
Sensitivity analysis of Mixed Tempered Stable parameters with implications in portfolio optimization

Asmerilda Hitaj · Lorenzo Mercuri · Edit Rroji

Abstract This paper investigates the use, in practical financial problems, of the Mixed Tempered Stable distribution both in its univariate and multivariate formulation. In the univariate context, we study the dependence of a given coherent risk measure on the distribution parameters. The latter allows to identify the parameters that seem to have a greater influence on the given measure of risk.

The multivariate Mixed Tempered Stable distribution enters in a portfolio optimization problem built considering a real market dataset of seventeen hedge fund indexes. We combine the flexibility of the multivariate Mixed Tempered Stable distribution, in capturing different tail behaviors, with the ability of the ARMA-GARCH model in capturing the time dependence observed in the data.

Keywords Mixed Tempered Stable distribution · sensitivity analysis · portfolio optimization

1 Introduction

The seminal approach of Markowitz in [21] changed the direction of research on portfolio selection. Assumptions within his framework do not hold in practice. For instance, the Markowitz approach considers only the first two moments of the distribution and therefore implies that the utility of the investor is quadratic or alternatively that the asset returns are jointly elliptical symmetric distributed with finite second moment. Classical examples for the latter distributions are the multivariate Normal and the multivariate t with more than two degrees of freedom. The general concept that the wealth allocation should be based on the statistical properties of assets and the risk return trade-off is well accepted. Starting from this trade-off, a number of frameworks appeared with less restrictive requirements and more realistic assumptions. Two main streams of research can be identified. One is the extension of the utility function to higher moments (see among others [11, 12, 15]), while the other considers alternative risk measures to variance in the objective function such as Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$). Artzner in [1] introduced the concept of a coherent risk measure and showed that the VaR is not coherent as it does not satisfy the subadditivity property. The increasing attention given to the efficient frontier obtained from the mean- $CVaR$ optimization problem proposed in [28] is justified with the need of a risk measure that is influenced from extreme losses and implementation simplicity of the optimization routine.

Different distributions have been proposed in literature to capture skewness and heavy-tails of asset returns. Normal Variance Mean Mixtures represent a large family of distributions that include the Normal Inverse Gaussian [3], the Variance Gamma [19], the Normal Tempered Stable [16, 26] and the Generalized Hyperbolic [6]. A new distribution called Mixed Tempered Stable (MixedTS) has been introduced in [29] as a generalization of the Normal Variance Mean Mixtures. In the same paper, it is shown to be more flexible

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than Normal Variance Mean Mixtures in modeling skewness and kurtosis observed in asset returns. The MixedTS distribution has been used for portfolio selection in [13] where the dependence structure is captured using the Independent Component Analysis. Recently, the multivariate MixedTS has been proposed in [10] where the number of parameters increases linearly with the number of components. The idea in this paper is to consider an ARMA-GARCH model where the dependence structure of innovations is described through the multivariate MixedTS in order to obtain a more parsimonious framework compared to that proposed in [13], since we do not need to estimate the mixing matrix of the Independent Components for reproducing asset returns.

We first study the impact of the parameters of the MixedTS on the *CVaR* risk measure, following the same idea as in [30]. This study is helpful in scenario stressing since it gives an intuition of the model parameters contribution on the risk measure. Moreover, we perform a sensitivity analysis of the multivariate MixedTS parameters on the mean-*CVaR* efficient frontier.

In the empirical analysis, we consider a portfolio of hedge fund indexes and model asset returns using the multivariate MixedTS distribution. Well known portfolio benchmarks are constructed and compared in an out-of-sample perspective.

The paper is organized as follows. Section 2 recalls the definition of the univariate MixedTS and its multivariate extension. Section 3 is a review of the risk measures and properties that it must satisfy in order to be a coherent risk measure. This section illustrates also the sensitivity analysis of the parameters of the MixedTS on *CVaR*. Section 4 describes the portfolio selection strategies taken into account in this paper. Section 5 describes the dataset used for a sensitivity analysis on the mean-*CVaR* efficient frontier and the out of sample results of different portfolio strategies. Section 6 draws some conclusions.

2 Mixed Tempered Stable distribution

In this section, we review the univariate Mixed Tempered Stable distribution (MixedTS) introduced in [29] and its multivariate extension proposed in [10]. In particular, we focus on those results that are useful for the construction of optimal portfolio strategies and the risk measure sensitivity analysis.

The univariate MixedTS is a generalization of the Normal Variance Mean Mixture (NVMM) where the normality assumption is substituted by the standardized Classical Tempered Stable (CTS). The resulting distribution preserves the infinitely divisible property when the mixing r.v. is itself infinitely divisible and is more flexible in modeling tails and skewness than the NVMM with the same mixing random variable. Applications in finance can be found in [23] for option pricing purpose, in [13] for portfolio selection with CARA utility function and in [24] for a risk parity portfolio selection.

Here we focus on the MixedTS with Gamma mixing density (*MixedTS* – Γ). Following notation in [10], Y is a *MixedTS* ($\mu, \beta, \alpha, \lambda_+, \lambda_-$) – $\Gamma(a, b)$ if we have:

$$Y = \mu + \beta V + \sqrt{V}X \quad (1)$$

where parameters $\mu, \beta \in \mathbb{R}$, V is a Gamma r.v. with shape parameter $a > 0$ and scale parameter $b > 0$. The r.v. X given V is a Classical Tempered Stable with parameters $(\alpha, \lambda_+ \sqrt{V}, \lambda_- \sqrt{V})$ with $\alpha \in (0, 2]$ and $\lambda_+, \lambda_- > 0$. For this distribution it is possible to obtain analytically the first four moments. The mixture representation becomes very clear for cumulant generating functions. Indeed, let

$$\Phi_Y(u) = \log E[e^{uY}], \quad \Phi_V(u) = \log E[e^{uV}], \quad (2)$$

and

$$\Phi_H(u) = \frac{(\lambda_+ - u)^\alpha - \lambda_+^\alpha + (\lambda_- + u)^\alpha - \lambda_-^\alpha}{\alpha(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} + \frac{(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})u}{(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})}, \quad (3)$$

where $\Phi_H(u)$ is the cumulant generating function of a random variable $H \sim CTS(\alpha, \lambda_+, \lambda_-)$. Then we have

$$\Phi_Y(u) = \mu u + \Phi_V(\beta u + \Phi_H(u)). \quad (4)$$

As shown in [29], we get some well-known distributions used for modeling financial returns as special cases. For instance if $\alpha = 2$ the Variance Gamma distribution is obtained. Fixing $b = \frac{1}{a}$ and letting a go to infinity leads to the Standardized Classical Tempered Stable [17].

The study of the left tail behavior for a MixedTS with Gamma mixing density has been presented in [10]. We briefly review the results that will be investigated more in details in the empirical analysis presented in the paper.

- Proposition 1** 1. If $\max\{-\beta\lambda_- + \Phi_H(-\lambda_-), \beta\lambda_+ + \Phi_H(\lambda_+)\} < b$ or $-\beta\lambda_- + \Phi_H(-\lambda_-) < b < \beta\lambda_+ + \Phi_H(\lambda_+)$ then $\log F((-\infty, -y]) \sim -y\lambda_-$.
2. If $-\beta\lambda_- + \Phi_H(-\lambda_-) > b > \beta\lambda_+ + \Phi_H(\lambda_+)$ then u_- is the unique real solution of $\beta u + \Phi_H(u) = b$ and $\log F((-\infty, -y]) \sim -yu_-$.
3. If $b < \min\{-\beta\lambda_- + \Phi_H(-\lambda_-), \beta\lambda_+ + \Phi_H(\lambda_+)\}$ $u_- < u_+$ are the two real solutions of $\beta u + \Phi_H(u) = b$ and $\log F((-\infty, -y]) \sim -yu_-$.

The generalization of the MixedTS in a multivariate context is not unique. Here following [10], we discuss a Multivariate MixedTS model where the number of parameters grows linearly as the components.

Proposition 2 A random vector $Y \in \mathcal{R}^N$ follows a multivariate MixedTS- Γ distribution if the i^{th} component is defined as:

$$Y_i = \mu_i + \beta_i V_i + \sqrt{V_i} X_i,$$

where V_i is the i^{th} component of multivariate Gamma random vector V , defined as:

$$V_i = G_i + a_i Z,$$

where $G_i \sim \Gamma(l_i, m_i)$ and $Z \sim \Gamma(n, k)$, with $\{G_i\}_{i=1}^N$ and Z independent, while

$$X_i | V_i \sim stdCTS(\alpha_i, \lambda_{+,i}\sqrt{V_i}, \lambda_{-,i}\sqrt{V_i}).$$

In order to ensure that the components of vector Y are MixedTS- Γ distributed we have to impose the following restrictions:

$$a_i = \frac{k}{m_i} \rightarrow a_i Z \sim \Gamma(n, m_i) \quad \forall i = 1, \dots, N.$$

Proposition 3 The characteristic function $\varphi_Y(u) = E[\exp(iuY)]$ of the multivariate MixedTS is:

$$\varphi_Y(u) = e^{i \sum_{h=1}^N u_h \mu_h + \Phi_Z\left(\sum_{h=1}^N (i a_h u_h \beta_h + a_h L_{stdCTS}(u_h; \lambda_{+,h}, \lambda_{-,h}, \alpha_h))\right)} \prod_{h=1}^N e^{\Phi_{G_h}(i u_h \beta_h + L_{stdCTS}(u_h; \lambda_{+,h}, \lambda_{-,h}, \alpha_h))}, \quad (5)$$

where the $L_{stdCTS}(u; \alpha, \lambda_+, \lambda_-)$ is the characteristic exponent of a standardized Classical Tempered Stable r.v. defined as:

$$L_{stdCTS}(u; \lambda_+, \lambda_-, \alpha) = \frac{(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha}{\alpha(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} + \frac{iu(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})}{(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})}.$$

The joint characteristic function in (5) gives us the possibility to estimate the multivariate MixedTS using the Generalized Method of Moment in [9].

2.1 COVARIANCE MixedTS

The characteristic function of the multivariate MixedTS has a closed form formula as reported in the following proposition (for the derivation see Appendix ??).

Proposition 4 The characteristic function of the multivariate MixedTS is:

$$\begin{aligned} \varphi_Y(u) &= E[\exp(iuY)] \\ &= e^{i \sum_{h=1}^N u_h \mu_h} e^{\Phi_Z\left(\sum_{h=1}^N (i a_h u_h \beta_h + a_h L_{stdCTS}(u_h; \lambda_{+,h}, \lambda_{-,h}, \alpha_h))\right)} \\ &\quad * \prod_{h=1}^N e^{\Phi_{G_h}(i u_h \beta_h + L_{stdCTS}(u_h; \lambda_{+,h}, \lambda_{-,h}, \alpha_h))}, \end{aligned} \quad (6)$$

where the $L_{stdCTS}(u; \alpha, \lambda_+, \lambda_-)$ is the characteristic exponent of a standardized Classical Tempered Stable r.v. defined as:

$$L_{stdCTS}(u; \lambda_+, \lambda_-, \alpha) = \frac{(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha}{\alpha(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} + \frac{iu(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})}{(\alpha - 1)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})}.$$

Proposition 5 Consider a random vector \mathbf{Y} where the distribution of each component is $Y_i \sim \text{MixedTS} - \Gamma(l_i + n, m_i)$ for $i = 1, \dots, N$. The formulas for the moments are:

– Mean of the general i^{th} element:

$$E[Y_i] = \mu_i + \beta_i \frac{l_i + n}{m_i}. \quad (7)$$

– Variance σ_i^2 of the i^{th} element:

$$\sigma_i^2 = \left(1 + \frac{\beta_i^2}{m_i}\right) \frac{(l_i + n)}{m_i}. \quad (8)$$

– Covariance σ_{ij} between the i^{th} and j^{th} elements:

$$\sigma_{ij} = \frac{\beta_i \beta_j}{m_i m_j} n. \quad (9)$$

– Third central moment of the i^{th} component:

$$m_3 = \left[(2 - \alpha_i) \frac{\lambda_{+,i}^{\alpha_i-3} - \lambda_{-,i}^{\alpha_i-3}}{\lambda_{+,i}^{\alpha_i-2} + \lambda_{-,i}^{\alpha_i-2}} + \left(3 + 2 \frac{\beta_i^2}{m_i}\right) \frac{\beta_i}{m_i} \right] \frac{(l_i + n)}{m_i}. \quad (10)$$

– Fourth central moment of the i^{th} element:

$$\begin{aligned} m_4 = & \beta_i^4 \left(3 + \frac{6}{l_i + n}\right) \frac{(l_i + n)^2}{m_i^4} + 6\beta_i^2 \frac{l_i + n}{m_i^3} (l_i + n + 2) + \\ & + 4\beta_i (2 - \alpha_i) \left(\frac{\lambda_{+,i}^{\alpha_i-3} - \lambda_{-,i}^{\alpha_i-3}}{\lambda_{+,i}^{\alpha_i-2} + \lambda_{-,i}^{\alpha_i-2}} \right) \frac{l_i + n}{m_i^2} + (3 - \alpha_i) (2 - \alpha_i) \left(\frac{\lambda_{+,i}^{\alpha_i-4} + \lambda_{-,i}^{\alpha_i-4}}{\lambda_{+,i}^{\alpha_i-2} + \lambda_{-,i}^{\alpha_i-2}} \right) \frac{l_i + n}{m_i}. \end{aligned} \quad (11)$$

See Appendix ?? for details on moment derivation. From (??) and (??) is evident that the multivariate *MixedTS* – Γ overcomes the limits of the multivariate Variance Gamma distribution in capturing the dependence structure between components (see [?]). Indeed, the relation that exists between the sign of the skewness of two marginals and the sign of their covariance in the multivariate Variance Gamma, is broken up by the tempering parameters in the multivariate *MixedTS* – Γ .

In particular the following result determines the existence of upper and lower bounds for the covariance depending on the tempering parameters. Here we consider the cases that the Semeraro model is not able to capture.

Theorem 1 Let Y_i and Y_j be two components of a multivariate *MixedTS*– Γ , the following results hold:

- 1 $\underline{\sigma}_{ij} := \frac{\beta_i^* \beta_j^*}{m_i m_j} n < \sigma_{ij}$ where σ_{ij} is defined in (??) and $\underline{\sigma}_{ij} < 0$ if $\text{skew}(Y_i) \geq 0$, $\text{skew}(Y_j) \geq 0$ and $\lambda_{+,i} \geq \lambda_{-,i} \wedge \lambda_{+,j} \leq \lambda_{-,j}$.
- 2 $\bar{\sigma}_{ij} := \frac{\beta_i^* \beta_j^*}{m_i m_j} n \geq \sigma_{ij}$ and $\bar{\sigma}_{ij} < 0$ if $\text{skew}(Y_i) \leq 0$, $\text{skew}(Y_j) \leq 0$ and $\lambda_{+,i} \leq \lambda_{-,i} \wedge \lambda_{+,j} \geq \lambda_{-,j}$.
- 3 $\underline{\sigma}_{ij} = -\infty$ and $\bar{\sigma}_{ij} = +\infty$ if $\text{skew}(Y_i) \leq 0$, $\text{skew}(Y_j) \geq 0$ or $\text{skew}(Y_i) \geq 0$, $\text{skew}(Y_j) \leq 0$.

Proof Let us first discuss the case where both components have positive skewness. In this case the lower bound of the covariance exists if the following problem admits a solution:

$$\begin{aligned} \underline{\sigma}_{ij} := & \min_{\beta_i \beta_j} \frac{\beta_i \beta_j}{m_i m_j} n \\ & \text{skew}(Y_i) \geq 0 \\ & \text{skew}(Y_j) \geq 0 \end{aligned} \quad (12)$$

The sign of skewness depends on the sign of the following quantities:

$$\begin{aligned} (2 - \alpha_i) \left(\frac{\lambda_{+,i}^{\alpha_i-3} - \lambda_{-,i}^{\alpha_i-3}}{\lambda_{+,i}^{\alpha_i-2} + \lambda_{-,i}^{\alpha_i-2}} \right) + 3 \frac{\beta_i}{m_i} + 2 \frac{\beta_i^3}{m_i^2} & \geq 0 \\ (2 - \alpha_j) \left(\frac{\lambda_{+,j}^{\alpha_j-3} - \lambda_{-,j}^{\alpha_j-3}}{\lambda_{+,j}^{\alpha_j-2} + \lambda_{-,j}^{\alpha_j-2}} \right) + 3 \frac{\beta_j}{m_j} + 2 \frac{\beta_j^3}{m_j^2} & \geq 0 \end{aligned}$$

The feasible region S_a of the minimization problem in (??) depends on the difference between tempering parameters. We observe that the cubic function $g(\beta_i) := (2 - \alpha_i) \left(\frac{\lambda_{+,i}^{\alpha_i-3} - \lambda_{-,i}^{\alpha_i-3}}{\lambda_{+,i}^{\alpha_i-2} + \lambda_{-,i}^{\alpha_i-2}} \right) + 3 \frac{\beta_i}{m_i} + 2 \frac{\beta_i^3}{m_i^2}$ is strictly increasing and satisfies the following limits:

$$\begin{aligned} \lim_{\beta_i \rightarrow +\infty} g(\beta_i) & = +\infty \\ \lim_{\beta_i \rightarrow -\infty} g(\beta_i) & = -\infty. \end{aligned}$$

Therefore exists only one β_i^* such that $g(\beta_i^*) = 0$. The sign of β_i^* is determined by the following implications:

$$\begin{aligned}\lambda_{+,i} = \lambda_{-,i} &\implies g(0) = 0 \implies \beta_i^* = 0 \\ \lambda_{+,i} > \lambda_{-,i} &\implies g(0) < 0 \implies \beta_i^* > 0 \\ \lambda_{+,i} < \lambda_{-,i} &\implies g(0) > 0 \implies \beta_i^* < 0.\end{aligned}$$

The feasible region can be written as:

$$S_a = \{(\beta_i, \beta_j) : \beta_i \geq \beta_i^* \wedge \beta_j \geq \beta_j^*\}$$

and the lower bound is $\underline{\sigma}_{ij} = \frac{\beta_i^* \beta_j^*}{m_i m_j} n$ while the upper bound is $\bar{\sigma}_{ij} = +\infty$. In this case the lower bound is negative when

$$\lambda_{+,i} > \lambda_{-,i} \wedge \lambda_{+,j} < \lambda_{-,j}$$

or

$$\lambda_{+,j} > \lambda_{-,j} \wedge \lambda_{+,i} < \lambda_{-,i}.$$

Now we consider the case when both skewnesses are negative. Following a similar procedure the feasible region becomes:

$$S_a = \{(\beta_i, \beta_j) : \beta_i \leq \beta_i^* \wedge \beta_j \leq \beta_j^*\}.$$

The $\underline{\sigma}_{ij} = -\infty$ and the upper bound is $\bar{\sigma}_{ij} = \frac{\beta_i^* \beta_j^*}{m_i m_j} n$. The upper bound is negative when

$$\lambda_{+,i} > \lambda_{-,i} \wedge \lambda_{+,j} < \lambda_{-,j}$$

or

$$\lambda_{+,j} > \lambda_{-,j} \wedge \lambda_{+,i} < \lambda_{-,i}.$$

The last case refers to the context when the skewnesses have different signs and following the same procedure as above we have $\underline{\sigma}_{ij} = -\infty$ and $\bar{\sigma}_{ij} = +\infty$.

3 Univariate MixedTS and Risk Measures

3.1 Review of risk measures

Let χ be a family of r. v.'s defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ describing the profit and loss (or return) of a given portfolio (or asset). The risk measure associated to $X \in \chi$ is defined as a map $\rho : \chi \rightarrow \mathbb{R}$ meaning that $\rho(X) \in \mathbb{R}$. From a theoretical point of view, the class of coherent risk measures is very appealing. Following the axiomatic approach introduced in [1], $\rho(X)$ is a coherent risk measure if it satisfies the following properties:

- *Translation Invariance* For all $\lambda \in \mathbb{R}$ and for all $X \in \chi$ we have $\rho(X - \lambda) = \rho(X) - \lambda$.
- *Monotonicity* For all $X, Y \in \chi$ such that $X \leq Y$ we have $\rho(X) \leq \rho(Y)$.
- *Positive Homogeneity* For all $\lambda \geq 0$ and for all $X \in \chi$ we have $\rho(\lambda X) = \lambda \rho(X)$.
- *Subadditivity* For all $X, Y \in \chi$ we have $\rho(X + Y) \leq \rho(X) + \rho(Y)$

We assume Y to be a continuous r.v. with characteristic function ϕ_Y . The distribution function is computed using the Inverse Fourier Transform (IFT) as follows:

$$F_Y(y) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{[e^{-ity} \phi_Y(t)]}{it} dt.$$

Consequently the most used risk measures, Value at Risk (VaR_ζ) at the confidence level ζ and Conditional Value at Risk ($CVaR_\zeta$) used in [27], are obtained respectively as:

$$VaR_\zeta(Y) = -F_Y^{-1}(\zeta), \tag{13}$$

$$CVaR_\zeta(Y) = -E[Y | Y \leq -VaR_\zeta(Y)]. \tag{14}$$

Notice that if Y has an analytical expression for the characteristic function ϕ_Y both measures (6) (7) can be evaluated efficiently through the IFT, see for instance [18].

The choice of the $CVaR$ as a risk measure in this paper comes out from the following observations: VaR is not subadditive, its estimators are stable and backtesting is straightforward. On the other side we have that $CVaR$ is coherent, is sensitive to extreme values but its estimators are less stable than the VaR estimators as shown in [31]. In the following we will study the behaviour of the $CVaR_\zeta(Y)$ when the r.v. Y is a *MixedTS* $(\mu, \beta, \alpha, \lambda_+, \lambda_-) - \Gamma(a, b)$.

3.2 Risk measure sensitivity

In this section we present a sensitivity analysis of the $CVaR$ on the parameters of the univariate Mixed Tempered Stable distribution. We notice that for the parameters μ and β , it is possible to exploit the results presented in [30] for the Normal Variance Mean Mixtures while $CVaR$'s dependence on the other parameters that control tail behavior is more complex and it is based on the study of the fundamental strip of a MixedTS presented in [10].

Proposition 6 *For a univariate MixedTS r.v. Y , defined as in (1), and given a coherent risk measure ρ we have:*

- $\mu \mapsto \rho(Y)$ is decreasing in \mathbb{R} ;

Proof: Observe that $\rho(Y) = \rho(\beta V + \sqrt{V}X) - \mu$.

- $\beta \mapsto \rho(Y)$ is non-increasing in \mathbb{R} ;

Proof: For $\Delta\beta \geq 0$ we have:

$$\rho((\beta + \Delta\beta)V + \sqrt{V}X) \leq \rho(\beta V + \sqrt{V}X) + \rho(\Delta\beta V).$$

Since $\Delta\beta V \geq 0$ a.s. we have

$$\rho((\beta + \Delta\beta)V + \sqrt{V}X) \leq \rho(\beta V + \sqrt{V}X) + \rho(0).$$

The influence of the scale parameter b in the Gamma r.v. V and of the tempering parameter λ_- is still to be investigated. Here, we perform a sensitivity analysis in order to get insights of possible theoretical results. In particular, we first fix the parameters in the MixedTS as $\mu = 0$, $\beta = 0$, $b = 1$, $a = 1$, $\alpha = 1.2$, $\lambda_+ = 1$, $\lambda_- = 1$ and then investigate the changes in the $CVaR$ value for small variation of parameters. Figure 1 clearly reproduces the results stated in Proposition 4 for the monotonicity of $CVaR$ with respect to μ and β .

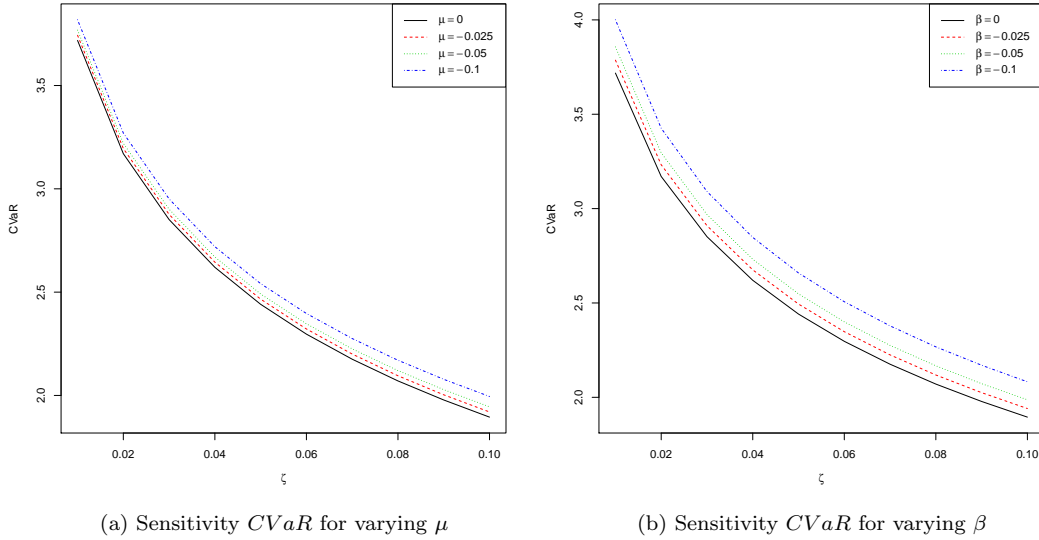
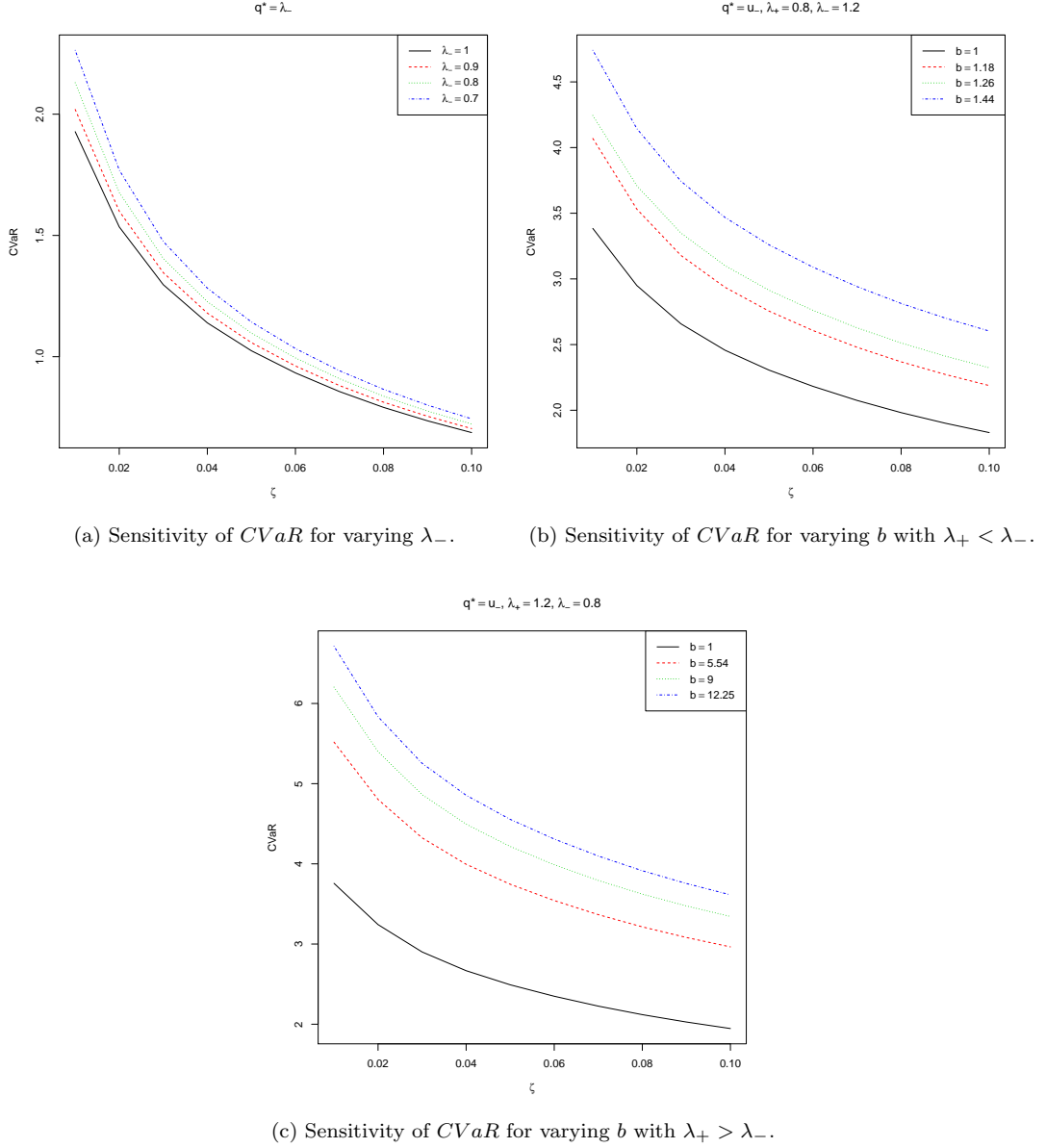


Fig. 1: $CVaR$ sensitivity w.r.t μ and β .

In Figure 2 we focus mainly on the parameters that influence left tail behavior. From Proposition 1, we have that $\log F((-\infty, -y]) \sim -yq^*$ where for specific restriction on parameters we can have either $q^* = \lambda_-$ or $q^* = u_-$. In Figure 2 are presented sensitivity results for both cases from where it seems that $CVaR$ is a decreasing function of λ_- and an increasing function of b . The monotonicity w.r.t λ_- , although not straightforward analytically, goes in line with financial interpretation. In fact, a lower value for the tempering parameter λ_- suggests a heavier left tail for return distributions that implies a higher value for the $CVaR$ computed on this distribution. Monotonicity of the $CVaR$ on the scale parameter b comes out numerically but it does not seem to have a direct financial intuition.

Fig. 2: $CVaR$ sensitivity w.r.t λ_- and b .

4 Portfolio optimization problem

We consider an investor who selects his portfolio from N risky assets, where there are no transaction costs and short selling is not allowed. Let us denote with $\mathbf{r} = (r_1, r_2, \dots, r_N)$ the return vector, where r_i indicates the returns of asset i and N is the number of assets available; by $\mathbf{w} = (w_1, w_2, \dots, w_N)$ the vector of weights, where w_i is the fraction of the initial endowment invested in the i -th asset. Any portfolio strategy based on an optimization problem requires a feasible region defined as follows:

$$\mathcal{A} := \left\{ \mathbf{w} \in \mathbb{R}^N \mid \sum_{i=1}^N w_i = 1 \wedge lb \leq w_i \leq ub \right\},$$

where lb and ub stand respectively for the lower and upper bound of each weight. lb and ub are usually set in $[0, 1]$ to avoid short selling and concentrated portfolios. In particular, to ensure $\mathcal{A} \neq \emptyset$, lb belongs to the interval $[0, \frac{1}{N}]$. Choosing $lb = \frac{1}{N}$, set \mathcal{A} is a singleton and contains only the equally weighted portfolio (EW). In this case, the strategy ignores completely the data and does not require any optimization or

estimation procedure. In the empirical part we use $lb = 0$, meaning that we do not allow for short selling, and $ub = 1$. Therefore the feasible region becomes:

$$\mathcal{A}_0 := \left\{ \mathbf{w} \in \mathbb{R}^N \mid \sum_{i=1}^N w_i = 1 \wedge 0 \leq w_i \leq 1 \right\}.$$

In the following we review the optimization problems in the alternative strategies considered in the empirical analysis.

4.1 Global minimum variance portfolio (GMV)

The *GMV* portfolio minimizes the overall variance and is not focused on the expected return of the portfolio. Mathematically the investor's problem can be written as:

$$\begin{cases} \min_{\mathbf{w}} \sigma^2 = \mathbf{w}' M_2 \mathbf{w} \\ \text{s.t.} \quad \mathbf{w} \in \mathcal{A}_0 \end{cases}$$

where M_2 is the covariance matrix of the portfolio components.

4.2 Equal risk contribution portfolio (ERC)

Equal risk contribution strategy, proposed by [25], introduces a risk budget in portfolio allocation, where weights are such that each asset provides the same contribution to portfolio risk. The properties of an unconstrained analytical solution of the *ERC* are analyzed by [20].

If we consider volatility σ as the portfolio risk measure, the marginal risk contribution of asset i is defined as:

$$\partial_{w_i} \sigma = \frac{\partial \sigma}{\partial w_i} = \frac{(M_2 \mathbf{w})_i}{\sqrt{\mathbf{w}' M_2 \mathbf{w}}}.$$

Let $\sigma_i(w) = w_i \partial_{w_i} \sigma$ denote the risk contribution of the i^{th} asset, the portfolio risk can be seen as the sum of risk contributions $\sigma = \sum_i^N \sigma_i(w)$ (see [20]). A feature of this strategy is that:

$$w_i \partial_{w_i} \sigma = w_j \partial_{w_j} \sigma \quad \forall i, j \quad \text{where } i, j = 1, \dots, N.$$

The algorithm used for the identification portfolio's weights can be written as:

$$\begin{cases} \min_{\mathbf{w}} \sum_{i=1}^N \sum_{j=1}^N (w_i (M_2 \mathbf{w})_i - w_j (M_2 \mathbf{w})_j)^2 \\ \text{s.t.} \quad \mathbf{w} \in \mathcal{A}_0 \end{cases}$$

4.3 Maximum diversified portfolio (MDP)

The basic idea behind the maximum diversification approach is to construct a portfolio that maximizes the benefits from diversification. [5] proposed the so-called diversification ratio (*DR*), which is the ratio of the weighted average asset volatility to portfolio actual volatility, defined as $DR = \frac{\sum_{i=1}^N w_i \sigma_i}{\sqrt{\mathbf{w}' M_2 \mathbf{w}}}$. Since different asset classes are not perfectly correlated to each other, this ratio in general is greater than 1. The investor problem when the MDP approach is used is:

$$\begin{cases} \max_{\mathbf{w}} DR = \frac{\sum_{i=1}^N w_i \sigma_i}{\sqrt{\mathbf{w}' M_2 \mathbf{w}}} \\ \text{s.t.} \quad \mathbf{w} \in \mathcal{A}_0 \end{cases}$$

4.4 Global minimum CVaR portfolio

Global minimum *CVaR* portfolio optimization problem follows the same idea as the *GMV* considering as a risk measure the *CVaR*. Therefore the investor problem reads:

$$\begin{cases} \min_{\mathbf{w}} \text{CVaR} \left(\sum_{i=1}^N w_i r_i \right) \\ \text{s.t.} \quad \mathbf{w} \in \mathcal{A}_0 \end{cases}$$

5 Multivariate analysis

This section is composed of two parts. The first part is devoted to the sensitivity analysis of the mean-*CVaR* efficient frontier obtained in a ARMA-GARCH model where residuals are generated from a multivariate MixedTS. Indeed, from the generation of scenarios for residuals we are able to generate also future log returns for the assets in our portfolio. In the second part we present some out-of-sample performance results obtained selecting different portfolio strategies.

5.1 Joint modeling and portfolio optimization

In order to analyze the efficient frontier resulting from an optimization problem that considers the Multivariate MixedTS distribution for modeling the joint dynamics of asset returns we consider a portfolio of seventeen hedge funds indexes¹. The observations are daily log-returns and span the period April 2012 to June 2017. Table 1 reports general statistics and Jarque-Bera test for the components of the portfolio. As we can observe, most of hedge fund indexes are negatively skewed. The kurtosis values are well above 3 (the lowest value is around 5.2) for all components, indicating deviations from the normal distribution and the presence of fat tails. This is also confirmed by the JB test, which rejects the normality assumption at 1% significance level for all the hedge fund indexes under investigation. Therefore, we can conclude that returns of our portfolio components are not normally distributed.

Hedge fund name	abbr	Annual mean	Annual std	Skewness	Kurtosis	p-value	JB-Test
HFRX Absolute Return Index	HF1	0.025	0.038	0.339	10.851	0.001	2639.197
HFRX ED: Distressed Restructuring Index	HF2	0.043	0.020	0.422	9.264	0.001	1698.140
HFRX ED: Merger Arbitrage Index	HF3	0.012	0.031	-0.450	5.219	0.001	243.660
HFRX EH: Equity Market Neutral Index	HF4	0.005	0.026	-0.854	8.305	0.001	1320.040
HFRX Equal Weighted Strategies EUR Index	HF5	0.015	0.025	-0.788	6.055	0.001	502.205
HFRX Equal Weighted Strategies Index	HF6	0.007	0.056	-0.668	6.171	0.001	503.154
HFRX Equity Hedge EUR Index	HF7	0.017	0.056	-0.719	6.471	0.001	599.973
HFRX Equity Hedge Index	HF8	0.011	0.049	-0.743	5.596	0.001	380.103
HFRX Event Driven EUR Index	HF9	0.021	0.048	-0.715	5.569	0.001	367.360
HFRX Event Driven Index	HF10	0.010	0.036	-0.756	6.059	0.001	494.877
HFRX Global Hedge Fund CAD Index	HF11	-0.003	0.035	-0.681	6.109	0.001	489.666
HFRX Global Hedge Fund EUR Index	HF12	0.007	0.035	-0.692	5.596	0.001	367.834
HFRX Global Hedge Fund Index	HF13	-0.010	0.047	-0.415	5.546	0.001	304.743
HFRX Macro/CTA EUR Index	HF14	-0.002	0.047	-0.480	5.890	0.001	394.043
HFRX Macro/CTA Index	HF15	-0.019	0.028	-0.344	16.485	0.001	7748.255
HFRX Relative Value Arbitrage EUR Index	HF16	-0.0092	0.0263	0.0014	11.8893	0.001	3358.314
HFRX Relative Value Arbitrage Index	HF17	0.0039	0.0491	-1.7714	23.2866	0.001	18024.13

Table 1: Statistics of the portfolio components in the out-of-sample period

To the log-returns of each portfolio component i we fit an ARMA(1,1)-GARCH(1,1) model, defined as follows:

$$\begin{aligned}
 r_{t,i} &= \bar{\mu}_i + \theta_{1,i}(r_{t-1,i} - \mu_i) + \theta_{2,i}z_{t-1,i} + z_{t,i} \\
 z_{t,i} &= \sigma_{t,i}\epsilon_{t,i} \\
 \sigma_{t,i}^2 &= \omega_{0,i} + \alpha_{1,i}z_{t-1,i}^2 + \beta_{1,i}\sigma_{t-1,i}^2
 \end{aligned}$$

and employ the multivariate MixedTS for the joint dynamics of the sequences of residuals $\epsilon_{t,i}$.

In practice, we consider a two-step procedure. In the first step we estimate the ARMA-GARCH(1,1) parameters and the residual $\epsilon_{t,i}$ using the quasi-likelihood method (see [8]). In the second step we estimate the parameters of a multivariate MixedTS on the sequences of residuals $\epsilon_{t,i}$. Due to the lack of an explicit formula for the joint density function of residuals, the classical maximum likelihood procedure is cumbersome since it involves a multidimensional Fourier Transform. In order to avoid these numerical issues, we apply the GMM proposed in [9] where the score function is defined as the difference between empirical and theoretical multivariate characteristic function. In Figure 3 are given both empirical and fitted marginal distributions for the 17 hedge funds.

We report in Table 2 the estimated parameters of the Multivariate MixedTS with the respective standard errors obtained using the bootstrap methodology, see [7].

¹The dataset is taken from www.hedgefundresearch.com.

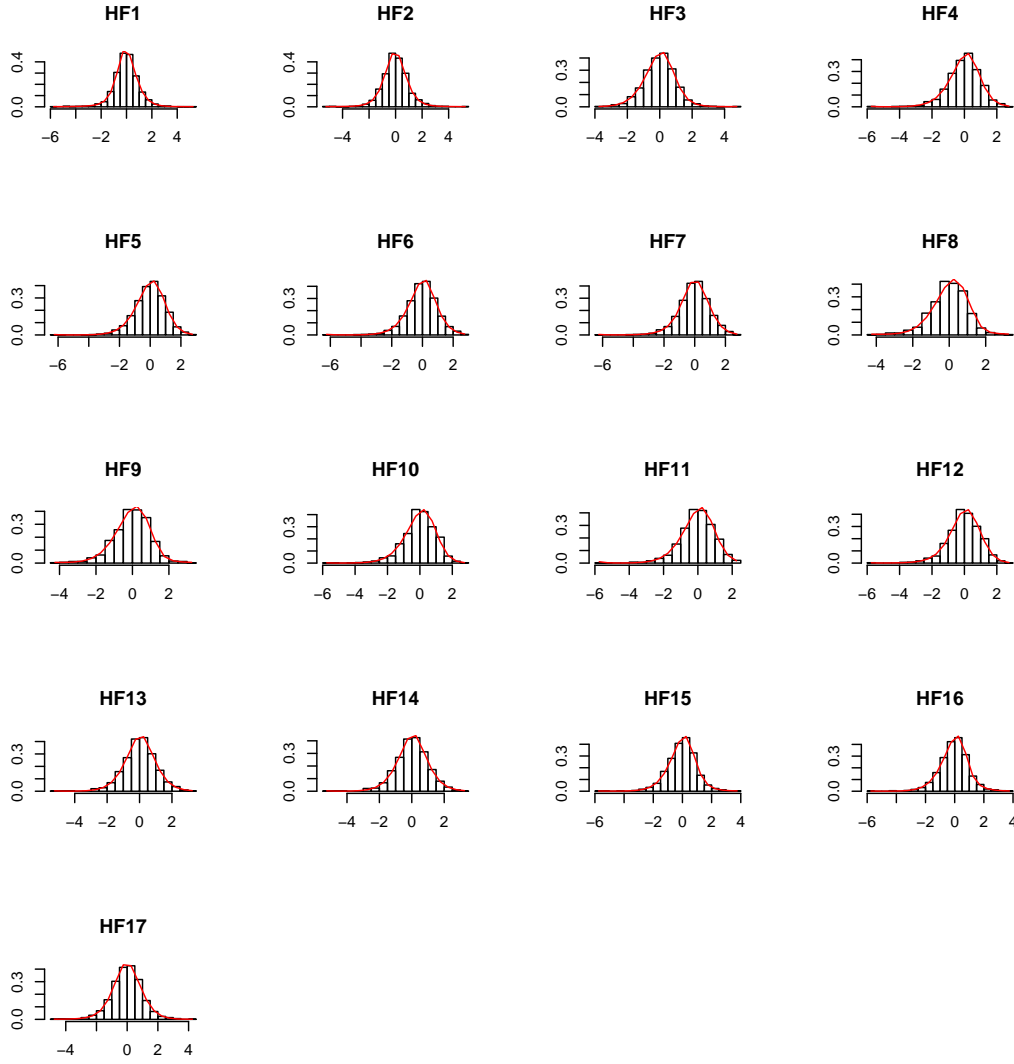


Fig. 3: Empirical and fitted MixedTS distributions for the innovations of the seventeen hedge funds. These densities are obtained estimating the multivariate MixedTS distribution through the GMM procedure.

We conclude this section by showing the behaviour of the mean- $CVaR$ frontier when, for each asset, some additive shifts in the same parameters analyzed in the univariate case are considered.

First, we assume that, in the market, there are no assets dominated according the mean- $CVaR$ criterion i.e. an asset with a higher expected return has also an higher level of risk in term of $CVaR$.

The construction of the frontier is very intuitive and mimics the Markowitz efficient frontier. Indeed, we need to solve a sequence of optimization problems where the minimization of the portfolio variance is substituted by the minimization of the portfolio $CVaR$. More precisely, problems have the same mathematical formulation of the Global minimal $CVaR$ described above with an additional constraint that ensures a minimal expected portfolio return \bar{r}_p . \bar{r}_p varies from the expected portfolio return of the $GVCVaR$ to the expected return of the asset with the highest $CVaR$.

In Figure 4 we report the behavior of the mean- $CVaR$ efficient frontier when, for each asset, we consider an additive shift for μ , β , λ_- and b parameters. We get an upward shifted frontier when, ceteris paribus, b decreases. For the remaining parameters we have an upward shift of the frontier for increasing parameter values. These numerical results seem to be coherent with the movements in Figures 1 and 2.

	μ_i	β_i	m_i	l_i	α_i	$\lambda_{+,i}$	$\lambda_{-,i}$
HF1	-0.391 (0.034)	0.172 (0.034)	3.149 (2.986)	5.45 (0.239)	1.659 (0.034)	0.42 (0.368)	0.007 (0.082)
HF2	-0.107 (0.054)	0.045 (0.024)	9.233 (3.409)	6.492 (0.175)	1.779 (0.838)	0.004 (0.734)	0.006 (0.036)
HF3	0.544 (0.023)	-0.448 (0.135)	3.889 (2.905)	4.322 (0.481)	1.898 (0.024)	0.001 (0.025)	18.012 (0.043)
HF4	0.454 (0.013)	-0.526 (0.085)	4.272 (3.254)	3.31 (0.175)	2 (0.076)	0.391 (0.434)	1.408 (0.024)
HF5	0.565 (0.034)	-0.657 (0.01)	5.519 (3.108)	4.424 (1.417)	2 (0.038)	10.879 (3.434)	27.573 (0.034)
HF6	0.262 (0.096)	-0.315 (0.073)	3.56 (1.767)	2.848 (0.912)	2 (0.434)	2.272 (0.356)	2.044 (0.034)
HF7	-1.931 (0.021)	2.54 (0.186)	20.147 (0.771)	14.502 (0.477)	0.749 (0.076)	9.309 (0.054)	1.186 (0.094)
HF8	2.794 (0.998)	-1.295 (0.034)	5.612 (1.017)	11.198 (0.992)	1.228 (0.034)	0.018 (0.734)	5.752 (0.343)
HF9	2.365 (1.067)	-0.983 (0.032)	4.499 (1.259)	10.041 (0.787)	1.5 (0.053)	0.005 (0.034)	1.911 (0.334)
HF10	0.733 (0.134)	-0.893 (0.156)	5.992 (2.186)	4.551 (0.484)	1.556 (0.015)	65.064 (14.343)	9.407 (0.134)
HF11	0.635 (0.126)	-0.758 (0.096)	5.591 (2.987)	4.323 (0.176)	1.833 (0.034)	231.946 (10.034)	42.972 (1.034)
HF12	-1.485 (0.846)	1.614 (0.734)	30.535 (1.447)	27.127 (1.259)	1.279 (0.073)	15.099 (1.143)	0.846 (0.234)
HF13	0.117 (0.047)	-0.082 (0.014)	15.244 (3.647)	3.095 (0.253)	1.979 (0.075)	0.157 (0.134)	2.725 (0.934)
HF14	0.339 (0.168)	-0.314 (0.084)	9.675 (3.734)	3.214 (0.451)	0.054 (0.062)	7.331 (0.037)	15.327 (1.034)
HF15	0.557 (0.267)	-0.332 (0.021)	2.456 (1.871)	3.519 (0.266)	1.815 (0.186)	0.001 (0.034)	2.443 (0.538)
HF16	1.157 (0.985)	-1.79 (0.281)	6.903 (1.986)	4.08 (0.302)	1.265 (0.038)	0.711 (0.034)	96.078 (5.034)
HF17	-0.275 (0.0125)	0.193 (0.086)	4.681 (2.111)	5.185 (1.549)	1.844 (0.043)	6.297 (2.273)	0.018 (0.008)

Table 2: Parameters of the Multivariate MixedTS fitted to a dataset of 17 hedge funds, $n = 0.512$ with standard deviation (0.121). In brackets is reported the standard deviation for each parameter.

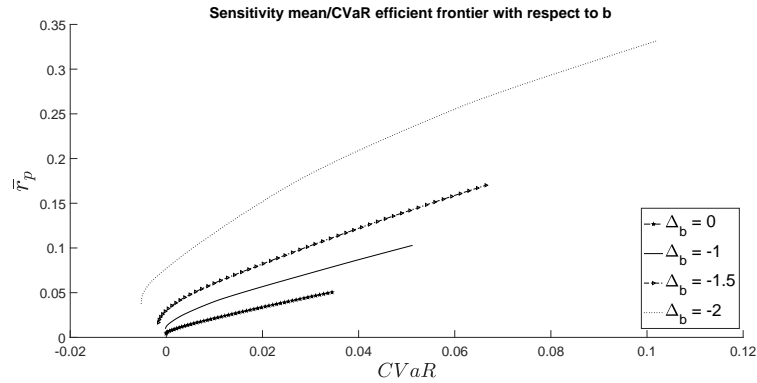
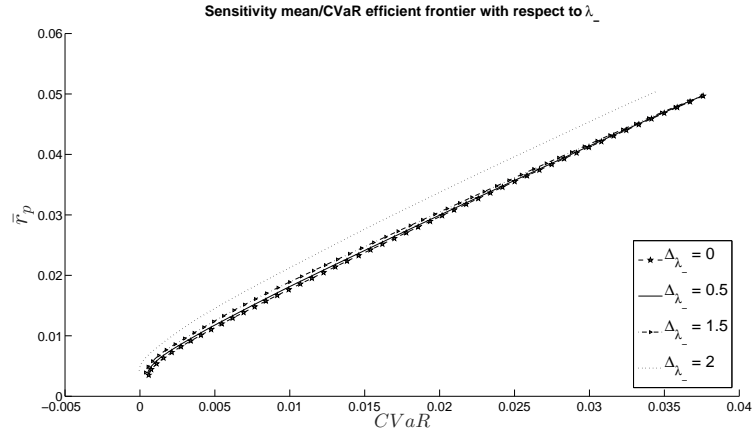
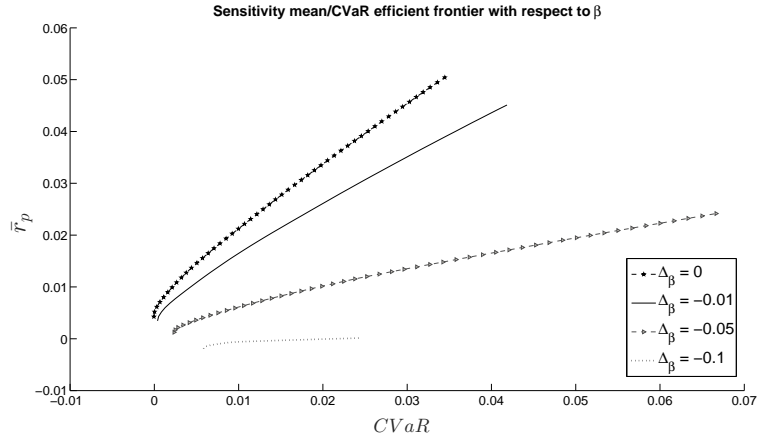
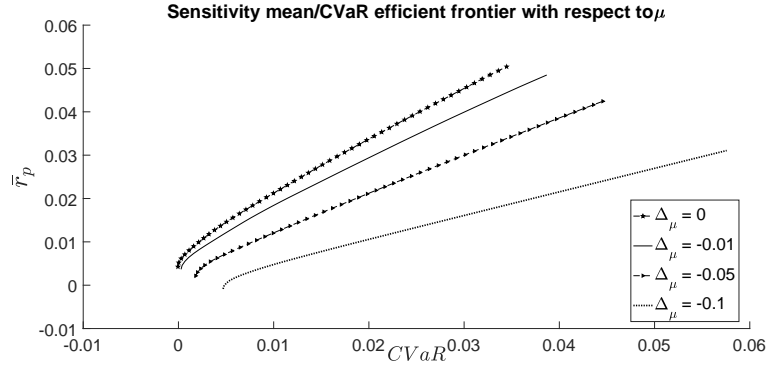


Fig. 4: Efficient mean-*CVaR* frontier sensitivity when the same additive shift, in a given parameter of the MixedTS, is considered for each asset.

5.2 Performance measurement

We use a buy and hold rolling window strategy with in-sample period of one year and out-of-sample period of one month. We find the optimal weights in each in-sample period and keep these constant until a new rebalance takes place. Given the time series of the out-of-sample returns generated for each portfolio strategy, presented in Section 4, we compute two different Risk Adjusted Performance Measures (RAPM), the *Omega ratio* and the *Information ratio*. For general review on different risk adjusted performance measure see [2].

The *Omega Ratio* (Ω) is defined as:

$$\Omega(\tau) = \frac{E(r_p - \tau)^+}{E(\tau - r_p)^+},$$

where τ is a fixed threshold value and r_p is the portfolio return. For a given level of τ the number $\Omega(\tau)$ is the probability-weighted ratio of gains to losses relative to the chosen threshold, as for any investor returns above the threshold are considered as gains and returns below as losses. For an investor the choice of the threshold reflects a particular risk preference, therefore no threshold level is “better” than another.

Information Ratio (IR) is defined as:

$$IR = \frac{\bar{r}_p - \bar{r}_{ref}}{\sigma_{r_p - r_{ref}}}.$$

where \bar{r}_{ref} is the average return of the reference portfolio. In the empirical part we use as reference portfolio the five strategies explained in Section 4 and compare the remaining portfolios. Once the reference portfolio is fixed, managers seek to maximize IR , i.e. to reconcile a high residual return and a low tracking error. This ratio allows us to check that the risk taken by the manager in deviating from the reference portfolio is sufficiently rewarded.

In Figure 5 we report the out-of-sample performances in terms of portfolio wealth for each considered strategy while synthetic statistics are given in Table 3.

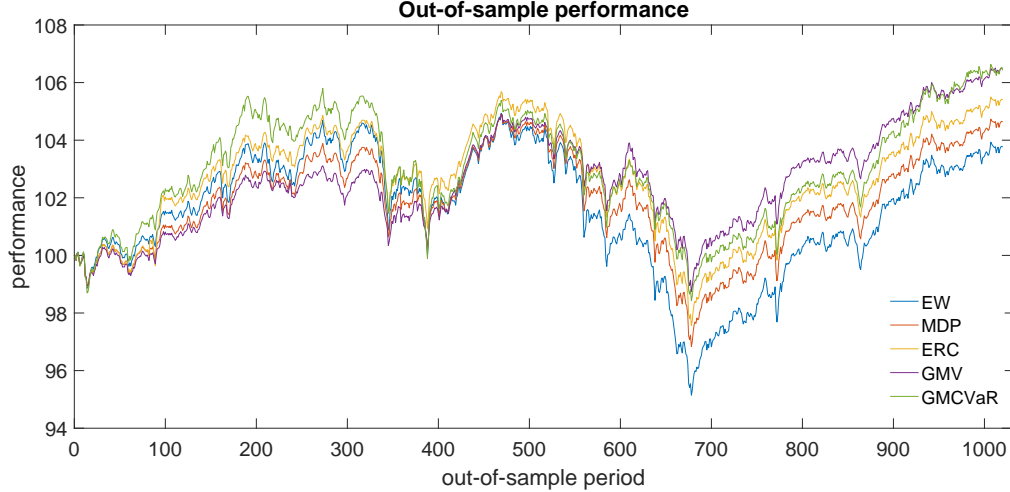


Fig. 5: Out-of-sample performance obtained using different strategies for portfolio selection.

It is well known that an investor will prefer the portfolio with the highest mean and skewness and lower variance and kurtosis [15]. From the results in Table 3 we can observe that the portfolios with the highest mean are the *GMV* and *GMCVaR*, the portfolio with the lowest variance is the *GMV*. The portfolio with the highest skewness is the *ERC* while the lowest kurtosis is reached, in our analysis, with the *EW* strategy. From the statistics in Table 3 is not evident which portfolio is preferred, thus we compare portfolios in an out-of-sample perspective considering the two RAPM measures: Information Ratio and the Omega Ratio. In Table 4 the Out-of-sample Information ratio is reported using as reference portfolio one of the strategies considered in Section 4. For example, the first column indicates the IR obtained using as reference portfolio the *EW*. Results, based on IR , clearly show that the *EW* strategy is outperformed by the others, as the IR

	Annual mean	Annual std	Skewness	Kurtosis	p-value	JB-Test
<i>EW</i>	0.009	0.028	-0.738	5.829	0.001	432.72
<i>MDP</i>	0.011	0.024	-0.777	5.912	0.001	463.09
<i>ERC</i>	0.012	0.025	-0.625	5.925	0.001	430.03
<i>GMV</i>	0.015	0.023	-0.823	6.182	0.001	545.54
<i>GMCVaR</i>	0.015	0.024	-0.681	5.831	0.001	419.48

Table 3: Statistics of the out-of-sample portfolios.

$r_p \backslash r_{ref}$	<i>EW</i>	<i>MDP</i>	<i>ERC</i>	<i>GMV</i>	<i>GMCVaR</i>
<i>EW</i>	...	-0.025	-0.040	-0.040	-0.049
<i>MDP</i>	0.025	...	-0.031	-0.052	-0.032
<i>ERC</i>	0.040	0.031	...	-0.024	-0.017
<i>GMV</i>	0.040	0.045	0.027	...	0.001
<i>GMCVaR</i>	0.042	0.044	0.025	-0.001	...

Table 4: Out-of-sample Information ratio for different reference portfolios.

obtained if *EW* is used as a reference portfolio, is always positive. The second column reports the *IR* using as reference portfolio the *MDP*. Based on *IR* we observe that *ERC*, *GMV* and *GMCVaR*, perform better than the *MDP*. From the results in the third column is evident that *GMV* and *CVaR* perform better than the *ERC*. Summarizing all results, we have that based on the *IR* the *GMV* and *GMCVaR* provide better out-of-sample results.

	$\tau = 0.001$	$\tau = 0.0015$	$\tau = 0.002$
<i>EW</i>	0.190	0.084	0.036
<i>MDP</i>	0.169	0.066	0.027
<i>ERC</i>	0.181	0.075	0.033
<i>GMV</i>	0.145	0.052	0.020
<i>GMCVaR</i>	0.212	0.092	0.041

Table 5: Out-of-sample Omega ratio for all the portfolios considering different threshold levels.

Since the Omega Ratio considers the whole distribution of assets returns, we decided to use it as an alternative to the *IR*. In Table 5 the out-of-sample Omega ratio is reported for different thresholds ($\tau = 0.001$, $\tau = 0.0015$ and $\tau = 0.002$). From these results is clear that for the considered threshold levels the best out-of-sample Omega Ratio is obtained for the *GMCVaR*. The ranking of the portfolios according to the Omega Ratio is *GMCVaR* \succ *EW* \succ *ERC* \succ *MDP* \succ *GMV* where \succ denotes preference ordering.

6 Conclusions

In this paper, we considered two main topics. First we performed a sensitivity analysis on the Mixed Tempered Stable distribution parameters with particular attention devoted to those parameters that in previous literature were suggested to control tail behavior. In particular, we investigated the sensitivity of univariate MixedTS parameters on *CVaR*. Then we presented a portfolio optimization problem based on the use of the Multivariate Mixed Tempered Stable distribution for the joint modeling of univariate residuals obtained by fitting an ARMA(1,1)-GARCH(1,1) to the log return series of each hedge fund index in the dataset. We plotted the mean-*CVaR* efficient frontier and showed its sensitivity to some parameters of the multivariate distribution. The out-of-sample results for competing portfolio strategies seem to suggest that Global minimum *CVaR* portfolio presents better and more stable performances in all the time framework considered.

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