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On the Fučík spectrum of the Laplacian on a torus

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Abstract

We study the Fučík spectrum of the Laplacian on a two-dimensional torus T^2. Exploiting the invariance properties of the domain T^2 with respect to translations we obtain a good description of large parts of the spectrum. In particular, for each eigenvalue of the Laplacian we will find an explicit global curve in the Fučík spectrum which passes through this eigenvalue; these curves are ordered, and we will show that their asymptotic limits are positive. On the other hand, using a topological index based on the mentioned group invariance, we will obtain a variational characterization of global curves in the Fučík spectrum; also these curves emanate from the eigenvalues of the Laplacian, and we will show that they tend asymptotically to zero. Thus, we infer that the variational and the explicit curves cannot coincide globally, and that in fact many curve crossings must occur. We will give a bifurcation result which partially explains these phenomena.

Keywords: Fučík spectrum; Variational characterization; Secondary bifurcation; Geometrical T^2-index

1. Introduction

The notion of Fučík spectrum for the Laplacian was introduced in [13] and [10]: it is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a nontrivial solution of the problem

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$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & \text{in } \Omega, \\ Bu = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, Bu stands for the considered boundary conditions, and $u^\pm(x) = \max\{0, \pm u(x)\}$.

If $\Omega = (0, 1)$ and Bu denotes either Dirichlet, Neumann or periodic boundary conditions, then the Fučík spectrum can be explicitly determined: it consists of global curves in $\mathbb{R}^2$, emanating from the points (λ_k, λ_k), where (λ_k) are the eigenvalues of $-u''$ with boundary conditions Bu. For instance, for periodic boundary conditions, the Fučík spectrum is given by the following curves, arising from the eigenvalues $(\lambda_k, \lambda_k) = (k^2 4\pi^2, k^2 4\pi^2)$:

$$\begin{aligned} &\Sigma_0\colon \quad \{\lambda^+ = \lambda_0 (= 0)\} \cup \{\lambda^- = \lambda_0 (= 0)\}, \\ &\Sigma_k\colon \quad \frac{1}{\sqrt{\lambda^+}} + \frac{1}{\sqrt{\lambda^-}} = \frac{1}{k\pi} = \frac{2}{\sqrt{\lambda_k}}, \quad k = 1, 2, 3, \ldots. \end{aligned} \tag{1.2}$$

In the case of higher dimensions there exist various results, mainly for Dirichlet boundary conditions, but the results are much less complete; it is known that

- Σ is a closed set;
- the lines $\{\lambda^+ = \lambda_0\}$ and $\{\lambda^- = \lambda_0\}$ belong to Σ (we will refer to this part of Σ as the trivial part), and Σ does not contain points with $\lambda^+ < \lambda_0$ or $\lambda^- < \lambda_0$;
- in each square $(\lambda_{k-1}, \lambda_{k+m+1})^2$, where $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$, from the point $(\lambda_k, \lambda_k) \in \Sigma$ arises a continuum composed by a lower and an upper curve, both decreasing (and maybe coincident), see for example [14,16,18,19];
- other points in $\Sigma \cap (\lambda_{k-1}, \lambda_{k+m+1})^2$ can only lie between the two curves (and hence in the open squares $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_{k+m}, \lambda_{k+m+1})^2$ there never are points of Σ).

Something more can be said about the lower part of the continuum Σ_1 arising from (λ_1, λ_1), the "first nontrivial curve in Σ": a variational characterization was found in [11], then developed in [9] and applied to the Neumann case in [1]. In these works it was also proved that for Neumann boundary conditions the asymptotic behavior of this first curve depends on the spatial dimension of the problem: it is asymptotic to the lines $\{\lambda^\pm = \lambda_0(= 0)\}$ for $N > 1$, while it is bounded away from $\{\lambda^\pm = \lambda_0(= 0)\}$ only for $N = 1$.

In a recent paper, Horák and Reichel [15] have combined analytical and numerical methods in the study of the Fučík spectrum for Eq. (1.1) with Dirichlet boundary conditions. They give a new variational characterization for the lower part of the first curve Σ_1, and show numerically the occurrence of secondary bifurcation on this curve and of curve crossings.

In [12], the periodic problem in an interval was considered: taking advantage of the intrinsic S^1-symmetry of the problem, a variational characterization of Fučík curves parting from the eigenvalues of the problem is given. The continuity of the characterization and the complete knowledge of the Fučík spectrum allow in this case to assert that the variational curves actually describe all the curves of the Fučík spectrum.

The difficulties encountered in characterizing the Fučík spectrum for the Laplacian in higher dimensions suggest that it probably has a complicated structure. On the other hand, the knowledge of the Fučík spectrum is important in the study of nonlinear elliptic equations, for example in the study of problems with "jumping nonlinearities," that is nonlinearities which are asymptotically linear at both $+\infty$ and $-\infty$, but with different slopes. If in addition one has also a variational characterization of the Fučík spectrum, then other interesting results can be obtained, cf. [8,9,11,12].

In this paper we consider the Fučík spectrum $\Sigma \subset \mathbb{R}^2$ of the Laplacian on a two-dimensional torus $T^2 = (0,1) \times (0,r)$, that is

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & \text{in } \mathbb{R}^2, \\ u(x,y) = u(x+1,y) = u(x,y+r) & \text{for all } (x,y) \in \mathbb{R}^2. \end{cases} \tag{1.3}$$

An important feature of this problem is its invariance under a compact group action given by

$$g \cdot \big[u(x,y)\big] = u(x+s, y+t), \quad g = (s,t) \in G = [0,1) \times [0,r).$$

More precisely, denoting $F(u) := -\Delta u - (\lambda^+ u^+ - \lambda^- u^-)$, we have that F is equivariant with respect to the action of G, i.e.

$$F(g \cdot u) = g \cdot F(u) \quad \text{for all } g \in G.$$

We note that the eigenvalues of the Laplacian on T^2 are explicit, given by

$$\lambda_{j,k} = j^2 4\pi^2 + k^2 4\pi^2 / r^2, \quad j,k = 0,1,2,\ldots. \tag{1.4}$$

In this paper, using the mentioned invariance properties of the Laplacian, we will be able to characterize large parts of the Fučík spectrum, and we will see that it is remarkably complex:

In our *first result* we prove that from every eigenvalue $(\lambda_{j,k}, \lambda_{j,k})$ there emanates an explicit global curve $\Sigma_{j,k}^{\mathrm{expl}} \subset \Sigma$ belonging to the Fučík spectrum.

As already mentioned, it is useful to have a variational characterization of the Fučík spectrum. In the case of the ODE with periodic boundary conditions, such a characterization was obtained in [12] by using the S^1-index due to V. Benci [2], see also [3]. Recently, an analogous G-index was introduced by W. Marzantowicz [17] for general compact groups.

In our *second result* we will use this G-index, more precisely the T^2-index, to prove that from each eigenvalue $\lambda_{j,k}$ there emanates a global branch of values $\Sigma_{j,k}^{\mathrm{var}} \subset \Sigma$ which can be characterized variationally.

Having proved the existence of an explicit global branch $\Sigma_{j,k}^{\mathrm{expl}}$ and a global variational branch $\Sigma_{j,k}^{\mathrm{var}}$ emanating from the same eigenvalue, one may ask whether these two branches coincide (as is the case in the one-dimensional problem mentioned above). Surprisingly, the answer is *no*.

Indeed, in our *third result* we will show that all the variational eigenvalues tend asymptotically to zero. Since the explicit branches have positive asymptotes, we conclude that many branch crossings occur: in fact, every explicit curve gets crossed by all variational curves starting above it, i.e. by infinitely many curves.

In our *fourth result*, we give a (partial) explanation regarding the separation of the variational curve from the explicit branch: we will show that on the first explicit branch there exist infinitely many points of secondary bifurcation. Thus, it is plausible that the variational branch initially follows the explicit branch (as we will show), and then, at the first branching point on the explicit curve, it will follow the branch of secondary bifurcation which will asymptotically go to zero.

In addition, we prove that all these secondary bifurcations are *symmetry breaking*: the solutions on the explicit curves depend (after a change of variables) on a single variable, and hence have an S^1-symmetry, while the solutions on the secondary bifurcation branch break this symmetry, and hence their orbit is homeomorphic to the full group T^2.

2. Results

In this section we give the precise statements of the results which we will prove in this paper.

In the following we will call H the space $H^1(T^2)$, the standard Sobolev space over the domain $T^2=(0,1)\times(0,r)$, with periodic boundary conditions as stated in (1.3). We will denote by $0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k\leqslant\cdots$ the (ordered) eigenvalues of $-\Delta$ in H, while we continue to write $\lambda_{j,k}$ when we refer to the explicit form of an eigenvalue given in (1.4) (see Section 3.1 for more details about the notations used).

First, in Section 4, we prove that from *every eigenvalue* λ_k there departs a *global explicit curve* belonging to Σ. This is somewhat surprising: in dimension $N\geqslant 2$ explicit global curves are only known in special domains departing from *certain eigenvalues*.

Theorem 2.1. *Let λ_k, $k\geqslant 0$, be an eigenvalue of the Laplacian on H; then*

(i) *if $k=0$, then the lines $\{\lambda^+=\lambda_0\}$ and $\{\lambda^-=\lambda_0\}$ are in Σ;*
(ii) *if $k\geqslant 1$, then the curve*

$$\Sigma_k^{\mathrm{expl}}:\quad \frac{1}{\sqrt{\lambda^+}}+\frac{1}{\sqrt{\lambda^-}}=\frac{2}{\sqrt{\lambda_k}}$$

belongs to Σ.

Remark 2.2. The above curves form an infinite family of curves in Σ, one for each eigenvalue. All these curves are similar, so that they never cross, and all have asymptotes at the value equal to one quarter of the corresponding eigenvalue.

In some regions, namely near the diagonal points (λ_k,λ_k) with λ_k corresponding to a two-dimensional eigenspace, we may guarantee that these are the only points in Σ:

Theorem 2.3. *Let λ_k be an eigenvalue associated to a two-dimensional eigenspace, and let λ_{k-1}, λ_{k+1}, resp., denote the nearest eigenvalues below and above λ_k: then all the points in $\Sigma\cap(\lambda_{k-1},\lambda_{k+1})^2$ are on the curve Σ_k^{expl} given in Theorem* 2.1.

In Section 5 we consider a different approach: we use variational methods and the mentioned T^2-index by W. Marzantowicz [17] to prove

Theorem 2.4. *For every $\mu\geqslant 0$ and $k\geqslant 1$, one can characterize variationally values $\lambda_k(\mu)>0$ with the following properties*:

- $\lambda_k(0)=\lambda_k$;
- $\Sigma_k^{\mathrm{var}}=\{(\lambda_k(\mu)+\mu,\lambda_k(\mu))\colon\ \mu\geqslant 0\}\subset\Sigma$;
- *each $\lambda_k(\mu)$ depends continuously and monotone decreasingly on μ;*
- *if $k>h$ then $\lambda_k(\mu)\geqslant\lambda_h(\mu)$.*

Moreover, $\lambda_1(\mu)$ describes the first nontrivial curve, in the sense that for a given μ, no point in Σ of the form $(\xi+\mu,\xi)$ exists with $\xi\in(0,\lambda_1(\mu))$.

Theorem 2.4 characterizes a family of curves Σ_k^{var} in Σ, each one passing through a diagonal point (λ_k, λ_k).

Remark 2.5. Observe that Theorem 2.3 implies that if λ_k has a two-dimensional eigenspace, then as long as $\Sigma_k^{\mathrm{var}} \subset (\lambda_{k-1}, \lambda_{k+1})^2$, it coincides with Σ_k^{expl}.

However, this is not always the case, as a consequence of the following theorem:

Theorem 2.6. *Let $\lambda_k(\mu)$, $k = 1, 2, \ldots$, denote the variational values obtained in Theorem* 2.4. *Then*

$$\lim_{\mu \to +\infty} \lambda_k(\mu) = 0.$$

This theorem says that all variational curves Σ_k^{var} have asymptotes in 0. Since the explicit curves Σ_k^{expl} have asymptotes in $\lambda_k/4$, it follows that Σ_k^{var} and Σ_k^{expl} cannot coincide globally. In fact, it also implies

Corollary 2.7. *Each explicit branch Σ_k^{expl} gets crossed by all variational curves Σ_j^{var}, with $j > k$.*

Recall that Theorem 2.3 says that in a neighborhood of the eigenvalue λ_1 the variational curve and the explicit branch coincide, while Theorem 2.6 implies that these curves cannot coincide globally. The following theorem gives an explanation for this:

Theorem 2.8. *There exists a sequence of secondary bifurcation points γ_j on Σ_1^{expl}, from which bifurcate global branches $\sigma_{1,j}$, $j \in \mathbb{N}$, consisting of T^2-tori of solutions.*

3. The linear spectrum of $-\Delta$ in $H = H^1(T^2)$

Let the domain T^2 be parameterized as $[0, 1] \times [0, r]$, then it is simple to see that the functions

$$\phi_{j,k}(x, y) = \cos(j2\pi x)\cos\left(k\frac{2\pi}{r}y\right), \quad j, k \geqslant 0,$$

and their translates are eigenfunctions corresponding to the eigenvalues

$$\lambda_{j,k} = j^2 4\pi^2 + k^2\frac{4\pi^2}{r^2}, \quad j, k \geqslant 0.$$

Since the functions above (with their translates) form a complete orthogonal system in $H = H^1(T^2)$, it follows that they are all the possible eigenfunctions. In particular, the first eigenvalue is $\lambda_0 = \lambda_{0,0} = 0$ and its eigenspace is the (one-dimensional) subspace of constant functions. The order of the subsequent eigenvalues depends on the value of r, however "in general" (for example if $r^2 \notin \mathbb{Q}$), the values $\lambda_{j,k}$ will be all distinct, with multiplicity 4 if $j \neq 0 \neq k$, and with multiplicity 2 for $\lambda_{j,0}$, $\lambda_{0,k}$. In the particular cases in which some of the $\lambda_{j,k}$ coincide, the multiplicity will be the sum of the corresponding multiplicities.

3.1. Notation for the spectrum

We will use the following notation: $\lambda_0 = \lambda_{0,0} = 0$ is the first eigenvalue, corresponding to the eigenspace generated by a constant positive function ϕ_0; then, since the following eigenvalues are always of even multiplicity, we will denote by λ_i $(i > 0)$, the nondecreasing sequence of eigenvalues, repeated accordingly to the half of their multiplicity. Moreover, one may always choose the eigenfunctions in each eigenspace in such a way that they are mutually orthogonal and that to each i correspond two orthogonal eigenfunctions (differing just by a translation) which we will denote by ϕ_i^a and ϕ_i^b; we will also maintain the notation with two indices when needed, denoting the corresponding eigenfunctions with the 4 indices a, b, c, d. For example, let $\lambda_i = \lambda_{j,k}$ with $j, k \neq 0$, then one possible choice of the eigenfunctions is

$$\phi_{j,k}^a = \phi_i^a = \frac{2}{\sqrt{r}} \cos(j2\pi x) \cos\left(k\frac{2\pi}{r} y\right), \qquad \phi_{j,k}^b = \phi_i^b = \frac{2}{\sqrt{r}} \sin(j2\pi x) \cos\left(k\frac{2\pi}{r} y\right),$$
$$\phi_{j,k}^c = \phi_{i+1}^a = \frac{2}{\sqrt{r}} \cos(j2\pi x) \sin\left(k\frac{2\pi}{r} y\right), \qquad \phi_{j,k}^d = \phi_{i+1}^b = \frac{2}{\sqrt{r}} \sin(j2\pi x) \sin\left(k\frac{2\pi}{r} y\right).$$

Also, we will assume that these eigenfunctions are chosen with unitary L^2 norm.

4. Explicit curves in the Fučík spectrum (proof of Theorem 2.1)

In this section we will obtain the explicit curves claimed in Theorem 2.1. These curves will always correspond to nontrivial solutions having a one-dimensional behavior, that is they will depend on a unique variable after a suitable reparameterization of T^2.

For point (i) in Theorem 2.1, it is simple to see that the vertical and the horizontal line through $(\lambda_0, \lambda_0) = (0, 0)$ are in Σ, actually ϕ_0 satisfies Eq. (1.3) with $\lambda^+ = 0$ and any λ^-, while $-\phi_0$ satisfies it with $\lambda^- = 0$ and any λ^+.

We now look for other elements in Σ, in order to obtain point (ii) in Theorem 2.1.

Recall that the equation $-u'' = \lambda^+ u^+ - \lambda^- u^-$ has 1-periodic nontrivial solutions for (λ^+, λ^-) satisfying

$$\frac{1}{\sqrt{\lambda^+}} + \frac{1}{\sqrt{\lambda^-}} = \frac{1}{n\pi}, \tag{4.1}$$

corresponding to functions of the form $\sin(\lambda^+ x)$ where positive and $\sin(\lambda^- x)$ where negative, having $2n$ zeros (and thus having minimal period $1/n$).

In the torus $[0, 1] \times [0, r]$, we may use the change of variables

$$\begin{cases} z = jx + \dfrac{ky}{r}, \\ w = jx - \dfrac{ky}{r}, \end{cases} \quad \text{with } k, j \in \mathbb{N} \text{ mutually prime;}$$

observe that the periodicity condition

$$u(x, y) = u(x + 1, y) = u(x, y + r) \quad \text{for any } x, y \in \mathbb{R},$$

becomes

$$U(z,w) = U(z+j, w+j) = U(z+k, w-k) \quad \text{for any } z, w \in \mathbb{R}.$$

We look now for solutions depending on only one of these two variables.

If $u(x,y) = U(z) = U(jx + \frac{ky}{r})$, then $\Delta u = U''(z)(j^2 + \frac{k^2}{r^2})$ and $U(z) = U(z+j) = U(z+k)$: since we chose j, k mutually prime this implies $U(z) = U(z+1)$.

In the same way, if $u(x,y) = U(w) = U(jx - \frac{ky}{r})$, then $\Delta u = U''(j^2 + \frac{k^2}{r^2})$ and $U(w) = U(w+1)$.

We conclude that any solution of the one-dimensional problem

$$\begin{cases} -U'' = \left(j^2 + \dfrac{k^2}{r^2}\right)^{-1} (\lambda^+ U^+ - \lambda^- U^-), \\ U \text{ 1-periodic} \end{cases} \tag{4.2}$$

will correspond to the two solutions of $-\Delta u = \lambda^+ u^- - \lambda^- u^-$ in the torus of the form $u(x,y) = U(jx \pm \frac{ky}{r})$.

These explicit solutions provide the explicit curves in Σ claimed in Theorem 2.1: actually (4.2) has solution for $\frac{\lambda^\pm}{j^2 + \frac{k^2}{r^2}}$ satisfying (4.1), so we obtain, for $n, j, k \in \mathbb{N}$ (j, k mutually prime) the curves

$$\frac{1}{\sqrt{\lambda^+}} + \frac{1}{\sqrt{\lambda^-}} = \frac{1}{n\pi\sqrt{j^2 + \frac{k^2}{r^2}}} = \frac{2}{\sqrt{\lambda_{nj,nk}}}.$$

As claimed in the theorem, each of these curves passes through the diagonal point corresponding to the eigenvalue $\lambda_{nj,nk} = 4n^2\pi^2(j^2 + \frac{k^2}{r^2})$, with eigenfunctions $\cos(2\pi((nj)x \pm \frac{(nk)y}{r}))$: actually, these split as $\cos(2\pi njx)\cos(2\pi\frac{nky}{r}) \mp \sin(2\pi njx)\sin(2\pi\frac{nky}{r})$, that is, they are a linear combination of the four separated variable eigenfunctions $\phi_{nj,nk}^{a,b,c,d}$.

Finally, for j or k equal to zero, one does not need any change of variable to obtain the claim by the same technique.

5. The variational characterization

In this section we will obtain the variational characterization of curves in Σ as claimed in Theorem 2.4.

We will follow the ideas of [12], and for this we need a suitable index for T^2-actions. We will use the index for general compact Lie groups in [17], whose definition and main properties we recall here, with some simplifications due to our setting.

5.1. The geometrical G-index of [17]

Let G be a compact Lie group and A a separable metric G-space (we will denote the action of an element $g \in G$ on $a \in A$ as $g \cdot a$).

First, one defines an index related to the fixed point set $A^G = \{a \in A\colon\ g \cdot a = a\ \forall g \in G\}$:

$$\gamma_e\bigl(A^G\bigr) = \inf\bigl\{k \geqslant 0\colon\ \bigl[A^G, S^n\bigr] = * \text{ for any } n \geqslant k\bigr\},$$

where by $[A^G, S^n]$ we mean the set of all the homotopy classes of maps from A^G to S^n, and by $*$ the class of those homotopic to a constant (if $A^G = \emptyset$ we put $\gamma_e(A^G) = 0$).

Then, one considers all representations V of the group G, such that

$$\begin{aligned}&\textit{there exists a } G\textit{-map } f : A \to V \setminus \{0\} \textit{ where}\\ &\bullet\ \dim_{\mathbb{R}} V^G = \gamma_e\bigl(A^G\bigr), \qquad f\bigl(A^G\bigr) \subseteq V^G \setminus \{0\},\\ &\bullet\ f|_{A^G} \textit{ is not homotopic to the constant function as a map into } V^G \setminus \{0\},\end{aligned} \tag{5.1}$$

and defines

$$\gamma_G^0(A) = \inf\bigl\{\dim_{\mathbb{C}} V_G\colon\ V \text{ as in (5.1)}\bigr\}, \tag{5.2}$$

where V_G is the complement of V^G in V.

We give some useful properties of this index in the following

Proposition 5.1.

1. *If A, B are G-metric spaces and there exists a G-equivariant map $\phi : A \to B$ such that $\phi|_{A^G}$ is a homotopy equivalence between A^G and B^G, then $\gamma_G^0(A) \leqslant \gamma_G^0(B)$ (see point* 2 *in Proposition* 3.7 *of* [17]).
2. *In particular, if $\phi : A \to B$ is a G-equivariant homeomorphism, then $\gamma_G^0(A) = \gamma_G^0(B)$ (see point* 3 *in Proposition* 3.7 *of* [17]).
3. *If V is an orthogonal representation of G and $S(V)$ the unit sphere in V, then $\gamma_G^0(S(V)) = \dim_{\mathbb{C}} V_G$ and $\gamma_e(S(V^G)) = \dim_{\mathbb{R}} V^G$ (see point* 10 *in Proposition* 3.7 *of* [17]).

Remark 5.2. In the case of our interest (see in the next section), A^G will always be homeomorphic to a subset of $\mathbb{R}$, then we have the following two possibilities:

$\gamma_e(A^G) = 0$, if A^G is homeomorphic to a connected subset of $\mathbb{R}$ or if $A^G = \emptyset$.
$\gamma_e(A^G) = 1$, if A^G is homeomorphic to a subset of $\mathbb{R}$ having more than one component.

5.2. The variational characterization (proof of Theorem 2.4)

In our application, we will consider the natural action of the group $G = T^2$ on $H = H^1(T^2)$ given by:

$$\text{if } g = (s,t) \in T^2 \text{ and } u = u(x,y) \in H, \quad \text{then } g \cdot u = u(x+s, y+t). \tag{5.3}$$

Observe that then $H^G = \{const\}$, so it is the one-dimensional eigenspace of the eigenvalue λ_0.

Like in [12] we define, for $k \geqslant 1$, $\mu \geqslant 0$,

$$\lambda_k(\mu) = \inf_{A \in \Gamma_k} \sup_{u \in A} I_\mu(u), \tag{5.4}$$

where

$$I_\mu : H \to \mathbb{R} : u \mapsto I_\mu(u) = \int_{T^2} |\nabla u|^2 - \mu \int_{T^2} |u^+|^2 \tag{5.5}$$

and

$$\Gamma_k = \{A \subseteq \partial B:\ A \text{ closed},\ A\ G\text{-invariant};\ \pm\phi_0 \in A;\ \gamma_G^0(A) \geqslant k\} \tag{5.6}$$

(here B is the L^2 ball in H).

Note that critical points $u \in H$ at level λ of I_μ constrained to ∂B are nontrivial solutions in H of the equation $-\Delta u = (\lambda + \mu)u^+ - \lambda u^-$, implying that $(\lambda + \mu, \lambda) \in \Sigma$.

Observe also that for $A \in \Gamma_k$, one always has $A^G = \pm\phi_0$, so that, by Remark 5.2, $\gamma_e(A^G) = 1$.

Theorem 2.4, except for the last claim which will be proved in Section 6, is a consequence of the following

Proposition 5.3. *For $k \geqslant 1$, $\mu \geqslant 0$, the values $\lambda_k(\mu)$ are well defined, positive, are critical values for I_μ constrained to ∂B, depend continuously and monotone decreasingly on μ, and $\lambda_k(0) = \lambda_k$.*

Finally, if $k > h$ then $\lambda_k(\mu) \geqslant \lambda_h(\mu)$.

Proof. The proof is standard, and goes trough the following points:

(1) $I_\mu|_{\partial B}$ is G-invariant and satisfies the PS condition.

(2) For each $k \geqslant 1$, $\Gamma_k \neq \emptyset$ and $\lambda_k(\mu)$ is well defined.

Actually, let $\widetilde{E}_k = \operatorname{span}\{\phi_0, \phi_1^a, \phi_1^b, \ldots, \phi_k^a, \phi_k^b\}$: this is a representation of G of (real) dimension $2k + 1$ with $\dim_{\mathbb{R}}(\widetilde{E}_k^G) = 1$, then $\gamma_G^0(\widetilde{S}_k) = k$, where $\widetilde{S}_k = \widetilde{E}_k \cap \partial B$ (by point 3 in Proposition 5.1), that is $\widetilde{S}_k \in \Gamma_k$.

Since $I_\mu(-\phi_0) = 0$ and $\widetilde{S}_k \in \Gamma_k$ is compact, we have that the infsup in (5.4) is finite and nonnegative (we refer to Section 6 for the proof that it is in fact strictly positive).

(3) $\lambda_k(\mu)$ is critical.

Indeed, let $A_\varepsilon \in \Gamma_k$ with $\sup_{u \in A_\varepsilon} I_\mu(u) < \lambda_k(\mu) + \varepsilon$: if $\lambda_k(\mu)$ were not critical then, using a G-equivariant deformation lemma in ∂B, $\sup_{u \in \eta(A_\varepsilon)} I_\mu(u) < \lambda_k(\mu) - \varepsilon$, where η is an equivariant homeomorphism satisfying $\eta(\pm\phi_0) = \pm\phi_0$ and then, by point 2 in Proposition 5.1, $\eta(A_\varepsilon) \in \Gamma_k$, which gives a contradiction.

(4) Continuity and monotonicity follow easily by the variational formulation, as in [12].

(5) Monotonicity in the index k is a consequence of the fact that if $k > h$ then $\Gamma_k \subseteq \Gamma_h$.

(6a) $\lambda_k(0) \leqslant \lambda_k$, actually $\widetilde{S}_k \in \Gamma_k$ and $\sup_{u \in \widetilde{S}_k} I_0(u) = \lambda_k$.

(6b) $\lambda_k(0) \geqslant \lambda_k$, actually by Lemma 5.4 below, if $A \in \Gamma_k$, then there exists $\hat{u} \in A \cap \widetilde{E}_{k-1}^\perp$, and then $I_0(\hat{u}) \geqslant \lambda_k$. □

Lemma 5.4. *If $A \in \Gamma_k$, then $A \cap \widetilde{E}_{k-1}^\perp \neq \emptyset$.*

Proof. Write H as $H = H^G \oplus F_1 \oplus F_2$ where F_1 and F_2 are invariant orthogonal subspaces. The lemma is a consequence of the following claim:

Let $A \subseteq \partial B$, A closed, A G-invariant, $\pm\phi_0 \in A$; if $A \cap F_2 = \emptyset$, then $\gamma^0(A) \leqslant \dim_{\mathbb{C}}(F_1)$.

In fact, consider $V = H^G \oplus F_1$ as a representation of G: then $V^G = H^G$ and $V_G = F_1$. Let $Q : H \to H^G \oplus F_1$ be the orthogonal projection:

- since $A \cap F_2 = \emptyset$, we obtain $Q(A) \subseteq (H^G \oplus F_1) \setminus \{0\}$;
- since $A^G = \{\pm\phi_0\}$ one has $\gamma_e(A^G) = 1 = \dim_{\mathbb{R}} H^G$, moreover $Q|_{A^G}$ is the identity, and then it is nontrivial as a map into $H^G \setminus \{0\}$.

This proves that Q is a G-map satisfying the properties in definition (5.1), and then implies, by (5.2), that $\gamma_G^0(A) \leqslant \dim_{\mathbb{C}}(F_1)$. □

6. Proof of Theorem 2.3 and end of proof of Theorem 2.4

In this section we will use some results from [16] and from [9] in order to prove Theorem 2.3 and to conclude the proof of Theorem 2.4.

The required result from [16] is summarized in the following proposition:

Proposition 6.1. *(See [16].) Let*

- *V be the eigenspace associated to the eigenvalue λ_h, W its complement in H and ∂B_V the unitary L^2 sphere in V;*
- *Λ be the open square $(\underline{\lambda}, \bar{\lambda})^2$ where $\underline{\lambda}$ (resp. $\bar{\lambda}$) is the nearest eigenvalue below (resp. above) λ_h;*
- *$I_{\lambda^+,\lambda^-}(u) = \int_\Omega |\nabla u|^2 - \lambda^+ \int_\Omega |u^+|^2 - \lambda^- \int_\Omega |u^-|^2$ be the functional defined in H associated to the Fučík problem with coefficients λ^+, λ^-;*
- *$\theta : V \to W$ be such that $\theta(v)$ is the (unique) solution of the equation $-\Delta w = P_W(g(v+w))$, where $g(t) = \lambda^+ t^+ - \lambda^- t^-$ and P_W is the orthogonal projection onto W.*

Then the curves in Λ given by

$$\Big\{ (\lambda^+, \lambda^-) \in \Lambda\colon \inf_{v \in \partial B_V} I_{\lambda^+,\lambda^-}\big(v + \theta(v)\big) = 0 \Big\},$$
$$\Big\{ (\lambda^+, \lambda^-) \in \Lambda\colon \sup_{v \in \partial B_V} I_{\lambda^+,\lambda^-}\big(v + \theta(v)\big) = 0 \Big\} \tag{6.1}$$

are continua in Σ, and any other point in $\Sigma \cap \Lambda$ must lie between them.

With this result, we may give the

Proof of Theorem 2.3. It is clear by the invariance of problem (1.3) that $\theta(g \cdot v) = g \cdot \theta(v)$ and then $I_{\lambda^+,\lambda^-}(g \cdot v + \theta(g \cdot v)) = I_{\lambda^+,\lambda^-}(g \cdot (v + \theta(v))) = I_{\lambda^+,\lambda^-}(v + \theta(v))$. Since in the hypotheses of Theorem 2.3 the eigenspace V contains a unique orbit and its positive multiples, $I_{\lambda^+,\lambda^-}(v + \theta(v))$ is constant in ∂B_V, which implies that the two curves defined in (6.1) coincide, and then no other point in $\Sigma \cap \Lambda$ may exist. □

In [9], as we commented in Section 1, a variational characterization of the first nontrivial curve was given and many interesting properties were proved for this characterization: we will summarize these results below, and will then establish a connection with our variational characterization for $\lambda_1(\mu)$ given in (5.4); indeed, we will see that the characterized curves coincide, and this will allow to extend to our characterization some of the properties proved there.

The variational characterization given in [9] is

$$\nu_1(\mu) = \inf_{h \in \Lambda_1} \sup_{u \in h([-1,1])} I_\mu(u), \tag{6.2}$$

where

$$\Lambda_1 = \{h : [-1, 1] \to \partial B \text{ continuous, with } h(\pm 1) = \pm\phi_0\},$$

and it was proved that

Proposition 6.2. *(See [9].)*

- *For each $\mu \geqslant 0$ the level $\nu_1(\mu) > 0$ is critical for the restriction to ∂B of the functional I_μ, that is $(\nu_1(\mu) + \mu, \nu_1(\mu)) \in \Sigma$ (Theorem 2.10).*
- $\nu_1(0) = \lambda_1$ (*Corollary* 3.2).
- *No other critical point may lie at level lower than $\nu_1(\mu)$ except for $\pm\phi_0$, which implies that $(\nu_1(\mu) + \mu, \nu_1(\mu))$ is the first nontrivial point of Σ on the parallel to the diagonal through $(\mu, 0)$ (Theorem 3.1).*
- *The curve described is continuous and strictly decreasing (Proposition 4.1).*

The following proposition will extend all these properties to our characterization $\lambda_1(\mu)$, and then imply the last claim in Theorem 2.4 and the strict positivity of $\lambda_k(\mu)$ which was not proved in Section 5.

Proposition 6.3. $\nu_1(\mu) = \lambda_1(\mu)$ *for all* $\mu \geqslant 0$.

Proof. For a given $h \in \Lambda_1$, let $Gh([-1, 1])$ be the union of the orbits of the points in $h([-1, 1])$: by the invariance of I_μ with respect to the action of G we have

$$\nu_1(\mu) = \inf_{h \in \Lambda_1} \sup_{u \in h([-1,1])} I_\mu(u) = \inf_{h \in \Lambda_1} \sup_{u \in Gh([-1,1])} I_\mu(u).$$

However, $Gh([-1, 1]) \in \Gamma_1$, since it is G-invariant, closed, contains $\pm\phi_0$, and also contains a path joining $\pm\phi_0$, so that no continuous function may map it in $H_0 \setminus \{0\}$ if the images of $\pm\phi_0$ are in different components, implying that a representation V as in definition (5.1) necessarily has $\dim_{\mathbb{C}} V_G \geqslant 1$. This implies that $\nu_1(\mu) \geqslant \lambda_1(\mu)$.

For $\mu = 0$ we already know that $\nu_1(0) = \lambda_1(0) = \lambda_1$.

Finally, since both characterizations are continuous and $\nu_1(\mu) > 0$ we have that if they were not the same, then there would exist $\mu > 0$ such that $\nu_1(\mu) > \lambda_1(\mu) > 0$, contradicting the fact that $\nu_1(\mu)$ is the first nontrivial curve. $\square$

Remark 6.4. In Section 1, we also recalled the results in [9] and in [1] about the asymptotic behavior of the characterized curve $\nu_1(\mu)$: we could use Proposition 6.3 to prove that $\lim_{\mu \to +\infty} \lambda_1(\mu) = \lambda_0 = 0$, however, we refer to the next section where we will use our characterization to obtain a more general result.

7. Study of the asymptotical behavior of the variational curves

The aim of this section is to construct suitable sets with a given value of γ_G^0 and then to use them in order to obtain estimates on the critical levels (5.4), which will result in the proof of Theorem 2.6.

For this purpose, we recall some useful definitions (see for example in [4]): the join $Y_1 * \cdots * Y_k$ of k nonempty G-spaces Y_i, is defined as the quotient $E/\sim$ of the space

$$E = \left\{ (a_1 y_1, \ldots, a_k y_k) \text{ with } y_i \in Y_i,\ a_i \in [0,1]\ (i = 1, \ldots, k),\ \sum_{i=1}^{k} a_i = 1 \right\},$$

with respect to the equivalence relation

$$\begin{aligned} \sim: \quad & (a_1 y_1, \ldots, 0 y_j, \ldots, a_k y_k) \sim \left(a_1 y_1, \ldots, 0 y_j', \ldots, a_k y_k\right) \\ & \text{for any } y_j, y_j' \in Y_j\ (j = 1, \ldots, k), \end{aligned}$$

where the G-action on the join is given by $g \cdot (a_1 y_1, \ldots, a_k y_k) = (a_1 g \cdot y_1, \ldots, a_k g \cdot y_k)$. We will denote by $J_k G$ the join $G * G * \cdots * G$, k times.

Also, the join of G-maps $\phi_1 * \cdots * \phi_k$ where $\phi_i : Y_i \to Z_i$ is defined as

$$\phi_1 * \cdots * \phi_k : Y_1 * \cdots * Y_k \to Z_1 * \cdots * Z_k : (a_1 y_1, \ldots, a_k y_k) \mapsto \left(a_1 \phi_1(y_1), \ldots, a_k \phi_k(y_k)\right).$$

We will need the following proposition, which is a consequence of [5]:

Proposition 7.1. *If G is a torus, consider the G-space $S^m * J_k G$ with the trivial action on S^m: then there does not exist a G-map $\phi : S^m * J_k G \to S(V)$ if*

- *V is a representation of G with $\dim_{\mathbb{R}} V^G = m + 1$ and $\dim_{\mathbb{C}} V_G = j < k$,*
- *ϕ induces a homotopy equivalence between the spheres S^m and $S(V^G)$.*

Sketch of the proof. As claimed in the proof of Corollary 2 in [5], there exists a G-map

$$\tau : S(V_G) \to G/K_1 * \cdots * G/K_j,$$

where each K_i is a closed proper subgroup of G; then consider the G-map

$$\tau' := \mathrm{id}_{S(V^G)} * \tau : S(V) \to S\left(V^G\right) * G/K_1 * \cdots * G/K_j;$$

if the G-map ϕ in the claim existed, then the composition

$$\tau' \circ \phi : S^m * J_k G \to S\left(V^G\right) * G/K_1 * \cdots * G/K_j$$

would be a G-map too; however, by Proposition 6 in [5], no such G-map exists if $j < k$. $\square$

7.1. Sets with given γ_G^0

We may now proceed to the construction of a set in the class Γ_k defined in (5.6): let $f_1, \ldots, f_k$ be k functions in $H \setminus H^G$ satisfying the following hypothesis:

(Hf) If $a_i \geqslant 0$, $g_i \in G$ $(i = 1, \ldots, k)$, with $a_j > 0$ for at least one $j \in \{1, \ldots, k\}$, then $\sum_{i=1}^k a_i g_i \cdot f_i \notin H^G$.

Remark 7.2. The above condition (Hf) is not difficult to be achieved, by choosing a "different shape" for the functions $f_1, \ldots, f_k$. In particular, the condition is satisfied if the f_i are nonconstant eigenfunctions taken in k distinct eigenspaces.

Another possible choice of the functions f_i will be described in the next section, in the proof of Theorem 2.6.

We define the set

$$W_{\{f_i\}_{i=1,\ldots,k}} = \left\{ \frac{a_0 s\phi_0 + \sum_{i=1}^k a_i g_i \cdot f_i}{\|a_0 s\phi_0 + \sum_{i=1}^k a_i g_i \cdot f_i\|_{L^2}} : \right.$$
$$\left. s \in \{\pm 1\},\ a_0, a_i \in [0,1],\ g_i \in G\ (i = 1, \ldots, k),\ a_0 + \sum_{i=1}^k a_i = 1 \right\} \tag{7.1}$$

and we prove

Lemma 7.3. *Provided that hypothesis* (Hf) *is satisfied, the set $W_{\{f_i\}_{i=1,\ldots,k}}$ is a compact G-invariant set such that $W_{\{f_i\}_{i=1,\ldots,k}} \in \Gamma_k$ (in fact, $\gamma_G^0(W_{\{f_i\}_{i=1,\ldots,k}}) \geqslant k$).*

Proof. First, one has to check the wellposedness of the definition, that is $\|a_0 s\phi_0 + \sum a_i g_i \cdot f_i\|_{L^2} \neq 0$: this is guaranteed by hypothesis (Hf).

Then, we see that there exists the natural G-map

$$A_k : S^0 * J_k G \to W_{\{f_i\}_{i=1,\ldots,k}} : (a_0 s, a_1 g_1, \ldots, a_k g_k) \mapsto \frac{a_0 s\phi_0 + \sum_{i=1}^k a_i g_i \cdot f_i}{\|a_0 s\phi_0 + \sum_{i=1}^k a_i g_i \cdot f_i\|_{L^2}}. \tag{7.2}$$

Since $S^0 * J_k G$ is a compact set, then $W_{\{f_i\}_{i=1,\ldots,k}}$ is compact too; also, it is a G-invariant subset of ∂B such that $W_{\{f_i\}_{i=1,\ldots,k}} \cap H^G = \{\pm\phi_0\}$, then $W_{\{f_i\}_{i=1,\ldots,k}} \in \Gamma_k$ provided $\gamma_G^0(W_{\{f_i\}_{i=1,\ldots,k}}) \geqslant k$.

Then suppose there exists a G-map $M : W_{\{f_i\}_{i=1,\ldots,k}} \to V \setminus \{0\}$ satisfying

- $\dim_{\mathbb{R}} V^G = \gamma_e(W^G_{\{f_i\}_{i=1,\ldots,k}}) = 1$ and $\dim_{\mathbb{C}} V_G < k$,
- $M(W^G_{\{f_i\}_{i=1,\ldots,k}}) \subseteq V^G \setminus \{0\}$, and $M|_{W^G_{\{f_i\}_{i=1,\ldots,k}}}$ is not homotopic to the constant function as a map into $V^G \setminus \{0\}$;

by composing $M \circ A_k$ and projecting on $S(V)$, one would obtain a G-map $: S^0 * J_k G \to S(V)$ which induces homotopy equivalence between S^0 and $S(V^G)$: this contradicts Proposition 7.1, hence such a map M cannot exist and then $\gamma_G^0(W_{\{f_i\}_{i=1,\ldots,k}}) \geqslant k$. $\square$

7.2. Every variational curve is asymptotic to 0

In order to prove Theorem 2.6, we will first produce a suitable function $f \in H$ and then use it in the construction above in order to build a suitable set with a given index k, which will allow us to estimate the infsup values in (5.4).

Among some other technical conditions, the main property required of this function f is to change sign, but having a suitably small ratio $\frac{\int_{T^2} |\nabla f|^2}{\int_{T^2} f^2}$: we remark that it is impossible to achieve this property in one spatial dimension, but it is always possible in higher dimension.

In particular, f will be defined to be the constant $-h < 0$ outside of a small ball, and with a spike in this ball which reaches the level $H > 0$: using a vector coordinate x in the square $[-1/2, 1/2] \times [-r/2, r/2]$ representing T^2 we set

$$f = \begin{cases} -h, & \text{if } |x| > \eta, \\ H - (h+H)\frac{|x|^\delta}{\eta^\delta}, & \text{if } |x| \leqslant \eta, \end{cases}$$

where h, H are two positive reals, and $\eta, \delta > 0$ are suitably small.

We claim that

Lemma 7.4. *Given $k \in \mathbb{N}$, $\varepsilon > 0$, it is possible to choose $\eta, \delta, H, h > 0$ in such a way that the following requirements are satisfied*:

(1) $\int_{T^2} f = 0$;
(2) $\frac{\int_{T^2} |\nabla f|^2}{\int_{T^2} f^2} < \varepsilon / k^2$;
(3) $\int_{T^2} f(g \cdot f) \geqslant -\frac{1}{k} \int_{T^2} f^2$ *for any* $g \in G$;
(4) $2k\eta < \min\{r; 1\}$.

Proof. First, straightforward computations give (we set $A = r$: the area of T^2)

$$\int_{T^2} f = -Ah + \pi\eta^2(h+H)\frac{\delta}{\delta+2}, \tag{7.3}$$

$$\int_{T^2} f^2 = h^2\left(A - \pi\eta^2\right) + 2\pi\eta^2\left(\frac{H^2}{2} + (h+H)^2\frac{\eta^2}{2\delta+2} - 2H(h+H)\frac{\eta^2}{\delta+2}\right), \tag{7.4}$$

$$\int_{T^2} |\nabla f|^2 = \pi\delta(h+H)^2. \tag{7.5}$$

Also (from now on we will assume $\eta, \delta > 0$ suitably small), given $g \in G$, one has $-h \leqslant f \leqslant H$ and then $f(g \cdot f) \geqslant -hH$; however, $f(g \cdot f) \equiv h^2$ in a region of area at least $A - 2\pi\eta^2$, then we may estimate

$$\int_{T^2} f(g \cdot f) \geqslant h^2\left(A - 2\pi\eta^2\right) - hH2\pi\eta^2. \tag{7.6}$$

Now, we choose the ratio H/h in order to obtain property (1): again a simple computation gives $\frac{H}{h} = \frac{A-\pi\eta^2\frac{\delta}{\delta+2}}{\frac{\pi\eta^2\delta}{\delta+2}}$ and $\frac{h+H}{h} = \frac{A(\delta+2)}{\pi\eta^2\delta}$, which we estimate as

$$\frac{3A}{\pi\eta^2\delta} \geqslant \frac{H+h}{h} \geqslant \frac{H}{h} \geqslant \frac{A}{\pi\eta^2\delta}. \tag{7.7}$$

With (7.7) we estimate (observe that the term in parentheses in (7.4) is larger than $\frac{H^2}{4}$ for small η)

$$\int_{T^2} f^2 \geqslant h^2 A/2 + \pi\eta^2 H^2/2 \geqslant h^2\left(A/2 + \pi\eta^2\frac{A^2}{2(\pi\eta^2\delta)^2}\right) \geqslant h^2\frac{A^2}{2\pi\eta^2\delta^2}, \tag{7.8}$$

$$\int_{T^2} f(g\cdot f) \geqslant \frac{h^2 A}{2} - h^2\frac{3A}{\pi\eta^2\delta}2\pi\eta^2 \geqslant -\frac{6h^2A}{\delta}, \tag{7.9}$$

$$\int_{T^2} |\nabla f|^2 \leqslant h^2\pi\delta\left(\frac{3A}{\pi\eta^2\delta}\right)^2 \leqslant 3h^2\frac{A^2}{\eta^4\delta}. \tag{7.10}$$

Now we analyze the requirements in the lemma, which will be achieved by choosing $\delta, \eta > 0$ small enough: (1) has already been enforced, (4) is straightforward and (3) is equivalent to $-6h^2A \geqslant -\frac{1}{k}\frac{h^2A^2}{2\pi\eta^2\delta}$, then it is possible to be achieved by the choice of η once that δ is small. So at this point we fix the value η so that the above requirements are achieved for suitably small (but still free) $\delta > 0$.

Finally, (2) is equivalent to $\frac{3h^2\frac{A^2}{\eta^4\delta}}{h^2\frac{A^2}{2\pi\eta^2\delta^2}} = 6\frac{\pi\delta}{\eta^2} < \varepsilon/k^2$, then we may set $\delta > 0$ small enough and conclude the proof. $\square$

Now we are in the position to prove the main result of this section:

Proof of Theorem 2.6. Since the function $\lambda_k(\mu)$ is decreasing and positive, we suppose, for sake of contradiction, that $\lambda_k(\mu) \geqslant \varepsilon > 0$ and with these values of ε, k we obtain from Lemma 7.4 a corresponding function f, then we set $f_i = f$ for $i = 1, \ldots, k$ and we consider the set

$$W := W_{\{f_i\}_{i=1,\ldots,k}} \tag{7.11}$$

as defined in (7.1).

First we verify hypothesis (Hf): condition (4) implies that k disks of radius η may not cover the whole of T^2, then for any choice of $g_1, \ldots, g_k \in G$ there exists a point p where $(g_i \cdot f)(p) = -h$ for any $i = 1, \ldots, k$ and then $\sum_{i=1}^k a_i(g_i\cdot f)(p) = -h\sum_{i=1}^k a_i$; however, let $a_j > 0$ and $(g_j \cdot f)(t) = H$, then $\sum_{i=1}^k a_i(g_i\cdot f)(t) \geqslant a_j(H+h) - h\sum_{i=1}^k a_i$. We conclude that $\sum_{i=1}^k a_i g_i \cdot f$ is not a constant function and then (Hf) is satisfied.

Since $W \in \Gamma_k$ by Lemma 7.3, we have, by (5.4),

$$\lambda_k(\mu) \leqslant \max_{u\in W} I_\mu(u); \tag{7.12}$$

let then $v(\mu) \in W$ be such that the maximum in (7.12) is assumed in $v(\mu)$, consider any sequence $\mu_n \to +\infty$ and let $v_n = v(\mu_n)$: up to a subsequence we have

$$v_n \to v_0 \in W, \quad \text{strongly in } H;$$

from (7.12) we get

$$I_{\mu_n}(v_n) \geqslant \lambda_k(\mu_n), \tag{7.13}$$

that is

$$\|\nabla v_n\|_{L^2}^2 - \mu_n \|v_n^+\|_{L^2}^2 \geqslant \lambda_k(\mu_n). \tag{7.14}$$

Taking the limit in (7.14), since we assumed that $\lambda_k(\mu_n) \geqslant \varepsilon$, gives

$$\|\nabla v_0\|_{L^2}^2 \geqslant \varepsilon. \tag{7.15}$$

Writing $v_0 = A_k(a_0 s, a_1 g_1, \dots, a_k g_k)$ in the notation of Eq. (7.2), since $\nabla\phi_0 = 0$ and $\|v_0\|_{L^2} = 1$, this becomes

$$\int_{T^2} \left| \sum_{i=1}^k a_i \nabla(g_i \cdot f) \right|^2 \geqslant \varepsilon \int_{T^2} \left(a_0 s \phi_0 + \sum_{i=1}^k a_i g_i \cdot f \right)^2, \tag{7.16}$$

where $\int_{T^2} (a_0 s \phi_0 + \sum_{i=1}^k a_i g_i \cdot f)^2 = \int_{T^2} (a_0 s \phi_0)^2 + \int_{T^2} (\sum_{i=1}^k a_i g_i \cdot f)^2$ since f is orthogonal to ϕ_0 (condition (1) in Lemma 7.4).

Now, if $a_0 = 1$ (that is, if all the other coefficients are zero), (7.16) gives $0 \geqslant \varepsilon$, contradiction; otherwise we collect as

$$\int_{T^2} \left| \sum_{i=1}^k a_i \nabla(g_i \cdot f) \right|^2 - \varepsilon \int_{T^2} \left(\sum_{i=1}^k a_i g_i \cdot f \right)^2 \geqslant \varepsilon \int_{T^2} (a_0 \phi_0)^2 \geqslant 0, \tag{7.17}$$

and then we get

$$\frac{\int_{T^2} |\sum_{i=1}^k a_i \nabla(g_i \cdot f)|^2}{\int_{T^2} (\sum_{i=1}^k a_i g_i \cdot f)^2} \geqslant \varepsilon. \tag{7.18}$$

Using the estimate $(\sum_{i=1}^k x_i)^2 \leqslant k \sum_{i=1}^k x_i^2$, one obtains

$$\int_{T^2} \left| \sum_{i=1}^k a_i \nabla(g_i \cdot f) \right|^2 \leqslant k \int_{T^2} \sum_{i=1}^k \left| a_i \nabla(g_i \cdot f) \right|^2 = \left[k \sum_{i=1}^k a_i^2 \right] \int_{T^2} |\nabla f|^2. \tag{7.19}$$

Writing $\int_{T^2}(\sum_{i=1}^k a_i g_i \cdot f)^2$ as

$$\int_{T^2}\sum_{i=1}^k (a_i g_i \cdot f)^2 + \int_{T^2}\sum_{\substack{i,j=1\\ i\neq j}}^k a_i a_j (g_i \cdot f)(g_j \cdot f) = \sum_{i=1}^k a_i^2 \int_{T^2} f^2 + \sum_{\substack{i,j=1\\ i\neq j}}^k a_i a_j \int_{T^2} (g_i \cdot f)(g_j \cdot f)$$

and using property (3) in Lemma 7.4 and the estimate $\sum_{i,j=1,\, i\neq j}^k x_i x_j \leqslant (k-1)\sum_{i=1}^k x_i^2$, we conclude

$$\int_{T^2}\left(\sum_{i=1}^k a_i g_i \cdot f\right)^2 \geqslant \left[\sum_{i=1}^k a_i^2 - \frac{k-1}{k}\sum_{i=1}^k a_i^2\right]\int_{T^2} f^2 = \left[\frac{1}{k}\sum_{i=1}^k a_i^2\right]\int_{T^2} f^2. \tag{7.20}$$

Inserting (7.19) and (7.20) into (7.18) and using property (2) in Lemma 7.4, one gets

$$\varepsilon \leqslant \frac{[k\sum_{i=1}^k a_i^2]\int_{T^2}|\nabla f|^2}{[\frac{1}{k}\sum_{i=1}^k a_i^2]\int_{T^2} f^2} = k^2\frac{\int_{T^2}|\nabla f|^2}{\int_{T^2} f^2} < k^2\varepsilon/k^2;$$

this contradiction concludes the proof. □

8. Secondary bifurcation from the first curve

By comparing Theorems 2.6 and 2.3, we deduce that the variational characterizations $\lambda_k(\mu)$ follow initially the explicit curves (at least in the case when λ_k has multiplicity two, to which Theorem 2.3 applies), but eventually separate from them to go asymptotically to 0.

This observation implies that the variational curves Σ_k^{var} described by $\lambda_k(\mu)$ cross every explicit curve Σ_j^{expl} with $1 \leqslant j < k$, and also suggests the presence of bifurcation points along the explicit curves: we investigate in this section the bifurcation from the first explicit curve $\Sigma_{1,0}^{\mathrm{expl}}$; for this we will impose $r < 1$ so that $\Sigma_{1,0}^{\mathrm{expl}}$ is in fact the first explicit curve and is distinct from $\Sigma_{0,1}^{\mathrm{expl}}$.

The result (which implies Theorem 2.8) is in the following

Theorem 8.1. *If $r < 1$, then along the explicit curve $\Sigma_{1,0}^{\mathrm{expl}}$ there exist infinitely many points of bifurcation, in the sense of the following Definition* 8.2.

Definition 8.2. If we define a continuous function $(0,+\infty) \ni \mu \mapsto (\lambda_\mu, u_\mu)$ such that $(\lambda_\mu + \mu, \lambda_\mu) \in \Sigma_{1,0}^{\mathrm{expl}}$ and u_μ is a related solution with $\|u_\mu\|_{L^2} = 1$, then we say that a point $(\lambda_\mu + \mu, \lambda_\mu) \in \Sigma_{1,0}^{\mathrm{expl}}$ is a bifurcation point if there exists a sequence $(\mu_j, \lambda_j, u_j) \to (\mu, \lambda_\mu, u_\mu)$ where $(\lambda_j + \mu_j, \lambda_j) \in \Sigma$, u_j is a corresponding solution with $\|u_j\|_{L^2} = 1$ and $u_j \notin Gu_{\mu_j}$.

Remark 8.3. (a) We remark that results of bifurcations from the curves of the Fučík spectrum were obtained for the Dirichlet problem on a square in [15]. In the same work, the authors found, through numerical approximations, examples where curves arising from different eigenvalues cross each other: this behavior was already known in a rectangular domain with Dirichlet boundary conditions between explicitly calculated curves (see [7]); we remark that our result differs from the cited ones, since it is obtained through analytical tools and since the crossings happen between curves arising from arbitrarily distant eigenvalues (one of which is not explicitly known).

(b) It is interesting to observe that the explicit nontrivial solutions u_μ corresponding to a point along $\Sigma_{1,0}^{\mathrm{expl}}$ only depend on the variable x, and then their orbit is homeomorphic to S^1. Once that we prove that there exists a bifurcation, since the solutions depending on just one variable are known, we obtain that the bifurcating solutions u_j break this symmetry and then their orbit is homeomorphic to T^2.

(c) In Remark 8.7 we show that in fact a result analogous to Theorem 8.1 holds for the curve $\Sigma_{0,1}^{\mathrm{expl}}$ too; also, we suggest that it should be true "in general" for any curve $\Sigma_{h,0}^{\mathrm{expl}}$ or $\Sigma_{0,h}^{\mathrm{expl}}$.

In order to prove Theorem 8.1, we will simplify the problem by getting rid of some of its symmetries: we consider the Neumann problem on a rectangular domain R having dimension $1/2$ and $r/2$ and we call Σ_R the corresponding Fučík spectrum: actually, any solution of such a problem may be extended by two subsequent reflections to a periodic solution in the rectangle of dimension 1 and r, corresponding to a solution on our torus (in general, the converse will not be true, and so we have the inclusion $\Sigma_R \subseteq \Sigma_{T^2}$). Also, it is straightforward that all the explicit curves $\Sigma_{h,0}^{\mathrm{expl}}$ and $\Sigma_{0,h}^{\mathrm{expl}}$ that we found in Section 4 are also in Σ_R.

If we find a bifurcation point for Σ_R, then it will correspond to a bifurcation point for Σ_{T^2} in the sense of Definition 8.2.

In the context of this simpler problem, we may proceed in a similar way as in [15] in order to investigate bifurcation points along the explicit curves: first, we reformulate our problem as the search for solutions of $F(\mu,\lambda,u)=0$ where

$$F:\mathbb{R}^2\times H\to\mathbb{R}\times H:\ F(\mu,\lambda,u)=\left(\|u\|_{L^2}^2-1,\,u-K\left[u+\lambda u+\mu u^+\right]\right), \tag{8.1}$$

where $H=H^1(R)$ and $K:H\to H$ is the inverse of the operator $-\Delta+1$, in the sense that $\langle Ku,v\rangle_H=\int_R\nabla(Ku)\nabla v+\int_R(Ku)v=\int_R uv$.

Since we are interested in bifurcations from a known solution with $(\lambda+\mu,\lambda)\in\Sigma_{1,0}^{\mathrm{expl}}$, we again define a continuous function $(0,+\infty)\ni\mu\mapsto(\lambda_\mu,u_\mu)$ such that $(\lambda_\mu+\mu,\lambda_\mu)\in\Sigma_{1,0}^{\mathrm{expl}}$ and u_μ is a related solution with $\|u_\mu\|_{L^2}=1$. Like in Theorem 12 in [15] one may prove that a sufficient condition in order to have a bifurcation point is that 0 is a simple eigenvalue of the derivative $F_{(\lambda,u)}(\mu,\lambda_\mu,u_\mu)$.

Also, the above condition turns out to be equivalent to the problem of determining when the eigenvalue $\lambda=0$ of the following equation (with Neumann boundary conditions) has multiplicity 2:

$$-\Delta v-\mu\chi_{u_\mu}v-\lambda_\mu v=\lambda v\quad\text{in }(0,1/2)\times(0,r/2), \tag{8.2}$$

where χ_{u_μ} is the characteristic function of the set $\{u_\mu>0\}$; actually, one considers

$$F_{(\lambda,u)}(\mu,\lambda_\mu,u_\mu)[l,v]=\left(\int_R u_\mu v,\,v-K[v+\lambda_\mu v+\mu\chi_{u_\mu>0}v+lu_\mu]\right)=(0,0);$$

by testing the second equation against u_μ one gets $l\|u_\mu\|_{L^2}^2 = 0$, that is, $l = 0$: then the second equation is equivalent to (8.2) with $\lambda = 0$ and the first one rules out the function u_μ which is always an eigenfunction of the zero eigenvalue for (8.2).

The spectrum of problem (8.2) is described in the following

Lemma 8.4. *The eigenvalues λ of* (8.2) *are $\lambda_{i,j}(\mu) = \rho_i(\mu) + k_j$ with corresponding eigenfunctions $v^\mu_{i,j}(x,y) = V^\mu_i(x)W_j(y)$ where $k_j = 4\pi^2 j^2/r^2$ and W_j $(j \geqslant 0)$ are eigenvalues and eigenfunctions of*

$$-W'' = kW \quad \text{in } (0, r/2), \qquad W'(0) = W'(r/2) = 0 \tag{8.3}$$

and $\rho_i(\mu)$, V^μ_i $(i \geqslant 0)$ are eigenvalues and eigenfunctions of

$$-V'' - \mu\chi_{u_\mu}V - \lambda_\mu V = \rho V \quad \text{in } (0, 1/2), \qquad V'(0) = V'(1/2) = 0. \tag{8.4}$$

Moreover, the eigenvalues $\rho_i(\mu)$ and k_j are all simple and one has that $\rho_1(\mu) = 0$, $V^\mu_1(x) = u_\mu(x)$, $\rho_0(\mu) < 0$, $V^\mu_0(x) > 0$ and $\rho_i(\mu) > 0$ for $i \geqslant 2$.

Corollary 8.5.

- *The eigenvalue $\lambda = 0$ of* (8.2) *is double if $k_j = -\rho_0(\mu)$ for some $j \geqslant 1$ and is simple otherwise*; *in fact, $v(x,y) = u_\mu(x)$ is always in the eigenspace.*
- *The number of negative eigenvalues of* (8.2) *is the number of $j \geqslant 0$ such that $k_j < -\rho_0(\mu)$.*

Proof. Using classical arguments (see for example in [6]), one performs separation of variables looking for solutions of (8.2) of the form $v(x,y) = V(x)W(y)$ and obtains Eqs. (8.3) and (8.4) where $\rho = \lambda - k$: since both equations have an unbounded increasing sequence of simple eigenvalues and a complete orthogonal system of eigenfunctions, then the product functions $V^\mu_i(x)W_j(y)$ form a complete orthogonal system and then the analysis of the separated variable equations (8.3)–(8.4) is sufficient for the analysis of Eq. (8.2).

Also, since $u_\mu(x)$ is a solution of (8.4) when $\rho = 0$, and since it changes sign once, we deduce that it has to be the second eigenfunction, that is $\rho_1(\mu) = 0$. As a consequence $\rho_0(\mu) < 0$, and it is known that the corresponding eigenfunction $V^\mu_0(x)$ is positive, while $\rho_i(\mu) > 0$ for $i \geqslant 2$.

The claims in the corollary follow straightforward, since $\lambda = 0$ may be obtained just by the combination $k_0 + \rho_1(\mu) = 0$ and (when possible) $k_j + \rho_0(\mu) = 0$, while $\lambda < 0$ may only come from $k_j + \rho_0(\mu) < 0$. $\square$

In order to prove the existence of the bifurcation points, and then to prove Theorem 8.1, we need to show that the eigenvalue zero of (8.2) is double in infinite points of $\Sigma^{\mathrm{expl}}_{1,0}$, in fact we prove the following

Lemma 8.6. *For $r < 1$, the function $\rho_0(\mu)$ defined in Lemma* 8.4 *crosses all the values $-k_i$: $i \geqslant 1$ as $\mu \to +\infty$.*

Proof. When $\mu = 0$, one has $\lambda_\mu = 4\pi^2$ and the constant function is the principal eigenvalue of (8.4) corresponding to $\rho_0(0) = -4\pi^2$: then we have $\rho_0(0) > -k_1 = -4\pi^2/r^2$.

It is known that the principal eigenvalue may be characterized variationally (see for example in [6]) as

$$\rho_0(\mu) = \inf_{V \in H^1(0,1)\setminus\{0\}} \frac{\int (V')^2 - \mu \int \chi_{u_\mu} V^2}{\int V^2} - \lambda_\mu; \tag{8.5}$$

by using $V = const$ in (8.5) and observing that $u_\mu > 0$ in a set of length $\frac{\pi}{2\sqrt{\mu+\lambda_\mu}}$, one may estimate

$$\rho_0(\mu) \leqslant -\frac{\pi}{2} \frac{\mu}{\sqrt{\mu+\lambda_\mu}} - \lambda_\mu;$$

since λ_μ is bounded between π^2 and $4\pi^2$, this implies that $\lim_{\mu\to+\infty} \rho_0(\mu) = -\infty$ and then, since the function $\rho_0(\mu)$ is continuous, the claim is proved. $\square$

Remark 8.7. We observe that if we consider the curve $\Sigma_{0,1}^{\mathrm{expl}}$ (that is, since we chose $r < 1$, a curve higher than $\Sigma_{1,0}^{\mathrm{expl}}$), then in Eq. (8.2) the function χ_{u_μ} depends on the variable y and we are able to obtain a result analogous to that in Lemma 8.4 and in Corollary 8.5, where the equation for V becomes like (8.3) in $(0, 1/2)$ and that for W like (8.4) in $(0, r/2)$.

A difference arises in Lemma 8.6 since in this case the principal (negative) eigenvalue of the equation for W when $\mu = 0$ is already below $-k_1$, but still it goes to $-\infty$ as $\mu \to +\infty$ and then we get infinite points of bifurcation along $\Sigma_{0,1}^{\mathrm{expl}}$ too.

Finally, if we consider a higher curve $\Sigma_{h,0}^{\mathrm{expl}}$ or $\Sigma_{0,h}^{\mathrm{expl}}$ with $h > 1$, we still are able to find an infinity of points where the zero-eigenspace of (8.2) has dimension higher than 1, but we can no more guarantee that this dimension is exactly 2, since in this case zero is still an eigenvalue of (8.4), but it is the hth one, so that we have more than one negative eigenvalue for (8.4) and then it may happen that $\rho_i(\mu) + k_j = 0$ for more than one couple $(i, j) \neq (h, 0)$; again $\rho_0(\mu)$ will cross infinite values $-k_i$, so we conclude that for such curves there should still be bifurcation points, but also more complicated phenomenons might arise.

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