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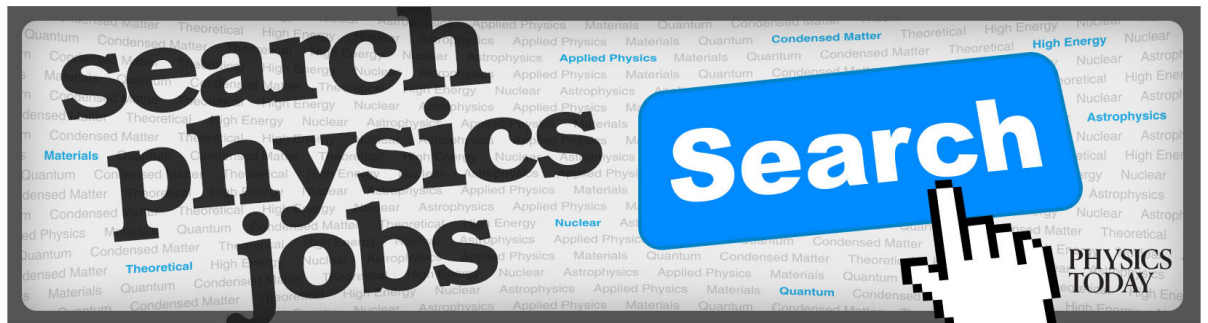
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Random Lie-point symmetries of stochastic differential equations

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We study the invariance of stochastic differential equations under random diffeomorphisms and establish the determining equations for random Lie-point symmetries of stochastic differential equations, both in Ito and in Stratonovich forms. We also discuss relations with previous results in the literature. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4982639>]

I. INTRODUCTION

Symmetry analysis of differential equations is a powerful and by now a rather standard tool in the study of *deterministic* nonlinear problems;^{2–4} its use in the context of *stochastic* differential equations (SDEs)^{5–8} is comparatively much less developed (a partial exception being the versions of Noether's theorem^{2,9} for stochastic variational problems¹⁰).

Albeit some concrete results exist, in particular concerned with *strongly conserved quantities* related to symmetries^{12–14} and to the *linearization* problem,¹⁵ a great deal of activity has been so far devoted to discussing what the suitable definition of symmetry would be in the case of stochastic differential equations (SDEs in the following); here we refer, e.g., to Refs. 12–20. See also Ref. 21 for a review.

As in the case of deterministic differential equations, these works considered smooth vector fields in the space of independent and dependent variables (also called the phase space or manifold); in fact, dealing with smooth vector fields is at the heart of the Sophus Lie approach, in that it allows us to deal with infinitesimal transformations and hence with linearized problems.

On the other hand, when dealing with SDEs one is from the beginning considering an object which is *not* just a smooth vector field; the evolution described by an SDE can be described in terms of random diffeomorphisms, i.e., a diffeomorphism which depends on a random process. Thus in this context, it would be quite natural to consider invariance under the same class of transformations.

Actually, this is exactly what has been done by Arnold and Imkeller in their seminal work on *normal forms* for SDEs¹¹ (see also the book by Arnold⁷); as it is well known, the theory of (Poincaré-Dulac in the case of general dynamical systems or Birkhoff-Gustavsson for Hamiltonian ones) normal forms^{22,23} is intimately connected with symmetry properties,^{4,22,24} so that the success of their approach suggests that one can follow the same path in discussing general symmetry properties of SDEs outside the perturbation approach.

The goal of the present paper is indeed to apply the Arnold-Imkeller approach to the analysis of symmetries of SDEs. We will see that this can be done without difficulties, and explicit *determining equations* for the symmetry of a given SDE can be obtained. The formulation of these is the main contribution of our paper. We will also consider some concrete examples and determine symmetries for them, choosing equations which have physical relevance.

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We will assume that the reader has some familiarity with the basic concepts in the theory of symmetry of (deterministic) differential equations (see, e.g., Refs. 2–4) and also with the basics of stochastic differential equations (see, e.g., Ref. 5); as the former may not be so familiar to readers primarily interested in SDEs, we will very briefly go over basic concepts for standard (that is, deterministic) symmetries of SDEs.

We will first consider the simple class of symmetries, in order to focus on the main point of our contribution, and only later on discuss the most general case. This will make the paper a little longer than it would be going directly to the most general case, but we trust it will help the reader—not to say that simple symmetries seem to be the most useful in applications.

The *plan of the paper* is as follows. After briefly introducing the class of maps to be considered, i.e., random diffeomorphisms (Sec. II), we will first discuss symmetries of SDEs in the Ito form (Secs. III and IV) in increasing generality and examples of these (Sec. V), passing then to discuss the case of Stratonovich SDEs (Sec. VI) and examples of these (Sec. VII). We then discuss the relation between symmetries of an Ito equation and of its Stratonovich counterpart (Sec. VIII); we also discuss the (lack of) simple algebraic structure of symmetry generators for a given Ito equation and the (existence of) the same structure for a given Stratonovich equation (Sec. IX). Finally we draw our brief conclusions in Sec. X.

All the functions and other mathematical objects (manifolds and vector fields) to be considered will be assumed—unless differently stated—to be smooth; by this we will always mean C^∞ .

We will always use (unless differently stated) the Einstein summation convention; we will usually denote partial derivatives with respect to the t and x variables and later on also with respect to the w^k variables, by the shorthand notation,

$$\partial_t := (\partial/\partial t), \quad \partial_i := (\partial/\partial x^i), \quad \widehat{\partial}_k := (\partial/\partial w^k).$$

II. ALLOWED MAPS

A. Random diffeomorphisms

Arnold and Imkeller¹¹ define a *near-identity random map* $h: \Omega \times M \rightarrow M$, with M a smooth manifold and Ω a probability space, as a measurable map such that

- (i) $h(\omega, \cdot) \in C^\infty(M)$;
- (ii) $h(\omega, 0) = 0$;
- (iii) $(Dh)(\omega, 0) = id$.

Property (i) means that we can consider this as a family of diffeomorphisms (i.e., passing to generators, of vector fields) on M , depending on elements ω of the probability space Ω . The dependence is rather arbitrary, i.e., no request of smoothness is present.

We will also refer to the generator of such a map, with a slight abuse of notation, as a *random diffeomorphism*. Note that random diffeomorphisms (as well as random maps) only act in M , i.e., they do not act on the elements of Ω .

In our case, $M = \mathbf{R} \times M_0$, with \mathbf{R} corresponding to the time coordinate, is the phase manifold for the system, while Ω will be the path space for the n -dimensional Wiener process $W(t) = \{w^1(t), \dots, w^n(t)\}$.

Moreover, as suggested by the notation above, we should consider M as a fiber bundle over \mathbf{R} (the fibers being M_0) and h should not act on \mathbf{R} .

In the end, introducing local coordinates x^i on M_0 , we want to consider random diffeomorphisms generated by vector fields of the form

$$X = \tau(x, t; w) \partial_t + \varphi^i(x, t; w) \partial_i. \quad (1)$$

A *time-preserving* random diffeomorphism will be characterized by having $\tau = 0$, while the *fibration-preserving* ones (with reference to the fibration $M \rightarrow \mathbf{R}$) will be characterized by $\tau = \tau(t)$. One should also mention that special care is needed when considering time changes which depend on x (which is itself a stochastic process) and/or w , which are *random time changes*.¹

We will start by considering “simple” (i.e., time-preserving) random symmetries in order to tackle the key problem in the simplest setting; later on (see Section IV) we will consider the general case.

Remark 1. In the literature one considers also transformations directly²⁸ acting on the Wiener processes as well; this is related to the so-called “W-symmetries.”²⁰ We will consider also this class of transformations, in which case one considers diffeomorphisms (in the extended space $(x, t; w)$) generated by vector fields of the form $X = \tau(x, t; w)\partial_t + \varphi^i(x, t; w)\partial_i + h^k(x, t; w)\widehat{\partial}_k$. \odot

B. Maps acting on the time variable

If we consider vector fields which act on the time variable as well, we should take into account that the Wiener processes $w^k(t)$ are affected by a change in t . In the simplest case, this action on t will be just a “global” reparametrization of time, i.e., will not depend on the $x^i(t)$ and $w^k(t)$ variables.²⁹

This situation was discussed, in the context of symmetries for SDEs, in Ref. 19 (see Appendix A there); we give a short account of this discussion here for the sake of completeness.

The probability that a Wiener process $w(t)$ undergoes a change $dw = z$ in the time interval $\theta = dt$ has a density

$$dp(z, \theta) = [1/\sqrt{2\pi\theta}] e^{-z^2/\theta} dz.$$

Under a near-identity map (we will assume $\tau' < 1/\varepsilon$ for all t)

$$t \rightarrow s = t + \varepsilon \tau(t), \tag{2}$$

we have $\theta = dt = [1/(1 + \varepsilon\tau')]ds$; thus the density dp should now be expressed in terms of $\widehat{\theta} = ds = (1 + \varepsilon\tau')dt$.

Instead of going through computations, we note that if we consider $\zeta = \sqrt{1 + \varepsilon\tau'}z$ and the stochastic process

$$\widehat{w}(s) = \sqrt{1 + \varepsilon\tau'} w(s), \tag{3}$$

the probability that $\widehat{w}(s)$ undergoes a change $\zeta = d\widehat{w}$ in the time interval $\theta = ds$ has a density

$$d\widehat{p}(\zeta, \theta) = [1/\sqrt{2\pi\theta}] e^{-\zeta^2/\theta} d\zeta.$$

Thus we conclude that map (2) induces map (3) on the standard Wiener process.

In the case of $\tau = \tau(x, t)$, extra care should be paid: in general this would produce a random non-smooth map, and only those expressed as integrals should be allowed¹ (the integration has a regularizing role); proceeding in a formal way as we will do in the following has indeed in general a formal value, and the actual well-posedness of the considered maps should be verified in each case.

When the considered non-autonomous map is acceptable, with $\tau = \tau(x, t)$ (see, e.g., Theorem 8.20 in the book by Oksendal⁶) or even $\tau = \tau(x, t; w)$, one proceeds in a similar way and obtains exactly the same result (see Section IV). This implies in particular that under (2),

$$dw^k \rightarrow dw^k + \varepsilon \frac{1}{2} \left(\frac{d\tau}{dt} \right) dw^k := dw^k + \varepsilon \delta w^k. \tag{4}$$

III. ITO EQUATIONS—SIMPLE SYMMETRIES

We will consider stochastic differential equations in the Ito form, i.e.,

$$dx^i = f^i(x, t) dt + \sigma^i_k(x, t) dw^k. \tag{5}$$

In the following it will be convenient to use the notation

$$\Delta u := \sum_{k=1}^n \frac{\partial^2 u}{\partial w^k \partial w^k} + \sum_{j,k=1}^n (\sigma \sigma^T)^{jk} \frac{\partial^2 u}{\partial x^j \partial x^k} := \Delta_w u + \Delta_x u. \tag{6}$$

For a function depending only on the (x, t) variables—as in the case of deterministic symmetries—the first term vanishes identically.

A. Deterministic symmetries

We will start by considering *simple symmetries*; in the deterministic case these are generated by vector fields,

$$X = \varphi^i(x, t) \partial_i, \tag{7}$$

while when we look for simple random symmetries we mean those generated by a vector field

$$Y = \varphi^i(x, t; w) \partial_i. \tag{8}$$

The *determining equations for simple deterministic symmetries* of Ito equations (that is, for $\tau = 0$ and $\varphi = \varphi(x, t)$) were determined in Ref. 19 (see also Ref. 20 for extensions) and turned out to be, in the present notation,

$$\begin{cases} \partial_t \varphi^i + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) = -\frac{1}{2} \Delta(\varphi^i), \\ \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) = 0. \end{cases} \tag{9}$$

B. Random symmetries

We will now consider the case of simple random symmetries, i.e., for vector fields of form (8); under this we have $x^i \rightarrow x^i + \varepsilon \varphi^i(x, w)$, and hence

$$\begin{aligned} dx^i &\rightarrow dx^i + \varepsilon d\varphi^i \\ &= dx^i + \varepsilon \left[(\partial_j \varphi^i) dx^j + (\partial_t \varphi^i) dt + (\widehat{\partial}_k \varphi^i) dw^k + \frac{1}{2} (\Delta \varphi^i) dt \right], \\ f^i(x, t) &\rightarrow f^i(x, t) + \varepsilon (\partial_j f^i) \varphi^j, \\ \sigma^i_k(x, t) &\rightarrow \sigma^i_k(x, t) + \varepsilon (\partial_j \sigma^i_k) \varphi^j. \end{aligned}$$

Plugging these into (5), the latter is mapped into a new Ito equation

$$dx^i = [f^i(x, t) + \varepsilon (\delta f)^i(x, t)] dt + \sigma^i_k(x, t) + \varepsilon (\delta \sigma)^i_k(x, t) dw^k, \tag{10}$$

where the variations are given by

$$\begin{aligned} (\delta f)^i(x, t) &= [\varphi^j (\partial_j f^i) - f^j (\partial_j \varphi^i) - \frac{1}{2} (\Delta \varphi^i) - (\partial_t \varphi^i)], \\ (\delta \sigma^i_k)(x, t) &= [\varphi^j (\partial_j \sigma^i_k) - \sigma^j_k (\partial_j \varphi^i) - (\widehat{\partial}_k \varphi^i)]. \end{aligned}$$

Thus the equations remain invariant if and only if, for all i and k ,

$$\begin{cases} (\partial_t \varphi^i) + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) = -\frac{1}{2} (\Delta \varphi^i), \\ (\widehat{\partial}_k \varphi^i) + \sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k) = 0. \end{cases} \tag{11}$$

These are the *determining equations for simple random symmetries*—of form (8)—for the Ito equation (5).

Note that introducing the vector fields

$$X := \partial_t + f^j \partial_j, \quad \widehat{X} := \widehat{\partial}_k + \sigma^j_k \partial_j, \quad Y_\varphi := \varphi^j \partial_j, \quad Z_\varphi := (\Delta \varphi^i) \partial_j, \tag{12}$$

(11) are simply rewritten as

$$[X, Y_\varphi] = -\frac{1}{2} Z_\varphi, \quad [\widehat{X}, Y_\varphi] = 0. \tag{13}$$

Remark 2. The only difference with respect to the determining equations for deterministic symmetries (9) is the presence of the $\partial_k \varphi^i$ term in the second equation; but one should however recall that—despite the formal analogy—the term $\Delta \varphi^i$ does now also include derivatives with respect to the w^k variables (i.e., the $\Delta_w \varphi^i$ term), which are of course absent in (9), where actually $\Delta \varphi^i = \Delta_x \varphi^i$. ◻

IV. ITO EQUATIONS—GENERAL RANDOM SYMMETRIES

So far we have considered (invariance of SDEs under) maps generated by vector fields of the special forms (7) or (8). We want now to remove this limitation and consider general vector fields in the $(x, t; w)$ space, i.e.,

$$Y = \tau(x, t; w) \partial_t + \varphi^i(x, t; w) \partial_i + h^k(x, t; w) \widehat{\partial}_k. \tag{14}$$

Here we started to use, as mentioned above, the shorthand notation

$$\widehat{\partial}_k := \partial / \partial w^k. \tag{15}$$

We also write $X = \tau \partial_t + \varphi^i \partial_i$ for the restriction of Y to the (x, t) space.

Remark 3. Note that in (14) we are considering also the possibility of direct action on the w^k variables (apart from the action induced by a change in time), as in the approach to W-symmetries.²⁰ As already pointed out there, the requirement that the transformed processes $\widehat{w}^k(t) = w^k(t) + \varepsilon h^k(x, t, w)$ are still Wiener processes implies that $\widehat{w}^k = M_\ell^k w^\ell$ with M the orthogonal matrix, and hence that necessarily

$$h^k = B_\ell^k(x, t; w) w^\ell \tag{16}$$

with B a (real) antisymmetric matrix; see Ref. 20 for details. This will be assumed from now on. (Note moreover that if B does not depend on w , then $\Delta(h^k)$ reduces to its “deterministic” part.) \odot

Remark 4. On physical grounds one would be specially interested in the case where the change of time does not depend on either the realization of the stochastic processes $w^k(t)$ or on the spatial coordinates x^i , i.e., on fiber-preserving maps. These will be obtained from the general case by simply setting $\tau = \tau(t)$. It should also be noted that, beside any physical considerations, a (non-trivially) space dependent time change would provide a process which is not absolutely continuous with respect to the original one—thus definitely not of interest in the present context. See also the brief discussion in Sec. II B. \odot

A. The general case

The vector field (14) induces—taking into account also the discussion of Section II B and in particular Eq. (4)—the infinitesimal map

$$\begin{aligned} x^i &\rightarrow x^i + \varepsilon \varphi^i(x, t; w), \\ t &\rightarrow t + \varepsilon \tau(x, t; w), \\ w^k &\rightarrow w^k + \varepsilon h^k(x, t; w) + \varepsilon \delta w^k. \end{aligned}$$

With this, the Ito equation (5) will read

$$dx^i = \left[f^i(x, t) + \varepsilon (\delta f)^i(x, t, w) \right] dt + \left[\sigma_k^i(x, t) + \varepsilon (\delta \sigma)_k^i(x, t, w) \right] dw^k. \tag{17}$$

We do of course aim at obtaining explicit expressions for δf and for $\delta \sigma$.

Working, as always, at the first order in ε , we have

$$\begin{aligned} f^i[x + \varepsilon \varphi, t + \varepsilon \tau] &= f^i(x, t) + \varepsilon \left(\tau \frac{\partial f^i}{\partial t} + \varphi^j \frac{\partial f^i}{\partial x^j} \right) := f^i(x, t) + \varepsilon X[f^i(x, t)], \\ \sigma_k^i[x + \varepsilon \varphi, t + \varepsilon \tau] &= \sigma_k^i(x, t) + \varepsilon \left(\tau \frac{\partial \sigma_k^i}{\partial t} + \varphi^j \frac{\partial \sigma_k^i}{\partial x^j} \right) := \sigma_k^i(x, t) + \varepsilon X[\sigma_k^i(x, t)]. \end{aligned} \tag{18}$$

The differentials $d\varphi^i$, $d\tau$, and dh^k should be computed by the Ito formula; for a generic function $F(x, t; w)$ we have, making use of (5),

$$\begin{aligned} dF &= (\partial_t F)dt + (\partial_j F)dx^j + (\widehat{\partial}_k F)dw^k + \frac{1}{2}(\Delta F)dt \\ &= (\partial_t F)dt + (\partial_j F)[f^j dt + \sigma_k^j dw^k] + (\widehat{\partial}_k F)dw^k + \frac{1}{2}(\Delta F)dt \\ &= \left[(\partial_t F) + f^j (\partial_j F) + \frac{1}{2}(\Delta F) \right] dt + \left[(\widehat{\partial}_k F) + \sigma_k^j (\partial_j F) \right] dw^k \\ &= L[F]dt + Y_k(F)dw^k, \end{aligned} \tag{19}$$

where we have defined the Misawa vector fields Y_μ and the second order operator L by

$$Y_0 := \partial_t + f^j \partial_j, \quad Y_k := \widehat{\partial}_k + \sigma_k^j \partial_j, \quad L := Y_0 + \frac{1}{2}\Delta. \tag{20}$$

The expressions for $d\varphi^i$, $d\tau$, and dh^k are immediately obtained specializing (19),

$$d\varphi^i = L[\varphi^i]dt + Y_k(\varphi^i)dw^k, \quad d\tau = L[\tau]dt + Y_k(\tau)dw^k, \quad dh^k = L[h^k]dt + Y_k(h^k)dw^k. \tag{21}$$

Using (18) and (21) we can rewrite (17) in the form

$$dx^i + \varepsilon d\varphi^i = [f^i + \varepsilon X(f^i)](dt + \varepsilon d\tau) + [\sigma_k^i + \varepsilon X(\sigma_k^i)](dw^k + \varepsilon \delta w^k + \varepsilon dh^k). \tag{22}$$

We like to write this in the form

$$dx^i = f^i(x, t)dt + \sigma_k^i(x, t)dw^k + \varepsilon \delta F^i. \tag{23}$$

Here, setting $\delta w^k = \psi dw^k$ (with $\psi = (1/2)(\partial_t \tau)$, see (4)), we have

$$\begin{aligned} \delta F^i &= -d\varphi^i + f^i d\tau + X(f^i)dt + \sigma_k^i dh^k + \psi \sigma_k^i dw^k + X(\sigma_k^i)dw^k \\ &= f^i [L(\tau)dt + Y_k(\tau)dw^k] - [L(\varphi^i) + Y_k(\varphi^i)] + X(f^i)dt \\ &\quad + X(\sigma_k^i)dw^k + \psi \sigma_k^i dw^k + \sigma_m^i [L(h^m)dt + Y_k(h^m)dw^k] \\ &= [X(f^i) - L(\varphi^i) + f^i L(\tau) + \sigma_k^i L(h^k)] dt \\ &\quad + [X(\sigma_k^i) - Y_k(\varphi^i) + f^i Y_k(\tau) + \sigma_m^i Y_k(h^m)] dw^k. \end{aligned}$$

We thus conclude that the determining equation for (random) symmetries of the Ito equation (5) is

$$\begin{cases} X(f^i) - L(\varphi^i) + f^i L(\tau) + \sigma_k^i L(h^k) = 0, \\ X(\sigma_k^i) - Y_k(\varphi^i) + f^i Y_k(\tau) + \sigma_m^i Y_k(h^m) = -\frac{1}{2}(\partial_t \tau) \sigma_k^i. \end{cases} \tag{24}$$

These can also be finally rewritten, using the explicit form of L and ψ , as

$$\begin{cases} X(f^i) - Y_0(\varphi^i) + f^i Y_0(\tau) + \sigma_k^i Y_0(h^k) = \frac{1}{2} [\Delta(\varphi^i) + f^i \Delta(\tau) + \sigma_k^i \Delta(h^k)], \\ X(\sigma_k^i) - Y_k(\varphi^i) + f^i Y_k(\tau) + \sigma_m^i Y_k(h^m) = -\frac{1}{2}(\partial_t \tau) \sigma_k^i. \end{cases} \tag{25}$$

Several special cases are considered in the following.

Remark 5. This is a system of $n + n^2$ linear equations for the $2n + 1$ unknown functions $\{\tau, \varphi^1, \dots, \varphi^n; h^1, \dots, h^n\}$; these reduce to n or $n + 1$ functions if we consider simple symmetries or at least symmetries not acting directly on the w variables. Thus the system is over-determined for all $n > 1$, and in general we will have no symmetries; even in the case there are symmetries, the equations are not always easy to deal with, despite being linear, due to the dimension. For $n = 1$ the counting of equations and unknown functions would suggest that we always have symmetries, but the solutions could be only local in some of the variables. \odot

Remark 6. The solutions to the determining equations should then be evaluated on the flow of the evolution equation (the Ito SDE); this can lead some functions to get less general or even trivial; see Example 1. \odot

Remark 7. We focused on the definition of random symmetries of an SDE and on the determining equations for these; on the other hand, we have not considered how the symmetries can be used in the study of the SDE. The first use of symmetries for SDEs should be through the introduction of symmetry-adapted coordinates; (see Remark 8 in this respect). A more structured approach, relating simple symmetries to reduction pretty much as for deterministic equations, has been developed by Kozlov¹⁸ in the case of deterministic symmetries of SDEs; we postpone the investigation of the possibility to extend his results to the framework of random symmetries to future work. \odot

B. Special cases

It is interesting to consider some special (simpler) cases.

- (1) In the case of deterministic simple (time preserving) vector fields, i.e., $\varphi = \varphi(x, t)$, $\tau = h = 0$, Equations (24) reduce to (9) seen above.
- (2) Similarly, in the case of simple random symmetries, i.e., $\varphi = \varphi(x, t; w)$, $\tau = h = 0$, we get Equations (11) derived above.
- (3) If we consider the case of deterministic fiber-preserving symmetries, i.e., $\varphi = \varphi(x, t)$, $\tau = \tau(t)$, $h = 0$, Equations (9) reduce to

$$\begin{cases} \partial_t \varphi^i - \partial_t(\tau f^i) + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) = -\frac{1}{2} \Delta \varphi^i, \\ \tau (\partial_t \sigma_k^i) + \varphi^j (\partial_j \sigma_k^i) - \sigma_k^j \partial_j \varphi^i = -\frac{1}{2} (\partial_t \tau) \sigma_k^i. \end{cases} \tag{26}$$

These equations coincide with those derived in Ref. 19, see Theorem 2 there.

- (4) When considering W-symmetries of SDEs²⁰ one considered vector fields with, in the present notation, $\varphi = \varphi(x, t)$, $\tau = \tau(t)$, $h = h(t, w)$. In this case, Equations (9) reduce to

$$\begin{cases} \partial_t \varphi^i - \partial_t(\tau f^i) + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) - \sigma_k^i (\partial_t h^k) = \frac{1}{2} \sigma_k^i \Delta(h^k) - \frac{1}{2} \Delta \varphi^i, \\ \tau (\partial_t \sigma_k^i) + \varphi^j (\partial_j \sigma_k^i) - \sigma_k^j \partial_j \varphi^i + \sigma_m^i (\widehat{\partial}_k h^m) = -\frac{1}{2} (\partial_t \tau) \sigma_k^i. \end{cases} \tag{27}$$

These equations were already obtained in Ref. 20, see the corollary to Proposition 1 there.

- (5) Let us consider the general case with $\varphi = \varphi(x, t; w)$, $\tau = \tau(t, w)$, $h = 0$. Equations (9) are in this case

$$\begin{cases} \partial_t \varphi^i + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) + \tau (\partial_t f^i) - f^i (\partial_t \tau) = \frac{1}{2} [f^i \Delta(\tau) - \Delta(\varphi^i)] , \\ \widehat{\partial}_k \varphi^i + \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) - \tau (\partial_t \sigma_k^i) - f^i (\widehat{\partial}_k \tau) = \frac{1}{2} (\partial_t \tau) \sigma_k^i. \end{cases} \tag{28}$$

- (6) As mentioned above (see Remark 4) we are specially interested in the case where $\tau = \tau(t)$ while φ and h are in the general form (up to the restriction on h discussed in Remark 3). In this case the only simplifications in (9) are, of course, in the terms involving τ and amount to $Y_0(\tau) = (\partial_t \tau)$, $Y_k(\tau) = 0$, and $\Delta(\tau) = 0$. Thus in this case (9) reduce to

$$\begin{cases} X(f^i) - Y_0(\varphi^i) + f^i (\partial_t \tau) + \sigma_k^i Y_0(h^k) = \frac{1}{2} [\Delta(\varphi^i) + \sigma_k^i \Delta(h^k)] , \\ X(\sigma_k^i) - Y_k(\varphi^i) + \sigma_m^i Y_k(h^m) = -\frac{1}{2} (\partial_t \tau) \sigma_k^i. \end{cases} \tag{29}$$

For $h = 0$ (i.e., excluding W-symmetries) these further reduce to

$$\begin{cases} \partial_t \varphi^i + f^j \partial_j \varphi^i - \varphi^j \partial_j f^i - \tau \partial_t f^i - (\partial_t \tau) f^i + \frac{1}{2} \Delta \varphi^i = 0, \\ \widehat{\partial}_k \varphi^i + \sigma_k^j \partial_j \varphi^i - \varphi^j \partial_j \sigma_k^i - \tau \partial_t \sigma_k^i - \frac{1}{2} (\partial_t \tau) \sigma_k^i = 0. \end{cases} \tag{30}$$

- (7) Finally, in applications one is often faced with n -dimensional system, depending on n Wiener processes,

$$dx^i = (M^i_j x^j) dt + \sigma^i_j dw^j(t) ,$$

with M and σ constant matrices. It is also frequent that σ is diagonal.

In this case (for general, i.e., non-necessarily diagonal, σ) the determining equations for simple random symmetries read

$$\begin{cases} (\partial_t \varphi^i) + M^i_q x^q (\partial_j \varphi^i) - M^i_j \varphi^j + \frac{1}{2} \Delta \varphi^i = 0, \\ (\widehat{\partial}_k \varphi^i) + \sigma^j_k (\partial_j \varphi^i) = 0. \end{cases} \tag{31}$$

We start by considering the second set of equations; assuming moreover that σ is diagonal, $\sigma = \text{diag}(\lambda_1, \dots, \lambda_n)$, these yield

$$\varphi^i = \varphi^i(z^1, \dots, z^n; t),$$

where we have defined $z^k := x^k - \lambda_k w^k$ (no sum on k). For functions of this form we get immediately (using again the *ansatz* on σ) that $\Delta\varphi = 0$, and hence the first set of determining equations reads simply as

$$\frac{\partial\varphi^i}{\partial t} + (M^j_k x^k) \left(\frac{\partial\varphi^i}{\partial z^j} \right) = M^i_j \varphi^j. \tag{32}$$

This is equivalent to

$$\frac{\partial\varphi^i}{\partial t} = M^i_j \varphi^j \text{ and } (M^T)_p^j \left(\frac{\partial\varphi^i}{\partial z^j} \right) = 0. \tag{33}$$

The first set of equations implies that

$$\varphi^i(z^1, \dots, z^n; t) = e^{(t-t_0)M} \varphi^i(z^1, \dots, z^n; t_0),$$

while the second one states that $\nabla\varphi^i$ is in the kernel of M^T .

V. EXAMPLES I: SYMMETRIES OF ITO EQUATIONS

A. Simple random symmetries

We start by considering simple random symmetries of Ito equations; in this case the relevant determining equations are (11). We will consider examples which were already studied—for what concerns deterministic symmetries—in Ref. 19, so that comparison with results in the deterministic case is immediate.

Example 1. We start by considering a rather trivial example, i.e., $n = 1$ and

$$dx = \sigma_0 dw(t) \tag{34}$$

with $\sigma_0 \neq 0$. In this case we just have a system of two equations for the single function $\varphi = \varphi(x, t; w)$, and (11) read

$$\begin{aligned} \partial_t \varphi &= -(1/2) \Delta\varphi, \\ \partial_w \varphi &= -\sigma_0 (\partial_x \varphi). \end{aligned}$$

The solution to the second of these is $\varphi = F(z, t)$, where F is an arbitrary (smooth) function of $z := x - \sigma_0 w$ and t . Plugging this into the first equation, we get

$$\partial_t F = -\sigma_0^2 \partial_z^2 F.$$

This is an autonomous linear equation, and it is readily solved (e.g., by considering the Fourier transform of F), showing that there are nontrivial simple random symmetries. Note however, these will grow exponentially fast in time.

It should also be noted that $dz = 0$ in solutions to Equation (34), see Remark 6.

Example 2. We consider another one-dimensional example, i.e.,

$$dx = dt + x dw. \tag{35}$$

This was considered in Ref. 19, where it was shown that it admits no deterministic symmetries. (11) read now

$$\begin{aligned} \partial_t \varphi + \partial_x \varphi &= -(1/2) \Delta\varphi, \\ \varphi - x(\partial_x \varphi) - (\partial_w \varphi) &= 0. \end{aligned}$$

The second equation yields

$$\varphi(x, t, w) = x \psi(z, t), \quad z := x e^{-w}.$$

Inserting this in the first equation, and recalling that now the coefficients of different powers of x must vanish separately, as $\psi = \psi(t, z)$, we get two equations,

$$\psi + z\psi_z = 0, \quad 2\psi_t + 3z\psi_z + 2z^2\psi_{zz} = 0.$$

Solving these, we have $\psi(z, t) = (c_1/z)\exp[-t/2]$, with c_1 the arbitrary constant; and hence we conclude that Equation (35) admits a simple random symmetry,

$$\varphi(x, t, w) = e^{w-t/2}.$$

Example 3. We pass to consider examples in dimension two; we will write the vector indices (in x, w, φ) as lower ones in order to avoid any misunderstanding. The first case we consider is a system related to the work by Finkel,²⁵ i.e.,

$$\begin{aligned} dx_1 &= (a_1/x_1)dt + dw_1, \\ dx_2 &= a_2 dt + dw_2. \end{aligned} \tag{36}$$

Here a_1, a_2 are two non-zero real constants.

The first set of (11) reads in this case

$$\begin{aligned} \frac{a_1}{x_1^2}\varphi_1 + \partial_t\varphi_1 + \frac{a_1}{x_1}\partial_1\varphi_1 + a_2\partial_2\varphi_1 + \frac{1}{2}\Delta\varphi_1 &= 0, \\ \partial_t\varphi_2 + \frac{a_1}{x_1}\partial_1\varphi_2 + a_2\partial_2\varphi_2 + \frac{1}{2}\Delta\varphi_2 &= 0, \end{aligned} \tag{37}$$

while the second set of determining equation (11) reads

$$\frac{\partial\varphi_1}{\partial w_1} + \frac{\partial\varphi_1}{\partial x_1} = 0, \quad \frac{\partial\varphi_1}{\partial w_2} + \frac{\partial\varphi_1}{\partial x_2} = 0, \quad \frac{\partial\varphi_2}{\partial w_1} + \frac{\partial\varphi_2}{\partial x_1} = 0, \quad \frac{\partial\varphi_2}{\partial w_2} + \frac{\partial\varphi_2}{\partial x_2} = 0.$$

These of course imply that, setting $z_k := x_k - w_k$,

$$\varphi_1(x_1, x_2, t; w_1, w_2) = \eta_1(t, z_1, z_2), \quad \varphi_2(x_1, x_2, t; w_1, w_2) = \eta_2(t, z_1, z_2).$$

Plugging these into Equations (37), and again recalling that—as $\eta_i = \eta_i(t, z_1, z_2)$ —the coefficient of different powers of x_1 must vanish separately, the first of those equations enforces

$$\eta_1(t, z_1, z_2) = 0,$$

while in the second we get $\partial\eta_2/\partial z_1 = 0$ and the equation reads

$$\frac{\partial\eta_2}{\partial t} + a_2\frac{\partial\eta_2}{\partial z_2} + \frac{\partial^2\eta_2}{\partial z_2^2} = 0.$$

Again this autonomous linear equation is readily solved, showing that there are simple random symmetries.

Example 4. Finally we will consider another two-dimensional example, which is an Ornstein-Uhlenbeck type process related to the Kramers equation,

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= -K^2 x_2 dt + \sqrt{2K^2} dw(t). \end{aligned} \tag{38}$$

Note that we have here a single Wiener process $w(t)$, and correspondingly we will look for solutions $\varphi^i = \varphi^i(x_1, x_2, t; w)$.

It was shown in Ref. 19 that this system admits some deterministic symmetries; in particular there are the symmetries

$$X_0 = \partial_t, \quad X_1 = \partial_1, \quad X_2 = e^{-K^2 t} [\partial_1 + K^2 \partial_2].$$

As in the previous example, we will start from the second set of equations in (11); for our system these read

$$(\partial\varphi_1/\partial w) = 0, \quad (\partial\varphi_1/\partial x_2) = 0, \quad (\partial\varphi_2/\partial w) = 0, \quad (\partial\varphi_2/\partial x_2) = 0.$$

These of course rule out any possible dependence on w , i.e., show that there is no simple random symmetry.

B. General random symmetries

Example 5. We will consider again the equations of Example 2, i.e.,

$$dx = dt + x dw. \tag{39}$$

We have seen that this does not admit any deterministic symmetry, but it admits one simple random symmetry. We will now check whether this admits some more general random symmetries; in order to keep computations simple, we will restrict to the time-independent case $\tau = 0$ and $\varphi_t = h_t = 0$.

In this case, Equations (24) read

$$\begin{aligned} xh_x + x^2h_{xx} - \varphi_x - \frac{1}{2}(\varphi_{ww} + x^2\varphi_{xx} + xh_{ww}) &= 0, \\ \varphi - \varphi_w - x\varphi_x + xh_w + x^2h_x &= 0. \end{aligned}$$

The second equation requires

$$\varphi(x, w) = x (h(x, w) + \eta(z)), \quad z := w - \log(|x|).$$

Plugging this into the first one we get

$$-\eta(z) + \eta'(z) + \frac{1}{2}\eta'(z) - x, \eta''(z) = h(x, w) - x^2 h_x(x, w).$$

Solutions to these are provided by

$$h(x, w) = e^{1/x} \beta(w) + k, \quad \eta(z) = -k,$$

with k the arbitrary constant and β the arbitrary smooth function.

The random symmetries we obtained in this way are

$$Y = [x e^{1/x} \beta(w)] \partial_x + [e^{1/x} \beta(w) + k] \partial_w. \tag{40}$$

Example 6. We consider the system

$$\begin{aligned} dx_1 &= [1 - (x_1^2 + x_2^2)] x_1 dt + dw_1, \\ dx_2 &= [1 - (x_1^2 + x_2^2)] x_2 dt + dw_2. \end{aligned} \tag{41}$$

This is manifestly covariant under simultaneous rotations in the (x_1, x_2) and the (w_1, w_2) planes.²⁰

In order to simplify (slightly) the computations, we will look for symmetries which are time-preserving and time-independent; that is, we assume $\tau = 0$, $(\partial_t \varphi^i) = 0 = (\partial_t h^k)$. The first set of (24) provides now

$$\begin{aligned} \frac{\partial\varphi_1}{\partial w_1} + \frac{\partial\varphi_1}{\partial x_1} = \frac{\partial h_1}{\partial w_1} + \frac{\partial h_1}{\partial x_1}, \quad \frac{\partial\varphi_1}{\partial w_2} + \frac{\partial\varphi_1}{\partial x_2} = \frac{\partial h_1}{\partial w_2} + \frac{\partial h_1}{\partial x_2}, \\ \frac{\partial\varphi_2}{\partial w_1} + \frac{\partial\varphi_2}{\partial x_1} = \frac{\partial h_2}{\partial w_1} + \frac{\partial h_2}{\partial x_1}, \quad \frac{\partial\varphi_2}{\partial w_2} + \frac{\partial\varphi_2}{\partial x_2} = \frac{\partial h_2}{\partial w_2} + \frac{\partial h_2}{\partial x_2}. \end{aligned}$$

Setting $z_k := x_k - w_k$, these give

$$\begin{aligned} h_1(x_1, x_2, w_1, w_2) &= \varphi_1(x_1, x_2, w_1, w_2) + \rho_1(z_1, z_2), \\ h_2(x_1, x_2, w_1, w_2) &= \varphi_2(x_1, x_2, w_1, w_2) + \rho_2(z_1, z_2), \end{aligned}$$

where ρ_i are arbitrary smooth functions of (z_1, z_2) .

Plugging these into the first set of (24) we obtain two equations involving φ^i and derivatives of ρ^i ,

$$(1 - 3x_1^2 - x_2^2)\varphi_1 - 2x_1x_2\varphi_2 + x_1(1 - x_1^2 - x_2^2)\frac{\partial\rho_1}{\partial z_1} + x_2(1 - x_1^2 - x_2^2)\frac{\partial\rho_1}{\partial z_2} + \frac{\partial^2\rho_1}{\partial z_1^2} + \frac{\partial^2\rho_1}{\partial z_2^2} = 0,$$

$$(1 - 3x_1^2 - x_2^2)\varphi_2 - 2x_1x_2\varphi_1 + x_1(1 - x_1^2 - x_2^2)\frac{\partial\rho_2}{\partial z_1} + x_2(1 - x_1^2 - x_2^2)\frac{\partial\rho_2}{\partial z_2} + \frac{\partial^2\rho_2}{\partial z_1^2} + \frac{\partial^2\rho_2}{\partial z_2^2} = 0.$$

These equations can then be solved for φ^i in terms of ρ^i , yielding some complicated expressions we do not report. This shows that we have random symmetries in correspondence with arbitrary functions $\rho_i(z_1, z_2)$.

When these are linear,

$$\rho_1 = r_{10} + r_{11}z_1 + r_{12}z_2, \quad \rho_2 = r_{20} + r_{21}z_1 + r_{22}z_2,$$

and writing $\chi := [-1 + 3(x_1^2 + x_2^2)]$, the resulting random symmetries are identified by

$$\begin{aligned} \varphi_1 &= (1/\chi) \left[x_1(1 - x_1^2 - 3x_2^2)r_{11} + x_2(1 - x_1^2 - 3x_2^2)r_{12} + 2x_1^2x_2r_{21} + 2x_1x_2^2r_{22} \right], \\ \varphi_2 &= (1/\chi) \left[2x_1^2x_2r_{11} + 2x_1x_2^2r_{12} + x_1(1 - 3x_1^2 - x_2^2)r_{21} + x_2(1 - 3x_1^2 - x_2^2)r_{22} \right], \\ h_1 &= (1/\chi) \left[r_{10}\chi + r_{11}(w_1 + 2x_1^3 - 3w_1(x_1^2 + x_2^2)) \right. \\ &\quad \left. + r_{12}(w_2 + 2x_1^2x_2 - 3w_2(x_1^2 + x_2^2)) + 2x_1^2x_2r_{21} + 2x_1x_2^2r_{22} \right], \\ h_2 &= (1/\chi) \left[r_{20}\chi + 2x_1^2x_2r_{11} + 2x_1x_2^2r_{12} + (w_1 + 2x_1x_2^2 - 3w_1(x_1^2 + x_2^2))r_{21} \right. \\ &\quad \left. + (w_2 + 2x_2^3 - 3w_2(x_1^2 + x_2^2)) \right]. \end{aligned}$$

With the choice

$$r_{10} = 0, \quad r_{20} = 0, \quad r_{11} = 0, \quad r_{12} = 1, \quad r_{21} = -1, \quad r_{22} = 0,$$

we get just simultaneous rotations in the (x_1, x_2) and (w_1, w_2) planes.²⁰

Remark 8. It may be interesting, also in the view of Remark 7, to change coordinates as suggested by the symmetry; we will set $x_1 = \rho \cos(\vartheta)$, $x_2 = \rho \sin(\vartheta)$; and similarly $w_1 = \chi \cos(\lambda)$, $w_2 = \chi \sin(\lambda)$. With these coordinates, Equations (41) read simply

$$\begin{aligned} d\rho &= (1 - \rho^2)\rho dt + \cos(\lambda - \vartheta)d\chi - \chi \sin(\lambda - \vartheta)d\lambda, \\ d\vartheta &= (1/\rho) [\sin(\lambda - \vartheta)d\chi + \chi \cos(\lambda - \vartheta)d\lambda]. \end{aligned}$$

The invariance under simultaneous rotations in the (x_1, x_2) and (w_1, w_2) planes (i.e., simultaneous shifts in ϑ and λ) is now completely explicit. ⊙

VI. SYMMETRIES OF STRATONOVICH EQUATIONS

So far we have considered SDE in the Ito form; as it is well known, in some framework it is convenient to consider instead SDE in the Stratonovich form,

$$dx^i = b^i(x, t) dt + \sigma^i_k(x, t) \circ dw^k(t). \tag{42}$$

In particular, these behave “normally” under the change of coordinates (on the other hand, the Stratonovich integral is not a martingale and its rigorous meaning is not immediate).

Stratonovich equations were also considered by pioneers in the symmetry analysis of differential equations;^{12–14} we are only aware of works dealing with *deterministic* symmetries of Stratonovich equations, so we believe a short discussion of their random symmetries is also of interest; this is given in the present section.

A. Simple symmetries

1. Simple deterministic symmetries

We will first consider (also in order to familiarize with the notation) the action of a *deterministic* vector field (7) on Stratonovich equations.

Under the action of X , Equation (42) is mapped into

$$dx^i + \varepsilon d\varphi^i = (b^i + \varepsilon\varphi^j \partial_j b^i) dt + (\sigma_k^i + \varepsilon\varphi^j \partial_j \sigma_k^i) \circ dw^k. \quad (43)$$

Taking into account (42) and expanding the term $d\varphi$, we have that terms of first order in ε cancel out if and only if

$$(\partial_t \varphi^i) dt + (\partial_j \varphi^i) dx^j = (\varphi^j \partial_j b^i) dt + (\varphi^j \partial_j \sigma_k^i) \circ dw^k. \quad (44)$$

If now we substitute for dx according to (42), this yields

$$(\partial_t \varphi^i) dt + (\partial_j \varphi^i) (b^j dt + \sigma_m^j \circ dw^m) = (\varphi^j \partial_j b^i) dt + (\varphi^j \partial_j \sigma_k^i) \circ dw^k,$$

which is finally rewritten as

$$\left[\partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) \right] dt + \left[\sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) \right] \circ dw^k = 0.$$

The vanishing of this (for all realizations of the Wiener processes w^k) is possible if and only if the $(n + n^2)$ equations

$$\begin{cases} \partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) = 0 & (i = 1, \dots, n), \\ \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) = 0 & (i, k = 1, \dots, n) \end{cases} \quad (45)$$

are satisfied. These are the *determining equations* for the simple deterministic symmetry generators—of form (7)—of the Stratonovich SDE (42).

Remark 9. We can introduce, as suggested by Misawa,¹² the $(n + 1)$ vector fields

$$Z_0 := \partial_t + b^i(x, t) \partial_i; \quad Z_k := \sigma_k^i(x, t) \partial_i. \quad (46)$$

With this notation, the determining equations (45) read simply

$$[X, Z_\mu] = 0 \quad (\mu = 0, 1, \dots, n). \quad (47)$$

2. Simple random symmetries

We will now consider again Equation (42), but now discuss its variation under a vector field of form (8); we will go through the same computation as in Subsection VI A 1.

Under the action of Y , Equation (42) is mapped into

$$dx^i + \varepsilon d\varphi^i = (b^i + \varepsilon\varphi^j \partial_j b^i) dt + (\sigma_k^i + \varepsilon\varphi^j \partial_j \sigma_k^i) \circ dw^k. \quad (48)$$

Taking into account (42) and expanding the term $d\varphi$, we have that terms of first order in ε cancel out if and only if

$$(\partial_t \varphi^i) dt + (\partial_j \varphi^i) dx^j + (\widehat{\partial}_k \varphi^i) \circ dw^k = (\varphi^j \partial_j b^i) dt + (\varphi^j \partial_j \sigma_k^i) \circ dw^k. \quad (49)$$

The last term in the l.h.s. is the only difference with respect to the computation in the deterministic case. Considering x in the solutions to (42), we get

$$(\partial_t \varphi^i) dt + (\partial_j \varphi^i) (b^j dt + \sigma_m^j \circ dw^m) + (\widehat{\partial}_k \varphi^i) \circ dw^k = (\varphi^j \partial_j b^i) dt + (\varphi^j \partial_j \sigma_k^i) \circ dw^k,$$

which is also rewritten as

$$\left[\partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) \right] dt + \left[(\widehat{\partial}_k \varphi^i) + \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) \right] \circ dw^k = 0,$$

and the determining equations for the *random* simple symmetry generators (of form (8) of the Stratonovich SDE (42) are therefore

$$\begin{cases} \partial_t \varphi^i + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) = 0 & (i = 1, \dots, n), \\ \widehat{\partial}_k \varphi^i + \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) = 0 & (i, k = 1, \dots, n). \end{cases} \quad (50)$$

Remark 10. In order to express this in compact terms, it is convenient to modify slightly the definition of the (Misawa) vector fields associated with the SDE; we will now write

$$Y_0 = \partial_t + b^i(x, t) \partial_i = Z_0, \quad Y_k = \widehat{\partial}_k + \sigma_k^i(x, t) \partial_i = \widehat{\partial}_k + Z_k. \quad (51)$$

Then the determining equations (50) read simply

$$[Y, Y_\mu] = 0 \quad (\mu = 0, 1, \dots, n). \quad (52)$$

B. Symmetries acting on the time variable

The computations presented in Section VI A above can be extended to cover the case where the considered transformations act on time as well; in this case the discussion in Section II B should be taken into account.

1. Deterministic symmetries

In the simpler case, i.e., a smooth transformation not depending on the random variables (deterministic symmetries), the role of X in (7) will be taken by

$$Z = \tau(t) \partial_t + \varphi^i(x, t) \partial_i. \quad (53)$$

Note that under this we get $t \rightarrow s = t + \varepsilon \tau(t)$.

As discussed above (see Section II B), $w^k(t)$ is mapped into $\widetilde{w}^k(t) = \sqrt{1 + \varepsilon \tau_t} w^k(t)$, and hence $d\widetilde{w}^k = [1 + \varepsilon(\tau_t/2)]dw^k$. Making use of this fact, and proceeding in the same way as above, we get at first order in ε

$$\begin{aligned} (\partial_j \varphi^i) dx^j + (\partial_t \varphi^i) dt &= [\varphi^j (\partial_j b^i) + \tau (\partial_t b^i) + (\partial_t \tau) b^i] dt + [\varphi^j (\partial_j \sigma_k^i) + \tau (\partial_t \sigma_k^i) \\ &\quad + (1/2)(\partial_t \tau) \sigma_k^i] \circ dw^k. \end{aligned}$$

Substituting now for dx^j according to (42), we get

$$\begin{aligned} [b^j (\partial_j \varphi^i) + (\partial_t \varphi^i) - \varphi^j (\partial_j b^i) - \tau (\partial_t b^i) - (\partial_t \tau) b^i] dt \\ + [\sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) - \tau (\partial_t \sigma_k^i) - (1/2) \tau' \sigma_k^i] \circ dw^k = 0. \end{aligned}$$

The determining equations are therefore

$$\begin{cases} (\partial_t \varphi^i) + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) - \tau (\partial_t b^i) - (\partial_t \tau) b^i = 0, \\ \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) - \tau (\partial_t \sigma_k^i) - (1/2) (\partial_t \tau) \sigma_k^i = 0. \end{cases} \quad (54)$$

2. Random symmetries

When we consider random symmetries, the computations are slightly more complex. Proceeding in the same way as earlier on, we obtain the determining equations in the form

$$\begin{cases} (\partial_t \varphi^i) + b^j (\partial_j \varphi^i) - \varphi^j (\partial_j b^i) - \tau (\partial_t b^i) - (\partial_t \tau) b^i = 0, \\ \widehat{\partial}_k \varphi^i + \sigma_k^j (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma_k^i) - \tau (\partial_t \sigma_k^i) - (1/2) (\partial_t \tau) \sigma_k^i = 0. \end{cases} \quad (55)$$

Remark 11. If we want to express this in terms of commutation properties, we introduce the vector fields

$$Z_0 = \partial_t + b^i(x, t; w) \partial_i, \quad Z_k = \widehat{\partial}_k + \sigma_k^i(x, t; w) \partial_i. \quad (56)$$

Then the determining equations are rewritten as

$$\begin{aligned} [Z_0, Z] &= \tau_t (\partial_t + b^i \partial_i), \\ [Z_k, Z] &= (1/2) \tau_t \sigma_k^i \partial_i. \end{aligned} \quad (57)$$

VII. EXAMPLES II: STRATONOVICH EQUATIONS

Example 7. Let us consider the equation

$$dx = -x dt + x \circ dw.$$

In this case the Misawa vector fields are

$$Y_0 = \partial_t - x \partial_x, \quad Y_1 = \partial_w + x \partial_x.$$

The requirement that $X := \varphi(x, t, w) \partial_x$ commutes with both Y_0 and Y_1 yields

$$\varphi(x, t, w) = e^{-t} \eta(z), \quad z := (e^w/x).$$

Example 8. Let us consider the system

$$\begin{aligned} dx_1 &= -x_2 dt + \alpha x_1 \circ dw_1, \\ dx_2 &= -x_1 dt + \alpha x_2 \circ dw_2. \end{aligned}$$

The Misawa vector fields are now

$$Y_0 = \partial_t - x_2 \partial_1 + x_1 \partial_2, \quad Y_1 = \widehat{\partial}_1 + \alpha r \partial_1, \quad Y_2 = \widehat{\partial}_2 + \alpha r \partial_2.$$

Requiring the vector field

$$X = \varphi^1(x_1, x_2, t, w_1, w_2) \partial_1 + \varphi^2(x_1, x_2, t, w_1, w_2) \partial_2$$

to commute with Y_1 and Y_2 enforces

$$\varphi^1 = x_1 \eta^1(z_1, z_2, t), \quad \varphi^2 = x_2 \eta^2(z_1, z_2, t),$$

where we have defined $z_k := [(aw_k - \log |x_k|)/a]$. Requiring now that X also commutes with Y_0 , we get that actually it must be $\eta^1 = \eta^2 = c$; thus in conclusion the only simple random symmetry of the system under consideration is

$$X = \partial_1 + \partial_2.$$

This is actually, obviously, a simple *deterministic* symmetry.

Example 9. We consider again the equation

$$dx = dt + x dw,$$

as in Example 2. The corresponding Stratonovich equation is

$$dx = \left[1 - \frac{x}{2} \right] dt + x \circ dw.$$

The determining equations (50) for simple random symmetries of this Stratonovich equation read

$$\begin{aligned} \partial_t \varphi + [1 - (x/2)] (\partial_x \varphi) + (1/2) \varphi &= 0, \\ \partial_w \varphi + x (\partial_x \varphi) - \varphi &= 0. \end{aligned}$$

It is immediate to check that these, or more precisely the first of these, do *not* correspond to the equations obtained in Example 2. But, this set of equations does admit as a solution,

$$\varphi(x, t, w) = c_0 \exp[w - t/2],$$

which is just the same solution we found in Example 2.

Example 10. When dealing with symmetries of Stratonovich equations, it is customary to consider the system, first introduced by Misawa,¹²

$$\begin{aligned} dx_1 &= (x_3 - x_2) dt + (x_3 - x_2) \circ dw, \\ dx_2 &= (x_1 - x_3) dt + (x_1 - x_3) \circ dw, \\ dx_3 &= (x_2 - x_1) dt + (x_2 - x_1) \circ dw. \end{aligned}$$

It is well known—and immediately apparent—that this admits the simple symmetry generated by

$$X = (1/2)(x_1^2 + x_2^2 + x_3^2) (\partial_1 + \partial_2 + \partial_3)$$

(and many others, as discussed by Albeverio and Fei¹⁴). Note that this involves only one Wiener process, which will induce a non-symmetric expression for the equivalent Ito system.

Using (58), the equivalent system of Ito equations turns out to be

$$\begin{aligned} dx_1 &= (1/2)(3x_3 - x_2 - 2x_1) dt + (x_3 - x_2) dw, \\ dx_2 &= (x_1 - x_3) dt + (x_1 - x_3) dw, \\ dx_3 &= (x_2 - x_1) dt + (x_2 - x_1) dw. \end{aligned}$$

It is immediate to check that the determining equations (11) are *not* satisfied by X ; more precisely, the second set of (11) is (of course) satisfied, while the first set is not: in fact, we get (for all $i = 1, 2, 3$)

$$\partial_t \varphi^i + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) + \frac{1}{2} (\Delta \varphi^i) = F(x),$$

where we have written

$$F(x) = 2x_1^2 + 3x_2^2 + 3x_3^2 - \left(\frac{5}{2} x_1 x_2 + 3x_2 x_3 + \frac{5}{2} x_1 x_3 \right).$$

VIII. SYMMETRIES OF STRATONOVICH VS. ITO EQUATIONS

As it is well known, there is a correspondence between stochastic differential equations in the Stratonovich and Ito form. In particular, the Stratonovich equation (42) and the Ito equation (5) are equivalent if and only if the coefficients b and f satisfy the relation

$$f^i(x, t) = b^i(x, t) + \frac{1}{2} \left[\frac{\partial}{\partial x^k} (\sigma^T)^i_j(x, t) \right] \sigma^{kj} := b^i(x, t) + \rho^i(x, t). \tag{58}$$

Note that this involves implicitly the metric (to raise the index in σ); as we work in \mathbf{R}^n we do not need to worry about this. Moreover, for σ (and hence also σ^T) a constant matrix, we get $\rho = 0$, i.e., $b^i = f^i$.

Note also that σ is the same in (42) and (5); thus (58) can be used in both directions. In particular, we can immediately use it to rewrite the determining equations for symmetries (of different types) of the Stratonovich equation (42) in terms of the coefficients in the equivalent Ito equation.

One would be tempted to study symmetries of an Ito equation by studying the symmetries of the corresponding Stratonovich one. This would be particularly attractive in the view of the fact that the determining equations (55) for random symmetries of Stratonovich equations are substantially simpler than the determining equations (24) for symmetries of Ito equations; the same holds at the level of determining equations for simple random symmetries, as seen by comparing (50) and (11).

Unfortunately, this way of proceeding would give incorrect results (as also shown by Example 10); this is already clear in the case of simple random—and actually even deterministic—symmetries, so that we will just discuss this case.

In fact, the determining equations (50) for random symmetries of (42) are immediately rewritten in terms of the coefficients f^i of the equivalent Ito equation as ($i, k = 1, \dots, n$)

$$\begin{cases} \partial_t \varphi^i + [f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i)] - [\rho^j (\partial_j \varphi^i) - \varphi^j (\partial_j \rho^i)] = 0, \\ \widehat{\partial}_k \varphi^i + [\sigma^j_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i_k)] = 0, \end{cases} \tag{59}$$

where $\rho^i(x, t)$ is defined in (58).

Note that Equations (59) can be expressed in the compact form (52) of commutation with the vector fields Y_μ defined in (51), except that now the same vector field Y_0 should now better (but equivalently) be defined as

$$Y_0 = \partial_t + [f^i(x, t) + \rho^i(x, t)] \partial_i. \tag{60}$$

The determining equations for deterministic symmetries of (5) are also obtained in the same way from (45), or directly from the above (59) by setting to zero the derivatives with respect to the w^k variables.

However, it is immediate to check that Equations (59) do *not* coincide with the correct Equations (11). The difference is due to

$$\delta^i := \varphi^j (\partial_j \rho^i) - \rho^j (\partial_j \varphi^i) - \frac{1}{2} \Delta \varphi^i \neq 0. \tag{61}$$

Note that this inequality generally holds (for $\sigma_x \neq 0$) even in one dimension, and even for *deterministic* vector fields (i.e., for $\partial_k \varphi^i \equiv 0$).

In fact, in the one-dimensional deterministic case we get

$$\delta = \frac{1}{2} \left[\frac{\partial}{\partial x} (\varphi \sigma \sigma_x - \phi \sigma^2) \right] = \frac{1}{2} \left[\varphi^2 \left(\frac{\partial^2}{\partial x^2} \frac{\sigma^2}{\varphi} \right) \right]. \tag{62}$$

The non-correspondence between the symmetries of an Ito equation and of the corresponding Stratonovich equation might seem rather surprising at first; however, first of all the notion of correspondence between an Ito and the associated Stratonovich equation is not so trivial, as discussed, e.g., in the last chapter of the book by Stroock⁸ (see in particular Sec. 8.1.2 there), and second one should in any case not expect the identity of symmetries, but rather a *correspondence* between the two; thus the difference between the symmetries of the two is not so strange.

On the other hand, an Ito equation and the associated Stratonovich equation do carry the same statistical information. In the view of the discussion and results in Ref. 19, we would expect that there is a correspondence between symmetries of the Fokker-Planck equation (symmetries of scalar Fokker-Planck equations were classified in Ref. 26, see also Ref. 27) which are also symmetries of the Ito equation and symmetries of the equivalent Stratonovich equation. This is indeed the case.

Proposition. Given an Ito equation and the associated Fokker-Planck equation, the symmetries of the latter which are also symmetries of the Ito equation are also symmetries of the associated Stratonovich equation.

Proof. In Ref. 19 it was shown that symmetries of the Fokker-Planck equation

$$\partial_t u + A^{ij} \partial_{ij}^2 u + B^i \partial_i u + C u = 0 \tag{63}$$

with $A = -[(1/2)\sigma\sigma^T]$, $B^i = f^i + 2\partial_j A^{ij}$, and $C = (\partial_i \cdot f^i) + \partial_{ij}^2 A^{ij}$ have to satisfy the system

$$\begin{aligned} \sigma_j^i \Gamma_s^k \delta^{js} + \sigma^k j \Gamma_s^i \delta^{js} &= 0, \\ \Lambda^i + 2 [A^{ik} \partial_k \beta + A^{im} \partial_{mk}^2 \xi^k] &= 0, \\ [\partial_t + f^i \partial_i - A^{ik} \partial_{ik}^2] [\beta + \partial_m \xi^m] &= 0, \end{aligned} \tag{64}$$

where

$$\begin{aligned} \Gamma_j^k &= \sigma_j^m \partial_m \xi^k - \xi^m \partial_m \sigma_j^k - \tau \partial_t \sigma_j^k - \frac{1}{2} \sigma_j^k \partial_t \tau, \\ \Lambda^i &= - [\partial_t (\xi^i - \tau f^i) + \{f, \xi\}^i - A^{mk} \partial_{mk}^2 \xi^i]. \end{aligned} \tag{65}$$

The symmetry of the Fokker-Planck is also a symmetry of the Ito equation if and only if $\Gamma_j^k = 0$ for all j, k , since the condition for a symmetry of the Ito equation is given by

$$\Lambda^i = 0, \quad \Gamma_j^k = 0.$$

On the other hand, the symmetries of the Stratonovich equation are given by (59), which can now be written as

$$\begin{aligned} \partial_t (\xi^i - \tau f^i) + \partial_t (\tau \rho^i) + \{f, \xi\}^i - \{\rho, \xi\}^i &= 0, \\ \Gamma_j^k &= 0, \end{aligned} \tag{66}$$

where $\rho^i = (1/2)(\partial/\partial x^k)[(\sigma^T)_j \sigma^{kj}]$.

Thus it will suffice to show that

$$A^{mk} \partial_{mk}^2 \xi^i + \partial_t(\tau \rho^i) - \{\rho, \xi\}^i = 0. \tag{67}$$

An explicit computation shows that

$$A^{mk} \partial_{mk}^2 \xi^i + \partial_t(\tau \rho^i) - \{\rho, \xi\}^i = \frac{1}{2} \sum_j [\sigma_j, \Gamma_j]^i := \frac{1}{2} \sum_j \sigma_j^k \partial_k \Gamma_j^i + \Gamma_j^k \partial_k \sigma_j^i. \tag{68}$$

This completes the proof. △

IX. ALGEBRAIC STRUCTURE OF SYMMETRIES

It is well known that the Lie-point symmetries (or more precisely the Lie-point symmetry generators) for a given deterministic differential equation form a Lie algebra.²⁻⁴ One may wonder if the same holds in the case of a stochastic differential equation. The question was answered in the negative by Wafo Soh and Mahomed¹⁶ for (first order) Ito equations.

Here we want to discuss this point for Ito equations—confirming of course the result of Ref. 16—and the analogous problem for Stratonovich ones; we will be satisfied with a discussion in the framework of simple symmetries (the negative result in the Ito case will *a fortiori* hold for general symmetries).

A. Ito equations

We will consider vector fields $Y_\xi = \xi^j \partial_j$ and $Y_\eta = \eta^j \partial_j$, which are symmetries for a given SDE; to these we associate the vector fields Z_ξ and Z_η as in (12). By assumption we have

$$[X, Y_\xi] = \frac{1}{2} Z_\xi, [\widehat{X}, Y_\xi] = 0, [X, Y_\eta] = \frac{1}{2} Z_\eta, [\widehat{X}, Y_\eta] = 0. \tag{69}$$

We now want to consider

$$Y_\varphi := [Y_\xi, Y_\eta] \tag{70}$$

and wonder if this is also a symmetry for the same SDE.

It is immediate to check that $[\widehat{X}, Y_\varphi] = 0$, just by the Jacobi identity. As for the first of (13), here the Jacobi identity implies that $[X, Y_\varphi] = [X, [Y_\xi, Y_\eta]] = [Y_\xi, [X, Y_\eta]] - [Y_\eta, [X, Y_\xi]]$; using now (69), this reads

$$[X, Y_\varphi] = \frac{1}{2} \{ [Y_\xi, Z_\eta] - [Y_\eta, Z_\xi] \}. \tag{71}$$

In order to check if the first of (13) is satisfied, we must express Z_φ in terms of $\{Y_\xi, Y_\eta, Z_\varphi, Z_\eta\}$. Using (70) and some simple algebra, we get

$$\Delta \varphi^i = (\Delta \xi^j) \partial_j \eta^i - (\Delta \eta^j) \partial_j \xi^i + \xi^j \partial_j (\Delta \eta^i) - \eta^j \partial_j (\Delta \xi^i) + 2 W_{(\xi, \eta)},$$

where we have defined

$$W_{(\xi, \eta)} := [(\widehat{\partial}_k \xi^j) \partial_j (\widehat{\partial}_k \eta^i) + (\partial_k \xi^j) \partial_j (\partial_k \eta^i) - (\widehat{\partial}_k \eta^j) \partial_j (\widehat{\partial}_k \xi^i) - (\partial_k \eta^j) \partial_j (\partial_k \xi^i)] \partial_i.$$

This computation shows that

$$Z_\varphi = [Y_\xi, Z_\eta] - [Y_\eta, Z_\xi] + 2 W_{(\xi, \eta)}. \tag{72}$$

Combining this with (71), the first of (13) reads simply

$$W_{(\xi, \eta)} = 0. \tag{73}$$

But we have already used the condition that Y_ξ, Y_η are symmetries of the SDE identified by X , hence (73) has no reason to be true in general.

Example 11. In order to check and substantiate this claim, we can consider Example 5, i.e., the Ito equation (39); in that case we have seen that symmetry generators are written in form (40). Let us consider two different symmetries Y_i ($i = 1, 2$) given by

$$Y_i = [x e^{1/x} \beta_i(w)] \partial_x + [e^{1/x} \beta_i(w) + k_i] \partial_w.$$

By explicit computation, we have

$$[Y_1, Y_2] = -e^{1/x} \left((k_2 + e^{1/x} \beta_2) \beta_1' - (k_1 + e^{1/x} \beta_1) \beta_2' \right) (x \partial_x + \partial_w).$$

This is (in general) *not* in form (40), and hence it is (in general) *not* a symmetry for Equation (39).

B. Stratonovich equations

The situation is quite different for equations in the Stratonovich form. This is rather evident comparing (11) and (50) (with our previous computation in hindsight).

Lemma. *The Lie-point simple symmetry generators of a given Stratonovich SDE form a Lie algebra.*

Proof. We can proceed as above and define now the vector fields

$$X_0 := \partial_t + b^j \partial_j, \quad \widehat{X}_k := \widehat{\partial}_k + \sigma_k^j \partial_j. \quad (74)$$

(Note here we change slightly our notation with respect to Sec. VI, in order to keep uniformity with Subsection IX A and to have a notation better suited to the present task.)

The determining equations (50) are now written as

$$[X_0, Y_\varphi] = 0, \quad [\widehat{X}_k, Y_\varphi] = 0. \quad (75)$$

It is then immediate to check that if Y_φ is given by (70) and Y_ξ, Y_η satisfy the determining equations

$$[X_0, Y_\xi] = 0 = [X_0, Y_\eta], \quad [\widehat{X}_k, Y_\xi] = 0 = [\widehat{X}_k, Y_\eta],$$

then—just by the Jacobi identity—(75) is also satisfied. \triangle

X. DISCUSSION AND CONCLUSIONS

Symmetry methods are widely recognized as one of our most effective tools in studying nonlinear deterministic equations;^{2,3} the literature devoted to symmetry methods for stochastic differential equations is comparatively smaller and moreover only considers the invariance of SDEs (in the Ito or Stratonovich form) only under deterministic transformations.

In this note we have considered—following the approach by Arnold and Imkeller in their analysis^{7,11} of normal forms for SDEs transformations—the transformations of SDEs under random diffeomorphisms, i.e., diffeomorphisms depending on a random (multi-dimensional Wiener) process and obtained the determining equations for random Lie-point symmetries of Ito stochastic differential equations.

The case of Stratonovich equations is also treated, in Section VI, and the determining equations are also obtained in this case.

We have also discussed the relation between symmetries of an Ito equation and those of the corresponding Stratonovich one; we have shown that in general—in particular, at the exception of the case where the matrix $\sigma(x, t)$ is actually independent of the space variables x^i —these do *not* admit the same symmetries. The reason for this lies in the actual meaning of the “correspondence” between Ito and Stratonovich equations.⁸ On the other hand, an Ito equation and the corresponding Stratonovich one do carry the same statistical information, so that one would expect correspondence between symmetries to hold when considering symmetries of the associated Fokker-Planck equation. This is indeed the case, in a sense made precise by our Proposition in Sec. VIII.

We have considered a number of concrete examples (both in the Ito and the Stratonovich case), choosing equations with a physical significance, and explicitly shown that the determining equations we have written down can be analyzed and explicitly solved, i.e., that our theory is concretely applicable.

As stressed above (see Remark 7), here we only focused on the proper definition of random symmetries of an SDE and on the equations which have to be solved to constructively determine

them; that is, we have not considered how the symmetries can be used in the study of the SDE (this appears to be a common feature of a large part of the literature devoted to the symmetry of SDEs).

On the other hand, it seems that the use of symmetries in the framework of SDEs should go through the same general ideas as in the case of deterministic equations; that is, beyond any specific technique, the presence of symmetries suggests first of all that the analysis will be simpler if using symmetry-adapted coordinates. A glimpse of this is provided in Example 6 (and Remark 8).

More structured results do exist in the case of deterministic symmetries of SDEs,¹⁸ we will investigate in future work how these results can be extended to the framework of the random symmetries introduced here.

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- ²⁸ As opposed to the "indirect" action due to the modification of the Wiener process induced by the action on the time variable; see below.
- ²⁹ Actually, in order to end up with an equation possibly of the same type (not to say about it being exactly the same equation as the original one), the transformed processes \tilde{w} should be only a function of the (transformed) time \tilde{t} , i.e., $\tilde{w} = \tilde{w}(\tilde{t})$; this amounts to requiring that the transformation of time does *not* depend on the space coordinates x^i . On the other hand, it could depend on the Wiener processes themselves.