

# A SIMPLE PROPERTY OF THE WEYL TENSOR FOR A SHEAR, VORTICITY AND ACCELERATION-FREE VELOCITY FIELD

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ABSTRACT. We prove that, in a space-time of dimension  $n > 3$  with a velocity field that is shear-free, vorticity-free and acceleration-free, the covariant divergence of the Weyl tensor is zero if and only if the contraction of the Weyl tensor with the velocity is zero. This extends a property found in Generalised Robertson-Walker spacetimes, where the velocity is also eigenvector of the Ricci tensor. Despite the simplicity of the statement, the proof is involved. As a product of the same calculation, we introduce a curvature tensor with an interesting recurrence property.

## 1. INTRODUCTION

A shear-free, vorticity-free and acceleration-free velocity field  $u_k$ , has covariant derivative

$$(1) \quad \nabla_i u_j = \varphi (g_{ij} + u_i u_j)$$

where  $\varphi$  is a scalar field, and  $u_k u^k = -1$ . For such a vector field we prove the following results for the Weyl tensor, in space-time dimension  $n > 3$ :

**Theorem 1.1.**

$$(2) \quad \nabla_m C_{jkl}{}^m = 0 \iff u_m C_{jkl}{}^m = 0$$

Next, we introduce the following tensor, where  $E_{kl} = u^j u^m C_{jklm}$  is the electric part of the Weyl tensor:

$$(3) \quad \Gamma_{iklm} = C_{iklm} - \frac{n-2}{n-3} (u_i u_m E_{kl} - u_k u_m E_{il} - u_i u_l E_{km} + u_k u_l E_{im}) \\ - \frac{1}{n-3} (g_{im} E_{kl} - g_{km} E_{il} - g_{il} E_{km} + g_{kl} E_{im})$$

**Theorem 1.2.**  $\Gamma_{jklm}$  is a generalised curvature tensor, it is totally trace-less and:

$$(4) \quad u^m \Gamma_{jklm} = 0$$

$$(5) \quad u^p \nabla_p \Gamma_{jklm} = -2\varphi \Gamma_{jklm}$$

The tensor is zero in  $n = 4$ .

The proofs make use of various properties of “twisted” space-times, that were introduced by B. Y. Chen [3] as a generalisation of warped space-times:

$$(6) \quad ds^2 = -dt^2 + f^2(\vec{x}, t) g_{\mu\nu}^*(\vec{x}) dx^\mu dx^\nu$$

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$f > 0$  is the scale factor and  $g_{\mu\nu}^*$  is the metric tensor of a Riemannian sub-manifold of dimension  $n - 1$ . If  $f$  only depends on time, the metric is warped and the space-time is a Generalized Robertson-Walker (GRW) space-time [2, 4, 10]. Chen [5] and the authors [11] gave covariant characterisations of twisted space-times; the latter reads: *a space-time is twisted if and only if there exists a time-like unit vector field  $u^i$  with the property (1).*

The space-time is GRW if  $u^i$  is also eigenvector of the Ricci tensor [10]; it is RW with the further condition that the Weyl tensor is zero,  $C_{jklm} = 0$ .

The next two short sections collect useful results on twisted space-times, and about the Weyl tensor in  $n = 4$ .

## 2. TWISTED SPACE-TIMES

We summarise some results on twisted space-times, taken from ref. [11]:

- i) the vector field  $u_j$  is unique (up to reflection).
- ii) the vector field  $u_j$  is Weyl compatible (see [8] for a general presentation):

$$(7) \quad (u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm}) u^m = 0.$$

This classifies the Weyl tensor as purely electric with respect to  $u_j$  [6].

A contraction gives the useful property:

$$(8) \quad C_{jklm} u^m = u_k E_{jl} - u_j E_{kl}$$

where  $E_{jk} = C_{ijkl} u^i u^l$ . It follows that  $C_{jklm} u^m = 0$  if and only if  $E_{ij} = 0$ .

- iii) the Ricci tensor has the general form

$$(9) \quad R_{jk} = \frac{R - n\xi}{n-1} u_j u_k + \frac{R - \xi}{n-1} g_{jk} + (n-2)(u_j v_k + u_k v_j - E_{jk})$$

where  $R = R^k_k$ ,  $\xi = (n-1)(u^p \nabla_p \varphi + \varphi^2)$ , and  $v^k = (g^{km} + u^k u^m) \nabla_m \varphi$  is a space-like vector.

- iv) A twisted space-time is a GRW space-time if and only if  $v_j = 0$ .

## 3. THE WEYL TENSOR IN FOUR-DIMENSIONAL SPACE-TIMES

The following algebraic identity by Lovelock holds in  $n = 4$  ([7], ex. 4.9):

$$(10) \quad \begin{aligned} 0 = & g_{ar} C_{bcst} + g_{br} C_{cast} + g_{cr} C_{abst} \\ & + g_{at} C_{bcrs} + g_{bt} C_{cars} + g_{ct} C_{abrs} \\ & + g_{as} C_{bctr} + g_{bs} C_{catr} + g_{cs} C_{abtr} \end{aligned}$$

It implies that  $C_{abcr} C^{abcs} = \frac{1}{4} \delta_r^s C^2$ , where  $C^2 = C_{abcd} C^{abcd}$ .

The contraction of (10) with  $u^c u^r$ , where  $u^j$  is any time-like unit vector, gives the Weyl tensor in terms of its contractions  $u^d C_{abcd}$  and  $E_{ad} = u^b u^c C_{abcd}$ :

$$(11) \quad \begin{aligned} C_{abcd} = & -u^m (u_a C_{mbcd} + u_b C_{amcd} + u_c C_{abmd} + u_d C_{abcm}) \\ & + g_{ad} E_{bc} - g_{bd} E_{ac} - g_{ac} E_{bd} + g_{bc} E_{ad} \end{aligned}$$

**Proposition 3.1.** *If  $u^m$  is Weyl compatible, (7), in  $n = 4$  the Weyl tensor is wholly given by its electric component:*

$$(12) \quad \begin{aligned} C_{abcd} = & 2(u_a u_d E_{bc} - u_a u_c E_{bd} + u_b u_c E_{ad} - u_b u_d E_{ac}) \\ & + g_{ad} E_{bc} - g_{ac} E_{bd} + g_{bc} E_{ad} - g_{bd} E_{ac} \end{aligned}$$

and  $C^2 = 8 E^2$ , where  $E^2 = E_{ab} E^{ab}$ .

*Proof.* The property (8) is used to simplify (11). Contraction with  $u^i u^j$  of the identity  $\frac{1}{4}C^2 g_{ij} = C_{iabc}C_j^{abc}$  and (8) give:  $-\frac{1}{4}C^2 = (u^i C_{iabc})(u_j C^{jabc}) = (u_b E_{ca} - u_c E_{ba})(u^b E^{ca} - u^c E^{ba})$ . Since  $E_{ca}u^c = 0$ , the result is  $-E_{ca}E^{ca} - E_{ba}E^{ba} = -2E^2$ .  $\square$

**Corollary 3.2.** *In a twisted space-time in  $n = 4$ ,  $C_{abcd} = 0$  if and only if  $E_{ab} = 0$ .*

#### 4. THE MAIN RESULTS

In  $n > 3$  the second Bianchi identity for the Riemann tensor translates to an identity for the Weyl tensor [1]:

$$(13) \quad \begin{aligned} \nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = & \frac{1}{n-3} \nabla_p (g_{jm} C_{kil}^p + g_{km} C_{ijl}^p \\ & + g_{im} C_{jkl}^p + g_{kl} C_{jim}^p + g_{il} C_{kjm}^p + g_{jl} C_{ikm}^p). \end{aligned}$$

As a consequence of (13), as shown in the Appendix, we obtain the intermediate result:

**Proposition 4.1.** *In a twisted space-time the divergence of the Weyl tensor is:*

$$(14) \quad \begin{aligned} \nabla_p C_{ikm}^p = & (n-3)(\nabla_i E_{km} - \nabla_k E_{im}) \\ & + (n-2)[u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2\varphi(u_i E_{km} - u_k E_{im})] \\ & + (2u_k u_m + g_{km})\nabla_p E_i^p - (2u_i u_m + g_{im})\nabla_p E_k^p. \end{aligned}$$

**Corollary 4.2.** *In a twisted space-time, if  $\nabla^p C_{klp} = 0$  then*

$$(15) \quad \nabla_p E^{pk} = 0 \quad \text{and} \quad u^p \nabla_p E_{km} = -\varphi(n-1)E_{km}$$

*Proof.* Note the identity:  $u^m \nabla_p C_{jkm}^p = \nabla_p (u^m C_{jkm}^p) = \nabla_p (u_j E_k^p - u_k E_j^p) = u_j \nabla_p E_k^p - u_k \nabla_p E_j^p$ . Then:  $u^k u^m \nabla_p C_{jkm}^p = \nabla_p E_j^p$ .

Another identity is:  $u^j \nabla_p C_{jkm}^p = \nabla_p (u^j C_{jkm}^p) - \varphi E_{km} = \nabla_p (u_m E^p_k - u^p E_{mk}) - \varphi E_{km} = u_m \nabla_p E^p_k - \varphi(n-1)E_{km} - u^p \nabla_p E_{km}$ .

Together, the two identities imply the statements.  $\square$

Now, we are able to extend to twisted space-times a property of GRW space-times (Theorem 3.4, [9]):

**Theorem 1.1:** In a twisted space-time of dimension  $n > 3$ :

$$(16) \quad \nabla_m C_{jkl}^m = 0 \iff u_m C_{jkl}^m = 0$$

*Proof.* If  $u^m C_{jklm} = 0$  then  $E_{kl} = 0$  and  $\nabla_m C_{jkl}^m = 0$  follows from (14).

If  $\nabla_m C_{jkl}^m = 0$ , the identities (15) simplify eq.(14) as follows:

$$0 = (n-3)[(\nabla_i E_{km} - \nabla_k E_{im}) - (n-2)\varphi(u_i E_{km} - u_k E_{im})]$$

If  $n > 3$ , a contraction with  $u^i$  gives:  $0 = u^i \nabla_i E_{km} + \varphi E_{km}$ . This and the second implication in (15) mean that  $E_{kl} = 0$  i.e.  $u^m C_{jklm} = 0$  by (8).  $\square$

The final result (20) in the Appendix, suggests the introduction of the new tensor (3), that combines the Weyl tensor with the generalized curvature tensors obtained as Kulkarni-Nomizu products of  $E_{ij}$  with  $u_i u_j$  or  $g_{ij}$ .

It has the symmetries of the Weyl tensor for exchange and contraction of indices, as well as the first Bianchi identity (it is a generalized curvature tensor). Moreover

it is traceless,  $\Gamma_{mbc}{}^m = 0$ , and any contraction with  $u$  is zero. The associated scalar  $\Gamma^2 = \Gamma_{abcd}\Gamma^{abcd}$  is evaluated:

$$(17) \quad \Gamma^2 = C^2 - 4\frac{n-2}{n-3}E^2$$

By Prop. 3.1 this tensor is identically zero in  $n = 4$ .

In dimension  $n > 4$ , **Theorem 1.2** is basically the result (20) of the long calculation in the Appendix.

**Remark 4.3.** The property  $\Gamma_{abcd}u^d = 0$  means that in the frame (6), where  $u^0 = 1$  and space components  $u^\mu$  vanish, the components  $\Gamma_{abcd}$  where at least one index is time, are zero. Therefore,  $\Gamma^2 > 0$  in  $n > 4$  and, for the same reason,  $E^2 \geq 0$ . We conclude that the Weyl scalar is positive:

$$(18) \quad C^2 = 4\frac{n-2}{n-3}E^2 + \Gamma^2 \geq 0$$

#### APPENDIX

**Proposition 4.4.** In a twisted space the following identities hold among the Weyl tensor and the contracted Weyl tensor:

$$(19) \quad \begin{aligned} \nabla_p C_{ikm}{}^p &= (n-3)(\nabla_i E_{km} - \nabla_k E_{im}) \\ &+ (n-2)[u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2\varphi(u_i E_{km} - u_k E_{im})] \\ &+ (2u_k u_m + g_{km})\nabla_p E_i{}^p - (2u_i u_m + g_{im})\nabla_p E_k{}^p \end{aligned}$$

$$(20) \quad \begin{aligned} &(n-3)(u^p \nabla_p C_{iklm} + 2\varphi C_{iklm}) \\ &= (n-2)[u^p \nabla_p (u_i u_m E_{kl} - u_k u_m E_{il} - u_i u_l E_{km} + u_k u_l E_{im}) \\ &\quad + 2\varphi(u_i u_m E_{kl} - u_k u_m E_{il} - u_i u_l E_{km} + u_k u_l E_{im})] \\ &+ [u^p \nabla_p (g_{im} E_{kl} - g_{km} E_{il} - g_{il} C_{km} + g_{kl} E_{im}) \\ &\quad + 2\varphi(g_{im} E_{kl} - g_{km} E_{il} - g_{il} E_{km} + g_{kl} E_{im})] \end{aligned}$$

*Proof.* Contraction of (13) with  $u^j$  is:

$$\begin{aligned} u^j \nabla_i C_{jklm} + u^j \nabla_j C_{kil m} + u^j \nabla_k C_{ijl m} &= \frac{1}{n-3}(u_m \nabla_p C_{kil}{}^p + u_l \nabla_p C_{ikm}{}^p) \\ &+ \frac{1}{n-3} \nabla_p [u^j (g_{km} C_{ijl}{}^p + g_{im} C_{jkl}{}^p + g_{kl} C_{jim}{}^p + g_{il} C_{kjm}{}^p)] \\ &- \frac{1}{n-3} \varphi u_p u^j (g_{km} C_{ijl}{}^p + g_{im} C_{jkl}{}^p + g_{kl} C_{jim}{}^p + g_{il} C_{kjm}{}^p) \end{aligned}$$

Where possible, the vector  $u^k$  is taken inside covariant derivatives to take advantage of property (8)

$$\begin{aligned} &\nabla_i (u^j C_{jklm}) - \varphi h_i^j C_{jklm} + u^j \nabla_j C_{kil m} + \nabla_k (u^j C_{ijl m}) - \varphi h_k^j C_{ijl m} \\ &= \frac{1}{n-3}(u_m \nabla_p C_{kil}{}^p + u_l \nabla_p C_{ikm}{}^p) + \frac{1}{n-3} \nabla^p [g_{km} (u_p E_{li} - u_l E_{pi}) \\ &+ g_{im} (u_l E_{pk} - u_p E_{lk}) + g_{kl} (u_m E_{pi} - u_p E_{mi}) + g_{il} (u_p C_{mk} - u_m E_{pk})] \\ &+ \frac{1}{n-3} \varphi [g_{km} E_{il} - g_{im} E_{kl} - g_{kl} E_{im} + g_{il} E_{km}] \end{aligned}$$

$$\begin{aligned}
& \nabla_i(u_l E_{mk} - u_m E_{lk}) - \varphi C_{iklm} - \varphi u_i(u_l E_{mk} - u_m E_{lk}) + u^j \nabla_j C_{kilm} \\
& + \nabla_k(u_m E_{li} - u_l C_{mi}) - \varphi C_{iklm} - \varphi u_k(u_m E_{li} - u_l C_{mi}) \\
& = \frac{1}{n-3}(u_m \nabla_p C_{kil}^p + u_l \nabla_p C_{ikm}^p) \\
& + \frac{1}{n-3} u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{kl} E_{mi} + g_{il} C_{mk}] \\
& + \frac{1}{n-3} \nabla^p [-g_{km} u_l E_{pi} + g_{im} u_l E_{pk} + g_{kl} u_m E_{pi} - g_{il} u_m E_{pk}] \\
& + \frac{n}{n-3} \varphi [g_{km} E_{il} - g_{im} E_{kl} - g_{kl} E_{im} + g_{il} E_{km}] \\
& (n-3)[u_l(\nabla_i E_{mk} - \nabla_k E_{mi}) - u_m(\nabla_i E_{lk} - \nabla_k E_{li}) - 2\varphi C_{iklm} + u^j \nabla_j C_{kilm}] \\
& = (u_m \nabla_p C_{kil}^p + u_l \nabla_p C_{ikm}^p) + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{kl} E_{mi} + g_{il} C_{mk}] \\
& - g_{km} u_l \nabla^p E_{pi} + g_{im} u_l \nabla^p E_{pk} + g_{kl} u_m \nabla^p E_{pi} - g_{il} u_m \nabla^p E_{pk} \\
& + 2\varphi [g_{km} E_{il} - g_{im} E_{kl} - g_{kl} E_{im} + g_{il} E_{km}]
\end{aligned}$$

Contraction with  $u^l$  yields the first result, (19):

$$\begin{aligned}
& \nabla_p C_{ikm}^p = (n-3)(\nabla_i E_{km} - \nabla_k E_{im}) \\
& + (n-2)[u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2\varphi(u_i E_{km} - u_k E_{im})] \\
& + (2u_k u_m + g_{km}) \nabla_p E_i^p - (2u_i u_m + g_{im}) \nabla_p E_k^p
\end{aligned}$$

which is used to replace the covariant divergences  $\nabla_p C_{jkl}^p$  in the previous expression

$$\begin{aligned}
& (n-3)[u_l(\nabla_i E_{mk} - \nabla_k E_{mi}) - u_m(\nabla_i E_{lk} - \nabla_k E_{li}) - 2\varphi C_{iklm} + u^j \nabla_j C_{kilm}] \\
& = -u_m \{ (n-3)(\nabla_i E_{kl} - \nabla_k E_{il}) + (n-2)[u^p \nabla_p (u_i E_{kl} - u_k E_{il}) + 2\varphi(u_i E_{kl} - u_k E_{il})] \\
& \quad + (2u_k u_l + g_{kl}) \nabla_p E_i^p - (2u_i u_l + g_{il}) \nabla_p E_k^p \} \\
& + u_l \{ (n-3)(\nabla_i E_{km} - \nabla_k E_{im}) + (n-2)[u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2\varphi(u_i E_{km} - u_k E_{im})] \\
& \quad + (2u_k u_m + g_{km}) \nabla_p E_i^p - (2u_i u_m + g_{im}) \nabla_p E_k^p \} \\
& \quad + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{kl} E_{mi} + g_{il} C_{mk}] \\
& \quad - g_{km} u_l \nabla^p E_{pi} + g_{im} u_l \nabla^p E_{pk} + g_{kl} u_m \nabla^p E_{pi} - g_{il} u_m \nabla^p E_{pk} \\
& \quad + 2\varphi [g_{km} E_{il} - g_{im} E_{kl} - g_{kl} E_{im} + g_{il} E_{km}]
\end{aligned}$$

Some derivatives cancel, and we are left with

$$\begin{aligned}
& (n-3)[-2\varphi C_{iklm} - u^p \nabla_p C_{iklm}] \\
& = -u_m \{ (n-2)[u^p \nabla_p (u_i E_{kl} - u_k E_{il}) + 2\varphi(u_i E_{kl} - u_k E_{il})] \} \\
& + u_l \{ (n-2)[u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2\varphi(u_i E_{km} - u_k E_{im})] \} \\
& \quad + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{kl} E_{mi} + g_{il} C_{mk}] \\
& \quad + 2\varphi [g_{km} E_{il} - g_{im} E_{kl} - g_{kl} E_{im} + g_{il} E_{km}]
\end{aligned}$$

The final equation is obtained.  $\square$

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