On the nonexistence of degenerate phase-shift discrete solitons in a dNLS nonlocal lattice.

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December 13, 2017

Abstract

We consider a one-dimensional discrete nonlinear Schrödinger (dNLS) model featuring interactions beyond nearest neighbors. We are interested in the existence (or nonexistence) of phase-shift discrete solitons, which correspond to four-sites vortex solutions in the standard two-dimensional dNLS model (square lattice), of which this is a simpler variant. Due to the specific choice of lengths of the inter-site interactions, the vortex configurations considered present a degeneracy which causes the standard continuation techniques to be non-applicable.

In the present one-dimensional case, the existence of a conserved quantity for the soliton profile (the so-called density current), together with a perturbative construction, leads to the nonexistence of any phase-shift discrete soliton which is at least C^2 with respect to the small coupling ϵ, in the limit of vanishing ϵ. If we assume the solution to be only C^0 in the same limit of ϵ, nonexistence is instead proved by studying the bifurcation equation of a Lyapunov-Schmidt reduction, expanded to suitably high orders. Specifically, we produce a nonexistence criterion whose efficiency we reveal in the cases of partial and full degeneracy of approximate solutions obtained via a leading order expansion.

KEYWORDS: discrete Non-Linear Schrödinger, discrete solitons, discrete vortex, current conservation, perturbation theory, Lyapunov-Schmidt decomposition.

1 Introduction

The discrete nonlinear Schrödinger (DNLS) is a prototypical nonlinear lattice dynamical model whose analytical and numerical tractability has enabled a considerable amount of progress towards understanding lattice solitary waves/coherent structures [19]. Its apparent simplicity in incorporating the interplay of nonlinearity and discrete dispersion, together with its relevance as an approximation of optical waveguide systems [8,23,39] and atomic systems in optical lattices [29] have significantly contributed to the popularity of the model. Moreover,

its ability to capture numerous linear and nonlinear experimentally observed features has made it a useful playground for a diverse host of phenomena. Such examples include, but are not limited to discrete diffraction [12] and its management [11], discrete solitons [12, 27, 28] and vortices [13, 30], Talbot revivals [16], and $\mathcal{PT}$-symmetry and its breaking [38], among many others.

Among the solutions that are of particular interest within the dNLS model are the so-called phase-shift ones, for which the solution does not bear a simply real profile (with phases 0 and π, or sites that are in- and out-of phase), but rather is genuinely complex featuring nontrivial phases [19]. Such solutions are more widely known as "discrete vortices" due to the fact that in order to ensure single-valuedness of the solution, upon rotation over a closed discrete contour, they involve variation of the phase by a multiple of 2π. Such waveforms were originally proposed theoretically in [17, 26] and subsequently observed experimentally, especially in the setting of optically induced photorefractive crystals [14, 31]. A systematic analysis of their potential existence in the standard nearest-neighbor dNLS model was provided in [33] and the relevant results, not only for 2d square lattices, but also for lattices of different types (triangular, honeycomb, etc.) were subsequently summarized in [19]. One of the main findings in this context are that the (symmetric) square vortices (with $\pi/2$ phase shifts between adjacent excited nodes) indeed persist at high order in case of the square lattice for different topological charges. Another relevant conclusion was that hexagonal and honeycomb lattices present the potential for vortices of topological charge both $S = 1$ and $S = 2$. Intriguingly, among the two and for focusing nonlinearity, the latter was found to be more stable than the former (whereas the stability conclusions were reversed in the case of self-defocusing nonlinearity.

However, the relevant analysis poses some intriguing mathematical questions. In particular, the consideration of the most canonical 4-site vortex in the 2d nearest-neighbor dNLS reveals (due to the relative phase change between adjacent sites of $\pi/2$) a degeneracy of the relevant waveforms and of their potential persistence. By degeneracy, here we mean that the standard approach of looking for critical points of the averaged (along the unperturbed periodic solution) perturbation (see [18, 20]), produces one-parameter families of solutions, where the Implicit Function Theorem cannot be applied. The tangent direction to the family represents a direction of degeneracy. Typically this calculation proceeds from the so-called anti-continuum limit [25] in powers of the coupling. Given then the degeneracy of these vortex states, a lingering question is whether such states will persist *to all orders* or whether they may be destroyed (i.e., the relevant persistence conditions will not be satisfied) at a sufficiently high order. While to all the leading orders considered in [19, 33] these solutions persist, in case of high degeneracy expansions up to high orders (ϵ^6 in the $\pi/2$-vortex, ϵ being the perturbation parameter) are necessary to reach a definitive answer to this question.

Inspired by the inherent difficulty of tackling the 2d problem, here we will opt to examine a simpler 1d problem. As a "caricature" of the 2d interaction, where the fourth site of a given square contour couples *back* into the first site, we choose to examine a *one-dimensional* model involving interactions not only of nearest neighbors (NN), but also of neighbors that are next-to-next-nearest (NNNN) ones. In principle, isolating a quadruplet of sites, we reconstruct a geometry similar to the 2d contour. The question that we ask in this simpler (per its 1d nature) setting is whether phase-shift solutions will exist. Surprisingly, and differently from the 2d scenario where at least the $\pi/2$ vortex was shown to exist, the answer that we find here is always in the negative. Using both a more direct, yet more restrictive, method involving a conserved flux quantity, as well as a more elaborate, yet less restrictive technique based

on Lyapunov-Schmidt reductions, we illustrate that such vortical states are always precluded from existence at a sufficiently high order. Since the continuation problem requires to explore the persistence of the solution at high orders in ϵ, where the differences with respect to the 2d model, in terms of lattice shape and interaction among sites, play a role, it is perhaps not surprising that we report here a different result in comparison to the $\pi/2$-vortex solution of the 2d lattice.

Our presentation will be structured as follows. In section 2, we will discuss the model and the principal result. In section 3, we will provide a perturbative approach which, combined with the conservation of the density current, leads towards the formulation of a finite regularity $(\mathcal{C}^2)$ version of this result. In section 4, we will overcome the technical limitation of the above regularity requirement by extending considerations to Lyapunov-Schmidt decompositions. Finally, in section 5, we will summarize our findings and present our conclusions, as well as some emerging questions for future work.

2   Theoretical Setup and Principal Results

As explained in the previous section, the aim of the work is to investigate the existence of discrete solitons in the NN and NNNN dNLS model of the form:

$$i\dot{\psi}_j = \psi_j - \frac{\epsilon}{2}[(\Delta_1 + \Delta_3)\psi]_j + \frac{3}{4}\psi_j|\psi_j|^2 \ , \qquad (\Delta_l\psi)_j := \psi_{j+l} - 2\psi_j + \psi_{j-l} \ , \qquad (1)$$

with vanishing boundary conditions at infinity $\psi \in \ell^2(\mathbb{C})$, in the anti-continuum limit, namely $\epsilon \to 0$. The equations can be written in the Hamiltonian form $i\dot{\psi}_j = \frac{\partial \mathcal{K}}{\partial \bar{\psi}_j}$ with

$$\mathcal{K} = \sum_{j\in\mathbb{Z}} |\psi_j|^2 + \frac{\epsilon}{2}\sum_{j\in\mathbb{Z}} \left(|\psi_{j+1} - \psi_j|^2 + |\psi_{j+3} - \psi_j|^2\right) + \frac{3}{8}\sum_{j\in\mathbb{Z}} |\psi_j|^4 \ . \qquad (2)$$

The original motivation leading to the study of the above model was the continuation of 4-site multibreathers in the Klein-Gordon model with NN and NNNN linear interactions of equal strength (once again motivated by the 2d problem), namely

$$H = \frac{1}{2}\sum_{j\in\mathbb{Z}} \left(y_j^2 + x_j^2\right) + \frac{\epsilon}{2}\sum_{j\in\mathbb{Z}} \left((x_{j+1} - x_j)^2 + (x_{j+3} - x_j)^2\right) + \frac{1}{4}\sum_{j\in\mathbb{Z}} x_j^4 \ . \qquad (3)$$

On one hand, this Hamiltonian represents a special case of the three-parameters family of up to next-to-next-nearest neighbor Klein-Gordon Hamiltonians

$$H_{\kappa_1,\kappa_2,\kappa_3} = \frac{1}{2}\sum_{j\in\mathbb{Z}} \left(y_j^2 + x_j^2\right) + \frac{\epsilon}{2}\sum_{j\in\mathbb{Z}}\sum_{m=1}^{3} \kappa_m(x_{j+m} - x_j)^2 + \frac{1}{4}\sum_{j\in\mathbb{Z}} x_j^4 \ ,$$

where the value of κ_m controls the range m interaction strength. The emergence of phase-shift multibreathers for values of κ_m large enough has been recently investigated [21, 37], while, the nonexistence of phase-shift multibreathers in the standard Klein-Gordon model with only nearest-neighbours interactions has been proved in [22].

On the other hand, the particular choice in (3), i.e. $\kappa_2 = 0$, $\kappa_1 = \kappa_3 = 1$, comes from the idea of reproducing in the 1D setting those vortex-like 4-sites interactions which are peculiar of a squared-lattice 2D model, with a standard nearest-neighbours interaction. In

the latter case the approximate 4-sites vortex solutions (obtained with $\epsilon = 0$) are degenerate objects [9], as a standard averaging analysis shows (see [1, 2, 4, 25]). Thus it is not possible to apply the usual implicit function theorem for their continuation. Nevertheless, it is well known (see [32, 35]) that for sufficiently low energies ($E \ll 1$) and in a suitable anti-continuum limit regime (namely $\epsilon \ll E$), (3) can be approximated by (2). Indeed, the dNLS Hamiltonian (2) turns out to be a resonant normal form of (3); this is evident by averaging both the coupling term and the nonlinear term with respect to the periodic flow given by the harmonic part of the Hamiltonian (3). The canonical change of coordinates which averages the KG model (3), generates both the normal form Hamiltonian (2) and also some remainder terms: the energy and small coupling regimes we are mentioning are those which guarantee the remainder terms to be much smaller, hence negligible, than the normal form (2). The interesting point in this normal form perspective is that the degeneracy is present also in the corresponding 4-sites discrete vortices of the normal form; the latter, being a model of the dNLS type, allows for an easier, thus more accurate and complete, analysis via perturbation theory. We plan to exploit the present results in order to transfer such an analysis to the original model in a forthcoming paper [36].

Finally, besides our original motivation of establishing a connection between solutions of the Klein-Gordon model with those of the corresponding dNLS normal form, we stress that the study of discrete solitons in beyond nearest neighbor 1-dimensional dNLS models has received some attention in the recent literature [6, 7, 20]. We expect that the results we are going to present can provide some additional insight on this and similar topics, as, e.g., for discrete solitons in zigzag dNLS models [10].

In order to state the results of the present paper, we recall that we are interested in periodic solutions of (1) of the form

$$\psi_j = e^{-i\lambda t}\phi_j \, , \qquad \{\phi_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \, ; \tag{4}$$

by inserting the previous ansatz in (1) one gets the *stationary equation*

$$\omega\phi_j = -\frac{\epsilon}{2}\Big[(\Delta_1 + \Delta_3)\phi\Big]_j + \frac{3}{4}\phi_j|\phi_j|^2 \, , \qquad \text{where } \omega := \lambda - 1 \, . \tag{5}$$

Specifically, being interested in many-site discrete solitons and in particular in vortex-like 4-site solutions, among the infinitely many trivial solutions of the unperturbed case, i.e., (5) with $\epsilon = 0$, we investigate a 4-dimensional torus (bearing, once again, in mind the analogy with the 2d lattice). Furthermore, since the discrete soliton solutions we are considering are single frequency solutions, namely in the form of standing waves as in (4), we need to set in the anti-continuum limit a fixed common amplitude R (which is fixed by the frequency λ, see (12)), so that the unperturbed solutions read

$$\phi_j^{(0)} = \begin{cases} Re^{i\theta_j} \, , & j \in S \, , \\ 0 \, , & j \notin S \, , \end{cases} \qquad \text{where } S = \{1, 2, 3, 4\} \text{ and } R > 0 \, . \tag{6}$$

All these orbits are uniquely defined except for a phase shift, due to the action of the symmetry $e^{i\sigma}$ along the orbit, which corresponds to a change of the initial configuration in the ansatz (4).

We are interested in the investigation of the *breaking of such a completely resonant lower dimensional torus*, i.e. we want to determine which solutions are going to survive as $\epsilon \neq 0$,

at fixed ω (and hence at fixed period). Before continue any further, we introduce the phase shifts between successive sites as

$$\varphi_j := \theta_{j+1} - \theta_j \ . \tag{7}$$

Our principal finding can be encapsulated then in the following statement, which provides a negative answer to the possible existence of phase-shift discrete solitons:

Theorem 2.1 *For ϵ small enough ($\epsilon \neq 0$), the only unperturbed solutions* (6) *that can be continued at fixed period to solutions $\phi(\epsilon)$ of* (5)*, correspond to $\varphi_j \in \{0, \pi\}$ and $j \in S' = \{1, 2, 3\}$.*

We stress that the existence of the trivial phase-shift solutions $\varphi_j \in \{0, \pi\}$ with $j \in S'$ can be obtained with standard arguments, restricting to real[1] solutions ϕ_j of the stationary equation (5) (see [34] Proposition 2.1 and references therein).

In order to prove a nonexistence result like the one stated above, a natural strategy could be the use a perturbative approach, i.e. expand both the candidate solution and the stationary equation in powers of ϵ and look for an obstruction to the solution order-by-order. The first issue in such a procedure is that the obstruction may appear at a high order; indeed that's the case in our model. We overcome this problem by exploiting the existence of a conserved quantity, the so-called density current (see [15, 19]),

$$J_j = \mathfrak{Im}\left(\phi_{j-1}\overline{\phi}_j + \phi_{j-3}\overline{\phi}_j + \phi_{j-2}\overline{\phi}_{j+1} + \phi_{j-1}\overline{\phi}_{j+2} \right) \ , \quad \text{for } j \in \mathbb{Z} \ , \tag{8}$$

that allows us to explore the implications of the obstruction (towards this conservation law), greatly simplifying the calculations and reveal them in a much lower order. A detailed description is presented in the first part of the paper (Section 3, see Theorem 3.1). We remark that this idea is expected to be applicable more broadly to several different one-dimensional models; clearly, due to the lack of a straightforward analogue of the density current (8) in two or three dimensions, there is a natural difficulty towards plainly extending this approach to higher (e.g., two or three) dimensional models. We nevertheless believe that the present analysis is useful to show how degenerate objects, which are likely to exist at lower order expansions, might have some obstruction to their continuation at higher order in perturbation theory.

Besides the technical difficulty, there is an intrinsic problem in the perturbative approach: it cannot provide a *complete* non existence result. Indeed one has to assume enough regularity of the solution to perform its expansion; in our case, due to the use of the density current, obstructions show up at the second order, thus Theorem 3.1 states that there cannot be any $\mathcal{C}^2$ discrete soliton solution in the perturbation parameter ϵ, except for the standard in-phase or out-of-phase ones. The complete proof of Theorem 2.1 requires to develop a Lyapunov-Schmidt decomposition which allows to rely on the regularity of the equations instead of the regularity of the solution, as shown in the second part of the paper (Section 4). An interesting remark is that the unsolvable systems of either linear or quadratic equations which appear in the density current expansion of Section 3, are essentially included at the level of either

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[1]Briefly, the idea is that restricting to real solutions of (5), the kernel directions (which are purely imaginary) are removed and the implicit function theorem can be applied. From a variational perspective, the restriction provides a critical point on the real phase space $\ell^2(\mathbb{R})$ which is indeed a critical point also on the complex phase space $\ell^2(\mathbb{C})$, due to the invariance of the Hamiltonian under conjugacy (see [5], Section 5).

the linearized (as in [33]) or quadratic bifurcation[2] equation. Hence, *a posteriori*, we could say that the obstructions to the continuation which arise in the density current expansion, *represent an effective non-existence criterion*.

3   A perturbative approach: finite regularity result

In the present Section we tackle the problem with a perturbative approach. For this purpose, we assume to deal with a continuation of $\{\phi_j(\epsilon)\}_{j\in\mathbb{Z}}$ which is at least $\mathcal{C}^2$ in ϵ and write

$$\phi_j = \phi_j^{(0)} + \epsilon\phi_j^{(1)} + \epsilon^2\phi_j^{(2)} + o(\epsilon^2) \ . \tag{9}$$

The continuation is assumed to be performed at fixed period (frequency). The results (that are weaker than Theorem 2.1) are collected in the following

Theorem 3.1 *For ϵ small enough ($\epsilon \neq 0$), the only unperturbed solutions* (6) *that can be continued at fixed period to $\mathcal{C}^2$ solutions $\phi(\epsilon)$ of* (5) *correspond to $\varphi_j \in \{0, \pi\}, j \in S'$.*

As anticipated in the Introduction, a key point is the fact that (5) preserves the density current, precisely

Lemma 3.1 *Let $\{\phi_j(\epsilon)\}_{j\in\mathbb{Z}}$ solve* (5), *then*

$$J_j := \mathfrak{Im}\left(\phi_{j-1}\overline{\phi}_j + \phi_{j-3}\overline{\phi}_j + \phi_{j-2}\overline{\phi}_{j+1} + \phi_{j-1}\overline{\phi}_{j+2}\right) \equiv 0 \ , \quad \forall j \in \mathbb{Z} \ .$$

proof: Let us define

$$a_n := \mathfrak{Im}(\phi_{n-1}\overline{\phi}_n) \quad \text{and} \quad b_n := \mathfrak{Im}(\phi_{n-3}\overline{\phi}_n) \ ;$$

then it is easy to see, e.g. by multiplying (5) by $\overline{\phi}_j$ and exploiting the reality of some of the terms thus obtained, that

$$a_n + b_n = a_{n+1} + b_{n+3} \ . \tag{10}$$

If we define the density current as $J_n := a_n + b_n + b_{n+1} + b_{n+2}$, by adding the quantity $b_{n+1} + b_{n+2}$ to both the l.h.t. and r.h.t. of (10), we get $J_n = J_{n+1}$. The hypothesis $\{\phi_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C})$, imply $J_j = 0, \forall j \in \mathbb{Z}$. $\square$

We present the detailed proof of Theorem 3.1 in the rest of this Section. First, starting from the zero order expansion, we determine the candidate solutions. Then we use the expansion of the stationary equation (5), together with the conserved quantity (8), in order to find an incompatibility condition and exclude all the solutions prohibited by Theorem 3.1. The advantage and novelty of this approach lies in the fact that such an incompatibility is revealed at a considerably lower order in the conserved quantity, in comparison to the original equations of motion.

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[2]By bifurcation equation we mean, as usual in this approach, the kernel equation of the Lyapunov-Schmidt decomposition. We stress here that the linearized bifurcation equation we get is the same as obtained in [33]. Instead of working with the variational formulation of the problem, we here prefer to perform the Taylor expansion at the level of the stationary equation, since we are not interested in the linear stability but only in the continuation of all the candidate orbits.

3.1 Zero order expansion and candidate solutions

The stationary equation (5) at order zero gives the uncoupled system

$$\omega \phi_j^{(0)} = \frac{3}{4} \phi_j^{(0)} |\phi_j^{(0)}|^2 \ , \tag{11}$$

which is trivialy invariant under the action of $e^{i\sigma}$. By using (6), it provides the frequency λ of the orbit, and its detuning ω from the linear frequency 1, namely

$$\omega = \frac{3}{4} R^2 \quad \text{and} \quad \lambda = 1 + \frac{3}{4} R^2 \ . \tag{12}$$

The conservation law (8) at order zero gives

$$J_j^{(0)} := \mathfrak{Im} \left(\phi_{j-1}^{(0)} \overline{\phi}_j^{(0)} + \phi_{j-3}^{(0)} \overline{\phi}_j^{(0)} + \phi_{j-2}^{(0)} \overline{\phi}_{j+1}^{(0)} + \phi_{j-1}^{(0)} \overline{\phi}_{j+2}^{(0)} \right) = 0 \ . \tag{13}$$

Recalling the form of the ansatz solution (6), eq. (13) is identically satisfied for $j \notin S$. Instead, for $j \in S$ we get

$$\sin(\varphi_1) = \sin(\varphi_2) = \sin(\varphi_3) = -\sin(\varphi_1 + \varphi_2 + \varphi_3) \ . \tag{14}$$

Remark 3.1 *The above systems of equations for the phase shifts can be obtained with a different procedure, namely the leading order approximation of a variational argument (see [18, 20]), which applies to system with symmetries. The solutions can indeed be obtained as critical points of the perturbed energy*

$$H_1(\psi) = \epsilon \sum_j \left[|\psi_{j+1} - \psi_j|^2 + |\psi_{j+3} - \psi_j|^2 \right] \ ,$$

restricted to the unperturbed solution $\phi^{(0)} e^{-i\lambda t}$ (or averaged over the unperturbed periodic orbit $\phi^{(0)} e^{-i\lambda t}$), giving in our case

$$F(\varphi_1, \varphi_2, \varphi_3) := H_1(\phi^{(0)} e^{-i\lambda t}) = -2R^2(\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) + \cos(\varphi_1 + \varphi_2 + \varphi_3)) \ , \tag{15}$$

apart from useless constant terms depending on R. However, in our perspective, it is more important to stress that the system (14) also represents the first term of the bifurcation equation (in the Lyapunov-Schmidt decomposition) expanded both in ϵ and in the kernel variables (see (55)).

The first of (14) provides four, one-parameter, families of solutions $(\varphi_1, \varphi_2(\varphi_1), \varphi_3(\varphi_1))$. By replacing φ_1 by φ we get precisely

$$(\varphi, \varphi, \varphi) \ , \quad (\varphi, \varphi, \pi - \varphi) \ , \quad (\varphi, \pi - \varphi, \varphi) \ , \quad (\varphi, \pi - \varphi, \pi - \varphi) \ . \tag{16}$$

By plugging the first one of (16), i.e., $(\varphi, \varphi, \varphi)$ in the second equation of (14) we get $\sin(\varphi) = -\sin(3\varphi)$, thus we obtain

$$(\varphi_1, \varphi_2, \varphi_3) \in \left\{ (0, 0, 0) \ , (\pi, \pi, \pi) \ , \pm \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right) \right\} \ .$$

The first two solutions represent the discrete in-phase and the alternating-phase solitons, while the other two are the discrete vortex solitons (the phase shift solutions). In the latter

cases, i.e. the solution we call symmetric vortices, we will conveniently use the following choice of phases:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \pm \left(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right) . \tag{17}$$

The remaining three families, when plugged in the second of (14) give the identity $\sin(\varphi_1) = \sin(\varphi_1)$, hence we get the following three 1-parameter families of solutions which are referred to as asymmetric vortices (see [33] for a similar example in the nearest neighbor case)

$$F_1 : (\varphi, \pi - \varphi, \pi - \varphi) , \quad F_2 : (\varphi, \varphi, \pi - \varphi) , \quad F_3 : (\varphi, \pi - \varphi, \varphi) . \tag{18}$$

Remarkably, the three families intersect in the two previously obtained symmetric vortex solutions $\pm\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$, while the in-phase/alternating phase solutions do not belong to any of these families, thus being isolated solutions. Moreover, the above families carry also all the other 3 couples of solutions with phase shifts $\varphi_j \in \{0, \pi\}$.

The above considerations provide us with the complete list of the phase shift, discrete vortex solutions that are admissible for the continuation with respect to ϵ. On a three-dimensional torus $\mathbb{T}^3$, representing all the possible phase differences (which means, the original $\mathbb{T}^4$ with a quotient with respect to the gauge symmetry), we have 2 isolated solutions, and three 1-parameter families intersecting in the 2 symmetric vortex solutions. Thus, the 2 vortex solutions will be fully degenerate, while any other asymmetric vortex solution on a family will be partially degenerate. This can be seen explicitly by using the approach developed in [18]. Indeed, it turns out that $(0, 0, 0)$ and (π, π, π) are non degenerate (absolute) extrema – respectively a maximum and a minimum – of F (see (15) above) on the torus $\mathbb{T}^3$, while the mixed standard solution are partially degenerate relative extrema. The two symmetric vortex solutions are fully degenerate, since the Hessian $\left(D^2 F\right)\left(\pm\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)\right)$ is the null matrix. This scenario in the 2d, nearest-neighbor dNLS was dubbed the super-symmetric case [33]. In all the other asymmetric vortex solutions, the Hessian $\left(D^2 F\right)$ has only 1 zero eigenvalue which corresponds to the tangent direction to the family one considers Notice that the gauge symmetry direction is absent from F, given its dependence on the variables φ. This fact can also be verified by a direct calculation.

Remark 3.2 *The above analysis bears extensive similarities with the study of the vortex solutions on a squared-lattice 2D dNLS model, performed in [33]. The above classifications of 1-parameter families (18) represents indeed the families (3.8)-(3.10) in that work.*

3.2   First and second order expansions

The strategy we are going to follow is based on the expansion in ϵ up to second order of both the stationary equation (5) and the density current (8). Hence we start providing here the explicit expansions that will be used in the following parts.

3.2.1   First order equation's structure:

The equation (5) at first order reads

$$\omega \phi_j^{(1)} = -\frac{1}{2}\left(\phi_{j+3}^{(0)} + \phi_{j+1}^{(0)} + \phi_{j-1}^{(0)} + \phi_{j-3}^{(0)}\right) + 2\phi_j^{(0)} + \frac{3}{4}\left(2\phi_j^{(1)}|\phi_j^{(0)}|^2 + \left(\phi_j^{(0)}\right)^2 \overline{\phi}_j^{(1)}\right) ;$$

which, separating between the core sites of the discrete soliton and the external ones, becomes

$$-\omega\left(\phi_j^{(1)} + e^{2i\theta_j}\overline{\phi}_j^{(1)}\right) = -\frac{1}{2}\left(\phi_{j+3}^{(0)} + \phi_{j+1}^{(0)} + \phi_{j-1}^{(0)} + \phi_{j-3}^{(0)}\right) + 2\phi_j^{(0)}, \qquad j \in S, \qquad (19)$$

$$\omega\phi_j^{(1)} = -\frac{1}{2}\left(\phi_{j+3}^{(0)} + \phi_{j+1}^{(0)} + \phi_{j-1}^{(0)} + \phi_{j-3}^{(0)}\right), \qquad j \notin S. \qquad (20)$$

Equation (19) is a linear equation in $\phi_j^{(1)}$ with a 1-dimensional kernel for any $j \in S$ given by

$$\phi_{j,Ker}^{(1)} = ie^{i\theta_j}, \qquad (21)$$

and following the ideas of [33], part 2, we can find a particular solution of (19) in the form

$$\phi_j^{(1)} = e^{i\theta_j}u_j^{(1)}, \qquad u_j^{(1)} \in \mathbb{R}. \qquad (22)$$

3.2.2 Second order equation's structure:

Equation (5) at second order, directly split between core and external sites, reads

$$-\omega\left(\phi_j^{(2)} + e^{2i\theta_j}\overline{\phi}_j^{(2)}\right) = -\frac{1}{2}\left(\phi_{j+3}^{(1)} + \phi_{j+1}^{(1)} + \phi_{j-1}^{(1)} + \phi_{j-3}^{(1)}\right) + 2\phi_j^{(1)} +$$
$$-\frac{3}{4}\left(2\phi_j^{(0)}|\phi_j^{(1)}|^2 + \left(\phi_j^{(1)}\right)^2\overline{\phi}_j^{(0)}\right), \qquad j \in S, \qquad (23)$$

$$\omega\phi_j^{(2)} = -\frac{1}{2}\left(\phi_{j+3}^{(1)} + \phi_{j+1}^{(1)} + \phi_{j-1}^{(1)} + \phi_{j-3}^{(1)}\right) + 2\phi_j^{(1)}, \qquad j \notin S. \qquad (24)$$

The kernel part associated to (23) being the same as in (19), gives

$$\phi_j^{(2)} = e^{i\theta_j}u_j^{(2)}, \qquad u_j^{(2)} \in \mathbb{R}. \qquad (25)$$

3.2.3 Density current expansion:

The conservation law (8) at first and second orders, respectively reads

$$J_j^{(1)} := \mathfrak{Im}\left(\phi_{j-1}^{(0)}\overline{\phi}_j^{(1)} + \phi_{j-3}^{(0)}\overline{\phi}_j^{(1)} + \phi_{j-2}^{(0)}\overline{\phi}_{j+1}^{(1)} + \phi_{j-1}^{(0)}\overline{\phi}_{j+2}^{(1)} + \right.$$
$$\left. + \phi_{j-1}^{(1)}\overline{\phi}_j^{(0)} + \phi_{j-3}^{(1)}\overline{\phi}_j^{(0)} + \phi_{j-2}^{(1)}\overline{\phi}_{j+1}^{(0)} + \phi_{j-1}^{(1)}\overline{\phi}_{j+2}^{(0)}\right) = 0, \qquad (26)$$

$$J_j^{(2)} := \mathfrak{Im}\left(\phi_{j-1}^{(0)}\overline{\phi}_j^{(2)} + \phi_{j-1}^{(1)}\overline{\phi}_j^{(1)} + \phi_{j-1}^{(2)}\overline{\phi}_j^{(0)} + \phi_{j-3}^{(0)}\overline{\phi}_j^{(2)} + \phi_{j-3}^{(1)}\overline{\phi}_j^{(1)} + \phi_{j-3}^{(2)}\overline{\phi}_j^{(0)} + \right.$$
$$\left. + \phi_{j-1}^{(0)}\overline{\phi}_{j+2}^{(2)} + \phi_{j-1}^{(1)}\overline{\phi}_{j+2}^{(1)} + \phi_{j-1}^{(2)}\overline{\phi}_{j+2}^{(0)} + \phi_{j-2}^{(0)}\overline{\phi}_{j+1}^{(2)} + \phi_{j-2}^{(1)}\overline{\phi}_{j+2}^{(1)} + \phi_{j-2}^{(2)}\overline{\phi}_{j+2}^{(0)}\right) = 0. \qquad (27)$$

3.3 The $(0,0,0)$ solution

In what follows, we start showing how our approach is implemented in the easiest case of a non degenerate discrete soliton, corresponding to $\theta_1 = \theta_2 = \theta_3 = \theta_4$ where the compatibility of the solution expansion with the density current expansion persists. This is a case where the implicit function theorem can be applied to continue the unperturbed solution, even without exploiting the restriction to real solutions of (5). However, we think it is instructive to start to apply here our approach, in order to show the differences with respect to the (partial and fully) degenerate cases which will follow.

Order 1: For the sake of simplicity we may chose $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, 0)$. By inserting the zeroth order of the solution (6) into (19) and (20), for the specific choice of θ_j, we get e.g. for $j \in S$

$$-2\omega\mathfrak{Re}(\phi_j^{(1)}) = R \Leftrightarrow \mathfrak{Re}(\phi_j^{(1)}) = -\frac{2}{3R}.$$

In a similar way we get the first order of the solution for the rest of the sites. The results for all the sites are

j	$\cdots$	-2	-1	0	$j \in S$	5	6	7	$\cdots$
$-\phi_j^{(1)}$	0	$\frac{2}{3R}$	$\frac{2}{3R}$	$\frac{4}{3R}$	$\frac{2}{3R} - i\alpha_j$	$\frac{4}{3R}$	$\frac{2}{3R}$	$\frac{2}{3R}$	0

,

where the four $\alpha_j \in \mathbb{R}$ represent the four independent kernel directions. The conservation law (26) is sastisfied, provided the α_j fulfill the following set of linear homogeneous equations

$$\begin{cases} 2\alpha_1 - \alpha_2 - \alpha_4 = 0 \\ \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0 \\ \alpha_1 + \alpha_3 - 2\alpha_4 = 0 \end{cases} \implies \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \ .$$

Thus, the first order expansion of the density current removes all the kernel directions but one, as expected by the non-degeneracy. The remaining kernel direction $\alpha = \alpha_1(1, 1, 1, 1)$ represents the effect of the rotational symmetry, which cannot be removed from the perturbation expansion.

Order 2: As before, using now (23) and (24), one has (limiting to the sites $-2 \le j \le 7$)

j	-2	-1	0	$j \in S$	5	6	7
$\phi_j^{(2)}$	$-\frac{(8+6i\alpha_1 R)}{9R^3}$	$-\frac{2i\alpha_1}{3R^2}$	$-\frac{(20+12i\alpha_1 R)}{9R^3}$	$-\frac{(16+9i\alpha_j^2 R^2)}{18R^3} + i\alpha_j'$	$-\frac{(20+12i\alpha_1 R)}{9R^3}$	$-\frac{2i\alpha_1}{3R^2}$	$-\frac{(8+6i\alpha_1 R)}{9R^3}$

If we insert the second order corrections in the conservation law (27), we get the same system of linear homogeneous equations obtained at first order in the new kernel variables α_j'

$$\begin{cases} 2\alpha_1' - \alpha_2' - \alpha_4' = 0 \\ \alpha_1' + \alpha_2' - \alpha_3' - \alpha_4' = 0 \\ \alpha_1' + \alpha_3' - 2\alpha_4' = 0 \end{cases} \implies \alpha_1' = \alpha_2' = \alpha_3' = \alpha_4' \ ,$$

whose solution is independent of α_1 and again leaves the symmetry direction $\alpha' = \alpha_1'(1, 1, 1, 1)$ as the only kernel direction.

Although it is not feasible to proceed explicitly in the expansion, in this non-degenerate case it is easy to figure out that, at any order, the conservation law would produce always the same system of linear homogeneous equation in the new variables. Thus one can recursively and uniquely determine all the needed coefficients, leaving as free parameter only the gauge direction, as expected from the symmetry of the system.

Remark 3.3 The (π, π, π) family. *By using similar arguments as with the $(0, 0, 0)$ family we also conclude that this family is also non-degenerate and thus it can be iteratively constructed order by order.*

3.4   The asymmetric $(\varphi, \pi - \varphi, \pi - \varphi)$ and $(\varphi, \varphi, \pi - \varphi)$ vortex solutions

We now deal with the problem of continuing any of the phase-shift solutions of the first two families of (18), except $\pm(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. Concerning the first one the perturbation expansion goes as follows.

Order 1:   As before we represent the solution in the following table

j	-2	-1	0	$\cdots$	5	6	7
$\phi_j^{(1)}$	$-\frac{2}{3R}$	$-\frac{2}{3R}e^{i\varphi}$	0	$\cdots$	$-\frac{4}{3R}\cos(\varphi)$	$\frac{2}{3R}$	$-\frac{2}{3R}e^{-i\varphi}$

1	2	3	4
$-\frac{4}{3R} + \frac{2}{3R}\cos(\varphi) + i\alpha_1$	$-\frac{4}{3R}e^{i\varphi} + i\alpha_2 e^{i\varphi}$	$\frac{4}{3R} + \frac{2}{3R}\cos(\varphi) - i\alpha_3$	$-\frac{4}{3R}e^{-i\varphi} + i\alpha_4 e^{-i\varphi}$

The conservation law (26) is satisfied, provided α_j fulfill

$$M \cdot \boldsymbol{\alpha} = 0 , \qquad M := \cos(\varphi) \begin{bmatrix} 2 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} , \qquad \boldsymbol{\alpha} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} . \tag{28}$$

It is immediate to notice that for $\varphi = \pm\frac{\pi}{2}$ the system is identically satisfied. This is the effect of the full degeneracy of the vortex solutions, that will be treated separately later. So, assuming $\varphi \neq \pm\frac{\pi}{2}$, we get

$$\alpha_3 = \alpha_1 , \quad \alpha_4 = 2\alpha_1 - \alpha_2 ,$$

which leaves two Kernel directions in the problem: the gauge direction and the tangent direction to the φ-family.

Order 2:   The expansion now proceeds as in the previous example, computing the second order corrections $\phi_j^{(2)}$. For $j \in S$ in particular the solutions can be taken in the form

$$\phi_j^{(2)} = u_j^{(2)}(\alpha_1, \alpha_2)e^{i\theta_j} + i\alpha_j' e^{i\theta_j} ,$$

with four new kernel directions α_j', as in the previous example. We omit listing here the explicit expressions; once inserted them into (27), we are left with three linear equations, in this case nonhomogeneous, corresponding again to $J_{2,3,4}^{(2)} = 0$, which take the form

$$M \cdot \boldsymbol{\alpha'} = \mathbf{a} + \mathbf{b}(\boldsymbol{\alpha}) , \qquad \mathbf{a} := \frac{8}{9R^3}\sin(\varphi)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \mathbf{b}(\boldsymbol{\alpha}) := \frac{4}{3R^2}\cos(\varphi)^2\alpha_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \tag{29}$$

The rank of M is equal to the rank of the augmented matrix $M|\mathbf{a} + \mathbf{b}(\boldsymbol{\alpha})$ only if $\sin(\varphi) = 0$, i.e. $\varphi = \{0, \pi\}$, hence any possible $\mathcal{C}^2$ asymmetric vortex solution in this family is excluded and only trivial phase families are allowed.

Taking into account the second family, all the calculations are essentially the same except from some permutation of indices, i.e. in this case the matrix M is

$$M := \cos(\varphi) \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & -2 \end{bmatrix} , \tag{30}$$

and **a** has a change in the sign, but the conclusions are the same as before.

3.5 The asymmetric $(\varphi, \pi - \varphi, \varphi)$ vortex solution

We proceed dealing with the problem of continuing any of the phase-shift solutions of the last family of (18), again with the exclusion of the solutions $\pm(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. With the convenient choice of $\theta = (0, \varphi, \pi, \pi + \varphi)$, the perturbation expansion goes as follows.

Order 1:

j	-2	-1	0	$\cdots$	5	6	7
$\phi_j^{(1)}$	$-\frac{2}{3R}$	$-\frac{2}{3R}e^{i\varphi}$	0	$\cdots$	0	$\frac{2}{3R}$	$\frac{2}{3R}e^{i\varphi}$

1	2	3	4
$-\frac{4}{3R} + i\alpha_1$	$-\frac{4}{3R}e^{i\varphi} + i\alpha_2 e^{i\varphi}$	$\frac{4}{3R} - i\alpha_3$	$\frac{4}{3R}e^{i\varphi} - i\alpha_4 e^{i\varphi}$

The conservation law (26) is satisfied, provided α_j fulfill

$$M \cdot \boldsymbol{\alpha} = 0 , \qquad M := \cos(\varphi) \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} . \tag{31}$$

It is immediate to notice that for $\varphi = \pm\frac{\pi}{2}$ the system is again identically satisfied, as a result of the full degeneracy of the vortex solutions. So, assuming $\varphi \neq \pm\frac{\pi}{2}$, we get

$$\alpha_3 = \alpha_1 , \quad \alpha_4 = \alpha_2 .$$

Order 2: As in the previous subsection we omit the explicit expression of the solution $\phi_j^{(2)}$; we directly give the result of their use into the density current conservation law (27) which reduces to the three equations corresponding to $J_{2,3,4}^{(2)} = 0$, which take the form

$$M \cdot \boldsymbol{\alpha}' = \mathbf{a} , \qquad \mathbf{a} := -\frac{8}{9R^3} \sin(\varphi) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \tag{32}$$

Again, the presence of the vector **a** implies $\sin(\varphi) = 0$, hence $\varphi \in \{0, \pi\}$, thus any possible $\mathcal{C}^2$ asymmetric vortex solution in the third family is excluded.

3.6 The vortex solutions

The previous analysis has shown that only the phase shifts, i.e., $\varphi_j \in \{0, \pi\}$, or the vortex solutions, i.e., $\varphi_j = \pm\frac{\pi}{2}$, have a chance to be continued as $\mathcal{C}^2$ solutions. To complete our analysis, we wish to now show that also the vortex solutions cannot be continued in ϵ. This, however, requires a separate expansion of the first two perturbation orders, due to the complete degeneracy in (28) and (31).

Order 1: Let, for the sake of simplicity, $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, \pi/2, \pi, 3\pi/2)$ for $j \in S$; i.e. the equations for $\phi_j^{(1)}$ are in a form such that only their real or imaginary parts appear, thus only half of the solution is determined. For example, for $j = 1$, remarking that $\phi_2^{(0)} + \phi_4^{(0)} = 0$, we have

$$-\omega\left(\phi_1^{(1)} + \overline{\phi}_1^{(1)}\right) = 2R - \frac{1}{2}\left[\phi_2^{(0)} + \phi_4^{(0)}\right] \qquad \Longrightarrow \qquad \phi_1^{(1)} = -\frac{4}{3R} + i\alpha_1 \; ;$$

for the remaining values of j the solution is completely determined. As a result we have

j	-2	-1	0	1	2	3	4	5	6	7
$\phi_j^{(1)}$	$-\frac{2}{3R}$	$-\frac{2i}{3R}$	0	$-\frac{4}{3R} + i\alpha_1$	$\alpha_2 - i\frac{4}{3R}$	$\frac{4}{3R} + i\alpha_3$	$\alpha_4 + i\frac{4}{3R}$	0	$\frac{2}{3R}$	$\frac{2i}{3R}$

Remark 3.4 *The four free parameters α_j represent the first order expansion in ϵ of the kernel directions in the forthcoming Lyapunov-Schmidt decomposition. Indeed, at this order, the equations we are able to solve represent the range equations in the same decomposition.*

In this case, and in contrast with the previous one, the density current equations $J_j^{(1)} = 0$ do not provide any further information on the first order solutions $\phi_j^{(1)}$, with $j \in S$. Indeed, the three equations $J_{2,3,4}^{(1)} = 0$ give the trivial system

$$M\boldsymbol{\alpha} = 0 \;, \qquad \text{with} \quad M := 0 \;, \tag{33}$$

so that the unknown α_j remain undetermined (see also (28), (30) and (31) with $\varphi = \pm\frac{\pi}{2}$).

Order 2: We face a similar situation as before, with the $\phi_j^{(2)}$, for $j \in S$, appearing into the equations only through their real or imaginary parts: thus we are left with 4 new parameters α_j', corresponding to the real part of $\phi_{2,4}^{(2)}$ and to the imaginary part of $\phi_{1,3}^{(2)}$, which are not determined, as before. The other four components appear instead as functions of the previous four free parameters α_j. As before, for $j \notin S$ no issues arise. To summarize, defining

$$f(x, y) := \frac{x + y}{3R^2} \;, \qquad g(x) := \frac{10}{9R^3} + \frac{x^2}{2R} \;.$$

and factorizing some constants in the values of the external sites, we have

j	-5	-4	-3	-2	-1	0
$\frac{9}{4}R^3\phi_j^{(2)}$	1	i	1	$-2+i\left(1-\frac{3R\alpha_1}{2}\right)$	$\left(1-\frac{3R\alpha_2}{2}\right)-2i$	$i\left(1-\frac{3R(\alpha_1+\alpha_3)}{2}\right)$

j	1	2
$\phi_j^{(2)}$	$(f(\alpha_2,\alpha_4)-g(\alpha_1))+i\alpha_1'$	$\alpha_2'+i(f(\alpha_1,\alpha_3)-g(\alpha_2))$

j	3	4
$\phi_j^{(2)}$	$(f(\alpha_2,\alpha_4)+g(\alpha_3))+i\alpha_3'$	$\alpha_4'+i(f(\alpha_1,\alpha_3)+g(\alpha_4))$

j	5	6	7
$\frac{9}{4}R^3\phi_j^{(2)}$	$-\left(1+\frac{3R(\alpha_2+\alpha_4)}{2}\right)$	$2-i\left(1+\frac{3R\alpha_3}{2}\right)$	$-\left(1+\frac{3R\alpha_4}{2}\right)+2i$

We consider now the second order of the expansion of the conservation law (8). It turns out that once again all the equations $J_n^{(2)}=0$, for $n\notin\{2,3,4\}$, are identically satisfied, providing no information on any of the eight free parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$. The only non trivial equations are again

$$J_2^{(2)}=0\ ,\quad J_3^{(2)}=0\ ,\quad J_4^{(2)}=0\ .$$

After long and tedious manipulation that we here omit, one reaches the following system of three quadratic equations which remarkably depend only on the first order kernel variables α_j, and not on the second order variables α_j'

$$
\begin{cases}
C=(b+d)(b-d+2a)\ ,\\
C=(a+c)(c-a+2d)\ ,\\
C=b^2-d^2+c^2-a^2+2(ad-bc)\ ,
\end{cases}
\quad\text{where}\quad
C:=\frac{16}{9R^2}\ ,\quad
\begin{matrix}
a\equiv\alpha_1\ ,&b\equiv\alpha_2\\
c\equiv\alpha_3\ ,&d\equiv\alpha_4
\end{matrix}\ .
\tag{34}
$$

The above system, by taking the differences of its equations, assume the form

$$
\begin{cases}
0=(b+d)(d-b+2c)\ ,\\
0=(a+c)(a-c+2b)\ ,\\
0=(a+b)^2-(c+d)^2\ .
\end{cases}
$$

The third equation implies either $a+b=c+d$ or $a+b=-(c+d)$, call them case 1) and 2) respectively. Moreover in the first two equations one has to exclude $b+d=0$ and $a+c=0$ because those would imply $C=0$ in (34). Thus necessarily we have

$$
\begin{cases}
b-d=2c\ ,\\
c-a=2b\ ,
\end{cases}
\quad\Longrightarrow\quad
\begin{cases}
b+c=-(a+d)\ ,\\
a+b-(c+d)=2(c-b)\ .
\end{cases}
$$

In case 1), the first on the right imply $a+b=c+d=0$, so that using the second we have $b=c$, which gives $a=d$, so that we end up with $b+d=a+c=0$, which we already excluded above since it is equivalent to $C=0$. We are thus left with case 2). Here

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the contradiction is obtained putting the system on the left in one of the first two equations of (34) to get $C = 2(a+c)(b+d)$. Since in case 2) we have $a+c = -(b+d)$ we would get $C < 0$.

To summarize the system turns out to be impossible, thus proving the incompatibility of the vortex solution.

Remark 3.5 *After concluding the presentation of the perturbative approach of the problem, the reader could assume that this methodology is only applicable to the present system because of the uniquenss of the form of* (8). *Actually, this is not true, since this approach can be very easily adapted in order to be used in every system of the dNLS type with interactions beyond the nearest-neighbor ones and the corresponding density current can be easily calculated. In fact the method has been used in the zigzag [36] configuration, where the corresponding vortex configurations also are excluded: in this case the degeneracy occurs, however, in lower order than the present example. Anyway, although the density current method is applicable to any kind of linear interaction, the kind of degeneracy is highly dependent on the number of sites considered in the configuration.*

4 Proof of Theorem 2.1 via Lyapunov-Schmidt decomposition

In the present Section we give the proof of Theorem 2.1, without the C^2 regularity restriction of the previous section. Consider again the stationary equation (5), now written as

$$\omega\phi + \epsilon L\phi - \frac{3}{4}\phi|\phi|^2 = 0 \;, \qquad \text{with} \quad L := \frac{1}{2}[\Delta_1 + \Delta_3] \;, \tag{35}$$

and $\phi \in \ell^2(\mathbb{C})$. Since we are interested in the continuation of vortex-like solutions from the anti-continuum limit, we introduce the translation

$$w(\epsilon) := \phi(\epsilon) - v \;, \tag{36}$$

where v is a fixed solution of the unperturbed equation, as in formula (6), and $w(\epsilon)$ represents a small displacement around it, namely

$$v_j = \begin{cases} Re^{i\theta_j} & j \in S \;, \\ 0 & j \notin S \;, \end{cases} \qquad \text{and} \quad \|w\| \ll \|v\| \;. \tag{37}$$

Remark 4.1 *The change of coordinate* (36) *is one of the places where a key difference appears with respect to the previous section. Although it might be seen as the analogue of the decomposition* (9) *— with v taking in the present Section the place of $\phi^{(0)}$, and w the role of the higher order terms of such an expansion — at variance with* (9), *in* (36) *no regularity is assumed (apart from the obvious continuity), as w is simply a small displacement. We will instead exploit the regularity at the level of the equations.*

Exploiting (i) that v is a generic solution of the unperturbed problem and (ii) that w is

small in ϵ, we split (35) in powers of w and define the following four functions

$$F(v; w, \epsilon) := \epsilon L v + \big(\Lambda w + \epsilon L w\big) - \big(N_2(v; w) + N_3(w)\big) ,$$

$$N_3(w) := \frac{3}{4} |w|^2 w$$

$$N_2(v; w) := \frac{3}{4} \big(w^2 \overline{v} + 2v|w|^2\big) \tag{38}$$

$$\Lambda w := \omega w - \frac{3}{4} \big(v^2 \overline{w} + 2|v|^2 w\big) .$$

Thus (35) takes the form

$$F(v; w(\epsilon), \epsilon) = 0 . \tag{39}$$

The linear part has been split into two terms since

$$\Lambda = \big(D_w F\big)(v; 0, 0) , \tag{40}$$

being independent of ϵ, is the operator one has to look at for the continuation of v as a solution $F(v; 0, 0) = 0$. It is useful, considering the shape of v given in (37), to represent Λ in a matrix form

$$\Lambda = \begin{pmatrix} \omega \mathbb{I} & 0 & 0 \\ 0 & -\omega M_S & 0 \\ 0 & 0 & \omega \mathbb{I} \end{pmatrix} \tag{41}$$

where M_S is a block matrix, composed of 4 blocks $M_{S,j}$, each in the form

$$M_{S,j} := \begin{pmatrix} 2\cos^2(\theta_j) & \sin(2\theta_j) \\ \sin(2\theta_j) & 2\sin^2(\theta_j) \end{pmatrix} . \tag{42}$$

Each block $M_{S,j}$ has zero determinant, its one dimensional kernel being given by

$$Ker(M_{S,j}) = \langle e_j \rangle , \qquad e_j := \begin{pmatrix} -\sin(\theta_j) \\ \cos(\theta_j) \end{pmatrix} \equiv i e^{i\theta_j} ; \tag{43}$$

hence, the differential Λ has a four dimensional kernel, as expected by the four dimensional tangent space of $\mathbb{T}^4$, which is given by

$$Ker(\Lambda) = \langle f_1, f_2, f_3, f_4 \rangle ,$$

where each f_j is the embedding of the corresponding e_j in the $2N$ dimensional real phase space (or in the N dimensional complex phase space). Hence Λ is not invertible.

4.1 Lyapunov-Schmidt decomposition

Given the above consideration on the operator Λ, instead of the standard implicit function theorem approach, we need to perform a Lyapunov-Schmidt decomposition.

We denote the kernel of Λ as K, and its range as H, and Π_K and Π_H the corresponding projectors. Due to the structure of the phase space[3] $\mathcal{P}$, we can identify $H = K^\perp$ and $\mathcal{P} = K \oplus H$. We consider the decomposition

$$w = k + h , \qquad \text{with} \quad k \in K \quad \text{and} \quad h \in H ,$$

---

[3]This is trivial in the finite dimensional case; in the infinite dimensional case, one has to notice that $F : \ell^2 \to \ell^2$.

and we project (39) onto the two spaces

$$F_H(v; h, k, \epsilon) := \Pi_H F(v; h + k, \epsilon) = 0 ,$$
$$F_K(v; h, k, \epsilon) := \Pi_K F(v; h + k, \epsilon) = 0 .$$

The strategy is, as usual, to first solve the range equation, exploiting the inveritbility of Λ on H, so as to find an $h(v; k, \epsilon)$ such that $F_H(v; h(v; k, \epsilon), k, \epsilon) = 0$; then to insert such a solution into the kernel equation.

Remark 4.2 *As anticipated in Remark 4.1, we stress once more that here we are not assuming any regularity of the solution. We will nevertheless expand h since in the range equation $F_H = 0$, the function F_H is regular, and thus a solution obtained via the implicit function theorem preserves such a regularity. The possible non-regularity of the solution may take place in the dependence of k on ϵ at the level of the kernel equation.*

We perform a preliminary simplification of the two equations above by observing the following elementary facts. First of all, we clearly have $\Lambda(h + k) = \Lambda h$; moreover the orthogonality of the kernel and range subspaces implies that we have $\mathfrak{Re}(k\bar{h}) = 0$, simplifying the nonlinear terms; and in particular, from (43) and (37), we get $\mathfrak{Re}(e_j \bar{v}_j) = 0$ for $j \in S$, which confirms that $v \in H$. Furthermore, exploiting again that within S one has that v_j and h_j are directed as $e^{i\theta_j}$ and k_j is parallel to $ie^{i\theta_j}$, it is easy to make the projections, e.g. as in $\bar{v}(h^2 + k^2) = v(|h|^2 - |k|^2) \in H$ and $2\bar{v}hk = 2\mathfrak{Re}(v\bar{h})k \in K$. Thus we have

$$\Lambda h + \epsilon \Pi_H \big(L(h + k + v) \big) - \frac{3}{4} \Big(\big(3|h|^2 + |k|^2 \big) v + \big(|h|^2 + |k|^2 \big) h \Big) = 0 , \qquad \text{(R)}$$

$$\epsilon \Pi_K \big(L(h + k + v) \big) - \frac{3}{4} \Big(|h|^2 + |k|^2 + 2\mathfrak{Re}\big(v\bar{h} \big) \Big) k = 0 . \qquad \text{(K)}$$

4.2 Range equation

As we commented above, the solvability of the range equation is not an issue. The interesting point is instead to shed some light on the general structure of the solution that we will exploit later on. According to such a purpose, and recalling that $h(v; k, \epsilon)$ is regular in k and ϵ, we start by expanding it as follows

$$h = h^{(0)}(v; k) + \epsilon h^{(1)}(v; k) + \epsilon^2 h^{(2)}(v; k) + \tilde{h}^{(3)}(v; k, \epsilon) . \qquad (44)$$

Inserting the above equation in (R), we can produce explicit expressions for the above expansion by solving the range equation iteratively. Before proceeding into such a task, it is also useful to notice here that the local terms can be further simplified splitting between core sites ($j \in S$) and the other ones: in particular the operator Λ, once applied to an element of the range $h \in H$, simply becomes

$$\Lambda h = \begin{cases} \omega h & j \notin S , \\ -2\omega h & j \in S , \end{cases}$$

and the nonlinear part of the equation takes the form

$$\Pi_H(N_2 + N_3) = \frac{3}{4} \begin{cases} |h|^2 h & j \notin S , \\ \big(|h|^2 + |k|^2 \big)(v + h) + 2|h|^2 v & j \in S . \end{cases}$$

Order 0: The order 0 component of the range equation takes the form

$$\Lambda h^{(0)} - \frac{3}{4}\Big(\big(|h^{(0)}|^2 + |k|^2\big)\big(v + h^{(0)}\big) + 2\big|h^{(0)}\big|^2 v\Big) = 0 \ ,$$

which can be split, after further simplifications, and introducing for the core sites $(j \in S)$ the temporary notation $h_j^{(0)} = \ell e^{i\theta_j}$, with $\ell \in \mathbb{R}$, as

$$
\begin{aligned}
h^{(0)}\big(R^2 - \ell^2\big) &= 0 & j \notin S, \\
\big(v + h^{(0)}\big)\big(\ell^2 + |k|^2 + 2R\ell\big) &= 0 & j \in S.
\end{aligned}
\tag{45}
$$

Recalling the smallness condition (37), we get

$$
\begin{aligned}
h^{(0)} &= 0 & j \notin S, \\
\ell &= \sqrt{R^2 - |k|^2} - R & j \in S,
\end{aligned}
$$

so that, introducing an expansion in $|k|$, we may conveniently write

$$
\begin{aligned}
h^{(0)}(v; k) &= h^{(0,2)} + \mathcal{O}(|k|^3) \ , \\
h^{(0,2)} &= -\frac{3/4|k|^2 v}{2\omega} = \frac{3}{4}\Lambda^{-1}\big(|k|^2 v\big) \qquad j \in S \ .
\end{aligned}
\tag{46}
$$

Order 1: At order 1 we have

$$\Lambda h^{(1)} + \Pi_H L\big(h^{(0)} + k + v\big) - \frac{3}{4}\Big(2\Re\mathfrak{e}\big(h^{(0)}\overline{h}^{(1)}\big)\big(3v + h^{(0)}\big) + \big(|h^{(0)}|^2 + |k|^2\big)h^{(1)}\Big) = 0 \ . \tag{47}$$

We now expand $h^{(1)}$ up to order one in the variables k , namely

$$h^{(1)}(v; k) = h^{(1,0)}(v) + h^{(1,1)}(v; k) + \mathcal{O}(|k|^2) \ ;$$

replacing $h^{(1)}$ in (47) with the latter expansion and recalling that we already obtained that $h^{(0)}$ is zero up to order two in k, we are left with the following equations,

$$
\begin{aligned}
\Lambda h^{(1,0)} + \Pi_H L v = 0 & \implies & h^{(1,0)} = -\Lambda^{-1}\Pi_H L v \ ; \tag{48} \\
\Lambda h^{(1,1)} + \Pi_H L k = 0 & \implies & h^{(1,1)} = -\Lambda^{-1}\Pi_H L k \ . \tag{49}
\end{aligned}
$$

Remark 4.3 *The leading order solution $h^{(1,0)}(v)$ is nothing but the first term $\phi^{(1)}$ obtained as solution of (19) in the first part of the paper, with the exception of the kernel directions α_j that cannot appear in the range part.*

Order 2: The expansion of the range equation at order two in ϵ gives

$$\Lambda h^{(2)} + \Pi_H L h^{(1)} - \frac{3}{4}\bigg(\Big(|h^{(1)}|^2 + 2\Re\mathfrak{e}\big(h^{(0)}\overline{h}^{(2)}\big)\Big)\Big(3v + h^{(0)}\Big)$$
$$+ 2\Re\mathfrak{e}\big(h^{(0)}\overline{h}^{(1)}\big)h^{(1)} + \big(|h^{(0)}|^2 + |k|^2\big)h^{(2)}\bigg) = 0 \ . \tag{50}$$

Here, we need only the leading order of the expansion of $h^{(2)}$ in powers of k, namely

$$h^{(2)} = h^{(2,0)}(v) + \mathcal{O}(|k|) ; \tag{51}$$

and recalling that

$$h^{(0)} = \mathcal{O}(|k|^2) , \qquad h^{(1)} = h^{(1,0)}(v) + \mathcal{O}(|k|) ,$$

from (50) we obtain, at leading order,

$$h^{(2,0)}(v) = -\Lambda^{-1}\left(\Pi_H L h^{(1,0)} - \frac{9}{4}|h^{(1,0)}|^2 v \right) . \tag{52}$$

4.3   Kernel equation

We consider now the kernel equation (K), where we insert the solution $h(v; k, \epsilon)$ of the range equation (R), so that we have to deal with

$$F_K(v; k, \epsilon) := \Pi_K F(v; k, h(v; k, \epsilon), \epsilon) = 0 , \qquad F_K(v; \cdot, \cdot) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 .$$

We recall that, given a solution of such an equation for $\epsilon = 0$, our aim is to look for its continuation for $\epsilon \neq 0$. In particular, taking v as in (37), we require $w(\epsilon)$ as introduced in (36) to solve (39) with the property $\lim_{\epsilon \to 0} w(\epsilon) = 0$ to guarantee that our solution is indeed a continuation from v. The first relevant remark is given by the following

Lemma 4.1 *Let $v \in \mathbb{T}^4$ and $k \in \mathbb{R}^4$, then*

$$F_K(v; k, 0) = 0 . \tag{53}$$

Thus, trivially, all the derivatives in k vanish in zero, i.e., $D_k^{(m)} F_K(v; 0, 0) = 0$.

proof: Since the dependence on ϵ of F_K is given also by the corresponding dependence of $h(v; k, \epsilon)$, we insert the expansion (44) of h in powers of ϵ, and we perform the corresponding expansion of the above function as $F_K(v; k, \epsilon) = F_K(v; k, 0) + \mathcal{O}(\epsilon)$. It turns out that $F_K(v; k, 0) = -\frac{3}{4}\left(|h^{(0)}|^2 + |k|^2 + 2\Re\mathfrak{e}(v\overline{h}^{(0)}) \right)k$, which is identically zero since $h^{(0)}$ solves (45), and the thesis follows. $\square$

From the above Lemma it follows that all v as in (37) represent possible candidates for the continuation, since we have $F_K(v; 0, 0) = 0$. Furthermore, for $\epsilon \neq 0$ the kernel equation takes the form

$$P(v; k, \epsilon) = 0 , \qquad \epsilon P(v; k, \epsilon) := F_K(v; k, \epsilon) , \tag{54}$$

where obviously $P(v; \cdot, \cdot) : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, and in $\epsilon = 0$ one has to read the definition of P as $P(v; k, 0) = \partial_\epsilon F_K(v; k, 0)$. Thus, we get a restriction on the possible bifurcation points, as follows;

Lemma 4.2 *A necessary condition for v^* to be a bifurcation point of the kernel equation is the following*

$$\Pi_K(Lv^*) = 0 . \tag{55}$$

proof: As we said, for $\epsilon \neq 0$, the kernel equation is equivalent to formula (54), so we want a v^* and a $k(\epsilon)$ such that $P(v^*; k(\epsilon), \epsilon) = 0$, but with $\lim_{\epsilon \to 0} k(\epsilon) = 0$. Thus, given the regularity of P, we need

$$P(v^*, 0, 0) = 0 , \tag{56}$$

and using as before the expansion of h and of F_K, one easily gets

$$P(v, 0, 0) = \partial_\epsilon F_K(v, 0, 0) = \Pi_K(Lv) . \tag{57}$$

□

The compatibility condition (55) is equivalent to the variational energy method introduced in [18] (see also Remark 2.1). Using the notation of the phase shifts φ (see (7)), one easily gets as candidate bifurcation points the same ones shown in Section 3.1, i.e. the two isolated points $\varphi \in \{(0, 0, 0), (\pi, \pi, \pi)\}$, the three families (18), and their intersections $\varphi \in \{\pm(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\}$ that we call symmetric vortices.

The next step is of course to test the applicability of the implicit function theorem, i.e. to check whether $dP(v^*; 0, 0)$ has the correct rank[4]. Introducing the linear operator

$$Tk := \Pi_K Lk - \frac{3}{2} \mathfrak{Re} \left(v^* \overline{h}^{(1,0)}(v^*; 0, 0) \right) k , \tag{58}$$

we have the following

Lemma 4.3 *Let v^* satisfy (55), then a sufficient condition for its continuation for $\epsilon \neq 0$ is*

$$\mathrm{rk}(T_1) = 3 \tag{59}$$

where the linear transformation T_1 is given by

$$T_1 \begin{pmatrix} k \\ \epsilon \end{pmatrix} := \left(Tk \ \Big| \ \Pi_K Lh^{(1,0)}(v^*; 0, 0)\epsilon \right) . \tag{60}$$

proof: Looking again at the expansion of F_K, one recognizes in (58) and (60) the expressions of $D_k P(v^*, 0, 0)[k]$ and $\partial_\epsilon P(v^*, 0, 0)\epsilon$, i.e. the differential applied to its increment, so that $T_1 = dP(v^*, 0, 0) : \mathbb{R}^5 \to \mathbb{R}^4$, and condition (59) is the standard one of the implicit function theorem, once we factor out the zero eigenvalue that comes from the gauge invariance. □

If the above Lemma does not apply we enter in the field of the degenerate implicit function theorems where a plethora of possible subcases exist (see, e.g., [24]) without clear and easily unifying statements. In particular if the rank is zero then clearly the function P starts with order two terms which have to be checked explicitly. For the intermediate values of the rank, in general all the possibilities are present, i.e. both to be able to directly prove existence, or non-existence, and the necessity to look at higher order terms to overcome the degeneracy. There is nevertheless the easier situation when the point we are interested in belongs to a

---

[4]It might be useful to remind that, with P being regular in (k, ϵ), so will be the implicit function, if the corresponding theorem applies. Thus, if the dependence of k with respect to ϵ in the implicit function is at least linear in the directions tranversal to the gauge, we will have $\mathrm{rk}(D_k P(v^*, 0, 0)) = 3$, and a regular $k(\epsilon)$. Nevertheless, in principle, we could have a regular implicit function with ϵ being a regular function of one of the k, but $k(\epsilon)$ sublinear and thus nonregular in the origin, as in the trivial lower dimensional example $f(y, \epsilon) = y^2 - \epsilon = 0$.

family, i.e. besides the gauge symmetry coming from the original problem, one has that the v^* solving (55) is not isolated, as for our three families (18). In such a case, a necessary condition for the continuation can be given (see, e.g., Proposition 2.10 of [33]). In the following Lemma we report a formulation suitably adapted to our system. Let us denote by $\mathcal{R}$ and $\mathcal{K}$ respectively the range and the kernel of T, then we have

Lemma 4.4 *Let v^* belong to a 2-dimensional family $v^*(\varphi)$ satisfying (55). Assume that $rk(T) = 2$, $\Pi_K L h^{(1,0)}(v^*(\varphi); 0, 0) \equiv 0$ on the whole family and that $\Pi_K L h^{(2,0)}(v^*; 0, 0) \not\equiv 0$. A necessary condition for the continuation of v^* for $\epsilon \neq 0$ is that*

$$\Pi_K L h^{(2,0)}(v^*; 0, 0) \in \mathcal{R} \ . \tag{61}$$

Remark 4.4 *If the (complex) matrix T is self-adjoint, then the necessary condition (61) is equivalent to*

$$\Pi_K L h^{(2,0)}(v^*; 0, 0) \perp \mathcal{K} \ . \tag{62}$$

We will exploit this equivalence as a nonexistence argument for the continuation of the three families.

proof: We are interested in a solution of $P(v^*; k(\epsilon), \epsilon) = 0$ as a continuation of $P(v^*; 0, 0) = 0$. Given that v^* belong to a 2-dimensional family (one dimension from the gauge, and the other being the proper family dimension), then $T = D_k P(v^*; 0, 0)$ has a zero eigenvalue of multiplicity at least 2. By the hypothesis on its rank, it follows that the multiplicity is exactly 2. The idea is clearly to perform a Lyapunov-Schmidt decomposition, so let us denote by $k_{\mathcal{K}}$ and $k_{\mathcal{R}}$ the kernel component of k and its orthogonal one, respectively, and by $[\,\cdot\,]_{\mathcal{R}}$ and $[\,\cdot\,]_{\mathcal{K}}$ the projections onto $\mathcal{R}$ and its orthogonal space, respectively. Our problem is then written as

$$\begin{aligned} \Big[P(v^*; k_{\mathcal{R}} + k_{\mathcal{K}}, \epsilon)\Big]_{\mathcal{R}} &= 0 \\ \Big[P(v^*; k_{\mathcal{R}} + k_{\mathcal{K}}, \epsilon)\Big]_{\mathcal{K}} &= 0 \end{aligned} \tag{63}$$

It is also useful to expand P with respect to k and ϵ, namely

$$\begin{array}{llll} P(v^*; 0, 0) & + \\ D_k P(v^*; 0, 0)[k] & + & \epsilon \partial_\epsilon P(v^*, 0, 0) \\ D_{kk} P(v^*; 0, 0)[kk] & + & \epsilon D_k \partial_\epsilon P(v^*, 0, 0)[k] & + & \epsilon^2 \partial_\epsilon^2 P(v^*, 0, 0) \\ \quad\vdots & + & \quad\vdots & + & \quad\vdots & + & \ddots \end{array} \tag{64}$$

The n-th column contains homogeneous terms of order ϵ^n and the r-th line gather the homogeneous terms of order r in (k, ϵ). In the present case, $P(v^*; 0, 0) = 0$ since v^* is the solution we are going to continue, $\partial_\epsilon P(v^*, 0, 0) = \Pi_K L h^{(1,0)}(v^*; 0, 0) = 0$ by hypothesis, $D_k P(v^*; 0, 0) = T$ so that $T k_{\mathcal{K}} = 0$. Moreover, since $P(v^*(\varphi); 0, 0) = 0$ on the whole family, expanding $k = k_{\mathcal{K}} + k_{\mathcal{R}}$, each term in the first column vanishes when applied only to $k_{\mathcal{K}}$ (which is characterized exactly by the family directions due the hypothesis $rk(T) = 2$); and the same applies to the second column, since $\partial_\epsilon P(v^*(\varphi), 0, 0) = 0$ on the whole family too. Thus the kernel directions of T are kernel directions for $D_k \partial_\epsilon P(v^*, 0, 0)$ and also for the higher

order terms in the k expansion. We thus have

$$
\begin{array}{ccccccc}
0 & + \\
Tk_\mathcal{R} & + & 0 \\
B(k_\mathcal{R}, k_\mathcal{R}) + B(k_\mathcal{R}, k_\mathcal{K}) & + & \epsilon M k_\mathcal{R} & + & \epsilon^2 \partial_\epsilon^2 P(v^*, 0, 0) \\
\vdots & + & \vdots & + & \vdots & + & \ddots
\end{array}
\tag{65}
$$

where B denotes the bilinear form $D_{kk}P(v^*; 0, 0)$ and $M = D_k \partial_\epsilon P(v^*, 0, 0)$. Recalling that k has to vanish with ϵ, the leading order of the range projection is

$$
Tk_\mathcal{R} + \epsilon^2 \Big[\partial_\epsilon^2 P(v^*, 0, 0) \Big]_\mathcal{R} = 0 \ . \tag{66}
$$

Thus have that $k_\mathcal{R}$ can be solved as a function of $k_\mathcal{K}$ and is of order ϵ^2 plus corrections. Since $\Big[Tk_\mathcal{R} \Big]_\mathcal{K} = 0$, the leading order of the kernel equation of the Lyapunov-Schmidt decomposition is

$$
\Big[B(k_\mathcal{R}, k_\mathcal{K}) + \epsilon^2 \partial_\epsilon^2 P(v^*, 0, 0) \Big]_\mathcal{K} = 0 \ , \tag{67}
$$

where we omitted the terms $\epsilon L k_\mathcal{R}$ and $B(k_\mathcal{R}, k_\mathcal{R})$ respectively of order 3 and 4 in ϵ. But in (67), the term $B(k_\mathcal{R}, k_\mathcal{K})$ is of order higher than 2, since $k_\mathcal{K}$ has to vanish with ϵ: thus the necessary condition $\Big[\partial_\epsilon^2 P(v^*, 0, 0) \Big]_\mathcal{K} = 0$, which is exactly (61) once we recall that $\partial_\epsilon^2 P(v^*, 0, 0) = \Pi_K L h^{(2,0)}(v^*; 0, 0)$. $\square$

As we said before, for the totally degenerate cases, i.e. when $dP(v^*; 0, 0) \equiv 0$, we have to consider higher orders terms of P. We report below the second order ones (dropping the dependence on $(v; k, \epsilon)$ in the various $h^{(m,n)}$),

$$
\begin{aligned}
P_2(v; k, \epsilon) = \ & \epsilon^2 \ \Pi_K L h^{(2,0)} + \\
& + \epsilon \Big(\Pi_K L h^{(1,1)} - \frac{3}{4} \Big(2 \mathfrak{Re} \Big(v \overline{h}^{(2,0)} \Big) + \big| h^{(1,0)} \big|^2 \Big) k \Big) + \\
& + \ \Big(\Pi_K L h^{(0,2)} - \frac{3}{2} \mathfrak{Re} \Big(v \overline{h}^{(1,1)} \Big) k \Big) \ .
\end{aligned}
\tag{68}
$$

4.4   Existence and nonexistence

Considering all the candidate v^* satisfying (55), we analyze here their continuation to solution of the full kernel equation, and thus of our original problem. The first step is to check if it is possible to apply Lemma 4.3. It is tedious yet straightforward to check that

$$
\Pi_K L h^{(1,0)}(v^*; 0, 0) = 0 \ , \tag{69}
$$

i.e. $\partial_\epsilon P(v^*, 0, 0) = 0$. For these as well as the forthcoming calculations one could check also the Appendix. As a consequence, condition (59) of Lemma 4.3 reduces to

$$
\mathrm{rk}(T) = 3 \ . \tag{70}
$$

Existence of the continuation for $(0,0,0)$ and (π,π,π). It is straightforward to verify that for the two isolated candidates, i.e. those with phase shifts $(0,0,0)$ and (π,π,π), the matrices representing T are respectively

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

both with a single zero eigenvalue associated to the gauge direction $e^{i\theta}(1,1,1,1)$, and thus rank equal to 3, so that Lemma 4.3 applies.

Non existence of the continuation for the family $(\varphi, \varphi, \pi - \varphi)$. Let us start with the first of the three families of candidates, the asymmetric vortices, ignoring the straightforward cases $\varphi \in \{0, \pi\}$, and $\varphi = \pm\frac{\pi}{2}$ of the symmetric vortices left for a subsequent analysis. The matrix representing T is

$$T = \frac{\cos(\varphi)}{2} \begin{pmatrix} 0 & e^{-i\varphi} & 0 & e^{-i\varphi} \\ e^{i\varphi} & -2 & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & 0 & e^{i\varphi} \\ e^{i\varphi} & 0 & e^{-i\varphi} & 2 \end{pmatrix}.$$

Such a matrix has a double zero eigenvalue, associated as expected to the gauge and the family directions, respectively $e^{i\theta}(1, e^{i\varphi}, e^{i2\varphi}, -e^{i\varphi})$ and $\partial_\varphi v^*(\varphi) = e^{i(\theta+\varphi)}(0, 1, 2e^{i\varphi}, -1)$. Therefore its rank is equal to 2, so that Lemma 4.3 does not apply. We therefore check for non existence via Lemma 4.4: recalling (69), one has to check only the necessary condition (62), and the calculations show that

$$\Pi_K L h^{(2,0)} = \frac{R \sin(\varphi)}{4\omega^2} i e^{i\theta}(1, 0, 0, e^{i\varphi}),$$

which is indeed non orthogonal to $\partial_\varphi v^*(\varphi)$.

Non existence of the continuation for the family $(\varphi, \pi - \varphi, \pi - \varphi)$. As before, ignoring the special cases $\varphi \in \{0, \pi\}$ and $\varphi = \pm\frac{\pi}{2}$, the matrix representing T is

$$T = \frac{\cos(\varphi)}{2} \begin{pmatrix} -2 & e^{-i\varphi} & 0 & e^{i\varphi} \\ e^{i\varphi} & 0 & e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} & 2 & e^{i\varphi} \\ e^{-i\varphi} & 0 & e^{-i\varphi} & 0 \end{pmatrix}.$$

Such a matrix has a double zero eigenvalue, associated as expected to the gauge and the family directions, respectively $e^{i\theta}(1, e^{i\varphi}, -1, e^{-i\varphi})$ and $\partial_\varphi v^*(\varphi) = i e^{i(\theta+\varphi)}(0, 1, 0, -e^{-i2\varphi})$. Therefore its rank is equal to 2, so that Lemma 4.3 does not apply. Looking for non existence via Lemma 4.4, we again has to check only the necessary condition (62), and the calculations show that

$$\Pi_K L h^{(2,0)} = \frac{R \sin(\varphi)}{4\omega^2} i e^{i\theta}(1, 0, 0, -e^{-i\varphi}),$$

which is indeed non orthogonal to $\partial_\varphi v^*(\varphi)$.

Non existence of the continuation for the family $(\varphi, \pi - \varphi, \varphi)$. As before, ignoring the special cases $\varphi \in \{0, \pi\}$ and $\varphi = \pm\frac{\pi}{2}$, the matrix representing T is

$$T = \frac{\cos(\varphi)}{2} \begin{pmatrix} 0 & e^{-i\varphi} & 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 & e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} & 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 & e^{i\varphi} & 0 \end{pmatrix} .$$

Such a matrix has a double zero eigenvalue, associated as expected to the gauge and the family directions, respectively $e^{i\theta}(1, e^{i\varphi}, -1, -e^{i\varphi})$ and $\partial_\varphi v^*(\varphi) = ie^{i(\theta+\varphi)}(0, 1, 0, -1)$. Therefore its rank is equal to 2, so that Lemma 4.3 does not apply. Looking for non-existence via Lemma 4.4, we again has to check only the necessary condition (62), and the calculations show that

$$\Pi_K L h^{(2,0)} = \frac{R \sin(\varphi)}{4\omega^2} ie^{i\theta}(1, 0, 0, e^{i\varphi}) ,$$

which is indeed non orthogonal to $\partial_\varphi v^*(\varphi)$.

Non existence of the continuation for the vortices $\pm\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$. The two vortices are located at the intersection of the above analyzed three families, and due to this fact the degeneracy is complete. It is thus not surprising that T turns out to be the null matrix, i.e. $dP(v^*; 0, 0) = 0$. The terms we have to investigate are those given in the second order expansion of P, written in (68).

After some calculations, we get that

$$\Pi_K L h^{(2,0)} = \frac{R}{4\omega^2} e^{i\theta}(i, 0, 0, -1) ,$$

$$\Pi_K L h^{(1,1)} = -\frac{k}{4\omega} ,$$

$$\Pi_K L h^{(0,2)} = \frac{3R}{4\omega} e^{i\theta}(k_1^2, ik_2^2, -k_3^2, -ik_4^2) ,$$

and

$$-\frac{3}{4}\left(2\mathfrak{Re}\left(v\overline{h}^{(2,0)}\right) + |h^{(1,0)}|^2\right)k = \frac{k}{4\omega} ,$$

$$-\frac{3}{2}\mathfrak{Re}\left(v\overline{h}^{(1,1)}\right)k = \frac{3R}{8\omega} e^{i\theta}(ik_1(k_2 - k_4), k_2(k_1 - k_3), ik_3(k_2 - k_4), k_4(k_1 - k_3)) .$$

In particular, the mixed derivatives $D_k \partial_\epsilon P(v^*; k, h)[k]$ vanish. Setting $P_2(v^*; k, h) = 0$, one obtains the system

$$\begin{cases} 2k_1^2 + ik_1(k_2 - k_4) = -\dfrac{2i}{3\omega} \\ 2ik_2^2 + k_2(k_1 - k_3) = 0 \\ -2k_3^2 + ik_3(k_2 - k_4) = 0 \\ -2ik_4^2 + k_4(k_1 - k_3) = \dfrac{2}{3\omega} \end{cases} , \tag{71}$$

which is clearly impossible: indeed, keeping in mind that k_j are real variables, it is evident in the first equation that the l.h.t. can be pure imaginary only when $k_1 = 0$, which however

implies $0 = -2i/3\omega$. A similar argument can be developed on the last equation, playing with k_4. Since for the vortices we have $P = P_2 + \text{h.o.t.}$, the nonexistence of solutions for $P_2 = 0$ implies the non existence of solutions for the full equation $P = 0$.

Remark 4.5 *The proof of Theorem 2.1 based on the Lyapunov-Schmidt decomposition, assuming only the continuity of the solution, is more general than the one obtained through the perturbative approach, that requires a C^2 regularity. Conversely, from a practical point of view, the perturbative method allows a direct approach that can be easily and efficiently implemented via algebraic manipulation. This is maybe even more evident in the analogue problem on other dNLS models, like the zigzag one.*

5   Discussion and Conclusions

In the present work, we have revisited the topic of discrete solitons and vortices in lattice models of the discrete nonlinear Schrödinger type. Motivated by the interest in examining 2d asymmetric and super-symmetric configurations, but also by the desire to have a setting that is more analytically tractable, we came up with a 1d toy model. The latter emulates a key feature of the 2d lattice through the inclusion (in a 1d chain) of interactions with the next-to-next-nearest neighbor. To leading order, the persistence conditions for a four-site configuration suggest the possibility of, not only "standard" solutions with phase differences of 0, π, but also of ones involving relative phases of $\pi/2$ (super-symmetric vortex-like states), and of asymmetric ones involving a free parameter φ, strongly reminiscent of those explored in [33] at the discrete level and in [3] at the level of continuum problems with periodic potentials.

To address the question of persistence of such states, we have utilized two different methods; one involved a conserved quantity upon the assumption of $\mathcal{C}^2$ regularity of the solutions with respect to the small coupling ϵ, while the second one required only continuity, using a Lyapunov-Schmidt reduction (and projections to the resulting kernel and range equations) to derive necessary and sufficient conditions for the persistence of the different solution families. The surprising finding is that among all the possible solutions *only* the ones with trivial phase shifts of 0 and π can be found to persist in a four-sites configuration.

This result raises some intriguing questions. A natural one is how general is this result. Admittedly, in the case of every model, the relevant conserved quantity or LS reduction need to be re-performed and the answer has to be given on a case by case basis. However, our experience so far with 1d chains suggests that for generalized dNLS models, such a conservation law should be traceable more generally and the conclusions are likely to be similar for super-symmetric and asymmetric families of vortices. On the other hand, we realize that this topic requires considerable additional examination for a conclusive closure. At the same time, understanding the similarities and differences of the present setting with that of the 2d problem and the supersymmetric vortices therein (as well as how these considerations extend to higher dimensional settings) is a particularly intriguing question. Such topics are currently under consideration and will be reported in future publications.

Acknowledgments:

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