J. Eur. Math. Soc. 20, 2607–2641

DOI 10.4171/JEMS/820

Alberto Canonaco · Paolo Stellari

Uniqueness of dg enhancements for the derived category of a Grothendieck category

Received September 22, 2015 and in revised form January 1, 2016 and October 18, 2016

Abstract. We prove that the derived category of a Grothendieck abelian category has a unique dg enhancement. Under some additional assumptions, we show that the same result holds true for its subcategory of compact objects. As a consequence, we deduce that the unbounded derived category of quasi-coherent sheaves on an algebraic stack and the category of perfect complexes on a noetherian concentrated algebraic stack with quasi-finite affine diagonal and enough perfect coherent sheaves have a unique dg enhancement. In particular, the category of perfect complexes on a noetherian scheme with enough locally free sheaves has a unique dg enhancement.

Keywords. Dg categories, dg enhancements, triangulated categories

Contents

Introduction

The relation between triangulated categories and higher categorical structures is highly non-trivial, very rich in nature and with various appearances in the recent developments of derived algebraic geometry. The easiest thing we can do is to produce a triangulated category $\mathbf{T}$ out of a pretriangulated dg category $\mathbf{C}$ by taking the homotopy category of $\mathbf{C}$.

A Canonaco: Dipartimento di Matematica “F. Casorati”, Università degli Studi di Pavia, Via Ferrata 5, 27100 Pavia, Italy; e-mail: alberto.canonaco@unipv.it

P. Stellari: Dipartimento di Matematica “F. Enriques”, Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy; e-mail: paolo.stellari@unimi.it, http://users.unimi.it/stellari

Mathematics Subject Classification (2010): 14F05, 18E10, 18E30

Roughly speaking, a pretriangulated dg category $\mathbf{C}$ whose homotopy category is equivalent to a triangulated category $\mathbf{T}$ is called a *dg enhancement* (or *enhancement*, for short) of $\mathbf{T}$.

Now, there exist triangulated categories with no enhancements at all. For example, this happens to some triangulated categories naturally arising in topology (see [36] or [17, Section 3.6] for a discussion). Triangulated categories admitting an enhancement are called *algebraic* (as is explained, for example, in [36, Section 3], algebraic triangulated categories are often defined in other equivalent ways). In practice, all triangulated categories one usually encounters in algebra or algebraic geometry are algebraic. For instance, the derived category of a Grothendieck category, as well as its full subcategory of compact objects, are algebraic. Recall that a *Grothendieck category* is an abelian category $\mathbf{G}$ which is closed under small coproducts, has a small set $\mathbf{S}$ of generators, and the direct limits of short exact sequences are exact. The objects in $\mathbf{S}$ are generators in the sense that, for any C in $\mathbf{G}$, there exists an epimorphism $S \twoheadrightarrow C$ in $\mathbf{G}$, where S is a small coproduct of objects in $\mathbf{S}$.

In particular, if X is a scheme or, more generally, an algebraic stack, it is not difficult to construct explicit enhancements of the derived category $\mathrm{D}(\mathbf{Qcoh}(X))$ of the Grothendieck category of quasi-coherent sheaves on X, of the bounded derived categories $\mathrm{D}^b(X)$ of coherent sheaves and of the category $\mathbf{Perf}(X)$ of perfect complexes on X. For example, this can be achieved either by taking complexes of injective sheaves or, under mild assumptions, Čech resolutions or chain complexes of sheaves in the corresponding categories or perfect complexes (see [3] and [25]).

Even when we know that an enhancement exists, one may wonder whether it is unique. Roughly, we say that a triangulated category $\mathbf{T}$ has a *unique enhancement* $\mathbf{C}$ if any other enhancement is related to $\mathbf{C}$ by a sequence of quasi-equivalences. These are the analogue, at the dg level, of the exact equivalences in the triangulated setting. Actually, at this level of generality, we cannot expect a positive answer to the above question. Indeed, the result of Dugger and Shipley [11] easily yields an example of two $\mathbb{Z}$-linear pretriangulated dg categories which are not quasi-equivalent but whose homotopy categories are equivalent. Finding a similar example over a field rather than a commutative ring is still a challenge.

Again, if we move to the geometric setting, then for a long while it was expected that any of the three triangulated categories $\mathrm{D}(\mathbf{Qcoh}(X))$, $\mathrm{D}^b(X)$ and $\mathbf{Perf}(X)$ should have unique enhancements, when X is a (quasi-)projective scheme. This was formally stated as a conjecture (even in a stronger form) by Bondal, Larsen and Lunts [3].

As we will explain later, this conjecture was confirmed by Lunts and Orlov in their seminal paper [24]. It should be noted that the quest for uniqueness of enhancements has a foundational relevance that cannot be overestimated by the 'working algebraic geometer'. Let us just mention an instance where the fact of having a unique enhancement has interesting consequences. The homological version of the so called *Mirror Symmetry Conjecture* by Kontsevich [18] predicts the existence of an A_∞-equivalence between a dg enhancement of $\mathrm{D}^b(X)$, for X a smooth projective scheme, and the Fukaya category of the mirror Y of X, which is actually an A_∞-category. The fact that the dg enhancements are unique allows us to conclude that finding an A_∞-equivalence (or rather a sequence of them) is the same as finding an exact equivalence between the corresponding homotopy categories. More generally, several geometric problems can be lifted to the dg level and treated there in a universal way (e.g. moduli problems or the characterization of exact functors). Having bridges between the different dg incarnations of the same triangulated or geometric problem is then crucial.

Let us now explain the contributions of this paper to the problem of showing the uniqueness of dg enhancements in geometric settings. The first point to make, which should be clear from now on, is that the analysis of these questions about $\mathrm{D}(\mathbf{Qcoh}(X))$ or $\mathbf{Perf}(X)$ (or $\mathrm{D}^b(X)$) follows slightly different paths. In particular, they can be deduced from two different general criteria whose statements are similar but whose proofs are rather different in nature.

We first consider the case of $\mathrm{D}(\mathbf{Qcoh}(X))$ and, setting the problem at a more abstract level, we first prove the following general result.

Theorem A. *If* $\mathbf{G}$ *is a Grothendieck category, then* $\mathrm{D}(\mathbf{G})$ *has a unique enhancement.*

We will later explain some key features in the proof. For the moment, we just recall that the main geometric applications are the following:

- If X is an algebraic stack, $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement (Corollary 5.4).
- If X is scheme and α is an element in the Brauer group $\mathrm{Br}(X)$ of X, then the twisted derived category $\mathrm{D}(\mathbf{Qcoh}(X, \alpha))$ has a unique enhancement (Corollary 5.7).

Now, if we want to study the enhancements of $\mathbf{Perf}(X)$ (and consequently of $\mathrm{D}^b(X)$), we should keep in mind that if X is a quasi-compact and semi-separated scheme, then $\mathbf{Perf}(X)$ is the triangulated subcategory of $\mathrm{D}(\mathbf{Qcoh}(X))$ consisting of compact objects. Our general result in this direction is then the following.

Theorem B. *Let* $\mathbf{G}$ *be a Grothendieck category with a small set* $\mathbf{A}$ *of generators such that*

(1) $\mathbf{A}$ *is closed under finite coproducts;*
(2) *every object of* $\mathbf{A}$ *is a noetherian object in* $\mathbf{G}$*;*
(3) *if* $f\colon A' \twoheadrightarrow A$ *is an epimorphism of* $\mathbf{G}$ *with* $A, A' \in \mathbf{A}$*, then* $\ker f \in \mathbf{A}$*;*
(4) *for every* $A \in \mathbf{A}$ *there exists* $N(A) > 0$ *such that* $\mathrm{D}(\mathbf{G})(A, A'[N(A)]) = 0$ *for every* $A' \in \mathbf{A}$.

Then $\mathrm{D}(\mathbf{G})^c$ *has a unique enhancement.*

Here $\mathrm{D}(\mathbf{G})^c$ denotes the subcategory of compact objects in $\mathrm{D}(\mathbf{G})$. One may wonder why the result above is conditional while Theorem A does not include any specific assumptions on $\mathbf{G}$. We will try to explain later that this is, in a sense, unavoidable but to reassure the reader about the mildness of (1)–(4), let us now discuss some geometric cases where Theorem B applies:

- If X is a noetherian concentrated algebraic stack with quasi-finite affine diagonal and with enough perfect coherent sheaves, then $\mathbf{Perf}(X)$ has a unique enhancement (Proposition 6.10).

- As a special (but maybe easier to understand) instance of the above case, if X is a noetherian scheme with enough locally free sheaves, then $\mathbf{Perf}(X)$ has a unique enhancement (Corollary 6.11).
- Under the same assumptions on the scheme X, the category $\mathrm{D}^b(X)$ has a unique enhancement (Corollary 7.2).

The (more or less) standard terminology involved in the above statements will be briefly recalled in Section 6.3.

It is very likely that Theorems A and B may be used in other geometric contexts. One direct application of the circle of ideas appearing in the proofs of these two results concerns the existence of exact equivalences. In particular, if X_1 and X_2 are noetherian schemes with enough locally free sheaves then the set of equivalences between $\mathbf{Perf}(X_1)$ and $\mathbf{Perf}(X_2)$ is not empty if and only if the same is true for the set of equivalences between $\mathrm{D}(\mathbf{Qcoh}(X_1))$ and $\mathrm{D}(\mathbf{Qcoh}(X_2))$ (Proposition 7.4).

The strategy of the proof

Before entering into some details of the proof it is worth pointing out the general approach to Theorems A and B. Even if these results have a dg flavour, the idea is to reduce them to questions about Verdier quotients of triangulated categories. Unfortunately, some of these latter questions are highly non-trivial and involve deep problems concerning the description of the subcategory of compact objects of a quotient. This is the reason why our proofs, which are conceptually quite simple, become technically rather involved.

Let us try to make this more precise and consider first Theorem A. The key observation is that the derived category $\mathrm{D}(\mathbf{G})$ of a Grothendieck category $\mathbf{G}$ is *well generated* in the sense of Neeman [30]. Thus one can choose a small set $\mathbf{A}$ of generators for $\mathbf{G}$ such that $\mathrm{D}(\mathbf{G})$ is naturally equivalent to the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$, where $\mathcal{D}(\mathbf{A})$ is the derived category of $\mathbf{A}$, seen as a dg category, and $\mathbf{L}$ is an appropriate localizing subcategory of $\mathcal{D}(\mathbf{A})$.

This is carried out in Section 5.1, where we also explain that Theorem A follows easily once we prove the following general criterion.

Theorem C. *Let $\mathbf{A}$ be a small category considered as a dg category concentrated in degree 0 and let $\mathbf{L}$ be a localizing subcategory of $\mathcal{D}(\mathbf{A})$ such that*

(a) *the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category;*
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_{i\in I}\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_i))[k_i]) = 0$ *for all $A, A_i \in \mathbf{A}$ (with I a small set) and all integers $k_i < 0$.*

Then $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement.

Here $\mathsf{Y}^{\mathbf{A}}\colon \mathbf{A} \to \mathcal{D}(\mathbf{A})$ denotes the Yoneda functor, $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ is the natural quotient functor, while $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(-,-)$ denotes the Hom-space in the category $\mathcal{D}(\mathbf{A})/\mathbf{L}$. We will give the precise definition of a well generated triangulated category in Section 1. For the moment, it is enough to keep in mind that it is a natural generalization of the usual notion of compactly generated triangulated category. The idea of the proof, which occupies the whole of Section 4, is very much inspired by the proof of [24, Theorem 2.7] but it differs at some technical steps. We will try to clarify them in a while when comparing our results to those in [24]. The geometric applications mentioned above and discussed in Section 5.2 can be deduced easily from the fact that, in all those cases, the category of quasi-coherent sheaves is a Grothendieck category, under our assumptions on X.

Once we have the equivalence $\mathrm{D}(\mathbf{G}) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}$ as above, it is clear that to prove Theorem B we have to show that the triangulated subcategory $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ of compact objects in $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement, for a smart choice of $\mathbf{A}$ (see Section 6.1). For this, one may hope to use Theorem 6.1 which was proved by Lunts and Orlov [24]. Indeed, this criterion for uniqueness asserts that if we change (a) in Theorem C to

(a′) $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ and $\mathbf{L}$ is generated by $\mathbf{L}^c$,

and we keep (b), then we can deduce that $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ has a unique enhancement as well. In this case, the proof is not too difficult, as we can use the fact that $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ and $\mathcal{D}(\mathbf{A})^c/\mathbf{L}^c$ are nicely related, as explained in [28] (see Theorem 1.5).

The issue here is that (a′) is not easily verified. Indeed, if $\mathbf{L}$ satisfies (a′), then the inclusion functor $\mathbf{L} \hookrightarrow \mathbf{T}$ has a right adjoint which preserves small coproducts. In general, given a compactly generated triangulated category $\mathbf{T}$ closed under small coproducts, a localizing subcategory $\mathbf{L}$ of $\mathbf{T}$ such that the inclusion $\mathbf{L} \hookrightarrow \mathbf{T}$ has the above property is called a *smashing subcategory*.

For a while, it was conjectured that all smashing subcategories $\mathbf{L}$ of a triangulated category $\mathbf{T}$ as above should verify (a′). This goes under the name of the *Telescope Conjecture* (see [33, 1.33] and [5, 3.4]). Unfortunately, the Telescope Conjecture is known to be false in this generality [15] and to be true in very few examples (see, for example, [29]). This shows that we cannot expect that (a′) holds true in general or easily.

In view of this discussion, the main task, which is carried out in Section 6.2, is to identify the correct choice for $\mathbf{A}$ such that (a′) holds for the corresponding localizing subcategory $\mathbf{L}$. To get this, one has to impose some additional assumptions on $\mathbf{G}$ and $\mathbf{A}$. This is the reason why the hypotheses (1)–(4) appear in Theorem B. Assuming this, the proof of Theorem B is contained in Section 6.2 and the core of the argument is then Theorem 6.6.

The applications concerning the uniqueness of enhancements for $\mathbf{Perf}(X)$ (Proposition 6.10 and Corollary 6.11) and $\mathrm{D}^b(X)$ (Corollary 7.2) are rather easy consequences once Theorem B and, more precisely, Theorem 6.6 are established.

Related work

As we recalled before, Bondal, Larsen and Lunts [3] first conjectured that all enhancements of $\mathrm{D}^b(X)$, for X a smooth projective scheme, should be unique. In the same paper, they show that all 'standard' enhancements are related by quasi-equivalences, giving the first evidence to their conjecture.

After that, the main reference is [24] which is certainly the principal source of inspiration for this paper as well. Let us briefly summarize the results contained in that paper and compare them to ours. For $\mathbf{A}$ a small category as in Theorem C, Lunts and Orlov show that $\mathcal{D}(\mathbf{A})/\mathbf{L}$ has a unique enhancement if (b) holds and (a) is replaced by

(a.1) Q sends $\mathcal{D}(\mathbf{A})^c$ to $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$, where $(-)^c$ denotes the full subcategory of compact objects;
(a.2) Q has a right adjoint.

This is [24, Theorem 2.7]. It should be noted that (a.1) and (a.2) together imply that $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is compactly generated. This is a special instance of our assumption (a) in Theorem C. Moreover, by [27], there are examples of Grothendieck categories whose derived category is not compactly generated but is well generated. Hence Theorem C is certainly a generalization of [24, Theorem 2.7]. The geometric consequences of [24, Theorem 2.7], which are discussed in the same paper, are then:

- For a Grothendieck category $\mathbf{G}$, the derived category $\mathrm{D}(\mathbf{G})$ has a unique enhancement, if $\mathbf{G}$ has a small set of generators which are compact in $\mathrm{D}(\mathbf{G})$ (see [24, Theorem 2.9]).
- This implies that if X is a quasi-compact and separated scheme that has enough locally free sheaves, then $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement (see [24, Theorem 2.10]).

As a second step, Lunts and Orlov deduce from [24, Theorem 2.8] that if X is a quasi-projective scheme, then both $\mathbf{Perf}(X)$ and $\mathrm{D}^b(X)$ have unique enhancements. A strong version of uniqueness is then discussed. Namely, they prove that these two categories have strongly unique enhancements when X is projective and another technical assumption is satisfied. This is out of the scope of this paper but we believe that the techniques discussed here might have applications to show the strong uniqueness of dg enhancements in new cases. Indeed, Theorem 6.6 has already been applied to prove the strong uniqueness of the category of perfect supported complexes (see [8, Theorem 1.2]).

New interesting enhancements of geometric nature have recently been introduced by Lunts and Schnürer [25]. Roughly speaking, they were used to show that the dg notion of Fourier–Mukai functor and the triangulated one agree, under some assumptions on the schemes. This important result was previously stated in [39] but without a rigorous proof.

Plan of the paper

This paper starts with a quick recollection of results about localizations of triangulated categories and of some properties of well generated triangulated categories (see Section 1).

Sections 2 and 3 have a rather abstract nature. They cover some basic material about dg categories and dg enhancements with an emphasis on the case of enhancements of well generated triangulated categories. Section 3 provides some properties of special functors which are used in the proof of Theorem C.

In Section 4 we prove Theorem C, while Theorem A, together with its geometric applications, is proved in Section 5. The proofs of Theorem B, of Proposition 6.10 and of Corollary 6.11 are the contents of Section 6.

Section 7 contains two further applications. The first one, concerning the uniqueness of enhancements for $\mathrm{D}^b(X)$, is proved in Section 7.1. The second one, about Fourier–Mukai functors, is explained in Section 7.2.

Notation

All categories and functors are assumed to be $\Bbbk$-linear, for a fixed commutative ring $\Bbbk$. By a $\Bbbk$-*linear category* we mean a category whose Hom-spaces are $\Bbbk$-modules and compositions are $\Bbbk$-bilinear, without assuming that finite coproducts exist.

Throughout the paper, we assume that a universe containing an infinite set is fixed. Several definitions concerning dg categories need special care because they may, in principle, require a change of universe. All possible subtle logical issues in this sense can be overcome in view of [24, Appendix A]. A careful reader should have a look at it. After these warnings and to simplify the notation, throughout the rest of the paper we will not explicitly mention the universe we are working in, as it should be clear from the context. The members of this universe will be called small sets. For example, when we speak about small coproducts in a category, we mean coproducts indexed by a small set. If not stated otherwise, we always assume that the Hom-spaces in a category form a small set. A category is called *small* if the isomorphism classes of its objects form a small set.

If $\mathbf{T}$ is a triangulated category and $\mathbf{S}$ a full triangulated subcategory of $\mathbf{T}$, we denote by $\mathbf{T}/\mathbf{S}$ the Verdier quotient of $\mathbf{T}$ by $\mathbf{S}$. In general, $\mathbf{T}/\mathbf{S}$ is not a category according to our convention (namely, the Hom-spaces in $\mathbf{T}/\mathbf{S}$ need not be small sets), but it is in many common situations, for instance when $\mathbf{T}$ is small.

Given a category $\mathbf{C}$ and two objects C_1 and C_2 in $\mathbf{C}$, we denote by $\mathbf{C}(C_1, C_2)$ the Hom-space between C_1 and C_2. If $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ is a functor and C_1 and C_2 are objects of $\mathbf{C}$, then we denote by F_{C_1,C_2} the induced map $\mathbf{C}(C_1, C_2) \to \mathbf{D}(\mathsf{F}(C_1), \mathsf{F}(C_2))$.

If I is a set, $|I|$ denotes its cardinality.

1. Well generated triangulated categories and localizations

In this section we use Krause's equivalent treatment (see [20]) of Neeman's notion of well generated triangulated category (see [30]). For a very clear survey of this subject, the reader can look up [21].

From now on, we assume that $\mathbf{T}$ is a triangulated category with small coproducts. Given a cardinal α, an object S of $\mathbf{T}$ is α-*small* if every map $S \to \coprod_{i\in I} X_i$ in $\mathbf{T}$ (where I is a small set) factors through $\coprod_{i\in J} X_i$ for some $J \subseteq I$ with $|J| < \alpha$. Recall that a cardinal α is called *regular* if it is not the sum of fewer than α cardinals, all of them smaller than α.

Definition 1.1. The category $\mathbf{T}$ is *well generated* if there exists a small set $\mathbf{S}$ of objects in $\mathbf{T}$ satisfying the following properties:

(G1) An object $X \in \mathbf{T}$ is isomorphic to 0 if and only if $\mathbf{T}(S, X[j]) = 0$ for all $S \in \mathbf{S}$ and all $j \in \mathbb{Z}$.

(G2) For every small set $\{X_i \to Y_i\}_{i\in I}$ of maps in $\mathbf{T}$, the induced map $\mathbf{T}(S, \coprod_i X_i) \to \mathbf{T}(S, \coprod_i Y_i)$ is surjective for all $S \in \mathbf{S}$ if $\mathbf{T}(S, X_i) \to \mathbf{T}(S, Y_i)$ is surjective for all $i \in I$ and all $S \in \mathbf{S}$.

(G3) There exists a regular cardinal α such that every object of $\mathbf{S}$ is α-small.

When the category $\mathbf{T}$ is well generated and we want to put emphasis on the cardinal α in (G3), we say that $\mathbf{T}$ is *α-well generated* by the set $\mathbf{S}$. In this situation, following [20], we denote by $\mathbf{T}^\alpha$ the smallest α-localizing subcategory of $\mathbf{T}$ containing $\mathbf{S}$. Recall that a full triangulated subcategory $\mathbf{L}$ of $\mathbf{T}$ is *α-localizing* if it is closed under α-coproducts and under direct summands (the latter condition is actually redundant if $\alpha > \aleph_0$). By definition, an α-coproduct is a coproduct of strictly less than α summands. On the other hand, $\mathbf{L}$ is *localizing* if it is closed under small coproducts in $\mathbf{T}$. The objects in $\mathbf{T}^\alpha$ are called *α-compact*. Thus we will sometimes say that $\mathbf{T}$ is *α-compactly generated* by the set $\mathbf{S}$ of *α-compact generators*.

Remark 1.2. (i) It is easy to observe that the objects in $\mathbf{T}^\alpha$ are α-small (see, for example, [20, Lemma 5]).

(ii) As alluded by the notation and explained in [20, 30], the subcategory $\mathbf{T}^\alpha$ does not depend on the choice of the set $\mathbf{S}$ of α-compact generators. Moreover, for any well generated triangulated category $\mathbf{T}$, one has $\mathbf{T} = \bigcup_\beta \mathbf{T}^\beta$, where β runs through all sufficiently large regular cardinals.

(iii) When $\alpha = \aleph_0$, then $\mathbf{T}^\alpha = \mathbf{T}^c$, the full triangulated subcategory of compact objects in $\mathbf{T}$. Notice that, in this case, $\mathbf{T}$ is $\aleph_0$-compactly generated by $\mathbf{S} \subseteq \mathbf{T}^c$ if (G1) holds (indeed, (G3) holds by definition of compact object, whereas (G2) is automatically satisfied). Following the usual convention, we simply say that $\mathbf{T}$ is *compactly generated* by $\mathbf{S}$.

Example 1.3. Let $\mathbf{G}$ be a Grothendieck category. Then the derived category $\mathrm{D}(\mathbf{G})$ is well generated (see [27, Theorem 0.2] and [21, Example 7.7]).

Given a small set $\mathbf{S}$ of objects in $\mathbf{T}$, we say that $\mathbf{S}$ *generates* $\mathbf{T}$ if $\mathbf{T}$ is the smallest localizing subcategory of $\mathbf{T}$ containing $\mathbf{S}$.

Proposition 1.4 ([32, Proposition 5.1]). *Let $\mathbf{T}$ be a well generated triangulated category. Then a small set $\mathbf{S}$ of objects in $\mathbf{T}$ satisfies* (G1) *if and only if $\mathbf{S}$ generates $\mathbf{T}$.*

Let us investigate a bit more when quotients by localizing subcategories can be well generated. When $\alpha = \aleph_0$, we have the following result which we will need later.

Theorem 1.5 ([28, Theorem 2.1]). *Let $\mathbf{T}$ be a compactly generated triangulated category and let $\mathbf{L}$ be a localizing subcategory which is generated by a small set of compact objects. Then*

(i) *$\mathbf{T}/\mathbf{L}$ has small Hom-sets and it is compactly generated;*
(ii) *$\mathbf{L}^c = \mathbf{L} \cap \mathbf{T}^c$;*
(iii) *the quotient functor $\mathsf{Q}\colon \mathbf{T} \to \mathbf{T}/\mathbf{L}$ sends $\mathbf{T}^c$ to $(\mathbf{T}/\mathbf{L})^c$;*
(iv) *the induced functor $\mathbf{T}^c/\mathbf{L}^c \to (\mathbf{T}/\mathbf{L})^c$ is fully faithful and identifies $(\mathbf{T}/\mathbf{L})^c$ with the idempotent completion of $\mathbf{T}^c/\mathbf{L}^c$.*

Recall that the fact that $(\mathbf{T}/\mathbf{L})^c$ is the idempotent completion of $\mathbf{T}^c/\mathbf{L}^c$ simply means that any object in $(\mathbf{T}/\mathbf{L})^c$ is isomorphic to a summand of an object in $\mathbf{T}^c/\mathbf{L}^c$. A similar result holds for well generated triangulated categories (see, for example, [21, Theorem 7.2.1]).

In general, assume that $\mathbf{T}$ is well generated by a small set $\mathbf{S}$. Let $\mathbf{L}$ be a localizing subcategory of $\mathbf{T}$ such that the quotient $\mathbf{T}/\mathbf{L}$ is well generated. Denote by

$$\mathsf{Q}\colon \mathbf{T} \to \mathbf{T}/\mathbf{L}$$

the quotient functor.

Remark 1.6. As we assume that $\mathbf{T}/\mathbf{L}$ is well generated, in particular, it has small Hom-sets. Moreover, $\mathbf{T}/\mathbf{L}$ has small coproducts and the quotient functor Q commutes with them by [30, Corollary 3.2.11]. Then it follows from [21, Theorem 5.1.1 and Proposition 2.3.1] that the functor Q has a fully faithful right adjoint Q^R (hence $\mathsf{Q} \circ \mathsf{Q}^R \cong \mathrm{id}$).

Although in general $\mathbf{T}/\mathbf{L}$ is not well generated by the set $\mathsf{Q}(\mathbf{S})$ because (G2) does not hold, we have the following result.

Proposition 1.7. *For $\mathbf{T}$, $\mathbf{S}$ and $\mathbf{L}$ as above, the set $\mathsf{Q}(\mathbf{S})$ satisfies* (G1) *and* (G3) *in $\mathbf{T}/\mathbf{L}$.*

Proof. Let $X \in \mathbf{T}/\mathbf{L}$ be such that $\mathbf{T}/\mathbf{L}\,(\mathsf{Q}(S), X[j]) = 0$ for all $S \in \mathbf{S}$ and all $j \in \mathbb{Z}$. Denoting by Q^R the right adjoint of Q (see Remark 1.6), we have

$$\mathbf{T}/\mathbf{L}\,(\mathsf{Q}(S), X[j]) \cong \mathbf{T}(S, \mathsf{Q}^R(X)[j]).$$

As $\mathbf{S}$ satisfies (G1), it follows that $\mathsf{Q}^R(X) \cong 0$. Since Q^R is fully faithful, this implies that $X \cong 0$. Therefore $\mathsf{Q}(\mathbf{S})$ satisfies (G1).

Observe moreover that, by Remark 1.2(ii), the (small) set $\mathsf{Q}(\mathbf{S})$ is contained in $(\mathbf{T}/\mathbf{L})^\alpha$ for some regular cardinal α. Hence it satisfies (G3), by Remark 1.2(i). $\square$

2. Dg categories and dg enhancements

In this section, we recall some general facts about dg categories and stick to the description of dg enhancements for well generated triangulated categories.

2.1. A quick tour of dg categories

An excellent survey of dg categories is [17]. Nevertheless, we briefly summarize here what we need in the rest of the paper.

First of all, recall that a *dg category* is a $\Bbbk$-linear category $\mathbf{C}$ such that the morphism spaces $\mathbf{C}(A, B)$ are $\mathbb{Z}$-graded $\Bbbk$-modules with a differential $d\colon \mathbf{C}(A, B) \to \mathbf{C}(A, B)$ of degree 1 and the composition maps $\mathbf{C}(B, C) \otimes_\Bbbk \mathbf{C}(A, B) \to \mathbf{C}(A, C)$ are morphisms of complexes for all A, B, C in $\mathbf{C}$. By definition, the identity of each object is a closed morphism of degree 0.

Example 2.1. (i) Any $\Bbbk$-linear category has a (trivial) structure of dg category, with morphism spaces concentrated in degree 0.

(ii) For a dg category $\mathbf{C}$, one defines the opposite dg category $\mathbf{C}^\circ$ with the same objects as $\mathbf{C}$ while $\mathbf{C}^\circ(A, B) := \mathbf{C}(B, A)$. One should notice that, given two homogeneous elements $f \in \mathbf{C}^\circ(A, B)$ and $g \in \mathbf{C}^\circ(B, C)$, the composition $g \circ f$ in $\mathbf{C}^\circ$ is defined as the composition $(-1)^{\deg(f)\deg(g)} f \circ g$ in $\mathbf{C}$.

(iii) Following [10], given a dg category $\mathbf{C}$ and a full dg subcategory $\mathbf{B}$ of $\mathbf{C}$, one can take the quotient $\mathbf{C}/\mathbf{B}$, which is again a dg category.

Given a dg category $\mathbf{C}$ we denote by $\mathrm{H}^0(\mathbf{C})$ its *homotopy category*. To be precise, the objects of $\mathrm{H}^0(\mathbf{C})$ are the same as those of $\mathbf{C}$ while the morphisms from A to B are obtained by taking the 0-th cohomology $H^0(\mathbf{C}(A, B))$ of the complex $\mathbf{C}(A, B)$.

A *dg functor* $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ between two dg categories is the datum of a map $\mathrm{Ob}(\mathbf{C}_1) \to \mathrm{Ob}(\mathbf{C}_2)$ and of morphisms of complexes of $\Bbbk$-modules $\mathbf{C}_1(A, B) \to \mathbf{C}_2(\mathsf{F}(A), \mathsf{F}(B))$, for $A, B \in \mathbf{C}_1$, which are compatible with the compositions and the units. Clearly, a dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ induces a functor $\mathrm{H}^0(\mathsf{F})\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$.

A dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ is a *quasi-equivalence* if the maps $\mathbf{C}_1(A, B) \to \mathbf{C}_2(\mathsf{F}(A), \mathsf{F}(B))$ are quasi-isomorphisms for every $A, B \in \mathbf{C}_1$, and $\mathrm{H}^0(\mathsf{F})$ is an equivalence.

One can consider the localization **Hqe** of the category of (small) dg categories with respect to quasi-equivalences. Given a dg functor F, we will denote by the same symbol its image in **Hqe**. A morphism in **Hqe** is called a *quasi-functor*. By the general theory of localizations and model categories (see, for example, [17, 39]), a quasi-functor between two dg categories $\mathbf{C}_1$ and $\mathbf{C}_2$ can be represented by a roof

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, "\mathsf{I}"'] \arrow[dr, "\mathsf{F}"] & \\ \mathbf{C}_1 & & \mathbf{C}_2 \end{tikzcd}$$

where $\mathbf{C}$ is a (cofibrant) dg category, I is a quasi-equivalence and F is a dg functor. A quasi-functor F in **Hqe** between the dg categories $\mathbf{C}_1$ and $\mathbf{C}_2$ induces a functor $\mathrm{H}^0(\mathsf{F})\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$, well defined up to isomorphism.

Given a small dg category $\mathbf{C}$, one can consider the dg category $\mathrm{dgMod}(\mathbf{C})$ of *right dg* $\mathbf{C}$*-modules*. A right dg $\mathbf{C}$-module is a dg functor $\mathsf{M}\colon \mathbf{C}^\circ \to \mathrm{dgMod}(\Bbbk)$, where $\mathrm{dgMod}(\Bbbk)$ is the dg category of dg $\Bbbk$-modules. It is known that $\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))$ is, in a natural way, a triangulated category with small coproducts (see, for example, [17]).

The full dg subcategory of acyclic right dg modules is denoted by $\mathrm{Ac}(\mathbf{C})$, and $\mathrm{H}^0(\mathrm{Ac}(\mathbf{C}))$ is a localizing subcategory of the homotopy category $\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))$. The objects of $\mathrm{Ac}(\mathbf{C})$ are the dg $\mathbf{C}$-modules M such that the complex $\mathsf{M}(C)$ of $\Bbbk$-modules is acyclic for all C in $\mathbf{C}$. A right dg $\mathbf{C}$-module is *representable* if it is contained in the image of the Yoneda dg functor

$$\mathsf{Y}^{\mathbf{C}}_{\mathrm{dg}}\colon \mathbf{C} \to \mathrm{dgMod}(\mathbf{C}), \qquad A \mapsto \mathbf{C}(-, A).$$

The *derived category* of the dg category $\mathbf{C}$ is the Verdier quotient

$$\mathcal{D}(\mathbf{C}) := \mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))/\mathrm{H}^0(\mathrm{Ac}(\mathbf{C})),$$

which turns out to be a triangulated category with small coproducts. Following [10], one could first take the quotient $\mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})$ of the corresponding dg categories. Again by [10], there is a natural exact equivalence

$$\mathcal{D}(\mathbf{C}) = \mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))/\mathrm{H}^0(\mathrm{Ac}(\mathbf{C})) \cong \mathrm{H}^0(\mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})). \tag{2.1}$$

A right dg $\mathbf{C}$-module is *free* if it is isomorphic to a small coproduct of dg modules of the form $\mathsf{Y}^{\mathbf{C}}_{\mathrm{dg}}(A)[m]$, where $A \in \mathbf{C}$ and $m \in \mathbb{Z}$. A right dg $\mathbf{C}$-module M is *semi-free* if it has a filtration

$$0 = \mathsf{M}_0 \subseteq \mathsf{M}_1 \subseteq \cdots \subseteq \mathsf{M} \tag{2.2}$$

such that M_j is a dg $\mathbf{C}$-module, $\mathsf{M}_j/\mathsf{M}_{j-1}$ is free for all $j > 0$, and M is the colimit of the M_j's. We denote by $\mathrm{SF}(\mathbf{C})$ the full dg subcategory of $\mathrm{dgMod}(\mathbf{C})$ consisting of semi-free dg modules. Obviously the image of $\mathsf{Y}^{\mathbf{C}}_{\mathrm{dg}}\colon \mathbf{C} \to \mathrm{dgMod}(\mathbf{C})$ is contained in $\mathrm{SF}(\mathbf{C})$.

Remark 2.2. (i) It is easy to see that, for a dg category $\mathbf{C}$, the homotopy category $\mathrm{H}^0(\mathrm{SF}(\mathbf{C}))$ is a full triangulated subcategory of $\mathrm{H}^0(\mathrm{dgMod}(\mathbf{C}))$. The dg category $\mathbf{C}$ is called *pretriangulated* if the essential image of the functor $\mathrm{H}^0(\mathsf{Y}^{\mathbf{C}}_{\mathrm{dg}})\colon \mathrm{H}^0(\mathbf{C}) \to \mathrm{H}^0(\mathrm{SF}(\mathbf{C}))$ is a triangulated subcategory.

(ii) Given a quasi-functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ between two pretriangulated dg categories, the induced functor $\mathrm{H}^0(\mathsf{F})\colon \mathrm{H}^0(\mathbf{C}_1) \to \mathrm{H}^0(\mathbf{C}_2)$ is an exact functor between triangulated categories.

(iii) By [10, Lemma B.3], there is a natural equivalence of triangulated categories $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$. We can actually be more precise about it. Indeed, the composition of natural dg functors

$$\mathsf{H}\colon \mathrm{SF}(\mathbf{C}) \hookrightarrow \mathrm{dgMod}(\mathbf{C}) \to \mathrm{dgMod}(\mathbf{C})/\mathrm{Ac}(\mathbf{C})$$

is a quasi-equivalence. So, up to composing with (2.1), $\mathrm{H}^0(\mathsf{H})$ provides the exact equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$ mentioned above.

If we are given a dg functor $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$, there exist dg functors

$$\mathsf{Ind}(\mathsf{F})\colon \mathrm{dgMod}(\mathbf{C}_1) \to \mathrm{dgMod}(\mathbf{C}_2), \qquad \mathsf{Res}(\mathsf{F})\colon \mathrm{dgMod}(\mathbf{C}_2) \to \mathrm{dgMod}(\mathbf{C}_1).$$

While $\mathsf{Res}(\mathsf{F})$ is simply defined by $\mathsf{M} \mapsto \mathsf{M} \circ \mathsf{F}^{\circ}$, the reader can look up [10, Sect. 14] for the explicit definition and properties of $\mathsf{Ind}(\mathsf{F})$. Let us just observe that $\mathsf{Ind}(\mathsf{F})$ preserves semi-free dg modules and $\mathsf{Ind}(\mathsf{F})\colon \mathrm{SF}(\mathbf{C}_1) \to \mathrm{SF}(\mathbf{C}_2)$ is a quasi-equivalence if $\mathsf{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ is. Moreover, $\mathsf{Ind}(\mathsf{F})$ commutes with the Yoneda embeddings up to dg isomorphism.

Example 2.3. Let $\mathbf{C}$ be a dg category and $\mathbf{B}$ a full dg subcategory of $\mathbf{C}$. Denote by $\mathsf{I}\colon \mathbf{B} \hookrightarrow \mathbf{C}$ the inclusion dg functor. Then the composition of dg functors

$$\mathbf{C} \xrightarrow{\mathsf{Y}^{\mathbf{C}}_{\mathrm{dg}}} \mathrm{dgMod}(\mathbf{C}) \xrightarrow{\mathsf{Res}(\mathsf{I})} \mathrm{dgMod}(\mathbf{B}) \to \mathrm{dgMod}(\mathbf{B})/\mathrm{Ac}(\mathbf{B})$$

yields, in view of Remark 2.2(iii), a natural quasi-functor $\mathbf{C} \to \mathrm{SF}(\mathbf{B})$.

Let us now give the key definition for this paper.

Definition 2.4. A *dg enhancement* (or simply an *enhancement*) of a triangulated category $\mathbf{T}$ is a pair $(\mathbf{C}, \mathsf{E})$, where $\mathbf{C}$ is a pretriangulated dg category and $\mathsf{E}\colon \mathrm{H}^0(\mathbf{C}) \to \mathbf{T}$ is an exact equivalence.

A priori, one may have 'different' enhancements for the same triangulated category. To make this precise, we need the following.

Definition 2.5. A triangulated category $\mathbf{T}$ *has a unique enhancement* if, given two enhancements $(\mathbf{C}, \mathsf{E})$ and $(\mathbf{C}', \mathsf{E}')$ of $\mathbf{T}$, there exists a quasi-functor $\mathsf{F}\colon \mathbf{C} \to \mathbf{C}'$ such that $\mathrm{H}^0(\mathsf{F})$ is an exact equivalence.

More concisely, a triangulated category $\mathbf{T}$ has a unique enhancement if, for any two enhancements $(\mathbf{C}, \mathsf{E})$ and $(\mathbf{C}', \mathsf{E}')$ of $\mathbf{T}$, the dg categories $\mathbf{C}$ and $\mathbf{C}'$ are isomorphic in $\mathbf{Hqe}$. It is clear that the notion of uniqueness of dg enhancements forgets about part of the data in the definition of enhancement. In particular, the equivalence E does not play a role. So, by abuse of notation, we will often simply say that $\mathbf{C}$ is an enhancement of $\mathbf{T}$ if there exists an exact equivalence $\mathrm{H}^0(\mathbf{C}) \cong \mathbf{T}$.

Nevertheless, there are stronger versions of the notion of uniqueness of dg enhancements. Indeed, we say that $\mathbf{T}$ *has a strongly unique* (respectively, *semi-strongly unique*) *enhancement* if moreover F can be chosen so that there is an isomorphism of exact functors $\mathsf{E} \cong \mathsf{E}' \circ \mathrm{H}^0(\mathsf{F})$ (respectively, there is an isomorphism $\mathsf{E}(C) \cong \mathsf{E}'(\mathrm{H}^0(\mathsf{F})(C))$ in $\mathbf{T}$ for every $C \in \mathbf{C}$).

Example 2.6. (i) If $\mathbf{C}$ is a dg category, $\mathrm{SF}(\mathbf{C})$ is an enhancement of $\mathcal{D}(\mathbf{C})$.

(ii) Let $\mathbf{C}$ be a pretriangulated dg category and let $\mathbf{B}$ be a full pretriangulated dg subcategory of $\mathbf{C}$. We have already mentioned that, by the main result of [10], we have a natural exact equivalence between the Verdier quotient $\mathrm{H}^0(\mathbf{C})/\mathrm{H}^0(\mathbf{B})$ and $\mathrm{H}^0(\mathbf{C}/\mathbf{B})$. Hence $\mathbf{C}/\mathbf{B}$, with the above equivalence, is an enhancement of $\mathrm{H}^0(\mathbf{C})/\mathrm{H}^0(\mathbf{B})$.

2.2. *Dg enhancements for well generated triangulated categories*

If $\mathbf{C}$ is a small dg category such that $\mathrm{H}^0(\mathbf{C})$ has α-coproducts, we denote by $\mathcal{D}_\alpha(\mathbf{C})$ the *α-continuous derived category* of $\mathbf{C}$, which is defined as the full subcategory of $\mathcal{D}(\mathbf{C})$ with objects those $M \in \mathrm{dgMod}(\mathbf{C})$ such that the natural map

$$(H^*(M))\Big(\coprod_{i\in I} C_i\Big) \to \prod_{i\in I}(H^*(M))(C_i)$$

(where the coproduct is intended in $\mathrm{H}^0(\mathbf{C})$) is an isomorphism for all objects $C_i \in \mathbf{C}$, where $|I| < \alpha$. It is useful to know that $\mathcal{D}_\alpha(\mathbf{C})$ is also equivalent to a quotient of $\mathcal{D}(\mathbf{C})$. More precisely, there is a localizing subcategory $\mathbf{N}$ of $\mathcal{D}(\mathbf{C})$ such that the quotient functor $\mathcal{D}(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$ restricts to an exact equivalence $\mathcal{D}_\alpha(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$ (see [32, Sect. 6] for details). By [32, Theorem 6.4], the triangulated category $\mathcal{D}_\alpha(\mathbf{C})$ is α-compactly generated.

Remark 2.7. The triangulated category $\mathcal{D}_\alpha(\mathbf{C})$ has an obvious enhancement $\mathrm{SF}_\alpha(\mathbf{C})$ given as the full dg subcategory of $\mathrm{SF}(\mathbf{C})$ whose objects correspond to those in $\mathcal{D}_\alpha(\mathbf{C})$, under the equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{C})) \cong \mathcal{D}(\mathbf{C})$ (see Remark 2.2(iii)). On the other hand, in a similar way, there is a an enhancement $\mathbf{N}'$ of $\mathbf{N}$ and, by Example 2.6(ii), the composition of dg functors

$$\mathrm{SF}_\alpha(\mathbf{C}) \hookrightarrow \mathrm{SF}(\mathbf{C}) \to \mathrm{SF}(\mathbf{C})/\mathbf{N}'$$

is a quasi-equivalence inducing the exact equivalence $\mathcal{D}_\alpha(\mathbf{C}) \to \mathcal{D}(\mathbf{C})/\mathbf{N}$. It follows that there is a natural quasi-functor $\mathrm{SF}(\mathbf{C}) \to \mathrm{SF}_\alpha(\mathbf{C})$.

The essential step in the proof of [32, Theorem 7.2] can be reformulated (with a slight modification) as follows.

Theorem 2.8. *Let* $\mathbf{C}$ *be a pretriangulated dg category such that* $\mathrm{H}^0(\mathbf{C})$ *is well generated and let* $\mathbf{B}_0$ *be a small set of objects in* $\mathbf{C}$. *Then there exist a regular cardinal* α *and a small and full dg subcategory* $\mathbf{B}$ *of* $\mathbf{C}$ *containing* $\mathbf{B}_0$ *such that* $\mathrm{H}^0(\mathbf{B})$ *is closed under* α*-coproducts and the natural quasi-functor* $\mathbf{C} \to \mathrm{SF}(\mathbf{B})$ *(see Example 2.3) induces an exact equivalence* $\mathsf{Y}' \colon \mathrm{H}^0(\mathbf{C}) \to \mathcal{D}_\alpha(\mathbf{B})$.

Proof. It is shown in the proof of [32, Theorem 7.2] that all the required properties are satisfied, except possibly $\mathbf{B}_0 \subseteq \mathbf{B}$, if we take α such that $\mathrm{H}^0(\mathbf{C})$ is α-compactly generated and $\mathbf{B}$ such that $\mathrm{H}^0(\mathbf{B}) \subseteq \mathrm{H}^0(\mathbf{C})^\alpha$ is a small subcategory and this inclusion is an equivalence. The conclusion then follows from Remark 1.2(ii). □

3. Some abstract results about exact functors

In this section, we go back to the triangulated setting and prove some abstract results about exact functors which will be crucial in the rest of the paper. This should be thought of as a rather technical but essential interlude towards the proof of Theorem C.

Let $\mathbf{A}$ be a small category which we see here as a dg category sitting all in degree 0 (see Example 2.1(i)). With a slight abuse of notation, we will identify $\mathcal{D}(\mathbf{A})$ with the homotopy category $\mathrm{H}^0(\mathrm{SF}(\mathbf{A}))$ (see Remark 2.2(iii)).

The fact that $\mathbf{A}$ is in degree 0 implies that an object of $\mathrm{dgMod}(\mathbf{A})$ can be regarded as a complex

$$C = \{\cdots \to C^{j-1} \xrightarrow{d^{j-1}} C^j \xrightarrow{d^j} C^{j+1} \to \cdots\}$$

in the abelian category $\mathrm{Mod}(\mathbf{A})$ of ($\Bbbk$-linear) functors $\mathbf{A}^\circ \to \mathrm{Mod}(\Bbbk)$ (where $\mathrm{Mod}(\Bbbk)$ is the abelian category of $\Bbbk$-modules). Moreover, there is a natural exact equivalence $\mathcal{D}(\mathbf{A}) \cong \mathrm{D}(\mathrm{Mod}(\mathbf{A}))$ (see, for example, the beginning of [24, Section 7] for a brief discussion). Under this natural identification, $\mathrm{H}^0(\mathsf{Y}^{\mathbf{A}}_{\mathrm{dg}}) \colon \mathbf{A} = \mathrm{H}^0(\mathbf{A}) \to \mathcal{D}(\mathbf{A})$ is actually the composition of the usual Yoneda functor $\mathsf{Y}^{\mathbf{A}} \colon \mathbf{A} \to \mathrm{Mod}(\mathbf{A})$ with the natural inclusion $\mathrm{Mod}(\mathbf{A}) \hookrightarrow \mathrm{D}(\mathrm{Mod}(\mathbf{A})) \cong \mathcal{D}(\mathbf{A})$. Therefore, for simplicity, we denote by

$$\mathsf{Y}^{\mathbf{A}} \colon \mathbf{A} \to \mathcal{D}(\mathbf{A})$$

the functor $\mathrm{H}^0(\mathsf{Y}^{\mathbf{A}}_{\mathrm{dg}})$.

As a consequence of the discussion above, it makes sense to say that $C \in \mathrm{dgMod}(\mathbf{A})$ is bounded (above or below), and for every integer n we can define the *stupid truncations* $\sigma_{\leq n}(C)$, $\sigma_{\geq n}(C)$ as

$$\sigma_{\leq n}(C) := \{\cdots \to C^j \xrightarrow{d^j} C^{j+1} \to \cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \to 0 \to \cdots\},$$
$$\sigma_{\geq n}(C) := \{\cdots \to 0 \to C^n \xrightarrow{d^n} C^{n+1} \to \cdots \to C^j \xrightarrow{d^j} C^{j+1} \to \cdots\}.$$

It is easy to see that if $C \in \mathrm{SF}(\mathbf{A})$, then each C^j is a free $\mathbf{A}$-module and $\sigma_{\leq n}(C)$, $\sigma_{\geq n}(C)$ are in $\mathrm{SF}(\mathbf{A})$. Now we can prove the following results, which should be compared to Lemma 3.2, Corollary 3.3 and Proposition 3.4 of [24].

Lemma 3.1. *Let* $\mathbf{T}$ *be a triangulated category with small coproducts and let* $\mathsf{F}\colon \mathcal{D}(\mathbf{A}) \to \mathbf{T}$ *be an exact functor which preserves small coproducts. Assume moreover that*

$$\mathbf{T}\Big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_{i\in I}\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A_i))[k_i]\Big) = 0 \tag{3.1}$$

for all $A, A_i \in \mathbf{A}$ *(with* I *a small set) and all integers* $k_i < 0$*. Then, given* $A \in \mathbf{A}$ *and* $C \in \mathcal{D}(\mathbf{A})$*,*

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq n}(C))[k]\big) = 0 \tag{3.2}$$

for all integers $k < n-1$*, and also for* $k = n-1$ *if* C *is bounded above.*

Proof. If $C \in \mathrm{SF}(\mathbf{A})$ has a filtration $\{C_j\}$, the induced filtration $\{C'_j\}$ of $\sigma_{\geq n}(C)$ has clearly the property that each quotient C'_j/C'_{j-1} is isomorphic to a small coproduct of objects of the form $\mathsf{Y}^{\mathbf{A}}(A)[s]$ for $A \in \mathbf{A}$ and $s \leq -n$. As F preserves small coproducts and satisfies (3.1), it follows immediately that

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(C'_j/C'_{j-1})[k]\big) = 0$$

for $k < n$. Applying the cohomological functor $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(-))$ to the distinguished triangle $C'_{j-1} \to C'_j \to C'_j/C'_{j-1}$ of $\mathcal{D}(\mathbf{A})$, we deduce by induction that

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(C'_j)[k]\big) = 0 \tag{3.3}$$

for every $j \geq 0$ and for $k < n$.

Now we use the fact that $\sigma_{\geq n}(C) \cong \mathrm{hocolim}\,(C'_j)$. Recall that if we denote by $s_j\colon C'_j \to C'_{j+1}$ the inclusion morphisms, then $\mathrm{hocolim}\,(C'_j)$ is, by definition, the cone in $\mathcal{D}(\mathbf{A})$ of the morphism

$$\sum_{j\geq 0}(\mathrm{id}_{C'_j} - s_j)\colon \coprod_{j\geq 0} C'_j \to \coprod_{j\geq 0} C'_j.$$

As F preserves small coproducts, we have an isomorphism in $\mathbf{T}$

$$\mathsf{F}(\sigma_{\geq n}(C)) \cong \mathsf{F}(\mathrm{hocolim}\,(C'_j)) \cong \mathrm{hocolim}\,\mathsf{F}(C'_j),$$

hence a distinguished triangle

$$\coprod_{j\geq 0}\mathsf{F}(C'_j) \to \coprod_{j\geq 0}\mathsf{F}(C'_j) \to \mathsf{F}(\sigma_{\geq n}(C)).$$

Applying the cohomological functor $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), -)$ to it and using (3.3), we conclude that (3.2) holds for $k < n-1$.

If C is bounded above, we can take $C_j = \sigma_{\geq t-j}(C)$ for some integer t. Then $\sigma_{\geq n}(C) = C_{t-n} = C'_{t-n}$ (meaning 0 if $t-n < 0$), so (3.2) also holds for $k = n-1$ by (3.3). □

Corollary 3.2. *Let $F\colon \mathcal{D}(\mathbf{A}) \to \mathbf{T}$ be an exact functor satisfying the same assumptions as in Lemma 3.1. Then, given $A \in \mathbf{A}$ and $C \in \mathcal{D}(\mathbf{A})$, the map (induced by the natural morphism $C \to \sigma_{\leq m}(C)$)*

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(C)\big) \to \mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\leq m}(C))\big)$$

is injective for every integer $m > 0$, and also for $m = 0$ if C is bounded above. Moreover, the map is an isomorphism for $m > 1$, and also for $m = 1$ if C is bounded above.

Proof. For every integer m we have a distinguished triangle

$$\sigma_{\geq m+1}(C) \to C \to \sigma_{\leq m}(C)$$

in $\mathcal{D}(\mathbf{A})$. It is then enough to apply the cohomological functor $\mathbf{T}(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(-))$ to it, taking into account that, by Lemma 3.1,

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq m+1}(C))\big) = 0$$

for $m > 0$ (also for $m = 0$ if C is bounded above) and

$$\mathbf{T}\big(\mathsf{F}(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}(\sigma_{\geq m+1}(C))[1]\big) = 0$$

for $m > 1$ (also for $m = 1$ if C is bounded above). □

Proposition 3.3. *Let $\mathbf{T}$ be a triangulated category with small coproducts and let $\mathsf{F}_1, \mathsf{F}_2\colon \mathcal{D}(\mathbf{A}) \to \mathbf{T}$ be exact functors which preserve small coproducts. Assume moreover that F_1 and F_2 both satisfy (3.1) and that there is an isomorphism of functors $\theta\colon \mathsf{F}_1 \circ \mathsf{Y}^{\mathbf{A}} \to \mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}}$. Then, for every $C \in \mathcal{D}(\mathbf{A})$ bounded above, there exists an isomorphism $\theta'_C : \mathsf{F}_1(C) \to \mathsf{F}_2(C)$ such that, for every $A \in \mathbf{A}$, every $k \in \mathbb{Z}$ and every $f \in \mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A)[k], C)$, the diagram*

$$\begin{tikzcd}
\mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)[k]) \arrow[r, "\mathsf{F}_1(f)"] \arrow[d, "\theta_A{[k]}"'] & \mathsf{F}_1(C) \arrow[d, "\theta'_C"] \\
\mathsf{F}_2(\mathsf{Y}^{\mathbf{A}}(A)[k]) \arrow[r, "\mathsf{F}_2(f)"] & \mathsf{F}_2(C)
\end{tikzcd} \tag{3.4}$$

commutes in $\mathbf{T}$.

Proof. The proof proceeds verbatim as the proof of [24, Proposition 3.4]. Indeed, it is enough to observe that [24, Corollary 3.3] can be replaced by Corollary 3.2 for $C \in \mathrm{SF}(\mathbf{A})$ bounded above. It should be noted that in [24] the authors assume further that $\mathsf{F}_i(\mathsf{Y}^{\mathbf{A}}(A))$ is a compact object for all $A \in \mathbf{A}$ and $i = 1, 2$. But this hypothesis is never used in the proof of [24, Proposition 3.4]. □

4. Uniqueness of enhancements: a general criterion

This section is completely devoted to the proof of Theorem C. Hence, let $\mathbf{A}$ be a small category, which we see here as a dg category sitting all in degree 0, and let $\mathbf{L}$ be a localizing subcategory of $\mathcal{D}(\mathbf{A})$. We will always assume that

(a) the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category;
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_{i\in I}\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_i))[k_i]) = 0$ for all $A, A_i \in \mathbf{A}$ (with I a small set) and all integers $k_i < 0$,

as in the hypotheses of Theorem C.

4.1. The quasi-functor

Assume that there exists an exact equivalence $\mathsf{E}\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathrm{H}^0(\mathbf{C})$ for some pretriangulated dg category $\mathbf{C}$. Notice that (a) clearly implies that $\mathrm{H}^0(\mathbf{C})$ is also a well generated triangulated category. Consider the composition of functors

$$\mathsf{H}\colon \mathbf{A} \xrightarrow{\mathsf{Y}^{\mathbf{A}}} \mathcal{D}(\mathbf{A}) \xrightarrow{\mathsf{Q}} \mathcal{D}(\mathbf{A})/\mathbf{L} \xrightarrow{\mathsf{E}} \mathrm{H}^0(\mathbf{C}).$$

Remark 4.1. As $\mathsf{Y}^{\mathbf{A}}(\mathbf{A})$ is a small set of compact generators of $\mathcal{D}(\mathbf{A})$ (see [24, Example 1.9]), it follows from Proposition 1.7 that $\mathsf{Q}\circ\mathsf{Y}^{\mathbf{A}}(\mathbf{A})$ satisfies (G1) in $\mathcal{D}(\mathbf{A})/\mathbf{L}$. Since E is an exact equivalence, also $\mathsf{H}(\mathbf{A})$ satisfies (G1) in $\mathrm{H}^0(\mathbf{C})$.

Denoting by $\mathbf{B}_0$ the full dg subcategory of $\mathbf{C}$ such that $\mathrm{H}^0(\mathbf{B}_0) = \mathsf{H}(\mathbf{A})$, we can clearly regard H as a functor $\mathbf{A} \to \mathrm{H}^0(\mathbf{B}_0)$.

Let $\tau_{\leq 0}(\mathbf{B}_0)$ be the dg category with the same objects as $\mathbf{B}_0$ and with

$$\tau_{\leq 0}(\mathbf{B}_0)(B_1, B_2) := \tau_{\leq 0}(\mathbf{B}_0(B_1, B_2))$$

for all B_1 and B_2 in $\mathbf{B}_0$. Here, for a complex of $\Bbbk$-modules (or, more generally, of objects in an abelian category)

$$C = \{\cdots \to C^{j-1} \xrightarrow{d^{j-1}} C^j \xrightarrow{d^j} C^{j+1} \to \cdots\},$$

and for every integer n, we define

$$\tau_{\leq n}(C) := \{\cdots \to C^j \xrightarrow{d^j} C^{j+1} \to \cdots \to C^{n-1} \to \ker d^n \to 0 \to \cdots\},$$

$$\tau_{\geq n}(C) := \{\cdots \to 0 \to \operatorname{coker} d^{n-1} \to C^{n+1} \to \cdots \to C^j \xrightarrow{d^j} C^{j+1} \to \cdots\}.$$

There are obvious dg functors $\tau_{\leq 0}(\mathbf{B}_0) \to \mathrm{H}^0(\mathbf{B}_0)$ and $\tau_{\leq 0}(\mathbf{B}_0) \to \mathbf{B}_0$, and the former is a quasi-equivalence thanks to (b) (taking into account that E is an exact equivalence). Thus we obtain a quasi-functor $\mathrm{H}^0(\mathbf{B}_0) \to \mathbf{B}_0$.

By Theorem 2.8 there exist a regular cardinal α and a small and full dg subcategory $\mathbf{B}$ of $\mathbf{C}$ containing $\mathbf{B}_0$ such that $\mathrm{H}^0(\mathbf{B})$ is closed under α-coproducts and the natural quasi-functor $\mathbf{C} \to \mathrm{SF}(\mathbf{B})$ induces an exact equivalence $\mathsf{Y}'\colon \mathrm{H}^0(\mathbf{C}) \to \mathcal{D}_\alpha(\mathbf{B})$.

If we compose $\mathsf{H}\colon \mathbf{A} \to \mathrm{H}^0(\mathbf{B}_0)$ with the quasi-functor $\mathrm{H}^0(\mathbf{B}_0) \to \mathbf{B}_0$ and the natural inclusion $\mathbf{B}_0 \hookrightarrow \mathbf{B}$, we get a quasi-functor $\mathsf{H}'\colon \mathbf{A} \to \mathbf{B}$. From it we finally obtain a quasi-functor

$$\mathsf{G}_1\colon \mathrm{SF}(\mathbf{A}) \xrightarrow{\mathsf{Ind}(\mathsf{H}')} \mathrm{SF}(\mathbf{B}) \to \mathrm{SF}_\alpha(\mathbf{B}),$$

where the arrow on the right denotes the natural quasi-functor described in Remark 2.7. By passing to the homotopy categories, we also have the exact functor

$$\mathsf{F}_1 := \mathrm{H}^0(\mathsf{G}_1)\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}_\alpha(\mathbf{B}).$$

On the other hand, we can proceed differently and take the exact functor

$$\mathsf{F}_2\colon \mathcal{D}(\mathbf{A}) \xrightarrow{\mathsf{Q}} \mathcal{D}(\mathbf{A})/\mathbf{L} \xrightarrow{\mathsf{E}} \mathrm{H}^0(\mathbf{C}) \xrightarrow{\mathsf{Y}'} \mathcal{D}_\alpha(\mathbf{B}).$$

The following results will be used later.

Lemma 4.2. *The functors* F_1 *and* F_2 *satisfy the assumptions of Proposition* 3.3.

Proof. It is clear from the definitions and from (b) that both functors preserve small coproducts. Indeed, it is obvious that F_2 commutes with small coproducts if and only if Q does. But this last fact was already observed in Remark 1.6. Moreover, F_2 satisfies (3.1) and $\mathsf{F}_1 \circ \mathsf{Y}^{\mathbf{A}} \cong \mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}}$. This last fact then clearly implies that also F_1 satisfies (3.1). □

Corollary 4.3. *The set* $\mathsf{F}_1 \circ \mathsf{Y}^{\mathbf{A}}(\mathbf{A})$ *satisfies* (G1) *in* $\mathcal{D}_\alpha(\mathbf{B})$.

Proof. As $\mathsf{F}_1 \circ \mathsf{Y}^{\mathbf{A}} \cong \mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}}$, it is enough to show that $\mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}}(\mathbf{A})$ satisfies (G1) in $\mathcal{D}_\alpha(\mathbf{B})$. Since $\mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}} = \mathsf{Y}' \circ \mathsf{H}$ and Y' is an exact equivalence, this follows from Remark 4.1. □

4.2. The proof of Theorem C

Let $\mathbf{C}$ and $\mathsf{E}\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathrm{H}^0(\mathbf{C})$ be as in Section 4.1. Denote by $\mathbf{L}'$ the full dg subcategory of $\mathrm{SF}(\mathbf{A})$ such that $\mathrm{H}^0(\mathbf{L}') \cong \mathbf{L}$ under the equivalence $\mathrm{H}^0(\mathrm{SF}(\mathbf{A})) \cong \mathcal{D}(\mathbf{A})$.

Lemma 4.4. *The quasi-functor* G_1 *factors through the quotient dg functor* $\mathrm{SF}(\mathbf{A}) \to \mathrm{SF}(\mathbf{A})/\mathbf{L}'$.

Proof. The proof is very similar to the one of [24, Lemma 5.2] with the required adjustments due to the more general setting we are working in. More precisely, in view of the main result of [10], it is enough to show that F_1 factors through the quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$, i.e. $\mathsf{F}_1(L) \cong 0$ for all L in $\mathbf{L}$.

By Corollary 4.3, we have just to show that

$$\mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}_1(L)) = 0$$

for all $A \in \mathbf{A}$ and $L \in \mathbf{L}$ (since $\mathbf{L}$ is closed under shifts).

By Lemma 4.2, we can apply the results of Section 3 to the functors F_1 and F_2. In particular, by Corollary 3.2, there are isomorphisms (for $i = 1, 2$)

$$\mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}_i(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}_i(L)) \cong \mathcal{D}_\alpha(\mathbf{B})\big(\mathsf{F}_i(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}_i(\sigma_{\le m}(L))\big)$$

for $m > 1$. On the other hand, by Proposition 3.3, there is an isomorphism $\mathsf{F}_1(\sigma_{\le m}(L)) \cong \mathsf{F}_2(\sigma_{\le m}(L))$. It follows that

$$\mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}_1(L)) \cong \mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}_2(\mathsf{Y}^{\mathbf{A}}(A)), \mathsf{F}_2(L)),$$

and the latter Hom-space is naturally isomorphic to 0, since $\mathsf{F}_2(L) \cong 0$. □

Hence G_1 factors through a quasi-functor $\mathsf{G}\colon \mathrm{SF}(\mathbf{A})/\mathbf{L}' \to \mathrm{SF}_\alpha(\mathbf{B})$. If we show that it defines an isomorphism in **Hqe**, Theorem C would follow immediately if we take into account Theorem 2.8. This is the content of the next proposition.

Proposition 4.5. *In the above situation,* $\mathsf{G}\colon \mathrm{SF}(\mathbf{A})/\mathbf{L}' \to \mathrm{SF}_\alpha(\mathbf{B})$ *defines an isomorphism in* **Hqe**.

Proof. Set $\mathsf{F} := \mathrm{H}^0(\mathsf{G})\colon \mathcal{D}(\mathbf{A})/\mathbf{L} \to \mathcal{D}_\alpha(\mathbf{B})$. It is enough to show that F is an equivalence. Notice that F preserves small coproducts, since the same is true for $\mathsf{F} \circ \mathsf{Q} \cong \mathsf{F}_1$ by Lemma 4.2.

We first prove that F is fully faithful, namely that the map

$$\mathsf{F}_{B,C}\colon \mathcal{D}(\mathbf{A})/\mathbf{L}\,(B,C) \to \mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}(B),\mathsf{F}(C))$$

is an isomorphism for all $B, C \in \mathcal{D}(\mathbf{A})/\mathbf{L}$. Now, it is easy to see that the full subcategory $\mathbf{S}$ of $\mathcal{D}(\mathbf{A})/\mathbf{L}$ which consists of the objects S for which $\mathsf{F}_{S,C}$ is an isomorphism for every $C \in \mathcal{D}(\mathbf{A})/\mathbf{L}$ is a localizing subcategory of $\mathcal{D}(\mathbf{A})/\mathbf{L}$. In view of Remark 4.1 and Proposition 1.4, it is therefore enough to prove that $\mathsf{F}_{\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),C}$ is an isomorphism for all $A \in \mathbf{A}$ and $C \in \mathcal{D}(\mathbf{A})/\mathbf{L}$. The proof proceeds as in [24, Lemma 5.3]. Let us outline the argument for the convenience of the reader.

Setting $P := \mathsf{Q}^R(C)$, we have $\mathsf{Q}(P) \cong C$, and so $\mathsf{F}_{\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),C}$ is an isomorphism if and only if $\mathsf{F}_{\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{Q}(P)}$ is. Moreover, as the map

$$\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),P}\colon \mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A),P) \to \mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{Q}(P))$$

is an isomorphism by adjunction (since $\mathsf{Q}^R(\mathsf{Q}(P)) \cong P$) and $\mathsf{F}_1 \cong \mathsf{F} \circ \mathsf{Q}$, we can just prove that

$$(\mathsf{F}_1)_{\mathsf{Y}^{\mathbf{A}}(A),P}\colon \mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A),P) \to \mathcal{D}_\alpha(\mathbf{B})(\mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{F}_1(P))$$

is an isomorphism.

By Lemma 4.2 the functors F_1 and F_2 satisfy the hypotheses of Corollary 3.2, and the same is clearly true of Q. In particular, there is an isomorphism

$$\mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{Q}(P)) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}\left(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{Q}(\sigma_{\le m}(P))\right),$$

for $m > 1$. Moreover, this is compatible with the natural isomorphism

$$\mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A),P) \cong \mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)).$$

As $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),P}$ is an isomorphism, so is $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)}$.

The same argument applies to the functor F_1, and then it is enough to check that $(\mathsf{F}_1)_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)}$ is an isomorphism. To this end, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{D}(\mathbf{A})(\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)) \arrow[r, "(\mathsf{F}_1)_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)}"] \arrow[dr, "(\mathsf{F}_2)_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\le m}(P)}"'] & \mathcal{D}_\alpha(\mathbf{B})\left(\mathsf{F}_1(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{F}_1(\sigma_{\le m}(P))\right) \arrow[d, "\gamma"] \\
& \mathcal{D}_\alpha(\mathbf{B})\left(\mathsf{F}_2(\mathsf{Y}^{\mathbf{A}}(A)),\mathsf{F}_2(\sigma_{\le m}(P))\right)
\end{tikzcd}$$

where the existence of an isomorphism γ is ensured by Proposition 3.3 (which again applies due to Lemma 4.2). Since $\mathsf{F}_2 = \mathsf{Y}' \circ \mathsf{E} \circ \mathsf{Q}$, the fact that $\mathsf{Q}_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m}(P)}$ is an isomorphism implies that so is $(\mathsf{F}_2)_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m}(P)}$, by taking into account that $\mathsf{Y}' \circ \mathsf{E}$ is an equivalence. In conclusion, $(\mathsf{F}_1)_{\mathsf{Y}^{\mathbf{A}}(A),\sigma_{\leq m}(P)}$ is an isomorphism as well.

Finally, the essential image of F is a localizing subcategory of $\mathcal{D}_\alpha(\mathbf{B})$ (because F preserves small coproducts) which contains $\mathsf{Y}' \circ \mathsf{H}(\mathbf{A})$ (as $\mathsf{Y}' \circ \mathsf{H} = \mathsf{F}_2 \circ \mathsf{Y}^{\mathbf{A}} \cong \mathsf{F} \circ \mathsf{Q} \circ \mathsf{Y}^{\mathbf{A}}$). Since Y' is an exact equivalence, it follows from Remark 4.1 and Proposition 1.4 that F is essentially surjective. □

5. The case of the derived category of a Grothendieck category

In this section, we prove Theorem A and discuss some geometric applications of this abstract criterion for Grothendieck categories.

5.1. The abstract result

Let $\mathbf{G}$ be a Grothendieck category and let $\mathbf{A}$ be a full subcategory of $\mathbf{G}$ whose objects form a small set of generators of $\mathbf{G}$. Setting $\mathbf{M} := \mathrm{Mod}(\mathbf{A})$, we will denote by $\mathsf{S}\colon \mathbf{G} \to \mathbf{M}$ the natural functor defined by

$$\mathsf{S}(C)(A) := \mathbf{G}(A, C) \quad \text{for } C \in \mathbf{G} \text{ and } A \in \mathbf{A}.$$

We can first prove the following result which should be compared to [24, Theorem 7.4].

Proposition 5.1. *The functor* $\mathsf{S}\colon \mathbf{G} \to \mathbf{M}$ *admits a left adjoint* $\mathsf{T}\colon \mathbf{M} \to \mathbf{G}$. *Moreover,* T *is exact,* $\mathsf{T} \circ \mathsf{S} \cong \mathrm{id}_{\mathbf{G}}$, $\mathbf{N} := \ker \mathsf{T}$ *is a localizing Serre subcategory of* $\mathbf{M}$, *and* T *induces an equivalence* $\mathsf{T}'\colon \mathbf{M}/\mathbf{N} \to \mathbf{G}$ *such that* $\mathsf{T} \cong \mathsf{T}' \circ \Pi$, *where* $\Pi\colon \mathbf{M} \to \mathbf{M}/\mathbf{N}$ *is the projection functor.*

Proof. In [9, Theorem 2.2] the analogous statement is proved for the functor $\mathsf{S}'\colon \mathbf{G} \to \text{MOD-}R$, which we are going to define. Consider the object $U := \coprod_{A \in \mathbf{A}} A$ of $\mathbf{G}$ and denote, for every $A \in \mathbf{A}$, by $\iota_A\colon A \hookrightarrow U$ and $\rho_A\colon U \twoheadrightarrow A$ the natural inclusion and projection morphisms, respectively. Let S be the ring (with unit) $\mathbf{G}(U, U)$ and R the subring of S consisting of those $s \in S$ for which $s \circ \iota_A \neq 0$ only for a finite number of $A \in \mathbf{A}$. Notice that R is a ring with unit if and only if $\mathbf{A}$ has a finite number of objects, in which case obviously $R = S$. Let moreover MOD-R be the full subcategory of $\mathrm{Mod}(R)$ having as objects those $P \in \mathrm{Mod}(R)$ for which $PR = P$ (clearly MOD-$R = \mathrm{Mod}(R) = \mathrm{Mod}(S)$ if $\mathbf{A}$ has a finite number of objects). Then S' is simply given as the composition of $\mathbf{G}(U, -)\colon \mathbf{G} \to \mathrm{Mod}(S)$ with the natural functor $\mathrm{Mod}(S) \to \text{MOD-}R$ defined on objects by $P \mapsto PR$. To deduce our statement from [9, Theorem 2.2] it is therefore enough to show that there is an equivalence of categories $\mathsf{E}\colon \mathbf{M} \to \text{MOD-}R$ such that $\mathsf{S}' \cong \mathsf{E} \circ \mathsf{S}$.

In order to define E, consider first an object M of $\mathbf{M}$, namely a ($\Bbbk$-linear) functor $M\colon \mathbf{A}^\circ \to \mathrm{Mod}(\Bbbk)$. As a $\Bbbk$-module, $\mathsf{E}(M)$ is just $\coprod_{A \in \mathbf{A}} M(A)$, whereas the R-module structure is defined as follows. Given $r \in R$ and $m \in \mathsf{E}(M)$ with components $m_A \in M(A)$ for every $A \in \mathbf{A}$, the element $mr \in \mathsf{E}(M)$ has components $(mr)_A =$

$\sum_{B\in\mathbf{A}} M(\rho_B \circ r \circ \iota_A)(m_B)$. It is easy to prove that this actually defines an object $\mathsf{E}(M)$ of MOD-R. As for morphisms, given $M, M' \in \mathrm{Mod}(\mathbf{A})$ and a natural transformation $\gamma\colon M \to M'$, the morphism of R-modules $\mathsf{E}(\gamma)\colon \mathsf{E}(M) \to \mathsf{E}(M')$ sends m to m', where $m'_A := \gamma(A)(m_A)$ for every $A \in \mathbf{A}$. It is not difficult to check that this really defines a functor $\mathsf{E}\colon \mathbf{M} \to$ MOD-R and that $\mathsf{S}' \cong \mathsf{E} \circ \mathsf{S}$.

It remains to prove that E is an equivalence. It is clear by definition that E is faithful. As for fullness, given $M, M' \in \mathrm{Mod}(\mathbf{A})$ and a morphism $\phi\colon \mathsf{E}(M) \to \mathsf{E}(M')$ in MOD-R, it is easy to see that $\phi = \mathsf{E}(\gamma)$, where $\gamma\colon M \to M'$ is the natural transformation defined as follows. For every $A \in \mathbf{A}$ and for every $a \in M(A)$, denoting by m the element of $\mathsf{E}(M)$ such that $m_A = a$ and $m_B = 0$ for $A \neq B \in \mathbf{A}$, we set $\gamma(A)(a) := \phi(m)_A$. Finally, E is essentially surjective because it is not difficult to prove that for every $P \in$ MOD-R we have $P \cong \mathsf{E}(M)$ with $M \in \mathbf{M}$ defined in the following way. Setting $r_f := \iota_B \circ f \circ \rho_A \in R$ for every morphism $f\colon A \to B$ of $\mathbf{A}$, we define $M(A) := Pr_{\mathrm{id}_A}$ for every $A \in \mathbf{A}$, whereas $M(f)\colon M(B) = Pr_{\mathrm{id}_B} \to M(A) = Pr_{\mathrm{id}_A}$ for every morphism $f\colon A \to B$ of $\mathbf{A}$ is given by $pr_{\mathrm{id}_B} \mapsto pr_f = (pr_f)r_{\mathrm{id}_A}$ for every $p \in P$. □

Remark 5.2. It should be noted that while, by Proposition 5.1, the functor T is exact, S is only left-exact in general. On the other hand, the fact that $\mathsf{T} \circ \mathsf{S} \cong \mathrm{id}_{\mathbf{G}}$ implies that S is fully faithful and that $\mathsf{T} \circ \mathsf{Y}^{\mathbf{A}}$ is isomorphic to the inclusion $\mathbf{A} \hookrightarrow \mathbf{G}$ (since $\mathsf{S}|_{\mathbf{A}} \cong \mathsf{Y}^{\mathbf{A}}$ by definition). Here, as in the previous sections, $\mathsf{Y}^{\mathbf{A}}\colon \mathbf{A} \hookrightarrow \mathbf{M} \hookrightarrow \mathrm{D}(\mathbf{M})$ is the Yoneda embedding.

Passing from $\mathbf{G}$ to its derived category $\mathrm{D}(\mathbf{G})$, we observe that the functors T, T' and Π being exact, we can denote by the same letters the corresponding derived functors.

Denote by $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ the full triangulated subcategory of $\mathrm{D}(\mathbf{M})$ consisting of complexes with cohomology in $\mathbf{N}$. Let moreover $\mathsf{Q}\colon \mathrm{D}(\mathbf{M}) \to \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ be the projection functor.

Corollary 5.3. *The functor Π induces an exact equivalence $\Pi'\colon \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \to \mathrm{D}(\mathbf{M}/\mathbf{N})$ such that $\Pi \cong \Pi' \circ \mathsf{Q}$. Moreover, $\mathsf{T}' \circ \Pi' \circ \mathsf{Q} \circ \mathsf{Y}^{\mathbf{A}}$ is isomorphic to the inclusion $\mathbf{A} \hookrightarrow \mathbf{G} \hookrightarrow \mathrm{D}(\mathbf{G})$.*

Proof. By Proposition 5.1, $\Pi\colon \mathbf{M} \to \mathbf{M}/\mathbf{N}$ admits a right adjoint, so the first part of the statement follows from [19, Lemma 5.9]. Hence the diagram

$$\begin{tikzcd}
 & \mathrm{D}(\mathbf{M}) \arrow[dl, "\mathsf{Q}"'] \arrow[d, "\Pi"] \arrow[dr, "\mathsf{T}"] & \\
\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \arrow[r, "\Pi'"] & \mathrm{D}(\mathbf{M}/\mathbf{N}) \arrow[r, "\mathsf{T}'"] & \mathrm{D}(\mathbf{G})
\end{tikzcd}$$

commutes up to isomorphism. By the above commutativity, the second part of the statement follows from the fact that the inclusion $\mathbf{A} \hookrightarrow \mathbf{G}$ is isomorphic to $\mathsf{T} \circ \mathsf{Y}^{\mathbf{A}}$ (see Remark 5.2). □

We are now ready to prove our first result.

Proof of Theorem A. Given a Grothendieck category **G**, by Proposition 5.1 and Corollary 5.3 we know that $\mathrm{D}(\mathbf{G}) \cong \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ for $\mathbf{M} = \mathrm{Mod}(\mathbf{A})$ and **N** defined as above.

Consider **A**, consisting of a small set of generators of **G**, as a small dg category all sitting in degree 0. As noted at the beginning of Section 3, there is a natural exact equivalence $\mathrm{D}(\mathbf{M}) \cong \mathcal{D}(\mathbf{A})$. By setting **L** to be the full localizing subcategory of $\mathcal{D}(\mathbf{A})$ which is the image of $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ under the above equivalence, we see that $\mathrm{D}(\mathbf{G}) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}$.

Let us observe the following:

(a) The quotient $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is a well generated triangulated category. This is because $\mathrm{D}(\mathbf{G})$, which is naturally equivalent to $\mathcal{D}(\mathbf{A})/\mathbf{L}$, is well generated by Example 1.3, and well generation is obviously preserved under exact equivalences.

(b) Consider the objects $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$ for A in **A**. By Corollary 5.3, they are mapped to objects in the abelian category **G** by the composition of the equivalences $\mathcal{D}(\mathbf{A})/\mathbf{L} \cong \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cong \mathrm{D}(\mathbf{G})$ described above. This implies that

$$\mathcal{D}(\mathbf{A})/\mathbf{L}\Big(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A)), \coprod_{i\in I} \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_i))[k_i]\Big) = 0$$

for all $A, A_i \in \mathbf{A}$ (with I a small set) and all integers $k_i < 0$.

In particular, the assumptions of Theorem C are satisfied and so the triangulated category $\mathcal{D}(\mathbf{A})/\mathbf{L}$ (and hence $\mathrm{D}(\mathbf{G})$) has a unique enhancement. □

5.2. *The geometric examples*

We now discuss some geometric incarnations of Theorem A. There are certainly many interesting geometric triangulated categories which are equivalent to the derived category of a Grothendieck category and which are not considered here. So we do not claim that our list of applications is complete. Notice that, beyond the geometric situations studied in [24] and described in the introduction, the uniqueness of enhancements has been investigated in other cases, e.g. for the derived categories of supported quasi-coherent sheaves in special situations (see [8, Lemma 4.6]).

Algebraic stacks. Let X be an algebraic stack. For general facts about these geometric objects, we refer to [22] and [37].

We can consider the abelian categories $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules on X and $\mathbf{Qcoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules on X. The fact that $\mathbf{Qcoh}(X)$ is a Grothendieck category is proved in [37, Tag 06WU]. Passing to the derived categories, we can consider $\mathrm{D}(\mathbf{Qcoh}(X))$ and the full triangulated subcategory $\mathrm{D}_{\mathrm{qc}}(X)$ of $\mathrm{D}(\mathrm{Mod}(\mathcal{O}_X))$ consisting of complexes with quasi-coherent cohomology. The relation between these two triangulated categories is delicate, as pointed out in [13, Theorem 1.2].

We then have the following.

Corollary 5.4. *If X is an algebraic stack, then* $\mathrm{D}(\mathbf{Qcoh}(X))$ *has a unique enhancement. If X is also quasi-compact and with quasi-finite affine diagonal, then* $\mathrm{D}_{\mathrm{qc}}(X)$ *has a unique enhancement.*

Proof. The first part of the statement is an obvious consequence of Theorem A. For the second part, observe that, by [14, Theorem A], the category $\mathrm{D}_{\mathrm{qc}}(X)$ is compactly generated by a single object. Hence, by [13, Theorem 1.1], the natural functor $\mathrm{D}(\mathbf{Qcoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an exact equivalence. □

Remark 5.5. The above result specializes to the case of schemes. In particular, $\mathrm{D}(\mathbf{Qcoh}(X))$ has a unique enhancement for any scheme X. If X is quasi-compact and semi-separated (i.e. the diagonal is affine or, equivalently, the intersection of two open affine subschemes in X is affine), then $\mathrm{D}(\mathbf{Qcoh}(X)) \cong \mathrm{D}_{\mathrm{qc}}(X)$ (see [2, Corollary 5.5]) and the same uniqueness result holds for $\mathrm{D}_{\mathrm{qc}}(X)$. This extends vastly the results in [24], where the uniqueness results for both categories are proved only for quasi-compact, semi-separated schemes with enough locally free sheaves. This last condition means that for any finitely presented sheaf F there is an epimorphism $E \twoheadrightarrow F$ in $\mathbf{Qcoh}(X)$, where E is locally free of finite type.

As in [24, Remark 7.7], we should observe here that we can take X to be a semi-separated scheme rather than separated, because the proof of [2, Corollary 5.5] works for a semi-separated scheme as well.

Twisted sheaves. Let X be a scheme and pick $\alpha \in H^2_{\text{ét}}(X, \mathcal{O}_X^*)$, i.e. an element in the Brauer group $\mathrm{Br}(X)$ of X. We may represent α by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$ with $X = \bigcup_{i\in I} U_i$ an appropriate open cover in the étale topology. An *α-twisted quasi-coherent sheaf* E consists of pairs $(\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ such that the E_i are quasi-coherent sheaves on U_i and $\varphi_{ij} : E_j|_{U_i\cap U_j} \to E_i|_{U_i\cap U_j}$ are isomorphisms satisfying the following conditions:

- $\varphi_{ii} = \mathrm{id}$;
- $\varphi_{ji} = \varphi_{ij}^{-1}$;
- $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

We denote by $\mathbf{Qcoh}(X, \alpha)$ the abelian category of such α-twisted quasi-coherent sheaves on X. It is proved in [23, Proposition 2.1.3.3] that this definition coincides with the alternative one in terms of quasi-coherent sheaves on the gerbe $\mathcal{X} \to X$ on X associated to α.

Proposition 5.6. *If X is a scheme and $\alpha \in \mathrm{Br}(X)$, then $\mathbf{Qcoh}(X, \alpha)$ is a Grothendieck abelian category.*

Proof. The same argument used in the proof of [1, Proposition 3.2] (where X is assumed to be quasi-compact and quasi-separated) works in this greater generality.[1] Indeed, denoting by $\mathcal{X} \to X$ the gerbe associated to α, one just needs to know that $\mathbf{Qcoh}(\mathcal{X})$ is a Grothendieck category, which is true because in any case $\mathcal{X}$ is an algebraic stack. □

It is then clear from Theorem A that we can deduce the following.

Corollary 5.7. *If X is a scheme and $\alpha \in \mathrm{Br}(X)$, then the triangulated category $\mathrm{D}(\mathbf{Qcoh}(X, \alpha))$ has a unique enhancement.*

[1] We thank Benjamin Antieau for pointing this out.

6. The case of the category of compact objects

In this section we prove Theorem B. This needs some preparation. In particular, using the arguments in Section 5.1, we construct an exact equivalence $\mathcal{D}(\mathbf{A})/\mathbf{L} \cong \mathrm{D}(\mathbf{G})$ for some localizing subcategory $\mathbf{L}$ of $\mathcal{D}(\mathbf{A})$ and reduce to the criterion for uniqueness due to Lunts and Orlov (see [24, Theorem 2]). Verifying that the assumptions of Lunts–Orlov's result are satisfied is the main and most delicate task of this section.

6.1. *The first reduction*

If $\mathbf{G}$ is a Grothendieck category and $\mathbf{A}$ is a small set of generators of $\mathbf{G}$ which we think of as a full subcategory of $\mathbf{G}$, we know from Section 5.1 that there is a pair of adjoint functors

$$\mathsf{T}\colon \mathbf{M} \to \mathbf{G}, \qquad \mathsf{S}\colon \mathbf{G} \to \mathbf{M}$$

where $\mathbf{M} := \mathrm{Mod}(\mathbf{A})$.

As already explained in the proof of Theorem A, the quotient $\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ (where $\mathbf{N} := \ker \mathsf{T}$) is naturally equivalent to $\mathcal{D}(\mathbf{A})/\mathbf{L}$. For this, we think of $\mathbf{A}$ as a dg category sitting in degree 0 and we take $\mathbf{L}$ to be the localizing subcategory corresponding to $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ under the natural equivalence $\mathrm{D}(\mathbf{M}) \cong \mathcal{D}(\mathbf{A})$. Moreover, there is an exact equivalence $\mathrm{D}(\mathbf{G}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ such that A in $\mathbf{A}$, seen as a subcategory of $\mathbf{G}$, is mapped to $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$, where $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ is the quotient functor. As a consequence,

$$\mathcal{D}(\mathbf{A})/\mathbf{L}\left(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]\right) = 0$$

for all $A_1, A_2 \in \mathbf{A}$ and all integers $i < 0$.

Consider now the following result.

Theorem 6.1 ([24, Theorem 2]). *Let* $\mathbf{A}$ *be a small category and let* $\mathbf{L}$ *be a localizing subcategory of* $\mathcal{D}(\mathbf{A})$ *such that*

(a) $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ *and* $\mathbf{L}^c$ *satisfies* (G1) *in* $\mathbf{L}$*;*
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) = 0$ *for all* $A_1, A_2 \in \mathbf{A}$ *and all integers* $i < 0$.

Then $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ *has a unique enhancement.*

By the discussion above, (b) holds. If we could prove (a), then this theorem would immediately imply that $\mathrm{D}(\mathbf{G})^c$ has a unique enhancement. Thus, in order to prove Theorem B, it is enough to show that the assumptions (1)–(4) in the statement imply that (a) holds. This delicate check will be the content of the next section.

6.2. *Verifying assumption (a) in Theorem 6.1*

In the following we need to know the precise definition of $\mathsf{T}\colon \mathbf{M} \to \mathbf{G}$. To this end, we first fix some notation. For $M \in \mathbf{M}$, let $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$ be the comma category whose objects are pairs (A, a) with $A \in \mathbf{A}$ and $a \in \mathbf{M}(\mathsf{Y}^{\mathbf{A}}(A), M)$, and whose morphisms are given by

$$(\mathsf{Y}^{\mathbf{A}} \downarrow M)((A', a'), (A, a)) := \{f \in \mathbf{A}(A', A) : a' = a \circ \mathsf{Y}^{\mathbf{A}}(f)\}.$$

Observe that, by Yoneda's lemma, $\mathbf{M}(\mathsf{Y}^{\mathbf{A}}(A), M)$ can be identified with $M(A)$ and that, in this way, the above equality $a' = a \circ \mathsf{Y}^{\mathbf{A}}(f)$ becomes $a' = M(f)(a)$; in what follows we will freely use these identifications. Denote by $\mathsf{F}_M : (\mathsf{Y}^{\mathbf{A}} \downarrow M) \to \mathbf{A}$ the forgetful functor. It is well known (see, for instance, [26, Section III.7]) that

$$M \cong \varinjlim((\mathsf{Y}^{\mathbf{A}} \downarrow M) \xrightarrow{\mathsf{F}_M} \mathbf{A} \xrightarrow{\mathsf{Y}^{\mathbf{A}}} \mathbf{M}),$$

where the colimit is taken over the composition $\mathsf{Y}^{\mathbf{A}} \circ \mathsf{F}_M$. Since T (being a left adjoint) preserves colimits and $\mathsf{T} \circ \mathsf{Y}^{\mathbf{A}}$ is isomorphic to the inclusion $\mathbf{A} \hookrightarrow \mathbf{G}$ (see Remark 5.2), we obtain

$$\mathsf{T}(M) \cong \varinjlim((\mathsf{Y}^{\mathbf{A}} \downarrow M) \xrightarrow{\mathsf{F}_M} \mathbf{A} \hookrightarrow \mathbf{G}).$$

More explicitly, consider the objects of $\mathbf{G}$

$$Y_M := \coprod_{(A,a) \in (\mathsf{Y}^{\mathbf{A}} \downarrow M)} A, \qquad X_M := \coprod_{(f : (A',a') \to (A,a)) \in \mathrm{Mor}(\mathsf{Y}^{\mathbf{A}} \downarrow M)} A',$$

and denote by $\iota_{(A,a)} : A \hookrightarrow Y_M$ (for every object (A, a) of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$) and $\iota_f : A' \hookrightarrow X_M$ (for every morphism $f : (A', a') \to (A, a)$ of $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$) the natural morphisms. Then, by (the dual version of) [26, Theorem 2, p. 113], we have

Lemma 6.2. *There is a natural isomorphism*

$$\mathsf{T}(M) \cong \mathrm{coker}(\alpha_M : X_M \to Y_M),$$

where, for every morphism $f : (A', a') \to (A, a)$ *of* $(\mathsf{Y}^{\mathbf{A}} \downarrow M)$*,*

$$\alpha_M \circ \iota_f := \iota_{(A',a')} - \iota_{(A,a)} \circ f. \tag{6.1}$$

Let us now move to the core of the proof that assumption (a) in Theorem 6.1 holds in our situation. As explained in the introduction, we do not expect it to hold true in general. This is the reason why we need the further assumptions (1)–(4) in Theorem B. For the convenience of the reader, we list them again here:

(1) $\mathbf{A}$ is closed under finite coproducts;
(2) every object of $\mathbf{A}$ is noetherian in $\mathbf{G}$;
(3) if $f : A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{G}$ with $A, A' \in \mathbf{A}$, then $\ker f \in \mathbf{A}$;
(4) for every $A \in \mathbf{A}$ there exists $N(A) > 0$ such that $\mathrm{D}(\mathbf{G})(A, A'[N(A)]) = 0$ for every $A' \in \mathbf{A}$.

Remark 6.3. If $f : \coprod_{i \in I} C_i \to C$ (with I a small set) is a morphism in $\mathbf{G}$ and B is a noetherian subobject of C such that $B \subseteq \mathrm{im}\, f$, then there exists a finite subset I' of I such that $B \subseteq f(\coprod_{i \in I'} C_i)$ (for otherwise we could find elements $i_1, i_2, \dots$ in I such that $f(\coprod_{j=1}^{n} C_{i_j}) \cap B$ for $n > 0$ form a strictly increasing sequence of subobjects of B).

Lemma 6.4. *Assume that conditions* (1) *and* (2) *above are satisfied. If* $f : C \twoheadrightarrow A$ *is an epimorphism of* $\mathbf{G}$ *with* $A \in \mathbf{A}$*, then there exists a morphism* $g : A' \to C$ *with* $A' \in \mathbf{A}$ *such that* $f \circ g : A' \twoheadrightarrow A$ *is again an epimorphism of* $\mathbf{G}$*.*

Proof. Given f as in the statement, there exist a small set I and an epimorphism $g'\colon \coprod_{i\in I} A_i \twoheadrightarrow C$ (so that $f\circ g'$ is also an epimorphism) with $A_i \in \mathbf{A}$ for every $i\in I$ (because the objects of $\mathbf{A}$ form a small set of generators of $\mathbf{G}$). As A is noetherian in $\mathbf{G}$ by condition (2), Remark 6.3 implies that there exists a finite subset I' of I such that if we set $A' := \coprod_{i\in I'} A_i$ (which is an object of $\mathbf{A}$ thanks to condition (1)) and $g := g'|_{A'}$, the composition $f\circ g\colon A' \twoheadrightarrow A$ is an epimorphism of $\mathbf{G}$. □

Proposition 6.5. *If conditions* (1) *and* (2) *are satisfied, then* $\mathbf{N}$ *coincides with the full subcategory* $\mathbf{N}'$ *of* $\mathbf{M}$ *having as objects those* $M\in\mathbf{M}$ *satisfying the following property: for every object* (A,a) *of* $(\mathsf{Y}^{\mathbf{A}}\downarrow M)$ *there exists an epimorphism* $f\colon A'\twoheadrightarrow A$ *of* $\mathbf{G}$ *with* $A'\in\mathbf{A}$ *such that* $a\circ\mathsf{Y}^{\mathbf{A}}(f)=0$.

Proof. Given $M\in\mathbf{N}'$, we have to prove that $\mathsf{T}(M)\cong 0$. By Lemma 6.2, this is true if and only if α_M is an epimorphism. So, given a morphism $g\colon Y_M\to C$ in $\mathbf{G}$ such that $g\circ\alpha_M = 0$, we need to show that $g=0$. Now, if g is given by morphisms $g_{(A,a)}\colon A\to C$ for every $(A,a)\in(\mathsf{Y}^{\mathbf{A}}\downarrow M)$, then $g\circ\alpha_M=0$ is equivalent, by (6.1), to $g_{(A',a')} = g_{(A,a)}\circ f$ for every morphism $f\colon (A',a')\to(A,a)$ of $(\mathsf{Y}^{\mathbf{A}}\downarrow M)$. Since $M\in\mathbf{N}'$, for every $(A,a)\in(\mathsf{Y}^{\mathbf{A}}\downarrow M)$ there exists an epimorphism $f\colon A'\twoheadrightarrow A$ of $\mathbf{G}$ with $A'\in\mathbf{A}$ such that $a\circ\mathsf{Y}^{\mathbf{A}}(f)=0$. Then $f,0\colon (A',0)\to(A,a)$ are morphisms of $(\mathsf{Y}^{\mathbf{A}}\downarrow M)$, whence

$$g_{(A,a)}\circ f = g_{(A',0)} = g_{(A,a)}\circ 0 = 0.$$

As f is an epimorphism, we conclude that $g_{(A,a)}=0$, thus proving that $g=0$.

Conversely, assume that $N\in\mathbf{N}$, and fix an object (A,a) of $(\mathsf{Y}^{\mathbf{A}}\downarrow N)$. Since α_N is an epimorphism (again by Lemma 6.2) and A is a noetherian object of $\mathbf{G}$, by Remark 6.3 we can find a finite number of distinct morphisms of $(\mathsf{Y}^{\mathbf{A}}\downarrow N)$, say $f_i\colon (A'_i,a'_i)\to(A_i,a_i)$ for $i=1,\dots,n$, such that, setting

$$A'_0 := \coprod_{i=1}^{n} A'_i \subset X_N,$$

we have $\iota_{(A,a)}(A)\subseteq \alpha_N(A'_0)$. Moreover,

$$\alpha_N(A'_0)\subseteq A_0 := \coprod_{(A',a')\in I} A' \subset Y_N,$$

where I is the (finite) subset of the objects of $(\mathsf{Y}^{\mathbf{A}}\downarrow N)$ consisting of those (A',a') which are equal to (A'_i,a'_i) or (A_i,a_i) for some $i=1,\dots,n$. Note that $A_0, A'_0\in\mathbf{A}$ by (1). In the cartesian diagram in $\mathbf{G}$

$$\begin{tikzcd} B \arrow[r, "f'"] \arrow[d, "g'"'] & A \arrow[d, "\iota_{(A,a)}"] \\ A'_0 \arrow[r, "\alpha_N|_{A'_0}"] & A_0 \end{tikzcd}$$

the morphism f' is an epimorphism because $\iota_{(A,a)}(A)\subseteq\alpha_N(A'_0)$. So, by Lemma 6.4, there exists a morphism $k\colon A'\to B$ with $A'\in\mathbf{A}$ such that $f:=f'\circ k\colon A'\twoheadrightarrow A$ is an epimorphism of $\mathbf{G}$. Set also $g := g' \circ k \colon A' \to A'_0$ and denote by

$$a_0 \colon \mathsf{Y}^{\mathbf{A}}(A_0) \cong \coprod_{(A',a') \in I} \mathsf{Y}^{\mathbf{A}}(A') \to N$$

the morphism of $\mathbf{M}$ whose components are given by a' for every $(A', a') \in I$. Then the diagram

$$\begin{tikzcd}
\mathsf{Y}^{\mathbf{A}}(A') \arrow[r, "\mathsf{Y}^{\mathbf{A}}(f)"] \arrow[d, "\mathsf{Y}^{\mathbf{A}}(g)"'] & \mathsf{Y}^{\mathbf{A}}(A) \arrow[r, "a"] \arrow[d, "\mathsf{Y}^{\mathbf{A}}(\iota_{(A,a)})"'] & N \\
\mathsf{Y}^{\mathbf{A}}(A'_0) \arrow[r, "\mathsf{Y}^{\mathbf{A}}(\alpha_N|_{A'_0})"] & \mathsf{Y}^{\mathbf{A}}(A_0) \arrow[ur, "a_0"'] &
\end{tikzcd}$$

commutes in $\mathbf{M}$. As $\mathsf{Y}^{\mathbf{A}}(A'_0) \cong \coprod_{i=1}^{n} \mathsf{Y}^{\mathbf{A}}(A'_i)$ and

$$a_0 \circ \mathsf{Y}^{\mathbf{A}}(\alpha_N|_{A'_0} \circ \iota_{f_i}) = a'_i - a_i \circ \mathsf{Y}^{\mathbf{A}}(f_i) = 0$$

for every $i = 1, \ldots, n$ (by (6.1) and by definition of morphism in $(\mathsf{Y}^{\mathbf{A}} \downarrow N)$), we see that $a_0 \circ \mathsf{Y}^{\mathbf{A}}(\alpha_N|_{A'_0}) = 0$. This clearly implies $a \circ \mathsf{Y}^{\mathbf{A}}(f) = 0$, which proves $N \in \mathbf{N}'$. □

Theorem 6.6. *Assume that conditions* (1)–(4) *are satisfied. Then* $\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$ *satisfies* (G1) *in* $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$.

Proof. In the triangulated category $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$, consider an object

$$M = (\cdots \to M^0 \xrightarrow{m^0} M^1 \to \cdots)$$

such that $M \not\cong 0$. We must find a morphism $0 \neq x \colon P \to M$ with P in $\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$. Set $N^i := H^i(M)$; by definition $N^i \in \mathbf{N}$ for every $i \in \mathbb{Z}$ and $N^i \neq 0$ for at least one i. Without loss of generality we can assume that $N^0 \neq 0$, hence there exists $(A^0, \bar{a}^0)$ in $(\mathsf{Y}^{\mathbf{A}} \downarrow N^0)$ with $\bar{a}^0 \neq 0$.

We claim that we can find a complex

$$A = (0 \to A^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} A^0 \to 0)$$

of $\mathbf{A} \subseteq \mathbf{G}$ with $n = N(A^0)$ such that $H^i(A) = 0$ for every $i \neq -n$. Here $N(A^0)$ is the integer whose existence is prescribed by (4) applied to A^0. Furthermore, we will show that there is a morphism $a \colon \mathsf{Y}^{\mathbf{A}}(A) \to M$ of complexes of $\mathbf{M}$ (with components $a^i \colon \mathsf{Y}^{\mathbf{A}}(A^i) \to M^i$) such that, with $p^i \colon \ker m^i \twoheadrightarrow N^i$ denoting the natural projection morphism for every $i \in \mathbb{Z}$, we have $p^0 \circ a^0 = \bar{a}^0$. Notice that since $m^0 \circ a^0 = 0$, we can regard a^0 as a morphism $\mathsf{Y}^{\mathbf{A}}(A^0) \to \ker m^0$. Moreover, observe that for such a complex A the objects $K^i := \ker d^i$ of $\mathbf{G}$ are actually in $\mathbf{A}$. Indeed, this is clear for $i \geq 0$ or $i < -n$, whereas for $-n \leq i < 0$ there is a short exact sequence

$$0 \to K^i \xrightarrow{j^i} A^i \to K^{i+1} \to 0$$

in $\mathbf{G}$ (because $H^{i+1}(A) = 0$), hence one can prove that $K^i \in \mathbf{A}$ by descending induction on i using condition (3).

In order to prove the claim, we define the morphisms a^i and d^i again by descending induction on i. For $i = 0$, we can find $a^0\colon \mathsf{Y}^{\mathbf{A}}(A^0) \to \ker m^0 \subseteq M^0$ such that $p^0 \circ a^0 = \bar{a}^0$ because p^0 is an epimorphism in $\mathbf{M}$. For the inductive step, assume that $-n \leq i < 0$ and that suitable $a^{i'}$ and $d^{i'}$ have already been defined for $i' > i$. There exists a (unique) $k^{i+1}\colon \mathsf{Y}^{\mathbf{A}}(K^{i+1}) \to \ker m^{i+1}$ such that the diagram

$$\begin{tikzcd}
\mathsf{Y}^{\mathbf{A}}(K^{i+1}) \arrow[r, "\mathsf{Y}^{\mathbf{A}}(j^{i+1})"] \arrow[d, "k^{i+1}"] & \mathsf{Y}^{\mathbf{A}}(A^{i+1}) \arrow[r, "\mathsf{Y}^{\mathbf{A}}(d^{i+1})"] \arrow[d, "a^{i+1}"] & \mathsf{Y}^{\mathbf{A}}(A^{i+2}) \arrow[d, "a^{i+2}"] \\
\ker m^{i+1} \arrow[r, hook] & M^{i+1} \arrow[r, "m^{i+1}"] & M^{i+2}
\end{tikzcd}$$

commutes (because $d^{i+1} \circ j^{i+1} = 0$ and the square on the right commutes by induction). Consider the object $(K^{i+1}, p^{i+1} \circ k^{i+1})$ of $(\mathsf{Y}^{\mathbf{A}} \downarrow N^{i+1})$. Since $N^{i+1} \in \mathbf{N}$, by Proposition 6.5 there exists an epimorphism $q^i : A^i \twoheadrightarrow K^{i+1}$ such that $p^{i+1} \circ k^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(q^i) = 0$. So

$$k^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(q^i)\colon \mathsf{Y}^{\mathbf{A}}(A^i) \to \ker m^{i+1}$$

factors through $\operatorname{im} m^i \hookrightarrow \ker m^{i+1}$, and there exists a morphism a^i such that the diagram

$$\begin{tikzcd}
\mathsf{Y}^{\mathbf{A}}(A^i) \arrow[rr, "\mathsf{Y}^{\mathbf{A}}(q^i)"] \arrow[d, "a^i"] \arrow[dr] & & \mathsf{Y}^{\mathbf{A}}(K^{i+1}) \arrow[r, "\mathsf{Y}^{\mathbf{A}}(j^{i+1})"] \arrow[d, "k^{i+1}"] & \mathsf{Y}^{\mathbf{A}}(A^{i+1}) \arrow[d, "a^{i+1}"] \\
M^i \arrow[r, two heads] \arrow[rrr, bend right, "m^i"] & \operatorname{im} m^i \arrow[r, hook] & \ker m^{i+1} \arrow[r, hook] & M^{i+1}
\end{tikzcd}$$

commutes. Then, setting $d^i := j^{i+1} \circ q^i$, we clearly have

$$H^{i+1}(A) = 0 \quad \text{and} \quad a^{i+1} \circ \mathsf{Y}^{\mathbf{A}}(d^i) = m^i \circ a^i,$$

thus completing the proof of the inductive step.

Since $A \cong K^{-n}[n]$ in $\mathrm{D}(\mathbf{G})$ and $K^{-n} \in \mathbf{A}$, the natural morphism of complexes $l : A^0 \to A$ defined by $l^0 = \mathrm{id}_{A_0}$ is 0 in $\mathrm{D}(\mathbf{G})(A^0, A) \cong \mathrm{D}(\mathbf{G})(A^0, K^{-n}[n])$ by condition (4). Thus we can find a complex C of $\mathbf{G}$ and a quasi-isomorphism $r\colon C \to A^0$ such that $l \circ r \sim 0$, where $\sim$ denotes homotopy of morphisms of complexes. As $H^i(C)$ is isomorphic to an object of $\mathbf{A}$ for every $i \in \mathbb{Z}$ and is 0 for $i > 0$, there exists a quasi-isomorphism $s\colon B \to C$ with $B^i \in \mathbf{A}$ for every $i \in \mathbb{Z}$ and $B^i = 0$ for $i > 0$: this follows for instance from [38, Lemma 1.9.5] (applied with F the inclusion of $\mathbf{A}$ in $\mathbf{G}$ and $\mathcal{C}$ the full subcategory of the category of complexes in $\mathbf{G}$ having as objects the complexes whose cohomologies are bounded above and isomorphic to objects of $\mathbf{A}$), whose key condition 1.9.5.1 is satisfied due to Lemma 6.4. Then $t := r \circ s\colon B \to A^0$ is also a quasi-isomorphism and $l \circ t \sim 0$. It is straightforward to check that t factors through a quasi-isomorphism $\tilde{t}\colon \tilde{B} := \tau_{\geq -n}(B) \to A^0$ and that $l \circ \tilde{t} \sim 0$, too. Indeed, the same maps $B^i \to A^{i-1}$ which provide the homotopy $l \circ t \sim 0$ and are necessarily zero for $i \leq -n$ yield the desired homotopy $l \circ \tilde{t} \sim 0$.

Hence, if we denote by $\tilde{A}$ the mapping cone of $\tilde{\imath}$ and by $u\colon A^0 \to \tilde{A}$ the natural inclusion, there exists a morphism of complexes $f\colon \tilde{A} \to A$ such that $f \circ u \sim l$. It is easy to prove, applying (3) and the same argument used above to show that $K^i \in \mathbf{A}$, that $\tilde{B}^i \in \mathbf{A}$ for every integer i. It follows from (1) that the same is true for $\tilde{A}^i$.

Now we can take $P := \mathsf{Y}^{\mathbf{A}}(\tilde{A})$ and $x := a \circ \mathsf{Y}^{\mathbf{A}}(f)\colon P \to M$ (or, better, its image in $\mathrm{D}(\mathbf{M})$). Indeed, $x \circ \mathsf{Y}^{\mathbf{A}}(u) \sim a \circ \mathsf{Y}^{\mathbf{A}}(l) = a^0$, which implies

$$H^0(x \circ \mathsf{Y}^{\mathbf{A}}(u)) = H^0(a^0) = \bar{a}^0 \neq 0.$$

Therefore $x \circ \mathsf{Y}^{\mathbf{A}}(u) \neq 0$, whence $x \neq 0$ in $\mathrm{D}(\mathbf{M})$. Moreover, $P \in \mathrm{D}(\mathbf{M})^c$ because $\mathrm{D}(\mathbf{M})^c$ is a triangulated subcategory of $\mathrm{D}(\mathbf{M})$ containing the image of $\mathsf{Y}^{\mathbf{A}}$ (see Remark 4.1), and $\tilde{A}$ is a bounded complex of objects of $\mathbf{A}$. Finally, we have $\mathsf{T}(P) \cong \tilde{A}$ by Remark 5.2. Remembering that T is exact and observing that $\tilde{A}$ is an acyclic complex (being the mapping cone of the quasi-isomorphism $\tilde{\imath}$), we conclude that

$$\mathsf{T}(H^i(P)) \cong H^i(\mathsf{T}(P)) \cong H^i(\tilde{A}) = 0$$

for every $i \in \mathbb{Z}$, which means that $P \in \mathrm{D}_{\mathbf{N}}(\mathbf{M})$. □

An easy application of the above result is the following.

Corollary 6.7. *Assume that conditions (1)–(4) are satisfied. Then*

(i) $\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c = \mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$;
(ii) *the quotient functor* $\mathrm{D}(\mathbf{M}) \to \mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ *sends* $\mathrm{D}(\mathbf{M})^c$ *to* $(\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$;
(iii) *the induced functor* $\mathrm{D}(\mathbf{M})^c/\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c \to (\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$ *is fully faithful and identifies* $(\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}))^c$ *with the idempotent completion of* $\mathrm{D}(\mathbf{M})^c/\mathrm{D}_{\mathbf{N}}(\mathbf{M})^c$.

Proof. The triangulated category $\mathrm{D}(\mathbf{M})$ is compactly generated by Remark 4.1, and so the isomorphism classes of objects in $\mathrm{D}(\mathbf{M})^c$ form a small set (for this use, for example, [20, Lemma 5]). Thus we can choose a small set $\mathbf{S}$ of representatives of the isomorphism classes of objects in $\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cap \mathrm{D}(\mathbf{M})^c$, and Theorem 6.6 clearly implies that $\mathbf{S}$ satisfies (G1) in $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$. Since $\mathrm{D}(\mathbf{M})/\mathrm{D}_{\mathbf{N}}(\mathbf{M}) \cong \mathrm{D}(\mathbf{G})$ is well generated (see Example 1.3), the localizing subcategory $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ is well generated as well (see, for example, [21, Theorem 7.4.1]). Hence, by Proposition 1.4, the category $\mathrm{D}_{\mathbf{N}}(\mathbf{M})$ is generated by $\mathbf{S}$. Now we just apply Theorem 1.5. □

Theorem 6.6 and Corollary 6.7(i) imply that the assumption (a) of Theorem 6.1 is satisfied in our specific situation (i.e. when (1)–(4) are satisfied). Hence the proof of Theorem B is complete.

Remark 6.8. It should be noted that, under the same assumptions (1)–(4) of Theorem B, one can actually prove that the triangulated category $\mathrm{D}(\mathbf{G})^c$ has a semi-strongly unique enhancement. This result follows again from Theorem 6.6 and Corollary 6.7(i) by using [24, Theorem 6.4], rather than Theorem 6.1.

6.3. The geometric examples

In this section we describe an easy geometric application of Theorem B in the case of perfect complexes on some algebraic stacks. For this we need to recall some definitions.

Let R be a commutative ring. A complex $P \in \mathrm{D}(\mathrm{Mod}(R))$ is *perfect* if it is quasi-isomorphic to a bounded complex of projective R-modules of finite presentation. Following [14], if X is an algebraic stack, a complex $P \in \mathrm{D}_{\mathrm{qc}}(X)$ is *perfect* if for any smooth morphism $\mathrm{Spec}(R) \to X$, where R is a commutative ring, the complex of R-modules $\mathbf{R}\Gamma(\mathrm{Spec}(R), P|_{\mathrm{Spec}(R)})$ is perfect. We denote by $\mathbf{Perf}(X)$ the full subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$ consisting of perfect complexes.

A quasi-compact and quasi-separated algebraic stack X is *concentrated* if $\mathbf{Perf}(X) \subseteq \mathrm{D}_{\mathrm{qc}}(X)^c$. On the other hand, if X has also quasi-finite affine diagonal, then the other inclusion $\mathrm{D}_{\mathrm{qc}}(X)^c \subseteq \mathbf{Perf}(X)$ holds as well, as a direct consequence of [14, Theorem A]. Moreover, we already observed in the proof of Corollary 5.4 that, under the same assumptions, the natural functor $\mathrm{D}(\mathbf{Qcoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an exact equivalence.

Summing up, if X is a concentrated algebraic stack with quasi-finite affine diagonal, then there is a natural exact equivalence

$$\mathbf{Perf}(X) \cong \mathrm{D}(\mathbf{Qcoh}(X))^c. \tag{6.2}$$

When a stack X has the property that $\mathbf{Qcoh}(X)$ is generated, as a Grothendieck category, by a small set of objects contained in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$, we say that X *has enough perfect coherent sheaves*.

Example 6.9. Suppose that a scheme X has enough locally free sheaves, according to the definition given in Remark 5.5. This yields a small set of generators of $\mathbf{Qcoh}(X)$ contained in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. Indeed, we can take a set of representatives for the isomorphism classes of locally free sheaves, as every sheaf in $\mathbf{Qcoh}(X)$ is a filtered colimit of finitely presented $\mathcal{O}_X$-modules (see [12, 9.4.9]). Hence a scheme with enough locally free sheaves has enough perfect coherent sheaves as well.

As an application of Theorem B, we get the following.

Proposition 6.10. *Let X be a noetherian concentrated algebraic stack with quasi-finite affine diagonal and enough perfect coherent sheaves. Then* $\mathbf{Perf}(X)$ *has a unique enhancement.*

Proof. Consider the isomorphism classes of objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. It is clear that they form a small set. Define then $\mathbf{A}$ to be the full subcategory of $\mathbf{Qcoh}(X)$ whose set of objects is obtained by taking a representative in each isomorphism class of objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. Since, by assumption, a subset of $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$ generates $\mathbf{Qcoh}(X)$, so does $\mathbf{A}$.

Let us now observe that $\mathbf{A}$ satisfies (1)–(4) in Theorem B. Indeed, (1) is obvious and (2) holds true because X is noetherian. To prove (3), observe that the kernel is defined in $\mathbf{Coh}(X)$ up to isomorphism, and moreover the kernel of an epimorphism $A \twoheadrightarrow A'$ in $\mathbf{A}$ is isomorphic to the shift of the cone of f in $\mathbf{Perf}(X)$. Hence it is (up to isomorphism) an object in $\mathbf{A}$. Finally, since X is concentrated, (4) is verified as well. Indeed, in view of [14, Remark 4.12], a concentrated scheme has finite homological dimension, and then (4) follows from the fact that the objects of $\mathbf{A}$ are in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$ rather than just in $\mathbf{Coh}(X)$.

At this point, the result follows directly from Theorem B and (6.2). $\square$

As a direct consequence, we get the following.

Corollary 6.11. *If X is a noetherian scheme with enough locally free sheaves, then $\mathbf{Perf}(X)$ has a unique enhancement.*

Proof. A scheme that is noetherian is concentrated (see [4, Theorem 3.1.1]). Moreover, by [40, Proposition 1.3], any noetherian scheme with enough locally free sheaves is semi-separated. By Example 6.9 and Proposition 6.10, the result is then clear. $\square$

7. Applications

In this section we discuss two easy applications of the circle of ideas concerning the uniqueness of enhancements for the category of perfect complexes. The first one is about a uniqueness result for the enhancements of the bounded derived category of coherent sheaves. The second one concerns some basic questions related to exact functors between the categories of perfect complexes or complexes of quasi-coherent sheaves.

7.1. The bounded derived category of coherent sheaves

Assume again that X is a noetherian scheme with enough locally free sheaves (which is again automatically semi-separated in view of [40, Proposition 1.3]). Let $\mathbf{A}$ be a full subcategory of $\mathbf{Qcoh}(X)$ whose objects are obtained by picking a representative in each isomorphism class of the objects in $\mathbf{Coh}(X) \cap \mathbf{Perf}(X)$. As we observed in the proof of Corollary 6.11, $\mathbf{A}$ is a small set of generators of $\mathbf{Qcoh}(X)$. Hence, we can apply the discussion in Section 5.1, getting a natural exact equivalence

$$\mathrm{D}(\mathbf{Qcoh}(X)) \cong \mathcal{D}(\mathbf{A})/\mathbf{L}, \tag{7.1}$$

where $\mathbf{L}$ is an explicit localizing subcategory of $\mathcal{D}(\mathbf{A})$. Remember that, under this equivalence, every object $A \in \mathbf{A}$ is mapped to $\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A))$, where, as usual, $\mathsf{Q}\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{A})/\mathbf{L}$ denotes the quotient functor (see the discussion in Section 6.1). Since $\mathbf{A} \subseteq \mathrm{D}(\mathbf{Qcoh}(X))^c$ in view of (6.2), it follows from [24, Remark 1.20] that $\mathbf{S} := \mathsf{Q} \circ \mathsf{Y}^{\mathbf{A}}(\mathbf{A})$ is a small set of compact generators of $\mathcal{D}(\mathbf{A})/\mathbf{L}$.

Following [24, Section 8], we say that an object B in $\mathcal{D}(\mathbf{A})/\mathbf{L}$ is *compactly approximated by the objects in* $\mathbf{S}$ if

(1) there is $m \in \mathbb{Z}$ such that, for any $S \in \mathbf{S}$, we have $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(S, B[i]) \cong 0$ when $i < m$;
(2) for any $k \in \mathbb{Z}$, there are P_k in $(\mathcal{D}(\mathbf{A})/\mathbf{L})^c$ and a morphism $f_k\colon P_k \to B$ such that, for every $S \in \mathbf{S}$, the canonical map

$$\mathcal{D}(\mathbf{A})/\mathbf{L}\,(S, P_k[i]) \to \mathcal{D}(\mathbf{A})/\mathbf{L}\,(S, B[i])$$

is an isomorphism when $i \geq k$.

We denote by $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}$ the full subcategory of $\mathcal{D}(\mathbf{A})/\mathbf{L}$ consisting of the objects which are compactly approximated by $\mathbf{S}$. It must be noted that this definition actually depends on the choice of $\mathbf{S}$.

Denote by $\mathrm{D}^b(X)$ the bounded derived category of the abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X. We now show that the equivalence (7.1) induces an exact equivalence

$$\mathrm{D}^b(X) \cong (\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}. \tag{7.2}$$

Indeed, the proof of [24, Proposition 8.9] can be repeated line by line and the same argument applies in our setting. The only delicate issue is that [24, Lemma 8.10] has to be replaced by the following statement: Let X be a scheme as above and let E be the image under (7.1) of an object in $\mathbf{S}$. Then there exists an integer $N(E)$ such that, for all $k \geq N(E)$ and all quasi-coherent sheaves F, we have $\mathrm{Ext}^k(E, F) = 0$. This follows easily from the fact that E is, by definition, a coherent sheaf in $\mathbf{Perf}(X)$. Thus the results of [38, Sections B.11 and B.12] apply.

Consider now the following result.

Theorem 7.1 ([24, Theorem 8.8]). *Let* $\mathbf{A}$ *be a small category and let* $\mathbf{L}$ *be a localizing subcategory of* $\mathcal{D}(\mathbf{A})$ *such that*

(a) $\mathbf{L}^c = \mathbf{L} \cap \mathcal{D}(\mathbf{A})^c$ *and* $\mathbf{L}^c$ *satisfies* (G1) *in* $\mathbf{L}$*;*
(b) $\mathcal{D}(\mathbf{A})/\mathbf{L}\,(\mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_1)), \mathsf{Q}(\mathsf{Y}^{\mathbf{A}}(A_2))[i]) = 0$ *for all* $A_1, A_2 \in \mathbf{A}$ *and all integers* $i < 0$.

Then $(\mathcal{D}(\mathbf{A})/\mathbf{L})^{ca}$ *has a unique enhancement.*

This has the following easy consequence.

Corollary 7.2. *If X is a noetherian scheme with enough locally free sheaves, then* $\mathrm{D}^b(X)$ *has a unique enhancement.*

Proof. The proofs of Proposition 6.10 and Corollary 6.11 actually show that, with these assumptions on X and our choice of $\mathbf{A}$, hypotheses (a) and (b) of Theorem 6.1 are satisfied. As they coincide with (a) and (b) of Theorem 7.1, we conclude by (7.2). □

7.2. *Fourier–Mukai functors*

Assume that X_1 and X_2 are noetherian schemes. Given $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$, we define the exact functor $\Phi_{\mathcal{E}} \colon \mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ as

$$\Phi_{\mathcal{E}}(-) := \mathbf{R}(p_2)_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)),$$

where $p_i \colon X_1 \times X_2 \to X_i$ is the natural projection.

Definition 7.3. An exact functor $\mathsf{F} \colon \mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ ($\mathsf{G} \colon \mathbf{Perf}(X_1) \to \mathbf{Perf}(X_2)$, respectively) is a *Fourier–Mukai functor* (or of *Fourier–Mukai type*) if there exists an object $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$ ($\mathsf{G} \cong \Phi_{\mathcal{E}}$, respectively).

These functors are ubiquitous in algebraic geometry (see [7] for a survey) and for a long while it was believed by some people that all exact functors between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$, with X_i a smooth projective scheme, had to be of Fourier–Mukai type. A beautiful counterexample by Rizzardo and Van den Bergh [34] showed this to be false. Moreover, if X_1 and X_2 are not smooth projective it is not even clear if the celebrated result of Orlov [31] asserting that all exact equivalences between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$ are of Fourier–Mukai type holds true.

A much weaker question can now be formulated as follows. For two triangulated categories $\mathbf{T}_1$ and $\mathbf{T}_2$, we denote by $\mathbf{Eq}(\mathbf{T}_1,\mathbf{T}_2)$ the set of isomorphism classes of exact equivalences between $\mathbf{T}_1$ and $\mathbf{T}_2$. When $\mathbf{T}_i$ is either $\mathrm{D}(\mathbf{Qcoh}(X_i))$ or $\mathbf{Perf}(X_i)$, for X_i a noetherian scheme, we can further define the subset $\mathbf{Eq}^{\mathrm{FM}}(\mathbf{T}_1,\mathbf{T}_2)$ consisting of all equivalences of Fourier–Mukai type.

As an application of the results in the previous section, we get the following.

Proposition 7.4. *Let X_1 and X_2 be noetherian schemes with enough locally free sheaves. Then $\mathbf{Eq}(\mathbf{Perf}(X_1),\mathbf{Perf}(X_2))\neq\emptyset$ if and only if $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))\neq\emptyset$. Moreover, in that case $\mathbf{Eq}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))\neq\emptyset$.*

Proof. In view of (6.2), an exact equivalence $\mathrm{D}(\mathbf{Qcoh}(X_1))\to\mathrm{D}(\mathbf{Qcoh}(X_2))$ restricts to an exact equivalence $\mathbf{Perf}(X_1)\to\mathbf{Perf}(X_2)$, since the subcategories of compact objects are clearly preserved. Hence, $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))\neq\emptyset$ implies that the same is true for the categories of perfect complexes.

On the other hand, assume that $\mathbf{Eq}(\mathbf{Perf}(X_1),\mathbf{Perf}(X_2))\neq\emptyset$. Denote by $\mathrm{Perf}^{\mathrm{dg}}(X_i)$ a dg enhancement of $\mathbf{Perf}(X_i)$, for $i=1,2$. By Corollary 6.11, $\mathrm{Perf}^{\mathrm{dg}}(X_1)\cong\mathrm{Perf}^{\mathrm{dg}}(X_2)$ in $\mathbf{Hqe}$. This clearly implies that there is an exact equivalence between $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_1))$ and $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_2))$. By [24, Proposition 1.16] (see also [24, proof of Corollary 9.13]), there is an exact equivalence between $\mathcal{D}(\mathrm{Perf}^{\mathrm{dg}}(X_i))$ and $\mathrm{D}(\mathbf{Qcoh}(X_i))$ for $i=1,2$. Thus $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))\neq\emptyset$.

As for the last statement, assume (without loss of generality by the previous part) that there is F in $\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))$. By [35, Proposition 6.9], the functor F sends the subcategory $\mathrm{D}^b(\mathbf{Qcoh}(X_1))$ of cohomologically bounded complexes to $\mathrm{D}^b(\mathbf{Qcoh}(X_2))$. By using the same argument as above, we see that F induces an exact equivalence

$$\mathrm{D}^b(\mathbf{Qcoh}(X_1))^c\to\mathrm{D}^b(\mathbf{Qcoh}(X_2))^c.$$

Then we conclude that $\mathbf{Eq}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))\neq\emptyset$, since $\mathrm{D}^b(\mathbf{Qcoh}(X_i))^c\cong\mathrm{D}^b(X_i)$ for $i=1,2$ by [35, Corollary 6.16]. □

Notice that if we assume further that $X_1\times X_2$ is noetherian and that any complex in $\mathbf{Perf}(X_i)$ is isomorphic to a bounded complex of vector bundles, then [39, Corollary 8.12] and [25, Theorem 1.1] imply that

$$\mathbf{Eq}(\mathbf{Perf}(X_1),\mathbf{Perf}(X_2))\neq\emptyset\quad\text{iff}\quad\mathbf{Eq}^{\mathrm{FM}}(\mathbf{Perf}(X_1),\mathbf{Perf}(X_2))\neq\emptyset$$

and

$$\mathbf{Eq}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))\neq\emptyset\ \text{ iff }\ \mathbf{Eq}^{\mathrm{FM}}(\mathrm{D}(\mathbf{Qcoh}(X_1)),\mathrm{D}(\mathbf{Qcoh}(X_2)))\neq\emptyset.$$

Hence Proposition 7.4 can be reformulated in terms of the sets of Fourier–Mukai equivalences.

Remark 7.5. By using the observation in Remark 6.8 and the strategy in the proof of [24, Corollary 9.12], we can make the above remarks more precise when dealing with perfect complexes. Indeed, pick $\mathsf{F} \in \mathbf{Eq}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2))$ for X_i noetherian with enough locally free sheaves and such that $X_1 \times X_2$ is noetherian and any complex in $\mathbf{Perf}(X_i)$ is isomorphic to a bounded complex of vector bundles. Then there exists G in $\mathbf{Eq}^{\mathrm{FM}}(\mathbf{Perf}(X_1), \mathbf{Perf}(X_2))$ such that $\mathsf{F}(C) \cong \mathsf{G}(C)$ for any C in $\mathbf{Perf}(X_1)$.

Acknowledgments. Parts of this paper were written while P.S. was visiting the Institut de Mathématiques de Jussieu, Université de Paris 7. The warm hospitality and the financial support of this institution is gratefully acknowledged. The second author was benefitting from the financial support by the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA–INDAM).

It is a great pleasure to thank Benjamin Antieau, Pieter Belmans, Valery Lunts, Dmitri Orlov and Marco Porta for very useful conversations and comments. We thank Pawel Sosna for carefully reading this paper in a preliminary version and for suggesting several improvements in the exposition. Various email exchanges with Henning Krause were crucial for this work. We warmly thank him for kindly answering all our questions. We are grateful to Olaf Schnürer who pointed out that all noetherian schemes with enough locally free sheaves are semi-separated. Our deepest thanks goes to the anonymous referee who pointed out a mistake in the previous version of the proof of Theorem A.

A. C. was partially supported by the national research project "Spazi di Moduli e Teoria di Lie" (PRIN 2012). P. S. is partially supported by the grants FIRB 2012 "Moduli Spaces and Their Applications" and the national research project "Geometria delle Varietà Proiettive" (PRIN 2010-11).

References

[1] Antieau, B.: A reconstruction theorem for abelian categories of twisted sheaves. J. Reine Angew. Math. **712**, 175–188 (2016) Zbl 1375.14072 MR 3466552

[2] Bökstedt, M., Neeman, A.: Homotopy limits in triangulated categories. Compos. Math. **86**, 209–234 (1993) Zbl 0802.18008 MR 1214458

[3] Bondal, A., Larsen, M., Lunts, V.: Grothendieck ring of pretriangulated categories. Int. Math. Res. Notices **2004**, 1461–1495 Zbl 1079.18008 MR 2051435

[4] Bondal, A., Van den Bergh, M.: Generators and representability of functors in commutative and noncommutative geometry. Moscow Math. J. **3**, 1–36 (2003) Zbl 1135.18302 MR 1996800

[5] Bousfield, A. K.: The localization of spectra with respect to homology. Topology **18**, 257–281 (1979) Zbl 0417.55007 MR 0551009

[6] Căldăraru, A.: Derived categories of twisted sheaves on Calabi–Yau manifolds. Ph.D. thesis, Cornell Univ. (2000) MR 2700538

[7] Canonaco, A., Stellari, P.: Fourier–Mukai functors: a survey. In: Derived Categories in Algebraic Geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., 27–60 (2012) Zbl 1283.14001 MR 2050699

[8] Canonaco, A., Stellari, P.: Fourier–Mukai functors in the supported case. Compos. Math. **150**, 1349–1383 (2014) Zbl 1326.14039 MR 3252022

[9] Castaño Iglesias, F., Enache, P., Năstăsescu, C., Torrecillas, B.: Gabriel–Popescu type theorems and applications. Bull. Sci. Math. **128**, 323–332 (2004) Zbl 1074.18002

[10] Drinfeld, V.: DG quotients of DG categories. J. Algebra **272**, 643–691 (2004) Zbl 1064.18009 MR 2028075

[11] Dugger, D., Shipley, B.: A curious example of triangulated-equivalent model categories which are not Quillen equivalent. Algebr. Geom. Topol. **9**, 135–166 (2009) Zbl 1159.18305 MR 2482071

[12] Grothendieck, A., Dieudonné, J.: Éléments de géométrie algébrique. I. Inst. Hautes Études Sci. Publ. Math. **4**, 228 pp. (1960) Zbl 0118.36206 MR 0217083

[13] Hall, J., Neeman, A., Rydh, D.: One positive and two negative results for derived categories of algebraic stacks. J. Inst. Math. Jussieu, to appear; arXiv:1405.1888

[14] Hall, J., Rydh, D.: Perfect complexes on algebraic stacks. Compos. Math. **153**, 2318–2367 (2017) Zbl 06810455 MR 3705292

[15] Keller, B.: A remark on the generalized smashing conjecture. Manuscripta Math. **84**, 193–198 (1994) Zbl 0826.18004 MR 1285956

[16] Keller, B.: Deriving DG categories. Ann. Sci. École Norm. Sup. **27**, 63–102 (1995) Zbl 0799.18007 MR 1258406

[17] Keller, B.: On differential graded categories. In: International Congress of Mathematicians (Madrid, 2006), Vol. II, Eur. Math. Soc., Zürich, 151–190 (2006) Zbl 1140.18008 MR 2275593

[18] Kontsevich, M.: Homological algebra of Mirror Symmetry. In: Proc. International Congress of Mathematicians (Zürich, 1994), Birkhäuser, Basel, 120–139 (1995) Zbl 0846.53021 MR 1403918

[19] Krause, H.: Deriving Auslander's formula. Documenta Math. **20**, 669–688 (2015) Zbl 1348.18018 MR 3398723

[20] Krause, H.: On Neeman's well generated triangulated categories. Documenta Math. **6**, 119–125 (2001) Zbl 0987.18012 MR 1836046

[21] Krause, H.: Localization theory for triangulated categories. In: Triangulated Categories, London Math. Soc. Lecture Note Ser. 375, Cambridge Univ. Press, 161–235 (2010) Zbl 1232.18012 MR 2681709

[22] Laumon, G., Moret-Bailly, L.: Champs algébriques. Ergeb. Math. Grenzgeb. 39, Springer (2000) Zbl 0945.14005 MR 1771927

[23] Lieblich, M.: Moduli of twisted sheaves. Duke Math. J. **138**, 23–118 (2007) Zbl 1122.14012 MR 2309155

[24] Lunts, V., Orlov, D.: Uniqueness of enhancements for triangulated categories. J. Amer. Math. Soc. **23**, 853–908 (2010) Zbl 1197.14014 MR 2629991

[25] Lunts, V., Schnürer, O. M.: New enhancements of derived categories of coherent sheaves and applications. J. Algebra **446**, 203–274 (2016) Zbl 1386.14076 MR 3421093

[26] Mac Lane, S.: Categories for the Working Mathematician. 2nd ed., Grad. Texts in Math. 5, Springer (1998) Zbl 0906.18001 MR 1712872

[27] Neeman, A.: On the derived category of sheaves on a manifold. Documenta Math. **6**, 483–488 (2001) Zbl 1026.18006 MR 1874232

[28] Neeman, A.: The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École Norm. Sup. **25**, 547–566 (1992) Zbl 0868.19001 MR 1191736

[29] Neeman, A.: The chromatic tower for $\mathrm{D}(R)$ (with an appendix by M. Bökstedt). Topology **31**, 519–532 (1992) Zbl 0793.18008 MR 1174255

[30] Neeman, A.: Triangulated Categories. Ann. of Math. Stud. 148, Princeton Univ. Press (2001) Zbl 0974.18008 MR 1812507

[31] Orlov, D.: Equivalences of derived categories and K3 surfaces. J. Math. Sci. **84**, 1361–1381 Zbl 0938(1997)14019 MR 1465519

[32] Porta, M.: The Popescu–Gabriel theorem for triangulated categories. Adv. Math. **225**, 1669–1715 (2010) Zbl 1227.18011 MR 2673743

[33] Ravenel, D. C.: Localization with respect to certain periodic homology theories. Amer. J. Math. **105**, 351–414 (1984) Zbl 0586.55003 MR 0737778

[34] Rizzardo, A., Van den Bergh, M.: An example of a non-Fourier–Mukai functor between derived categories of coherent sheaves. arXiv:1410.4039 (2014)

[35] Rouquier, R.: Dimensions of triangulated categories. J. K-theory **1**, 193–258 (2008) Zbl 1165.18008 MR 2434187

[36] Schwede, S.: Topological triangulated categories. arXiv:1201.0899 (2012)

[37] The Stacks Project Authors, The Stacks Project. http://stacks.math.columbia.edu/

[38] Thomason, R. W., Trobaugh, T.: Higher algebraic K-theory of schemes and of derived categories. In: The Grothendieck Festschrift III, Birkhäuser, Boston, 247–436 (1990) Zbl 0731.14001 MR 1106918

[39] Toën, B.: The homotopy theory of dg-categories and derived Morita theory. Invent. Math. **167**, 615–667 (2007) Zbl 1118.18010 MR 2276263

[40] Totaro, B.: The resolution property for schemes and stacks. J. Reine Angew. Math. **577**, 1–22 (2004) Zbl 1077.14004 MR 2108211