FUZZY NORMS, DEFAULT REASONING AND EQUILIBRIUM SELECTION IN GAMES UNDER UNFORESEEN CONTINGENCIES AND INCOMPLETE KNOWLEDGE. [1][2]

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1. Introduction and motivation

This paper focuses on the role that norms play in the selection of equilibrium points seen as social conventions under unforeseen contingencies – that is, their role in the emergence of regularities of behaviour which are self-enforcing and effectively adhered to by bloodedly rational agents due to their self-policing incentives.

We are not concerned here with the explanation that norms induce equilibrium points in games because of their structuring effects on payoffs and incentives. Equilibrium points are seen as pre-existing possible combinations of strategies in a given game situation. Norms, on the other hand, are primarily seen as sets of logically consistent normative statements prescribing a single action for each player in any game situation of a given class of games. Hence, these statements simply ask players to adopt a given equilibrium behaviour within the existing equilibrium set of the given game situation. Their effectiveness does not reside in the enforcement of some set of actions by means of external incentives or sanctions, but rather, and only, in their functioning as cognitive devices which induce the appropriate system of expectations about equilibrium behaviours. Given the appropriate system of expectations, players have endogenous incentives to adopt the prescribed behaviour, as long as they expect the other players to execute the corresponding actions.

To explain, we shall split the reasoning into steps. First, by 'norm' is meant a set of normative statements asserting that a given rule of behaviour must be obeyed. Second, strategies that comply with norms are recognised as reciprocal best responses - the components of an equilibrium point. Third, adoption of the corresponding equilibrium strategies is supported by the belief that (for whatever reason) the norm is generally accepted. If players have these beliefs, they will expect that all of them will act accordingly. Then, given these mutual expectations, they will find the endogenous incentives to follow the strategies recommended because of their equilibrium properties.

Note, however, that any non-trivial non-cooperative game has numerous equilibrium points. Mixed-interest coordination games or the typical iterated Prisoners' Dilemma, where the Folk theorem applies, can be regarded as the natural classes of game situations to which norms refer. Consequently, given multiple equilibria, there are also multiple norms in the meaning just defined. There may be even more norms, in the sense of sets of statements prescribing individual actions for the players if some of them can also recommend the endorsement of behaviours that do not satisfy the equilibrium property. Even if the possibility of such utterances cannot be excluded in principle, we shall disregard them here, as they do not satisfy the definition of norms as a set of sentences prescribing compliance with pre-existing equilibrium strategies in a game. We may term these further norms "prima-facie norms" – so to speak. The main multiplicity problem still remains, however: given a multiplicity of equilibria - such that the expectation of each equilibrium supports the decision to comply with a different norm in turn - which norm should be chosen or followed? We assume that an equilibrium point gains its justificatory power in terms of individual rational choice, since shared knowledge that the equilibrium itself will be adopted as the solution of a game has gained general acceptance. Hence, multiple equilibria are situations such that, even if we know that an equilibrium point will be adopted, there is no implication as to which equilibrium will be adopted, and as a consequence we have no reason to act according to any of the possible equilibria. Hence no incentive-based reason for complying with any norm is effective. Unless we know which equilibrium is to be adopted in order to solve the game, we cannot say that we have the belief system concerning the equilibrium behaviour that grounds a given norm in terms of the incentive-based reasons to act upon it.

What we are looking for relates to the explanation that equilibrium points are selected because they are focal points or exhibit what is usually called salience (Shelling 1960, Lewis 1969). In those approaches which view focal points not as primitive brute psychological facts attached to behaviours but rather as the result of existing social norms (Kreps 1990), norms work as the determinants of the state of knowledge that makes an equilibrium the predictable outcomes of the game, providing in their turn endogenous incentives to comply with the relevant norm. There is, however, a sort of circle here: equilibrium points are selected because of their focality, which in turn is due to social norms that give "salience" to that equilibrium. But on the other hand this may be so only because some of the possible multiple norms are focal points on their own account, ones able to co-ordinate expectations in the relevant class of games. Which requires no less explanation than the salience of equilibrium points as such.

Thus more hypotheses are needed than the simple one that norms are sets of statements demanding action according to a rule of behaviour which coincides with an equilibrium point of the game. This by no means suffices to predict that an equilibrium point will emerge from strategic interaction in a game. Besides the normative meaning ("do that") and the coincidence with an equilibrium of the game ("if all the player follow the rule then they have the incentives to do so because it coincides with an equilibrium point"), it must be also assumed that players expect that all of them will abide by the rule. In other words, there must be shared knowledge that the norm will be followed if it must be true that the norm itself will induce the appropriate set of expectations and incentives. It is only because there is shared knowledge that (i) all the players will comply with the norm, and hence that (ii) in the given game the strategies

recommended by the norm will be adopted by all players, that they then have the strategic reason to adopt the recommended strategy because – finally - it is the best response to the actions expected to be chosen by the others.

As just mentioned, there is a risk of circularity in this reasoning. Because there is a system of expectations such that everybody knows that the norm will be complied with in general (for a given class of game), also there is a system of reciprocal iterated expectations that everybody will act according to a particular equilibrium point in the game at hand. As a consequence, all players have the appropriate game theoretical reasons to act according to the given equilibrium solution (by using the "best response" reasoning). This train of argument can be duplicated by each player. Hence it follows that it is common knowledge that this equilibrium point is the solution of the game. But this seems to reverse the conclusion with its premise. Was it not already required that everybody should know that a norm coinciding with an equilibrium would be followed by all the players? In order to escape this apparent vicious circle, it should be borne in mind that the inference about common knowledge of the game solution presupposes some kind of independent shared knowledge or expectation about norm-following behaviour in the given game other than the knowledge of the best response property of each player's strategy given his expectation regarding the other players' choices.

The relevant question to ask is therefore the following: where does the basic shared knowledge about norm-following behaviour come from? There must be an independent process of reasoning which induces us to believe that everybody will comply with the norm and the prescribed behaviour during the given game in order to generate the system of expectations about the strategies adopted in the game which rationally justifies a player's adoption of his equilibrium strategy because of its best response property. We shall seek to deal with this problem by investigating how players come to believe that in a given game all the players will conform with a general norm before they begin calculating that, because it is known that all of the players will conform with the norm, they then play the given equilibrium. To simplify, let us introduce an ex ante/ex post distinction into players' reasoning processes: it is ex post reasoning to say that, because it is common knowledge that a given equilibrium is the solution of the game, then players have the incentive to comply with it. By contrast, it is ex ante reasoning that takes us from the knowledge of a pre-existing norm, with prescriptive meaning and generally accepted as the solution concept for a given game class, to the conclusion that this norm is in fact accepted as the solution theory also in the given game that concerns us, so that the ex post common knowledge-based reasoning may then emerge. By 'equilibrium selection process' is meant the ex ante reasoning that brings the players to a state of shared knowledge that a given equilibrium will be the solution of a given game since the game situation confronts player with a choice among many equilibrium strategies.

Equilibrium selection is a topic widely addressed in game theory (Harsanyi 1975, Harsanyi-Selten 1988, Binmore 1987/88, Fudenberg-Levine 1998, Samuelson 1997). Harsanyi and Selten put forward the first theory of equilibrium selection, baseing it on a model of the outguessing regress idea – which consists of the supposition that players reciprocally simulate the each others' reasoning processes by repeatedly assigning their own previous reasoning steps to the counterpart – an approach that Binmore (1987) calls "eductive" (and which we shall adopt in this work). However, they developed their theory

under overly strong epistemic assumptions - so strong that not even their equilibrium selection model could satisfy them. Common knowledge of the prior – which is something like the point of view of an external rational observer concerning the basic uncertainty characterizing all the players and which dictates the only appropriate way for them to formulate their uncertainty about initial equilibrium choices – is incompatible with the assumption of a common prior as the starting point of the equilibrium selection process, i.e. a shared way to formulate uncertainty about the possible initial equilibrium choices in the game (Sacconi 1995b).[6]

On the other hand, we are looking for a solution to the equilibrium selection problem which is not usually addressed because it ranges over games where equilibrium points must be selected by players with incomplete knowledge of the possible states of the world and consequently of the game in which they are involved. Multiplicity occurs in this context because of the ambiguity concerning the game that players are about to play. This can be construed as if they were able to predict the single equilibrium with which they could solve a game if there were no ambiguity about the game that they are going to play. Nevertheless, multiplicity returns because of the ambiguity about the game that they are to play occasioned by unforeseen contingencies. This is not the same as extending the standard approach of games under incomplete information (Harsanyi 1967/68). Our approach is to model how players come to the conclusion that a given game falls within the domain of a pre-existing general norm, while we must nevertheless assume that, as far as a game where this norm does apply is concerned, it is common knowledge that the norm is complied with by all the participants. The point to be understood is how players come to recognise that a given game situation is an exemplar of a general abstract norm so that they can reasonably believe that in the game to be played they must comply with that norm while everybody else must also follow the same norm.

In a sense this is not to eliminate the presumption that there exists some "common knowledge" in the world about how players will act given certain norms and forms of interaction. This presumption can be retained, but we want to minimise its impact. In fact, we break the just-mentioned apparently circular argument at the point where it says that because we know that a general norm is followed in some games class, then everybody knows that in the game that they are playing (which belongs to the class) the norm will be followed and this implies that a given equilibrium will be used to solve that particular game. The chain of reasoning is broken where it states that because there is a norm, and because it is shared knowledge that it is followed in a certain class of game situations, then it is also shared knowledge that it will be followed in our game. This inference cannot be made as a matter of pure logic, nor can it be taken for granted as a primitive psychological fact. It must be explained by explicit modelling of the limited reasoning process performed by the players when they face game situations and reason as to whether these games can be understood according to the pattern of a solution theory devised for a general abstract game form. It is by this reasoning that we can explain from whence derives the required "common knowledge" (or something less than common but which can perform the same role - call it "shared knowledge"[7]) in the concrete game situations that the players face.

We intend to show that such reasoning employs fuzzy pattern recognition and default logic.[8] These two building blocks of the equilibrium selection reasoning process can be viewed as providing a new

account of how deliberative rationality, together with general abstract norms, aids equilibrium selection in games by subsuming new games under the domain of norms when this concerns not logical consequences but genuine default extensions of the domain of the existing general norms. Of course, all this makes sense only in contexts where norms make sense: games played under incomplete knowledge about unforeseen states of the world (that is to say, a typical domain of bounded rationality).

The paper proceeds as follows. First (sec.2), we define two versions of a basic game such that in each of them it is assumed that players know that a given solution (norm) is shared knowledge (the Nash product for the "demand game" version and the typical non-cooperative solution for the "enlarged DP" version – both coinciding with equilibrium points of the two versions of the basic game). Then (sects. 3 and 4), we introduce the basic incomplete knowledge situation by assuming that a move of Nature selects states of world where one of the two versions of the basic game could be played. However, some of the states of the world that Nature may choose are unforeseen. Thus, when Nature has made its choice, players face states that they ex post know to occur but are vaguely described in terms of the players' ex ante knowledge about the rules of the game. Unforeseen contingencies are then modelled as states that belong to events (sets of states) defined in terms of the ex ante knowledge base through fuzzy membership functions (Sacconi 2000). Players must decide which strategy to adopt under this characteristic lack of clear information, i.e. under vagueness about the game that they are going to play. Their information is represented by fuzzy measures of membership that induce possibility distributions over states but do not furnish shared knowledge of the norm (solution) which must be complied with in the game to be played in the given state. Consequently there is no basis at this stage to infer that everybody knows that a given equilibrium will be played, nor to conclude that a player must use his best response belonging to a given equilibrium solution.

Default reasoning then enters the scene. We first (sec. 5) introduce default logic according to Reiter (1980) and outline a rough approach to the default reasoning that players perform in order to assign a unique solution concept (norm) to each unforeseen state of the world (sec.6). However, although this rough approach is instructive, it is inadequate for explanaining of equilibrium selection. Hence section 7 introduces a proper selection process based on the reformulation of default logic in terms of possibility theory (sec. 7.1) suggested - after Zadeh (1978) - by Dubois, Prade and Benferhat (Bubois-Prade 1995, 1996, Bubois-Prade-Benferhat 1992). As the initial step of a recursive reasoning process each player's first hypothetical choice, given the basic vague knowledge about the game that they are going to play, is calculated in terms of a fuzzy expected utility function (Yager 1979) - where fuzzy utility is combined with a possibility measure on unforeseen states (sec.7.2).

In the second step (sec. 7.3), each player guesses the reasoning step carried out by the counterpart. Maybe erroneously, each player attributes his own model of reasoning to the other player by default, because it appears to him to be the most "normal course of reasoning" - since he is not yet aware of any falsification of his own model of reasoning. To provide a formal model of this, we use the formulas of a formal language to encode the players' knowledge base about the game and their default rules of inference - enunciates like "normally players possess such and such information" or "normally a player who possesses such information adopts strategy such and such in a game like this". Over these formulas

it is possible to induce a possibility ordering. This ordering will represent constraints on what the players believe to be the "normal course of facts" imposed by the default rules that extend the players' knowledge base.

Then, in the third step (sec. 7.4), we calculate each player's second hypothetical choice given the reconstruction of the other player's symmetrical reasoning and the ensuing new possibility that the player may choose any action in each game, given any state. For this calculation to be performed, the player must combine two items of vague knowledge: the first-order vague information expressed by the possibility distribution over unforeseen states of the world and the second-order vague information represented by the possibility of each action in each game given any state, which is derived by a default inference simulating the counterpart's reasoning. This brings the players to a new best response for each of them in fuzzy expected utility terms. Iterating the procedure will not change the set of predicted choices, so that we can conclude (sec.7.5) that the default-possibilistic reasoning will stabilize in a couple of strategy choices that constitutes one of the two basic equilibrium points. We end by suggesting that the resulting equilibrium will be supported by what in default logic is called an "extension" of a given theory - which is usually characterised as a fixed point (Reiter 1980) - obtained by the iterate application onto itself of the set of accepted defaults, without introducing any contradiction. A remarkable feature of the result is that the equilibrium achieved rests essentially on the non-monotonicity of default logic, and hence allows for mistakes and revisions which seem to pertain to the very nature of bounded rationality.

2. Two games of reference

Consider the typical demand game. 100 dollars are to be divided into two shares according to demands made by two players. If shares sum to 100 or less, the players obtain what they demand, but if the demanded shares sum to more than 100, they do not obtain anything. The players participate in the game by announcing proposals to share that they accept and that they think can be compatible with the proposal to share that the other player is simultaneously making. In fact, they make demands and offers at the same time, and only once. They can also refuse to demand or offer anything, and this will end the game notwithstanding the other player's proposal, even though he would have been ready to concede the entire sum to the first player. If one or both refuse to agree, the result is the status quo (zero dollars). Thus players must guess demands that may be acceptable to the counterpart, under the condition that only couples of demands that sum to no more than 100 are allowed.

Let us assume (this is where the announced assumption of background shared knowledge comes in) that the players know that in games like this there is a unique rational solution: that is, maximising Nash bargaining function (N.b.f.) which, under the appropriate utility representation, coincides with the 50-50 sharing rule. We simply assume that it is shared knowledge that in this game players will adopt the Nash bargaining solution because they accept the bargaining axioms given by John Nash (or the equivalent Harsanyi-Zeuthen's axiomatisation). This is an abstract norm generally accepted for solving games like the one under consideration, and it is shared knowledge that it is so. It also requires one more condition: games like the demand game are cooperative bargaining games; which means that the demands and offers

made by a player are binding for him and, as far as they are accepted by the counterpart, the proponent cannot renege on them - i.e. he must comply with any agreement once it has been reached. Let us consider a situation in which two compatible proposals are made. A pair of demands and offers determining an agreement then results, and it will necessarily be executed. The player cannot make defection on the agreement: he cannot refuse to concede the share offered and cannot expropriate the full sum by cheating the other party to the agreement. This amounts to saying that, within the demand game, it is shared knowledge that the game is cooperative, which implies that one more rule - pacta sunt servanda – applies to it, and it is shared knowledge that it is binding. There may be many reasons why it is known that this is an effective rule of the game: it may be that neither player is physically able to breach announced agreements, or the players may dislike cheating in the sense that the payoffs which they attach to defection are zero. The game could be embedded in an un-modelled and even larger repeated game, where respecting pacts would be a self-enforcing convention made endogenously stable by the threat of being punished or losing reputation in subsequent iterations. Here we do not ask why this is so, and simply assume that inasmuch as the players understand the demand game - so that it is shared knowledge that N.b.f. is the accepted solution - they must also understand that players cannot successfully make defection.

Next, let us assume that the players also know that there are some games where this is not the case. As far as the opportunity to share 100 dollars by agreement is concerned, these further games are completely identical to the demand games. However they are non-co-operative – i.e. they are game situations that do not satisfy the requirement that all players accept the rule pacta sunt servanda.

Illustrated below are two quite similar matrix games, the difference between them being that in the second game players can make defection and this is consequential to their payoffs, whereas in the first – which is the demand game - defection is not possible. This inability of cheating is captured in G1 by the non-co-operating strategy D, which identifies with keeping to the status quo without gaining any advantage from non-cooperation. The remaining strategies should be read as follows: any player may propose three possible sharings of 100 dollars (33-66; 50-50; 66-33, the first of the two numbers being the proponent's payoff). If it turns out that the proposals are compatible, in the sense that the total sum of 100 is not exceeded, they are automatically accepted and translated into payoffs. If the proposals are incompatible the result is nil for each player (status quo). They may also deliberately refuse to agree (keeping the status quo) by playing D. This decision has no positive effect on the player's utility, whereas it may frustrate the counterpart willing to take advantage by participating in the game.

Game G1:

	33	50	66	D
33	0.33 0,33	0.5 0.5	0.66 0.33	0 0
50	0.33 0.5	0.50 0.5	0 0	0 0
66	0.33 0.66	0 0	0 0	0 0
D	0 0	0 0	0 0	0 0

The players' utility functions are taken to be identical to linear transformations of the monetary payoffs mapping money onto the real line [0,1], so that 100 corresponds in utility to 1, 66 corresponds to 0.66, 50 to 0.5, 33 to 0.33, 0 to 0. The reason for adopting these particular utility functions will become clear later. The game is given a matrix form in order to stress that the pairs of mutually adapting demands summing to 100 are Nash equilibria of a simultaneous moves non-cooperative tacit coordination phase (in fact there is no explicit bargaining in the game as the players have only one round in which to make proposals). Thus the problem of choosing one equilibrium point from the possible three is represented as non-cooperative in the tacit coordination (bargaining) phase.

Any pair of demands satisfying the 100 constraints can be enacted as an agreement in the un-modelled implementation phase, and this reflects the underlying cooperative assumption. However the underlying co-operative nature of the game is also pointed up by the fact that D (defection) is ineffective and simply coincides with the status quo. It is also assumed, in accordance with cooperative bargaining theory, that the players will accept 50-50 as the unique rational solution. In fact the matrix game can be seen as a rough approximation of the Nash bargaining game, where the status quo is zero for both and the payoffs space is the convex (compact) utility representation of any possible proposal to share that sums to 100 or less, also including any linear combination mixing any two or more pure proposals to share.

The main assumption in game G1 is that players accept the cooperative theory of bargaining (a set or normative statements): this is shared knowledge. They are then expected to adopt the corresponding equilibrium - the one which maximises the product $\Pi_{i=1,2}(u_i - d_i)$) where, in this context, the status quo d_i

to any player i is 0, that is, (50,50) = (0.5, 0.5). This yields the required system of mutual expectations that in the game makes the case for conforming to the solution. We may say that games like G1 belong to the domain of a general abstract norm: the solution theory $\sigma1$, i.e. N.b.f.

Game G2:

	33	50	66	D
33	0.33 0,33	0.5 0.5	0.66 0.33	1 0
50	0.33 0.5	0.50 0.5	0 0	1 0
66	0.33 0.66	0 0	0 0	1 0
D	0 1	0 1	0 1	0.21 0.21

Next consider the second game G2 consisting of a slight payoff change to the demand game G1. In G2 the defection strategy D has an effective role to play, because the players are now able to free-ride the counterpart by refusing to co-operate, and at the same time by appropriating the entire sum if the counterpart decides to make a proposal for agreement. This can be read as cheating the other player and obtaining the entire sum after the other player has made a proposal. For example, let the second player make any proposal. What he immediately sees is the first player refusing to co-operate by leaving the bargaining table and tipping it over. Next, however, the first player considers the situation more carefully and realises that the entire sum to be shared has been stolen. According to the rule of the demand game, that sum should be forgone by both players. On the contrary, in game G2 the defecting player may obtain the entire sum by taking unilateral advantage of the other player's disposition to agree.

There is no assumption of any co-operative structure underlying this game. Moreover, G2 is meant to represent that no agreement has any chance of being self-enforced because the defection strategy is dominant in the game (which is in fact an enlarged PD game). The explicit representation through the D strategy of the ability to frustrate any cooperative endeavour and gain advantage from this conduct shows that the underlying game situation no longer belongs to the domain of cooperative game theory. Pacta are

not seen by the players as being *servanda*, and there is shared knowledge that the rules of the game do not imply that any agreement will be enforced by some unrepresented mechanism on the back of the situation. Moreover there is no shared knowledge that N.b.f. is the accepted norm in this game. On the contrary, there is a quite different norm accepted by both the players, and whose acceptance by them is shared knowledge: the solution theory $\sigma 2$, i.e. "if there is an outcome in dominant strategies, it can be expected to be the solution of the game". Here this is enough to select the unique solution (D,D).[9]

3. Unforeseen states of the world: vagueness on the appropriate solution concept

The foregoing section was only preparatory to our main analysis. Consider the following game situation. Two players P1 and P2 are in a position to play a game under incomplete knowledge, which means that they can be subjected to choices by a Nature that selects states of the world which they were not yet able to imagine before the start of the game. We assume that they know the two abstract classes of games - the demand game and the enlarged DP game - depicted above. They have the necessary background in game theory, and consequently also know the accepted solution theories to be applied to the two prototypal game situations, i.e. the two solution theories $\sigma 1$ and $\sigma 2$ that, according to our point of view, are two established social norms. They of course have shared knowledge that if the game that Nature induces them to play were G1, it would be obvious for them to play according to the norm $\sigma 1$ (N.b.f.), but if Nature were to involve them in a G2 game then it would be obvious to them that they must adhere to the norm $\sigma 2$. That is to say, at the back of their minds P1 and P2 have the representation of a set of possible states of the world which all and only match one of two possible events - having to play G1 or G2 - and they expect to be confronted by Nature's choices that will put them in states where alternatively G1 or G2 are to be played.

The game starts with a Nature's move. Nature may select one of two states w_1 and w_2, in each of which it is true that either G1 or G2 (but not both) will be played. However, Nature may also select (or, more exactly, let the players discover) a set Ω of unforeseen (by the players) states of the world, which are states that do not conform with the properties and predicates in use in P1's and P2's language with which they in general describe games. Owing to their unforeseen characteristics, these states do not exactly match the foreseen events that the players are prepared to learn. That is to say, in each $\omega_i \in \Omega$ it is neither true nor false that G1 or G2 is to be played. Note that this is not the usual game under uncertainty: it is not the case that the players have information partitions defined over the possible states' set which are not refined enough to discriminate exactly the occurrence of states where it is true that the game G1 is to be played from the occurrence of states where on the contrary it is true that game G2 is to be played. After Nature's choice, the players discriminate perfectly among the states and there is no uncertainty about which state has been reached. The situation is meant on the contrary to represent the ambiguity or vagueness that the players face, as long as any state is scrutinised, about whether in that particular state the game G1 or G2 is to be played. Vagueness is the consequence of their lack of ex ante knowledge

about the entire set of states that ex post Nature reveals to be possible, which implies that these are unforeseen states. Thus vagueness is the consequence of the players' lack of the conceptual and linguistic tools with which to clearly describe each $\omega_i \in \Omega$ in any detail or, in other words, the consequence of the fact that each $\omega_i \in \Omega$ is unclear as to the game to be played in it. If it were clearly specified ex post whether in any state $\omega_i \in \Omega$ the game G1 or G2 is to be played, then the players would have been able to foresee these states at least as states of the world that ex ante they knew to be possible.

According to one of the present authors (Sacconi 2000, 2001), however, unforeseen states of the world can be analysed in terms of fuzzy events, which are the fuzzy sub-sets of the reference set Ω. A general norm, construed by the theory as the solution for a given abstract class of games, has a domain of application (a set) which is defined by the membership functions of each state in the set representing that domain. Foreseen states have crisp membership functions (from the beginning we are able to say whether these states, if they occurred, would make a case for a specific norm). Unforeseen states, by contrast, cannot a priori be classified as belonging to one or another. When they are revealed they make the judgment about their belonging vague. Hence, ex post they have fuzzy (graded) membership functions taking their values into the real line [0,1] and expressing the degree to which the ex post revealed states can be recognised as belonging to the domain of any given norm. Vagueness has nothing to do with the normative content of the norms, or solution theories – i.e. the prescription to use the strategy D or the strategy 50. It concerns the descriptive premises for those norms, for example whether the game can be described as cooperative or non cooperative, or whether the descriptive conditions necessary to say that pacta are seen as servanda are satisfied.[10] We capture this point by introducing vagueness into the ex post description of the "defect" strategy in each ω_i state of the world. As far as the D strategy is describable as in G2 (alternatively in G1), the game is non-cooperative (alternatively cooperative) and does not belong (alternatively belongs) to the domain of the solution theory $\sigma 1$. Conditional on the description of strategy D, we may say whether the event G1 or the event G2 occurs. However, this is a matter of vagueness under unforeseen states of the world.

So far, we have had two possible norms, each prescribing a unique equilibrium point of the game within which it is the accepted solution theory. But in the game under incomplete knowledge and unforeseen states of the world, players cannot say at first glance which equilibrium point will come about, because they cannot say which solution theory is appropriate to the game that they are going to play (or, better, in the current state of the world). We want to understand how general and abstract norms provide at least the starting point for a norm-based equilibrium selection reasoning procedure which in the end will be able to determine which equilibrium point, belonging either to game G1 or game G2, will be played as the unique solution of the game. As expected, the answer will prove to depend on Nature's choice of the state of the world. This does not introduce any uncertainty, nor does it make room for the probabilistic nature of the equilibrium selection procedure. Nature first makes its choice and then players learn it without any uncertainty before they reach their own move in the game. If Nature did not select unforeseen states of the world, players P1 and P2 would learn with no ambiguity whether they were going

to play G1 or G2, and they would solve each game according to its accepted solution theory. Consequently, we need only investigate what happens when any $\omega_i \in \Omega$ is chosen by Nature.

4. Solution domains as fuzzy sets

Let us start by assuming that Nature has two consecutive moves at the beginning of the game. First it chooses among w_1, w_2 or Ω. Under w_1 and w_2 it is true that games G1 or G2 are played respectively. We disregard these branches of the game tree because there is a completely obvious solution for each of them. Next, Nature has its second move among states in Ω.

Take Ω to be the set of all the possible, alternative, complete descriptions of the interaction situations between players P1 and P2, which can be worked out by affirmation or negation of any concrete property that may influence the structure of their interaction - expressible by predicates in the language that will evolve in order to describe the unexpected features of Ω. Assume that any $\omega_i \in \Omega$ typically specifies the characteristics of the players, the principles of ethics that they follow, the set of options open to them, the environment that surrounds them, and every other property that makes it possible to say whether the norm pacta sunt servanda holds or does not. Thus, Ω is the "universe of discourse" containing all the possible concrete game situations that the players may be involved in once Nature has made its second choice. Remember, however, that some of these concrete characteristics are genuinely "unexpected" and are not univocally expressible by means of terms belonging to the ex ante language which was in use when the description of the two games G1 and G2 was under consideration. This means in fact that, within the ex ante in use language, some predicates cannot be exactly affirmed or denied when the states of the world $\omega_i \in \Omega$ are considered. In particular, under each state $\omega_i \in \Omega$, some of their describable characteristics cannot be traced back unambiguously to the conditions established as necessary to state whether the game form G2 or the game form G1 is the appropriate abstraction from the concrete game situations under consideration. Thus, it is not clear whether possession of such characteristics or their negation implies that game G1 or game G2 will be played. [11]

Then, let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the set of ex ante unforeseen states that Nature may select at its second move. Hence define two fuzzy events

$$\underline{G}1 = \{\omega_i, \mu_{\underline{G}1}(\omega_i) \mid \forall \omega_i \in \Omega\}$$

$$\underline{G}2 = \{\omega_i, \mu_{\underline{G}2}(\omega_i) \mid \forall \omega_i \in \Omega\}$$

For example let them be

$$\underline{G}1 = \{(\omega_1, 0.9), (\omega_2, 0.5), (\omega_3, 0.2), (\omega_4, 0.1)\}$$

$$\underline{G}2 = \{(\omega_1, 0.2), (\omega_2, 0.4), (\omega_3, 0.5), (\omega_4, 0.8)\}$$

The fuzzy membership functions μ_{G1} and μ_{G2} represent the vagueness associated with the occurrence of each state concerning whether the game to be played is G1 or G2. As G1 and G2 are sets of states defined by associating each unforeseen state ω_i with its membership function in the relevant set, they may

be understood as fuzzy events that occur when unforeseen states are selected. These sets may also be understood as the vague domains of application of the two solution theories σ1 and σ2 respectively (for example σ1 applies to ω_1 at degree 0.9).

Note that we cannot interpret the fuzzy distribution over states as a probability distribution over states, nor can we sum the fuzzy degrees in order to compute the total probability of the fuzzy event. Instead, we can consider any fuzzy event as a flexible constraint on the possibility of states, according to the expression

$$\pi_{X is \underline{Gj}}(\omega_i) = \mu_{\underline{Gj}}(\omega_i), \qquad i=1,2,3,4; \quad j=1,2$$

Let X be a variable ranging over the set of states Ω constrained by the fuzzy information that "X is G_j": that is, the game played in the state selected out of Ω is G_j, j=1,2, according to the relevant membership function. Saying that X is a variable ranging over the universe of discourse Ω means that X takes its values from the set of concrete descriptions of the possible interaction situations between P1 and P2, which is in fact what any state in the set Ω describes. The players' information constrains the possibility that X takes any value in Ω: that is, the admitted values of X are restricted to those that are compatible with the information that any given ω_i belongs to G1 or G2. Given the unexpected nature of these states, however, variable X is constrained by fuzzy information about the identification of any interaction situation with the game form G1 (G2 respectively). Thus the possibility of any state ω_i - given the fuzzy information about whether the game played under ω_i belongs to G_j - sums to the degree of the state's belonging to G_j. This also admits the somewhat misleading wording that $\pi_{X is \underline{Gj}}(\omega_i)$ is the possibility of the state ω_i, taking for granted that the players find themselves involved in the game G_j. (We say "misleading" because this possibility depends on the fuzzy constraint representing our information that "X is G_j". Once we are aware that this possibility depends on the condition that ω_i belongs to G_j to a certain degree, we can use it without further hesitation). If all we want to represent by a possibility function is the fuzzy constraint just mentioned, and no more than that fuzzy constraint, we can also compute the possibility of an event G_j from the following property of possibility theory:

$\Pi(\cup_{i \in I} A_i) = \sup \Pi_{i \in I}(A_i)$ for any finite sequence of natural numbers I (where Π stands for possibility). That is, in our case

$$\Pi(\underline{G_j}) = \sup\{\pi_{X is \underline{Gj}}(\omega_i) | \forall \omega_i \in \Omega\}, \quad j=1,2$$

and in practice, in our example

$$\Pi(\underline{G_1}) = 0.9$$

$$\Pi(\underline{G_2}) = 0.8$$

This means that the overall possibility that event G_j will occur is the supremum of the set of possibility degrees attached to any state of the world conditional? on the fuzzy information that in the state occurring, whichever it may be, also the event G_j occurs.

Let us now define the strategic decision problem faced by the players in this game. First the players observe the state of the world selected by Nature. Then they have to decide simultaneously among the

four strategies depicted in games G1 or G2. Their strategy spaces are $\forall i \ S_i = \{33, 50, 66, D\}$, i=1,2,3,4. They must take? a decision under vagueness because the meaning of the strategies D is vague. In particular, each player considers the case that, if he chooses D, and the game is G2, he may successfully make defection and take advantage of the second player's co-operation without doing his own part, but if instead the game is G1 he may obtain nothing. Secondly, he considers the case that whenever a proposal to share is made, the adversary may successfully make defection, obtaining the entire sum if the game is G2; or alternatively he may fail, leaving both with zero payoffs if the game is G1. These cases are assessed according to the possibility that the state ω_i is selected such that game G1 or game G2 is to be played – that is, in terms of the respective fuzzy membership functions defined over unforeseen states. Under G1 and G2 it is shared knowledge that solution theories $\sigma 1$ and $\sigma 2$ are respectively employed. Let us assume that this is a deductive consequence of the theory of games as applied to the two games G1 and G1 that the players take for granted (consider all this as part of the "knowledge base" that the players have in mind). As a consequence, they attempt to understand whether it is correct to play each of the two solution concepts contingent upon the vague states of the world. Thus their decision set shrinks to $\{50, D\}$.

To explain, take the point of view of player P1 and assume that Nature chooses state ω_2. Player P1 knows that, if state ω_2 occurs, the other player will conform up to degree 0.5 with solution $\sigma 1$, while the other player will conform up to degree 0.4 with solution $\sigma 2$. In other words, when state ω_2 occurs, his knowledge is representable by possibility 0.5 that game G1 will be played, and possibility 0.4 that game G2 will be played. Thus, whenever in the current vague situation - when state ω_2 occurs - he plays D, his payoffs will be 0 under game G1 (with possibility $\pi_{XisG1}(\omega_2) = 0.5$) and 0.21 under game G2 (with possibility $\pi_{XisG2}(\omega_2) = 0.4$). On the other hand, whenever in the same vague situation - when state ω_2 occurs - he plays 50 (which is the only other strategy he can choose in order to conform with a solution concept), he will obtain payoff 0.5 under game G1 (with possibility $\pi_{XisG1}(\omega_2) = 0.5$) and payoff 0 under game G2 (with possibility $\pi_{XisG2}(\omega_2) = 0.4$).

5. Default reasoning

The deliberative procedure used by players is a kind of default reasoning based on fuzzy membership functions mapping states into the fuzzy domain of norms. Reiter (1980) uses the term 'defaults' to denote rules of inference like syllogisms which extend the set of statements proved under a given theory and knowledge base by adding to them new statements derived through the application of the default rules to the basic set of statements of the theory and its default consequences. Given an incomplete collection of facts about the world, defaults are a way to "complete" our belief system about the world by inferring what is allowed by basic beliefs (justified by our knowledge base) plus a set of "reasonable" conditionals that cannot be falsified given our current incomplete knowledge. Typical default rules take the following form

$$A(x): MA(x) \rightarrow B(x)$$

———————————————

$$\therefore \; B(x)$$

where A(x) is a precondition to the rule belonging to the knowledge base, MA(x) $\rightarrow$ B(x) is the conditional clause that is checked for consistency with the existing knowledge base (and is assumed by default), and B(x) is a consequence that is added to the base of justified beliefs if the clause is "consistent". According to the coherence interpretation of default logic (Reiter 1980), in fact, the modal operator M means "it is consistent to assume that…" and clause MA$\rightarrow$ B can be interpreted as follows: "in the absence of proof to the contrary it is coherent to assume that A$\rightarrow$ B …". (M could also be understood as "normally…" or as "according to our best knowledge of the matter, it is reasonable to think that…").

This illustrates the basic intuition of the logic of default reasoning. Default rules of inference permit the addition of more sentences to our base of knowledge and beliefs by assuming that any conditional is acceptable whenever we have an incomplete collection of positive examples of it and we have no constructive proof of the contrary to the conditional itself ensuing from the knowledge base. In the absence of a refutation, although we do not have conclusive proof of its truth but only some positive examples, we are permitted to assume the conditional sentence as part of the premises of an inference rule. By using the sentences of a knowledge base - a theory - as major premises of a syllogism, and by adding to them default conditionals, we derive new conclusions that add to the theory. In other words, as long as a proof of the contrary does not arise, the "consistent" clause put together with the base knowledge allows statements to be derived which constitute extensions of the basic theory. Of course, default reasoning is non-monotonic and its conclusions are defeasible: as more information emerges, some conclusions can be retracted in order to account for new information. This is what is shown by the typical case where A(x) stands for Bird(x), B(x) for Fly(x), and it is known that penguin(x) implies Bird(x). The consequence of the default inference is "penguin flies", which eventually proves to be false.

To understand how defaults operate to extend a given knowledge base or body of justified beliefs by supplementing them with new sentences - derived through the application of default rules of inference - some definitions are in order. Let us define a default rule of inference as? any expression of the following form

$$\alpha(x): M\beta_1(x),…,M\beta_n(x)$$

———————————————————— $\qquad$ (#)

$$w(x)$$

where $\alpha(x)$, $M\beta_1(x),…,M\beta_n(x)$, w(x) are first-order formulas, $\alpha(x)$ is a premise belonging to the knowledge base, the $M\beta_i(x)$ are default clauses, w(x) is a consequence, and $x = x_1,…,x_n$ are variables (which are not free under the foregoing expression of a default). Next, following Reiter (1980), let L denote a first-order language, S a set of formulas, w a formula, S a set of closed formulas, i.e. formulas of L without free variables. Finally, let $Th_L(S)$ denote the set of closed formulas $w \in L$ which may be proved ("theorems") from the premises S.

A closed default theory $\Theta = \langle \Delta, \Phi \rangle$ is then a pair where Δ is a set of defaults of the given form (#) within which any formula is a closed formula of L, and Φ is a set of first-order closed formulas of L belonging to the knowledge base.

For any set of closed formulas $S \subseteq L$, let us define an operator Γ such that $\Gamma(S)$ is the smaller set of formulas satisfying the three following properties

1) $\Phi \subseteq \Gamma(\underline{S})$

2) $\mathrm{Th}_L(\Gamma(\underline{S})) = \Gamma(\underline{S})$

3) If $(\alpha : M\beta_1, \ldots, M\beta_n / w) \in \Delta$, $\alpha \in \Gamma(\underline{S})$ and $(\neg\beta_{1,\ldots}, \neg\beta_n) \notin \underline{S}$, then $w \in \Gamma(\underline{S})$

That is, the set $\Gamma(S)$ includes all the consequences following by default inference from premises like α which are consistent with the formulas set S. Because $\Gamma(S)$ includes all the basic first-order formulas W and the logical consequences of any formulas in $\Gamma(S)$, this means that $\Gamma(S)$ contains all the consequences of defaults consistent with S.

Now let us take a set of closed formulas $E \subseteq L$ such that $\Gamma(E) = E$: that is, E is the minimum fixed point of the operator Γ. This is what Reiter calls an extension for the theory $\Theta = \langle \Delta, \Phi \rangle$. Of course, E satisfies the conditions

1') $\Phi \subseteq E$

2') $\mathrm{Th}_L(E) = E$

3') If $(\alpha : M\beta_1, \ldots, M\beta_n / w) \in \Delta$, $\alpha \in E$ and $(\neg\beta_{1,\ldots}, \neg\beta_n) \notin \underline{E}$, then $w \in E$

Thus an extension E contains all the closed formulas derivable by defaults from the theory $\Theta = \langle \Delta, \Phi \rangle$ in such a way that any set of formulas derivable by applying defaults rules to closed formulas belonging to the theory Δ or to closed formulas derived from them are already included in E.

Let us see what amounts to characterising extension E as fixed point (Reiter 1980, Ginsberg 1986). Consider the operator Γ_Δ that carries a set of closed formulas (a formal theory) E to a new set of closed formulas (an extended formal theory) in which the set of default rules Δ have been applied in a way consistent with E. By means of Γ_Δ we generate all the default consequences of the set of closed formulas E. Then, because E is already an extension of the basic theory $\Theta = \langle \Delta, \Phi \rangle$, it must be such that under the application of the entire set of consistent default rules Δ it remains unchanged - that is, E already contains every change that can be performed by Γ_Δ, so that it is a fixed point of the operator Γ_Δ

$$E = \Gamma_\Delta(E)$$

One important aspect of this definition is that it captures some sense of stability in the theory. The set of believable sentences of an agent under any iteration of the same set of default rules (the operator just defined) is unchanged if the agent's theory is in fact an extension. An extension is the set of sentences defining a theory enunciated by means of the formulas of the formal language that cannot be changed by means of the same default rules of inference which were employed to generate the extension, and thus remains stable.

6. A first account of the players' default equilibrium selection process

Our argument is that players' reasoning about how to act in unforeseen states of the worlds proceeds by default reasoning in jumping from what they know to what they do not know but is nonetheless reasonable to assume about the unforeseen states. In our context, a default rule of inference is based on the fuzzy information that the players have about the truth of various sentences in the unforeseen states of worlds. Note that we introduce the following shift in the meaning of default clauses. The condition "Given our best knowledge, until proof to the contrary is given, it is consistent to assume that…" is substituted with "If vagueness does not exceed a given threshold, it is consistent to assume that …". In fact, it has the same meaning in the fuzzy context as the typical coherence condition on a default clause. Although some statements are not clearly true, they have not been proved to be false. At least we are vague about them. Moreover, because evidence in their favour keeps vagueness below a given level, and because they are not definitely false, we accept them as the basis for deriving some believable consequences. We define a threshold by an α-cut set including all the states that have a membership function in Gj, j=1,2, at least equal to the threshold δ. We then take for granted that if a state is member of the set Gj at least to degree δ, then the state definitely belongs to the domain of the relevant solution concept.

Assume $\delta = 0.7$, so that the threshold of vagueness not to be exceeded is $1-\delta = 0.3$. The δ-cut set defining the event that we can take for granted as being in the domain of application of the solution concept $\sigma 1$ is

$$G1_\delta = \{(\omega_1, 1), (\omega_2, 0), (\omega_3, 0), (\omega_4, 0)\}$$

while the event that we can take for granted as being in the domain of the solution concept $\sigma 2$ is the δ-cut set

$$G2_\delta = \{(\omega_1, 0), (\omega_2, 0), (\omega_3, 0), (\omega_4, 1)\}$$

Note that these are not fuzzy sets but crisp sets: one state belongs or does not belong to any of them in a clear-cut manner. This also suggests where the default inference comes in. We have only to assume that if a state ω_i is a member of the δ-cut set $G_{1\delta}$ this is enough to conclude that the game to be played in ω_i is G1, while if it is a member of the δ-cut set $G_{2\delta}$ the game to be played in ω_i is G2, and the solutions concepts to be used in the two cases are the relevant σj, for j =1,2, respectively. To see how this can be done, let us encode our game's statements as follows:

$\alpha_1(\omega_i)$: = "the state ω_i belongs to the event $G1_\delta$"

$\alpha_2(\omega_i)$: = "the state ω_i belongs to the event $G2_\delta$"

$M\beta_1(\omega_i)$: = "if the state ω_i belongs to $G1_\delta$ then 'normally' the game played under the state ω_i is G1"

$M\beta_2(\omega_i)$: = "if the state ω_i belongs to $G2_\delta$ then 'normally' the game played under the state ω_i is G2"

$M\beta_3(\omega_i)$: = "if the state ω_i belongs to $G_{i\delta}$ then normally the solution of the game is σj"

$w_j(\omega_i)$: = "the solution to be played under the state ω_i is σj"

Then the default rule of inference defined for our situation is the following

$$\alpha_k(\omega_i): M\beta_1(\omega_i), M\beta_2(\omega_i), M\beta_3(\omega_i)$$

$$\overline{\qquad\qquad\qquad\qquad\qquad}$$

$$\therefore\ w_j(\omega_i)$$

where formulas $\alpha_k(\omega_i)$ (k =1,2), belonging to the knowledge base Φ of the game, stand for $\alpha_1(\omega_i)$ or $\alpha_2(\omega_i)$ but not for both. The first default clause, according to the meaning of the operator M, can be read as follows: "games G_i played in situations ω_i , satisfying the condition $\mu_{G1}(\omega_i) \geq \delta$, are normally understood to be exemplars of the game class G1". The operator M also means "coherent", i.e. it is coherent with our knowledge to treat state ω_i as an exemplar of the game class G1 whenever vagueness whether state ω_i belongs to the event G1 is positive but does not exceed the stated threshold. Similarly read the second default assumption as referring to the case when the condition $\mu_{G2}(\omega_i) \geq \delta$ is satisfied. Finally, the third default assumption is that "games G_j played in situations ω_i satisfying one of the two conditions $\mu_{Gi}(\omega_i) \geq \delta$ stated above are normally understood to be solved according to the relevant solution concept σj". Following the notation of the foregoing section we may write (α_k: $M\beta_1$, $M\beta_2$, $M\beta_3$ /w_j) $\in \Delta$ to denote that the three default clauses generate the default knowledge base of our theory. Assume that Nature selects the state ω_1. In this case, the default inference rule implies $w_1(\omega_i)$, which states that the solution of the game played in ω_1 is $\sigma1$. By contrast, if Nature selects ω_4 we infer by default $w_2(\omega_i)$, which states that the solution $\sigma2$ is played.

This first default rule applied to the belief set of the players considerably simplifies their task, because now, at least in two states of the world, ω_1 and ω_4, they jump to the conclusion that there is a definite solution concept. Note that we are assuming that fuzzy sets are the same between the two players because vagueness is not a kind of subjective uncertainty but rather an inter-subjective state of incomplete knowledge (which makes this assumption far less questionable than its probabilistic counterpart – the "Harsanyi doctrine" of a unique common prior). Thus both the players jump to the same conclusion in the same two states, and they come identically to treat these two states as if they were exactly cases of the typical G1 and G2 games. In these games there is shared knowledge of the appropriate solution theory, and as a consequence the players will employ 50 or D in the two states respectively. We have a simple extension of those solution theories to cover these two "new" states of the world. If no more information enters the scene (remember that default inferences generate beliefs which are defeasible and non-monotonic) the two players will both come to the conclusion that they have arrived at the typical equilibrium solutions of the two standard situations.

This is quite a direct way to generate equilibrium solutions for games under unforeseen contingencies. New states of the world are simply subsumed under the domain of given general norms (and solution theories) by means of a default inference based on fuzzy pattern recognition of "new" states in terms of the domains of application of "old" general abstract norms. This is only a somewhat deeper explanation than the assumption that in any situation there is a norm which is salient and which is shared knowledge

to be complied with by players. In fact, we model the cognitive process by which norm can subsume unforeseen states under their domain so that salience of a given equilibrium is no longer an assumption or a brute datum of psychology, but begins to be endogenously explained. In fact, it is produced by a default rule applied to the belief set of the two players based on fuzzy information about the domain of application of general norms.

However, this approach does not properly account for the default reasoning process leading to equilibrium selection in the ambiguous game situation under consideration. First of all, it fails to investigate how players manage to reach a unique equilibrium point under the remaining two states of world ω_2 and ω_3, where the conditions for a δ-cut are not satisfied and players cannot jump by this default rule to a definite conclusion about the solution theory to be adopted. This suggests that default rules should be attached not only to high degrees of membership (thresholds) but to any measure of fuzzy membership of the states to the possible game G1 and G2 (or any possibility value of the states as constrained by fuzzy information about their being cases of the game G1 or G2).

Secondly, even though we have considered an example in which this is not the case, in general there is no reason for the threshold condition $\mu_{Gj}(\omega_i) \geq \delta$, i=1,2,3,4 and j=1,2, not to be satisfied by the same state ω_i for both the crisp events $G1_\delta$ and $G2_\delta$. Hence, two simultaneous default inferences would be admitted together, leading to the contradictory conclusions that in the same state ω_i the solution to be played is $\sigma1$ and the solution to be played is $\sigma2$. Of course, far from solving the equilibrium selection problem, this would create one more multiplicity problem and give rise to a situation typical of non-monotonic systems of logic, in which some default consequences are contradictory and should be revised and contracted. This also suggests that default rules should be attached, not to high degrees of membership but to the calculations of a player's possibilistic utility based on the composition of the fuzzy utility of actions in a state, under the hypothesis that a given game is to be played in that state, and the possibility of the state (derived from the fuzzy information about that state being an exemplar of any given game). Once this calculation had been performed by a player under the current unforeseen state of the world, it would establish a model of reasoning useful to derive the solution strategy that any "normal" player would adopt in the current unforeseen state of the world. Assigning this model of reasoning to each player in the game by a default rule, under the hypothesis that he is a "normal" player, would make it possible to derive the possibility that each "normal" player assigns to every choice of other "normal" players. It also enables him to restart his calculation of the possibilistic utility attached to any choice, given the new possibilistic information. This looks like the typical outguessing regress modelled by some equilibrium selection procedures where each player simulates any layer n of the other players' reasoning through introspection on his own reasoning at layer n-1 - see for example Harsanyi and Selten's tracing procedure (Harsanyi-Selten 1988, Sacconi 1995).

Third, thus far we have not been able to provide a unified language with which to treat both statements concerning the fuzzy knowledge that the players have on states of the world, and statements concerning what the players conclude by default rules of inference about the strategy to be adopted in each state. This is unfortunate, as the premises of the default inference rule are given by a condition

expressed in the language of fuzzy membership functions, whereas clauses and consequents of the default inference apparently belong to a quite different language. A remedy for this drawback is provided by the unified language of possibility theory (see sec.7), since this may be employed to account for default reasoning. On the one hand, possibility measures over states, seen as exemplars of certain games, are derived from statements about the players' knowledge, i.e. membership functions associating the same states with games understood as fuzzy sets. On the other hand, default rules connecting statements about players' (fuzzy) knowledge of states to statements about players' actions induce possibility measures over players' choices and over their use of solution concepts associated with types of games, given their knowledge of the current states. Both possibility measures are involved in the same fuzzy utility calculation which induces any player to select his best response at any stage of the equilibrium selection reasoning process.

7. A possibilistic account of the equilibrium selection default reasoning

7.1. Default reasoning and possibility theory

Items of incomplete knowledge are described by possibilistic logic formulas, i.e. first-order logic formulas with a numerical weight between 0 and 1, which is a lower bound on a possibility measure Π (Dubois&Prade 1996). This is a basic feature of our model. For this reason, before describing the players' steps in the reasoning process, we provide a brief introduction to possibility theory in addition to the fuzzy definition of possibility given in sec.4. Here the focus is on the meaning of some axioms and some propositions suited to our purposes (for more details see [Benfherat et al. 1992; Benfherat et al. 1997; Dubois&Prade 1995; Dubois&Prade 1996]).

Assume that a propositional language L is used to express some base of knowledge. Then, the notion of possibility is given by a possibility distribution π, which is a mapping from the set U of the interpretations of the formulas given in our propositional language L?to the interval [0,1] and thus provides a complete ordering of interpretations where the most plausible ones get the highest value 1. Roughly speaking, a possibility degree $\Pi(\phi)$?of a formula ϕ, given the set of interpretations which verify this formula, evaluates to what extent ϕ is consistent with the available knowledge expressed by π. Given its fuzzy logic's base, Π satisfies the characteristic property:

$$\forall \phi, \forall \psi, \Pi(\phi \vee \psi) = \max(\Pi(\phi), \Pi(\psi)) \; ; \; \Pi(\bot) = 0;$$

where $\bot$ is any inconsistent formula.

Moreover, from the axioms governing possibility measures it follows directly that

$$\Pi(\phi \wedge \psi) \leq \min(\Pi(\phi), \Pi(\psi))$$

An ordinal notion of conditional possibility can be defined by means of the Bayesian-like equation

$$\Pi(\phi \wedge \psi) = \min(\Pi(\psi|\phi), \Pi(\phi)) \text{ when } \Pi(\phi) > 0.$$

$\prod(\psi|\phi)$ is then defined as the greatest solution to the previous equation in accordance with the minimum specificity principle. It leads to

$$\prod(\psi|\phi) = 1 \text{ if } \prod(\phi \wedge \psi) = \prod(\psi) \text{ (i.e., if } \prod(\phi \wedge \psi) \geq \prod(\phi \wedge \neg\psi)\});$$

$$\prod(\psi|\phi) = \prod(\phi \wedge \psi) \text{ otherwise (i.e., if } \prod(\phi \wedge \psi) < \prod(\phi \wedge \neg\psi)\}) \text{ when } \prod(\phi) > 0.$$

If $\prod(\phi) = 0$, then by convention $\prod(\psi|\phi) = 1, \forall \psi \neq \perp$.

Since $\prod(\phi) > 0$, (Benfherat et al. 1997) we have also that

$$N(\neg\psi|\phi) > 0 \Leftrightarrow \prod(\phi \wedge \neg\psi) > \prod(\phi \wedge \psi)$$

where the dual necessity (or certainty) degree $N(\phi) = 1 - \prod(\neg\phi)$ evaluates to what extent ϕ is entailed by the available knowledge. Therefore we have:

$$\forall\phi, \forall\psi, N(\phi \wedge \psi) = \min(N(\phi),N(\psi)).$$

Finally, regarding the deductive processes involved in possibility logic, we shall recall only the following deduction rules, which can be been proved sound and complete for refutation.

If p, q and r are interpretations of our language, and α,β are weights inducing an ordering among the interpretations, the following *resolution rule* yields:

$$\frac{(\neg p \vee q, \alpha) \, (p \vee r, \beta)}{(q \vee r, \min(\alpha,\beta))}$$

This rule will be employed in order to find a possibilistic interpretation of the conditional knowledge base (i.e., as will be explained thoroughly in what follows, a set of defaults). In particular, it will be employed to derive from our default rules a possibility ordering on the interpretations of our propositional language's formulas given by a set of the possible worlds.

7.2. First step in the equilibrium selection process: calculating the first hypothetical best response of a player

We now use the possibility approach to the study of the default equilibrium selection reasoning in a game situation where nature selects unforeseen states of the world, so that it is vague which of two games G1 and G2 has to be played. We confine our analysis to unforeseen states ω_2 and ω_3 (see sec.4), as these were not accounted for in the previous section. Thus the relevant events, meant as domains of the two solution concepts σ1 and σ2, are the two fuzzy sets $G_1 = \{(\omega_2, 0.5), (\omega_3, 0.2)\}$ and $G_2 = \{(\omega_2, 0.4), (\omega_3, 0.4)\}$. We start by considering the possibilistic expected utility calculation that each player performs given his information on the two prototypal game G1 and G2 and the fuzzy knowledge on states ω_2 and ω_3, in order to derive his first conjecture on the solution strategy to be used.

To begin with, comment is required about the form of the players' utility functions. These take the form of fuzzy utility functions understood as measures of the degree to which a decision satisfies a

flexible constraint in terms of intermediary levels of capability. Let the constraint be expressed by the following requirement: "take the largest part of 100 dollars under the feasibility condition that shares gained by the two players must sum to no more than 100". Utility functions express the rank of capability associated with any decision, conditionally on the other player's decision, to satisfy the given constraint. This is expressed by means of a membership function which maps the decision set on the real line [0,1] representing levels of satisfaction. Hence, utility values coincide with membership degrees of a fuzzy set defined over the strategy set of each player taken as the typical reference set. These utility functions have the form appropriate to enable us carrying out the exercise in possibilistic decision-making that shall follow (they have essentially the same form as the possibility distribution but are defined on the strategy set).

Possibilistic expected utility here takes the form of the max-min operation, given that the operation min is the one appropriate to representing the intersection or conjunction (logical multiplication) of fuzzy sets, and the max operation is the one appropriate to representing the set theoretic union or disjunction (the logical sum) of fuzzy sets. In fact possibility and utility are here coincident with fuzzy sets (or fuzzy distributions), the first defined over the set of unforeseen states of the world and the second over the strategy set of each player. Multiplying the possibility of any event by? the utility of a decision when that event occurs means, in this context, operating the set-intersection between two fuzzy sets. Summing the expected utility of a decision over different possible events means calculating the set union operation among fuzzy sets. Consider one state of the world at a time, starting from ω_3 according to the point of view of player P1. Thus player P1's expected utility from strategy D, given the possibility that in state ω_3 game G1 or game G2 is played, is as follows

$U_1(D \mid \omega_3) = \max\{\min[\pi_{XisG1}(\omega_3), u_1(D|G1)], \min[\pi_{XisG2}(\omega_3), u_1(D|G2)]\} =$

$= \max\{\min(0.2, 0), \min(0.5, 0.21)\} = 0.21$

where $u_1(D|G1)$ is player P1's utility from playing the solution strategy D under the hypothesis that the game played is G1, $\pi_{XisG1}(\omega_3)$ is the possibility of state ω_3 under the fuzzy information that the game G_j (i.e. the variable X ranging over the set Ω of possible descriptions of interaction situations) played in ω_3 is G1, and $U_1(D \mid \omega_3)$ is player P1's possibilistic expected fuzzy utility from playing D given that ω_3 has occurred.

In analogous manner we obtain player P1's expected utility from? strategy 50

$U_1(50|\omega_3) = \max\{\min[\pi_{XisG1}(\omega_3), u_1(50|G1)], \min[\pi_{X isG2}(\omega_3), u_1(50|G2)]\} =$
$= \max\{\min(0.2, 0.5), \min(0.5, 0.21)\} = 0.2$

No other strategies are allowed by the solution theories. In order to maximise his expected utility function contingent on ω_3, player P1 has to select the pure strategy D.

Now consider the case of state ω_2. Player P1's expected utilities for the two strategies prescribed by the two solution theories are now respectively

$U_1(D \mid \omega_2) = \max\{\min[\pi_{XisG1}(\omega_2), u_1(D|G1)], \min[\pi_{XisG2}(\omega_2), u_1(D|G2)]\} =$
$= \max\{\min(0.5, 0), \min(0.4, 0.21)\} = 0.21.$

$U_1(50 \mid \omega_2) = \max\{\min[\pi_{XisG1}(\omega_2), u_1(50|G1)], \min[\pi_{XisG2}(\omega_2), u_1(50|G2)]\} =$

$= \max \{\min (0.5, 0.5), \min (0.4, 0) \} = 0.5.$

In order to maximize his expected utility function contingent on ω_2, player P1 has to select the pure strategy 50.

Thus, we know that if Nature selects state ω_2 or ω_3, then player P1 at the first step of his reasoning process - having at his disposal only the information concerning the possibility that states display certain games - must select 50 or D respectively. This is only his "first glance reasoning", however. It does not account for the fact that if a solution strategy has to be rationally chosen, it must be a best response against the simultaneous choice made by the other party, who faces a similar problem.

7.3. Second step in the equilibrium selection process: inferring by default the second player's reasoning and generating an overall conjecture of his choice

The second step in player P1's reasoning process is his attempt to guess the first step in the reasoning process just executed by the counterpart. Being reflective and rational, even if boundedly so, player P1 endeavours to simulate player P2's reasoning process by introspection. Here default reasoning again enters the scene. Player P1 has at his disposal his own model of reasoning at the first step, but there is no evidence that this is also the first step in player's P2 reasoning process. He only knows that in two states of the world ω_2 and ω_3 a typical player (himself) has reasoned according to the maximisation rule just mentioned and derived the solution theories to be rationally used in these two cases. No proof exists that any other similarly boundedly rational players would not employ similar calculations. Of course such information may be forthcoming later, but as long as the information available to the player is that just described in his model of reasoning, player P1 (at the moment) does not obtain such contrary evidence. Thus, player P1 introduces a default rule of inference by assuming that it is coherent with his knowledge base that player P2 in states like ω_2 and ω_3 will perform exactly the same calculations that he performed in his first step in the reasoning process.

Let us define a propositional language by encoding the statements of our theory in the following formulas:

- $\Phi :=$ the first-order knowledge base, which encodes statements concerning

    a) the objective description of the states $\omega2$ and $\omega3$, including their membership functions to the two fuzzy events G1 and G2;

    b) the algebraic method to calculate fuzzy expected utility;

    c) the description of the two solution theory $\sigma1$ and $\sigma2$

- s := "the strategy D is chosen when the state ω_3 occurs *and* the strategy 50 is chosen when the state ω_2 occurs". This is player P1's first step scheme of behaviour, and it may be formally written as $[(\omega_2 \wedge 50) \wedge (\omega_3 \wedge D)]$.

Moreover, we encode in our prepositional language the characteristic description of the player i, for $i \in \{1,2\}$, as follows:

- $\Psi(i)$, whose interpretation is "player i owns Φ";

- $\Sigma(i)$, whose interpretation is "player i acts according to s";

- $\Gamma(i)$, whose interpretation is "player i is rational".

Let us consider the following set of default clauses: "if player i owns Φ then normally player i acts according to s", "if player i is rational then normally he owns Φ", symbolically written as the set of default $\Delta = \{\Psi(i) \rightarrow \Sigma(i), \Gamma(i) \rightarrow \Psi(i) \}$, where Δ is the default knowledge base of player P1. Together with the set of formulas constituting the first-order knowledge base of our theory Φ, this allows us to define a default theory of the game $\Theta = \langle \Delta, \Phi \rangle$

In order to infer, as a logical consequence of his default theory $\Theta = \langle \Delta, \Phi \rangle$, the decision that will be made by player P2 in each state and his own consequent best response, player P1 needs to combine the calculation of the expected fuzzy utility with his default knowledge base or conditional knowledge. In particular, player P1 needs to deduce the measure of possibility $\Pi_2(50|\omega_2)$ on player P2's choice 50 (i.e. the plausibility of the statement "player P2 acts according to s" when the state is ω_2) and, similarly, the measure of possibility $\Pi_2(D|\omega_3)$ on player P2's choice is D. It is in accomplishing this task that the possibilistic logic machinery and its connections with default logic come in.

The basic idea is to encode each default like "if A then normally B", denoted by (A→B), into a constraint expressing that the situation where (A∧B) is true has greater plausibility than the one where (A∧¬B) is true (Benferat et al. 1992, Dubois&Prade 1995, 1996). We follow this approach by encoding the defaults denoted by ($\Psi(i) \rightarrow \Sigma(i)$) and ($\Gamma(i) \rightarrow \Psi(i)$) into constraints expressing that the situation where ($\Psi(x) \wedge \Sigma(x)$) and ($\Gamma(x)? \Psi(x)$) are true has greater plausibility than the one where ($\Psi(x) \wedge \neg \Sigma(x)$) and ($\Gamma(x)? \neg \Psi(x)$) are true. Then we need a qualitative plausibility relation $>_\Pi$ for comparing plausibility levels and generating a possibility ranking of situations (Dubois 1986). In the finite case, the only numerical counterparts to the plausibility relations are possibility measures such that $\Pi(a \vee b) = \max(\Pi(a), \Pi(b))$ for all a, b in the conditional knowledge base. Thus, a default ($\Psi(i) \rightarrow \Sigma(i)$) may be understood formally as an ordinal constraint ($\Pi\Psi(i) \wedge \Sigma(x)) > (\Pi\Psi(i) \wedge \neg \Sigma(i)$) on the possibility measure Π introduced via its qualitative counterpart based on $>_\Pi$, and describing the semantic of the knowledge available to player i. The ordinal constraint on possibility can be also shown to be equivalent to the measure of necessity N($\Sigma(i)|\Psi(i)$) which estimates to what extent strategies different from s have a low likelihood of being chosen by player x given that player i owns $\Psi(i)$.

Following Dubois&Prade (1995) and considering the two defaults in player P1's default knowledge base, the set of defaults $\Delta = \{\Psi(x) \rightarrow \Sigma(x), \Gamma(x) \rightarrow \Psi(x)\}$ will be represented by the following set of qualitative constraints C:

$$c1: \Psi(x) \wedge \Sigma(x) >_\Pi \Psi(x) \wedge \neg \Sigma(x)$$

$$c2: \Gamma(x) \wedge \Psi(x) >_\Pi \Gamma(x) \wedge \neg \Psi(x).$$

As numerical counterparts to plausibility relations are possibility measures, the qualitative constraints translate into the following set of ordinal constraints on possibility C'

$$\text{c'1:} \quad \Pi(\Psi(i)\wedge \Sigma(i)) > \Pi(\Psi(i)\wedge \neg\Sigma(i))$$

$$\text{c'2:} \quad \Pi(\Gamma(i)\wedge \Psi(i)) > \Pi(\Gamma(i) \wedge \neg\Psi(i)).$$

Let U be the finite set of interpretations of *our propositional language* $\Psi(i)$, $\Gamma(i)$, $\Sigma(i)$, Φ, s, x. These interpretations are to artificial possible worlds that correspond to formulas formulated in the artificial propositional language just defined, in which the conjunctions $^*\Psi(x)\wedge^*\Gamma(x)\wedge^*\Sigma(x)$ are true, where $*$ stands for the presence of the negation sign $\neg$ or its absence. Hence the models of our formal language are

$$U = \{ u_0: \neg\Psi(x)\wedge \neg\Sigma(x)\wedge \neg\Gamma(x); u_1: \neg\Psi(x)\wedge \neg\Sigma(x)\wedge\Gamma(x);$$

$$u_2: \neg\Psi(x)\wedge\Sigma(x)\wedge \neg\Gamma(x); u_3: \neg\Psi(x)\wedge \Sigma(x)\wedge\Gamma(x);$$

$$u_4: \Psi(x)\wedge\neg\Sigma(x)\wedge \neg\Gamma(x); u_5: \Psi(x)\wedge \neg\Sigma(x)\wedge \Gamma(x);$$

$$u_6: \Psi(x)\wedge \Sigma(x)\wedge\neg p; \ u_7: \Psi(x)\wedge \Sigma(x)\wedge \Gamma(x) \}.$$

Then the set of ordinal constraints C' on possibility measures translates into the following set of constraints C" on the possibility order of models:

$$\text{c"1} : \max(\pi(u_6), \pi(u_7)) > \max(\pi(u_4), \pi(u_5))$$

$$\text{c"2} : \max(\pi(u_5), \pi(u_7)) > \max(\pi(u_1), \pi(u_3)).$$

Now let $>_\pi$ be a ranking of U, such that $u >_\pi u'$ iff $\pi(u) > \pi(u')$ for each u, u'$\in$U. Any finite consistent set of constraints like $(a \wedge b) >_\Pi (a \wedge\neg b)$ thus induces a partially defined ranking $>_\pi$ on U that can be completed according to the principle of minimum specificity. The idea is to try to assign to each world $u\in$U the highest possibility level (in forming a well-ordered partition of U) without violating the constraints. The ordered partition of U associated with $>_\pi$ using the minimum specificity principle can be easily obtained by the following procedure (Dubois&Prade 1995)

a. i = 0, F_i= $\{\varnothing\}$

b. As far as U is not empty, repeat b.1.-b.4.:

b.1. i $\leftarrow$ i + 1

b.2. Put in F_i every model which does not appear in the right side of any constraint in C",

b.3. Remove the elements of F_i from U

b.4. Remove from C" any constraint containing elements of F_i.

Let us now apply this algorithm. We obtain:

- i=1 : F_1=$\{u_0, u_2, u_6, u_7\}$, U=$\{u_1, u_3, u_4, u_5\}$, C"=$\{\varnothing\}$

- i=1 : F_2=$\{ u_1, u_3, u_4, u_5\}$, U=$\{\varnothing\}$, C'=$\{\varnothing\}$

Finally, the well-ordered partition of U is: $\{u_0, u_2, u_6, u_7\} >_\pi \{u_5, u_4, u_1, u_3\}$.

Note that $\Sigma(x) \Leftrightarrow u_2 \vee u_7$ and also $\Sigma(x) \Leftrightarrow u_3 \vee u_6$. Since $\Pi(a \vee b) = \max(\Pi(a), \Pi(b))$ and for each $u \in F_2$ it is true that $u_2 >_\pi u, \ u_6 >_\pi u, \ u_7 >_\pi u$, then

$$\Pi(\Sigma(i)) \geq \Pi(u \vee u') \text{ for all } u, u' \in U . \tag{1}$$

In particular also $\Pi(\Sigma(i)) \geq \Pi(\neg\Sigma(i))$ is satisfied.

Without loss of generality, we can write $\Pi(\Sigma(i)) = \Pi_i(s)$, with i=1,2, where $\Pi_i(s)$ is the measure of possibility that player i acts according to $(\omega_2 \wedge 50) \wedge (\omega_3 \wedge D)$. Moreover, from the axioms governing possibility measures it follows directly that

$$\Pi_i(s) = \Pi_i[(\omega_2 \wedge 50) \wedge (\omega_3 \wedge D)] \leq \min[\Pi_i(\omega_2 \wedge 50), \Pi_i(\omega_3 \wedge D)]$$

and hence directly from (**1**)

$$\min[\Pi_i(\omega_2 \wedge 50), \Pi_i(\omega_3 \wedge D)] \geq \Pi(\neg\Sigma(i)) = \quad \Pi_i(\neg s) \tag{2}$$

Consider $\Pi_i(\omega_2 \wedge 50)$. From (2) it follows that

$$\Pi_i(\omega_2 \wedge 50) \geq \Pi(\neg\Sigma(i)) = \Pi_i(\neg s) \tag{3}$$

Note that, since ω_2 and ω_3 are always true, we can write the following relation

$$\neg s \Leftrightarrow \neg((\omega_2 \wedge 50) \wedge (\omega_3 \wedge D)) \Leftrightarrow ((\omega_2 \wedge \neg 50) \vee (\omega_2 \wedge \neg D) \vee (\omega_3 \wedge \neg 50) \vee (\omega_3 \wedge \neg D))$$

Therefore

$$\Pi_i(\neg s) = \max(\Pi_i(\omega_2 \wedge \neg 50), \Pi_i(\omega_2 \wedge \neg D), \Pi_i(\omega_3 \wedge \neg 50), \Pi_i(\omega_3 \wedge \neg D)) \tag{4}$$

From (3) and (4) it follows directly that $\Pi_i(\omega_2 \wedge 50) \geq \Pi_i(\omega_2 \wedge \neg 50)$. Since $\Pi_i(\omega_2) > 0$, (Benfherat et al. 1997) we have also that

$$N_i(\neg 50 \mid \omega_2) > 0 \Leftrightarrow \Pi_i(\omega_2 \wedge \neg 50) > \Pi_i(\omega_2 \wedge 50)$$

where $N_i(\neg 50 \mid \omega_2)$ is the *conditional necessity measure* which estimates to what extent strategies different from s have a low degree of possibility of being enacted by player i given that the state ω_2 has occurred (Benfherat et al. 1997). Since the second member is false, we conclude that $N_i(\neg 50 \mid \omega_2) = 0$.

By definition $N_i(\neg 50 \mid \omega_2) = 1 - \Pi_i(50 \mid \omega_2)$, where $\Pi_i(50 \mid \omega_2)$ is the *conditional possibility measure* that player i plays 50 given that the state ω_2 has occurred; thus we obtain $\Pi_i(50 \mid \omega_2) = 1$. In a similar way we also obtain that $\Pi_i(D \mid \omega_3) = 1$.

It may be concluded that if player P1 uses the default rules represented by the constraints on the possibility ranking of interpretations (artificial worlds) given at the beginning of this section, then the possibility he that assigns to the choices of player P2 reproduces the behaviour that player P1 would have adopted at step 1 in his reasoning. However, in order to produce a reasonable overall guess of players P2's choices, player P1 has to combine two items of vague knowledge, i.e. the first-step fuzzy knowledge

about each state being an exemplar of any given possible game, and the second-step default conclusions concerning player P2's first step choice. A reasonable way to do this is by first considering the conjunction of any pair of events like "player P2 chooses strategy 50" and "the game played in the current state is Gj" i.e.$(G_j | \omega_h)$, for j=1,2 and h=2,3; and second by taking the possibility of the resulting conjoint conditioned events, i.e. $(50 \wedge (\underline{G}_j | \omega_h))$ - where it happens that when the state ω_h is the case, the game played is $\underline{G}_j$ and player P2 chooses strategy 50. Given the Bayes-like definition of the possibility of conjoint events in sec.7.1, we may write

$$\Pi_i[50 \wedge (\underline{G}_j | \omega_h)] = \min[\Pi_i[50 | (\underline{G}_j | \omega_h)], \Pi_i(\underline{G}_j | \omega_h)], (i=1,2; h=2,3; j=1,2)$$

The meaning of $(\underline{G}_j | \omega_h)$ is "occurrence of the game $\underline{G}_j$ in the state ω_h" that, given the vague knowledge context, corresponds to the fuzzy membership function stating how it is vague that state ω_h belongs to the event $\underline{G}_j$ - that is, how it is true that when state ω_h occurs also the event $\underline{G}_j$ occurs. The double conditioning of choice 50 is not problematic because the meaning of the conditioned event $50 | (\underline{G}_j | \omega_h)$ is the same as the conditioned event $(50 | \omega_h)$. In fact, the event that player P2 chooses 50 can be subordinated by player P1 only to the occurrence of a state ω_h, which is the only evidence that he can use, being definitely vague which game is to be played in ω_h. Moreover, because the membership function represents an inter-subjective state of vague information which is the same for all the players, it follows that we may write

$$\Pi_i(\underline{G}_j | \omega_h) = \mu_{\underline{G}j}(\omega_h) = \pi_{XisGj}(\omega_h)$$

Thus we obtain

$$\Pi_i[50 \wedge (\underline{G}_j | \omega_h)] = \min[\Pi_i(50 | \omega_h), \pi_{XisGj}(\omega_h)]$$

Then we may calculate

$$\Pi_i[50 \wedge (G_1 | \omega_2)] = \min[\Pi_i(50 | \omega_2), \pi_{XisG1}(\omega_2)] = \min(1, 0.5) = 0.5,$$

as a possibility measure that player i plays 50 in the game G_1 given the state is ω_2, and for the remaining Joint events

$$\Pi_i[(D \wedge (G_1 | \omega_3)] = \min[\Pi_i(D | \omega_3), \pi_{XisG1}(\omega_3)] = \min(1, 0.2) = 0.2,$$

$$\Pi_i[(50 \wedge (G_2 | \omega_2)] = \min[\Pi_i(50 | \omega_2), \pi_{XisG2}(\omega_2)] = \min(1, 0.4) = 0.4,$$

$$\Pi_i[(D \wedge (G_2 | \omega_3)] = \min[\Pi_i(D | \omega_3), \pi_{XisG2}(\omega_3)] = \min(1, 0.5) = 0.5.$$

These possibility measures conclude player P1's attempt to guess the reasoning performed by player P2 at the first step in his reasoning process.

7.4. Third step in the selection process: calculating a second hypothetical best response

In third step of his reasoning process, Player P1 may now check whether the conclusions that he reached in the first step, relative to the strategy to be used in each unforeseen state ω_2 and ω_3, are stable against the new beliefs that he has gained by default about player P2's action in each state.

A reasonable way to calculate expected utility for player P1 is

$U_1(D|\omega_2) =$

$= \max\{\min\{\Pi_2[(50 \wedge G_1)| \ \omega_2], u_1[D|(G_1 \wedge 50)]\}, \min\{\Pi_2[(50 \wedge G_2)| \ \omega_2], u_1[D|(G_2 \wedge 50)]\}\} =$

$= \max\{\min(0.5 , 0), \min(0.4 , 1)\} = 0.4$

where $u_1[D|(G_1 \wedge 50)]$ is the utility value of player P1 when player P2 plays 50 in the game G_1. Similarly

$U_1(50|\omega_2) =$

$= \max\{\min\{\Pi_2[(50 \wedge G_1)| \ \omega_2], u_1[50|(G_1 \wedge 50)]\}, \min\{\Pi_2[(50 \wedge G_2)|\omega_2], \ u_1[50|(G_2 \wedge 50)]\}\}$

$= \max\{\min(0.5, 0.5), \min(0.4 , 0.5)\} = 0.5$

$U_1(D|\omega_3) =$

$= \max\{\min\{\Pi_2[(D \wedge G_1)| \ \omega_3], u_1[D|(G_1 \wedge D)]\}, \min\{\Pi_2[(D \wedge G_2)| \ \omega_3], u_1[D|(G_2 \wedge D)]\}\} =$

$= \max\{\min(0.2 , 0), \min(0.5 , 0.21)\} = 0.21$

$U_1(50|\omega_3) =$

$\max\{\min\{\Pi_2[(D \wedge G_1)| \ \omega_3], u_1[50|(G_1 \wedge D)]\}, \min\{\Pi_2[(D \wedge G_2)| \ \omega_3], u_1[50|(G_2 \wedge D)]\}\} =$

$= \max\{\min(0.2 , 0), \min(0.5 , 0)\} = 0$

The expected fuzzy utility of playing 50 when the state ω_2 occurs is greater than the expected fuzzy utility of playing D when the same state occurs, whereas the fuzzy utility of playing D when the state is ω_3 occurs is greater than that of playing 50 when ω_3 occurs. Remember that at step two, player P1 stated a set of default rules (the conditional knowledge base Δ) in order - as far as his understanding of the matter was concerned - to account for the reasoning of a "normal" rational player. He must now check whether the conclusions that he reached at step three (by calculating his "second guess" about his highest fuzzy expected utility strategy given his prediction of player P2's choices) are not inconsistent with the default model of "normal" reasoning. Note that a default theory is not monotonic. The third deductive step in player P1's reasoning process could produce new conclusions about normal rational reasoning that could be incompatible with the conclusions derived by default in his foregoing steps. If it does, he would be required to retract some of the conclusions derived from his default base of knowledge Δ. This amounts to checking whether the possibility measures derivable from the last calculation of his best reply are

consistent with the constraints imposed on possibility ordering by the defaults contained in the default knowledge base Δ.

To begin with, we can argue that the conditional measure of possibility that player P1 plays 50 given that the state ω_2 occurs is higher than the conditional measure of possibility that player P1 plays D given that the same state occurs. This is a natural consequence of the reasoning performed by player P1 in his third step. Hence we can say in formulas

$$\Pi_1(50 \mid \omega_2) > \Pi_1(D \mid \omega_2)$$

Since $\Pi_1(50 \mid \omega_2) = 1 - N_1(\neg 50 \mid \omega_2)$ and $\Pi_1(D \mid \omega_2) = 1 - N_1(\neg D \mid \omega_2)$, then

$$N_1(\neg D \mid \omega_2) > N_1(\neg 50 \mid \omega_2) \geq 0.$$

Moreover, we note that $N_1(\neg D \mid \omega_2) > 0 \Rightarrow \Pi_1(\omega_2 \wedge \neg D) > \Pi_1(\omega_2 \wedge D)$.

Since $\omega_2 \Leftrightarrow (\omega_2 \wedge \neg D) \vee (\omega_2 \wedge D)$, then

$$\Pi_1(\omega_2) = \max(\Pi_1(\omega_2 \wedge \neg D), \Pi_1(\omega_2 \wedge D)) = \Pi_1(\omega_2 \wedge \neg D).$$

On the other hand, we have already seen from (3) and (4) that $\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\neg s)$, and that $\Pi_1(\omega_2 \wedge \neg D)$ is at most as high as $\Pi_1(\neg s)$. It follows that

$$\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\omega_2 \wedge \neg D) = \Pi_1(\omega_2) \tag{5}$$

and by definition

$$\Pi_1(\omega_2 \wedge 50) \leq \min(\Pi_1(\neg 50), \Pi_1(\omega_2)) \leq \Pi_1(\omega_2) \tag{6}$$

Hence, from (5) and (6) we obtain that

$$\Pi_1(\omega_2 \wedge 50) = \Pi_1(\omega_2) \tag{7}$$

Note that, for $\Pi_1(\omega_2) > 0$, the conditional possibility measure $\Pi_1(50 \mid \omega_2)$ is defined (Benfherat et al. 1997) as the greatest solution to the equation $\Pi_1(\omega_2 \wedge 50) = \min(\Pi_1(50 \mid \omega_2), \Pi_1(\omega_2))$ in accordance with the minimum specificity principle. Therefore

$$\Pi_1(50 \mid \omega_2) = 1 \text{ if } \Pi_1(\omega_2 \wedge 50) = \Pi_1(\omega_2)$$

and

$$\Pi_1(50 \mid \omega_2) = \Pi_1(\omega_2 \wedge 50) \text{ otherwise.}$$

Since from (7) we have $\Pi_1(\omega_2 \wedge 50) = \Pi_1(\omega_2)$, thus $\Pi_1(50 \mid \omega_2) = 1$ and

$$N_1(\neg 50 \mid \omega_2) = 1 - \Pi_1(50 \mid \omega_2) = 0.$$

As we saw above, it can be easily checked that for $\Pi_1(\omega_2) > 0$

$$N_1(\neg 50 \mid \omega_2) > 0 \Leftrightarrow \Pi_1(\omega_2 \wedge \neg 50) > \Pi_1(\omega_2 \wedge 50)$$

Since the first member is false, we conclude that

$$\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\omega_2 \wedge \neg 50) \tag{8}$$

Note also that from (5) we have

$$\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\omega_2 \wedge \neg D) \tag{9}$$

By similar arguments we can also deduce that

$$\Pi_1(\omega_3 \wedge D) \geq \Pi_1(\omega_3 \wedge \neg D) \tag{10}$$

and

$$\Pi_1(\omega_3 \wedge D) \geq \Pi_1(\omega_3 \wedge \neg 50). \tag{11}$$

Comparing (8),(9),(10),(11) with (4), i.e. comparing the possibility of actions encoded by s with the possibility of actions encoded by ¬s, player P1 does not find any inconsistency with the defaults contained in the conditional knowledge base Δ. Therefore player P1 has no reason to change his previous conditional knowledge base. Moreover, in this case too, player P1 must continue to use the strategy s that he calculated as his best choice at the first step in his reasoning process. This means that, for example, insofar as player P2 is predicted by default reasoning to use his strategy D in the state ω_3, player P1's possibilistic best response is strategy D, because this is player P1's best response calculated given the first-step fuzzy knowledge on states and the second-step default knowledge on player P2 choices. Of course (D,D) is the equilibrium pair dictated by the solution theory σ2 of game G_2. However, at this stage the players do not have definite knowledge that the game to be played in the current state is G_2. Nevertheless, this is the equilibrium pair which tends to emerge form the reasoning process that the players perform in the ongoing game under unforeseen contingencies, when they only know that ω_3 is the case in point. Conversely, the strategy pair (50, 50) is the emerging solution when they only know that the state ω_2 is the case in point.

7. 5. Further steps: equilibrium selection

Little work is needed to verify that if by analogous default rules player P1's reasoning at step 3 is assigned to player P2, in order to recalculate his strategy choice at the third step of his reasoning process, player P2 can only be predicted to reach the same conclusions as player P1. In other words, player P2 can be predicted by default to be recalculating his fuzzy expected utility given similar defaults over player P1 reasoning and reaching the symmetrical conclusion that strategy 50 is to be played when ω_2 is the case and strategy D is to be played when ω_3 is the case. This would be taken as the fourth step in player P1's default reasoning process. Moreover, if player P1 uses this information in order to recalculate his further best response, it is fairly clear that, given that the data are unchanged, the results cannot change with respect to any states of the world ω_2 and ω_3. That is to say, on applying iterately the same default rules in order to deduce new extensions of the players' theory of the game and new reasonable beliefs based on default inferences, the set of statements of the theory that the players believe no longer changes. Let us state this result as the following

PROPOSITION I: *Each player, by replicating the other player's reasoning process with iterated applications of the same set of defaults belonging to D, is induced to believe that each of them is going to play the solution strategy-pair $s1$ (respectively $s2$) if the current unforeseen state is w_2 (respectively w_3) even if he does not have incontrovertible proof of the truth of the statement that in the state w_2 (respectively w_3) the game G_1 (respectively G_2) is to be played.*

This takes us back to an important feature of an extension E meant as the consistent set of all the formulas derivable from a given first-order knowledge base conjoint to a set of defaults (a conditional knowledge base) - i.e. a default theory $\Theta = \langle \Delta, \Phi \rangle$ - in such a way that any set of formulas derived by applying default rules to closed formulas belonging to Θ or derived from them are already included in E. Let us consider again the definition of an extension E as (the minimal) fixed point of the default inference operator Γ_Δ

$$E = \Gamma_\Delta(E)$$

where Γ_Δ carries sets of closed formulas to new sets of closed formulas by applying the set of defaults Δ without introducing any inconsistency into them. This applies to our case because recursive applications of the same defaults to the reasoning of the adversary, in order to generate simulations of the other player's steps in his reasoning process, do not produce no new sentences of the theory that contradict the old ones. No changes occur in the calculated best responses, nor changes intervene in the set of beliefs each player holds about the other player's behaviour.

There is a natural game theoretical interpretation of an extension seen as a fixed point of the default operator applied in order to deduce any internally consistent set of conclusions from the base theory of an agent. Consider that defaults are employed to simulate repeatedly the reasoning of each player by means of the other's models of reasoning, each being concerned with a prediction of the other's behaviour. When the players reach an extension of the knowledge base of the game, which is enlarged by the applications of a basic set of defaults, their beliefs are at a fixed point. Further iterations of the reasoning process, as it is driven by these defaults, will reproduce the same set of beliefs. What emerges is a system of internally consistent expectations for each player, where any order layer of a player's beliefs confirms the lower one. A system of reciprocal expectations that are stable and confirm each other at any level, on the other hand, is an appropriate state of players' beliefs in order to say that, contingent on any given state of the world ω_i, they reasonably believe that one equilibrium point will constitute the unique solution of the game. As a consequence, if the hypothesis is accepted that both players reason identically by resorting to the same set of defaults, they will select the same solution concept relative to the given unforeseen state of the world. Let us state this result as follows.

PROPOSITION II: *Assuming that any player P_i will use the same default theory of the game $Q = áD, Fñ$ and that they, in order to guess each other's choice, will employ the default reasoning operator G_D naturally defined by recursively applying defaults belonging to D to sentences in Q or derived from Q,* then *they will eventually converge on the same solution concept strategies, which is an "equilibrium"*

(i.e. a state from which they do not have any incentive to move in terms of fuzzy utility) relative to the state of their knowledge represented by the fixed point *of their default reasoning operator*.

Some remarks are in order to point out how stable this result is. *First*, we are assuming inter-subjective vagueness, so that the players P_i have the same fuzzy-set representations of the relationships between unforeseen states and the *ex ante* defined classes of games G_j. *Second*, since it is quite obvious that the first-order knowledge base is identical between them, we also assume that they share the same default rules. That is, the model that they employ to guess the other player's reasoning is not only their own model of reasoning attributed by default to the other players but also the same model between them. While the first assumption seems to have some foundation in *fuzzy sets thinking* (Kosko 1993), the second is a simplifying one which should be relaxed in a more general context.

However this need not to be the most serious fragility in the "equilibrium" to which the players converge. The fact that by employing the same default reasoning operator the players eventually reach an *extension* which implies stability of the reciprocal guesses and induces the players to consistently adopt the same solution σ_j does not imply that they are *in fact* playing the proper equilibrium-solution strategy (in the Nash equilibrium sense) of the *effective* game situation that they confronted in the current unforeseen state. By recollecting more information, they could *ex post* discover that they were mistaken in thinking that in state ω_3, for example, strategy D was to be used. Thus the solution concept that the players may select as equilibrium of the game under unforeseen contingencies is only a *conjectural* solution concept, in the sense that it is conditional on a given default theory $\Theta = \langle \Delta, \Phi \rangle$ and it is non-monotonic with respect to the flux of facts that they may learn about the unforeseen states of the world. The possibility of discovering that previous beliefs, apparently justified by the "normal" course of facts, are mistaken is inherent in the non-monotonic nature of default reasoning. Given the players' extension of their theory derived by default rules, they see a solution concept as the rational way to play the game under unforeseen contingencies. They do so, however, without any assurance that *ex post* - when they have accumulated enough experience of the game played under the current state - they will not learn facts showing that in the current state the true game to be played was quite different.

References

Aumann R.J. (1976) "Agreeing to Disagree" *Annals of Statistics*, 4, pp.1236-9

Bacharach M (1987), "A Theory of Rational decision in Games", *Erkenntnis*, vol.27, pp.167-190

Bacharach M. (1994), "The Epistemic Structure of a Game", *Theory and Decisions*, 37, pp.7-48.

Benferhat S., D. Dubois, and H. Prade (1992), "Representing default rules in possibilistic logic". In B. Nebel, C. Rich and W. Swartout, editors. *Proc.of the 3rd Inter. Conf. on Principles of Knowledge Representation and Reasoning (KR'92)*, pages 673-684, Cambridge, MA, Oct. 25-29.

Benferhat S., D. Dubois, and H. Prade (1997), *Nonmonotonic reasoning, conditional objects and possibility theory*, Institut de Recherce en Informatique de Toulouse, Universitè Paul Sabatier, mimeo.

Bicchieri C. (1993), *Rationality and Coordination*, Cambridge U.P., Cambridge.

Binmore K and A. Brandenburger (1990) "Common Knowledge and Game theory" in Ken Binmore *Essays on the Foundations of Game Theory,* Basil Blackwell, Oxford, 1990

Binmore K (1987/88), "Modelling Rational Players", part I and II, *Economics and Philosophy*, 3, n.2-3.

Dubois D. and H.Prade (1995), Possibilistic Logic and Plausible Inference – *Mathematical Models for Handing Partial Knowledge in Artificial intelligence* – Ed. By G.Coletti et Al., Plenum Press, New York.

Dubois D. and H.Prade (1996), Fuzzy Sets in Approximate Reasoning: A Personal View ,mimeo

Dubois D. and H.Prade (1988), *Possibility Theory,* Plenum Press, New York.

Dubois D., H.Prade, R.Sabbadin (1997), "A Possibilistic Machinery for Qualitative Decision", in *Qualitative Preference in Deliberation and Practical Reasonoing,* Working Notes, March 24-26, , Stanford University.

Fudenberg D. and D. Levine (1998), *Learning in Games,* MIT Press.

Geffner D. (1996), *Default Reasoning, Causal and Conditional Theories,* CAmbridge Mas (MIT Press).

Ginsberg M. (ed.), (1987) *Readings in Non-monotonic Reasoning*, Morgan Kaufmann Publ. Los Altos.

Fagin R., J.Y. Halpern, Y.Moses, M. Y.Vardi, (1995), *Reasoning about Knowledge*, MIT Press, Cambridge Mass.

Harsanyi J.C. (1967-8), "Games with Incomplete information Played by *Bayesian* Players" Parts I, II and III, *Management Science,* 14, pp. 159-82, 320-32, 468-502.

Harsanyi J.C. (1975), "The Tracing Procedure. A Bayesian Approach to Defining a Solution for Non-cooperative Games" *International Journal of Game Theory*, vol.5, pp.61-94.

Harsanyi J.C. and R.Selten (1988), *A General Theory of Equilibrium Selection,* MIT Press.

Kreps D. (1990), *Games and Economic Modelling*, Oxford.

Kosko B., (1993), *Fuzzy Thinking: The New Science of Fuzzy Logic*, Hyperion.

Lewis D. (1969), *Conventions, A philosophical Study*. Harvard U.P.

Reiter R. (1980), "A Logic for Default Reasoning", *Artificial Intelligence*, 13, pp.81-132.

Reiter R. and Cresciuolo G. (1987), "On interacting Defaults" in Ginsberg (ed.) *Reading on Non-monotonic Reasoning*, Moragn Kaufmann Publ.

Sacconi L. (1995), "Equilibrio e giustizia, parte II: la selezione del contratto sociale", *Il Giornale degli economisti*, 10/12

Sacconi L. (2000), *The Social Contract of The Firm*, Springer, Berlin-Heidelberg

Sacconi L. (2001), *Incomplete Contracts and Corporate Ethics: a Game-theoretical Model under Fuzzy Information*, Luic papers n.91, ottobre 2001.

Samuelson L. (1997), *Evolutionary Games and Equilibrium Selection*, the Mit Press, Cambridge Mass.

Sugden R (1995) "A Theory of Focal Points", in *Economic Journal*, 105, 430.

Yager R.R. (1979), "Possibilistic decision making" IEEE *Trans. on Systems Man and Cybernetics*, 9, pp.388-392.

Zadeh L.A. (1978), "Fuzzy Sets as a Basis for a theory of possibility", *Fuzzy Sets and Systems*, 1, pp.3-28.

Zimmerman H.J. (1991) *Fuzzy Sets Theory and Its Applications*, 2nd revised edition, Kluwer Ac. Press.

Notes

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[6] This statement can be put in the form of a paradox. Assume that the starting point of any equilibrium selection process cannot be an equilibrium point itself. Then hypothesise that the players have common knowledge that (i.e. it is true that) there is a given common prior which synthesises the players' uncertainty about each others' possible equilibrium choices. Then, each player, knowing the common prior, will adopt (with probability one – leaving mixed strategies aside) a best response strategy which was not assigned probability one by the common prior (otherwise the players would have been in a state of equilibrium since the beginning of the selection process.) It follows that each player must know that both himself and all other players will adopt a best response strategy to the *common prior-behaviour* of the others, which does not exactly correspond to the one described by the common prior itself – i.e. he knows (and expects) that any other player's behaviour deviates from the common prior. But, according the definition of knowledge used in the epistemic logic employed to formalize common knowledge - Aumann (1976), Binmore and Brandenburger (1990), Fagin, Halpern, Moses and Vardi (1995) – everything which is common knowledge must be also true, and if a statement is logical consequence of something which is true and commonly known, then it must also be known (logical omniscience.) Hence, if the common prior is common knowledge it must be true that each player entertains the *common-prior-beliefs* concerning the choices of the others, but at the same time it must also be true that everybody knows that anybody adopts a strategy deviating from what the common prior predicts. Thus, if at the starting point of the equilibrium selection process there is common knowledge of the common prior, *it cannot be true that the common prior reflects what the players truly believe about the other players' choice in the game.* This means that for each player the state of knowledge in which he finds himself *contradicts* the common prior - i.e. it contradicts that the common prior is the *true state of believing of any player.* And, as a consequence, it also contradicts that the players have common knowledge of the common prior, which should be their true state of believing. This suggests that any "eductive" equilibrium selection process endowed with an initial common prior must assume limited reasoning and limited introspection of the players on their own cognitive starting point, and that it should evolve through reasoning steps like fallible and retractable conjectures concerning the other players' expectations and continuous revisions of such fallible inferences until they enter a state of "equilibrium expectations" (Sacconi 1995b).

[7] In fact we need only account for a less cumbersome state of knowledge - called here "shared knowledge" - such that "everybody knows that in game G the solution is σ, and everybody knows that everybody knows that in game G the solution is σ". Infinite iterations of this basic item of shared knowledge are not required to account for the selection of that an equilibrium point is selected in a context of boundedly rational players. Requiring common knowledge in the strict sense would be to ask for an infinite hierarchy of mutually consistent layers of beliefs or knowledge, and this in turn would imply the logical omniscience of the players, which seems an inappropriate assumption to make in our context.

[8] As far as I know, the first time that default logic was suggested as one possible approach to the foundations of some game theoretical concepts was Bacharach (1994) with reference to rationalisability. However, here I complement it with fuzzy logic and possibility theory and address a different game theoretical problem.

[9] Note that in game G2 there are not multiple equilibria, so that in this game *per se* there is no equilibrium selection problem. However, in this paper we generate the typical multiplicity problem in a way quite different from the standard one. I shall assume that there are different possible games, each endowed with a unique equilibrium solution (which in G2 is also in dominant strategies), and that the players have ambiguous information about the game they are going to play. Hence, they must decide which of the two equilibrium strategies to use.

[10] We may say that vagueness ranges across the questions whether the rules of game necessary to use the Nash bargaining solution apply, or whether the rules of game necessary to use the typical non-cooperative DP solution do so.

[11] One may think "as if" the language employed in describing concrete but unexpected properties of the unforeseen states belonging to Ω has undergone some sort of evolutionary pressure for practical descriptive reasons, so that it is to some extent changed with respect to the previous one of general and abstract terms employed to describe the general game forms G1 and G2. Thus there is some sort of ambiguity in classifying new objects expressed by the new evolved, concrete predicates in terms of the game concepts used to define G1 and G2.