K(\mathrm{D^b}(X)) = K(\mathrm{Coh}(X))$. We denote by $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q})$ the Chern character, which is defined by

$$\mathrm{ch}(E) = \left(\mathrm{ch}_0(E) := \mathrm{rk}(E), \mathrm{ch}_1(E) := c_1(E), \mathrm{ch}_2(E) := \frac{1}{2}c_1(E)^2 - c_2(E), \dots\right)$$

in terms of the Chern classes $c_i(E)$ (see [37], Appendix A for details). We fix the rank two lattice Λ^1_H, whose generators are vectors of the form

$$(H^n \mathrm{ch}_0(E), H^{n-1}\mathrm{ch}_1(E)) \in \mathbb{Q}^2 \quad \text{for } E \in \mathrm{Coh}(X).$$

The Chern character induces a natural surjection $v : K(X) \to \Lambda^1_H$. Then, the pair $\sigma_H = (\mathrm{Coh}(X), Z_H)$, where $Z_H : \Lambda^1_H \to \mathbb{C}$ is given by

$$Z_H(E) = -H^{n-1}\mathrm{ch}_1(E) + \sqrt{-1}H^n \mathrm{ch}_0(E),$$

is a weak stability condition. Indeed, since the degree $H^{n-1}\mathrm{ch}_1(E)$ of a torsion sheaf E is non negative, we have that $Z_H \circ v$ is a weak stability function. By [18], Lemma 2.4, the HN property holds (see also [68], Proposition 4.10). In [7], Remark 2.6, it is observed that if we have a rank two lattice Λ and $Z : \Lambda \to \mathbb{C}$ is injective, then any non negative quadratic form Q on $\Lambda \otimes \mathbb{R}$ satisfies the support property. Thus, considering the trivial form $Q = 0$, we deduce our claim.

Notice that if X is one-dimensional, then σ_H is a stability condition. Indeed, in this case if $\mathrm{rk}(E) = 0$, then $\deg(E) > 0$. This is not true in higher dimension, as Z_H vanishes on torsion objects supported in codimension $\geqslant 2$.

Actually, the slope defined by Z_H coincides with the classical notion of slope stability for sheaves and we denote it by μ_H. We point out that the classical Bogomolov-Gieseker inequality implies

$$\Delta_H(E) := (H^{n-1}\mathrm{ch}_1(E))^2 - 2(H^n \mathrm{ch}_0(E))(H^{n-2}\mathrm{ch}_2(E)) \geqslant 0$$

for every σ_H-semistable $E \in \mathrm{Coh}(X)$. We refer to $\Delta_H(E)$ as the **discriminant** of E.

The construction of Bridgeland stability conditions is in general a difficult task. However, starting from a weak stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{T}$, it is possible to produce a new heart of a bounded t-structure, by **tilting** $\mathcal{A}$. Let us recall this method. Let $\mu \in \mathbb{R}$; we define the following subcategories of $\mathcal{A}$:

$$\begin{aligned}\mathcal{T}^{\mu}_{\sigma} &:= \{E \in \mathcal{A} : \text{all HN factors } F \text{ of } E \text{ have slope } \mu_\sigma(F) > \mu\} \\ &= \langle E \in \mathcal{A} : E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) > \mu\rangle\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^{\mu}_{\sigma} &:= \{E \in \mathcal{A} : \text{all HN factors } F \text{ of } E \text{ have slope } \mu_\sigma(F) \leqslant \mu\} \\ &= \langle E \in \mathcal{A} : E \text{ is } \sigma\text{-semistable with } \mu_\sigma(E) \leqslant \mu\rangle.\end{aligned}$$

Here, the symbol $\langle - \rangle$ means the extension closure, i.e. the smallest full additive subcategory of $\mathcal{A}$ containing the objects in the brackets which is closed with respect to extensions.

Proposition 0.4.13 ([36])**.** *The category*

$$\mathcal{A}^\mu_\sigma := \langle \mathcal{T}^\mu_\sigma, \mathcal{F}^\mu_\sigma[1]\rangle = \{E \in \mathcal{T} : \mathcal{H}^0_{\mathcal{A}}(E) \in \mathcal{T}^\mu_\sigma, \mathcal{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F}^\mu_\sigma, \mathcal{H}^i_{\mathcal{A}}(E) = 0 \textit{ for } i \neq 0, -1\}$$

is the heart of a bounded t-structure on $\mathcal{T}$*.*

We say that the heart $\mathcal{A}^\mu_\sigma$ is obtained by tilting $\mathcal{A}$ with respect to the weak stability condition σ at the slope μ. In the next section, we will explain how to construct weak stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ by tilting $\mathrm{Coh}(X)$ with respect to slope stability.

0.4.2 Tilt stability on $\mathrm{D}^{\mathrm{b}}(X)$

Let us consider the case $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(X)$, where X is a smooth projective variety of dimension n. We fix $\beta \in \mathbb{R}$. By Example 0.4.12 and Proposition 0.4.13, we can consider the heart

$$\mathrm{Coh}^\beta(X) := \mathrm{Coh}(X)^\beta_{\sigma_H}$$

obtained by tilting $\mathrm{Coh}(X)$ with respect to the weak stability condition σ_H at slope β.

It is possible to to define weak stability conditions having $\mathrm{Coh}^\beta(X)$ as heart. Indeed, for $E \in \mathrm{D}^{\mathrm{b}}(X)$, we set

$$\mathrm{ch}^\beta(E) := e^{-\beta}\,\mathrm{ch}(E).$$

Explicitly, the first three terms are

$$\mathrm{ch}^\beta_0(E) := \mathrm{ch}_0(E) = \mathrm{rk}(E), \quad \mathrm{ch}^\beta_1(E) := \mathrm{ch}_1(E) - \beta H\,\mathrm{ch}_0(E)$$

and

$$\mathrm{ch}^\beta_2(E) := \mathrm{ch}_2(E) - \beta H\,\mathrm{ch}_1(E) + \frac{\beta^2 H^2}{2}\,\mathrm{ch}_0(E).$$

We consider the rank two lattice Λ^2_H generated by the vectors

$$(H^n\,\mathrm{ch}_0(E), H^{n-1}\,\mathrm{ch}_1(E), H^{n-2}\,\mathrm{ch}_2(E)) \in \mathbb{Q}^3$$

for $E \in \mathrm{Coh}(X)$ and we denote by $\mathrm{ch}_{\leqslant 2}(E) \in \Lambda^2_H$ the truncated Chern character till degree 2. The classical Bogomolov inequality recalled in Example 0.4.12 implies that Δ_H satisfies the second part of the support property. Thus, we have the following key result.

Proposition 0.4.14 ([7], Proposition 2.11)**.** *Given* $\alpha > 0$, $\beta \in \mathbb{R}$*, the pair* $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$*, where*

$$Z_{\alpha,\beta}(E) := -\left(H^{n-2}\,\mathrm{ch}^\beta_2(E) - \frac{1}{2}\alpha^2 H^n\,\mathrm{ch}^\beta_0(E)\right) + \sqrt{-1}H^{n-1}\,\mathrm{ch}^\beta_1(E),$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ *with respect to* Λ^2_H*. Moreover, these stability conditions vary continuosly as* $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ *varies, with a locally-finite wall and chamber structure.*

Let us explain the meaning of the last sentence of the proposition, whose proof is given in [11], Appendix B. We can visualize these weak stability conditions in the upper half plane

$$\{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha > 0\}.$$

Definition 0.4.15. Let v be a vector in Λ^2_H.

1. A **numerical wall** for v is the set of pairs $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that there is a vector $w \in \Lambda^2_H$ verifying the numerical relation
$$\mu_{\alpha,\beta}(v) = \mu_{\alpha,\beta}(w).$$

2. A **wall** for $F \in \mathrm{Coh}^\beta(X)$ is a numerical wall for $v := \mathrm{ch}_{\leqslant 2}(F)$ such that for every (α, β) on the wall there is an exact sequence of semistable objects $0 \to E \to F \to G \to 0$ in $\mathrm{Coh}^\beta(X)$ such that $\mu_{\alpha,\beta}(F) = \mu_{\alpha,\beta}(E) = \mu_{\alpha,\beta}(G)$ gives rise to the numerical wall.

3. A **chamber** is a connected component in the complement of the union of walls in the upper half plane.

The key point is that tilt stability conditions satisfy well-behaved wall-crossing:

- The function $\mathbb{R}_{>0} \times \mathbb{R} \to \mathrm{Stab}^{\mathrm{w}}(X)$ defined by

$$(\alpha, \beta) \mapsto (\mathrm{Coh}^\beta(X), Z_{\alpha,\beta})$$

  is continuous ([11], Proposition B.2).

- Walls with respect to a class $v \in \Lambda^2_H$ in the image of this map are locally finite. In particular, if $v = \mathrm{ch}_{\leqslant 2}(F)$ with $F \in \mathrm{Coh}^\beta(X)$, then the stability of F remains unchanged as (α, β) varies in a chamber ([11], Proposition B.5).

Remark 0.4.16. Conjecturally, tilt stability is the starting point to produce Bridgeland stability conditions. Here we summarize what is known till now.

- If X is a surface, then $\sigma_{\alpha,\beta}$ is a Bridgeland stability condition (see [18] for the case of K3 surfaces, [5] for its generalization to smooth projective surfaces, or [68], Section 6 for a survey).

- If X has dimension > 2, then $Z_{\alpha,\beta}$ vanishes on objects with support in codimension $\geqslant 3$.

- Assume that X has dimension 3. In [12], Section 3, the authors consider a new heart obtained by tilting $\mathrm{Coh}^\beta(X)$ with respect to the weak stability condition $\sigma_{\alpha,\beta}$, and they defined an appropriate central charge involving the third Chern character. By [11], Theorem 4.2, this construction defines a Bridgeland stability condition if and only if a generalized form of Bogomolov-Gieseker inequality holds. We recommend [68], Section 9 for details and references.

0.4.3 The $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane

The aim of this paragraph is to present an alternative way to the usual (α, β)-upper half plane in order to visualize tilt stability conditions. This method was introduced by Li and Zhao in [66], Section 1 in the case of stability conditions on $\mathbb{P}^2$.

Keeping the notation of the previous section, we consider the projective plane $\mathbb{P}(\Lambda^2_H)$ with homogeneous coordinates $[H^n \mathrm{ch}_0 : H^{n-1} \mathrm{ch}_1 : H^{n-2} \mathrm{ch}_2]$. We fix the line $\{H^n \mathrm{ch}_0 = 0\}$ and we define the affine plane

$$\mathbb{A}^2_H := \mathbb{P}(\Lambda^2_H) \backslash \{H^n \mathrm{ch}_0 = 0\}.$$

We will refer to $\mathbb{A}^2_H$ as the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane and to $\mathbb{P}(\Lambda^2_H)$ as the projective $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. We fix the affine coordinates

$$\left(\frac{H^{n-1} \mathrm{ch}_1}{H^n \mathrm{ch}_0}, \frac{H^{n-2} \mathrm{ch}_2}{H^n \mathrm{ch}_0} \right)$$

on $\mathbb{A}^2_H$.

A complex $E \in \mathrm{D^b}(X)$ such that $\mathrm{ch}_{\leqslant 2}(E) \neq (0,0,0)$ is represented by a point in the projective $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane. Moreover, if $\mathrm{ch}_0(E) \neq 0$, then E gives rise to a point in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane.

In $\mathbb{A}^2_H$, we consider the parabola Δ_H described by the equation

$$\frac{1}{2} \left(\frac{H^{n-1} \mathrm{ch}_1}{H^n \mathrm{ch}_0} \right)^2 - \frac{H^{n-2} \mathrm{ch}_2}{H^n \mathrm{ch}_0} = 0$$

and the area above the parabola $\bar{\Delta}_H$ given by

$$\frac{1}{2}\left(\frac{H^{n-1}\operatorname{ch}_1}{H^n\operatorname{ch}_0}\right)^2-\frac{H^{n-2}\operatorname{ch}_2}{H^n\operatorname{ch}_0}<0.$$

It is not difficult to see that a point above the parabola corresponds to a weak stability condition on $\mathrm{D^b}(X)$, as we explain in the following lemma.

Lemma 0.4.17. *For every point $(s,t)\in\mathbb{A}^2_H$ such that $t>1/2s^2$, the pair $\sigma'_{s,t}=(\mathrm{Coh}^s(X),Z'_{s,t})$, where*

$$Z'_{s,t}(E):=-\left(H^{n-2}\operatorname{ch}_2(E)-tH^n\operatorname{ch}_0(E)\right)+\sqrt{-1}\left(H^{n-1}\operatorname{ch}_1(E)-sH^n\operatorname{ch}_0(E)\right),$$

is a weak stability condition on $\mathrm{D^b}(X)$ with respect to Λ^2_H. Moreover, these stability conditions vary continuosly as $(s,t)\in\bar{\Delta}_H$ varies, with a locally-finite wall and chamber structure.

Proof. It is enough to notice that given $(\alpha,\beta)\in\mathbb{R}_{>0}\times\mathbb{R}$, the tilt stability $\sigma_{\alpha,\beta}$ is the same as $\sigma'_{\beta,\frac{\beta^2+\alpha^2}{2}}$, up to the action of an element in $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Indeed, we have that

$$\begin{pmatrix}1&-\beta\\0&1\end{pmatrix}\begin{pmatrix}\Re Z_{\alpha,\beta}\\ \Im Z_{\alpha,\beta}\end{pmatrix}=\begin{pmatrix}\Re Z'_{\beta,\frac{\beta^2+\alpha^2}{2}}\\ \Im Z'_{\beta,\frac{\beta^2+\alpha^2}{2}}\end{pmatrix}.$$

We point out that the morphism $Z'_{\beta,\frac{\beta^2+\alpha^2}{2}}$ defines a weak stability function for $\mathrm{Coh}^\beta(X)$, because $\Im Z'_{\beta,\frac{\beta^2+\alpha^2}{2}}=\Im Z_{\alpha,\beta}$ and, when the imaginary part vanishes, the real part does not change. In particular, for an object $E\in\mathrm{Coh}^\beta(X)$, we have $Z(E)=0$ if and only if $Z'(E)=0$. Moreover, notice that

$$\mu_{\alpha,\beta}=\mu'_{\beta,\frac{\beta^2+\alpha^2}{2}}-\beta,$$

where μ' is the slope with respect to Z' (and an object with infinite slope remains with infinite slope). It follows that a $\sigma_{\alpha,\beta}$-stable object is $\sigma'_{\beta,\frac{\beta^2+\alpha^2}{2}}$-stable. In particular, the HN-filtration exists and this action respects the order of the HN-factors'slopes, because we are just shifting by a constant. Finally, the quadratic form given by the equation of the parabola Δ_H satisfies the support property. Indeed, notice that the kernel of $Z'_{s,t}$ is given by the point $(s,t)\in\bar{\Delta}_H$. As $(\beta,\frac{\beta^2+\alpha^2}{2})$ is a point above the parabola and stable objects give points below the parabola by Bogomolov-Gieseker inequality, we conclude that $\sigma'_{\beta,\frac{\beta^2+\alpha^2}{2}}$ is a weak stability condition. It is easy to see that every point $(s,t)\in\bar{\Delta}_H$ comes from a point in the (α,β)-upper half plane. Thus, the claim follows from Proposition 0.4.14. □

It is interesting to remark the following properties of the $\{1,\frac{\operatorname{ch}_1}{\operatorname{ch}_0},\frac{\operatorname{ch}_2}{\operatorname{ch}_0}\}$-plane representation.

1. As already observed in proof of Lemma 0.4.17, the weak stability condition $\sigma'_{s,t}$ is identified with $\ker Z'_{s,t}$, which is the point $(s,t)\in\mathbb{A}^2_H$ over the parabola Δ_H.
2. Let $P=(s,t)$ be a point in $\bar{\Delta}_H$ and let E be a slope semistable vector bundle in $\mathrm{Coh}(X)$. Then, we have that E is in the heart $\mathrm{Coh}^s(X)$ if and only if it determines a point in the right-half plane

$$\left\{\left(\frac{H^{n-1}\operatorname{ch}_1}{H^n\operatorname{ch}_0},\frac{H^{n-2}\operatorname{ch}_2}{H^n\operatorname{ch}_0}\right)\in\mathbb{A}^2_H:\frac{H^{n-1}\operatorname{ch}_1}{H^n\operatorname{ch}_0}>s\right\},$$

while $E[1]$ is in $\mathrm{Coh}^s(X)$ if and only the character of E defines a point in

$$\left\{\left(\frac{H^{n-1}\operatorname{ch}_1}{H^n\operatorname{ch}_0},\frac{H^{n-2}\operatorname{ch}_2}{H^n\operatorname{ch}_0}\right)\in\mathbb{A}^2_H:\frac{H^{n-1}\operatorname{ch}_1}{H^n\operatorname{ch}_0}\leqslant s\right\}.$$

3. Fix a point $P = (s,t) \in \bar{\Delta}_H$ and an object $E \in \operatorname{Coh}^s(X)$ such that $\operatorname{ch}_0(E) \neq 0$. We still denote by E the point in $\mathbb{A}^2_H$ defined by the Chern character of E. Then, the slope of E can be represented on the plane $\mathbb{A}^2_H$ in the following way. We draw a vertical line passing through P and we consider the semiline l_- from P to $-\infty$. We denote by l_{EP} the semiline in the right half plane

$$\left\{ \left(\frac{H^{n-1}\operatorname{ch}_1}{H^n \operatorname{ch}_0}, \frac{H^{n-2}\operatorname{ch}_2}{H^n \operatorname{ch}_0} \right) \in \mathbb{A}^2_H : \frac{H^{n-1}\operatorname{ch}_1}{H^n \operatorname{ch}_0} > s, \text{ or } \frac{H^{n-1}\operatorname{ch}_1}{H^n \operatorname{ch}_0} = s \text{ and } \frac{H^{n-2}\operatorname{ch}_2}{H^n \operatorname{ch}_0} > t \right\}$$

lying on the line joining E and P. An easy computation shows that the phase with respect to $Z'_{s,t}$ of E is equal to the angle between the semilines l_{EP} and l_- divided by π. As a consequence, two objects E and F in $\operatorname{Coh}^s(X)$ with non zero rank satisfy

$$\mu'_{s,t}(E) > \mu'_{s,t}(F) \tag{4}$$

if and only if the ray l_{EP} is above l_{FP} (see [66], Lemma 1.17).

4. It is possible to represent the potential walls in the projective $\{1, \frac{\operatorname{ch}_1}{\operatorname{ch}_0}, \frac{\operatorname{ch}_2}{\operatorname{ch}_0}\}$-plane. Indeed, let $P = (s,t)$ be a point in $\bar{\Delta}_H$ and let E and F be two objects in $\operatorname{Coh}^s(X)$ having $\operatorname{ch}_{\leqslant 2} \neq (0,0,0)$. Then $Z'_{s,t}(E)$ and $Z'_{s,t}(F)$ are on the same ray if and only if E, F and P are collinear in $\mathbb{P}(\Lambda^2_H)$. Indeed, we observe that $Z'_{s,t}(E)$ and $Z'_{s,t}(F)$ are on the same ray if and only if $Z'_{s,t}(aE - bF) = 0$ for some $a, b \in \mathbb{R}_{>0}$.This is equivalent to have E, F and $P = \ker Z'_{s,t}$ on the same line in the projective $\{1, \frac{\operatorname{ch}_1}{\operatorname{ch}_0}, \frac{\operatorname{ch}_2}{\operatorname{ch}_0}\}$-plane (see [66], Lemma 1.16).

This kind of representation does not prove any new result about these tilt stability conditions, but it simplifies a lot the computations. First of all, the plane $\mathbb{A}^2_H$ is more complete with respect to the classical (α, β)-plane, because it allows to represent the characters of the objects and the weak stability conditions on the same plane. Moreover, the potential walls are essentialy described as straight lines on $\mathbb{A}^2_H$. Finally, we can compare the slope of two objects with different Chern character till degree 2 looking at their position on the $\{1, \frac{\operatorname{ch}_1}{\operatorname{ch}_0}, \frac{\operatorname{ch}_2}{\operatorname{ch}_0}\}$-plane. This allows to compare the slope with respect to different weak stability conditions and characters simultaneously. We will use this description in Chapter 4 to explain the computations.

Part I

Cubic fourfolds

Chapter 1

Introduction to Part I

The aim of this part is to present a detailed proof of some results about Fourier-Mukai partners of cubic fourfolds (see [87]), and concerning the Fano variety of lines and the LLSvS eightfold of a cubic fourfold, coming from a joint work with Chunyi Li and Xiaolei Zhao (see [65]).

A cubic fourfold Y is a smooth hypersurface of degree 3 in $\mathbb{P}^5_{\mathbb{C}}$. In [57], Kuznetsov studied the derived category $\mathrm{D}^{\mathrm{b}}(Y)$ of bounded complexes of coherent sheaves on Y to address the problem of the (non)rationality of the cubic fourfold. As recalled in Example 0.1.27, the derived category $\mathrm{D}^{\mathrm{b}}(Y)$ admits a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

where $\mathrm{Ku}(Y)$ is the right orthogonal of the subcategory of $\mathrm{D}^{\mathrm{b}}(Y)$ generated by $\{\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\}$. It turns out that the Kuznetsov component $\mathrm{Ku}(Y)$ has certain similarities with the bounded derived category of a K3 surface, e.g. the Serre functor on $\mathrm{Ku}(Y)$ is the homological shift [2] (see [54], Corollary 4.3).

The Kuznetsov component should carry the information about the birational type of the cubic hypersurface. Infact, it has been conjectured that Y is rational if and only if $\mathrm{Ku}(Y)$ is equivalent to the derived category of a K3 surface (see [57], Conjecture 1.1). To support this guess, Kuznetsov proved in [57] that the cubic fourfolds which were known to be rational satisfy this condition (see also [73]).

On the level of the Hodge theory, the existence of an associated K3 surface as an indicator of rationality was deeply studied (see [40], for a complete survey). Actually, Kuznetsov's conjecture would imply that a cubic fourfold with a Hodge associated K3 surface is rational, by results of Addington, Thomas and Bayer, Lahoz, Macrì, Nuer, Perry and Stellari, relating the categorical and the Hodge theoretical setting (see [4], Theorem 1.1 and [8] or [69], Theorem 3.7). Nevertheless, these conjectures have not been proved yet.

In [47], Huybrechts studied the category $\mathrm{Ku}(Y)$, in order to develop a theory for cubic fourfolds which parallels that of the derived category of a (twisted) K3 surface. In particular, he proved the analogous version for $\mathrm{Ku}(Y)$ of some results concerning Fourier-Mukai partners of a K3 surface. A cubic fourfold Y' is a **Fourier-Mukai partner** of Y if there exists an equivalence of categories

$$\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y')$$

which is of Fourier-Mukai type, i.e. such that the composition

$$\mathrm{D}^{\mathrm{b}}(Y) \xrightarrow{i^*} \mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y') \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y')$$

is a Fourier-Mukai functor; here, i^* denotes the left adjoint functor of the full inclusion $i : \mathrm{Ku}(Y) \hookrightarrow \mathrm{D^b}(Y)$. Usually in the literature this denomination is used to identify smooth projective varieties with equivalent derived categories. However, cubic fourfolds satisfying this condition are isomorphic by a result of Bondal and Orlov (see [17], Theorem 3.1). Thus, it becomes interesting to address the same problem by considering the Kuznetsov components.

Huybrechts showed that the number of (isomorphism classes of) Fourier-Mukai partners for a cubic fourfold Y is finite (see [47], Theorem 1.1), as in the case of Fourier-Mukai partners for a K3 surface (see [20], Proposition 5.3). Moreover, he proved that the very general cubic fourfold Y, i.e. such that $\mathrm{rk}(H^{2,2}(Y,\mathbb{Z})) = 1$, has no non-trivial Fourier-Mukai partners (see [47], Corollary 3.6).

It is natural to ask whether a *special* cubic fourfold Y, i.e. such that $\mathrm{rk}(H^{2,2}(Y,\mathbb{Z})) \geqslant 2$, admits Fourier-Mukai partners which are not isomorphic to Y. In particular, we may wonder if for special cubic fourfolds it is possible to prove a version of Theorem 1.7 and Corollary 1.8 of [81], which state that there are examples of K3 surfaces having a prescribed number of non-isomorphic Fourier-Mukai partners.

The first result is that the answer is positive in the case that the rank of $H^{2,2}(Y,\mathbb{Z})$ is exactly two and the cubic fourfold Y admits an associated K3 surface X with "enough" non-trivial Fourier-Mukai partners. More precisely, given a positive integer d, we denote by $\mathcal{C}_d$ the divisor parametrizing special cubic fourfolds with discriminant d (see Section 2.1). We recall that:

- (see [39], Theorem 1.0.1) the divisor $\mathcal{C}_d$ is non empty if and only if
$$d > 6 \text{ and } d \equiv 0, 2 (\mathrm{mod}\, 6); \tag{0}$$
- (see [39], Theorem 1.0.2 or Section 2.1) a cubic fourfold $Y \in \mathcal{C}_d$ has an *associated K3 surface* if and only if
$$4 \nmid d,\ 9 \nmid d,\ p \nmid d \text{ for any odd prime } p \text{ such that } p \equiv 2 (\mathrm{mod}\, 3). \tag{a}$$

The first result is a counting formula for the number of Fourier-Mukai partners for very general special cubic fourfolds admitting an associated K3 surface.

Theorem 1.0.1. *Let d be a positive integer satisfying* (0) *and* **(a)**. *Let Y be a very general cubic fourfold in $\mathcal{C}_d$ and let m be the number of non-isomorphic Fourier-Mukai partners of an associated K3 surface to Y. Then, the cubic fourfold Y has exactly m non-isomorphic Fourier-Mukai partners, when* $d \equiv 2(\mathrm{mod}\, 6)$; *otherwise, if* $d \equiv 0(\mathrm{mod}\, 6)$, *the number of non-isomorphic Fourier-Mukai partners of Y is equal to* $\lceil m/2 \rceil$.

As a consequence of Theorem 1.0.1, we deduce that there exist cubic fourfolds admitting an arbitrary number of Fourier-Mukai partners, depending on the number of distinct odd primes in the prime factorization of the discriminant (see Proposition 3.1.4).

More generally, we recall that a cubic fourfold $Y \in \mathcal{C}_d$ has an *associated twisted K3 surface* (see [47], Section 2.4 or Section 2.4) if and only if

$$n_i \equiv 0 (\mathrm{mod}\, 2) \text{ for all primes } p_i \equiv 2 (\mathrm{mod}\, 3) \text{ in } 2d = \prod p_i^{n_i}. \tag{a'}$$

A weaker formulation of Theorem 1.0.1 holds for very general cubic fourfolds Y in $\mathcal{C}_d$, admitting an associated twisted K3 surface (X,α), if 9 does not divide the discriminant d. Indeed, in Section 3.2, we show that the number of non-isomorphic twisted Fourier-Mukai partners of (X,α) with the Brauer class of the same order as α, gives a lower bound for the number of Fourier-Mukai partners of the cubic fourfold.

Theorem 1.0.2. *Let d be a positive integer satisfying* (0) *and* **(a')**. *Assume that* 9 *does not divide d. Let Y be a very general cubic fourfold in $\mathcal{C}_d$ with associated twisted K3 surface (X,α), where α has* *order κ; let m' be the number of non-isomorphic Fourier-Mukai partners of (X, α) with Brauer class of order κ. Then the cubic fourfold Y admits at least m' non-isomorphic Fourier-Mukai partners, when $d \equiv 2 \pmod 6$; otherwise, if $d \equiv 0 \pmod 6$, the number of non-isomorphic Fourier-Mukai partners of Y is at least $\lceil m'/2 \rceil$.*

In particular, under the hypotheses of Theorem 1.0.2, we have that m' is controlled by the number of distinct primes in the prime factorization of $d/2$ divided by the square of the order of the Brauer class α and by the Euler function evaluated in $\mathrm{ord}(\alpha)$, as we show in Proposition 3.2.8.

Notice that our construction represents the first example of non-trivial Fourier-Mukai partners for a cubic fourfold. Actually, these results complete the expected analogy between cubic fourfolds and K3 surfaces, stated in [47]. They also represent a first step in order to understand whether cubic fourfolds which are Fourier-Mukai partners are birational.

An other approach in order to better understand the Kuznetsov component and its relation with the geometry of Y is by looking at moduli spaces of stable objects in $\mathrm{Ku}(Y)$. This is now possible thanks to the result in [7], where Bayer, Lahoz, Macrì and Stellari provide a construction of Bridgeland stability conditions on $\mathrm{Ku}(Y)$ (see Section 4.1.1 for a summary of this construction). Notice that, since the component $\mathrm{Ku}(Y)$ is a K3 subcategory, moduli spaces of Bridgeland stable objects in $\mathrm{Ku}(Y)$ are naturally endowed with a symplectic 2-form, by the same argument explained in Example 0.3.8. Actually, in the forthcoming paper [8], the authors prove that under mild assumptions these moduli spaces are smooth projective irreducible holomorphic symplectic manifolds of K3 type (see Section 0.3.2 for the meaning). This gives a systematic way to produce complete families of (polarized) projective hyperkähler manifolds.

On the other hand, using classical techniques in algebraic geometry, two examples of K3 type hyperkähler minifolds are constructed out of lines and twisted cubic curves in a cubic fourfold. In particular, Beauville and Donagi showed in [14] that the Fano variety F_Y of lines on Y is a smooth projective hyperkähler fourfold, deformation equivalent to the Hilbert square of a K3 surface (see Example 0.3.9). More recently, in [64] Lehn, Lehn, Sorger and van Straten construted a hyperkähler eightfold M_Y from the irreducible component of the Hilbert scheme of twisted cubic curves on a cubic fourfold Y non containing a plane (see Example 0.3.10). An interesting question is then to understand the relation between the classical and the homological setting.

The main results in Chapter 4 give a description of F_Y and M_Y in terms of moduli spaces of stable objects in the Kuznetsov component, with respect to the Bridgeland stability conditions defined in [7].

In particular, recall that the algebraic Mukai lattice of $\mathrm{Ku}(Y)$ always contains an A_2 lattice spanned by two classes λ_1 and λ_2 (see Section 2.3). We denote by $M_\sigma(v)$ the moduli space of σ-stable objects in $\mathrm{Ku}(Y)$ with Mukai vector v, where σ is a stability condition as in [7]. To each line ℓ on Y, we can associate an object $P_\ell \in \mathrm{Ku}(Y)$, of Mukai vector $\lambda_1 + \lambda_2$ (see Section 4.3). The following result gives a reconstruction of F_Y as follows.

Theorem 1.0.3. *For any line ℓ in a cubic fourfold Y, the object P_ℓ is σ-stable and the moduli space $M_\sigma(\lambda_1 + \lambda_2)$ is isomorphic to the Fano variety F_Y.*

The case of twisted cubics on Y is even more interesting from many perspectives. Assume that Y does not contain a plane. Every twisted cubic curve C in Y has an associated object F'_C in $\mathrm{Ku}(Y)$ with Mukai vector $2\lambda_1 + \lambda_2$ (see Section 4.2.1). Then, we prove the following result.

Theorem 1.0.4 (Theorem 4.2.7 and Theorem 4.2.8)**.** *Let Y be a smooth cubic fourfold not containing a plane. If C is a twisted cubic on Y, then the object F'_C is σ-stable. Moreover, the projective hyperkähler eightfold $M_\sigma(2\lambda_1 + \lambda_2)$ parametrizes only objects of the form F'_C, and it is isomorphic to the LLSvS eightfold M_Y.*

The first advantage of our approach is that it only involves homological properties of twisted cubic curves, without requiring the detailed analysis of the singularities and the determinantal representations of the twisted cubics and the cubic surfaces in Y used in [64]. In particular, we interpret the contraction of the locus of non CM twisted cubic curves in LLSvS picture via wall-crossing in weak stability. Furthermore, our result provide a description of the birational models of M_Y, which are obtained by crossing a wall of Bridgeland stability. Finally, this description gives a more conceptual explanation of the existence of the holomorphic symplectic structure on F_Y and M_Y, as moduli spaces of stable complexes in a K3 category.

An other feature is that Theorem 1.0.3 and Theorem 1.0.4 can be used to address Torelli type questions. Indeed, by Theorem 1.0.1, we know that the Kuznetsov component does not determine the cubic fourfold. More generally, in [48] Huybrechts and Rennemo proved a categorical version of Torelli Theorem, which essentially states that two cubic fourfolds are isomorphic if and only if there is an equivalence between their Kuznetsov components which satisfies an additional property (see Section 4.4.1). As explained in the Appendix of [7], where they treated the case of very general cubic fourfolds, the interpretation of the Fano variety F_Y as a moduli space of stable objects in $\mathrm{Ku}(Y)$ can be used to give a different proof of the categorical version of Torelli Theorem for cubic fourfolds. Thus, Theorem 1.0.3 allows to apply this argument without assumptions on Y (Corollary 4.4.1).

On the other hand, a direct consequence of Theorem 1.0.4 is the identification of the period point of M_Y with that of F_Y and Y.

Proposition 1.0.5 (Proposition 4.4.2)**.** *For a cubic fourfold Y not containing a plane, the period point of M_Y is identified with the period point of the Fano variety F_Y.*

Finally, a still open question is the Derived Torelli Theorem, which essentially states, in analogy to the case of K3 surfaces, that two cubic fourfolds are Fourier-Mukai partners if and only if they have Hodge isometric Mukai lattices. An evidence of this conjecture is that it has been proved by Huybrechts in [47] under genericity assumptions (see Remark 2.4.2). Section 4.4.3 is an attempt to extend this result for every cubic fourfold. In particular, we show that our strategy works in the simple case of the identity on $\mathrm{Ku}(Y)$, as explained below.

Proposition 1.0.6 (Proposition 4.4.3)**.** *Let Y be a cubic fourfold not containing a plane. Then the composition of the projection functor on the Kuznetsov component of Y with the embedding $\mathrm{Ku}(Y) \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y)$ is a Fourier-Mukai functor with kernel given by the restriction of the (quasi-)universal family on $M_\sigma(2\lambda_1 + \lambda_2) \times Y$ to $Y \times Y$.*

Related works. The problem of finding Fourier-Mukai partners has already been studied in [16], in the case of cubic fourfolds containing a plane. In particular, they proved that the very general cubic fourfold in $\mathcal{C}_8$ has only one isomorphism class of Fourier-Mukai partners (see [16], Proposition 6.3).

In [62] the authors gave an interpretation of LLSvS geometric picture in the categorical setting. In particular, they described M'_Y and M_Y as components of moduli spaces of Gieseker stable sheaves on Y. For very general cubic fourfolds, they also realized the contraction from M'_Y to M_Y via wall-crossing in tilt-stability.

We point out that Theorem 1.0.3 and Theorem 1.0.4 were proved for very general cubic fourfolds in the Appendix of [7] and [62], respectively. In this situation, the algebraic Mukai lattice of $\mathrm{Ku}(Y)$ is exactly the A_2 lattice. This property rules out most of the potential walls, allowing to prove the theorems without going through the construction of the stability conditions. It was made clear in [3] and [62] that for each twisted cubic C, the object F'_C is the correct one to consider.

Notation. We use the following terminology: a cubic fourfold Y is very general if $\mathrm{rk}(H^{2,2}(Y,\mathbb{Z})) = 1$, while a very general special cubic fourfold Y (i.e. a very general Y in a divisor $\mathcal{C}_d$) has $\mathrm{rk}(H^{2,2}(Y,\mathbb{Z})) = 2$.

Chapter 2

Recollection of results

In this chapter we review some known aspects about Hodge theory for cubic fourfolds following [39], the definition of Mukai lattice for the Kuznetsov component given in [4] and the statement of the Derived Torelli Theorem for very general special cubic fourfolds as in [47]. Finally, we recall some results concerning Fourier-Mukai partners of (twisted) K3 surfaces stated in [81] and [67], which we will use in the next.

2.1 Special cubic fourfolds and associated K3 surface

Let Y be a cubic fourfold; the Hodge diamond of Y is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array} .$$

Indeed, the cohomology in degree $\leqslant 3$ is the same as that of $\mathbb{P}^5$ by Lefschetz Hyperplane Theorem. Moreover, since $\omega_Y = \mathcal{O}_Y(-3)$ is antiample, we have that $H^0(Y, \omega_Y) = 0$. The other Hodge numbers for $H^4(Y, \mathbb{Q})$ were computed by Hirzebruch in [42], Chapter 22.

By classical results of Hodge theory and classification of quadratic forms, we have that the lattice given by the degree four integral cohomology group $H^4(Y, \mathbb{Z})$, endowed with the intersection form $(\, , \,)$ with reversed sign, is isometric to the odd unimodular lattice $L := I_{2,21} = \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 21}$ (see Example 0.2.4). It contains an element h such that $(h, h) = -3$, corresponding to the square of the class of a hyperplane in Y. For reasons which will be clear later, we prefer to consider the group $H^4(Y, \mathbb{Z})(1)$, where (1) denotes the Tate twist, which carries a weight-two Hodge structure. By [39], Proposition 2.1.2, the twisted primitive lattice $H^4(Y, \mathbb{Z})_0(1)$ with intersection form of reversed sign is isometric to

$$L^0 := A_2(-1) \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}.$$

We set

$$Q := \{y \in \mathbb{P}(L^0 \otimes \mathbb{C}) : (y, y) = 0, (y, \bar{y}) > 0\}. \tag{2.1}$$

The choice of a connected component $\mathcal{D}'$ of Q determines the local period domain for cubic fourfolds. Let Γ^+ be the subgroup of the group of automorphism of L, preserving the class h and the component $\mathcal{D}'$. The *global period domain* of cubic fourfolds is the quotient $\mathcal{D} := \Gamma^+ \backslash \mathcal{D}'$. We denote by $\mathcal{C}$ the moduli space of cubic fourfolds and let

$$\tau : \mathcal{C} \to \mathcal{D}$$

be the *period map*. Voisin proved that τ is an open immersion, i.e. Torelli Theorem holds for cubic fourfolds (see [94]).

A cubic fourfold Y is *special* if there exists a rank-two (negative definite) primitive sublattice $(K, (\, , \,))$ of $H^4(Y,\mathbb{Z}) \cap H^{2,2}(Y)$, containing the class h. This lattice K is a *labelling* for Y. We will write K_d to underline the fact that the labelling has discriminant d. By Hassett's work, special Hodge structures with a labelling of discriminant d form a divisor $\mathcal{D}'_d$ in the local period domain. If $\mathcal{D}_d = \Gamma^+ \backslash \mathcal{D}'_d$, then $\mathcal{C}_d = \mathcal{C} \cap \mathcal{D}_d$ (via τ) is the irreducible divisor in $\mathcal{C}$ of *special cubic fourfolds of discriminant* d. By [39], Theorem 1.0.1, the divisor $\mathcal{C}_d$ is non empty if and only if

$$d > 6 \text{ and } d \equiv 0, 2 (\operatorname{mod} 6). \tag{0}$$

It turns out that there are numerical conditions on d which ensure the existence of an *associated K3 surface*, as we explain in the following proposition.

Theorem 2.1.1 ([39], Theorem 1.0.2)**.** *Let Y be a cubic fourfold in $\mathcal{C}_d$ with labelling K_d. There exist a K3 surface X with polarization class of degree d and an isometry of Hodge structures*

$$K_d^{\perp} \cong H^2(X,\mathbb{Z})_0$$

between the orthogonal sublattice to K_d in $H^4(Y,\mathbb{Z})(1)$ and the degree two primitive cohomology of the K3 surface, if and only if d satisfies the following condition:

$$4 \nmid d, \; 9 \nmid d, \; p \nmid d \; \text{ for any odd prime } p \text{ such that } p \equiv 2 (\operatorname{mod} 3). \tag{\textbf{a}}$$

We point out the following property concerning the discriminant group $d(K_d^{\perp})$ of $K_d^{\perp}$, endowed with the discriminant form $q_{K_d^{\perp}}$ induced by the intersection form.

Proposition 2.1.2 ([39], Proposition 3.2.6)**.** *If $d \equiv 0 (\operatorname{mod} 6)$, then $d(K_d^{\perp}) \cong \mathbb{Z}/\frac{d}{3}\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, which is cyclic unless nine divides d. Furthermore, we may choose this isomorphism so that*

$$q_{K_d^{\perp}}((0,1)) \equiv -\frac{2}{3} (\operatorname{mod} 2\mathbb{Z}) \quad \text{and} \quad q_{K_d^{\perp}}((1,0)) \equiv \frac{3}{d} (\operatorname{mod} 2\mathbb{Z}).$$

If $d \equiv 2 (\operatorname{mod} 6)$, then $d(K_d^{\perp}) \cong \mathbb{Z}/d\mathbb{Z}$. Furthermore, we may choose a generator g so that

$$q_{K_d^{\perp}}(g) \equiv \frac{1-2d}{3d} (\operatorname{mod} 2\mathbb{Z}).$$

2.2 Immersion into the moduli spaces of K3 surfaces

In [39], Section 5.3, Hassett proved that the existence of an isometry of Hodge structures as in Theorem 2.1.1 allows an identification between the moduli space of marked special cubic fourfolds of discriminant d and the moduli space of degree d polarized K3 surfaces. Let us explain this observation. We fix a rank-two, negative definite, primitive sublattice $K_d \subset L$ of discriminant d, containing h. We write Γ_d^+ to denote the subgroup of the group of automorphisms of L fixing the class h and preserving the labelling K_d. Let $\mathcal{D}_d^{\mathrm{lab}}$ be the global period domain which parametrizes Hodge structures $x \in \mathcal{D}'$ with $K_d \subset H^{2,2}(x) \cap L$, modulo the action of Γ_d^+, i.e.

$$\mathcal{D}_d^{\mathrm{lab}} := \Gamma_d^+ \backslash \mathcal{D}'_d.$$

We say that $\mathcal{D}_d^{\mathrm{lab}}$ is the global period domain of *labelled special Hodge structures with discriminant* d. Notice that $\mathcal{D}_d^{\mathrm{lab}}$ is birational to $\mathcal{D}_d$ via the morphism $\mathcal{D}_d^{\mathrm{lab}} \to \mathcal{D}$. Actually, a very general point in $\mathcal{D}_d$ has a unique labelling. In particular, $\mathcal{D}_d^{\mathrm{lab}}$ is the normalization of $\mathcal{D}_d$ (see [39], Section 3.1)

Let, now, G_d^+ be the subgroup of Γ_d^+ of automorphisms acting trivially on K_d. Then, the global period domain of *marked special Hodge structures of discriminant* d is the quotient

$$\mathcal{D}_d^{\mathrm{mar}} := G_d^+ \backslash \mathcal{D}'_d.$$

In this new space, two cubic fourfolds having the same labelling K_d which comes from different primitive embeddings in $H^{2,2}(-) \cap L$ are not identified. The relation between $\mathcal{D}_d^{\mathrm{mar}}$ and $\mathcal{D}_d^{\mathrm{lab}}$ is explained in the following proposition.

Proposition 2.2.1 ([39], Proposition 5.3.1)**.** *The group G_d^+ is equal to Γ_d^+ (resp. the group G_d^+ is an index-two subgroup of Γ_d^+), if $d \equiv 2 (\mathrm{mod}\, 6)$ (resp. if $d \equiv 0 (\mathrm{mod}\, 6)$).*

The forgetful map $\rho : \mathcal{D}_d^{mar} \to \mathcal{D}_d^{lab}$ is an isomorphism (resp. a double cover), if $d \equiv 2 (\mathrm{mod}\, 6)$ (resp. if $d \equiv 0 (\mathrm{mod}\, 6)$).

Remark 2.2.2. If $d \equiv 0 (\mathrm{mod}\, 6)$, then the Hodge structures on L^0 represented by elements in the same fiber of ρ are exchanged by the automorphism γ of Γ^+ such that $\gamma \notin G_d^+$ (see [39], the proof of Proposition 5.3.1).

On the other hand, let $\mathcal{N}'_d$ and $\mathcal{N}_d$ be respectively the local and the global period domains for K3 surfaces with polarization class of degree d. We have the following result.

Theorem 2.2.3 ([39], Theorem 5.3.2, 5.3.3)**.** *Let d be a positive integer satisfying conditions* (0) *and* **(a)***. Then, there exists an isomorphism*

$$j_d : \mathcal{D}_d^{mar} \to \mathcal{N}_d,$$

which is unique up to the choice of an element in $\mathrm{Iso}(d(K_d^\perp), d(\Lambda_d^0))/(\pm 1)$, which is the quotient of the set of all such isomorphisms of discriminant groups by the action of the group $\{n \in \mathbb{Z}/d\mathbb{Z} : n^2 = 1\}$.

2.3 Mukai lattice for Ku(Y)

Let us now consider the categorical framework. We have already mentioned in the introduction that the subcategory $\mathrm{Ku}(Y)$ of a cubic fourfold Y behaves in a certain way as the derived category of a K3 surface. In [57], Kuznetsov proved that for certain special cubic fourfolds Y, there exist a K3 surface X and an equivalence of categories $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{D^b}(X)$. In general, if this condition is satisfied, we say that $\mathrm{Ku}(Y)$ is *geometric*. In [4], Addington and Thomas explained the relation between Kuznetsov's K3 surface and Hassett's Hodge theoretic associated K3 surface. Let us recall the construction.

We denote by $K(Y)_{\mathrm{top}}$ the topological K-theory of Y, which in this case is just the Grothendieck group of topological complex vector bundles over Y. Recall that the Euler pairing on $K(Y)_{\mathrm{top}}$ is given by $\chi(E, F) = \sum_i (-1)^i \dim \mathrm{Hom}(E, F[i])$. Let

$$v : K(Y)_{\mathrm{top}} \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p=0}^{4} H^{2p}(Y, \mathbb{Q})(p),$$

be the isomorphism induced by the Mukai vector, which is defined by $v(-) = \mathrm{ch}(-).\sqrt{\mathrm{td}(X)}$ (see [44], Definition 5.28). Then v induces a weight-zero Hodge structure on

$$K(\mathrm{Ku}(Y))_{\mathrm{top}} := \{\kappa \in K(Y)_{\mathrm{top}} : \chi([\mathcal{O}_Y(i)], \kappa) = 0, \text{ for all } i = 0, 1, 2\}.$$

More precisely, we have that

$$K(\mathrm{Ku}(Y))_{\mathrm{top}} \otimes \mathbb{C} = \bigoplus_{p+q=2} \tilde{H}^{p,q}(\mathrm{Ku}(Y)),$$

where

$$\tilde{H}^{2,0}(\mathrm{Ku}(Y)) = v^{-1}(H^{3,1}(Y))$$

and

$$\tilde{H}^{1,1}(\mathrm{Ku}(Y)) = v^{-1}(H^0(Y,\mathbb{C}) \oplus H^{1,1}(Y) \oplus H^{2,2}(Y) \oplus H^{3,3}(Y) \oplus H^8(Y,\mathbb{C})).$$

We denote by $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$ the lattice $K(\mathrm{Ku}(Y))_{\mathrm{top}}(-1)$ with the induced weight-two Hodge structure: it is isomorphic to the lattice $\tilde{\Lambda} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$ and it is called the *Mukai lattice* of Y (see [4], Section 2.3). Let

$$N(\mathrm{Ku}(Y)) := \tilde{H}^{1,1}(\mathrm{Ku}(Y),\mathbb{Z}) = \tilde{H}^{1,1}(\mathrm{Ku}(Y)) \cap \tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$$

be the *generalized Néron-Severi lattice* of $\mathrm{Ku}(Y)$ and we denote by $T(\mathrm{Ku}(Y))$ its orthogonal complement in $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$, which is the *generalized trascendental lattice* of $\mathrm{Ku}(Y)$. Then, there exist two elements λ_1, λ_2 in $N(\mathrm{Ku}(Y)))$, corresponding to the projections in $\mathrm{Ku}(Y)$ of the structure sheaf of a line in Y twisted by 1 and 2 respectively, spanning a rank two sublattice with intersection matrix

$$A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Proposition 2.3.1 ([4], Proposition 2.3)**.** *The Mukai vector induces an isometry between the orthogonal complement $A_2^{\perp}$ of A_2 in $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$ and the primitive lattice $\langle h\rangle^{\perp} = H^4(Y,\mathbb{Z})_0(1)$. Moreover, if $\kappa_1,\dots,\kappa_n$ are elements of $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$, then v induces an isometry*

$$\langle \lambda_1, \lambda_2, \kappa_1, \dots, \kappa_n\rangle^{\perp} \cong \langle h, c_2(\kappa_1), \dots, c_2(\kappa_n)\rangle^{\perp}.$$

Remark 2.3.2. Since, by definition, the lattice A_2 is contained in $N(\mathrm{Ku}(Y))$, the orthogonality condition implies that $T(\mathrm{Ku}(Y))$ is in $A_2^{\perp}$. In particular, as observed in [47], Section 3.3, the orthogonal complement to the trascendental lattice in $A_2^{\perp}$ is $N(\mathrm{Ku}(Y)) \cap A_2^{\perp}$.

Theorem 2.3.3 ([4], Theorem 1.1)**.** *If $\mathrm{Ku}(Y)$ is geometric, then Y belongs to $\mathcal{C}_d$ for some d satisfying condition* **(a)** *of Theorem 2.1.1. Conversely, for each d satisfying* **(a)***, the set of cubic fourfolds Y in $\mathcal{C}_d$ for which $\mathrm{Ku}(Y)$ is geometric forms a Zariski open dense subset.*

Remark 2.3.4. In [8], the authors prove that every $Y \in \mathcal{C}_d$ for d satisfying **(a)** is geometric, extending Addington and Thomas' result to the whole divisor (see [69], Theorem 3.7).

In [47], Proposition 3.4, Huybrechts proved that, given two cubic fourfolds Y and Y', the existence of a Fourier-Mukai equivalence $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y')$ implies the existence of a Hodge isometry of the corresponding Mukai lattices. The surprising fact is that, under some assumptions, the category $\mathrm{Ku}(Y)$ is completely determined by the Hodge structure on $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z})$, as we recall in the next section.

2.4 Associated twisted K3 surface

In [47], Huybrechts generalized Theorem 2.1.1 and Theorem 2.3.3 to the case of cubic fourfolds admitting an *associated twisted K3 surface.* We recall that a twisted K3 surface is the data of a K3 surface X and a class in the Brauer group $H^2(X,\mathcal{O}_X^*)_{\mathrm{tors}}$ of X. Following [49], Section 2, let B be a rational class of $H^2(X,\mathbb{Q})$, which is sent to α through the composition

$$H^2(X,\mathbb{Q}) \to H^2(X,\mathcal{O}_X) \xrightarrow{\exp} H^2(X,\mathcal{O}_X^*).$$

We say that B is a B-field lift of α. We denote by $\tilde{H}(X,\alpha,\mathbb{Z})$ the cohomology ring $H^*(X,\mathbb{Z})$ with the Mukai pairing and the weight two Hodge structure defined by

$$\tilde{H}^{2,0}(X,\alpha) := \exp(B)H^{2,0}(X) \quad \text{and} \quad \tilde{H}^{1,1}(X,\alpha) := \exp(B)H^{1,1}(X).$$

We see that $\tilde{H}(X,\alpha,\mathbb{Z})$ is isomorphic as a lattice to $\tilde{\Lambda}$ and we call it the Mukai lattice of (X,α). We can consider the algebraic part

$$N(X,\alpha) = \tilde{H}^{1,1}(X,\alpha,\mathbb{Z}) := \tilde{H}^{1,1}(X,\alpha) \cap \tilde{H}(X,\alpha,\mathbb{Z})$$

and we define the generalized twisted trascendental lattice $T(X,\alpha)$ as the orthogonal complement of $N(X,\alpha)$ with respect to the Mukai pairing. On the other hand, using the intersection product with B, we can identify the class α with a surjective morphism $\alpha : T_X \to \mathbb{Z}/\mathrm{ord}(\alpha)\mathbb{Z}$. Then, the kernel of α is isomorphic via $\exp(B)$ to $T(X,\alpha)$ (see [43], Proposition 4.7). For this reason, we will use the same notation for $T(X,\alpha)$ and $\ker(\alpha)$ (resp. for $N(X,\alpha)$ and the orthogonal complement of $\ker(\alpha)$ in $\tilde{H}(X,\alpha,\mathbb{Z})$), even if the first one is primitively embedded in $\tilde{H}(X,\alpha,\mathbb{Z})$, while the second one is not.

As in the untwisted case, the condition of having an associated twisted K3 surface on the level of Hodge structures on the Mukai lattices depends only on the value of the discriminant d.

Theorem 2.4.1 ([47], Theorem 1.3). *Let Y be a cubic fourfold. There exist a twisted K3 surface (X,α) and a Hodge isometry $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z}) \cong \tilde{H}(X,\alpha,\mathbb{Z})$ if and only if Y belongs to $\mathcal{C}_d$ for d such that*

$$n_i \equiv 0 (\mathrm{mod}\, 2) \text{ for all primes } p_i \equiv 2 (\mathrm{mod}\, 3) \text{ in } 2d = \prod p_i^{n_i}. \qquad \textbf{(a')}$$

Moreover, Theorem 2.3.3 and Remark 2.3.4 have the following analogous in the twisted setting.

- If there exists a twisted K3 surface (X,α) such that the category $\mathrm{Ku}(Y)$ is equivalent to the derived category $\mathrm{D^b}(X,\alpha)$ of bounded complexes of α-twisted coherent sheaves on X, then the cubic fourfold Y belongs to $\mathcal{C}_d$ for d satisfying condition (**a'**) of Theorem 2.4.1 (see [47], Theorem 1.4(i)).
- In [47], Theorem 1.4(ii), Huybrechts proved that if d satisfies (**a'**), then a Zariski open subset of cubic fourfolds Y in the divisor $\mathcal{C}_d$ have $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{D^b}(X,\alpha)$. In [8], the authors extend this result to all cubic fourfolds in $\mathcal{C}_d$.

Remark 2.4.2 (Derived Torelli Theorem). In [47], Theorem 1.5(ii), Huybrechts proved that for d satisfying (**a'**) and a Zariski dense open set of cubics $Y \in \mathcal{C}_d$, there exists a Fourier-Mukai equivalence $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y')$ if and only if there exists a Hodge isometry $\tilde{H}(\mathrm{Ku}(Y),\mathbb{Z}) \cong \tilde{H}(\mathrm{Ku}(Y'),\mathbb{Z})$. Now, using that every Y in such a divisor has $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{D^b}(X,\alpha)$ by [8], we can extend this result to all the divisor $\mathcal{C}_d$ following the same proof of [47] (see [69], Theorem 3.27).

Remark 2.4.3. Set

$$\tilde{Q} := \{\varphi \in \mathbb{P}(\tilde{\Lambda} \otimes \mathbb{C}) : (\varphi,\varphi) = 0, (\varphi,\bar{\varphi}) > 0\}.$$

A point $\varphi \in \tilde{Q}$ is of K3 type (resp. of twisted K3 type) if there is a K3 surface X (resp. a twisted K3 surface (X,α)) such that the Hodge structure defined by φ on $\tilde{\Lambda}$ is Hodge isometric to $\tilde{H}(X,\mathbb{Z})$ (resp. $\tilde{H}(X,\alpha,\mathbb{Z})$) (see [47], Definition 2.5). We denote by Q_{K3} (resp. $Q_{\mathrm{K3}'}$) the set of points of K3 type (resp. of twisted K3 type) in $\tilde{Q}$.

Notice that $\mathcal{D}' \subset Q \subset \tilde{Q}$, as $L^0 \cong A_2^{\perp}$. Thus, we can consider the sets

$$\mathcal{D}_{\mathrm{K3}} := Q_{\mathrm{K3}} \cap \mathcal{D}' \quad \text{and} \quad \mathcal{D}_{\mathrm{K3}'} := Q_{\mathrm{K3}'} \cap \mathcal{D}',$$

containing period points in $\mathcal{D}'$ of (twisted) K3 type (see [47], Section 2.5).

2.5 Counting formulas for Fourier-Mukai partners of a K3 surface

The aim of this section is to recollect some known formulas which count the number of isomorphism classes of (twisted) Fourier-Mukai partners of a given (twisted) K3 surface. We recall that a *twisted Fourier-Mukai partner* of a K3 surface X (resp. of a twisted K3 surface (X,α)) is a twisted K3 surface (X',α') such that there exists an equivalence of categories $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X',\alpha')$ (resp. $\mathrm{D^b}(X,\alpha) \xrightarrow{\sim} \mathrm{D^b}(X',\alpha')$); if the Brauer class α' is trivial, we say that the Fourier-Mukai partner is untwisted.

The first result concerns the number of isomorphism classes of untwisted Fourier-Mukai partners of a very general polarized K3 surface, which is determined by the number of distinct primes in the factorization of the degree of the polarization class.

Theorem 2.5.1 ([81], Proposition 1.10). *Let X be a K3 surface with Néron-Severi lattice $\mathrm{NS}(X)$ of rank one generated by a polarization class l_X such that $l_X^2 = 2n$. Let m be the number of (isomorphism classes of) Fourier-Mukai partners of X; then we have:*

- $m = 1$, *if* $l_X^2 = 2$ *or* $l_X^2 = 2^a$,
- $m = 2^{h-1}$, *if* $l_X^2 = 2p_1^{e_1} \cdot \ldots p_h^{e_h}$,
- $m = 2^h$, *if* $l_X^2 = 2^a p_1^{e_1} \cdot \ldots p_h^{e_h}$,

where a, h and the e_i's are natural numbers with $a \geqslant 2$, the p_i's are different primes such that $p_i \geqslant 3$.

More generally, Ma proved in [67] a counting formula for isomorphism classes of twisted Fourier-Mukai partners of a twisted K3 surface (X,α) which admits an untwisted Fourier-Mukai partner (see [67], Theorem 1.1). Moreover, relaxing this hypothesis, he obtained an upper bound to the number of twisted Fourier-Mukai partners of (X,α). We conclude this paragraph by resuming Ma's construction, which will be useful in the next chapter.

Let (X,α) be a twisted K3 surface with $\mathrm{ord}(\alpha) = \kappa$. We recall that a twisted K3 surface (X',α') is isomorphic to (X,α) if there exists an isomorphism $F : X \cong X'$ such that $F^*\alpha' = \alpha$. We denote by $\mathrm{FM}^r(X,\alpha)$ the set of isomorphism classes of Fourier-Mukai partners (X',α') of (X,α), having α' of order r. We say that (X_1,α_1) and (X_2,α_2) in $\mathrm{FM}^r(X,\alpha)$ are $\sim$-equivalent if there exists a Hodge isometry $g : T_{X_1} \cong T_{X_2}$ such that $g^*\alpha_2 = \alpha_1$. We define the quotient

$$\mathcal{FM}^r(X,\alpha) := \mathrm{FM}^r(X,\alpha)/\sim$$

and we denote by $\pi : \mathrm{FM}^r(X,\alpha) \twoheadrightarrow \mathcal{FM}^r(X,\alpha)$ the quotient map. Let $I^r(d(T(X,\alpha)))$ be the set of all isotropic subgroups of order r of the discriminant group $(d(T(X,\alpha)), q_{T(X,\alpha)})$ of $T(X,\alpha)$, i.e.

$$I^r(d(T(X,\alpha))) := \left\{ x \in d(T(X,\alpha)) : q_{T(X,\alpha)}(x) = 0 \in \mathbb{Q}/2\mathbb{Z}, \mathrm{ord}(x) = r \right\}.$$

We define the map

$$\mu : \mathcal{FM}^r(X,\alpha) \to \mathrm{O_{Hdg}}(T(X,\alpha)) \backslash I^r(d(T(X,\alpha))), \tag{2.2}$$

where $\mathrm{O_{Hdg}}(T(X,\alpha))$ is the group of Hodge isometries of the generalized trascendental lattice, in the following way. For every (X_1,α_1) in $\mathrm{FM}^r(X,\alpha)$, there exists a Hodge isometry $g_1 : T(X_1,\alpha_1) \cong T(X,\alpha)$. Then

$$\frac{g_1^\vee(T_{X_1})}{T(X,\alpha)} \cong \frac{T_{X_1}}{T(X_1,\alpha_1)} \cong \frac{\mathbb{Z}}{r\mathbb{Z}}$$

is an isotropic, cyclic subgroup of $d(T(X,\alpha))$ of order r. Thus, for every class $[(X_1,\alpha_1)]$ in $\mathcal{FM}^r(X,\alpha)$, we set

$$\mu([(X_1,\alpha_1)]) = x := [g_1(\alpha_1^{-1}(\bar{1}))] \in \mathrm{O_{Hdg}}(T(X,\alpha)) \backslash I^r(d(T(X,\alpha))).$$

We have that:

1. The map μ is well-defined and injective (see [67], Lemma 3.2);

2. The image of μ is contained in $\mathrm{O}_{\mathrm{Hdg}}(T(X,\alpha))\backslash J^r(d(T(X,\alpha)))$, where
$$J^r(d(T(X,\alpha))) = \{x \in I^r(d(T(X,\alpha))) : \text{there exists an embedding } U \hookrightarrow \langle N(X,\alpha), \lambda(x)\rangle\},$$
for $\lambda : d(T(X,\alpha)) \cong d(N(X,\alpha))$ (see [67], Proposition 3.4).

On the other hand, for every (X_1,α_1) in $\mathrm{FM}^r(X,\alpha)$, we can define a map
$$\nu : \pi^{-1}(\pi(X_1,\alpha_1)) \to \Gamma(X_1,\alpha_1)^+\backslash\mathrm{Emb}(U, N(X_1)), \tag{2.3}$$
where $\mathrm{Emb}(U,N(X_1))$ is the set of the embeddings of U in $N(X_1) = H^0(X_1,\mathbb{Z})\oplus\mathrm{NS}(X_1)\oplus H^4(X_1,\mathbb{Z})$ and $\Gamma(X_1,\alpha_1)^+$ is the set of orientation-preserving isometries of $N(X_1)$, [1] which come from isometries of T_{X_1} fixing α_1 (see [67], Section 3.2). We have that:

1. The map ν is injective (see [67], Lemma 3.5);

2. The map ν is surjective if and only if the Căldăraru's Conjecture holds (see [67], Remark 3.7).

We recall the statement of *Căldăraru's Conjecture*, which was proposed for the first time in [23], Conjecture 5.5.5.

Conjecture 2.5.2 ([67], Question 3.8)**.** *Let (X,α) be a twisted K3 surface. For each untwisted Fourier-Mukai partner X' of X and each Hodge isometry $g : T_{X'} \cong T_X$, the twisted K3 surface $(X', g^*\alpha)$ is a Fourier-Mukai partner of (X,α).*

Remark 2.5.3. We point out that Conjecture 2.5.2 is related to an other conjecture due to Căldăraru, which asks whether two twisted K3 surfaces having Hodge isometric twisted trascendental lattices are Fourier-Mukai partners. This conjecture is known to be false in general by [49], Example 4.11.

To state Ma's formula, we need to introduce some notation. For every x in $I^r(d(T(X,\alpha)))$, we define the overlattice
$$T_x := \langle x, T(X,\alpha)\rangle$$
of $T(X,\alpha)$ and the morphism
$$\alpha_x : T_x \twoheadrightarrow \frac{T_x}{T(X,\alpha)} \cong \langle x\rangle \cong \frac{\mathbb{Z}}{r\mathbb{Z}}.$$
For a pair (x,M) such that
$$\langle\lambda(x), N(X,\alpha)\rangle \cong U \oplus M,$$
we define the number
$$\tau(x,M) := \#(\mathrm{O}_{\mathrm{Hdg}}(T_x,\alpha_x)\backslash\mathrm{O}(d(M))/\mathrm{O}(M)),$$
where $\mathrm{O}_{\mathrm{Hdg}}(T_x,\alpha_x)$ is the set of Hodge isometries g of T_x, such that $g^*\alpha_x = \alpha_x$. For a natural number r, we define
$$\varepsilon(r) = \begin{cases} 1, & \text{if } r = 1,2 \\ 2, & \text{if } r \geqslant 3. \end{cases}$$
Finally, if $\mathcal{G}(L)$ is the genus of a lattice L, $\mathrm{O}(L)_0$ is the kernel of the map $r_L : \mathrm{O}(L) \to \mathrm{O}(d(L))$ and $\mathrm{O}(L)_0^+$ is the subgroup of $\mathrm{O}(L)_0$ of orientation-preserving isometries, we define the subsets
$$\mathcal{G}_1(L) := \{L' \in \mathcal{G}(L) : \mathrm{O}(L')_0^+ \neq \mathrm{O}(L')_0\}, \quad \mathcal{G}_2(L) := \{L' \in \mathcal{G}(L) : \mathrm{O}(L')_0^+ = \mathrm{O}(L')_0\}.$$
Using the previous observations, Ma proved that the following inequality holds.

[1] In general, given a lattice L of signature (l_+, l_-) with $l_+ > 0$, we can consider the set of oriented positive definite l_+-planes in $L \otimes \mathbb{R}$. An orientation for L is the choice of an orientation for such a positive definite l_+-plane. For a subgroup Γ of $\mathrm{O}(L)$, we denote by Γ^+ the subgroup of isometries of Γ which preserve the given orientation (see [67], Section 2.1).

Theorem 2.5.4 ([67], Proposition 4.3). *We have the inequality*

$$\#\mathrm{FM}^r(X,\alpha) \leqslant \sum_x \left\{ \sum_M \tau(x,M) + \varepsilon(r) \sum_{M'} \tau(x,M') \right\}. \tag{2.4}$$

Here:

- *x runs over the set $\mathrm{O}_{\mathrm{Hdg}}(T(X,\alpha))\backslash J^r(d(T(X,\alpha)))$;*
- *the lattices M and M' run over the sets $\mathcal{G}_1(M_\varphi)$, $\mathcal{G}_2(M_\varphi)$ respectively, where M_φ is a lattice satisfying $\langle \lambda(x), N(X,\alpha)\rangle \cong U \oplus M_\varphi$.*

Chapter 3

Fourier-Mukai partners of cubic fourfolds

In this chapter we present the proof of Theorem 1.0.1 and Theorem 1.0.2, which are the main results of [87].

3.1 Construction of the examples (untwisted case)

The aim of this section is to prove Theorem 1.0.1, whose statement is the following.

Theorem 3.1.1. *Let d be a positive integer satisfying* (0) *and* **(a)**. *Let Y be a very general cubic fourfold in $\mathcal{C}_d$ and let m be the number of non-isomorphic Fourier-Mukai partners of an associated K3 surface to Y. Then, the cubic fourfold Y has exactly m non-isomorphic Fourier-Mukai partners, when $d \equiv 2 (\mathrm{mod}\, 6)$; otherwise, if $d \equiv 0 (\mathrm{mod}\, 6)$, the number of non-isomorphic Fourier-Mukai partners of Y is equal to $\lceil m/2 \rceil$.*

In the first paragraph we exhibit some preliminary computations on the level of the period domain of (marked) cubic fourfolds, while in the last section we provide the proof of the theorem.

3.1.1 Some preliminary computations

Let Y be a very general special cubic fourfold in $\mathcal{C}_d$ with d satisfying condition **(a)** of Theorem 2.1.1; let us choose a K3 surface X of degree d associated to Y. In this section we study the number of distinct points in the period domain $\mathcal{D}_d$ determined by the non-isomorphic representatives of the isomorphism classes of untwisted Fourier-Mukai partners of X.

We recall that $N(\mathrm{Ku}(Y))$ has rank 3, because Y is very general in $\mathcal{C}_d$ (see [47], Lemma 2.2). Let υ_Y be a generator of the rank one lattice $N(\mathrm{Ku}(Y)) \cap A_2^{\perp}$. Let m (possibly equal to 1) be the number of isomorphism classes of Fourier-Mukai partners of X. We fix a representative for each class of isomorphism and we denote them by $X_1, \dots, X_m$, choosing $X_1 := X$. By [83], Theorem 3.3, this is equivalent to ask that, for every index $2 \leqslant k \leqslant m$, there exists a Hodge isometry $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(X_k, \mathbb{Z})$. In particular, the Néron-Severi lattice of X_k has rank one with the polarization class of degree d. We denote by x_k the point in the local period domain $\mathcal{N}'_d$, which is determined by the Hodge structure on the trascendental lattices of the K3 surface X_k. These points also descend to different points in the global period domain $\mathcal{N}_d$, since they come from non-isomorphic polarized K3 surfaces.

Composing the isometries of Proposition 2.3.1 and of Theorem 2.1.1, we get the isometry of Hodge structures

$$\varphi : T(\mathrm{Ku}(Y)) = \langle \lambda_1, \lambda_2, \upsilon_Y \rangle^{\perp} \cong H^2(X, \mathbb{Z})_0 = T_X,$$

where the trascendental lattice $T(\mathrm{Ku}(Y))$ is defined in Section 2.3. This induces an isomorphism

$$j' : \mathcal{D}'_d \to \mathcal{N}'_d$$

between the local period domains. For every $1 \leqslant k \leqslant m$, we denote by y_k the preimage of x_k with respect to j'. By definition, the point y_k parametrizes a special Hodge structure with labelling of discriminant d on $A_2^{\perp}$. In particular, there exists a class υ_k in $A_2^{\perp}$ with $(\upsilon_k, \upsilon_k) = (\upsilon_Y, \upsilon_Y)$, such that if $T_k = (\mathbb{Z}\,\upsilon_k)^{\perp}$ in $A_2^{\perp}$, then there is an isometry of Hodge structures $\varphi_k : T_{X_k} \cong T_k$.

As verified in the proof of Theorem 5.3.2 of [39], the isomorphism j' descends to an isomorphism

$$j : \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d.$$

Thus, the points $y_1, \dots, y_m$ descends to distinct points, which we denote in the same way, in the period domain $\mathcal{D}_d^{\text{mar}}$.

Let us consider their images in the global period domain $\mathcal{D}_d^{\text{lab}}$; here, these points could be identified. By the way, we observe that, if some of them are not identified in $\mathcal{D}_d^{\text{lab}}$, then they correspond to distinct points in the global period domain $\mathcal{D}_d$. Indeed, the map sending $\mathcal{D}_d^{\text{lab}}$ in $\mathcal{D}_d$, which forgets the labelling, is an isomorphism on very general points of $\mathcal{D}_d$.

In particular, it is enough to study the behavior of the forgetful map $\rho : \mathcal{D}_d^{\text{mar}} \to \mathcal{D}_d^{\text{lab}}$ over the points $y_1, \dots, y_m$, to understand how many of them define different very general special Hodge structures of discriminant d. According to Proposition 2.2.1, we have to distinguish two cases depending on the value of the discriminant.

Case $d \equiv 2(\mathbf{mod}\, 6)$: by Theorem 2.2.3, we have that the map ρ is an isomorphism. Hence, $y_1, \dots, y_m$ are not identified by the action of Γ_d^+ and they determine m distinct very general special Hodge structures of discriminant d.

Case $d \equiv 0(\mathbf{mod}\, 6)$: by Theorem 2.2.3, the map ρ is a double cover. Thus, it is possible that there exist two indexes $1 \leqslant k_1 \neq k_2 \leqslant m$ such that y_{k_1} and y_{k_2} belong to the same fiber of ρ. As recalled in Remark 2.2.2, this is equivalent to asking that the diagram

$$\begin{array}{ccc} T_{k_1} & \longrightarrow & A_2^{\perp} \\ \cong\big\downarrow & & \big\downarrow\gamma \\ T_{k_2} & \longrightarrow & A_2^{\perp} \end{array} \tag{3.1}$$

commutes. Moreover, we have that γ induces an isometry of Hodge structures between $T_{X_{k_1}}$ and $T_{X_{k_2}}$, which we denote by γ', via φ_{k_1} and φ_{k_2}. The isometry γ' does not extend to an automorphism of $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$, as we prove in the next lemma.

Lemma 3.1.2. *The K3 surfaces X_{k_1} and X_{k_2} are not isomorphic.*

Proof. Keeping the notation used above, we have that the lattices T_{k_1}, T_{k_2} sit in the diagram (3.1), by hypothesis.

First of all, we prove that the isometry γ' does not extend to an automorphism of Λ. Indeed, let $\bar{\gamma}'$ be the isomorphism over the discriminant groups induced by γ', which respects the discriminant quadratic forms. By construction, we have $\gamma' := (\varphi_{k_2})^{-1} \circ \gamma \circ \varphi_{k_1}$; thus, passing to the discriminant groups, we have the following commutative diagram:

$$\begin{array}{ccc} d(T_{X_{k_1}}) & \xrightarrow{\bar{\varphi}_{k_1}} & d(T_{k_1}) \\ \bar{\gamma}'\big\downarrow & & \big\downarrow\bar{\gamma} \\ d(T_{X_{k_2}}) & \xrightarrow{\bar{\varphi}_{k_2}} & d(T_{k_2}) \end{array} . \tag{3.2}$$

By Proposition 0.2.7 ([76], Proposition 1.6.1), we have that $d(T_{X_{k_i}})$ is isomorphic to the discriminant group of the Néron-Severi lattice $d(\mathrm{NS}(X_{k_i}))$ for every $i = 1, 2$. As a consequence, there exists an induced isomorphism

$$\bar{\gamma}'_N : d(\mathrm{NS}(X_{k_1})) \to d(\mathrm{NS}(X_{k_2}).)$$

Now, by Proposition 0.2.8 ([76], Proposition 1.5.2), we have that the isometry γ' extends to the whole Λ if and only if the isomorphism $\bar{\gamma}'_N$ comes from an isometry of the form $\mathrm{NS}(X_{k_1}) \cong \mathrm{NS}(X_{k_2})$. But, we recall that $\mathrm{NS}(X_{k_i}) = \mathbb{Z}\, l_{k_i}$ for every $i = 1, 2$. Hence, there exist only two isometries between $\mathrm{NS}(X_{k_1})$ and $\mathrm{NS}(X_{k_2})$, defined by sending the polarization class l_{k_1} to l_{k_2} (resp. to $-l_{k_2}$).

Let us suppose that the isomorphism of discriminant groups $\bar{\gamma}'_N$ comes from one of these two isometries. Then, it has to act as the multiplication by 1 or -1 on the generators of the discriminant groups. By diagram (3.2), we deduce that the same property holds for the isomorphism $\bar{\gamma} : d(T_{k_1}) \cong d(T_{k_2})$. We recall that, for every $i = 1, 2$, the lattice T_{k_i} is isometric to the orthogonal complement in L of the labelling $K_d^{k_i}$. Thus, the induced isomorphism between the discriminant groups $d(K_d^{k_i \perp})$'s acts as the multiplication by ± 1 on the generators, in contradiction with the definition of γ (see Remark 2.2.2). Thus, we deduce that the isomorphism $\bar{\gamma}'_N$ does not arise from an isometry $\mathrm{NS}(X_{k_1}) \cong \mathrm{NS}(X_{k_2})$ and, hence, the isometry γ' does not extend to an isometry of Λ, as we stated.

Finally, we observe that there cannot exist an isometry between the cohomology groups $H^2(X_{k_1}, \mathbb{Z})$ and $H^2(X_{k_2}, \mathbb{Z})$, because it should be an extension of γ'. Hence, by Torelli Theorem for K3 surfaces, we deduce that the K3 surfaces X_{k_1} and X_{k_2} are not isomorphic, as we wanted. □

Anyway, the fibers of the map ρ contain two points. Hence, we deduce that our points $y_1, \dots, y_m$ descend to at least $\lceil m/2 \rceil$ different Hodge structures in $\mathcal{D}_d$.

On the other hand, we observe that if T is a sublattice of Λ which is Hodge isometric to T_X, then the lattice $\gamma(T)$, with the Hodge structure induced by that one on T through $\gamma_{\mathbb{C}}$, satisfies the same property. As a consequence, we obtain that the K3 surface $X_{\gamma(T)}$ with trascendental lattice $\gamma(T)$ is a Fourier-Mukai partner of X. Since by Lemma 3.1.2 they are non-isomorphic K3 surfaces, their corresponding period points in $\mathcal{N}_d$ define two distinct period points in $\mathcal{D}_d^{\mathrm{mar}}$, which belong to the same fiber of ρ. It follows that the m points $y_1, \dots, y_m$ determine exactly $\lceil m/2 \rceil$ different special Hodge structures of discriminant d.

3.1.2 Proof of Theorem 1.0.1

Keeping the notation introduced in Section 3.1, we set

$$p := \begin{cases} m & \text{if } d \equiv 2 (\operatorname{mod} 6) \\ \lceil m/2 \rceil & \text{if } d \equiv 0 (\operatorname{mod} 6). \end{cases}$$

Firstly, we prove that p is an upper bound to the number of Fourier-Mukai partners of Y. Actually, this represents an alternative way to prove the finiteness result of [47], Corollary 3.5, under the previous hypotheses.

Proposition 3.1.3. *Let Y be a very general cubic fourfold in $\mathcal{C}_d$ with d satisfying condition* **(a)** *of Theorem 2.1.1. If the associated K3 surface X admits m (possibly equal to one) non-isomorphic Fourier-Mukai partners, then the cubic fourfold Y cannot have more than m (resp. $\lceil m/2 \rceil$) Fourier-Mukai partners if $d \equiv 2 (\operatorname{mod} 6)$ (resp. if $d \equiv 0 (\operatorname{mod} 6)$).*

Proof. Consider the p distinct points $y_1, \dots, y_p \in \mathcal{D}_d$ defined in Section 3.1.1. We claim that y_k belongs to the image of the period map of cubic fourfolds for every $1 \leqslant k \leqslant p$. Indeed, we observe that d is not 2 or 6, because d satisfies condition (0), as $\mathcal{C}_d$ is not empty. Moreover, the point y_k is very general in $\mathcal{D}_d$, thus it has a unique labelling, as recalled in Section 2.2. It follows that y_k is a period point in the complement of $\mathcal{D}_2 \cup \mathcal{D}_6$. By [63], Theorem 1.1, there exists a cubic fourfold Y_k in $\mathcal{C}_d$ such that $\tau(Y_k) = y_k$, as we wanted.

Now, let Y' be a Fourier-Mukai partner of Y, i.e. such that there exists an equivalence $\mathrm{Ku}(Y) \xrightarrow{\sim} \mathrm{Ku}(Y')$ of Fourier-Mukai type. By [47], Proposition 3.4, this induces a Hodge isometry $\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) \cong \tilde{H}(\mathrm{Ku}(Y'), \mathbb{Z})$. Notice that Y' is a very general element in $\mathcal{C}_d$, as Y is. Thus its period point $\tau(Y') \in \mathcal{D}_d$ corresponds to a point in $\mathcal{D}_d^{\mathrm{lab}}$ which we denote in the same way. Let $y' \in \mathcal{D}_d^{\mathrm{mar}}$ be a point in the fiber $\rho^{-1}(\tau(Y'))$. We set $x' := j(y') \in \mathcal{N}_d$. The point x' corresponds to a very general K3 surface X' with unique primitive polarization $l_{X'}$ of degree d. In particular, since $\tilde{H}(\mathrm{Ku}(Y'), \mathbb{Z})$ is Hodge isometric to the Mukai lattice $\tilde{H}(X', \mathbb{Z})$, it follows from [83], Theorem 3.3 that X' is a Fourier-Mukai partner of X. Thus, there exists an index $k \in \{1, \dots, m\}$ such that, if l_{X_k} denotes the unique primitive polarization on X_k, then $(X', l_{X'}) \cong (X_k, l_{X_k})$ as polarized K3 surfaces. Equivalently, we have that the points x' and x_k are identified in $\mathcal{N}_d$. Since j is an isomorphism, it follows that $y_k = y'$ in $\mathcal{D}_d^{\mathrm{mar}}$. In particular, they represent the same point in $\mathcal{D}_d$: by the Torelli Theorem for cubic fourfolds, we conclude that Y' is isomorphic to Y_k. This implies the desired statement. □

We are ready to prove Theorem 1.0.1, which is formulated in a more precise way using Theorem 2.5.1.

Proposition 3.1.4 (Theorem 1.0.1)**.** *Let d be a positive integer satisfying conditions* (0) *and* **(a)**. *Then, the number of isomorphism classes of Fourier-Mukai partners for a very general cubic fourfold in $\mathcal{C}_d$ is*

- $p = 2^{h-1}$, *if $d \equiv 2 (\mathrm{mod}\, 6)$ and the prime factorization of d has $h > 1$ distinct odd primes;*
- $p = 2^{h-2}$, *if $d \equiv 0 (\mathrm{mod}\, 6)$ and the prime factorization of d has $h > 2$ distinct odd primes;*
- $p = 1$, *otherwise.*

Proof. Let Y be a very general cubic fourfold in $\mathcal{C}_d$ as in the statement. We consider the p distinct points $y_1, \dots, y_p$ in $\mathcal{D}_d$ defined in Section 3.1.1. Arguing as in the proof of Proposition 3.1.3, by [63], Theorem 1.1, there exist p very general special cubic fourfolds $Y_1, \dots, Y_p \in \mathcal{C}_d$ such that $\tau(Y_k) = y_k$ for $k = 1, \dots, p$. Notice that $Y_1 \cong Y$ and the cubic fourfolds $Y_1, \dots, Y_p$ are not isomorphic to each other by Torelli Theorem for cubic fourfolds.

By construction, for every $2 \leqslant k \leqslant p$, there is an isometry of Hodge structures

$$\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) \cong \tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(X_k, \mathbb{Z}) \cong \tilde{H}(\mathrm{Ku}(Y_k), \mathbb{Z}).$$

By Remark 2.4.2, the existence of such an isometry of Hodge structures implies the existence of a Fourier-Mukai equivalence between $\mathrm{Ku}(Y)$ and $\mathrm{Ku}(Y_k)$. On the other hand, by Proposition 3.1.3 every other Fourier-Mukai partner of Y is isomorphic to one of those we constructed. Finally, the counting formula of Theorem 2.5.1 implies the statement. □

Example 3.1.5. Using Proposition 3.1.4, it is easy to find the divisors in $\mathcal{C}$ whose very general element has non trivial Fourier-Mukai partners. For example, take $d = 182$, which is $\equiv 2 (\mathrm{mod}\, 6)$. By Proposition 3.1.4 the very general cubic fourfold in $\mathcal{C}_{182}$ has one non-isomorphic Fourier-Mukai partner. If $d = 546 \equiv 0 (\mathrm{mod}\, 6)$, then the very general element in $\mathcal{C}_{546}$ has one non-isomorphic Fourier-Mukai partner.

Remark 3.1.6. Notice that, to prove these results, we have fixed an associated K3 surface to Y and, consequently, an isomorphism between the period domains $\mathcal{D}_d^{\mathrm{mar}}$ and $\mathcal{N}_d$. Actually, we could choose a Fourier-Mukai partner of X as fixed associated K3 surface to Y: this would have given a different isomorphism $\tilde{j}$ on the level of period domains and a different identification of Fourier-Mukai partners of Y with Fourier-Mukai partners of X (see [40], Remark 27). Anyway, the considerations about the number of Fourier-Mukai partners hold in the same way.

3.2 Construction of the examples (twisted case)

This section is devoted to the proof of Theorem 1.0.2, whose statement is the following.

Theorem 3.2.1. *Let d be a positive integer satisfying* (0) *and* **(a')**. *Assume that* 9 *does not divide d. Let Y be a very general cubic fourfold in $\mathcal{C}_d$ with associated twisted K3 surface (X, α), where α has order κ; let m' be the number of non-isomorphic Fourier-Mukai partners of (X, α) with Brauer class of order κ. Then the cubic fourfold Y admits at least m' non-isomorphic Fourier-Mukai partners, when $d \equiv 2 (\mathrm{mod}\, 6)$; otherwise, if $d \equiv 0 (\mathrm{mod}\, 6)$, the number of non-isomorphic Fourier-Mukai partners of Y is at least $\lceil m'/2 \rceil$.*

In particular, in Section 3.2.2 and 3.2.3 we explicit the lower bound to the number of Fourier-Mukai partners of a cubic fourfold Y as in Theorem 1.0.2, in terms of the number of primes in the prime factorization of the discriminant of Y and the Euler function evaluated in the order of the Brauer class of the associated twisted K3 surface.

3.2.1 Proof of Theorem 1.0.2

Let Y be a very general special cubic fourfold in $\mathcal{C}_d$ such that condition **(a')** of Theorem 2.4.1 holds. If d satisfies in addition **(a)**, then we fix an associated untwisted K3 surface and the following construction provides the same period points constructed in Section 3. In the general case, the cubic fourfold Y has a twisted associated K3 surface, which we denote by (X, α) with α of order κ, and there is an isometry of Hodge structures

$$\phi : \tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) \cong \tilde{H}(X, \alpha, \mathbb{Z}).$$

Notice that ϕ induces a Hodge isometry $\phi_T : T(\mathrm{Ku}(Y)) \cong T(X, \alpha)$. Recall that $T(\mathrm{Ku}(Y))$ is isometric to the orthogonal complement $K_d^{\perp} \subset L^0 \subset L$ of a labelling $K_d \subset L$, as Y is very general in $\mathcal{C}_d$. On the other hand, we identify $T(X, \alpha)$ with an abstract sublattice T of $\tilde{\Lambda}$ such that $T(X, \alpha) \cong T$. In other words, the lattice T sits in the commutative diagram

$$\begin{array}{ccc} T(X,\alpha) & \hookrightarrow & \tilde{H}(X,\alpha,\mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ T & \hookrightarrow & \tilde{\Lambda}. \end{array}$$

We have that ϕ_T induces an isometry

$$j : K_d^{\perp} \cong T.$$

Assume in addition that

$$\text{9 does not divide the discriminant } d. \tag{b}$$

Notice that condition **(b)** implies that the discriminant group of $T(\mathrm{Ku}(Y))$ and, consequently, also that of $T(X, \alpha)$, are cyclic, by Proposition 2.3.1 and Proposition 2.1.2. As a consequence, by [76], Theorem 1.14.4, the natural embedding

$$T(X, \alpha) \hookrightarrow \tilde{H}(X, \alpha, \mathbb{Z}) \cong \tilde{\Lambda} \tag{3.3}$$

is unique up to isometry of $\tilde{\Lambda}$, because $\mathrm{rk}(N(X, \alpha)) \geqslant l(d(T(X, \alpha))) + 2 = 3$.

Now, let (X', α') be a twisted Fourier-Mukai partner of (X, α) of order κ. By [49], Proposition 4.3, there is an isometry of Hodge structures $\tilde{H}(X, \alpha, \mathbb{Z}) \cong \tilde{H}(X', \alpha', \mathbb{Z})$. This induces the Hodge isometry $T(X, \alpha) \cong T(X', \alpha')$. Since the embedding of (3.3) is unique in the above sense, we have that $T(X, \alpha)$

and $T(X',\alpha')$ are identified with the same sublattice T of $\tilde{\Lambda}$ and the weight two Hodge structures determined by $\tilde{H}^{2,0}(X,\alpha)$ and $\tilde{H}^{2,0}(X',\alpha')$ induce two Hodge structures on T, which are exchanged by an isometry $f_T \in \mathrm{O}(T)$. The situation is summarized by the following commutative diagram:

$$\begin{array}{ccccc} T(X,\alpha) & \xrightarrow{\cong} & T & \hookrightarrow & \tilde{\Lambda} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle f_T} & & \downarrow \\ T(X',\alpha') & \xrightarrow{\cong} & T & \hookrightarrow & \tilde{\Lambda} \end{array}.$$

Via j, the lattice $T(X',\alpha') \cong T$ with the Hodge structure induced by $\tilde{H}^{2,0}(X',\alpha')$ determines a Hodge structure on $K_d^{\perp}$. We have then an induced Hodge structure on L^0 having the generator of $T(X',\alpha')^{\perp} \subset L^0$ in its $(1,1)$ part. This corresponds to a period point y' in the quadric Q defined in (2.1). Up to exchanging $\tilde{H}^{2,0}(X',\alpha')$ with $\tilde{H}^{0,2}(X',\alpha')$, we can assume that y' is in $\mathcal{D}'_d$. This is a generalization of the argument used in [39], Section 5.3, to construct the isomorphism of period domains $\mathcal{D}'_d$ and $\mathcal{N}'_d$.

The image of y' in $\mathcal{D}_d$ (which we still denote by y') is equal to the period point $y := \tau(Y)$ if and only if there is an isometry of $K_d^{\perp}$ induced by $T(X,\alpha) \cong T(X',\alpha')$ which extends to an isometry of L^0, i.e. which sits in a commutative diagram of the form

$$\begin{array}{ccc} T(X,\alpha) \cong T \overset{j}{\cong} K_d^{\perp} & \hookrightarrow & L^0 \\ \downarrow{\scriptstyle\cong} & & \downarrow \\ T(X',\alpha') \cong T \overset{j}{\cong} K_d^{\perp} & \hookrightarrow & L^0 \end{array}.$$

In this case, we would have that the two Hodge structures on $K_d^{\perp}$ given by those on $T(X,\alpha)$ and $T(X',\alpha')$, respectively, induce the same Hodge structure on L^0.

As in the untwisted case, it is convenient to consider firstly the period domain $\mathcal{D}_d^{\mathrm{mar}}$. Here, the points y and y' are identified if and only if they are in the same orbit by the action of G_d^{+}. We recall that elements in G_d^{+} are isometries of $K_d^{\perp}$ acting trivially on the discriminant group $d(K_d^{\perp})$.

Assume that (X',α') is not isomorphic to (X,α). In the next lemma, we prove that y and y' are distinct in the period domain $\mathcal{D}_d^{\mathrm{mar}}$ under this assumption.

Lemma 3.2.2. *The period points y and y' are distinct in $\mathcal{D}_d^{\mathrm{mar}}$.*

Proof. We will actually prove that if $y = y'$ in $\mathcal{D}_d^{\mathrm{mar}}$, then the twisted K3 surfaces (X,α) and (X',α') are isomorphic, in contradiction with our assumption.

If y and y' are the same point in the period domain $\mathcal{D}_d^{\mathrm{mar}}$, then there exists an isometry of Hodge structures

$$\eta : T(X,\alpha) \cong T(X',\alpha'),$$

such that the induced isomorphism $\bar{\eta}$ between the discriminant groups $d(T(X,\alpha))$ and $d(T(X',\alpha'))$ is trivial. More precisely, there exists a lattice T, which is Hodge isometric to $T(X,\alpha)$ and $T(X',\alpha')$, such that the map η_T, which sits in the diagram

$$\begin{array}{ccc} T(X,\alpha) & \xrightarrow{\eta} & T(X',\alpha') \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ T & \xrightarrow{\eta_T} & T \end{array}$$

acts as the identity on the discriminant group $d(T)$.

First of all, we prove that the Hodge isometry η extends to a Hodge isometry of the trascendental lattices T_X and $T_{X'}$. Indeed, we set

$$H = \frac{T_X}{T} \quad \text{and} \quad H' = \frac{T_{X'}}{T},$$

which are cyclic subgroups of $d(T)$ of order κ. Thus, H and H' are the same subgroup, because they have the same order. Moreover, if $\bar{\eta}_T$ denotes the automorphism of $d(T)$ induced by η_T, then

$$\bar{\eta}_T(H) = \mathrm{id}_{d(T)}(H) = H.$$

By [76], Proposition 1.4.2, we conclude that the isometry η_T extends to an isometry $g : T_X \cong T_{X'}$. By construction, the isometry g preserves the Hodge structures on T_X and $T_{X'}$. If we define the embeddings $i : T \cong T(X, \alpha) \to T_X \cong S$ and $i' : T \cong T(X', \alpha') \to T_{X'} \cong S$, we have a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{i} & S \\ \downarrow{\scriptstyle \eta_T} & & \downarrow{\scriptstyle g_S} \\ T & \xrightarrow{i'} & S \end{array}$$

where g_S is the isometry induced by g via the identification of T_X and $T_{X'}$ with a lattice S.

Secondly, we prove that, if the isomorphism $\bar{\eta}_T$ acts as the identity on $d(T)$, then also $\bar{g}_S$, induced by g_S, is the identity on $d(S)$. Indeed, let us denote by $g_S^\vee$ (resp. $\eta_T^\vee$) the extension of g_S to $S^\vee$ (resp. of η_T) to $T^\vee$. We recall that $g_S^\vee$ and $\eta_T^\vee$ are defined by the precomposition with g_S and η_T, respectively, and they make the diagram

$$\begin{array}{ccccccc} T & \xrightarrow{i} & S & \longrightarrow & S^\vee & \longrightarrow & T^\vee \\ \downarrow{\scriptstyle \eta_T} & & \downarrow{\scriptstyle g_S} & & \downarrow{\scriptstyle g_S^\vee} & & \downarrow{\scriptstyle \eta_T^\vee} \\ T & \xrightarrow{i'} & S & \longrightarrow & S^\vee & \longrightarrow & T^\vee \end{array}$$

to commute. Next, we observe that we have the isomorphisms of groups

$$r : \frac{S^\vee / i(T)}{H} \cong \frac{S^\vee}{S} \quad \text{and} \quad r' : \frac{S^\vee / i'(T)}{H'} \cong \frac{S^\vee}{S},$$

where $H = S/i(T)$ and $H' = S/i'(T)$. We claim that the isomorphism

$$\bar{\tilde{g}} : \frac{S^\vee / i(T)}{H} \cong \frac{S^\vee / i'(T)}{H'},$$

induced by $g_S^\vee$, is identified with $\bar{g}_S$ via the isomorphisms r and r'. Indeed, we have that $g_S^\vee = \eta_T^\vee|_{S^\vee}$ induces the isomorphism

$$\tilde{g} : \frac{S^\vee}{i(T)} \to \frac{S^\vee}{i'(T)},$$

which is actually the restriction of $\bar{\eta}_T$ to $S^\vee/T$. Now, we denote by π and π' the quotient maps

$$\pi : \frac{S^\vee}{i(T)} \to \frac{S^\vee / i(T)}{H}, \quad \pi' : \frac{S^\vee}{i'(T)} \to \frac{S^\vee / i'(T)}{H'}.$$

The isomorphism $\bar{\tilde{g}}$, defined by $\tilde{g}$ passing to the quotient, is well defined, because

$$\pi'(\tilde{g}(H)) = \pi'(\bar{\eta}_T(H)) = \pi'(H') = 0.$$

Thus the diagram

$$\begin{array}{ccc} \frac{S^\vee/i(T)}{H} & \xrightarrow{\bar{\bar{g}}} & \frac{S^\vee/i'(T)}{H'} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ \frac{S^\vee}{S} & \xrightarrow{\bar{g}_S} & \frac{S^\vee}{S} \end{array} \tag{3.4}$$

commutes. Now, we observe that $\bar{\bar{g}}$ acts as the identity, since it is induced by $\bar{\eta}_T|_{(S^\vee/T)}$ which is the identity map by our hypothesis. Since the diagram (3.4) commutes, we conclude that also $\bar{g}_S$ acts as the identity map, as we stated.

Finally, we denote by $\mathbb{Z}\,l$ the rank one lattice which is the orthogonal complement of S in Λ. Since $d(S) \cong d(\mathbb{Z}\,l)$, by Proposition 0.2.8, we conclude that the isometry g_S extends to an isometry f_Λ of Λ and, therefore, the isometry g extends to $f : H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$. Furthermore, the restriction of f_Λ to $\mathbb{Z}\,l$ is the identity, because by construction it induces the identity on the discriminant group of $\mathbb{Z}\,l$. In particular, we deduce that the isometry f preserves the ample cones of X and X'. By Torelli Theorem, we have that there exists an isomorphism F between the K3 surfaces X' and X such that $F^* = f$. Since, by definition, the isometry f sends the class α to α', we conclude that (X,α) and (X',α') are isomorphic as twisted K3 surfaces, in contradiction with our assumption. Therefore, we conclude that y and y' are not the same point in $\mathcal{D}_d^{\text{mar}}$, as we wanted. □

Proof of Theorem 1.0.2. The representatives of the m' isomorphism classes of twisted Fourier-Mukai partners of order κ of (X,α) determine m' distinct period points $y_k \in \mathcal{D}_d^{\text{mar}}$ by Lemma 3.2.2. Arguing as in the untwisted case, the proof follows from Proposition 2.2.1, Theorem 1.1 of [63] and Remark 2.4.2. □

Remark 3.2.3. In analogy to the untwisted case, the identification between Hodge structures on the generalized trascendental lattice of twisted Fourier-Mukai partners of (X,α) with fixed order of the Brauer class and certain Hodge structures in $\mathcal{D}'_d$ depends on the choice of the isometry j, or equivalently of a Fourier-Mukai partner of (X,α). However, the number of Fourier-Mukai partners constructed in Theorem 1.0.2 does not depend on this choice.

Remark 3.2.4. Notice that it is necessary to assume that $\text{ord}(\alpha) = \text{ord}(\alpha')$, in order to extend the isometry η to the trascendental lattices. Indeed, if this condition is not satisfied, then the discriminant groups of T_X and $T_{X'}$ could not be isomorphic. Actually, we can prove that Lemma 3.2.2 does not hold in general without this assumption, by giving a counterexample in the untwisted case.

We set $d = 2 \cdot 13^2$, which is congruent to 2 modulo 6 and let Y be a very general cubic fourfold in $\mathcal{C}_d$. Since d satisfies condition **(a)**, there exists a K3 surface X, which is associated to Y. By the counting formula of Theorem 2.5.1, the K3 surface X admits $2^0 = 1$ isomorphism class of Fourier-Mukai partners. On the other hand, by [67], Proposition 5.1, there exist $\varphi(13) \cdot 2^{-1} = 6$ isomorphism classes of Fourier-Mukai partners of order 13 of X. We denote by (X',α') one of them. Assume that there is a cubic fourfold $Y' \in \mathcal{C}_d$ such that $\tilde{H}(\text{Ku}(Y'),\mathbb{Z}) \cong \tilde{H}(X',\alpha',\mathbb{Z})$. By Remark 2.4.2, we have that Y' is a Fourier-Mukai partner of Y. On the other hand, by the counting formula of Theorem 3.1.4, every Fourier-Mukai partner of Y is isomorphic to Y; it follows that $Y \cong Y'$. On the other hand, the K3 surfaces X and (X',α') cannot clearly be isomorphic.

This prevents us to have a well-defined map between $\mathcal{D}_d^{\text{mar}}$ and the period space of generalized Calabi-Yau structures of hyperkähler type (see [43] for the definition), and to generalize Theorem 5.3.2 and 5.3.3 of [39] to the twisted case.

3.2.2 Ma's formula in our setting

The aim of this paragraph is to prove that if we consider a very general cubic fourfold Y in $\mathcal{C}_d$ satisfying condition **(a')** and **(b)**, then formula (2.4) gives precisely the number of elements in the set $\text{FM}^r(X,\alpha)$, where (X, α) is a twisted K3 surface associated to Y. The key point of the proof is the fact that the Caldărăru Conjecture 2.5.2 holds in this particular case.

Proposition 3.2.5. *Let (X, α) be a twisted K3 surface such that there exist a special cubic fourfold Y of discriminant d and a Hodge isometry $\tilde{H}(X, \alpha, \mathbb{Z}) \cong \tilde{H}(\mathrm{Ku}(Y), \mathbb{Z})$. If X has $\mathrm{rk}(\mathrm{NS}(X)) = 1$, and $9 \nmid d$, then the number of (isomorphism classes of) Fourier-Mukai partners of (X, α) of order r is given by formula* (2.4).

Proof. Firstly, we observe that the Căldăraru's Conjecture 2.5.2 holds under our assumptions for every Fourier-Mukai partner (X_1, α_1) of (X, α). More precisely, we prove that if a K3 surface X_1' has the trascendental lattice $T_{X_1'}$ Hodge isometric to T_{X_1} via g_1, then the twisted K3 surface $(X_1', \alpha_1' := g_1^* \alpha_1)$ is a Fourier-Mukai partner of (X_1, α_1). Indeed, the isometry g_1 restricts to the isometry of Hodge structures

$$f := (g_1)|_{T(X_1', \alpha_1')} : T(X_1', \alpha_1') \cong T(X_1, \alpha_1).$$

Notice that there exists a Hodge isometry $T(X, \alpha) \cong T(X_1, \alpha_1)$; therefore, the discriminant group $d(T(X_1, \alpha_1))$ is cyclic. Thus, by Theorem 0.2.9 ([76], Theorem 1.14.1), the isometry f extends to an isometry of Hodge structures

$$\phi_1 : \tilde{H}(X_1', \alpha_1', \mathbb{Z}) \cong \tilde{H}(X_1, \alpha_1, \mathbb{Z}).$$

By [47], Lemma 2.3, we know that every Hodge structure on $\tilde{\Lambda}$ determined by a point in $\mathcal{D}'$ admits a Hodge isometry that reverses any given orientation of the four positive directions. As a consequence, up to composing with this isometry, we can assume that ϕ_1 is orientation-preserving: by [50], Theorem 0.1, we conclude that there exists an equivalence of categories $\mathrm{D^b}(X_1', \alpha_1') \xrightarrow{\sim} \mathrm{D^b}(X_1, \alpha_1)$. In particular, we obtain that the map ν of (2.3) is bijective.

To conclude the proof, we show that the map μ of (2.2) has image $\mathrm{O_{Hdg}}(T(X, \alpha)) \backslash J^r(d(T(X, \alpha)))$; in particular, this implies that we have an equality in formula (2.4).

Let x be in $J^r(d(T(X, \alpha)))$; by definition, x is an element of $I^r(d(T(X, \alpha)))$ such that there exists an embedding

$$\varphi : U \to \tilde{M}_x,$$

where

$$\tilde{M}_x := \langle \lambda(x), N(X, \alpha) \rangle \subset N(X, \alpha)^\vee$$

is an overlattice of $N(X, \alpha)$. By [76], Proposition 1.4.1, we have that

$$d(\tilde{M}_x) \cong \langle \lambda(x) \rangle^\perp / \langle \lambda(x) \rangle \cong \langle x \rangle^\perp / \langle x \rangle \cong d(T_x).$$

Thus, by [76], Proposition 1.6.1, we have an embedding $\tilde{M}_x \oplus T_x \hookrightarrow \tilde{\Lambda}$, with $\tilde{M}_x$ and T_x both embedded primitively. We define the lattice

$$\Lambda_\varphi := \varphi(U)^\perp \cap \tilde{\Lambda},$$

which is isometric to the K3 lattice Λ, with the Hodge structure induced from T_x. By the surjectivity of the period map, there exist a K3 surface X_φ and a Hodge isometry

$$h : H^2(X_\varphi, \mathbb{Z}) \cong \Lambda_\varphi.$$

We denote by α_φ the composition $\alpha_x \circ h|_{T_{X_\varphi}}$; then, we obtain a twisted K3 surface $(X_\varphi, \alpha_\varphi)$.

Now, we observe that the map h induces the isometry

$$f : T(X_\varphi, \alpha_\varphi) = \ker \alpha_\varphi \cong \ker \alpha_x = T(X, \alpha).$$

Moreover, since $d(T(X, \alpha))$ is a cyclic group, applying [76], Theorem 1.14.4, we conclude that f extends to a Hodge isometry

$$\tilde{f} : \tilde{H}(X_\varphi, \alpha_\varphi, \mathbb{Z}) \cong \tilde{H}(X, \alpha, \mathbb{Z}).$$

By [47], Lemma 2.3, we can assume that $\tilde{f}$ is orientation-preserving. By [50], Theorem 0.1, we conclude that $(X_\varphi, \alpha_\varphi)$ belongs to $\mathrm{FM}^r(X, \alpha)$. By construction, we have that $\mu([(X_\varphi, \alpha_\varphi)]) = [x]$.

Finally, we observe that if x and x' in $J^r(d(T(X, \alpha)))$ are in the same orbit for the action of $\mathrm{O}_{\mathrm{Hdg}}(T(X, \alpha))$, then the twisted K3 surfaces $(X_\varphi, \alpha_\varphi)$ and $(X'_\varphi, \alpha'_\varphi)$, such that $\mu([(X_\varphi, \alpha_\varphi)]) = [x]$ and $\mu([(X'_\varphi, \alpha'_\varphi)]) = [x']$, are $\sim$-equivalent. Indeed, by hypothesis, there exists a Hodge isometry η of $T(X, \alpha)$ which induces an isomorphism $\bar{\eta}$ on $d(T(X, \alpha))$ such that $\bar{\eta}(x) = x'$. Then, by Proposition 0.2.6 ([76], Proposition 1.4.2), the overlattices $\langle x, T(X, \alpha)\rangle \cong T_{X_\varphi}$ and $\langle x', T(X, \alpha)\rangle \cong T_{X'_\varphi}$ are isomorphic. Moreover, this isomorphism sends α_φ to α'_φ, because it is an extension of η; this observation completes the proof of the proposition. $\square$

3.2.3 Application of Proposition 3.2.5

Let Y be a very general special cubic fourfold of discriminant d satisfying conditions **(a')** and **(b)**. By Proposition 3.2.5, we have that the number of isomorphism classes of Fourier-Mukai partners of order κ of (X, α) is

$$m' = \sum_x \left\{ \sum_M \tau(x, M) + \varepsilon(r) \sum_{M'} \tau(x, M') \right\}.$$

Let us write m' in a more explicit way, in order to find numerical conditions on d and κ, which guarantee the existence of non-isomorphic Fourier-Mukai partners for Y. We consider only the case $\kappa \geqslant 2$, because we have already treated the untwisted case in Section 4.1. Let c be the degree of the polarization class on X. Notice that $d = \kappa^2 c$ (see [47], Lemma 2.13).

Lemma 3.2.6. *Let g be a generator of the cyclic group $d(T(X, \alpha))$ of order d. Then*

$$I^\kappa(d(T(X, \alpha))) = \{(a\kappa c)g : a \in (\mathbb{Z}/\kappa\mathbb{Z})^\times\}.$$

Proof. We observe that every element of the form $x = (a\kappa c)g$ with $a \in (\mathbb{Z}/\kappa\mathbb{Z})^\times$ belongs to the set $I^\kappa(d(T(X, \alpha)))$. Indeed, let g be a generator of $d(T(X, \alpha))$ as in Proposition 2.1.2. An easy computation shows that $q_{T(X,\alpha)}((a\kappa c)g) \in 2\mathbb{Z}$ and that $(a\kappa c)g$ has order κ. On the other hand, the elements of $I^\kappa(d(T(X, \alpha)))$ are all the possible generators of the unique subgroup of order κ of $d(T(X, \alpha)) \cong \mathbb{Z}/d\mathbb{Z}$. $\square$

For every $x = (a\kappa c)g$ in $I^\kappa(d(T(X, \alpha)))$, we set

$$\tilde{M}_x := \langle \lambda(x), N(X, \alpha)\rangle \quad \text{and} \quad H_x := \frac{\tilde{M}_x}{N(X, \alpha)}.$$

We point out that

$$J^\kappa(d(T(X, \alpha))) = \{x \in I^\kappa(d(T(X, \alpha))) : \tilde{M}_x \cong U \oplus \mathbb{Z}\, l \text{ with } l^2 = c\}.$$

Indeed, given $x \in J^\kappa(d(T(X, \alpha)))$, let (X_x, α_x) be the twisted K3 surface such that $\mu([(X_x, \alpha_x)]) = [x]$ (which exists because μ is surjective as showed in the proof of Proposition 3.2.5). Then, by definition, we have that

$$N(X_x) \cong \langle \lambda(x), N(X, \alpha)\rangle \quad \text{and} \quad T_{X_x} \cong \langle x, T(X, \alpha)\rangle.$$

Since $T(X_x, \alpha_x) \cong T(X, \alpha)$, we have that

$$d = |d(T(X_x, \alpha_x))| = \mathrm{ord}(\alpha_x)^2 |d(T_{X_x})| = \kappa^2 |d(T_{X_x})|,$$

which implies that

$$d(\tilde{M}_x) \cong d(T_{X_x}) \cong \mathbb{Z}/c\mathbb{Z}.$$

On the other hand, the opposite inclusion follows from the definition of $J^\kappa(d(T(X, \alpha)))$.

Lemma 3.2.7. *Every element x in $I^\kappa(d(T(X,\alpha)))$ belongs to $J^\kappa(d(T(X,\alpha)))$.*

Proof. Let $\bar{x} = (\bar{a}\kappa c)g$ be the image via μ of the isomorphism class of the K3 surface (X,α), with $\bar{a}$ in $(\mathbb{Z}/\kappa\mathbb{Z})^\times$. By definition, we have that

$$U \oplus \mathbb{Z}\, l \cong N(X) \cong \langle \lambda(\bar{x}), N(X,\alpha)\rangle,$$

with $l^2 = c$; in particular, the lattice $U \oplus \mathbb{Z}\, l$ is an overlattice of $N(X,\alpha)$. Let $x = (a\kappa c)g$ be an element in $I^\kappa(d(T(X,\alpha)))$. Since the groups H_x and $H_{\bar{x}}$ are cyclic subgroups of $d(N(X,\alpha))$ of the same order, they are the same subgroup. By [76], Theorem 1.4.1, we conclude that the overlattices $U \oplus \mathbb{Z}\, l$ and $\tilde{M}_x$ are isomorphic. In particular, the element x is in $J^\kappa(d(T(X,\alpha)))$. □

Proposition 3.2.8. *We have that*

$$m' := \#\mathrm{FM}^\kappa(X,\alpha) = \begin{cases} \varphi(\kappa)2^{h-2} & \textit{if } \kappa > 2 \textit{ and } c = 2 \\ \varphi(\kappa)2^{h-1} & \textit{if } \kappa = 2 \textit{ or } c > 2, \end{cases}$$

where h is the number of distinct prime factors in the prime factorization of $c/2$ if $c > 2$, and $h = 1$ if $c = 2$.

Proof. We observe that the lemmas of this subsection and the fact that $\mathrm{O}_{\mathrm{Hdg}}(T(X,\alpha)) = \{\pm\mathrm{id}\}$ imply that

$$\#(\mathrm{O}_{\mathrm{Hdg}}(T(X,\alpha)))\backslash J^\kappa(d(T(X,\alpha))) = \begin{cases} 1 & \text{if } \kappa = 2, \\ \frac{1}{2}\varphi(\kappa) & \text{if } \kappa > 2 \end{cases}$$

where φ denotes the Euler function. On the other hand, as we have already observed, the only lattice M_φ such that $\tilde{M}_x \cong U \oplus M_\varphi$ is $\mathbb{Z}\, l$ with $l^2 = c$. Thus, our computation is actually the same used in [67], to prove Proposition 5.1. Indeed, we have that

$$\mathcal{G}(\mathbb{Z}\, l) = \{\mathbb{Z}\, l\} = \begin{cases} \mathcal{G}_1(\mathbb{Z}\, l) & \text{if } c = 2, \\ \mathcal{G}_2(\mathbb{Z}\, l) & \text{if } c > 2. \end{cases}$$

Moreover, we notice that

$$\mathrm{O}(\mathbb{Z}\, l) = \{\pm\mathrm{id}\} \quad \text{and} \quad \mathrm{O}(d(\mathbb{Z}\, l)) = \begin{cases} \{\mathrm{id}\} & \text{if } c = 2, \\ \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^h & \text{if } c > 2. \end{cases}$$

In particular, the order of the set $\mathrm{O}(d(\mathbb{Z}\, l))$ is 2^h if $c > 2$. Finally, we observe that

$$\mathrm{O}_{\mathrm{Hdg}}(T_x, \alpha_x) = \begin{cases} \{\pm\mathrm{id}\} & \text{if } \kappa = 2, \\ \{\mathrm{id}\} & \text{if } \kappa > 2. \end{cases}$$

So, if $\kappa > 2$, then

$$m' = \begin{cases} \frac{1}{2}\varphi(\kappa) & \text{if } c = 2, \\ \frac{1}{2}\varphi(\kappa)2^h = \varphi(\kappa)2^{h-1} & \text{if } c > 2. \end{cases}$$

Otherwise, if $\kappa = 2$, then

$$m' = \begin{cases} 1 & \text{if } c = 2, \\ 2^{h-1} & \text{if } c > 2, \end{cases}$$

as we claimed. □

By Proposition 3.2.5 and Proposition 3.2.8, we have that the lower bound given by Theorem 1.0.2 is explicitely determined. In particular, it is easy to construct examples of very general twisted K3 surfaces and, consequently, of very general cubic fourfolds with an arbitrary big number of non-isomorphic Fourier-Mukai partners.

Example 3.2.9. Let us take $d = 50$, which satisfies condition (**a'**) and **(b)**. A cubic fourfold in $\mathcal{C}_{50}$ has a twisted associated K3 surface with Brauer class of order $\kappa = 5$. By Theorem 1.0.2 and Proposition 3.2.8, the very general element in $\mathcal{C}_{50}$ admits at least $\varphi(5)/2 = 4/2 = 2$ (isomorphim classes of) Fourier-Mukai partners.

Remark 3.2.10. A natural question is whether it is possible to count Fourier-Mukai partners for a very general special cubic fourfold $Y \in \mathcal{C}_d$ without associated (twisted) K3 surface. Following the proof in [81] for the case of K3 surfaces, one could reduce the problem to counting the number of overlattices $A \cong A_2^{\perp}$ of $S \oplus T$, where $S = \mathbb{Z}\, \upsilon_Y$ and $T = T(\mathrm{Ku}(Y))$ are primitively embedded in A. However, to argue in this way, we need that an isometry between the generalized trascendental lattices lifts to an isometry of the Mukai lattices. This holds if $9 \nmid d$, as we have already explained in Section 3.2.1. The second issue is that $A_2^{\perp}$ is not unimodular; for this reason, the computation performed in [81], Lemma 4.5 cannot be performed in the same way.

Chapter 4

Rational curves of low degree on cubic fourfolds

The aim of this chapter is to prove Theorem 1.0.3, Theorem 1.0.4 and to discuss some applications. This is the content of [65] which is a joint work with Chunyi Li and Xiaolei Zhao.

4.1 Stability conditions on Ku(Y)

In this section we review the construction of stability conditions on $\mathrm{Ku}(Y)$ following [7]. The new contribution is given by Proposition 4.1.5, where we prove that this construction does not depend on the line fixed at the very beginning, and Lemma 4.1.6, which is useful to characterize weak semistable objects with discriminant zero and negative rank by their Chern character.

4.1.1 Summary of the construction and line-change trick

Let us firstly recall the construction of Bridgeland stability conditions on $\mathrm{Ku}(Y)$ introduced in [7] by Bayer, Lahoz, Macrì and Stellari. The key idea of their strategy is to embed the Kuznetsov component into a "three dimensional" category, where it is easier to define weak stability conditions by tilting, as explained in Section 0.4.2. More concretely, let us fix a line $L \subset Y$ which is not contained in a plane in Y, and we denote by

$$\sigma : \tilde{Y} \to Y$$

the blow-up of L in Y. The projection from L to a disjoint $\mathbb{P}^3$ equips $\tilde{Y}$ with a natural conic fibration structure

$$\pi : \tilde{Y} \to \mathbb{P}^3.$$

In particular, we have an associated sheaf of Clifford algebras over $\mathbb{P}^3$, whose even part (resp. odd part) is denoted by $\mathcal{B}_0$ (resp. $\mathcal{B}_1$). Let h be the hyperplane class on $\mathbb{P}^3$ and we use the same notation for its pullback to $\tilde{Y}$. We consider the $\mathcal{B}_0$-bimodules

$$\mathcal{B}_{2j} := \mathcal{B}_0 \otimes \mathcal{O}_{\mathbb{P}^3}(jh) \quad \text{and} \quad \mathcal{B}_{2j+1} := \mathcal{B}_1 \otimes \mathcal{O}_{\mathbb{P}^3}(jh) \quad \text{for } j \in \mathbb{Z}.$$

As recalled in Example 0.1.32, by Kuznetsov's work, we have a semiorthogonal decomposition of $\mathrm{D^b}(\tilde{Y})$ with a component given by the essential image of $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$ via a fully faithful functor Φ. On the other hand, we can apply Orlov's blow-up formula reviewed in Example 0.1.31, in order to get a semiorthgonal decomposition of $\mathrm{D^b}(\tilde{Y})$ with a copy of the $\mathrm{D^b}(Y)$ and three copies of the exceptional divisor. Starting from these decompositions, in [7], Proposition 7.7, they proved that there is a semiorthogonal decomposition of the form

$$\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0) = \langle \Psi(\sigma^* \mathrm{Ku}(Y)), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle, \tag{4.1}$$

where $\Psi : \mathrm{D}^{\mathrm{b}}(\tilde{Y}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ is a fully faithful functor defined by

$$\Psi(-) = \pi_*(- \otimes \mathcal{O}_{\tilde{Y}}(h) \otimes \mathcal{E}[1]).$$

Here $\mathcal{E}$ is a sheaf of right $\pi^*\mathcal{B}_0$-modules on $\tilde{Y}$, constructed in Section 7 of [7]. Denote by

$$\mathrm{Forg} : \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$$

the forgetful functor; it is known that $\mathrm{Forg}(\mathcal{E})$ is a vector bundle of rank 2.

Now the first step is to construct weak stability conditions on the derived category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0))$, where $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ is the category of coherent sheaves on $\mathbb{P}^3$ with a right $\mathcal{B}_0$-modules structure. It turns out that, in order to obtain a suitable Bogomolov inequality for $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, it is necessary to modify the usual Chern character. More precisely, for $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, the modified Chern character is defined as

$$\mathrm{ch}_{\mathcal{B}_0}(\mathcal{F}) = \mathrm{ch}(\mathrm{Forg}(\mathcal{F}))(1 - \frac{11}{32}l),$$

where l denotes the class of a line in $\mathbb{P}^3$. Notice that it differs from the usual Chern character from degree $\geqslant 2$. Moreover, the twisted Chern character is given by

$$\mathrm{ch}^{\beta}_{\mathcal{B}_0} = e^{-\beta h}\,\mathrm{ch}_{\mathcal{B}_0} = (\mathrm{rk}, \mathrm{ch}_{\mathcal{B}_0,1} - \mathrm{rk}\,\beta h, \mathrm{ch}_{\mathcal{B}_0,2} - \beta h \cdot \mathrm{ch}_{\mathcal{B}_0,1} + \mathrm{rk}\,\frac{\beta^2}{2}h^2, \dots).$$

In the next, we will identify the Chern characters on $\mathbb{P}^3$ with rational numbers.

One useful property of $\mathrm{ch}_{\mathcal{B}_0}$ is that its image lattice is spanned by the modified Chern characters of λ_1, λ_2 and $\mathrm{ch}_{\mathcal{B}_0,\leqslant 2}(\mathcal{B}_i)$ for $i = 1, 2, 3$. See the proof of Proposition 9.10 of [7] for details.

We denote by $\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathcal{B}_0)$ the heart of a bounded t-structure obtained by tilting $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to the slope stability at slope β. We consider the rank three lattice $\Lambda^2_{\mathcal{B}_0}$ defined as in Section 0.4.2 just using the Mdified Chern character. Furthermore, the discriminant can be defined as

$$\Delta_{\mathcal{B}_0}(\mathcal{F}) = (\mathrm{ch}_{\mathcal{B}_0,1}(\mathcal{F}))^2 - 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{ch}_{\mathcal{B}_0,2}(\mathcal{F}) = (\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F}))^2 - 2\,\mathrm{rk}(\mathcal{F})\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,2}(\mathcal{F}).$$

Having this notation, we can state the following result, which is the analogous of Proposition 0.4.14 in our noncommutative setting.

Proposition 4.1.1 ([7], Proposition 9.3)**.** *Given $\alpha > 0$ and $\beta \in \mathbb{R}$, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathcal{B}_0), Z_{\alpha,\beta})$ with*

$$Z_{\alpha,\beta}(\mathcal{F}) = i\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F}) + \frac{1}{2}\alpha^2\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,0}(\mathcal{F}) - \mathrm{ch}^{\beta}_{\mathcal{B}_0,2}(\mathcal{F})$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. The quadratic form can be given by the discriminant $\Delta_{\mathcal{B}_0}$. In particular, for a $\sigma_{\alpha,\beta}$-semistable object $\mathcal{F}$, we have

$$\Delta_{\mathcal{B}_0}(\mathcal{F}) \geqslant 0.$$

Remark 4.1.2. We observe that the last part of Proposition 4.1.1 follows easily from [7], Theorem 8.3 arguing as in [11], Section 3.

We recall that when $\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F}) \neq 0$, the slope of $\mathcal{F}$ associated to $\sigma_{\alpha,\beta}$ is defined as

$$\mu_{\alpha,\beta}(\mathcal{F}) = \frac{-\Re(Z_{\alpha,\beta}(\mathcal{F}))}{\Im(Z_{\alpha,\beta}(\mathcal{F}))} = \frac{\mathrm{ch}^{\beta}_{\mathcal{B}_0,2}(\mathcal{F}) - \frac{1}{2}\alpha^2\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,0}(\mathcal{F})}{\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(\mathcal{F})}.$$

Remark 4.1.3. We can represent these weak stability conditions as explained in Section 0.4.3. The difference is that in this case we prefer to work with the homogeneous coordinates $[\mathrm{ch}^{-1}_{\mathcal{B}_0,0} : \mathrm{ch}^{-1}_{\mathcal{B}_0,1} : \mathrm{ch}^{-1}_{\mathcal{B}_0,2}]$ on $\mathbb{P}(\Lambda^2_{\mathcal{B}_0})$ and the corresponding affine coordinates on $\mathbb{A}^2_{\mathcal{B}_0} := \mathbb{P}(\Lambda^2_{\mathcal{B}_0})\backslash\{\mathrm{ch}_0 = 0\}$. Thus, a weak stability condition defined by the point $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ is identified with the weak stability condition corresponding to the point $(\beta + 1, \frac{\alpha^2+(\beta+1)^2}{2})$ above the parabola $\Delta_{\mathcal{B}_0}$ in $\mathbb{A}^2_{\mathcal{B}_0}$. We will use this description in the next sections.

The second step is to induce stability conditions on $\mathrm{Ku}(Y)$ from the weak stability conditions on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. We only sketch this part as details will not be used. We fix $\alpha < \frac{1}{4}$ and $\beta = -1$, and we consider the tilting of $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to $\mu_{\alpha,\beta} = 0$. This new heart is denoted by $\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0)$. Note that $\mathrm{Ku}(Y)$ embeds into $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. As shown in [7], Section 9, the pair

$$\sigma_\alpha = (\mathrm{Coh}^0_{\alpha,-1}(\mathbb{P}^3, \mathcal{B}_0) \cap \mathrm{Ku}(Y), -iZ_{\alpha,-1}) \tag{4.2}$$

defines a Bridgeland stability condition on $\mathrm{Ku}(Y)$.

Remark 4.1.4. Notice that these stability conditions are constructed with respect to the Néron-Severi lattice $N(\mathrm{Ku}(Y))$ by [7], Proposition 9.10. More precisely, consider the factorization $\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) \twoheadrightarrow N(\mathrm{Ku}(Y)) \overset{u}{\twoheadrightarrow} \Lambda^2_{\mathcal{B}_0}$. We define the element $\eta(\sigma) \in N(\mathrm{Ku}(Y))$ such that $Z(u(-)) = (\eta(\sigma), -)$. We denote by $\mathcal{P} \subset N(\mathrm{Ku}(Y))_{\mathbb{C}}$ the open subset consisting of those vectors whose real and imaginary parts span positive-definite two-planes in $N(\mathrm{Ku}(Y))_{\mathbb{R}}$. Then $\eta(\sigma)$ is in $(A_2)_{\mathbb{C}} \cap \mathcal{P}$.

One subtle issue is that the Clifford structure and the embedding of $\mathrm{Ku}(Y)$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ depend on the choice of the line L to blow up. However, for the induced stability conditions on the Kuznetsov component, we are able to prove the following result.

Proposition 4.1.5. *For a fixed $\alpha > 0$, the induced stability condition σ_α defined in* (4.2) *is independent of the choice of L.*

Proof. For simplicity, we denote the stability condition by the pair

$$\sigma_L = (\mathcal{A}_L, Z_L).$$

The central charge Z_L factors via $\mathrm{ch}^{\beta}_{\mathcal{B}_0}$, which is independent of the choice of L. We need to show that the heart $\mathcal{A}_L$ is constant.

Let F_Y be the Fano variety of lines on Y. If Y contains a plane, then we remove the set of lines over this plane from the Fano variety and we denote it by F_Y to simplify the notation. It is shown in [8] that σ_L is a family of stability conditions over F_Y, satisfying the openness of heart property. In particular, if an object $\mathcal{F}$ is σ_{L_0}-semistable for a line $L_0 \in F_Y$, then there exists an open set $U_0 \subset F_Y$, such that $\mathcal{F}$ is σ_L-semistable for any line $L \in U_0$.

Now we show that in our case, this implies that $\mathcal{F}$ is σ_L-semistable for any $L \in F_Y$. If not, assume that there exists a line L_1 such that $\mathcal{F}$ is not σ_{L_1}-semistable. Then we consider the Harder-Narasimhan filtration of $\mathcal{F}$ with respect to the slicing of σ_{L_1}:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}.$$

By our assumption, $\mathcal{F}_1$ is σ_{L_1}-semistable, and its phase satisfies $\phi(\mathcal{F}_1) > \phi(\mathcal{F})$.

Using the openness of heart property again, we know that there exists an open set $U_1 \subset F_Y$, such that for any $L \in U_1$, $\mathcal{F}_1$ is σ_L-semistable. In particular, if we take a line $L \in U_0 \cap U_1$, then $\mathcal{F}$ and $\mathcal{F}_1$ are both σ_L-semistable. Since the central charge is independent of L, we still have $\phi(\mathcal{F}_1) > \phi(\mathcal{F})$. On the other hand, by our construction there is a non-trivial morphism $\mathcal{F}_1 \to \mathcal{F}$, giving a contradiction (see (3) in Section 0.4.1). This concludes the proof of the statement. $\square$

4.1.2 Stable objects of discriminant zero

The following general lemma will be crucial in order to study the destabilizing objects by their Chern characters. The basic idea is that a stable object E of zero discriminant and negative rank has to be stable with respect to any weak stability condition $\sigma_{\alpha,\beta}$. Then, comparing the slopes of E and $\mathcal{B}_i$ with respect to different stability conditions, we get strong restrictions on $\mathrm{Hom}(\mathcal{B}_i, E[j])$, which can be used to show that $E = \mathcal{B}_i^{\oplus n}[1]$. We suggest to keep in mind the representation of weak stability conditions explained in Remark 4.1.3.

Lemma 4.1.6. *Let E be a $\sigma_{\alpha_0,\beta_0}$-semistable object in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3, \mathcal{B}_0)$ for some $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$. Assume that $\Delta_{\mathcal{B}_0}(E) = 0$ and $\mathrm{rk}(E) < 0$. Then*

$$E = \mathcal{B}_i^{\oplus n}[1] \quad \textit{for some } i \in \mathbb{Z} \textit{ and } n \in \mathbb{N}.$$

Proof. In order to simplify the notation, we set

$$\mu_E = \frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(E)}{\mathrm{rk}(E)}.$$

As we will compare the slopes of E with $\mathcal{B}_i$, it is helpful to keep in mind that

$$\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(\mathcal{B}_i)}{\mathrm{rk}(\mathcal{B}_i)} = \frac{i}{2} - \frac{1}{4}.$$

Without loss of generality, by considering $E \otimes_{\mathcal{B}_0} \mathcal{B}_k$ for suitable $k \in \mathbb{Z}$, we may assume that

$$\mu_E \in [-\frac{1}{4}, \frac{1}{4}).$$

By choosing a stable factor of E, we may first assume that E is actually $\sigma_{\alpha_0,\beta_0}$-stable. By [11], Lemma 3.9, when $\beta > \mu_E - 1$, the object E is in $\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathcal{B}_0)$ and can become strictly semistable only when each stable factor E_i satisfies $\Delta_{\mathcal{B}_0}(E_i) < \Delta_{\mathcal{B}_0}(E) = 0$, which is not possible. Therefore, we deduce that E is $\sigma_{\alpha,\beta}$-stable for $\beta > \mu_E - 1$. In particular, we have that E is σ_{0+,β_1} stable for

$$\mu_E < \beta_1 + 1 < \frac{1}{4}.$$

Since $\mathrm{rk}(E) < 0$, we have

$$\mu_{0+,\beta_1}(\mathcal{B}_{-2}[1]) < \mu_{0+,\beta_1}(E) < \mu_{0+,\beta_1}(\mathcal{B}_1).$$

Here and in the following, the notation μ_{0+,β_i} means that it is possible to find suitable values of $\alpha > 0$, realizing the relations between the slopes (use the property in (4) in Section 0.4.1). By comparing the slope using (3) in Section 0.4.1 and applying Serre duality, it follows that

$$\mathrm{Hom}(\mathcal{B}_1, E[j]) = 0$$

for $j \neq 1$. Therefore, $\chi(\mathcal{B}_1, E) \leqslant 0$.

Now we study the vertical wall. Suppose that E is strictly semistable when $\beta_2 = \mu_E - 1$. Then each stable factor E_i satisfies one of the two conditions:

$$\mathrm{rk}(E_i) < 0 \text{ or } \mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_i) = (0,0,0).$$

We study these two cases separately. Given a stable factor E_i with negative rank, by [11], Lemma 3.9, we have that $E_i[-1]$ is in the heart $\mathrm{Coh}^\beta(\mathbb{P}^3, \mathcal{B}_0)$ and it is $\sigma_{\alpha,\beta}$-stable for any $\beta + 1 < \mu_E$. In particular, $E_i[-1]$ is σ_{0+,β_3}-stable for

$$-\frac{3}{4} < \beta_3 + 1 < \mu_E.$$

Since $\mathrm{rk}(E_i) < 0$, we have

$$\mu_{0+,\beta_3}(\mathcal{B}_{-2}[1]) < \mu_{0+,\beta_3}(E_i[-1]) < \mu_{0+,\beta_3}(\mathcal{B}_1).$$

As a consequence, we get

$$\mathrm{Hom}(\mathcal{B}_1, E_i[1]) = \mathrm{Hom}(E_i[-1], \mathcal{B}_{-2}[1])^* = 0.$$

Since E_i is also $\sigma_{\alpha,\beta}$-stable for $\beta > \mu_E - 1$ (by the same argument used for E), we deduce that $\mathrm{Hom}(\mathcal{B}_1, E_i[j]) = 0$ for any $j \in \mathbb{Z}$, i.e. $E_i \in \mathcal{B}_1^\perp$. In particular, $\chi(\mathcal{B}_1, E_i) = 0$.

In the second case, we show that such a torsion stable factor cannot exist. Assume that E_i is a stable factor with $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_i) = (0,0,0)$; note that

$$\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_1, E_i[j]) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathrm{Forg}(E_i \otimes_{\mathcal{B}_0} \mathcal{B}_{-1})[j]) = 0$$

if and only if $j \neq 0$. This implies that $\chi(\mathcal{B}_1, E_i) > 0$. Since $\chi(\mathcal{B}_1, E_i)$ is also non positive by the previous computation, we conclude that E_i has to be zero. Hence, we may assume that each stable factor E_i satisfies $\mathrm{rk}(E_i) < 0$.

Now we want to show that $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_i[-1]) = c\,\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(\mathcal{B}_0)$ for some positive integer c. It suffices to show that $\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(E_i)}{\mathrm{rk}(E_i)} = -\frac{1}{4}$. Assume not, we may consider the tilt stability condition $\sigma_{0+,\beta_1'}$ for some

$$\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(\mathcal{B}_0)}{\mathrm{rk}(\mathcal{B}_0)} < \beta_1' + 1 < \frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(E_i)}{\mathrm{rk}(E_i)}.$$

In this case, we have

$$\mu_{0+,\beta_1'}(\mathcal{B}_{-1}[1]) < \mu_{0+,\beta_1'}(\mathcal{B}_0[1]) < \mu_{0+,\beta_1'}(E_i[-1]) < \mu_{0+,\beta_1'}(\mathcal{B}_2) < \mu_{0+,\beta_1'}(\mathcal{B}_3)$$

and

$$\mu_{0+,\beta_1}(\mathcal{B}_{-1}[1]) < \mu_{0+,\beta_1}(\mathcal{B}_0[1]) < \mu_{0+,\beta_1}(E_i) < \mu_{0+,\beta_1}(\mathcal{B}_2) < \mu_{0+,\beta_1}(\mathcal{B}_3).$$

Hence

$$\mathrm{Hom}(\mathcal{B}_2, E_i[j]) = \mathrm{Hom}(\mathcal{B}_3, E_i[j]) = 0$$

for any $j \in \mathbb{Z}$. This shows that E_i belongs to $\Psi(\sigma^* \mathrm{Ku}(Y))$. In particular, the twisted Chern character of E_i satisfies

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_i) = a\lambda_1 + b\lambda_2$$

for some $(a,b) \neq (0,0)$. Note that any E_i with such truncated twisted Chern character satisfies $\Delta_{\mathcal{B}_0}(E_i) \geqslant 7$. This leads to a contradiction with the assumption that E has zero discriminant.

We may now assume that $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_i[-1]) = c\,\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(\mathcal{B}_0)$ for some positive integer c. Since

$$\mu_{0+,\beta_3}(\mathcal{B}_{-3}[1]) < \mu_{0+,\beta_3}(\mathcal{B}_{-1}[1]) < \mu_{0+,\beta_3}(E_i[-1]) < \mu_{0+,\beta_3}(\mathcal{B}_2)$$

and

$$\mu_{0+,\beta_1}(\mathcal{B}_{-1}[1]) < \mu_{0+,\beta_1}(E_i) < \mu_{0+,\beta_1}(\mathcal{B}_2),$$

we have the vanishing $\mathrm{Hom}(\mathcal{B}_2, E_i[j]) = 0$ for any $j \in \mathbb{Z}$, and $\mathrm{Hom}(\mathcal{B}_0, E_i[j]) = 0$ for any $j \neq 0$ or -1. Therefore, we have that

$$\begin{aligned}
0 = \chi(\mathcal{B}_2, E_i) &= \chi_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(E_i)(-h)) \\
&= \mathrm{ch}_3(\mathrm{Forg}(E_i)(-h)) + 2\,\mathrm{ch}_2(\mathrm{Forg}(E_i)(-h)) + \frac{11}{6}\,\mathrm{ch}_1(\mathrm{Forg}(E_i)(-h)) + \mathrm{rk}(E_i) \\
&= \chi_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(E_i)) - \mathrm{ch}_2(\mathrm{Forg}(E_i)) - \frac{3}{2}\,\mathrm{ch}_1(\mathrm{Forg}(E_i)) - \mathrm{rk}(E_i) \\
&= \chi_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(E_i)) - \mathrm{ch}^{-1}_{\mathcal{B}_0,2}(E_i) - \frac{1}{2}\,\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(E_i) - \frac{11}{32}\,\mathrm{rk}(E_i) \\
&= \chi_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(E_i)) - \frac{1}{32}\,\mathrm{rk}(E_i) + \frac{1}{8}\,\mathrm{rk}(E_i) - \frac{11}{32}\,\mathrm{rk}(E_i) \\
&> \chi_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(E_i)) = -\hom(\mathcal{B}_0, E_i[-1]) + \hom(\mathcal{B}_0, E_i).
\end{aligned}$$

In particular, it follows that $\mathrm{Hom}(\mathcal{B}_0, E_i[-1]) \neq 0$. As both $\mathcal{B}_0$ and $E_i[-1]$ are σ_{0+,β_3}-stable with the same slope, we must have $E_i = \mathcal{B}_0[1]$. Since this condition holds for every stable factor and $\mathrm{Ext}^1(\mathcal{B}_0, \mathcal{B}_0) = 0$, we deduce that $E = \mathcal{B}_0^{\oplus n}[1]$ as desired. □

4.2 LLSvS eightfold and stability

This section is devoted to the proof of Theorem 1.0.4. The strategy is the following. We consider the object in $\mathrm{D^b}(Y)$ which can be associated to a twisted cubic curve and we prove that its image in $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$ is stable with respect the weak stability condition $\sigma_{\alpha,-1}$ for α large. Then we compute the walls where the stability can change. At the first wall the object of a non CM twisted cubic curve becomes unstable, but its projection in $\mathrm{Ku}(Y)$ is stable. Then we prove that this new object remains stable and we relate the moduli space obtained in this way to the eightfold M_Y.

4.2.1 Twisted cubics and objects

Let Y be a smooth cubic fourfold not containing a plane. We will use the notation introduced in Example 0.3.10. As in [62], given a twisted cubic curve C contained in a cubic surface $S \subset Y$, we denote by F_C the kernel of the evaluation map

$$\mathrm{H}^0(Y, \mathcal{I}_{C/S}(2H)) \otimes \mathcal{O}_Y \twoheadrightarrow \mathcal{I}_{C/S}(2H),$$

where $\mathcal{I}_{C/S}$ is the ideal sheaf of C in S and H is the class of a hyperplane in Y. Let F'_C be the projection of F_C in the Kuznetsov category $\mathrm{Ku}(Y)$. Explicitly, as the projection is the composition of the mutations $\mathbb{R}_{\mathcal{O}_Y(-H)}\mathbb{L}_{\mathcal{O}_Y}\mathbb{L}_{\mathcal{O}_Y(H)}$ (whose definition is recalled in Section 0.1.4), it is possible to compute that

$$F'_C := \mathbb{R}_{\mathcal{O}_Y(-H)} F_C.$$

Indeed, by [62], Lemma 2.3, if C is an aCM twisted cubic curve, then F_C is in $\mathrm{Ku}(Y)$; in this case, F_C and F'_C are identified. If C is a non CM curve, by the definition of F'_C, we have the triangle

$$F'_C \to F_C \to \mathcal{O}_Y(-H)[1] \oplus \mathcal{O}_Y(-H)[2].$$

Using the notation introduced in the previous section, we set

$$E_C := \Psi(\sigma^* F_C) \quad \text{and} \quad E'_C := \Psi(\sigma^* F'_C);$$

by (4.1) we have that E'_C is in $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle^\perp$. Applying σ^* and Ψ, for a non CM curve C, we get the triangle

$$E'_C \to E_C \to \mathcal{B}_{-1}[1] \oplus \mathcal{B}_{-1}[2]; \tag{4.3}$$

here we have used [7], Proposition 7.7. In particular, we note that

$$\mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}(E'_C) = \mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}(E_C) = \mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}(\Psi\sigma^*(2\lambda_1 + \lambda_2)) = (0, 6, 0).$$

Now, we recall that, since Y does not contain a plane, the cubic surface S, which is cut out by the $\mathbb{P}^3$ spanned by C, is irreducible. We will assume that the line L, which is blown up in the cubic fourfold, is disjoint from this $\mathbb{P}^3$. For such a choice of L, the blow-up σ and the projection π map S isomorphically to a cubic surface S' in the base $\mathbb{P}^3$. In this section and in the next section, for a fixed twisted cubic C, we will work with such a line L. By Proposition 4.1.5, this will not change the stability condition induced on $\mathrm{Ku}(Y)$.

Let $\sigma_{\alpha,\beta}$ be the weak stability condition on $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$ introduced in Proposition 4.1.1. In the next proposition we prove that E_C is $\sigma_{\alpha,-1}$-stable for α large enough.

Proposition 4.2.1. *The torsion sheaf E_C on $\mathbb{P}^3$ is slope-stable. In particular, E_C is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$.*

Proof. We compute E_C with respect to L. Recall that $\pi_*\sigma^*\mathcal{O}_Y = 0$; hence, by definition of F_C and E_C, we have

$$E_C = \pi_*(\sigma^* F_C \otimes \mathcal{O}_{\tilde{Y}}(h) \otimes \mathcal{E}[1]) = \pi_*(\sigma^*\mathcal{I}_{C/S}(2H) \otimes \mathcal{O}_{\tilde{Y}}(h) \otimes \mathcal{E}),$$

where $\mathcal{E}$ is a vector bundle supported on $\tilde{Y}$. As a consequence, the sheaf E_C is torsion free, supported over the irreducible cubic surface S' in $\mathbb{P}^3$.

Note that $\mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}(E_C) = (0, 6, 0)$. Let F be a torsion sheaf destabilizing E_C. Then we have that F has the same support of E_C and it has rank one as a sheaf over S'. It follows that $\mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 1}(F) = (0, 3)$. However, such an object cannot exist in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$, because this character is not in the lattice spanned by the characters of λ_1, λ_2 and $\mathcal{B}_i$ for $i = 1, 2, 3$. It follows that E_C is slope-stable, in the sense that any proper $\mathcal{B}_0$-subsheaf of E_C has a smaller slope $\mathrm{ch}^{-1}_{\mathcal{B}_0, 1} / \mathrm{rk}$. Since for $\alpha \to \infty$, the weak stability $\sigma_{\alpha,-1}$ converges to the slope stability, we deduce the desired statement. □

4.2.2 Computation of the walls with respect to $\sigma_{\alpha,-1}$

Having the stability of E_C for α large from Proposition 4.2.1, we are now interested in computing explicitly the walls where the object could potentially become strictly semistable. In this section, we list the character $\mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}$ of all possible destabilizing objects of E_C and E'_C with respect to the weak stability conditions $\sigma_{\alpha,-1}$.

We recall that by [7], Remark 8.4, the rank of $\mathcal{B}_0$-modules on $\mathbb{P}^3$ is always a multiple of 4. Thus, we write the characters of the destabilizing subobjects and quotient objects as

$$(0, 6, 0) = (4a, b, \frac{c}{8}) + (-4a, 6 - b, -\frac{c}{8}) \tag{4.4}$$

for $a, b, c \in \mathbb{Z}$. These characters have to satisfy several additional conditions:

1. The two characters have non-negative discriminant $\Delta_{\mathcal{B}_0}$ as recalled in Proposition 4.1.1.

2. There exists $\alpha > 0$ such that the two characters have the same slope with respect to $\sigma_{\alpha,-1}$.

3. The two characters should be integral combinations of the characters of λ_1 and λ_2, and $\mathrm{ch}^{-1}_{\mathcal{B}_0, \leqslant 2}(\mathcal{B}_i)$ for $i = 1, 2, 3$.

4. The ordinary Chern character of objects in $\mathrm{D^b}(\mathbb{P}^3)$ truncated to degree 2 is represented by a triple $(R, C, D/2)$, where C and D are integers of the same parity. Thus, the two characters have the form

$$(R, C, \frac{D}{2})(1, 0, -\frac{11}{32})(1, 1, \frac{1}{2}) = (R, C + R, \frac{D}{2} + C - \frac{5}{16}R).$$

Using these conditions, by a standard computation we obtain the following result.

Proposition 4.2.2. *The possible solutions of* (4.4) *are:*

1. *for* $\alpha = 3/4$, $a = 1$, $b = 3$, $c = 9$;
2. *for* $\alpha = 1/4$,
   (a) $a = \pm 1$, $b = 1$, $c = \pm 1$;
   (b) $a = \pm 2$, $b = 2$, $c = \pm 2$;
   (c) $a = \pm 3$, $b = 3$, $c = \pm 3$;
   (d) $a = 1$, $b = 3$, $c = 1$;
3. *a very small value* $\bar{\alpha} \approx 1/9$.

Note that the stability condition σ_α is constructed from $\sigma_{\alpha,-1}$ with $\alpha < 1/4$. In the rest of this section, we will study the stability of E_C. We will firstly prove that if C is an aCM curve, then E_C remains stable with respect to $\sigma_{\alpha,-1}$ after the first wall. On the other hand, if C is non CM, then E_C is destabilized. In particular, we need to consider the mutation E'_C of E_C, which instead becomes stable. Then we prove that the second wall can be crossed without changing the stability of E'_C. The third wall also does not change the stability of E'_C; this fact can be directly proved without using specific information about the destabilizing objects.

4.2.3 First wall: $\alpha = \frac{3}{4}$

By Proposition 4.2.1 and Proposition 4.2.2, we have that E_C is $\sigma_{\alpha,-1}$-stable for $\alpha > 3/4$. In this section, we study the stability of E_C after the first wall.

Proposition 4.2.3. *For* $1/4 < \alpha < 3/4$, *we have that* E'_C *is* $\sigma_{\alpha,-1}$*-stable. More precisely:*

- *If* C *is an aCM twisted cubic curve in* Y, *then* $E'_C = E_C$ *is* $\sigma_{\alpha,-1}$*-stable.*
- *If* C *is a non CM cubic curve, then* E_C *becomes strictly* $\sigma_{\alpha,-1}$*-semistable at the wall* $\alpha = 3/4$. *Instead, for* $1/4 < \alpha < 3/4$, *the object* E'_C *is* $\sigma_{\alpha,-1}$*-stable.*

Proof. Let us consider the destabilizing quotient object given by Proposition 4.2.2 with

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2} = (-4, 3, -9/8).$$

By Lemma 4.1.6, we know that this object is $\mathcal{B}_{-1}[1]$. Since the Serre functor on $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$ is

$$S(-) = (-) \otimes_{\mathcal{B}_0} \mathcal{B}_{-3}[3],$$

by (4.1) we have that

$$\mathrm{Hom}_{\mathcal{B}_0}(E'_C, \mathcal{B}_{-1}[1]) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_2, E'_C[2])^\vee = 0.$$

The first claim follows easily from the fact that $E_C \cong E'_C$ in the aCM case.

Assume now that C is a non CM twisted cubic curve. Then using the sequence (4.3) and the fact that

$$\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_2, \mathcal{B}_{-1}[3]) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_{-1}, \mathcal{B}_{-1})^\vee \cong \mathbb{C},$$

we get

$$\operatorname{Hom}_{\mathcal{B}_0}(E_C, \mathcal{B}_{-1}[1]) \cong \mathbb{C}\,.$$

In particular, for $\alpha = 3/4$, it follows that E_C is strictly $\sigma_{\alpha,-1}$-semistable and the Jordan-Hölder filtration in $\operatorname{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ is given by

$$0 \to M_C \to E_C \to \mathcal{B}_{-1}[1] \to 0.$$

Finally, for $1/4 < \alpha < 3/4$, using again the sequence (4.3), it is easy to see that the new stable object is E'_C, which fits into the sequence

$$0 \to \mathcal{B}_{-1}[1] \to E'_C \to M_C \to 0.$$

□

4.2.4 Second wall: $\alpha = \frac{1}{4}$

The aim of this section is to prove the following result.

Proposition 4.2.4. *Let $0 < \alpha < 1/4$. If C is a twisted cubic curve in Y, then E'_C is $\sigma_{\alpha,-1}$-stable.*

This proposition is a consequence of Lemma 4.2.5 and Lemma 4.2.6 below.

We firstly consider the objects given by the second part of Proposition 4.2.2 and we show that they cannot destabilize E'_C. The key observation is that if E'_C is destabilized, then a slope comparison argument implies that its stable factors have to be in $\Psi(\sigma^* \operatorname{Ku}(Y))$. This will lead to a contradiction, as such stable factors do not exist for the wall $\alpha = 1/4$.

Lemma 4.2.5. *Let E be a $\sigma_{\frac{1}{4}+\epsilon,-1}$-stable object in $\Psi(\sigma^*\operatorname{Ku}(Y))$ with $\operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E) = (0, 6, 0)$. Then E is $\sigma_{\frac{1}{4}-\epsilon,-1}$-stable.*

Proof. Suppose that E is not $\sigma_{\frac{1}{4}-\epsilon,-1}$-stable; we consider the Harder-Narasimham filtration of E with respect to $\sigma_{\frac{1}{4}-\epsilon,-1}$:

$$0 \to E_1 \to \cdots \to E_k = E.$$

Here each factor E_{i+1}/E_i is $\sigma_{\frac{1}{4}-\epsilon,-1}$-semistable with strictly decreasing slopes.

Assume that $\operatorname{Hom}(E_k/E_{k-1}, \mathcal{B}_0[1]) \neq 0$. Note that E_k/E_{k-1} is a quotient object of E in the heart $\operatorname{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$. Since $\mathcal{B}_0[1]$ is also an object in $\operatorname{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, the assumption above implies

$$\operatorname{Hom}(E, \mathcal{B}_0[1]) \neq 0.$$

By Serre duality, we obtain

$$\operatorname{Hom}(\mathcal{B}_3, E[2]) = (\operatorname{Hom}(E, \mathcal{B}_0[1]))^* \neq 0,$$

which contradicts the condition that $E \in \Psi(\sigma^* \operatorname{Ku}(Y))$. Therefore, it follows that

$$\operatorname{Hom}(\mathcal{B}_3, E_k/E_{k-1}[2]) = 0.$$

By a similar argument, we get

$$\operatorname{Hom}(\mathcal{B}_1, E_{k-1}) = 0.$$

Note that we have the following inequalities:

$$\begin{aligned}
&\mu_{\frac{1}{4},-1}(\mathcal{B}_{-2}[1]) < \mu_{\frac{1}{4},-1}(\mathcal{B}_{-1}[1]) < \mu_{\frac{1}{4},-1}(E_k/E_{k-1}) = \\
&\mu_{\frac{1}{4},-1}(E_{k-1}) < \mu_{\frac{1}{4},-1}(\mathcal{B}_2) < \mu_{\frac{1}{4},-1}(\mathcal{B}_3);\\
&\mu_{\frac{1}{4}-\epsilon,-1}(E_k/E_{k-1}) < \mu_{\frac{1}{4}-\epsilon,-1}(\mathcal{B}_1);\\
&\mu_{\frac{1}{4}-\epsilon,-1}(\mathcal{B}_0[1]) < \mu_{\frac{1}{4}-\epsilon,-1}(E_i/E_{i-1}) \text{ for every } 1 \leqslant i < k.
\end{aligned}$$

We point out that both E_{k-1} and E_k/E_{k-1} are $\mu_{\frac{1}{4},-1}$-semistable, and each E_i/E_{i-1} is $\mu_{\frac{1}{4}-\epsilon,-1}$-semistable. By Serre duality, we have

$$\operatorname{Hom}(\mathcal{B}_s, E_k/E_{k-1}[j]) = \operatorname{Hom}(\mathcal{B}_s, E_{k-1}[j]) = 0,$$

for $s = 1, 2, 3$ and every $j \neq 1$. Since $E \in \Psi(\sigma^* \operatorname{Ku}(Y))$,

$$\chi(\mathcal{B}_s, E_k/E_{k-1}) + \chi(\mathcal{B}_s, E_{k-1}) = \chi(\mathcal{B}_s, E) = 0$$

for $s = 1, 2, 3$. Therefore,

$$\operatorname{Hom}(\mathcal{B}_s, E_k/E_{k-1}[1]) = \operatorname{Hom}(\mathcal{B}_s, E_{k-1}[1]) = 0,$$

for $s = 1, 2, 3$. In particular, we deduce that E_{k-1} and E_k/E_{k-1} are in $\Psi(\sigma^* \operatorname{Ku}(Y))$. As a consequence, the twisted Chern character of E_{k-1} satisfies

$$\begin{aligned} \operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_{k-1}) &= a \operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(\mathcal{B}_1) + b(0,6,0) \in \{(x, y, \frac{1}{32}x)\} \\ \text{and } \operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_{k-1}) &= c\lambda_1 + d\lambda_2 \in \{(x, y, -\frac{7}{32}x)\}. \end{aligned}$$

We conclude that $\operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_{k-1})$ must be of the form $(0, y, 0)$. However, it would destabilize E with respect to $\sigma_{\frac{1}{4}+\epsilon,-1}$, which is a contradiction. This proves the stability of E as in the statement. □

Now we consider the third wall in Proposition 4.2.2. In this case, we obtain a slightly general result, showing that for $\alpha < 1/4$, the only stable objects are in $\Psi(\sigma^* \operatorname{Ku}(Y))$ and they cannot be destabilized. The argument is similar to the proof of Lemma 4.2.5.

Lemma 4.2.6. *For $0 < \alpha_0 < \frac{1}{4}$, let E be a $\sigma_{\alpha_0,-1}$ stable object such that $[E] = [E'_C]$ in the numerical Grothendieck group. Then E is in $\Psi(\sigma^*\operatorname{Ku}(Y))$ and it is $\sigma_{\alpha,-1}$ stable for any $0 < \alpha \leqslant \alpha_0$.*

Proof. We set $\mu = \mu_{\alpha_0,-1}$ for simplicity. As $[E] = [E'_C]$ in the numerical Grothendieck group, we observe that

$$\mu(\mathcal{B}_{-2}[1]) < \mu(\mathcal{B}_{-1}[1]) < \mu(\mathcal{B}_0[1]) < \mu(E) < \mu(\mathcal{B}_1) < \mu(\mathcal{B}_2) < \mu(\mathcal{B}_3).$$

By Serre duality we have that

$$\operatorname{Hom}(\mathcal{B}_s, E[j]) = 0$$

for any $s = 1, 2, 3$ and $j \neq 1$. Again, since $[E] = [E'_C]$ in the numerical Grothendieck group, we have

$$\chi(\mathcal{B}_s, E) = \chi(\mathcal{B}_s, E'_C) = 0$$

for $s = 1, 2, 3$. It follows that

$$\operatorname{Hom}(\mathcal{B}_s, E[1]) = 0$$

for any $s = 1, 2, 3$, proving that E belongs to $\Psi(\sigma^* \operatorname{Ku}(Y))$.

Suppose that E becomes strictly $\sigma_{\alpha,-1}$-semistable for some $\alpha < \alpha_0 < \frac{1}{4}$. We may consider the Harder-Narasimhan filtration of E with respect to $\sigma_{\alpha-\epsilon,-1}$:

$$0 \subset E_1 \subset \cdots \subset E_k = E.$$

By comparing $\mu_{\alpha-\epsilon,-1}$ of E_k/E_{k-1}, E_{k-1}, $\mathcal{B}_{-2}[1]$, $\mathcal{B}_{-1}[1]$, $\mathcal{B}_0[1]$, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$, using the same argument applied in the proof of Lemma 4.2.5, we get the conclusion that both E_k/E_{k-1} and E_{k-1} are in $\Psi(\sigma^* \operatorname{Ku}(Y))$. But this implies that

$$\{(x, y, \frac{1}{2}\alpha^2 x)\} \ni \operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_{k-1}) = a\lambda_1 + b\lambda_2 \in \{(x, y, -\frac{7}{32}x)\}.$$

Hence, we must have $\operatorname{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_{k-1}) = (0, y, 0)$, which leads to a contradiction. This proves the stability of E as we wanted. □

4.2.5 Stability after the second tilt and the moduli space

This section is devoted to the proof of Theorem 1.0.4. Firstly, we show that E'_C is $\sigma^0_{\alpha,-1}$-stable, where $\sigma^0_{\alpha,-1}$ is the weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ obtained by tilting $\sigma_{\alpha,-1}$ (see [7], the proof of Theorem 1.2). In particular, this implies the stability of F'_C with respect to the stability condition $\sigma := \sigma_\alpha$ on $\mathrm{Ku}(Y)$, defined in (4.2) and constructed in [7].

Theorem 4.2.7. *Let Y be a smooth cubic fourfold not containing a plane. If C is a twisted cubic curve on Y, then the object F'_C is σ-stable, with respect to $\sigma := \sigma_\alpha$ given in* (4.2)*.*

Proof. Note that by definition the stability function for $\sigma^0_{\alpha,-1}$ is $Z_{\alpha,-1}$ multiplied by $-\sqrt{-1}$. In particular, the new heart obtained through the second tilt is just the previous heart rotated by ninety degrees. It follows that the walls would correspond to those we have computed for $\sigma_{\alpha,-1}$ and the previous argument proves that these can be crossed preserving the stability of E'_C. This implies the stability of E'_C with respect to $\sigma^0_{\alpha,-1}$. As the stability conditions σ on $\mathrm{Ku}(Y)$ are induced from $\sigma^0_{\alpha,-1}$ for $\alpha < 1/4$, and F'_C is in the Kuznetsov component, we get the desired statement. □

Now we are able to describe the moduli space $M_\sigma(2\lambda_1 + \lambda_2)$ of σ-stable objects with Mukai vector $2\lambda_1 + \lambda_2$ and, in particular, its identification with the LLSvS eightfold M_Y constructed in [64]. We use a standard argument, which is very similar to [62], Section 5.3. We point out that the results in [8] implies that $M_\sigma(2\lambda_1 + \lambda_2)$ is a smooth, projective, irreducible hyperkähler eightfold.

Theorem 4.2.8. *The moduli space $M_\sigma(2\lambda_1 + \lambda_2)$ parametrizes only objects of the form F'_C. Moreover, $M_\sigma(2\lambda_1 + \lambda_2)$ is isomorphic to the LLSvS eightfold M_Y.*

Proof. Let M_3 be the irreducible component of the Hilbert scheme parameterizing twisted cubic curves on Y. Then there exists a quasi-universal family $\mathcal{F}$ on $Y \times M_3$ parametrizing the sheaves $\mathcal{I}_{C/Y}(2H)$. By [58], Theorem 6.4, we have a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(Y \times M_3) = \langle \mathrm{Ku}(Y \times M_3), \mathcal{O}_Y \boxtimes \mathrm{D}^{\mathrm{b}}(M_3), \mathcal{O}_Y(H) \boxtimes \mathrm{D}^{\mathrm{b}}(M_3), \mathcal{O}_Y(2H) \boxtimes \mathrm{D}^{\mathrm{b}}(M_3)\rangle.$$

Now consider the relative projection $\mathcal{F}'$ of $\mathcal{F}$ in $\mathrm{Ku}(Y \times M_3) := \mathrm{Ku}(Y) \boxtimes \mathrm{D}^{\mathrm{b}}(M_3)$. As in [3], it is possible to verify that the projection of $\mathcal{I}_{C/Y}(2H)$ in the Kuznetsov component is exactly F'_C (see Section 4.4.3 for the computation in the non CM case). So, Theorem 4.2.7 implies that $\mathcal{F}'$ is a quasi-universal family of σ-stable objects F'_C in $\mathrm{Ku}(Y)$. Then there is an induced dominant morphism $M_3 \to M_\sigma(2\lambda_1 + \lambda_2)$. As M_3 is projective, we know that this morphism is surjective. This concludes the first statement.

For the second statement, we just need to show that for two twisted cubic curves C_1 and C_2, we have $F'_{C_1} = F'_{C_2}$ if and only if C_1 and C_2 are contained in the same fiber of the morphism $M_3 \to M_Y$ constructed in [64]. This is exactly proved in [3], Proposition 2. Indeed, they consider the projection in the K3 subcategory $\langle \mathcal{O}_Y(-H), \mathcal{O}_Y, \mathcal{O}_Y(H)\rangle^\perp$, which is equivalent to $\mathrm{Ku}(Y)$. This ends the proof of the theorem. □

4.3 Fano variety and stability

In this section, we use a similar argument to that applied in the case of twisted cubic curves in order to describe the Fano variety F_Y parametrizing lines in a cubic fourfold Y as a moduli space of Bridgeland stable objects.

Recall that given a line ℓ in Y, we can associate an object P_ℓ in $\mathrm{Ku}(Y)$, which sits in the distinguished triangle

$$\mathcal{O}_Y(-H)[1] \to P_\ell \to \mathcal{I}_\ell,$$

where $\mathcal{I}_\ell$ denotes the ideal sheaf of ℓ in Y (see [62], Section 6.3). It is easy to compute that the Mukai vector of P_ℓ is $\lambda_1 + \lambda_2$.

By Proposition 4.1.5, we can assume that the line L used in the construction of stability conditions is disjoint from ℓ. [1] Let us compute explicitly the image $M_\ell = \Psi(\sigma^* P_\ell)$ in $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$. By [7], Proposition 7.7, we have that

$$\Psi(\mathcal{O}_{\tilde{Y}}(-H)) = \mathcal{B}_{-1}.$$

On the other hand, we consider the sequence

$$\mathcal{I}_\ell \to \mathcal{O}_Y \to \mathcal{O}_\ell.$$

We recall that

$$\Psi(\mathcal{O}_{\tilde{Y}}) = 0.$$

By our assumption, we know that ℓ maps isomorphically to a line in $\mathbb{P}^3$; hence we have that

$$\Psi(\sigma^*\mathcal{I}_\ell) = \Psi(\sigma^*\mathcal{O}_\ell)[-1] = \pi_*(\mathcal{E}(h)|_{\sigma^{-1}(\ell)})$$

is a torsion sheaf supported over the image of ℓ in $\mathbb{P}^3$. We denote it by $\mathcal{E}_\ell$. So we have the distinguished triangle

$$\mathcal{B}_{-1}[1] \to M_\ell \to \mathcal{E}_\ell \tag{4.5}$$

in $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$.

Note that

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(M_\ell) = (-4, 3, \frac{7}{8}).$$

The following lemma gives us the starting point of the wall crossing argument.

Lemma 4.3.1. *The object M_ℓ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$.*

Proof. Assume that M_ℓ is not stable with respect to $\sigma_{\alpha,-1}$ for $\alpha \gg 0$. Then there is a destabilizing sequence

$$P \to M_\ell \to Q$$

in the heart $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, where P, Q are $\sigma_{\alpha,-1}$-semistable for $\alpha \gg 0$, and $\mu_{\alpha,-1}(P) > \mu_{\alpha,-1}(Q)$. We have two possibilities for P: either it is torsion or it has rank equal to -4. If we are in the first case, then, for α going to infinity, the slope $\mu_{\alpha,-1}(P)$ is a finite number, while $\mu_{\alpha,-1}(Q) = +\infty$. Thus such a P cannot destabilize M_ℓ.

In the case $\mathrm{rk}(P) = -4$, let us consider the cohomology sequence

$$0 \to \mathcal{H}^{-1}(P) \to \mathcal{H}^{-1}(M_\ell) \to \mathcal{H}^{-1}(Q) \to \mathcal{H}^0(P) \to \mathcal{H}^0(M_\ell) \to \mathcal{H}^0(Q) \to 0.$$

By (4.5) we have that $\mathcal{H}^{-1}(M_\ell) = \mathcal{B}_{-1}$ and $\mathcal{H}^0(M_\ell) = \mathcal{E}_\ell$. Also, we know that $\mathcal{H}^{-1}(Q) = 0$, because Q is a torsion element in the heart. It follows that $\mathcal{H}^{-1}(P) = \mathcal{B}_{-1}$ and we have the sequence

$$0 \to \mathcal{H}^0(P) \to \mathcal{E}_\ell \to \mathcal{H}^0(Q) \to 0.$$

We recall that $\mathcal{E}_\ell$ is a rank two torsion free sheaf over its support. Since $\mathcal{H}^0(P)$ is a subsheaf of $\mathcal{E}_\ell$, it has the same support. There are three cases. If $\mathcal{H}^0(P)$ has the same rank of $\mathcal{E}_\ell$ as a sheaf on its support, then

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(P) = \mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(M_\ell),$$

and $\mu_{\alpha,-1}(Q) = +\infty$, so it is not a destabilizing sequence. The second possibility is that $\mathcal{H}^0(P)$ has rank 1 and it is torsion free as a sheaf over a line. In this case, we have $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(P) = (-4, 3, -1/8)$, whose slope $\mu_{\alpha,-1}$ is less than that of M_ℓ. The third case when $\mathcal{H}^0(P) = 0$ is similar. This proves the stability of M_ℓ for α big enough. ☐

[1] Notice that if Y contains a plane P, then it is possible to choose L such that $L \cap \ell = \varnothing$ and L is not on P. For example, we consider a $\mathbb{P}^3$ intersecting P in a point. We define the cubic surface $S = Y \cap \mathbb{P}^3$. Choosing a general $\mathbb{P}^3$, we have that S is smooth. Every line on S is not on P by definition. Then it is easy to find a line L on S with the desired properties.

Now an easy computation using the four conditions listed at the beginning of Section 4.2.2 shows that the only potential wall for M_ℓ is given by $\alpha_0 = \frac{\sqrt{5}}{4}$. In the following lemma, we prove that M_ℓ remains stable after crossing this wall.

Lemma 4.3.2. *Let $\alpha_0 \geqslant \frac{\sqrt{5}}{4}$. If E is a $\sigma_{\alpha_0,-1}$-stable object in $\Psi(\sigma^* Ku(Y))$ such that $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E) = (-4, 3, \frac{7}{8})$, then E is $\sigma_{\alpha,-1}$-stable for any $\alpha > 0$.*

Proof. A direct computation and [11], Lemma 3.9, imply that the object E can be strictly semistable only with respect to $\sigma_{\frac{\sqrt{5}}{4},-1}$. If this happens, the Harder-Narasimham filtration of E with respect to $\sigma_{\frac{\sqrt{5}}{4}-\epsilon,-1}$ would be of the form

$$0 \subset E_1 \subset E$$

with $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E_1) = (0, 2, 1)$ and $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leqslant 2}(E/E_1) = (-4, 1, -\frac{1}{8})$. By Lemma 4.1.6, we have that $E/E_1 \simeq \mathcal{B}_0[1]$. In particular, we get

$$\mathrm{Hom}(\mathcal{B}_3, E[3]) = (\mathrm{Hom}(E, \mathcal{B}_0))^* \neq 0,$$

which contradicts to the assumption that E is in $\Psi(\sigma^* \mathrm{Ku}(Y))$. This proves the stability of E as claimed. □

Proof of Theorem 1.0.3. The first part is a consequence of Lemma 4.3.1 and Lemma 4.3.2. The second part follows from the same argument explained in Section 4.2.5 for twisted cubics. We point out that by projecting the universal family, we get an isomorphism from F_Y to $M_\sigma(\lambda_1 + \lambda_2)$. Hence the projectivity of $M_\sigma(\lambda_1 + \lambda_2)$ follows from that of F_Y, without using the result in [8]. □

4.4 Applications

In this section we discuss some applications of Theorem 1.0.3 and Theorem 1.0.4, concerning the categorical version of Torelli Theorem and the derived Torelli Theorem for cubic fourfolds. We also explain the identification of the period point of M_Y with that of F_Y.

4.4.1 Torelli Theorem for cubic fourfolds

In the Appendix of [7] the authors gave a different proof of the categorical version of Torelli Theorem for cubic fourfolds introduced in [48], in the case that the algebraic Mukai lattice does not contain (-2)-classes, e.g. for very general cubic fourfolds. In particular, they deduce the classical version of Torelli Theorem for cubic fourfolds. The key point of their proof is the interpretation of the Fano variety of lines on a very general cubic fourfold as a moduli space of Bridgeland stable objects in the Kuznetsov component.

As a direct consequence of Theorem 1.0.3, we are able to reprove the categorical formulation of Torelli Theorem for cubic fourfolds without the generality assumption. We recall that the degree shift functor of a cubic fourfold Y is the autoequivalence (1) of $\mathrm{Ku}(Y)$ given by the composition of the tensor product with the line bundle $\mathcal{O}_Y(1)$ and the projection to $\mathrm{Ku}(Y)$.

Corollary 4.4.1 (Categorical Torelli Theorem)**.** *Two cubic fourfolds Y and Y' are isomorphic if and only if there is an equivalence between* $\mathrm{Ku}(Y)$ *and* $\mathrm{Ku}(Y')$*, whose induced map on the algebraic Mukai lattices commutes with the action of the degree shift functor* (1)*.*

Proof. Notice that Theorem 1.0.3 implies that the object P_ℓ associated to a line ℓ in Y is stable with respect to every stability condition σ such that $\eta(\sigma) \in (A_2)_{\mathbb{C}} \cap \mathcal{P}$. Then we apply the same argument of the proof of [7], Theorem A.1. □

4.4.2 Period point of M_Y

In this section we discuss the relation between the period point of the LLSvS eightfold M_Y associated to a cubic fourfold Y and the period point of Y. [2]

As observed in [30], Example 6.4, the period point of Y is identified with the period point of the Fano variety F_Y. More precisely, let ${}^2\mathcal{M}_6^{(2)}$ be the moduli space of smooth projective hyperkähler fourfolds with a fixed polarization class of degree 6 and divisibility 2, deformation equivalent to the Hilbert square of a K3 surface. The Fano variety F_Y with the Plücker polarization is an element in ${}^2\mathcal{M}_6^{(2)}$. Let

$$ {}^2p_6^{(2)} : {}^2\mathcal{M}_6^{(2)} \to {}^2\mathcal{P}_6^{(2)} $$

be the period map which is an open embedding by Verbitsky's Torelli Theorem (see [93]).

We recall that the embedding of Hodge structures

$$ H^2(F_Y, \mathbb{Z}) \to \langle \lambda_1 \rangle^\perp \subset \tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}), $$

identifies the polarization class with $\lambda_1 + 2\lambda_2$ and $H^2(F_Y, \mathbb{Z})_0$ is Hodge isometric to $\langle \lambda_1, \lambda_1 + 2\lambda_2 \rangle^\perp$ (see [2], Proposition 7).

Let ${}^4\mathcal{M}_2^{(2)}$ be the moduli space of smooth projective hyperkähler eightfolds with a fixed polarization class of degree 2 and divisibility 2, deformation equivalent to the Hilbert scheme of points of length four on a K3 surface. Let

$$ {}^4p_2^{(2)} : {}^4\mathcal{M}_2^{(2)} \to {}^4\mathcal{P}_2^{(2)} $$

be the period map of these eightfolds.

By a direct computation it is possible to show that M_Y carries a natural polarization class of degree 2 and divisibility 2. Actually, as observed in [62], Lemma 3.7, the eightfold M_Y admits a natural antisymplectic involution τ whose fixed locus contains the cubic fourfold Y. Thus, M_Y with the fixed polarization is an element of ${}^4\mathcal{M}_2^{(2)}$.

Proposition 4.4.2. *Given a cubic fourfold Y, we have that*

$$ {}^2p_6^{(2)}(F_Y) = {}^4p_6^{(2)}(M_Y) $$

and they coincide with the period point of Y.

Proof. In [8] the authors prove that if M is a moduli space of Bridgeland stable objects in $\mathrm{Ku}(Y)$ with Mukai vector v of dimension $2 + v^2 \geqslant 0$, then there is an embedding of Hodge structures

$$ H^2(M, \mathbb{Z}) \to \tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}). $$

More precisely, the image of $H^2(M, \mathbb{Z})$ is identified with the orthogonal complement $v^\perp$ of v in the Mukai lattice. Thus, by Theorem 1.0.4, we have the Hodge isometry

$$ H^2(M_Y, \mathbb{Z}) \cong \langle \lambda_1 + 2\lambda_2 \rangle^\perp. $$

In particular, we can identify the polarization class on M_Y with λ_1. Then, the primitive degree two lattice $H^2(M_Y, \mathbb{Z})_0$ is Hodge isometric to $\langle \lambda_1 + 2\lambda_2, \lambda_1 \rangle^\perp$. It follows that

$$ H^2(M_Y, \mathbb{Z})_0 \cong \langle \lambda_1, \lambda_2 \rangle^\perp \cong H^2(F_Y, \mathbb{Z})_0, $$

which implies the statement. □

As explained in [26], Section B.1, Proposition 4.4.2 can be used to reprove in a more direct way the result by Laza and Loojenga (see [63]) about the image of the period map of cubic fourfolds, excluding the divisor of cubic fourfolds containing a plane. This is a work in progress of Bayer and Mongardi.

[2] We point out that Franco Giovenzana is independently working on this problem.

4.4.3 Approach to Derived Torelli Theorem for cubic fourfolds

In [47], Theorem 1.5, Huybrechts proved a version of the derived Torelli Theorem for cubic fourfolds, which was extended by [8], as recalled in Remak 2.4.2

An interesting question would be to prove the theorem without assumptions on the cubic fourfold. Here we suggest a possible strategy which makes use of the description of the eightfold M_Y given by Theorem 4.2.7. For this reason, we need to assume that Y does not contain a plane (actually, in this case the derived Torelli Theorem holds as recalled above).

Assume that there is a Hodge isometry $\phi : \tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) \cong \tilde{H}(\mathrm{Ku}(Y'), \mathbb{Z})$. Let $v := 2\lambda_1 + \lambda_2$ and we set $v' := \phi(v)$. By [8], the moduli space $M_{\sigma'}(v')$ for $\sigma' \in \mathrm{Stab}(\mathrm{Ku}(Y'))$ is non empty and in particular is a hyperkähler eightfold. By the Birational Torelli Theorem for hyperkähler varieties (see for example [45], Corollary 6.1), we have that $M_\sigma(v)$ is birational to $M_{\sigma'}(v')$. Thus, by [9], Theorem 1.4 (which works in the same way in our setting), we can find a stability condition σ'' such that $M_\sigma(v)$ is isomorphic to $M_{\sigma''}(v')$. By the construction in [64], the cubic fourfold Y is embedded in $M_\sigma(v)$ as a Lagrangian submanifold. Thus, we can see Y inside $M_{\sigma''}(v')$. We denote by $\mathcal{F}$ the restriction of the universal family $\mathcal{E}'_C$ on $M_{\sigma''}(v') \times Y'$ to $Y \times Y'$.

Here we deal only with the simple case $Y = Y'$ and $\phi \in \mathrm{O}(\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}))$. In the next result we show that the Fourier-Mukai functor $\Phi_{\mathcal{F}} : \mathrm{D^b}(Y) \to \mathrm{D^b}(Y)$, defined by $\Phi_{\mathcal{F}}(-) = q_*(p^*(-) \otimes \mathcal{F})$, commutes with the identity over $\mathrm{Ku}(Y)$. In this case, it is convenient to denote by i^* the projection functor into the Kuznetsov component, changing the notation of the previous sections.

Proposition 4.4.3. *Let Y be a cubic fourfold which does not contain a plane. Let i be the inclusion of $\mathrm{Ku}(Y)$ in $\mathrm{D^b}(Y)$ and we denote by i^* its left adjoint functor. Then we have that $\Phi_{\mathcal{F}} = i \circ i^*$.*

Proof. By [56], Theorem 3.7 and Proposition 3.8, we have that the composition $i \circ i^*$ is a Fourier-Mukai functor with kernel given by $\mathcal{G} := \mathrm{pr}(\mathcal{O}_\Delta)$. Here $\mathcal{O}_\Delta$ denotes the structure sheaf of the diagonal in $Y \times Y$ and $\mathrm{pr} : \mathrm{D^b}(Y \times Y) \to \mathrm{D^b}(Y) \boxtimes \mathrm{Ku}(Y)$ is the projection functor. Moreover, we have that $\mathcal{G}$ belongs to $\mathrm{Ku}(Y)(-2) \boxtimes \mathrm{Ku}(Y)$. Note that this is precisely the condition for a Fourier-Mukai functor of $\mathrm{D^b}(Y)$ in itself in order to factorize to the Kuznetsov component (see [48], Corollary 1.6).

We claim that $\Phi_{\mathcal{G}}(\mathcal{O}_y) = \mathcal{G}_y$ is σ-stable for every $y \in Y$. Indeed, given a point y on the cubic fourfold, there is a non CM twisted cubic curve C on Y which has y as embedded point. In particular, we have the sequence

$$0 \to \mathcal{I}_{C/Y}(2) \to \mathcal{I}_{C_0/Y}(2) \to \mathcal{O}_y \to 0, \tag{4.6}$$

where C_0 is the plane cubic curve, singular in y, defined by C. The ideal sheaf of C_0 in Y has the following resolution:

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 3} \to \mathcal{O}_Y(1)^{\oplus 3} \to \mathcal{I}_{C_0/Y}(2) \to 0. \tag{4.7}$$

We recall that $i^* := \mathbb{R}_{\mathcal{O}_Y(-1)} \mathbb{L}_{\mathcal{O}_Y} \mathbb{L}_{\mathcal{O}_Y(1)}$. We observe that $i^*(\mathcal{I}_{C_0/Y}(2)) = 0$. Indeed, we split the sequence (4.7) in two exact sequences

$$0 \to \mathcal{K} \to \mathcal{O}_Y(1)^{\oplus 3} \to \mathcal{I}_{C_0/Y}(2) \to 0$$

and

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 3} \to \mathcal{K} \to 0.$$

From the first sequence we get $\mathbb{L}_{\mathcal{O}_Y(1)}(\mathcal{I}_{C_0/Y}(2)) \cong \mathbb{L}_{\mathcal{O}_Y(1)}(\mathcal{K})[1]$. On the other hand, $\mathbb{L}_{\mathcal{O}_Y(1)}$ has not effect on the second sequence, because the objects are in $\langle \mathcal{O}_Y(1) \rangle^\perp$. Applying $\mathbb{L}_{\mathcal{O}_Y}$, we obtain

$$\mathbb{L}_{\mathcal{O}_Y} \mathbb{L}_{\mathcal{O}_Y(1)}(\mathcal{K}) \cong \mathbb{L}_{\mathcal{O}_Y}(\mathcal{O}_Y(-1)) = \mathcal{O}_Y(-1)[1].$$

It follows that

$$\mathbb{L}_{\mathcal{O}_Y} \mathbb{L}_{\mathcal{O}_Y(1)}(\mathcal{I}_{C_0/Y}(2)) \cong \mathcal{O}_Y(-1)[2].$$

Since $\mathbb{R}_{\mathcal{O}_Y(-1)}(\mathcal{O}_Y(-1)) = 0$, we deduce that $i^*(\mathcal{I}_{C_0/Y}(2)) = 0$. Thus by the sequence (4.6), we deduce that $i^*(\mathcal{I}_{C/Y}(2)) \cong i^*(\mathcal{O}_y)[-1]$.

Now, note that $i^*(\mathcal{I}_{C/Y}(2)) \cong i^*(\mathcal{I}_{C/S}(2))$, where S is the cubic surface containing C. Indeed, by the resolution

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(1)^{\oplus 2} \to \mathcal{I}_{S/Y}(2) \to 0,$$

we see that $\mathcal{I}_{S/Y}(2)$ is in $\langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle$. Hence, $i^*(\mathcal{I}_{S/Y}(2)) = 0$. Using the exact sequence

$$0 \to \mathcal{I}_{S/Y}(2) \to \mathcal{I}_{C/Y}(2) \to \mathcal{I}_{C/S}(2) \to 0,$$

we get

$$i^*(\mathcal{I}_{C/Y}(2)) \cong i^*(\mathcal{I}_{C/S}(2)) = F'_C.$$

By the previous computation, we deduce that $i^*(\mathcal{O}_y) \cong F'_C[1]$, which is σ-stable by Theorem 1.0.4.

It follows that $\mathcal{G}$ defines an inclusion of Y in the eightfold $M_\sigma(v)$ by

$$y \mapsto \Phi_{\mathcal{G}}(\mathcal{O}_y).$$

Thus $\mathcal{G}$ has to be isomorphic to the restriction of the universal family $\mathcal{E}'_C$ of $M_\sigma(v) \times Y$ to $Y \times Y$. We conclude that $\mathcal{G} \cong \mathcal{F}$, which gives the statement. □

In the general case, it is expected that the Fourier-Mukai functor $\Phi_{\mathcal{F}}$ factorizes to an equivalence on the level of the Kuznetsov categories.

Part II

Gushel-Mukai fourfolds

Chapter 5

Introduction to Part II

This part is devoted to the study of the double EPW sextic of a Gushel-Mukai fourfold as a moduli space of (twisted) stable sheaves on a K3 surface (see [88]). Finally, we describe a conic fibration for ordinary Gushel-Mukai fourfolds (firstly appeared in [84]), which could provide the geometrical setting in order to construct Bridgeland stability conditions for their Kuznetsov component (this is a joint work in progress with Alex Perry and Xiaolei Zhao).

The geometry of Gushel-Mukai (GM) varieties has been recently studied by Debarre and Kuznetsov in [28], [29], and from a categorical point of view by Kuznetsov and Perry in [61]. Of particular interest is the case of GM fourfolds, which are smooth intersections of dimension four of the cone over the Grassmannian $\mathrm{Gr}(2,5)$ with a quadric hypersurface in a eight-dimensional linear space over $\mathbb{C}$. Indeed, these Fano fourfolds have a lot of similarities with cubic fourfolds; for instance, it is unknown if the very general GM fourfold is irrational, even if there are rational examples (see [91], Section 4, [89], Section 3, and [27], Section 7).

In [27] Debarre, Iliev and Manivel investigated the period map and the period domain of GM fourfolds, in analogy to the work done by Hassett for cubic fourfolds. In particular, they proved that period points of *Hodge-special* GM fourfolds (see Definition 6.2.2) form a countable union of irreducible divisors in the period domain, depending on the discriminant of the possible labellings (see Section 2.3). It is not difficult to check that the discriminant of a Hodge-special GM fourfold is an integer $\equiv 0, 2$ or $4 \pmod 8$ (see [27], Lemma 6.1). Furthermore, the non-special cohomology of a Hodge-special GM fourfold X is Hodge isometric (up to a Tate twist) to the degree two primitive cohomology of a polarized K3 surface if and only if the discriminant d of X satisfies also the following numerical condition:

$$8 \nmid d \text{ and the only odd primes which divide } d \text{ are } \equiv 1 \pmod 4. \qquad (**)$$

The first result of this part is a generalization of the previous property to the twisted case, as done by Huybrechts for cubic fourfolds in [47].

Theorem 5.0.1. *A GM fourfold X has an associated twisted K3 surface in the cohomological sense (see Definition 7.1.11) if and only if the discriminant d of X satisfies*

$$d = \prod_i p_i^{n_i} \text{ with } n_i \equiv 0 \pmod 2 \text{ for } p_i \equiv 3 \pmod 4. \qquad (**')$$

On the other hand, a general GM fourfold X has an associated hyperkähler variety, as cubic fourfolds have their Fano variety of lines. Indeed, X determines a triple (V_6, V_5, A) of Lagrangian data, where $V_6 \supset V_5$ are six and five dimensional vector spaces, respectively, and $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace with respect to the symplectic structure induced by the wedge product, with no decomposable vectors (see [28], Theorem 3.16). Conversely, it is possible to reconstruct an ordinary and a special GM variety from a Lagrangian data having A without decomposable vectors (see [28], Theorem 3.10 and Proposition 3.13). The data of A determines a stratification in subschemes of the form $Y_A^{\geqslant 3} \subset Y_A^{\geqslant 2} \subset Y_A^{\geqslant 1} \subset \mathbb{P}(V_6)$, where $Y_A^{\geqslant 1}$ is a Eisenbud-Popescu-Walter (EPW) sextic hypersurface (see Section 6.2). As recalled in Example 0.3.12, if $Y_A^{\geqslant 3}$ is empty, then the double cover $\tilde{Y}_A$ of the EPW sextic is a hyperkähler fourfold deformation equivalent to the Hilbert scheme of length-two subschemes on a K3 surface. Actually, in order to guarantee the smoothness of $\tilde{Y}_A$, it is enough to avoid the divisor $\mathcal{D}''_{10}$ in the period domain by [29], Remark 5.29.

The second main result is the following theorem, whose analogue for cubic fourfolds was proven by Addington in [2]. Let λ_1 and λ_2 be the classes in the topological K-theory of a GM fourfold defined in (6.4).

Theorem 5.0.2. *Let X be a Hodge-special GM fourfold such that $Y_A^{\geqslant 3} = \varnothing$. Consider the following propositions:*

(a) *X has discriminant d satisfying $(**)$;*

(b) *T_X is Hodge isometric to $T_S(-1)$ for some K3 surface S, or equivalently, there is a hyperbolic plane $U = \langle \kappa_1, \kappa_2 \rangle$ primitively embedded in the algebraic part of the Mukai lattice;*

(c) *$\tilde{Y}_A$ is birational to a moduli space of stable sheaves on S.*

Then we have that (a) *implies* (b)*, and* (b) *is equivalent to* (c)*.*

Moreover, (b) *implies* (a) *if either $H^{2,2}(X, \mathbb{Z})$ has rank 3, or there is an element τ in the hyperbolic plane U such that $\langle \lambda_1, \lambda_2, \tau \rangle$ has discriminant $\equiv 2$ or $4 \pmod 8$.*

In Section 7.1.3 we discuss a counterexample showing that the inverse implication of the second part of Theorem 5.0.2 does not hold in full generality. More precisely, we show that there are GM fourfolds satisfying condition (b), but without a Hodge-associated K3 surface. In particular, we deduce that property (b) is not always divisorial and that there are period points of K3 type corresponding to GM fourfolds without a Hodge-associated K3 surface.

We also prove its natural extension to the twisted case, as in [47] for cubic fourfolds.

Theorem 5.0.3. *Let X be a Hodge-special GM fourfold with discriminant d such that $Y_A^{\geqslant 3} = \varnothing$. Then $\tilde{Y}_A$ is birational to a moduli space of stable twisted sheaves on a K3 surface S if and only if d satisfies $(**')$.*

Finally, we determine the numerical condition on the discriminant d of a Hodge-special GM fourfold in order to have $\tilde{Y}_A$ birational to the Hilbert scheme $S^{[2]}$ on a K3 surface S; this condition is stricter than that of $(**)$, as proved in [2] for cubic fourfolds (see Remark 7.2.7).

Theorem 5.0.4. *Let X be a Hodge-special GM fourfold of discriminant d such that $Y_A^{\geqslant 3} = \varnothing$. Then $\tilde{Y}_A$ is birational to the Hilbert scheme $S^{[2]}$ on a K3 surface S if and only if d satisfies the condition*

$$a^2 d = 2n^2 + 2 \quad \text{for } a, n \in \mathbb{Z}. \tag{***}$$

The strategy to prove these results relies on the definition of the Mukai lattice for the *Kuznetsov component*, which is the K3 subcategory arising from the semiorthogonal decomposition of the derived category of a GM fourfold constructed in [61] (see also Example 0.1.30). The Mukai lattice is defined as done in [4] by Addington and Thomas for cubic fourfolds; actually, we can prove the analogue of their results, using the vanishing lattice of a GM fourfold instead of the primitive degree four lattice of cubic fourfolds. In particular, following the work of Addington, this allows us to apply Propositions 4 and 5 of [2] and, then, to prove Theorems 5.0.2 and 5.0.4. On the other hand, we obtain that if a very general GM fourfold has a homological associated K3 surface, then there is a Hodge-theoretic associated K3 surface (see Theorem 7.1.10 for a more precise statement).

It becomes evident that there are many information about the geometry of the GM fourfold which can be recovered from its Kuznetsov component. In particular, Bridgeland stability conditions would be a powerful tool for this kind of investigation. As an example, it would be possible to study moduli spaces of stable objects in $\mathrm{Ku}(X)$, as done in the case of cubic fourfolds. As explained in Section 4.1.1, Bayer, Lahoz, Macrì and Stellari develop a method to induce Bridgeland stability conditions on semiorthogonal decompositions (see [7]). As a consequence, they prove that there are Bridgeland stability conditions on the Kuznetsov component of many Fano threefolds and cubic fourfolds. In this last case, the starting point in order to apply their general method is the construction of a conic fibration induced by the blow up of a line in the cubic fourfold and the projection to $\mathbb{P}^3$.

The last result of this thesis is the construction of a flat conic fibration for ordinary GM fourfolds, obtained by blowing up of a degree four del Pezzo surface in the GM fourfold and projecting to a $\mathbb{P}^3$ (see Proposition 8.1.3). In particular, this geometric picture is obtained by the restriction to a hyperplane of a fibration constructed by Debarre and Kuznetsov in [28], Proposition 4.5. We point out that in [84], joint with Mattia Ornaghi, we used this fibration to prove Voevodsky's conjecture for general GM fourfolds.

In a joint work in progress with Alex Perry and Xiaolei Zhao, we are trying to use this result to induce Bridgeland stability conditions on the GM category.

Related works. In [51], Proposition 2.1, Iliev and Madonna prove that if a smooth double EPW sextic is birational to the Hilbert scheme $S^{[2]}$ on a K3 surface S with polarization of the degree d, then the negative Pell equation $\mathcal{P}_{d/2}(-1) : n^2 - \frac{d}{2}a^2 = -1$ is solvable. Thus Theorem 5.0.4 is consistent with this necessary condition (see also Remark 7.2.8).

Finally, in [30], Corollary 7.6, Debarre and Macrì prove that the Hilbert square of a general polarized K3 surface of degree d is isomorphic to a double EPW sextic if and only if the Pell equation $\mathcal{P}_{d/2}(-1)$ is solvable and the equation $\mathcal{P}_{2d}(5) : n^2 - (2d)a^2 = 5$ is not. By Theorem 5.0.4, we see that the birationality to this Hilbert scheme is obtained by relaxing the second condition on $\mathcal{P}_{2d}(5)$.

Chapter 6

Background material

The aim of this section is to recall some definitions and properties concerning Hodge-special GM fourfolds and to fix the notation. Our main references are [27], [28] [29] and [61].

6.1 Geometry of GM varieties

Let V_5 be a 5-dimensional complex vector space; we denote by $\mathrm{Gr}(2, V_5)$ the Grassmannian of 2-dimensional subspaces of V_5, viewed in $\mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9$ via the Plücker embeddig. Let $\mathrm{CGr}(2, V_5) \subset \mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5) \cong \mathbb{P}^{10}$ be the cone over $\mathrm{Gr}(2, V_5)$ of vertex $\nu := \mathbb{P}(\mathbb{C})$.

Definition 6.1.1. A **GM variety** of dimension $2 \leqslant n \leqslant 6$ is a smooth n-dimensional intersection

$$X = \mathrm{CGr}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

where W is a $n + 5$-dimensional vector subspace of $\mathbb{C} \oplus \bigwedge^2 V_5$ and Q is a quadric hypersurface in $\mathbb{P}(W) \cong \mathbb{P}^{n+4}$.

A **GM fourfold** is a GM variety of dimension 4.

Notice that ν does not belong to X, because X is smooth. Thus, the linear projection from ν defines a regular map

$$\gamma_X : X \to \mathrm{Gr}(2, V_5)$$

called the **Gushel map**. We denote by $\mathcal{U}_X$ the pullback via γ_X of the tautological rank-2 subbundle of $\mathrm{Gr}(2, V_5)$.

We can associate two hulls to X. The *Grassmannian hull* of X is the intersection

$$M_X := \mathrm{CGr}(2, V_5) \cap \mathbb{P}(W);$$

it is a variety of dimension $n + 1$, because X has dimension n, and, by definition, $X = M_X \cap Q$ is a quadric section of M_X. Let $W' = W/\mathbb{C}$ be the projection of W to $\bigwedge^2 V_5$. The intersection

$$M'_X := \mathrm{Gr}(2, V_5) \cap \mathbb{P}(W')$$

is called the *projected Grassmannian hull* of X. We can distinguish two cases:

- If the linear space $\mathbb{P}(W)$ does not contain the vertex ν, then the linear projection $\mathbb{P}(W) \to \mathbb{P}(\bigwedge^2 V_5)$ from ν is well-defined: indeed, we have $W \cong W'$. In particular, we have $M_X \cong M'_X$ via this map. Therefore, considering Q as a quadric hypersurface in $\mathbb{P}(W')$, we have that

  $$X \cong M'_X \cap Q \cong \mathrm{Gr}(2, V_5) \cap \mathbb{P}^{n+4} \cap Q \subset \mathbb{P}^9,$$

  i.e. X is a quadric section of a linear section of the Grassmannian $\mathrm{Gr}(2, V_5)$. Gushel-Mukai varieties of this form are called **ordinary**.

- If the vertex ν is contained in $\mathbb{P}(W)$, then the linear space $\mathbb{P}(W)$ is a cone over $\mathbb{P}(W')$; in particular, we have that $M_X = \mathrm{C}M'_X$. Since X is smooth by definition, the quadric Q does not contain the vertex of the cone. Thus, the projection from the vertex defines a double cover

$$\gamma_X : X \xrightarrow{2:1} M'_X.$$

In other words, the variety X is a double cover of a linear section of $\mathrm{Gr}(2, V_5)$: Gushel-Mukai varieties of this form are called **special**. Moreover, the branch divisor of the double cover γ_X is $X' := M'_X \cap Q'$, where $Q' = Q \cap \mathbb{P}(W')$. Since by [28], Proposition 2.20, the hull M'_X is smooth, we have that X' is a smooth ordinary Gushel-Mukai variety of dimension $n - 1$.

We denote by $\sigma_{i,j} \in H^{2(i+j)}(\mathrm{Gr}(2, V_5), \mathbb{Z})$ the Schubert cycles on $\mathrm{Gr}(2, V_5)$ for every $3 \geqslant i \geqslant j \geqslant 0$ and we set $\sigma_i := \sigma_{i,0}$. The restriction $h := \gamma_X^* \sigma_1$ of the hyperplane class $H := \sigma_1$ on $\mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5)$ defines a natural polarization of degree 10 on X. Indeed, $\mathrm{CGr}(2, V_5)$ has degree 5, the degree of Q is 2 and X is dimensionally trasverse. On the other hand, since the canonical class of $\mathrm{CGr}(2, V_5)$ is $-6H$, by adjunction formula we get

$$K_X = (-6 + (6 - n) + 2)h = -(n - 2)h.$$

Let $(\mathrm{Sch}/\mathbb{C})$ be the category of schemes over $\mathbb{C}$. For $2 \leqslant n \leqslant 6$, the *moduli stack* $\mathcal{M}_n$ of n-dimensional Gushel-Mukai varieties is the fibered category over $(\mathrm{Sch}/\mathbb{C})$ whose fiber over $S \in (\mathrm{Sch}/\mathbb{C})$ is the groupoid of pairs $(\pi : \chi \to S, \mathcal{L})$, where $\pi : \chi \to S$ is a smooth proper morphism of schemes and $\mathcal{L}$ belongs to $\mathrm{Pic}_{\chi/S}(S)$, such that for every geometric point $\bar{s} \in S$ the pair $(X_{\bar{s}}, \mathcal{L}_{\bar{s}})$ is isomorphic to an n-dimensional Gushel-Mukai variety with its natural polarization. A morphism from $(\pi' : \chi' \to S', \mathcal{L}')$ to $(\pi : \chi \to S, \mathcal{L})$ is a fiber product diagram

$$\begin{array}{ccc} \chi' & \xrightarrow{g'} & \chi \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S \end{array}$$

such that $(g')^*(\mathcal{L}) = \mathcal{L}' \in \mathrm{Pic}_{\chi'/S'}(S')$. By [61], Proposition 2.4, we have that $\mathcal{M}_n$ is a smooth and irreducible Deligne-Mumford stack of finite type over $\mathbb{C}$, of dimension $25 - (6 - n)(5 - n)/2$.

6.2 Period map and Hodge-special GM fourfolds

From now on, we restrict our investigation to GM fourfolds. By the previous section, we have that a GM fourfold X is a Fano fourfold with canonical class $-2h$. We recall the Hodge numbers of X:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 22 & & 1 & & 0 \end{array}$$

(see [51], Lemma 4.1). Notice that $H^\bullet(X, \mathbb{Z})$ is torsion free by [29], Proposition 3.3. The classes h^2 and $\gamma_X^* \sigma_2$ span the embedding of the rank-two lattice $H^4(\mathrm{Gr}(2, V_5), \mathbb{Z})$ in $H^4(X, \mathbb{Z})$. The **vanishing lattice** of X is the sublattice

$$H^4(X, \mathbb{Z})_{00} := \left\{ x \in H^4(X, \mathbb{Z}) : x \cdot \gamma_X^*(H^4(\mathrm{Gr}(2, V_5), \mathbb{Z})) = 0 \right\}.$$

By [27], Proposition 5.1, we have an isomorphism of lattices

$$H^4(X,\mathbb{Z})_{00} \cong E_8^2 \oplus U^2 \oplus I_{2,0}(2) =: \Lambda.$$

Let e and f be two classes in $I_{22,2}$ of square 2 and $e \cdot f = 0$, which generate the orthogonal of Λ in $I_{22,2}$. The choice of an isometry $\phi : H^4(X,\mathbb{Z}) \cong I_{22,2}$ sending $\gamma_X^*\sigma_{1,1}$ and $\gamma_X^*(\sigma_2 - \sigma_{1,1})$ to e and f respectively, and such that $\phi(H^4(X,\mathbb{Z})_{00}) = \Lambda$, determines a **marking** for X. Notice that the Hodge structure on the vanishing lattice is of K3 type. Let $\tilde{\mathrm{O}}(\Lambda)$ be the subgroup of automorphisms of $\mathrm{O}(\Lambda)$ acting trivially on the discriminant group $d(\Lambda)$. The groups $\tilde{\mathrm{O}}(\Lambda)$ and $\mathrm{O}(\Lambda)$ act properly and discontinuosly on the complex variety

$$\Omega := \{w \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : w \cdot w = 0, w \cdot \bar{w} < 0\}. \tag{6.1}$$

The **global period domain** is the quotient $\mathcal{D} := \tilde{\mathrm{O}}(\Lambda)\backslash\Omega$, which is an irreducible quasi-projective variety of dimension 20. We observe that two markings differ by the action of an element in $\tilde{\mathrm{O}}(\Lambda)$. It follows that the **period map** $p : \mathcal{M}_4 \to \mathcal{D}$, which sends X to the class of the one dimensional subspace $H^{3,1}(X)$, is well-defined. As a map of stacks, p is dominant with 4-dimensional smooth fibers (see [27], Theorem 4.4). The **period point** of X is the image $p(X)$ in $\mathcal{D}$.

As proved in [29], the period point of a general GM fourfold is determined by that of its associated double EPW sextic. More precisely, let (V_6, V_5, A) be the Lagrangian data of X, as defined in the introduction. As recalled in Example 0.3.12, by the work of O'Grady, we can consider the closed subschemes

$$Y_A^{\geqslant l} := \{[U_1] \in \mathbb{P}(V_6) : \dim(A \cap (U_1 \wedge \bigwedge^2 V_6)) \geqslant l\} \quad \text{for } l \geqslant 0.$$

Since A has no decomposable vectors, we have that $Y_A := Y_A^{\geqslant 1}$ is a normal sextic hypersurface, called EPW sextic, which is singular along the integral surface $Y_A^{\geqslant 2}$. Moreover, $Y_A^{\geqslant 3}$ is finite and it is the singular locus of $Y_A^{\geqslant 2}$, while $Y_A^{\geqslant 4}$ is empty (see [28], Proposition B.2). Let $\tilde{Y}_A$ be the double cover of the EPW sextic Y_A branched over $Y_A^{\geqslant 2}$. If $Y_A^{\geqslant 3}$ is empty, (e.g. for generic A), then the **double EPW sextic** $\tilde{Y}_A$ is a smooth hyperkähler fourfold of K3 type (see [79], Theorem 1.1). In this case, the period point of $\tilde{Y}_A$ coincides with $p(X)$, as explained in the following result.

Theorem 6.2.1 ([29], Theorem 5.1)**.** *Let X be a GM fourfold with associated Lagrangian data (V_6, V_5, A). Assume that the double EPW sextic $\tilde{Y}_A$ is smooth (i.e. $Y_A^{\geqslant 3} = \varnothing$). Then, there is an isometry of Hodge structures*

$$H^4(X,\mathbb{Z})_{00} \cong H^2(\tilde{Y}_A,\mathbb{Z})_0(-1),$$

where $H^2(\tilde{Y}_A,\mathbb{Z})_0$ is the degree two primitive cohomology of $\tilde{Y}_A$ equipped with the Beauville-Bogomolov-Fujiki form q.

As in the case of special cubic fourfolds, it is possible to consider GM fourfolds such that the rank of $H^4(X,\mathbb{Z})$ is not minimal. We call them Hodge-special in order to avoid confusion with special GM fourfolds defined before.

Definition 6.2.2. A GM fourfold X is **Hodge-special** if $H^{2,2}(X) \cap H^4(X,\mathbb{Q})_{00} \neq 0$.

Equivalently, X is Hodge-special if and only if $H^{2,2}(X,\mathbb{Z})$ contains a rank-three primitive sublattice K containing $\gamma_X^*(H^4(\mathrm{Gr}(2,V_5),\mathbb{Z}))$. Such a lattice K is a **labelling** for X and the discriminant of the labelling is the determinant of the intersection matrix on K. We say that X *has discriminant* d if it has a labelling of discriminant d.

We have that d is positive and $d \equiv 0, 2$ or $4 (\bmod 8)$ (see [27], Lemma 6.1). More precisely, the period point of a Hodge-special GM fourfold with discriminant d belongs to an irreducible divisor $\mathcal{D}_d$ in $\mathcal{D}$ if $d \equiv 0 (\bmod 4)$, or to the union of two irreducible divisors $\mathcal{D}'_d$ and $\mathcal{D}''_d$ in $\mathcal{D}$ if $d \equiv 2 (\bmod 8)$ (see

[27], Corollary 6.3). In particular, the hypersurfaces $\mathcal{D}'_d$ and $\mathcal{D}''_d$ are interchanged by the involution $r_{\mathcal{D}}$, defined on $\mathcal{D}$ by exchanging e and f.

Let X be a Hodge-special GM fourfold with a labeling K of discriminant d. The orthogonal $K^\perp$ of K in $I_{22,2}$ is the **non-special lattice** of X; it is equipped with a Hodge structure induced by the Hodge structure on $H^4(X,\mathbb{Z})$. A pseudo-polarized K3 surface S of degree d is **Hodge-associated** to (X,K) if it exists an isometry of Hodge structures between the non-special cohomology $K^\perp$ and the primitive cohomology lattice $H^2(S,\mathbb{Z})_0$ which reverses the sign. As already explained in the introduction, this is equivalent to have d satisfying $(**)$. Moreover, if $p(X)$ is not in $\mathcal{D}_8$, then the pseudo-polarization is a polarization (see [27], Proposition 6.5).

6.3 Kuznetsov component and K-theory

The analogy of GM fourfolds with cubic fourfold reflects also on their derived categories. Indeed, we denote by $\mathrm{D^b}(X)$ the derived category of bounded complexes of coherent sheaves on a GM fourfold X. As recalled in Example 0.1.30, by [61], Proposition 4.2, there exists a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1)\rangle,$$

where $\mathrm{Ku}(X)$ is the right orthogonal to the subcategory generated by the exceptional objects

$$\mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1), \tag{6.2}$$

in $\mathrm{D^b}(X)$. We refer to $\mathrm{Ku}(X)$ as the *Kuznetsov component* of X. The Kuznetsov component has the same Serre functor of the derived category of a K3 surface (see [61], Proposition 4.5). In particular, the category $\mathrm{Ku}(X)$ is a non commutative K3 surface. Moreover, if X is an ordinary GM fourfold containing a quintic del Pezzo surface, then there exists a K3 surface S realizing the equivalence $\mathrm{Ku}(X) \xrightarrow{\sim} \mathrm{D^b}(S)$ (see [61], Theorem 1.2).

We denote by $K_0(\mathrm{Ku}(X))$ the Grothendieck group of $\mathrm{Ku}(X)$ and let χ be the Euler pairing. The numerical Grothendieck group of $\mathrm{Ku}(X)$ is given by the quotient $K_0(\mathrm{Ku}(X))_{\mathrm{num}} := K_0(\mathrm{Ku}(X))/\ker\chi$. By the additivity with respect to semiorthogonal decompositions, we have the orthogonal direct sum

$$K_0(X)_{\mathrm{num}} = K_0(\mathrm{Ku}(X))_{\mathrm{num}} \oplus \langle [\mathcal{O}_X], [\mathcal{U}_X^*], [\mathcal{O}_X(1)], [\mathcal{U}_X^*(1)]\rangle_{\mathrm{num}} \cong K_0(\mathrm{Ku}(X))_{\mathrm{num}} \oplus \mathbb{Z}^4$$

with respect to χ. In particular, since the Hodge conjecture holds for X over $\mathbb{Q}$ (see [24]), it follows that

$$\mathrm{rank}(K_0(\mathrm{Ku}(X))_{\mathrm{num}}) = \sum_k h^{k,k}(X,\mathbb{Q}) - 4 = 4 + h^{2,2}(X,\mathbb{Q}) - 4 = h^{2,2}(X,\mathbb{Q}).$$

We recall the following lemma, which will be useful to study the relation between the Mukai lattice of $\mathrm{Ku}(X)$ and the vanishing cohomology of X.

Lemma 6.3.1 ([61], Lemma 5.14 and Lemma 5.16)**.** *If X is a non Hodge-special GM fourfold, then*

$$K_0(\mathrm{Ku}(X))_{\mathrm{num}} \cong \mathbb{Z}^2$$

and it admits a basis such that the Euler form with respect to this basis is given by

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

We end this section with the explicit computation of the basis of Lemma 6.3.1. The Todd class of a GM fourfold X is

$$\mathrm{td}(X) = 1 + h + \left(\frac{2}{3}h^2 - \frac{1}{12}\gamma_X^*\sigma_2\right) + \frac{17}{60}h^3 + \frac{1}{10}h^4. \tag{6.3}$$

Let P be a point in X, L be a line lying on X, Σ be the zero locus of a regular section of $\mathcal{U}_X^*$, S be the complete intersection of two hyperplanes in X and H be a hyperplane section of X. Since X is not Hodge-special, the structure sheaves of these subvarieties give a basis for the numerical Grothendieck group. Thus, an element κ in $K_0(X)_{\mathrm{num}}$ can be written as

$$\kappa = a[\mathcal{O}_X] + b[\mathcal{O}_H] + c[\mathcal{O}_S] + d[\mathcal{O}_\Sigma] + e[\mathcal{O}_L] + f[\mathcal{O}_P],$$

for $a, b, c, d, e, f \in \mathbb{Z}$. A computation using Riemann-Roch gives that κ belongs to $K_0(\mathrm{Ku}(X))$ if and only if it is a linear combination of the following classes:

$$\begin{aligned} \lambda_1 &:= -4[\mathcal{O}_X] + 2[\mathcal{O}_H] + [\mathcal{O}_S] + 5[\mathcal{O}_L] - 5[\mathcal{O}_P] \\ \lambda_2 &:= -2[\mathcal{O}_X] + [\mathcal{O}_\Sigma] + 2[\mathcal{O}_L] - [\mathcal{O}_P]. \end{aligned} \tag{6.4}$$

It is easy to verify that the matrix they define with respect to the Euler form is as in Lemma 6.3.1.

Remark 6.3.2. Let C be a generic conic in a GM fourfold X; we denote by $\mathcal{O}_C$ its structure sheaf. Notice that λ_1 is the class in the K-theory of X of the projection of $\mathcal{O}_C(1)$ in $\mathrm{Ku}(X)$. Indeed, the projection $\mathrm{pr} : \mathrm{D^b}(X) \to \mathrm{Ku}(X)$ is given by the composition $\mathrm{pr} := \mathbb{L}_{\mathcal{O}_X}\mathbb{L}_{\mathcal{U}_X^*}\mathbb{L}_{\mathcal{O}_X(1)}\mathbb{L}_{\mathcal{U}_X^*(1)}$ of the left mutation functors with respect to the exceptional objects. Performing this computation, we get that

$$[\mathrm{pr}(\mathcal{O}_C(1))] = [\mathcal{O}_C(1)] - [\mathcal{O}_X(1)] + [\mathcal{U}_X^*] + [\mathcal{O}_X],$$

which has the same Chern character of λ_1. The second element λ_2 should be the class of an object in $\mathrm{Ku}(X)$ obtained as the image of $\mathrm{pr}(\mathcal{O}_C(1))$ via an autoequivalence of $\mathrm{Ku}(X)$.

Chapter 7

Double EPW sextic of a Gushel-Mukai fourfold

In this chapter, we prove Theorems 5.0.1, 5.0.2, 5.0.3, 5.0.4. These results appear in [88].

7.1 Mukai lattice for the Kuznetsov component

In this section we describe the Mukai lattice of the GM category. The main results of Section 7.1.1 are Proposition 7.1.1, where we prove that the vanishing lattice is Hodge isometric to the orthogonal of the lattice generated by λ_1 and λ_2 in the Mukai lattice, and Corollary 7.1.5, where we determine Hodge-special GM fourfolds by their Mukai lattice. In Section 7.1.2 we relate the condition of having an associated K3 surface with the Mukai lattice (Theorem 7.1.6); as a consequence, we get Theorem 7.1.10, where we prove that the existence of a homological associated K3 surface implies that there is a Hodge-theoretic associated K3 surface for very general Hodge-special GM fourfolds. Then we prove Theorem 5.0.1. We follow the methods introduced in [4] and [47] for cubic fourfolds.

7.1.1 Mukai lattice and vanishing lattice

Let X be a GM fourfold. We denote by $K(X)_{\text{top}}$ the topological K-theory of X which is endowed with the Euler pairing χ. As recalled in Section 2.2, the group $H^\bullet(X,\mathbb{Z})$ is torsion-free; by [6], Section 2.5 (see also [4], Theorem 2.1), it follows that also $K(X)_{\text{top}}$ is torsion-free.

Inspired by [4], we define the **Mukai lattice** of the Kuznetsov component $\text{Ku}(X)$ as the abelian group

$$K(\text{Ku}(X))_{\text{top}} := \{\kappa \in K(X)_{\text{top}} : \chi([\mathcal{O}_X(i)],\kappa) = \chi([\mathcal{U}_X^*(i)],\kappa) = 0 \text{ for } i = 0,1\}$$

with the Euler form χ. We point out that $K(\text{Ku}(X))_{\text{top}}$ is torsion-free, because $K(X)_{\text{top}}$ is. We recall that the Mukai vector of an element κ of $K(X)_{\text{top}}$ is given by

$$v(\kappa) = \text{ch}(\kappa).\sqrt{\text{td}(X)}$$

and it induces an isomorphism of $\mathbb{Q}$-vector spaces $v : K(X)_{\text{top}} \otimes \mathbb{Q} \cong H^\bullet(X,\mathbb{Q})$. We define the weight-zero Hodge structure on the Mukai lattice given by pulling back via the isomorphism

$$K(\text{Ku}(X))_{\text{top}} \otimes \mathbb{C} \to \bigoplus_{p=0}^{4} H^{2p}(X,\mathbb{C})(p)$$

induced by v. It is also convenient to consider the Mukai lattice $K(\text{Ku}(X))_{\text{top}}(-1)$ with weight-two Hodge structure $\bigoplus_{p+q=2} \tilde{H}^{p,q}(\text{Ku}(X))$ and Euler form with reversed sign. In the following, we will use both conventions according to the situation. The *Néron-Severi lattice* of $\mathrm{Ku}(X)$ is

$$N(\mathrm{Ku}(X)) = \tilde{H}^{1,1}(\mathrm{Ku}(X), \mathbb{Z}) := \tilde{H}^{1,1}(\mathrm{Ku}(X)) \cap K(\mathrm{Ku}(X))_{\mathrm{top}}$$

and the *trascendental lattice* $T(\mathrm{Ku}(X))$ is the orthogonal complement of the Néron-Severi lattice with respect to χ.

We observe that by [61], Theorem 1.2, there exist GM fourfolds X such that the associated Kuznetsov component $\mathrm{Ku}(X)$ is equivalent to the derived category of a K3 surface S. Moreover, any equivalence $\mathrm{Ku}(X) \xrightarrow{\sim} \mathrm{D^b}(S)$ induces an isometry of Hodge structures $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1) \cong K(S)_{\mathrm{top}}$, by the same argument used in [4], Section 2.3. We set $\tilde{\Lambda} := U^4 \oplus E_8(-1)^2$ and we recall that $K(S)_{\mathrm{top}}$ is isomorphic as a lattice to $\tilde{\Lambda}$. Since the definition of $K(\mathrm{Ku}(X))_{\mathrm{top}}$ does not depend on X (any two GM fourfolds are deformation equivalent), we deduce that the Euler form is symmetric on $K(\mathrm{Ku}(X))_{\mathrm{top}}$ and $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is isomorphic as a lattice to $\tilde{\Lambda}(-1) = U^4 \oplus E_8^2$.

We denote by $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ the orthogonal complement with respect to the Euler pairing of the sublattice of $K(\mathrm{Ku}(X))_{\mathrm{top}}$ generated by the objects λ_1, λ_2 determined in (6.4). In the next result, we explain the relation of this lattice with the vanishing lattice $H^4(X, \mathbb{Z})_{00}$.

Proposition 7.1.1. *Let X be a GM fourfold. Then the Mukai vector v induces an isometry of Hodge structures*

$$\langle \lambda_1, \lambda_2 \rangle^{\perp} \cong H^4(X, \mathbb{Z})_{00}(2) = \langle h^2, \gamma_X^* \sigma_2 \rangle^{\perp}.$$

Moreover, for every set of n objects $\zeta_1, \dots, \zeta_n$ in $K(\mathrm{Ku}(X))_{\mathrm{top}}$, the Mukai vector induces the isometry

$$\langle \lambda_1, \lambda_2, \zeta_1, \dots, \zeta_n \rangle^{\perp} \cong \langle h^2, \gamma_X^* \sigma_2, c_2(\zeta_1), \dots, c_2(\zeta_n) \rangle^{\perp}.$$

Proof. By definition κ belongs to $\langle \lambda_1, \lambda_2 \rangle^{\perp} \subset K(\mathrm{Ku}(X))_{\mathrm{top}}$ if and only if

$$\begin{cases} \chi(\mathcal{O}_X, \kappa) = \chi(\mathcal{O}_X(1), \kappa) = \chi(\mathcal{U}_X^*, \kappa) = \chi(\mathcal{U}_X^*(1), \kappa) = 0 \\ \chi(\lambda_1, \kappa) = \chi(\lambda_2, \kappa) = 0. \end{cases} \tag{7.1}$$

The Chern character of κ has the form

$$\mathrm{ch}(\kappa) = k_0 + k_2 h + k_4 + k_6 h^3 + k_8 h^4 \quad \text{for } k_0, k_2, k_6, k_8 \in \mathbb{Q} \text{ and } k_4 \in H^4(X, \mathbb{Q}).$$

Thus, using Riemann-Roch, we can express the conditions (7.1) as a linear system in the variables $k_0, k_2, k_4 \cdot h^2, k_4 \cdot \gamma_X^* \sigma_{1,1}, k_6, k_8$. Since the equations are linearly independent, we obtain that the system (7.1) has a unique solution, i.e.

$$k_0 = k_2 = k_6 = k_8 = 0 \quad \text{and} \quad k_4 \cdot h^2 = k_4 \cdot \gamma_X^* \sigma_{1,1} = 0.$$

In particular, $\mathrm{ch}(\kappa)$ belongs to $\langle 1, h, h^2, \gamma_X^* \sigma_2, h^3, h^4 \rangle^{\perp} = \langle h^2, \gamma_X^* \sigma_2 \rangle^{\perp}$ in $H^4(X, \mathbb{Q})$. Since $k_4 \cdot h = 0$, $v(\kappa) = k_4$, i.e. $v(\kappa)$ is in the sublattice $\langle h^2, \gamma_X^* \sigma_2 \rangle^{\perp}$ of $H^4(X, \mathbb{Q})$. Since the lowest-degree term of the Mukai vector is integral (see [6], Section 2.5, and [29], Proposition 3.4), we conclude that κ belongs to $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ if and only if $v(\kappa)$ is in $H^4(X, \mathbb{Z})_{00}$.

By [6], Section 2.5, we have that $v : \langle \lambda_1, \lambda_2 \rangle^{\perp} \to H^4(X, \mathbb{Z})_{00}(2)$ is injective. It remains to prove the surjectivity. It is possible to argue as in the proof of [4], Proposition 2.3, using [6], Section 2.5. We propose an alternative way. We observe that the lattices $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ and $H^4(X, \mathbb{Z})_{00}$ have both rank 22. Notice that $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ has signature $(20, 2)$. Moreover, the discriminant group of $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, because the Mukai lattice is unimodular. On the other hand, by Section 2.2 (see [27]), Proposition 5.1, we deduce that $H^4(X, \mathbb{Z})_{00}$ and $\langle \lambda_1, \lambda_2 \rangle^{\perp}$ have the same signature and isomorphic discriminant groups. Since the genus of such a lattice contains only one element by [76], Theorem 1.14.2, we conclude that v is an isometry which preserves the Hodge structures, as we wanted.

For the second part of the proposition, let $v(\zeta_i) = z_0 + z_2 h + z_4 + z_6 h^3 + z_8 h^4$ with $z_0, z_2, z_6, z_8 \in \mathbb{Q}$ and $z_4 \in H^4(X, \mathbb{Q})$. Using the previous computation, we have that

$$0 = \chi(\zeta_i, \kappa) = \int_X \exp(h)\, v(\zeta_i)^* \cdot k_4 = k_4 \cdot z_4$$

for every κ in $\langle \lambda_1, \lambda_2, \zeta_1, \dots, \zeta_n \rangle^\perp$. Since z_4 is by definition a linear combination of $c_2(\zeta_i)$, h^2 and $\gamma_X^* \sigma_2$, using again that k_4 is in $H^4(X, \mathbb{Z})_{00}$, we deduce that $k_4 \cdot z_4 = 0$ if and only if $k_4 \cdot c_2(\zeta_i) = 0$. This completes the proof of the statement. □

We point out that the lattice $\langle \lambda_1, \lambda_2 \rangle$ has a primitive embedding in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ by [76], Corollary 1.12.3. By Proposition 7.1.1, we have the isomorphism of lattices

$$\langle \lambda_1, \lambda_2 \rangle^\perp \cong H^4(X, \mathbb{Z})_{00} \cong E_8^2 \oplus U^2 \oplus I_{2,0}(2).$$

On the other hand, the lattice $\langle \lambda_1, \lambda_2 \rangle$ is isomorphic to $I_{0,2}(2)$. Notice that by [76], Theorem 1.14.4, there exists a unique (up to isomorphism) primitive embedding

$$i : I_{0,2}(2) \hookrightarrow \tilde{\Lambda}(-1) = E_8^2 \oplus U^4.$$

Let us denote by f_1, f_2 the standard generators of $I_{0,2}(2)$ and by u_1, v_1 (resp. u_2, v_2) the standard basis of the first (resp. the second) hyperbolic plane U. Then, we define i setting

$$i(f_1) = u_1 - v_1 \quad i(f_2) = u_2 - v_2.$$

The orthogonal complement of $I_{0,2}(2)$ via i is

$$I_{0,2}(2)^\perp \cong E_8^2 \oplus U^2 \oplus I_{2,0}(2).$$

In particular, we have an isometry $\phi : K(\mathrm{Ku}(X))_{\mathrm{top}} \cong \tilde{\Lambda}(-1)$ such that

$$\phi(\lambda_1) = i(f_1), \quad \phi(\lambda_2) = i(f_2), \quad \phi(\langle \lambda_1, \lambda_2 \rangle^\perp) \cong I_{0,2}(2)^\perp \cong E_8^2 \oplus U^2 \oplus I_{2,0}(2), \tag{7.2}$$

which is equivalent to the data of a marking for X. Hence, we can write $p(X) = [\phi_{\mathbb{C}}(\tilde{H}^{2,0}(\mathrm{Ku}(X)))]$.

Now, we prove that the isomorphism of Proposition 7.1.1 extends to the quotients $K(\mathrm{Ku}(X))_{\mathrm{top}}/\langle \lambda_1, \lambda_2 \rangle$ and $H^4(X, \mathbb{Z})/\langle h^2, \gamma_X^* \sigma_2 \rangle$. The proof is analogous to that of [4], Proposition 2.4.

Proposition 7.1.2. *The second Chern class induces a group isomorphism*

$$\bar{c}_2 : \frac{K(\mathrm{Ku}(X))_{\mathrm{top}}}{\langle \lambda_1, \lambda_2 \rangle} \to \frac{H^4(X, \mathbb{Z})}{\langle h^2, \gamma_X^* \sigma_2 \rangle}.$$

Proof. The composition of the projection $p : H^4(X, \mathbb{Z}) \twoheadrightarrow H^4(X, \mathbb{Z})/\langle h^2, \gamma_X^* \sigma_2 \rangle$ with c_2 is a group homomorphism, because

$$c_2(\kappa_1 + \kappa_2) = c_2(\kappa_1) + c_1(\kappa_1) c_1(\kappa_2) + c_2(\kappa_2) = c_2(\kappa_1) + m h^2 + c_2(\kappa_2) \quad \text{for } m \in \mathbb{Z}.$$

Since the second Chern classes of λ_1 and λ_2 are respectively

$$c_2(\lambda_1) = 2h^2 \quad \text{and} \quad c_2(\lambda_2) = -\gamma_X^* \sigma_{1,1},$$

we have that $\langle \lambda_1, \lambda_2 \rangle$ is in the kernel of $p \circ c_2$. In particular, the induced morphism $\bar{c}_2$ of the statement is well-defined.

Notice that $\bar{c}_2$ is injective. Indeed, let κ be an element in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ such that $c_2(\kappa)$ belongs to the sublattice $\langle h^2, \gamma_X^* \sigma_2 \rangle$. In particular, κ is an element of $K(X)_{\mathrm{top}}$ such that $\mathrm{ch}(\kappa)$ belongs to

$H^0(X,\mathbb{Z})\oplus H^2(X,\mathbb{Z})\oplus\mathbb{Z}\langle h^2,\gamma_X^*\sigma_2\rangle\oplus H^6(X,\mathbb{Z})\oplus H^8(X,\mathbb{Z})$. Then we have that κ is a linear combination of $[\mathcal{O}_X],[\mathcal{O}_H],[\mathcal{O}_S],[\mathcal{O}_\Sigma],[\mathcal{O}_L],[\mathcal{O}_P]$ with the notation of Section 2.4, because X is AK-compatible (see [61], Section 5). Since it belongs to $K(\mathrm{Ku}(X))_{\mathrm{top}}$, by the same computation done in the end of Section 2.4, we deduce that κ is a linear combination of λ_1 and λ_2, as we claimed.

Finally, we show that $\bar{c}_2$ is surjective. Let T be a class in $H^4(X,\mathbb{Z})$. By [4], Theorem 2.1(3), there exists τ in $K(X)_{\mathrm{top}}$ such that $v(\tau)$ is the sum of $-T$ with highter degree terms. Then the projection $\mathrm{pr}(\tau)$ of τ in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is a linear combination of τ and the classes of the exceptional objects in (6.2). Since the Chern classes of the exceptional objects are all multiples of h^i and $\gamma_X^*\sigma_{1,1}$, it follows that $c_2(\mathrm{pr}(\tau))$ differs form $c_2(\tau)$ by a linear combination of h^2 and $\gamma_X^*\sigma_{1,1}$. We conclude that $\bar{c}_2(\mathrm{pr}(\tau))=c_2(\tau)=T$ in $H^4(X,\mathbb{Z})/\langle h^2,\gamma_X^*\sigma_2\rangle$. □

Remark 7.1.3. Notice that the image of the algebraic K-theory $K(\mathrm{Ku}(X))$ in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is contained in $N(\mathrm{Ku}(X))$. However, we do not know if the opposite inclusion holds, because it is not clear if every Hodge class in $H^{2,2}(X,\mathbb{Z})$ comes from an algebraic cycle with integral coefficients. In the case of cubic fourfolds the integral Hodge conjecture holds by the work of Voisin (see [94]); thus, in [4], Proposition 2.4, they use this fact to prove that the $(1,1)$ part of the Hodge structure on the Mukai lattice is identified with $K(\mathrm{Ku}(X))_{\mathrm{num}}$.

Voisin's argument should work also for GM fourfolds, but it requires to give a description of the intermediate Jacobian of a GM threefold, as done in [71], Theorem 5.6, and [31], Theorem 1.4, in the cubic threefolds case. An other approach could be firstly to construct Bridgeland stability conditions for the Kuznetsov component (e.g. as in [7] for the Kuznetsov component of a cubic fourfold). Then, to deduce the integral Hodge conjecture by an argument on moduli spaces of stable objects with given Mukai vector, along the same lines as in [8] where they develop the argument for cubic fourfolds.

Finally, we need the following lemma, which is a consequence of Proposition 7.1.2; the proof is the same as that of [4], Proposition 2.5, so we skip it.

Lemma 7.1.4. *Let $\kappa_1,\dots,\kappa_n$ be in $K(\mathrm{Ku}(X))_{\mathrm{top}}$; we define the sublattices*

$$M_K:=\langle\lambda_1,\lambda_2,\kappa_1,\dots,\kappa_n\rangle\subset K(\mathrm{Ku}(X))_{\mathrm{top}}$$

and

$$M_H:=\langle h^2,\gamma_X^*\sigma_2,c_2(\kappa_1),\dots,c_2(\kappa_n)\rangle\subset H^4(X,\mathbb{Z}).$$

1. *An element κ of $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is in M_K if and only if $c_2(\kappa)$ is in M_H.*
2. *M_H is primitive if and only if M_K is.*
3. *M_H is non degenerate if and only if M_K is.*
4. *If M_K is in $N(Ku(X))$, then M_K and M_H are non-degenerate.*
5. *If M_K and M_H are non-degenerate, then M_H has signature (r,s) if and only if M_K has signature $(r-2,s+2)$ and they have isomorphic discriminant groups.*

Corollary 7.1.5. *The period point of a Hodge-special GM fourfold X belongs to the divisor $\mathcal{D}_d$ (resp. to the union of the divisors $\mathcal{D}'_d$ and $\mathcal{D}''_d$) for $d\equiv 0 \pmod 4$ (resp. for $d\equiv 2 \pmod 8$) if and only if there exists a primitive sublattice M_K of $N(Ku(X))$ of rank 3 and discriminant d which contains $\langle\lambda_1,\lambda_2\rangle$.*

Proof. As recalled in Section 2.3, we have that the period point of X satisfies the condition of the statement if and only if there is a labelling M_H of $H^{2,2}(X,\mathbb{Z})$ with discriminant d. The claim follows from Lemma 7.1.4. □

7.1.2 Associated (twisted) K3 surface and Mukai lattice

The first result of this section characterizes period points of very general Hodge-special GM fourfolds by their Mukai lattice. It is analogous to [4], Theorem 3.1, for cubic fourfolds and the proof develops in a similar fashion.

Theorem 7.1.6. *Let X be a Hodge-special GM fourfold. If X admits a Hodge-associated K3 surface, then $N(Ku(X))$ contains a copy of the hyperbolic plane. Moreover, the converse holds assuming one of the following conditions:*

1. *X is very general (i.e. $H^{2,2}(X,\mathbb{Z})$ has rank 3);*

2. *There is an element τ in the hyperbolic plane such that $\langle\lambda_1,\lambda_2,\tau\rangle$ has discriminant $d \equiv 2$ or 4 (mod 8).*

Proof. Assume that X has a Hodge-associated K3 surface; as recalled in the introduction and in Section 2.3, there exists a labelling M_H whose discriminant d satisfies (∗∗). Equivalently, by Corollary 7.1.5, there exists a primitive sublattice M_K in $N(\mathrm{Ku}(X))$ of rank 3 containing $\langle\lambda_1,\lambda_2\rangle$, with same discriminant d. Thus, there exists a rank one primitive sublattice $\mathbb{Z}w$ and a primitive embedding $j:\mathbb{Z}w \hookrightarrow U^3\oplus E_8^2$ with $w^2=-d$, such that $M_K^\perp$ in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is isomorphic to $\mathbb{Z}w^\perp$. Adding U to both sides of j, we get the primitive embedding of $U\oplus\mathbb{Z}w$ in $\tilde{\Lambda}(-1)$. Since $U\oplus\mathbb{Z}w$ and $M_K\subset K(\mathrm{Ku}(X))_{\mathrm{top}}\cong\tilde{\Lambda}(-1)$ have isomorphic orthogonal complements, they have isomorphic discriminant groups by [76], Corollary 1.6.2. Since one contains U, they are isomorphic by [76], Corollary 1.13.4. In particular, we conclude that U is contained in $M_K\subset N(\mathrm{Ku}(X))$, as we wanted.

Conversely, let X be as in the second part of the statement and let κ_1,κ_2 be two classes in $N(\mathrm{Ku}(X))$ spanning a copy of U. Notice that $\langle\lambda_1,\lambda_2\rangle$ is negative definite and U is indefinite; hence, the lattice $\langle\lambda_1,\lambda_2,\kappa_1,\kappa_2\rangle$ has rank three or four. We distinguish these two cases.

Rank 3. Let M_K be the saturation of $\langle\lambda_1,\lambda_2,\kappa_1,\kappa_2\rangle$ and we denote by d its discriminant. We have the inclusions $U\subset M_K\subset K(\mathrm{Ku}(X))_{\mathrm{top}}\cong\tilde{\Lambda}(-1)$. Since U is unimodular, there exists a rank one sublattice $\mathbb{Z}w$ with $w^2=-d$ such that $M_K\cong U\oplus\mathbb{Z}w$. On the other hand, the orthogonal to U in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is an even unimodular lattice of signature $(19,3)$; thus it is isomorphic to $U^3\oplus E_8^2$. As a consequence, we have that $M_K^\perp$ in $K(\mathrm{Ku}(X))_{\mathrm{top}}$ is isomorphic to $\mathbb{Z}w^\perp$ in $U^3\oplus E_8^2$. As observed before, this is equivalent to the existence of a labelling M_H for X of discriminant d satisfying condition (∗∗). This ends the proof in the rank three. In particular, this proves the statement for X very general.

Rank 4. Consider the rank three lattices of the form $\langle\lambda_1,\lambda_2,x\kappa_1+y\kappa_2\rangle$, where x and y are intergers not both zero. We define the quadratic form

$$Q(x,y):=\begin{cases}\mathrm{disc}(\langle\lambda_1,\lambda_2,x\kappa_1+y\kappa_2\rangle) & \text{if } x\neq 0 \text{ or } y\neq 0\\ 0 & \text{if } x=y=0.\end{cases}$$

We observe that the second Chern class $c_2(x\kappa_1+y\kappa_2)$ is in $H^{2,2}(X)$; hence, by the Hodge-Riemann bilinear relations and Lemma 7.1.4, it follows that $Q(x,y)$ is positive unless $x=y=0$.

Let

$$\begin{pmatrix}-2 & 0 & k & m\\ 0 & -2 & l & n\\ k & l & 0 & 1\\ m & n & 1 & 0\end{pmatrix}$$

be the matrix defined by the Euler pairing on the lattice $\langle\lambda_1, \lambda_2, \kappa_1, \kappa_2\rangle$. We have that

$$\begin{aligned} Q(x,y) &= \begin{vmatrix} -2 & 0 & kx+my \\ 0 & -2 & lx+ny \\ kx+my & lx+ny & 2xy \end{vmatrix} \\ &= 8xy + 2(kx+my)^2 + 2(lx+ny)^2 \\ &= (2k^2+2l^2)x^2 + (8+4km+4ln)xy + (2m^2+2n^2)y^2. \end{aligned}$$

We set

$$A := 2k^2 + 2l^2, \quad B := 8 + 4km + 4ln, \quad C := 2m^2 + 2n^2.$$

We denote by h the highest common factor of A, B and C; notice that h is even. We set

$$a = A/h, \quad b = B/h, \quad c = C/h$$

and we have that $Q(x,y) = hq(x,y)$, where

$$q(x,y) = ax^2 + bxy + cy^2.$$

In the next lemmas we prove that h satisfies $(**)$ and that there exist integers x and y such that $q(x,y)$ represents a prime $p \equiv 1 \pmod 4$.

Lemma 7.1.7. *The only odd primes that divide the highest common factor h of the coefficients of Q are $\equiv 1 \pmod 4$. Moreover, we have that $8 \nmid h$.*

Proof. Let $\mathbb{Z}[\sqrt{-1}]$ be the domain of Gaussian integers with the Euclidean norm $|\,|$. We set

$$\alpha := k + l\sqrt{-1} \quad \text{and} \quad \gamma := m + n\sqrt{-1}.$$

We rewrite the coefficients of Q as

$$A = 2|\alpha|^2, \quad B = 4\mathrm{Re}(\alpha\bar{\gamma}) + 8, \quad C = 2|\gamma|^2.$$

Suppose that p is an odd prime which is not congruent to 1 modulo 4, i.e. $p \equiv 3 \pmod 4$. Then p is prime in $\mathbb{Z}[\sqrt{-1}]$ (see [25], Proposition 4.18). Thus if p divides $A = 2\alpha\bar{\alpha}$, then p divides α. In particular, p divides $\mathrm{Re}(\alpha\bar{\gamma})$; so p does not divide $\mathrm{Re}(\alpha\bar{\gamma}) + 2$. It follows that p does not divide B and we conclude that $p \nmid h$.

For the second part, we observe that $8 \mid h$ if and only if k, l, m, n are even. In this case, we have that $8 \mid Q(x,y)$ for every $x, y \in \mathbb{Z}$. However, the assumption we made in item 2 of the theorem exclude this possibility. □

Lemma 7.1.8. *We have that $a \not\equiv 3 \pmod 4$, $c \not\equiv 3 \pmod 4$, and b is even.*

Proof. By definition we have that

$$k^2 + l^2 = \frac{h}{2}a \quad \text{and} \quad m^2 + n^2 = \frac{h}{2}c.$$

Notice that if an odd prime $\equiv 3 \pmod 4$ divides the sum of two squares, then it has to appear with even exponent (see [77], Corollary 5.14). Since by Lemma 7.1.7 the only odd primes dividing h are $\equiv 1 \pmod 4$, we have that a prime $\equiv 3 \pmod 4$ appears in the prime factorization of a and c only with even exponent. This gives the first part of the claim.

Now, we prove that b is odd if and only if $8 \mid h$. This implies the desired statement by the second part of Lemma 7.1.7.

Assume that b is odd. Since

$$B = 4(2 + \mathrm{Re}(\alpha\bar{\gamma})) = hb,$$

we have that $4 \mid h$. Thus, 4 divides $A = 2|\alpha|^2$ and $C = 2|\gamma|^2$. It follows that $(1 + \sqrt{-1}) \mid \alpha$ and $(1 + \sqrt{-1}) \mid \gamma$, which implies that $2 \mid \alpha\bar{\gamma}$. We conclude that $8 \mid B$ and thus $8 \mid h$.

Conversely, assume that $8 \mid h$; arguing as above, we see that $8 \mid B$. Notice that $2 \nmid h/8$, because otherwise $2 \mid B/8$, in contradiction with the fact that $B = 8(1 + 2r)$. Since

$$\frac{B}{8} = 1 + 2r = \frac{h}{8}b,$$

we conclude that b is odd. □

Lemma 7.1.9. *There exist integers x and y such that $q(x, y)$ is a prime $p \equiv 1 \pmod 4$.*

Proof. We adapt part of the proof of [4], Proposition 3.3, to our case. Let us list all the possible forms $q(x, y)$ modulo 4, using the restrictions given by Lemma 7.1.8:

For $b \equiv 0 \pmod 4$:	
$a \equiv 0 \pmod 4$	$0, y^2, 2y^2$
$a \equiv 1 \pmod 4$	$x^2, x^2 + y^2$
$a \equiv 2 \pmod 4$	$2x^2, 2x^2 + 2y^2$

For $b \equiv 2 \pmod 4$:	
$a \equiv 0 \pmod 4$	$2xy, 2xy + 2y^2$
$a \equiv 1 \pmod 4$	$x^2 + 2xy + y^2, x^2 + 2xy + 2y^2$
$a \equiv 2 \pmod 4$	$2x^2 + 2xy, 2x^2 + 2xy + y^2, 2x^2 + 2xy + 2y^2$

Notice that we have excluded the cases

$$x^2 + 2y^2, \quad x^2 + 2xy, \quad 2x^2 + y^2, \quad 2xy + y^2,$$

because by completing the square we get

$$x^2 + 2xy = (x + y)^2 - y^2 \equiv (x + y)^2 + 3y^2 \pmod 4$$

and

$$x^2 + 2y^2 = (x + y)^2 - 2xy + y^2 \equiv (x + y)^2 + 2xy + y^2 \equiv 2(x + y)^2 + 3x^2 \pmod 4,$$

which is not possible by Lemma 7.1.8.

We exclude the cases corresponding to a non primitive form, i.e.

$$0, 2y^2, 2x^2, 2x^2 + 2y^2, 2xy, 2xy + 2y^2, 2x^2 + 2xy, 2x^2 + 2xy + 2y^2.$$

In the other cases, we find that q can represent only numbers which are $\equiv 0$ or $1 \pmod 4$ (i.e. $y^2, x^2, (x + y)^2, x^2 + 2xy + 2y^2, 2x^2 + 2xy + y^2$), or $\equiv 0, 1$ or $2 \pmod 4$ (i.e. $x^2 + y^2$). Since a primitive positive definite form represents infinitely many primes, it must represent a prime $\equiv 1 \pmod 4$. □

We observe that h satisfies $(**)$ by Lemma 7.1.7. Thus, by Lemma 7.1.9 we conclude that there exist some integers x and y not both zero such that the discriminant of the lattice $\langle \lambda_1, \lambda_2, x\kappa_1 + y\kappa_2 \rangle$ satisfies $(**)$. This observation implies the proof of the statement. Indeed, if M_K is the saturation of this lattice, then its discriminant still satifies condition $(**)$, because $\mathrm{discr}(\langle \lambda_1, \lambda_2, x\kappa_1 + y\kappa_2 \rangle) = i^2 \mathrm{discr}(M_K)$, and M_K has rank three. By the same argument used at the end of the rank three case, we deduce that X has a Hodge-associated K3 surface. □

In Section 3.3 we give examples of GM fourfolds having a primitively embedded hyperbolic plane in the algebraic part of the Mukai lattice, but without a Hodge-associated K3 surface.

A consequence of Theorem 7.1.6 is that the condition of having a homological associated K3 surface implies the existence of a Hodge-associated K3 surface for very general GM fourfolds. This is analogous to the easy implication of [4], Theorem 1.1.

Theorem 7.1.10. *Let X be a GM fourfold such that $\mathrm{Ku}(X)$ is equivalent to the derived category of a K3 surface S. Under the hypothesis of the second part of Theorem 7.1.6, we have that X has discriminant d with d satisfying* $(**)$.

Proof. Assume that there is an equivalence $\Phi : \mathrm{Ku}(X) \xrightarrow{\sim} \mathrm{D^b}(S)$ where S is a K3 surface. We observe that $K(S)_{\mathrm{num}}$ contains a copy of the hyperbolic plane spanned by the classes of the structure sheaf of a point and the ideal sheaf of a point. Since Φ induces an isometry of Hodge structures $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1) \cong K(S)_{\mathrm{top}}$, it follows that U is contained in $N(\mathrm{Ku}(X))$. Applying Theorem 7.1.6, we deduce the proof of the result. $\square$

In the last part of this section we show that period points of Hodge-special GM fourfolds with an associated twisted K3 surface are organized in a countable union of divisors determined by the value of the discriminant. The argument essentially follows [47], Section 2. To this end, given a GM fourfold X, we consider the Mukai lattice $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1)$ with the weight-two Hodge structure and Euler pairing with reversed sign. Accordingly, the local period domain is given by

$$\Omega := \{w \in \mathbb{P}(I_{2,0}(2)^{\perp} \otimes \mathbb{C}) : w \cdot w = 0, w \cdot \bar{w} > 0\}$$

changing the sign of the definition in (6.1) and identifying $\Lambda = I_{2,0}(2)^{\perp}$. We set $\mathcal{Q} = \{x \in \mathbb{P}(\tilde{\Lambda} \otimes \mathbb{C}) : x^2 = 0, (x.\bar{x}) > 0\}$ and we consider the canonical embedding of Ω in $\mathcal{Q}$.

We recall that a point x of $\mathcal{Q}$ is of *K3 type* (resp. *twisted K3 type*) if there exists a K3 surface S (resp. a twisted K3 surface (S, α)) such that $\tilde{\Lambda}$ with the Hodge structure defined by x is Hodge isometric to $\tilde{H}(S, \mathbb{Z})$ (resp. $\tilde{H}(S, \alpha, \mathbb{Z})$) (see [47], Definition 2.5). By [47], Lemma 2.6, we have that a point $x \in \mathcal{Q}$ is of K3 type (resp. of twisted K3 type) if and only if there exists a primitive embedding of U (resp. an embedding of $U(n)$) in the $(1,1)$-part of the Hodge structure defined by x on $\tilde{\Lambda}$. We denote by $\mathcal{D}_{\mathrm{K3}}$ (resp. $\mathcal{D}_{\mathrm{K3}'}$) the set of points of Ω of K3 type (resp. of twisted K3 type).

Definition 7.1.11. A GM fourfold X has an associated twisted K3 surface if the period point $p(X)$ comes from a point in $\mathcal{D}_{\mathrm{K3}'}$.

Remark 7.1.12. Notice that if X has a Hodge-associated K3 surface, then it corresponds to a point x of K3 type. In fact, it follows from the first part of Theorem 7.1.6 and [47], Lemma 2.6. Moreover, the converse holds for very general Hodge-special GM fourfolds and for GM fourfolds satisfying condition 2 in Theorem 7.1.6. On the other hand, in Section 3.3 we see that a GM fourfold with period point of K3 type does not necessarily have a Hodge-associated K3 surface.

Proof of Theorem 5.0.1. The proof is analogous to that of [47], Proposition 2.10. As done in [47], Proposition 2.8, we have that

$$\mathcal{D}_{\mathrm{K3}'} = \Omega \cap \bigcup_{\substack{0 \neq \varepsilon \in \tilde{\Lambda} \\ \chi(\varepsilon,\varepsilon)=0}} \varepsilon^{\perp}.$$

Assume that x is a twisted K3 type point. By the previous observation, there exists an isotropic non trivial element ε in $\tilde{\Lambda}$. We consider the lattice $\langle \lambda_1, \lambda_2, \varepsilon \rangle$ in $\tilde{\Lambda}$, with Euler pairing given by

$$\begin{pmatrix} -2 & 0 & x \\ 0 & -2 & y \\ x & y & 0 \end{pmatrix}.$$

Notice that $\langle \lambda_1, \lambda_2, \varepsilon \rangle$ has discriminant $2x^2 + 2y^2$, which satisfies condition $(**')$. Then, let M_K be the saturation of $\langle \lambda_1, \lambda_2, \varepsilon \rangle$ in $\tilde{\Lambda}$. If d is the discriminant of M_K and i is the index of $\langle \lambda_1, \lambda_2, \varepsilon \rangle$ in its saturation, then we have

$$2x^2 + 2y^2 = i^2 d.$$

It follows that also d verifies condition $(**')$, as we wanted.

The other implication of the statement is proved following the same argument in the opposite direction. □

7.1.3 Extending Theorem 7.1.6: a counterexample

In this section we show that there are examples of GM fourfolds having a primitively embedded hyperbolic plane in the Néron-Severi lattice, but which cannot have a Hodge-associated K3 surface. Consistently with Theorem 7.1.6, our examples have $\mathrm{rank}(N(\mathrm{Ku}(X))) = 4$ and their period points belong only to divisors corresponding to discriminants $\equiv 0 \pmod 8$.

Assume that X is a GM fourfold such that $N(\mathrm{Ku}(X)) = \langle \lambda_1, \lambda_2, \tau_1, \tau_2 \rangle$ with Euler form given by

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2(k^2+l^2) & 1-2km-2ln \\ 0 & 0 & 1-2km-2ln & -2(m^2+n^2) \end{pmatrix} \tag{7.3}$$

(here we consider the Mukai lattice $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1)$ with weight-two Hodge structure and quadratic form with reversed sign). Recall that by the Hodge Index Theorem and Lemma 7.1.4, the Néron-Severi lattice of $\mathrm{Ku}(X)$ has signature $(2, *)$. Thus, we have to choose $k, l, m, n \in \mathbb{Z}$ such that the form in (7.3) has signature $(2, 2)$. This happens if and only if

$$4(kn - lm)^2 + 4km + 4ln - 1 > 0,$$

or, equivalently, if and only if

$$(kn - lm)^2 + km + ln > 0.$$

It is not difficult to see that there are infinite many values for these integers satisfying this requirement, e.g. if

$$km + ln > 0. \tag{7.4}$$

For the rest of this section, we will require this stronger condition to simplify the computation.

Notice that the classes

$$\kappa_1 := k\lambda_1 + l\lambda_2 + \tau_1 \quad \text{and} \quad \kappa_2 := m\lambda_1 + n\lambda_2 + \tau_2$$

span a copy of U in $N(\mathrm{Ku}(X))$.

However, it is easy to see that every labelling of X will have discriminant congruent to 0 modulo 8; hence, we cannot find a labelling with discriminant satisfying $(**)$. It follows that X cannot have a Hodge-associated K3 surface.

It remains to prove that such a GM fourfold exists. We recall that the image of the period map is contained in the complement of the divisors $\mathcal{D}_2$, $\mathcal{D}_4$ and $\mathcal{D}_8$ (it is expected that they coincide). In particular, the period point of a nodal GM fourfold lies in $\mathcal{D}_8$ (see [27], Section 7.5). We know that the period points we are considering are not in $\mathcal{D}_2$ and $\mathcal{D}_4$. In the next lemma, we study the conditions on k, l, m, n in order to avoid the divisor $\mathcal{D}_8$.

Lemma 7.1.13. *The period point of a GM fourfold X with $N(Ku(X))$ as in* (7.3) *satisfying* (7.4) *is not in $\mathcal{D}_8$ if and only if either $k \neq m$ or $l \neq n$, and $(k, l) \neq (1, 0), (0, 1)$, $(m, n) \neq (1, 0), (0, 1)$.*

Proof. We actually prove that $p(X)$ is in $\mathcal{D}_8$ if and only if either $k = m$ and $l = n$, or $(k, l) = (1, 0)$, or $(k, l) = (0, 1)$, or $(m, n) = (1, 0)$, or $(m, n) = (0, 1)$.

First of all, we observe that X has period point in $\mathcal{D}_8$ if and only if there is a class τ in $N(\mathrm{Ku}(X))$ of selfintersection -2, which is orthogonal to λ_1 and λ_2. Indeed, by Corollary 7.1.5, we have that $p(X) \in \mathcal{D}_8$ if and only if there is a primitive sublattice $\langle\lambda_1, \lambda_2, \tau\rangle$ of $N(\mathrm{Ku}(X))$ with discriminant -8. Since $N(\mathrm{Ku}(X))$ is an even lattice, the matrix representing the Euler pairing in this basis is of the form

$$\begin{pmatrix} 2 & 0 & a \\ 0 & 2 & b \\ a & b & 2c \end{pmatrix}.$$

Since 8 divides the discriminant, we have that a and b are even. Diagonalizing the matrix, we obtain a basis whose form is given by

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2k \end{pmatrix}.$$

Thus the discriminant is -8 if and only if $k = -1$, as we claimed.

As τ is orthogonal to λ_1 and λ_2, we write $\tau = \gamma\tau_1 + \delta\tau_2$ and

$$\chi(\tau) = -2(k^2 + l^2)\gamma^2 - 2(m^2 + n^2)\delta^2 + 2\gamma\delta(1 - 2km - 2ln).$$

We search the values of k, l, m, n such that $\chi(\tau) = -2$ has a solution in γ and δ. Equivalently, we study the equation

$$(\gamma k + \delta m)^2 + (\gamma l + \delta n)^2 - \gamma\delta = 1. \tag{7.5}$$

It is easy to see that $(k, l) = (1, 0)$, or $(k, l) = (0, 1)$, or $(m, n) = (1, 0)$, or $(m, n) = (0, 1)$ if and only if one between τ_1 and τ_2 has square -2.

Assume we are not in the previous situation, i.e. $\gamma\delta \neq 0$. We observe that if $k = m$ and $l = n$, then the possible solutions for the equation (7.5) are $(\gamma, \delta) = \pm(1, -1)$. Indeed, if $\gamma \neq -\delta$, we have

$$k^2(\gamma + \delta)^2 + l^2(\gamma + \delta)^2 - \gamma\delta \geqslant 2(\gamma + \delta)^2 - \gamma\delta > 1.$$

So, the solutions of (7.5) have the form $\gamma = -\delta$, i.e. $(\gamma, \delta) = \pm(1, -1)$. Conversely, if $(\gamma, \delta) = \pm(1, -1)$, then $k = m$ and $l = n$.

In the next, we prove that these are the only possibilities for k, l, m, n such that equation (7.5) has solution in γ and δ. Indeed, assume that there are other values for k, l, m, n such that there exists a solution (γ, δ) for (7.5). Notice that either $\gamma\delta > 0$ or $\gamma\delta = 0$, or $(\gamma, \delta) = \pm(1, -1)$; since the last two cases are in the previous list, we assume $\gamma\delta > 0$. Then we have

$$(k^2 + l^2)\gamma^2 + (m^2 + n^2)\delta^2 + \gamma\delta(2km + 2ln - 1) > 1,$$

because $2km + 2ln - 1 > 0$ by condition (7.4). It follows that (7.5) is not satisfied, in contradiction with our assumption. This ends the proof of the claim. □

The aim of the following part is to prove that among all the Hodge structures on $\tilde{\Lambda}$ having algebraic part given by the lattice defined in (7.3) and satisfying the condition in Lemma 7.1.13, there is at least one which belongs to the image of the period map of GM fourfolds.

We denote by $\mathcal{S}_{k,l,m,n}$ the set of period points in $\mathcal{D}$ whose Hodge structure on $\tilde{\Lambda}$ with algebraic part containing the lattice $N_{k,l,m,n}$ defined in (7.3), for k, l, m, n satisfying (7.4) and the condition in Lemma 7.1.13. This is the locus in $\mathcal{D}$ coming from $\mathbb{P}(N_{k,l,m,n}^{\perp} \otimes \mathbb{C}) \subset \mathbb{P}(\tilde{\Lambda} \otimes \mathbb{C})$. We set

$$\mathcal{S} := \bigcup_{k,l,m,n} \mathcal{S}_{k,l,m,n}$$

and we denote by $\mathcal{S}' \subset \mathcal{S}$ the locus of period points with algebraic part of rank four. Thus, points in $\mathcal{S}'$ are very general points of $\mathcal{S}$ and their algebraic part is equal to a lattice $N_{k,l,m,n}$.

Lemma 7.1.14. *The intersection of $\mathcal{S}'$ with the image of the period map p is non empty.*

Proof. The argument is inspired by [30], from which we take the notation. Let $\mathcal{M}_2^{(1)}$ be the moduli space of (smooth) hyperkähler fourfolds deformation equivalent to the Hilbert square of a K3 surface, with polarization of degree 2 and divisibility 1, whose period domain is given by $\mathcal{D}$. By [30], Theorem 6.1 and Example 6.3, we have that the image of the period map $p_2^{(1)} : \mathcal{M}_2^{(1)} \to \mathcal{D}$ is equal to the complement of the divisors $\mathcal{D}_2$, $\mathcal{D}_8$ and $\mathcal{D}''_{10}$. Thus, by our assumption, it follows that $\mathcal{S}$ is contained in the image of the period map $p_2^{(1)}$.

We denote by $\mathcal{U}_2^{(1)}$ the Zariski open set of $\mathcal{M}_2^{(1)}$ parametrizing smooth double EPW sextics. By [27], Theorem 8.1, we have that $p_2^{(1)}(\mathcal{U}_2^{(1)})$ meets every component of $\mathcal{D}_d$ among the possible values of $d > 8$ (except $\mathcal{D}''_{10}$). [1] As a consequence, if we set

$$D_{2,d}^{(1)} := \mathcal{D}_d \cap p_2^{(1)}(\mathcal{M}_2^{(1)}),$$

which is a hypersurface in $p_2^{(1)}(\mathcal{M}_2^{(1)})$, then

$$U_{2,d}^{(1)} := D_{2,d}^{(1)} \cap p_2^{(1)}(\mathcal{U}_2^{(1)}) \neq \varnothing.$$

In particular, we have that $U_{2,d}^{(1)}$ is a Zariski open set in $D_{2,d}^{(1)}$.

Now, we fix k and l, and we set $d := -8(k^2 + l^2)$. We have that

$$U_{2,d}^{(1)} \cap \bigcup_{m,n} \mathcal{S}_{k,l,m,n} \neq \varnothing,$$

where m, n vary in the countable range of values given by (7.4) and Lemma 7.1.13. Indeed, the union $\bigcup_{m,n} \mathcal{S}_{k,l,m,n}$ is dense in $D_{2,d}^{(1)}$ by [?], Section 5.3.4. As $U_{2,d}^{(1)}$ is Zariski open in $D_{2,d}^{(1)}$, we deduce the claim. Thus, there exist m and n such that

$$U_{2,d}^{(1)} \cap \mathcal{S}_{k,l,m,n} \neq \varnothing.$$

As the set above is Zariski open in $\mathcal{S}_{k,l,m,n}$, it contains a very general point of $\mathcal{S}_{k,l,m,n}$, which belongs to $\mathcal{S}'$. It follows that

$$p_2^{(1)}(\mathcal{U}_2^{(1)}) \cap \mathcal{S}' \neq \varnothing.$$

For every $x \in p_2^{(1)}(\mathcal{U}_2^{(1)}) \cap \mathcal{S}'$, we denote by $\tilde{Y}_A$ a smooth double EPW sextic such that $p_2^{(1)}([\tilde{Y}_A]) = x$.

Finally, we observe that there exists a GM fourfold X such that its associated double EPW sextic is precisely $\tilde{Y}_A$. Indeed, $\tilde{Y}_A$ determines a six dimensional $\mathbb{C}$-vector space V_6 and a Lagrangian subspace $A \subset \bigwedge^3 V_6$ without decomposable vectors. The choice of a five dimensional subspace $V_5 \subset V_6$ with $A \cap \bigwedge^3 V_5$ of the right dimension defines a Lagrangian data, which by [28], Theorem 3.10, Proposition 3.13, and Theorem 3.16 determines a GM fourfold X, as we wanted. ☐

Applying Lemma 7.1.14 to a lattice as in (7.3) with the conditions given by (7.4) and Lemma 7.1.13, we deduce that there is a GM fourfold X having the desired properties. We point out that this example includes all the possible GM fourfolds X with $N(\mathrm{Ku}(X))$ of rank four, which do not satisfy the assumption we made in item 2 of Theorem 7.1.6. It follows that the condition of having an embedded U in $N(\mathrm{Ku}(X))$ is not divisorial, in contrast to what happens for cubic fourfolds.

[1] We point out that [27], Theorem 8.1 does not cover the case of the divisor $\mathcal{D}''_{18}$. Anyway, in the first version of [30], Footnote 4, they argue as follows. Having a GM fourfold with an associated double EPW sextic whose period point is in $\mathcal{D}''_{18}$, we consider its *dual* GM fourfold. By [28], Theorem 3.27, they have dual EPW sextics. Then, their period points are dual by [29], Section 5.4, which means that one is in $\mathcal{D}'_{18}$ and the other is in $\mathcal{D}''_{18}$.

7.2 Associated double EPW sextic

The aim of this section is to prove Theorems 5.0.2, 5.0.3 and 5.0.4 stated in the introduction. We follow the argument of [2] and of [47] for the twisted case; in particular, we define a Markman embedding for $H^2(\tilde{Y}_A,\mathbb{Z})$ in $\tilde{\Lambda}$ and we apply Propositions 4 and 5 of [2].

7.2.1 Proof of Theorem 5.0.2 and 5.0.3

Assume that X is a GM fourfold with Lagrangian data (V_6,V_5,A) such that $\tilde{Y}_A$ is smooth. Before starting with the proofs, we need the following lemma, which relates the sublattice $\langle\lambda_1\rangle^\perp$ of $K(\mathrm{Ku}(X))_{\mathrm{top}}$ (equipped with the induced Hodge structure) and $H^2(\tilde{Y}_A,\mathbb{Z})$.

Lemma 7.2.1. *There exists an isometry of Hodge structures between the lattices $\langle\lambda_1\rangle^\perp\subset K(\mathrm{Ku}(X))_{\mathrm{top}}$ and $H^2(\tilde{Y}_A,\mathbb{Z})(1)$.*

Proof. Composing the isometry of Proposition 7.1.1 with that of Theorem 6.2.1, we obtain the Hodge isometry

$$f:\langle\lambda_1,\lambda_2\rangle^\perp\cong H^2(\tilde{Y}_A,\mathbb{Z})_0(1).$$

Notice that twisting by 1, we have shifted by two the weight of the Hodge structure on the primitive cohomology and we have reversed the sign of q; in particular, f is an isometry of weight zero Hodge structures.

Now, we observe that $\langle\lambda_1\rangle^\perp$ is isomorphic to $E_8^2\oplus U^3\oplus\mathbb{Z}(u_1+v_1)$ via the marking ϕ defined in (7.2). On the other hand, by [79], Theorem 1.1, we have the isometry

$$H^2(\tilde{Y}_A,\mathbb{Z})\cong H^2(S,\mathbb{Z})\oplus\mathbb{Z}\,\delta\cong E_8(-1)^2\oplus U^3\oplus I_{0,1}(2),$$

where S is a degree-four K3 surface and $q(\delta)=-2$. Twisting by 1, we get

$$H^2(\tilde{Y}_A,\mathbb{Z})(1)\cong E_8^2\oplus U^3\oplus\mathbb{Z}\,\delta,\quad \text{with } q(\delta)=2,$$

using that $U(-1)\cong U$. In particular, $\langle\lambda_1\rangle^\perp$ and $H^2(\tilde{Y}_A,\mathbb{Z})(1)$ are isomorphic lattices.

Let h_A be the polarization class on $\tilde{Y}_A$ with satisfies $q(h_A)=-2$ in $H^2(\tilde{Y}_A,\mathbb{Z})(1)$ (see [79], eq. (1.3)). We define an isometry $g:\langle\lambda_1,\lambda_2\rangle^\perp\oplus\langle\lambda_2\rangle\cong H^2(\tilde{Y}_A,\mathbb{Z})_0(1)\oplus\langle h_A\rangle$ such that

$$g(\lambda_2)=h_A\quad\text{and}\quad g(\langle\lambda_1,\lambda_2\rangle^\perp)=f(\langle\lambda_1,\lambda_2\rangle^\perp)=H^2(\tilde{Y}_A,\mathbb{Z})_0(1).$$

Notice that g preserves the Hodge structures, because f does and g sends the $(0,0)$ class λ_2 to the $(0,0)$ class h_A. In particular, g defines an isomorphism of Hodge structures $\langle\lambda_1\rangle^\perp\cong H^2(\tilde{Y}_A,\mathbb{Q})(1)$ over $\mathbb{Q}$.

We claim that g extends to an isometry $\langle\lambda_1\rangle^\perp\cong H^2(\tilde{Y}_A,\mathbb{Z})(1)$ over $\mathbb{Z}$. Indeed, we set $S_1:=H^2(\tilde{Y}_A,\mathbb{Z})_0(1)$, $S_2:=\langle\lambda_1,\lambda_2\rangle^\perp$ and $L:=H^2(\tilde{Y}_A,\mathbb{Z})(1)$. We denote by K_1 and K_2 the orthogonal complements of S_1 and S_2 in L. By definition, we have $K_1\cong\langle h_A\rangle$ and $K_2\cong\langle\lambda_2\rangle$. We set

$$H_1:=\frac{L}{S_1\oplus K_1}\subset d(S_1)\oplus d(K_1)\quad\text{and}\quad H_2:=\frac{L}{S_2\oplus K_2}\subset d(S_2)\oplus d(K_2);$$

recall that

$$d(S_i)\cong\frac{\mathbb{Z}}{2\mathbb{Z}}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}}\quad\text{and}\quad d(K_i)\cong\frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Let $H_{i,S}$ and $H_{i,K}$ be the projections of H_i in $d(S_i)$ and $d(K_i)$, respectively. Then, there is an isomorphism $\gamma_i:H_{i,S}\cong H_{i,K}$, given by the composition of the inverse of the projection on the first factor with the projection to the second factor. By definition $H_{i,K}$ is a subgroup of $d(K_i)\cong\mathbb{Z}/2\mathbb{Z}$.

We exclude the case $H_{i,K} = 0$. Then we have $H_{i,K} = d(K_i)$. We list the generators of the subgroups of $d(S_i) \oplus d(K_i)$ mapping to $\mathbb{Z}/2\mathbb{Z}$ via the two projections:

$$(1,0,1), (0,1,1) \text{ and } (1,1,1).$$

Since H_i is isotropic with respect to $q := q_{S_i} \oplus q_{K_i}$, we exclude $(1,1,1)$, because

$$q((1,1,1)) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = -\frac{1}{2} \neq 0 \mod \mathbb{Z}.$$

Moreover, recall that by [76], Proposition 1.4.1(b), we have that

$$d(L) \cong \frac{H_i^{\perp}}{H_i},$$

where $H_i^{\perp}$ is the orthogonal to H_i in $d(S_i) \oplus d(K_i)$. This condition implies that $H_i = \langle (0,1,1) \rangle$. Indeed, assume that $H_i = \langle (1,0,1) \rangle$. Writing explicitely the generators of the discriminant groups we have

$$d(K_i) = \langle \frac{f_2}{2} \rangle, \quad d(S_i) = \langle \frac{g_1}{2}, \frac{g_2}{2} \rangle, \quad d(L) = \langle \frac{g_1}{2} \rangle, \quad H_i = \langle \frac{g_1}{2} + \frac{f_2}{2} \rangle.$$

However, we have

$$\frac{H_i^{\perp}}{H_i} = \frac{\langle \frac{g_1}{2} + \frac{f_2}{2}, \frac{g_2}{2} \rangle}{\langle \frac{g_1}{2} + \frac{f_2}{2} \rangle} = \langle \frac{g_2}{2} \rangle$$

giving a contradiction.

Now, recall that by [76], Corollary 1.5.2, the isometry f extends to an isometry of L if and only if there exists an isometry $f' : K_1 \to K_2$ such that the diagram

$$\begin{array}{ccccccc} d(S_1) & \overset{\supset}{\longleftarrow} & H_{1,S} & \underset{\cong}{\overset{\gamma_1}{\longrightarrow}} & H_{1,K} & \overset{=}{\longrightarrow} & d(K_1) \\ & & \downarrow \bar{f} & & \downarrow \bar{f}' & & \\ d(S_2) & \overset{\supset}{\longleftarrow} & H_{2,S} & \underset{\cong}{\overset{\gamma_2}{\longrightarrow}} & H_{2,K} & \overset{=}{\longrightarrow} & d(K_2). \end{array}$$

commutes, where $\bar{f}$ and $\bar{f}'$ are induced by f and f' on the discriminant groups. So, we consider the isometry $f' : K_1 \to K_2$ sending h_A to λ_2; we have that f' acts trivially on the discriminant group. On the other hand, the isometry f acts either as the identity on $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or it exchanges the two factors. Assume we are in the first case. Then, we have that $\bar{f}' \circ \gamma_1((0,1)) = \gamma_2 \circ \bar{f}((0,1))$.

In the second case, we change the marking ϕ with the marking $\phi' : K(\mathrm{Ku}(X))_{\mathrm{top}} \cong \tilde{\Lambda}(-1)$, such that $\phi'(\lambda_1) = f_2$ and $\phi'(\lambda_2) = f_1$. By the same argument explained above, we have that $H_2 = \langle (1,0,1) \rangle$ and $H_{2,S} = \langle (1,0) \rangle$. It follows that

$$\gamma_2 \circ \bar{f}((0,1)) = \gamma_2((1,0)) = H_{2,K} = \bar{f}' \circ \gamma_1((0,1)).$$

Then [76], Corollary 1.5.2 applies and we deduce that the isometry f extends to an isometry of L. It follows that g is well defined over $\mathbb{Z}$, which concludes the proof. □

Remark 7.2.2. In the same way we can prove that there is a Hodge isometry $\langle \lambda_2 \rangle^{\perp} \cong H^2(\tilde{Y}_A, \mathbb{Z})(1)$ which extends f and sending λ_1 to h_A.

By Lemma 7.2.1, it follows that there is a primitive embedding

$$H^2(\tilde{Y}_A, \mathbb{Z}) \hookrightarrow K(\mathrm{Ku}(X))_{\mathrm{top}}(-1).$$

By [70], Section 9, it is unique up to isometry of $\tilde{\Lambda}$. Thus it defines a **Markman embedding** as discussed in [2], Section 1.

Proof of Theorem 5.0.2. If d satisfies $(**)$, then $N(\mathrm{Ku}(X))$ contains a copy of the hyperbolic plane U by Theorem 7.1.6. This proves that (a) implies (b). Recall that T_X is Hodge isometric to $T_{\tilde{Y}_A}(-1)$ by Theorem 6.2.1. Then (b) is equivalent to (c) by Proposition 4 in [2].

Assume that X is as in the second part of the statement. Then by Theorem 7.1.6 we have that d satisfies $(**)$ if and only if $U \subset N(\mathrm{Ku}(X))$. The statement follows applying [2], Proposition 4 as before. □

Remark 7.2.3. As observed in [2] for cubic fourfolds, under the hypothesis of Theorem 5.0.2, we have that $\tilde{Y}_A$ is birational to a moduli space of Bridgeland stable objects if and only if d satisfies $(**)$, by [10], Theorem 1.2(c).

Remark 7.2.4. As observed in [29], Remark 5.29, the image of the closure of the locus of smooth GM fourfolds having singular associated double EPW sextic is precisely the divisor $\mathcal{D}''_{10}$. We claim that there exist Hodge-special GM fourfolds with smooth associated double EPW sextic. Indeed, by [27], Section 7.2, this is clear for general GM fourfolds containing a τ-plane: their period points lie in the divisor $\mathcal{D}_{12}$ and they do not belong to $\mathcal{D}''_{10}$, because of generality assumption. Now, let d be a positive integer $\equiv 0, 2, 4 \pmod 8$. Assume $d > 12$ if $d \equiv 0 \pmod 4$, resp. $d \geqslant 10$ if $d \equiv 2 \pmod 8$. By [27], Theorem 8.1, the image of the period map meets all divisors $\mathcal{D}_d$, $\mathcal{D}'_d$ and $\mathcal{D}''_d$ for the respective values of the discriminant. More precisely, for every d as before, they construct a GM fourfold X_0 whose period point $p(X_0)$ belongs to the intersection of $\mathcal{D}''_{10}$ with $\mathcal{D}_d$ (resp. $\mathcal{D}'_d$ or $\mathcal{D}''_d$) if $d \equiv 0 \pmod 4$ (resp. $d \equiv 2 \pmod 8$). Consider the case $d \equiv 0 \pmod 4$. Since the period map is dominant (see Section 2.2), there exists an open dense subset U of $\mathcal{D}$ contaning $p(X_0)$ such that $U \subseteq p(\mathcal{M}_4)$. Notice that $U \cap \mathcal{D}_d$ is open in $\mathcal{D}_d$ and it contains $p(X_0)$. Moreover, it is not contained in $\mathcal{D}_d \cap \mathcal{D}''_{10}$, because the latter has codimension 1 in $\mathcal{D}_d$. It follows that $(U \cap \mathcal{D}_d) \backslash \mathcal{D}''_{10} \neq \varnothing$. The same argument applies to the case $d \equiv 2 \pmod 8$ and it completes the proof of the claim.

Remark 7.2.5. Assume that X is a Hodge-special GM fourfold such that $\tilde{Y}_A$ is smooth. Notice that the period point defined by the Hodge structure on $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1)$ is of K3 type if and only if $\tilde{Y}_A$ is birational to a moduli space of stable sheaves on a K3 surface.

As in [47], Proposition 4.1, in the case of cubic fourfolds, we can prove the twisted version of Theorem 5.0.2.

Proof of Theorem 5.0.3. Assume that $\tilde{Y}_A$ is birational to a moduli space $M(v)$ of α-twisted stable sheaves on a K3 surface S, where v is primitive in $\tilde{H}^{1,1}(S, \alpha, \mathbb{Z})$ and $(v, v) = 2$. Using the embedding $H^2(\tilde{Y}_A, \mathbb{Z}) \hookrightarrow K(\mathrm{Ku}(X))_{\mathrm{top}}(-1)$ and Torelli Theorem for hyperkähler manifolds, this is equivalent to have an isometry of Hodge structures $K(\mathrm{Ku}(X))_{\mathrm{top}}(-1) \cong \tilde{H}(S, \alpha, \mathbb{Z})$, which restricts to

$$H^2(\tilde{Y}_A, \mathbb{Z}) \cong H^2(M(v), \mathbb{Z}) \cong v^\perp \hookrightarrow \tilde{H}(S, \alpha, \mathbb{Z}).$$

Equivalently, by Theorem 5.0.1, we have that X is Hodge-special with a labelling of discriminant d satisfying condition $(**')$. This proves one direction.

On the other hand, assume that $p(X)$ belongs to a divisor with d satisfying $(**')$. Then the image v of λ_1 through $H^2(\tilde{Y}_A, \mathbb{Z}) \hookrightarrow K(\mathrm{Ku}(X))_{\mathrm{top}}(-1) \cong \tilde{H}(S, \alpha, \mathbb{Z})$ is primitive. Since the induced moduli space $M(v)$ is non-empty and the Hodge isometry $H^2(\tilde{Y}_A, \mathbb{Z}) \cong v^\perp \cong H^2(M(v), \mathbb{Z})$ extends to $\tilde{\Lambda}$, we conclude the desired statement. □

7.2.2 Proof of Theorem 5.0.4

Firstly, we need the following lemma, which is analogous to [2], Lemma 9, and that we will use also in Section 4.2.

Lemma 7.2.6. *Let X be a Hodge-special GM fourfold of discriminant d such that $d \equiv 2$ or $4 \pmod 8$. Then there exists an element $\tilde{\tau}$ in $N(Ku(X))$ such that $\langle \lambda_1, \lambda_2, \tilde{\tau} \rangle$ is a primitive sublattice of discriminant d with Euler pairing given, respectively, by*

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 2k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix} \quad \text{with } d = 2 + 8k,$$

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 2k \end{pmatrix} \quad \text{with } d = 4 + 8k.$$

Proof. By Corollary 7.1.5, there exists an element $\tau \in N(\mathrm{Ku}(X))$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant d with Euler paring given by

$$\begin{pmatrix} -2 & 0 & a \\ 0 & -2 & b \\ a & b & c \end{pmatrix}.$$

Notice that c is even, because $N(\mathrm{Ku}(X))$ is an even lattice.

Assume that $d \equiv 2 \pmod 8$; then one of a and b is even and the other is odd. Assume that b is even. Substituting τ with $\tau' = \tau + b/2\lambda_2$, we get a new basis with Euler form

$$\begin{pmatrix} -2 & 0 & a \\ 0 & -2 & 0 \\ a & 0 & c' \end{pmatrix}.$$

We can write $a = 4d + e$ with $e = -1, 1$. Then, the basis $\lambda_1, \lambda_2, \tilde{\tau} = \tau' + 2d\lambda_1$ has Euler pairing

$$\begin{pmatrix} -2 & 0 & e \\ 0 & -2 & 0 \\ e & 0 & 2k \end{pmatrix}.$$

If $e = -1$, we change $\tilde{\tau}$ with $-\tilde{\tau}$ and we return to the case $e = 1$. If a is even, by the same argument we obtain a basis with the second matrix in the statement. This proves the claim in the case $d \equiv 2 \pmod 8$. The case $d \equiv 4 \pmod 8$ works in the same way. $\square$

Proof of Theorem 5.0.4. Assume that there exist a K3 surface S and a birational equivalence $\tilde{Y}_A \dashrightarrow S^{[2]}$. By Lemma 7.2.1 and [2], Proposition 5, this is equivalent to the existence of an element w in $N(\mathrm{Ku}(X))$ such that the Euler pairing of $K := \langle \lambda_1, \lambda_2, w \rangle$ has the form

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & n \\ 1 & n & 0 \end{pmatrix} \quad \text{for some } n \in \mathbb{Z}.$$

In particular, the discriminant of K is $2n^2 + 2$. Let M_K be the saturation of K; if a is the index of K in M_K and d is the discriminant of M_K, we have that $\mathrm{discr}(K) = a^2 d$, as we wanted.

Conversely, assume that d satisfies condition (∗∗∗). Then there exist integers n and a such that $a^2 d = 2n^2 + 2$. Firstly, we observe that $2n^2 + 2$ satisfies (∗∗). Indeed, every odd prime p dividing $n^2 + 1$ is $\equiv 1 \pmod 4$, and $8 \nmid 2n^2 + 2$. It follows that a is the product of odd primes $\equiv 1 \pmod 4$; in particular, $a \equiv 1 \pmod 4$.

Suppose firstly that $d \equiv 2 \pmod 8$; then n is even. Indeed, assume that $n \equiv 1 \pmod 4$ (resp. $n \equiv 3 \pmod 4$). It follows that $n^2 + 1 \equiv 2 \pmod 4$; then $d \equiv 4 \pmod 8$, which is impossible.

Furthermore, by Lemma 7.2.6, there is an element $\tau \in N(\mathrm{Ku}(X))$ such that the sublattice $\langle \lambda_1, \lambda_2, \tau \rangle$ has Euler pairing of the form

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 2k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix}.$$

Assume that we are in the first case. We set

$$w := \frac{a-1}{2}\lambda_1 + \frac{n}{2}\lambda_2 + a\tau \in N(\mathrm{Ku}(X)),$$

where $n/2$ is an integer, because n is even. An easy computation shows that

$$\chi(\lambda_1, w) = 1 \quad \text{and} \quad \chi(w) = 0.$$

By [2], Proposition 5, it follows that $\tilde{Y}_A$ is birational to $S^{[2]}$ for a K3 surface S.

If we are in the second case, we consider the Markman embedding $H^2(\tilde{Y}_A, \mathbb{Z}) \subset K(\mathrm{Ku}(X))_{\mathrm{top}}(-1)$ defined by the Hodge isometry $\langle \lambda_2 \rangle^{\perp} \cong H^2(\tilde{Y}_A, \mathbb{Z})(1)$ (see Remark 7.2.2). Setting

$$w := \frac{n}{2}\lambda_1 + \frac{a-1}{2}\lambda_2 + a\tau \in N(\mathrm{Ku}(X)),$$

the proof follows from [2], Proposition 5.

Now assume that $d \equiv 4 \pmod 8$; then n is odd. Indeed, if $n \equiv 0 \pmod 4$ (resp. $n \equiv 2 \pmod 4$), then $n^2 + 1 \equiv 1 \pmod 4$. Since $a^2 d/2 = n^2 + 1$ and $a \equiv 1 \pmod 4$, we conclude that $d/2 \equiv 1 \pmod 4$, which is impossible. By Lemma 7.2.6, there is an element $\tau \in N(\mathrm{Ku}(X))$ such that the sublattice $\langle \lambda_1, \lambda_2, \tau \rangle$ has Euler pairing of the form

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 2k \end{pmatrix} \text{ with } d = 4 + 8k.$$

We set

$$w := \frac{a-1}{2}\lambda_1 + \frac{a-n}{2}\lambda_2 + a\tau \in N(\mathrm{Ku}(X)).$$

Notice that $(a-n)/2$ is integral, because n is odd. Arguing as before, we conclude the proof of the result. □

Remark 7.2.7. As seen in the proof of Theorem 5.0.4, condition $(***)$ implies condition $(**)$. On the other hand, condition $(***)$ is stricter than condition $(**)$. For example, $d = 50$ satisfies $(**)$, but not $(***)$.

Remark 7.2.8. In [51], Proposition 2.1, they proved that if a smooth double EPW sextic is birational to the Hilbert scheme $S^{[2]}$ on a K3 surface S with polarization of the degree d, then the negative Pell equation

$$\mathcal{P}_{d/2}(-1) : n^2 - \frac{d}{2}a^2 = -1$$

is solvable in $\mathbb{Z}$. This condition is actually condition $(***)$ in the case of the double EPW associated to a GM fourfold. Notice also that the K3 surface S, realizing the birational equivalence between $\tilde{Y}_A$ and $S^{[2]}$ in Theorem 5.0.4, has a pseudo-polarization of degree d. Indeed, the hypothesis implies that there is a rank-three sublattice M_K of $N(\mathrm{Ku}(X))$. Moreover, it contains a copy of the hyperbolic plane and $H^2(S, \mathbb{Z}) \cong U^{\perp} \subset N(\mathrm{Ku}(X))$, as explained in the proof of [2], Proposition 5. Then, the generator of $U^{\perp} \cap M_K$ has degree d, as we wanted. Moreover, if $p(X) \notin \mathcal{D}_8$, then there are no classes of square 2 in $H^4(X, \mathbb{Z})_{00} \cap H^{2,2}(X, \mathbb{Z})$. In this case, the pseudo-polarization is a polarization class on S.

Chapter 8

Stability conditions on Ku(X) (work in progress)

In this section we describe a conic fibration over a $\mathbb{P}^3$ associated to an ordinary GM fourfold. This construction was firstly described in the joint work [84] with Mattia Ornaghi. This is also the starting point of a joint work in progress with Alex Perry and Xiaolei Zhao, where we are trying to construct stability conditions on the component $\mathrm{Ku}(X)$.

8.1 Conic fibration over $\mathbb{P}^3$

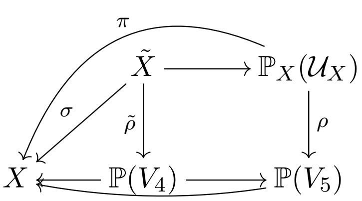
Let X be an ordinary GM fourfold. We denote by $\pi : \mathbb{P}_X(\mathcal{U}_X) \to X$ the projectivization of the bundle $\mathcal{U}_X$. We can consider the map

$$\rho : \mathbb{P}_X(\mathcal{U}_X) \to \mathbb{P}(V_5)$$

induced by the embedding $\mathcal{U}_X \hookrightarrow V_5 \otimes \mathcal{O}_X$. By [28], Proposition 4.5, we have that ρ is a fibration in quadrics. More precisely, by [28], Remark 3.15 and Remark B.4, the fibers of ρ are all conics in $\mathbb{P}^2$ except for the fiber over a point v_0 in $\mathbb{P}(V_5)$, which is a 2-dimensional quadric in $\mathbb{P}^3$. Let us fix a four-dimensional subvector space V_4 of V_5 such that the point v_0 is not contained in $\mathbb{P}(V_4)$. We set

$$\tilde{X} := \mathbb{P}_X(\mathcal{U}_X) \times_{\mathbb{P}(V_5)} \mathbb{P}(V_4)$$

and we denote by $\tilde{\rho}$ the restriction of ρ to $\tilde{X}$. Thus, we have the following commutative diagram



(8.1)

By the previous observations, we have that the restriction $\tilde{\rho}$ defines a flat conic fibration over $\mathbb{P}(V_4) \cong \mathbb{P}^3$. In the following, we prove that $\tilde{X}$ is smooth for a general choice of the subspace V_4.

Notice that for every x in X, the fiber of σ over x is equal to $\mathbb{P}(\mathcal{U}_{X,x} \cap V_4)$. In particular, we have that $\sigma^{-1}(x)$ is a point (resp. a line) if the dimension of $\mathcal{U}_{X,x} \cap V_4$ is equal to 1 (resp. if $\mathcal{U}_{X,x} \subset V_4$). It follows that the locus of non trivial fibers of σ is the intersection

$$S := \mathrm{Gr}(2, V_4) \cap X \subset \mathbb{P}(\bigwedge^2 V_5) \cong \mathbb{P}^9 \,. \tag{8.2}$$

Since the Grassmannian $\mathrm{Gr}(2, V_4)$ has degree 2, we have that the degree of S is at most 4. Moreover, the expected dimension of S is 2. On the other hand, by Lefschetz Theorem the fourfold X cannot contain a divisor with degree less than 10, because its class has to be cohomologous to the class of a hyperplane in X. Thus, we conclude that $\dim(S) \leqslant 2$. In the next lemma, we show that S is smooth if V_4 is general and v_0 is not contained in V_4; in this case, S is a del Pezzo surface of degree 4.

Lemma 8.1.1. *The locus S defined in (8.2) is smooth for a general subvector space V_4 of V_5 such that $v_0 \notin V_4$.*

Proof. Using the identification $H^0(\mathrm{Gr}(2,V_5),\mathcal{U}^*) \cong V_5^*$, we observe that the zero locus of a regular section of $\mathcal{U}^*$ is an embedded Grassmannian $\mathrm{Gr}(2,V_4) \subset \mathrm{Gr}(2,V_5)$. Indeed, a section $s \in H^0(\mathrm{Gr}(2,V_5),\mathcal{U}^*)$ corresponds to a linear form $\eta_s \in V_5^*$, which determines $V_4 = \ker(\eta_s) \subset V_5$. Thus, the zero locus $Z(s)$ of s contains points $x = [V_2]$ of $\mathrm{Gr}(2,V_5)$ such that the restriction of η_s to V_2 is trivial. This is equivalent to have $V_2 \subset V_4$ as we claimed.

Moreover, the condition that v_0 is not in V_4 is equivalent to the fact that the hyperplane in V_5^* defined by v_0 does not contain η_s. This assumption determines an open subset U of V_5^*. Since $\mathcal{U}^*$ is generated by its global sections, by [75], Theorem 1.10, we have that the zero locus of a general element in $H^0(\mathrm{Gr}(2,V_5),\mathcal{U}^*)$ is smooth of codimension 2. We point out that U contains a general element, i.e. a section whose zero locus is smooth of the expected codimension.

Now let us consider the map

$$\gamma_X^* : H^0(\mathrm{Gr}(2,V_5),\mathcal{U}^*) \to H^0(X,\mathcal{U}_X^*),$$

defined by pulling back sections of $\mathcal{U}^*$ via γ_X. Notice that γ_X^* is injective. Indeed, for $s \neq 0 \in H^0(\mathrm{Gr}(2,V_5),\mathcal{U}^*)$, the fact that $\gamma_X^*(s)$ is the zero section is equivalent to having $X \subset Z(s) = \mathrm{Gr}(2,V_4)$, in contradiction with the definition of X. We conclude that $s = 0$.

We prove that γ_X^* is actually bijective. To this end, it is enough to show that $H^0(X,\mathcal{U}_X^*)$ has dimension 5. The Todd class of X is computed in (6.3). By Riemann-Roch, we have that the Euler characteristic $\chi(X,\mathcal{U}_X^*) = 5$. In Lemma 8.1.2 we prove that all the higher cohomology groups of $\mathcal{U}_X^*$ vanish; it follows that $V_5^* \cong H^0(X,\mathcal{U}_X^*)$.

We conclude that the zero locus of a section s of $\mathcal{U}_X^*$ is represented by an intersection as in (8.2) and it is enough to choose s in the open subset U defined above, in order to guarantee the smoothness of $Z(s)$. ☐

Lemma 8.1.2. *For $i > 0$, we have that*

$$h^i(\mathcal{U}_X^*) := \dim(H^i(X,\mathcal{U}_X^*)) = 0.$$

Proof. We set $G := \mathrm{Gr}(2,V_5)$ and $M_X := G \cap \mathbb{P}(W)$; by definition we have that $X = M_X \cap Q$. To simplify the notation, given a sheaf $\mathcal{F}$ on G, we denote in the same way its pullback to M_X or to X and the pushforward of the pullback of $\mathcal{F}$ on G. Actually, by adjuction of pullback and pushforward, they have isomorphic cohomology groups.

Let us consider the exact sequences

$$0 \to \mathcal{O}_G(-1) \to \mathcal{O}_G \to \mathcal{O}_{M_X} \to 0$$

and

$$0 \to \mathcal{O}_{M_X}(-2) \to \mathcal{O}_{M_X} \to \mathcal{O}_X \to 0$$

of sheaves on G and on M_X respectively. We denote by $\mathcal{U}_{M_X}^*$ the restriction of $\mathcal{U}^*$ to M_X. Tensoring the first sequence by $\mathcal{U}^*$, the second by $\mathcal{U}_{M_X}^*$ and applying projection formula on the third element of the sequences, we get

$$0 \to \mathcal{U}^*(-1) \to \mathcal{U}^* \to \mathcal{U}_{M_X}^* \to 0 \tag{8.3}$$

and

$$0 \to \mathcal{U}_{M_X}^*(-2) \to \mathcal{U}_{M_X}^* \to \mathcal{U}_X^* \to 0. \tag{8.4}$$

In order to prove the claim, by (8.4) it is enough to show that

$$h^i(\mathcal{U}^*_{M_X}) = h^{i+1}(\mathcal{U}^*_{M_X}(-2)) = 0 \quad \text{for } i > 0.$$

Since

$$h^i(\mathcal{U}^*) = 0 \quad \text{for } i > 0$$

and

$$h^i(\mathcal{U}^*(-1)) = h^i(\mathcal{U}^*(-2)) = h^i(\mathcal{U}^*(-3)) = 0 \quad \text{for every } i$$

(see [33], Proposition 4.8), we deduce from (8.3) the desired vanishing. □

As a consequence, we obtain the smoothness of the blow up $\tilde{X}$.

Proposition 8.1.3 ([84], Proposition 2.3). *Let X be an ordinary GM fourfold. For a general vector space $V_4 \subset V_5$ such that the non flat point v_0 does not belong to $\mathbb{P}(V_4)$, we have that $\tilde{X}$ is the blow-up of X in S. In particular, the map $\tilde{\rho} : \tilde{X} \to \mathbb{P}(V_4)$ defined in* (8.1) *is a flat conic fibration, where $\tilde{X}$ is smooth.*

Proof. Choosing V_4 as in Lemma 8.1.1, we have that the locus S defined by (8.2) is smooth of codimension 2. Notice that $\sigma^{-1}(S)$ is by definition the projective bundle $\mathbb{P}_S(\mathcal{U}_X) \to S$. On the other hand, the exceptional divisor of the blow-up of X in S is isomorphic to the projectivized conormal bundle $\mathbb{P}_S(\mathcal{N}^*_{S|X})$. Since S is the zero locus of a regular section of $\mathcal{U}^*_X$, the conormal bundle of S in X is isomorphic to $\mathcal{U}_X$. Hence, we deduce that $\tilde{X}$ is the blow-up of X in S. It follows that $\tilde{X}$ is smooth, as we claimed. □

In conclusion, we have the following commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{i} & \tilde{X} & \xrightarrow{\alpha} & \mathbb{P}_{\mathbb{P}^3}(\mathcal{F}) \\ {\scriptstyle p}\swarrow & {\scriptstyle \sigma}\swarrow & & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle q} \\ S \longrightarrow X & & & & \mathbb{P}(V_4) \cong \mathbb{P}^3 \end{array} \tag{8.5}$$

where $\mathcal{F}$ is a rank three vector bundle over $\mathbb{P}^3$. We denote by H (resp. by h) the hyperplane class of X (resp. of $\mathbb{P}^3$) and we use the same notation for their pullback to $\tilde{X}$ and $\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})$. The rest of this section is devoted to prove that $\mathcal{F}$ is $\mathcal{T}_{\mathbb{P}^3}(-h)$, i.e. the tangent bundle over $\mathbb{P}^3$ twisted by the line bundle $\mathcal{O}_{\mathbb{P}^3}(-h)$.

Let $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2 V_4)$ be the five dimensional projective space containing $\mathrm{Gr}(2, V_4)$. Then the blow up of $\mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V_5)$ in $\mathbb{P}^5$ is the projective bundle $\tilde{\mathbb{P}} := \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}^{\oplus 6}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-h))$ over $\mathbb{P}^3 = \mathbb{P}(\bigwedge^2 V_5 / \bigwedge^2 V_4)$. We identify $\mathbb{P}^3$ with $\mathbb{P}(V_4)$. Indeed, if $v_1, \ldots, v_4$ is a basis for V_4 and we complete it to a basis of V_5 adding v_5, then $v_1 \wedge v_5, \ldots, v_4 \wedge v_5$ is a basis for the quotient $\bigwedge^2 V_5 / \bigwedge^2 V_4$. Then the natural identification gives the desired isomorphism.

Notice that the blow up of the Grassmannian $\mathrm{Gr}(2, V_5)$ in $\mathrm{Gr}(2, V_4)$ is contained in $\tilde{\mathbb{P}}$ and maps to $\mathbb{P}^3$. By the same argument used for X and its blow up in S, it can be described as the projective bundle $\mathbb{P}_{\mathrm{Gr}(2,V_5)}(\mathcal{U})$. In particular, if v is in V_4, then the fiber over v is $\mathbb{P}(v \wedge V_5)$ and we have the natural exact sequence of vector spaces

$$0 \to v \in V_4 \to v \in V_5 \to V_5/v \cong v \wedge V_5 \to 0.$$

This gives the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-h) \to \mathcal{O}^{\oplus 5}_{\mathbb{P}^3} \to \mathcal{E} \to 0,$$

where $\mathbb{P}_{\mathbb{P}^3}(\mathcal{E})$ is the blow up of $\mathrm{Gr}(2,V_5)$ in $\mathrm{Gr}(2,V_4)$. Writing $\mathcal{O}_{\mathbb{P}^3}^{\oplus 5} = \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^3}$, we have that $\mathcal{O}_{\mathbb{P}^3}(-h)$ maps into $\mathcal{O}_{\mathbb{P}^3}^{\oplus 4}$ by definition. Recognizing the Euler sequence from this construction, we deduce that $\mathcal{E} = \mathcal{T}_{\mathbb{P}^3}(-h) \oplus \mathcal{O}_{\mathbb{P}^3}$.

Now let us consider the intersection of $\mathrm{Gr}(2,V_5)$ with $\mathbb{P}(W) = \mathbb{P}^8$ and its blow up in $\mathrm{Gr}(2,V_4) \cap \mathbb{P}(W)$. By the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\mathrm{Gr}(2,V_4)\cap\mathbb{P}(W)}(\mathrm{Gr}(2,V_5)\cap\mathbb{P}(W)) & \longrightarrow & \mathrm{Bl}_{\mathbb{P}(\bigwedge^2 V_4)\cap\mathbb{P}(W)}(\mathbb{P}(W)) \\ \downarrow & & \downarrow \\ \mathrm{Bl}_{\mathrm{Gr}(2,V_4)}(\mathrm{Gr}(2,V_5)) & \longrightarrow & \mathrm{Bl}_{\mathbb{P}(\bigwedge^2 V_4)}(\mathbb{P}(\bigwedge^2 V_5)) \end{array} \tag{8.6}$$

we get

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^3}(-h)\ . \\ \downarrow & & \downarrow \\ \mathcal{T}_{\mathbb{P}^3}(-h) \oplus \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^3}(-h) \end{array} \tag{8.7}$$

This implies $\mathcal{F} = \mathcal{T}_{\mathbb{P}^3}(-h)$, as we wanted. Alternatively, we can consider the sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to 0,$$

since $\mathbb{P}(\mathcal{F})$ is a hyperplane section of $\mathbb{P}(\mathcal{E})$. Then applying the push-forward to $\mathbb{P}^3$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to (\mathcal{T}_{\mathbb{P}^3}(-h) \oplus \mathcal{O}_{\mathbb{P}^3})^\vee \to \mathcal{F}^\vee \to 0,$$

as desired.

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