Localization in the ground state of an interacting quasi-periodic fermionic chain

Vieri Mastropietro

Universitá di Milano, Via C. Saldini 50, 20133, Milano, Italy

We consider a one dimensional many body fermionic system with a large incommensurate external potential and a weak short range interaction. We prove, for chemical potentials in a gap of the non interacting spectrum, that the zero temperature thermodynamical correlations are exponentially decaying for large distances, with a decay rate much larger than the gap; this indicates the persistence of localization in the interacting ground state. The analysis is based on Renormalization Group, and convergence of the renormalized expansion is achieved using fermionic cancellations and controlling the small divisor problem assuming a Diophantine condition for the frequency.

1. INTRODUCTION AND MAIN RESULTS

A. Introduction

The properties of a fermionic system, like the conduction electrons in a metal, are determined, when the interaction between particles is not taken into account, by the eigenfunctions of the single particle hamiltonian. In the presence of an external periodic potential, the eigenfunctions are Bloch waves, and the zero temperature a.c. conductivity is vanishing (insulating behavior) or not (metallic behavior) on whether the Fermi level lies within a gap in the single particle spectrum or not. A different way in which an external potential can produce an insulating behavior is known as *Anderson localization* [1]; in the presence of certain potentials (like random ones, physically describing the presence of unavoidable impurities in the metal) the eigenfunctions of the single particle Hamiltonian can be exponentially localized and this produces an insulating behavior. Localization in the single particle Schroedinger equation with a random field has been indeed rigorously proved in various regimes of energy and disorder, starting from [2],[3]. Note that in one dimension *any* amount of disorder produces localization (the same is believed to happen in two dimensions as well), while in three dimensions the disorder has to be sufficiently strong and a metal to insulator transition is expected varying the strength of the random field. Localization does not necessarily require disorder, as it has long been known [4] that also nonrandom systems with quasi-periodic potentials (or incommensurate in the lattice case) can present single particle localization. The one dimensional quasi-periodic Schroedinger equation has *extended* Bloch-Floquet eigenfunctions in the weak coupling regime [5],[7] and *localized* eigenfunctions in the strong coupling regime, see [8],[9],[10], provided that some Diophantine condition is assumed on the frequency of the potential. In the lattice case with a cosine potential $\cos 2\pi(\theta + x\omega)$, x integer, the weak or strong coupling regime are connected by a duality transformation [4], and in this case it can be proved [11] that the spectrum is a Cantor set for *any* irrational frequency ω (not only Diophantine). The case of 1D quasi-periodic potential resembles the 3D random situation, as there is a transition between an extended and localized phase varying the strength of the potential.

A realistic description of metals must include the electron-electron interaction, so that the problem of the interplay between localization and interactions naturally arises [12]. In the physical literature the zero temperature thermodynamical properties of 1D interacting fermions with disorder has been analyzed in [13], [14], finding localized and delocalized regions; the quasi-periodic case has been studied in [15]. While such works concern the computation of the zero temperature thermodynamical quantities, in more recent times attention has been devoted also to the localization properties of excited states of interacting disordered many body systems, starting from [16], see [17]-[20]. Evidence has been found that in several interacting systems with disorder all the eigenfunctions are localized for weak interactions, while stronger interactions can destroy localization, leading to a so-called many-body localization transition; similar properties has been found also in the quasi-periodic case [21], [22].

It should however be remarked that not only the results about the excited states of the N-particle Hamiltonian but even the ground state properties (that is, the zero temperature thermodynamical quantities) are based on conjectures or approximations, and more quantitative results based on rigorous methods seem necessary. In particular, while there are examples of interacting disordered systems of quantum rotators in which ground state localization persists in the presence of interaction [23], exponential decay of ground state correlation for disordered fermionic systems has been proved so far only in the absence of interaction [24]. The mathematical tools used for single particle localization in the disordered case can usually treat only the case of a *finite* number of interacting disordered particles, see [25]. Using a sequence of unitary transformations, localization of most eigenstates (in the sense that the expectations of local observables are exponentially decaying) has been rigorously proved in [26] (see also [27]) in a many body interacting disordered fermionic chain , under a physically reasonable assumption that limits the amount of level attraction in the system. Evidence of localization for finite times in interacting disordered bosons has been found in [28].

There exist powerful methods, based on the version of Renormalization Group (RG) developed for constructive Quantum Field Theory, to compute the thermodynamical properties at zero temperature of interacting fermions. Such techniques encounter at the moment some difficulty in the application to random fermions, but can be successfully applied in the case quasi-periodic or incommensurate potentials; this is not surprising as quasi-periodic potentials produce *small divisors* similar to the ones in the KAM Lindstedt series, whose convergence was established by RG methods, see [29],[30]. We will therefore analyze the interplay of localization and interaction in the thermodynamical functions of interacting fermions with a quasi-periodic potential by RG methods. We consider a system of spinless fermions with Hamiltonian, $x \in \mathbb{Z}$

$$H_N = -\varepsilon \sum_{i=1}^{N} \Delta_{x_i} + u \sum_{i=1}^{N} \phi_{x_i} + \lambda \sum_{\substack{i,j=1 \\ i\neq j}}^{N} v(x_i - x_j) \tag{1}$$

where $\Delta_x f(x) = f(x+1) + f(x-1) - 2f(x)$, $\phi_x = \bar{\phi}(\omega x)$ with $\bar{\phi}(t) = \bar{\phi}(t+1)$, ω irrational and $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$. When $\phi_x = \cos(\omega x 2\pi)$ the above model is the interacting version of the Aubry-André model [4], and in recent times several experiments have been focused to systems modelized by it, see [34]. In the absence of interactions between particles ($\lambda = 0$) the eigenfunctions of H_N are Slater deteminants obtained by the single particle eigenfunctions of the Schroedinger equation

$$-\varepsilon\psi(x+1) - \varepsilon\psi(x-1) + u\phi_x\psi(x) = E\psi(x) \tag{2}$$

which were extensively analyzed, see for instance [5],[6],[7], [8],[9],[10]. In principle, the thermodynamical quantities could be obtained from such studies but, as a matter of fact, even in the $\lambda = 0$ case the only available results on the zero temperature properties of (1) were obtained by RG methods for functional integrals. Indeed in [31] the Grand canonical imaginary time correlations with $\lambda = 0$ were written in terms of an expansion plagued by a small divisor problem, and convergence was proved in [31], for small $\frac{u}{\varepsilon}$, suitable chemical potentials and assuming a Diophantine condition on the frequencies, that is $||2\pi n\omega||_{2\pi} \geq Cn^{-\tau}$ for any $n \in \mathbb{Z}/\{0\}$ where $||.||_{2\pi}$ is the norm on the one dimensional torus with period 2π. It was found a power law or an exponential decay of the zero temperature correlations at large distances depending on whether the chemical potential is inside a gap or not; that is metallic or a band insulator behavior. In the opposite limit when u/ε is *large* in the non interacting case $\lambda = 0$ it was proved in [32] that the correlations decay exponentially in the coordinates for suitable values of the chemical potential, in agreement with the localization properties of the single particle eigenfunction; the time decay is faster than any power if the chemical potential correspond to a gap in the spectrum.

The only rigorous result for quasi-periodic *interacting* fermions is in [33], in which it was proved that for small $\frac{u}{\varepsilon}$ and small λ there is still a power law decay of correlations for values of the chemical potential outside the gap, but the exponent is anomalous with a critical exponent signaling Luttinger liquid behavior. Therefore the metallic behavior, which was present in the non interacting case as consequence of the extended nature of the single particle eigenfunctions, persists also in the presence of interaction (but one has a Luttinger liquid instead than a Fermi liquid). In addition if the chemical potential is inside a gap one has exponentially decay of correlation, and an anomalous exponent appears in the decay rate.

In the present paper we finally consider a system of *interacting* fermions with a *large* incommensurate potential, a weak short range interaction and chemical potentials in a gap of the non interacting one particle spectrum. We prove that the zero temperature thermodynamical correlations are exponentially decaying for large distances, with a decay rate much larger than the gap; such property indicates the persistence of localization in the interacting ground state.

B. Thermodynamical quantities and solvable limits

We consider the Grand-canonical ensemble, in which one performs averages over the particle number. If Λ is a one dimensional lattice $\Lambda = \{x \in \mathbb{Z}, -L/2 \leq x \leq L/2\}$, L even, we introduce fermionic creation and annihilation operators a^+_x, a^-_x, $x \in \Lambda$ on the Fock space verifying $\{a^\varepsilon_x, a^{-\varepsilon'}_y\} = \delta_{\varepsilon,\varepsilon'}\delta_{x,y}$. The Fock space Hamiltonian corresponding to (1) can be written as

$$H = -\varepsilon(\sum_{x=-L/2}^{L/2-1} a^+_{x+1}a_x + \sum_{-L/2+1}^{L/2} a^+_x a^-_{x-1}) + \tag{3}$$
$$+u\sum_{x=-L/2}^{L/2} \phi_x a^+_x a^-_x - \mu \sum_{x=-L/2}^{L/2} a^+_x a^-_x + \lambda \sum_{x,y=-L/2}^{L/2} v(x-y) a^+_x a^-_x a^+_y a^-_y$$

ternal potential ϕ_x, the third Using the Jordan-Wigner transformation the model can be mapped in the XXZ model with a coordinate dependent magnetic field $h_x = \phi_x$.

Let us consider now the thermodynamical quantities in the grand-canonical ensemble. We consider the operators $a^\pm_{\mathbf{x}} = e^{x_0 H} a^\pm_x e^{-H x_0}$, with

$$\mathbf{x} = (x, x_0)\ , \quad 0 \le x_0 < \beta \tag{4}$$

for some $\beta > 0$ (β^{-1} is the temperature); x_0 is the imaginary time and on it antiperiodic boundary conditions are imposed, that is, if $a^\pm_{\mathbf{x}} = a^\pm_{x,x_0}$, then $a^\pm_{x,\beta} = -a^\pm_{x,0}$. The 2-point *Schwinger function* is defined as

$$\frac{\mathrm{Tr}\left[e^{-\beta H}\mathbf{T}(a^-_{\mathbf{x}} a^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} = \mathbf{I}(x_0 - y_0 > 0)\frac{\mathrm{Tr}[e^{-\beta H} a^-_{\mathbf{x}} a^+_{\mathbf{y}}]}{\mathrm{Tr}[e^{-\beta H}]} - \mathbf{I}(x_0 - y_0 \le 0)\frac{\mathrm{Tr}[e^{-\beta H} a^+_{\mathbf{y}} a^-_{\mathbf{y}}]}{\mathrm{Tr}[e^{-\beta H}]} \tag{5}$$

where $\mathbf{T}$ is the time order product. The above quantity cannot be exactly computed, so that one has to rely on a perturbative expansion around some solvable limit. In particular the model is solvable in the *free fermion* limit ($\lambda = u = 0$), which is an extended phase and in the *molecular limit* $\lambda = \varepsilon = 0$, which is a localized phase; in order to investigate the interplay of localization and interaction we will perform an expansion around the molecular limit. Before doing that, let us discuss the main properties of the solvable limits.

In the *free fermion limit*, corresponding to $u = \lambda = 0$, the Hamiltonian can be written in diagonal form in momentum space. If we assume $x = 0, 1, ...L$ and periodic boundary conditions and we set $a^\pm_x = \frac{1}{L}\sum_k e^{\pm ikx}\widehat{a}^\pm_k$, with $k = \frac{2\pi}{L}n$ and $\{\widehat{a}^\varepsilon, \widehat{a}^{-\varepsilon'}_{k'}\} = L\delta_{\varepsilon,\varepsilon'}\delta_{k,k'}$ then ($\varepsilon = 1$ for definiteness)

$$H_0 = \sum_k (-\cos k + \mu)\widehat{a}^+_k \widehat{a}^-_k \tag{6}$$

The two point Schwinger function is equal to

$$G(\mathbf{x}-\mathbf{y}) = \frac{\mathrm{Tr}\left[e^{-\beta H_0}\mathbf{T}(a^-_{\mathbf{x}}a^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} = \frac{1}{L}\sum_k e^{-ik(x-y)}\widehat{G}(k, x_0-y_0) = \tag{7}$$

$$= \frac{1}{L}\sum_k e^{-ik(x-y)}\left\{\frac{e^{-(x_0-y_0)e(k)}}{1+e^{-\beta e(k)}}\mathbf{I}(x_0-y_0>0) - \frac{e^{-(\beta+x_0-y_0)e(k)}}{1+e^{-\beta e(k)}}\mathbf{I}(x_0-y_0\le 0)\right\}$$

where $\varepsilon(k) = \mu - \cos k$ The function $\widehat{G}(k,\tau)$ is defined only for $-\beta < \tau \le \beta$, but we can extend it periodically over the whole real axis. The function $\widehat{G}(k,\tau)$ is antiperiodic in τ of period β; hence its Fourier series is of the form

$$\widehat{G}(k,\tau) = \frac{1}{\beta}\sum_{k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2})}\widehat{G}(k_0,k)e^{-ik_0\tau} \tag{8}$$

with

$$\widehat{G}(k,k_0) = \int_0^\beta d\tau e^{i\tau k_0}\frac{e^{-\tau e(k)}}{1+e^{-\beta e(k)}} = \frac{1}{-ik_0+e(k)} \tag{9}$$

Note that the function $\widehat{G}(\mathbf{k})$ is singular, in the limit $L\to\infty, \beta\to\infty$, at $k_0 = 0, k = \pm p_F$, with $\cos p_F = \mu$. $\pm p_F$ are the Fermi momenta and close to them, that is for k' small it behaves as

$$\widehat{G}(k'\pm p_F, k_0) \sim \frac{1}{-ik_0 \pm v_F k'} \tag{10}$$

Another solvable limit is the *Molecular limit* corresponding to $\lambda = \varepsilon = 0$. The Hamiltonian reduces to ($u = 1$ for definiteness)

$$H_0 = \sum_{x\in\Lambda}(\phi_x - \mu)a^+_x a^-_x \tag{11}$$

The 2–point function $g(\mathbf{x},\mathbf{y}) = \left\langle \mathbf{T}\{a^-_{\mathbf{x}}a^+_{\mathbf{y}}\}\right\rangle_{\beta,L}$ is equal to

$$g(\mathbf{x},\mathbf{y}) = \delta_{x,y}\left\{\frac{e^{-(x_0-y_0)(\phi_x-\mu)}}{1+e^{-\beta(\phi_x-\mu)}}\mathbf{I}(x_0-y_0>0) - \frac{e^{-(\beta+x_0-y_0)(\phi_x-\mu)}}{1+e^{-\beta(\phi_x-\mu)}}\mathbf{I}(x_0-y_0\le 0)\right\}$$

$$= \delta_{x,y}\bar{g}(x, x_0-y_0) \tag{12}$$

The function $\bar{g}(x,\tau)$ is defined only for $-\beta < \tau \le \beta$, but we can extend it periodically over the whole real axis. This periodic extension is smooth in τ for $\tau \ne n\beta, n\in\mathbb{Z}$, but has a jump discontinuity at $\tau = n\beta$ equal to $(-1)^n$, as for the two point function in the free fermion case.

The function $g(\mathbf{x},\mathbf{y})$ is antiperiodic in $x_0 - y_0$ of period β; hence its Fourier series is of the form

$$g(\mathbf{x},\mathbf{y}) = \delta_{x,y}\frac{1}{\beta}\sum_{k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2})}\widehat{g}(x,k_0)e^{-ik_0(x_0-y_0)} \tag{13}$$

with

$$\widehat{g}(x,k_0)=\int_0^\beta d\tau e^{i\tau k_0}\frac{e^{-\tau(\phi_x-\mu)}}{1+e^{-\beta(\phi_x-\mu)}}=\frac{1}{-ik_0+\phi_x-\mu} \tag{14}$$

Let $M\in\mathbb{N}$ and $\chi(t)$ a smooth compact support function that is 1 for $t\le 1$ and 0 for $t\ge\gamma$, with $\gamma>1$. Let $\mathcal{D}_\beta=D_\beta\cap\{k_0\,:\,\chi_0(\gamma^{-M}|k_0|)>0\}$, where $D_\beta=\{k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2}),n_0\in\mathbb{Z}\}$. If $x_0-y_0\neq n\beta$, we can write

$$g(\mathbf{x},\mathbf{y})=\lim_{M\to\infty}\delta_{x,y}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\chi(\gamma^{-M}|k_0|)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\phi_x-\mu}\equiv$$

$$\delta_{x,y}\frac{1}{\beta}\sum_{k_0\in D_\beta}e^{-ik_0(x_0-y_0)}\widehat{g}^{(\le M)}(x,k_0)\equiv\lim_{M\to\infty}g^{(\le M)}(\mathbf{x},\mathbf{y}) \tag{15}$$

Because of the jump discontinuities, $g^{(\le M)}(\mathbf{x},\mathbf{y})$ is not absolutely convergent but is pointwise convergent and the limit is given by $g(\mathbf{x},\mathbf{y})$ at the continuity points, while at the discontinuities it is given by the mean of the right and left limits.

In particular, the above equality is not true for $x_0-y_0=n\beta$, where the propagator is equal $\bar{g}(x,0^-)$ while the r.h.s. is equal to $\frac{\bar{g}(x,0^-)+\bar{g}(x,0^+)}{2}$. Note that $\lim_{\beta,L\to\infty}-\bar{g}(x,0^-)=\mathbf{I}(\phi_x-\mu\le 0)$, which is the occupation number of the ground state.

We assume $\phi_x=\bar{\phi}(\omega x)$ with $\bar{\phi}:\mathbb{R}\to\mathbb{R}$ a C^∞ periodic $\bar{\phi}(t)=\bar{\phi}(t+1)$ and even $\bar{\phi}(t)=\bar{\phi}(-t)$, $t\in\mathbb{R}$; we assume moreover that there is only one $(\omega\bar{x})_{\text{mod}.1}\in(0,\frac{1}{2})$ such that $\mu=\bar{\phi}(\omega\bar{x})$; therefore, for small $(\omega x')_{\text{mod}.1}$

$$\phi_{x'+\rho\bar{x}}-\mu=\rho v_0(\omega x')_{\text{mod}.1}+r_{\rho,x'}\quad v_0=\partial\bar{\phi}(\omega\bar{x}),\quad \rho=\pm \tag{16}$$

with $r_{\rho,x'}=O(((\omega x')_{\text{mod}.1})^2)$; therefore the 2-point function can be written as, for small $(\omega x')_{\text{mod}.1}$

$$\widehat{g}(x'\pm\bar{x},k_0)\sim\frac{1}{-ik_0\pm v_0(\omega x')_{\text{mod}.1}} \tag{17}$$

Note the similarity of (17) with (10); this analogy suggests to call $\pm\bar{x}$ as *Fermi coordinates*, in analogy with the Fermi momenta $\pm p_F$. In the special case of $\phi_x=\cos(2\pi\omega x)$ (Almost-Mathieu operator), setting $\varepsilon=u$

$$\widehat{G}(k,k_0)|_{k=2\pi\omega x}=\widehat{g}(x,k_0) \tag{18}$$

which is is a manifestation of the well known *Aubry-duality*.

C. Grassmann Integral representation

If $\mathcal{B}_{\beta,L}=\{\mathcal{D}_\beta\otimes\Lambda\}$, we consider the Grassmann algebra generated by the Grassmannian variables $\{\psi^{\pm}_{x,k_0}\}_{x,k_0\in\mathcal{B}_{\beta,L}}$ and a Grassmann integration $\int\big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}d\psi^-_{x,k_0}d\psi^+_{x,k_0}\big]$ defined as the linear operator on the Grassmann algebra such that, given a monomial $Q(\psi^-,\psi^+)$ in the variables $\psi^{\pm}_{x,k_0}$, its action on $Q(\psi^-,\psi^+)$ is 0 except in the case $Q(\psi^-,\psi^+)=\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\psi^-_{x,k_0}\psi^+_{x,k_0}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommuting properties of the variables, by the condition

$$\int\Big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}d\psi^+_{x,k_0}d\psi^-_{x,k_0}\Big]\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\psi^-_{x,k_0}\psi^+_{x,k_0}=1 \tag{19}$$

We define also Grassmanian field as $\psi^{\pm}_{\mathbf{x}}=\frac{1}{\beta}\sum_{k_0\in\mathcal{B}_{\beta,L}}e^{\pm ik_0x_0}\psi^{\pm}_{x,k_0}$ with $x_0=m_0\frac{\beta}{\gamma^M}$ and $m_0\in(0,1,...,\gamma^M-1)$. The "Gaussian Grassmann measure" (also called integration) is defined as

$$P(d\psi)=[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\beta d\psi^-_{x,k_0}d\psi^+_{x,k_0}\widehat{g}^{(\le M)}(x,k_0)]\exp\{-\sum_{x,k_0}(\widehat{g}^{(\le M)}(x,k_0))^{-1}\psi^+_{x,k_0}\psi^-_{x,k_0}\} \tag{20}$$

We introduce the generating functional $W_M(\phi)$ defined in terms of the following Grassmann integral (Dirichelet boundary conditions are imposed)

$$e^{W_M(\phi)}=\int P(d\psi)e^{-\mathcal{V}^{(M)}(\psi)-\mathcal{B}^{(M)}(\psi,\phi))} \tag{21}$$

where $\psi^{\pm}_{\mathbf{x}}$ and $\phi^{\pm}_{\mathbf{x}}$ are Grassmann variables, $P(d\psi)$ has propagator

$$g^{(\le M)}(\mathbf{x},\mathbf{y})=\delta_{x,y}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_{\beta,L}}\chi(\gamma^{-M}|k_0|)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\phi_x-\phi_{\bar{x}}} \tag{22}$$

and $\int d\mathbf{x}$ is a shorthand for $\sum_{x\in\Lambda}\frac{\beta}{\gamma^M}\sum_{x_0}$; moreover

$$\mathcal{V}^{(M)}=\lambda\int d\mathbf{x}d\mathbf{y}v(\mathbf{x},\mathbf{y})\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}}+\varepsilon\int d\mathbf{x}(t^1_x\psi^+_{\mathbf{x+e_1}}\psi^-_{\mathbf{x}}+t^2_x\psi^+_{\mathbf{x-e_1}}\psi^-_{\mathbf{x}}) \tag{23}$$
$$+\nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\int d\mathbf{x}d\mathbf{y}\lambda v(\mathbf{x},\mathbf{y})\nu_C(\mathbf{y})\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}$$

where $v(\mathbf{x}-\mathbf{y})=\delta(x_0-y_0)(\delta_{x,y+1}+\delta_{x,y-1})$, $t^1_{L/2}=t^2_{-L/2}=0$ and $t^1_x=t^2_x=1$ otherwise and

$$\nu_C(x)=\frac{1}{2}[\bar{g}(x,0^+)-\bar{g}(x,0^-)] \tag{24}$$

and $\bar{g}(x,0^-)$ was defined in (12). Finally

$$\mathcal{B}^{(M)}(\psi,\phi)=\int d\mathbf{x}[\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}] \tag{25}$$

Note that we expect that the chemical potential is modified by the interaction; in the analysis it is convenient to keep fixed the value of the Fermi coordinate in the free or interacting theory, therefore we write the chemical potential as $\phi_{\bar{x}} + \nu$, where ν is a counterterm to be fixed so that the free and interacting Fermi coordinate are the same.

Let us define

$$S_2^{M,\beta,L}(\mathbf{x},\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} W_M(\phi)|_0 \tag{26}$$

Note that $\lim_{M\to\infty} S_2^{M,\beta,L}$ can be written as a series in ε, λ coinciding order by order with the series expansion for the Schwinger functions (5) with chemical potential $\mu = \phi_{\bar{x}} + \nu$. Indeed each term of the series for (5) or $\lim_{M\to\infty} S_2^{M,\beta,L}$ can be expressed as a sum of integrals over propagators (respectively $g(\mathbf{x},\mathbf{y})$ (12) or $\lim_{M\to\infty} g^{(\leq M)}(\mathbf{x},\mathbf{y})$ (20)) which can be represented by Feynman graphs. The subset of graphs contributing to (5) and with no tadpoles coincides the the graphs contributing to $\lim_{M\to\infty} S_2^{M,\beta,L}$ and no vertices ν_C. The integrands are different, as the propagators $g(\mathbf{x},\mathbf{y})$ (12) or $\lim_{M\to\infty} g^{(\leq M)}(\mathbf{x},\mathbf{y})$ (20) are different at coinciding times. However the integrals are well defined and coincide, as the integrands of the graphs coincide except in a set of zero measure. Let us consider the remaining graphs. In the graphs with a tadpole in the expansion for $\lim_{M\to\infty} S_2^{M,\beta,L}$ there is a factor of the form

$$\int d\mathbf{y}\lambda v(\mathbf{x},\mathbf{y}) g(\mathbf{x}_1 - \mathbf{x})\nu_T(\mathbf{y}) g(\mathbf{x} - \mathbf{x}_2) \quad , \quad \nu_T(\mathbf{y}) = -\frac{1}{2}[\bar{g}(y,0^+) + \bar{g}(y,0^-)] \tag{27}$$

On the other hand, given a graph G of this type, there is another graph $\widetilde{G}$, which differs from it only because, in place of the term $\mathcal{V}(\psi)$ which produced the tadpole, there is a vertex ν_C. If we sum the values of G and $\widetilde{G}$, we get a number which is equal to the value of G, with $-\lambda\bar{g}(y,0^-)$ replacing $\nu_T(y)$, so that the terms coincide with the analogous term in the expansion for (5). Therefore the two perturbative expansions coincide. An analyticity argument, analogue to the one in Proposition 2.1 of [35], allows to conclude the coincidence of (5) and $\lim_{M\to\infty} S_2^{M,\beta,L}$ beyond perturbation theory, once that the limit exists and certain analyticity properties are proved; this is quite standard and will be not repeated here for brevity, so we state our main results directly for the Grassmann integral.

D. Main result: localization in the presence of interaction

We set $u = 1$ and we consider ε, λ small. We define $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the one dimensional torus and $||\theta||_1$ the distance on $\mathbb{T}$, that is the absolute value of θ modulo 1 so that $0 \leq ||\theta||_1 \leq \frac{1}{2}$. Our main result is the following.

Theorem 1.1 *Let us consider the 2-point function* $S_2^{M,\beta}(\mathbf{x},\mathbf{y})$ *(26) with* $\phi_x = \bar{\phi}(\omega x)$*, with* $\bar{\phi} \in C^\infty(\mathbb{T})$ *and such that* $\bar{\phi}(t) = \bar{\phi}(-t)$*,* $\bar{\phi}(t) = \bar{\phi}(t+1)$*,* $\bar{\phi}(t)$ *increasing for* $0 < t < \frac{1}{2}$ *and* ω *verifying*

$$||\omega x||_1 \geq C|x|^{-\tau}, \quad \text{for} \quad \text{any} \quad 0 \neq x \in \mathbb{Z} \tag{28}$$

for some constants $\tau > 1$ *and* $C_0 > 0$*. Assume* $\bar{x}$ *in (22) half integer* $\bar{x} = n + \frac{1}{2}$*,* $n \in \mathbb{N}$ *and that* $v_0 = \partial_t \bar{\phi}(\omega \bar{x}) > 0$*. Then there exists an* ε_0*, depending on* ω *and* $\bar{x}$*, such that for* $|\varepsilon| \leq \varepsilon_0$ *and* $|\lambda| \leq \varepsilon^{2\bar{x}+2}$ *there exists a continuous function* $\nu(\varepsilon, \lambda)$ *such that, for any* N*, the limit* $\lim_{\beta\to\infty} \lim_{L\to\infty} \lim_{M\to\infty} S_2^{M,\beta,L}(\mathbf{x},\mathbf{y}) = S_2(\mathbf{x},\mathbf{y})$ *exists and verifies, for any* $N \in \mathbb{N}$

$$|S_2(\mathbf{x},\mathbf{y})| \leq C_N \frac{e^{-\kappa \log|\varepsilon|^{-1}|x-y|}}{1 + (|\sigma||x_0 - y_0|)^N} \tag{29}$$

where $\sigma = a\varepsilon^{2\bar{x}} + O(\varepsilon^{2\bar{x}+1})$ *with* $a \neq 0$ κ, C_N *positive constants.*

A typical example of function ϕ_x verifying the condition of Theorem 1.1 is $\phi_x = \cos(2\pi\omega x)$; the condition $\bar{\phi}(t) = \bar{\phi}(-t)$ fixes naturally the origin of coordinates. As we discussed above, in the absence of many body interaction Anderson localization of the single particle Schroedinger equation implies exponential decay of the 2-point Schwinger function, as proven in [32]. The above Theorem says that, for suitable chemical potentials and ω diophantine, such exponential decay persists also in the presence of interaction at least for certain chemical potentials, provided that the hopping is smaller than $O(\bar{x}!^{-\gamma})$ for some positive γ, and the interaction is much smaller than the hopping; that is, if we remain sufficiently close to the molecular limit. The decay rate in the coordinate difference is much faster than the gap and this indicates the persistence of localization for the ground state eigenfunction of an interacting many body system.

Note that, even if one can guess that the condition on the smallness of λ can be improved, it is believed that large interactions can destroy Anderson localization; large ε destroys Anderson localization as well. The chemical potential is chosen in correspondence of a gap of the one particle non interacting spectrum; this assumption is the analogue of the filled band condition on the Fermi momentum in the extended regime and it is physically quite natural, as the non interacting system tends to have a dense set of gaps. The choice of the chemical potential ensures faster than any power decay in the time direction and this fact plays an important role in the proof, as it allows to bound certain terms (*loops*) in the perturbative expansion, which are generated by the presence of the many body interaction. The random case has far fewer resonances than the quasi periodic potential, a fact which simplifies the Renormalization Group analysis of the interacting quasi-periodic case.

A consequence of Theorem 1.1 combined with [33] is the existence of a *quantum phase transition* between an extended and a localized phase. Indeed it was proved in [33] in the small λ, u case that even in the presence of interaction the system has a metallic or a band insulating behavior depending on the chemical potential; that is, for small u and λ ($\varepsilon = 1$) if $\mu = 1 - \cos p_F$ then if $p_F = m\omega\pi$ then

$$|S_2(\mathbf{x}, \mathbf{y})| \leq \frac{C_N}{1 + (\Delta|\mathbf{x} - \mathbf{y}|)^N} \tag{30}$$

with $\Delta = \widehat{\phi}_m |u|^{1-\eta}(1 + O(\lambda))$ and $\eta = a\lambda + O(\lambda^2)$ with $a > 0$ suitable constant; the 2-point function has a faster than any power decay with the same rate in space and time, indicating *band insulating* behavior. On the other hand, if $||2p_F + 2\pi n\omega||_{2\pi} \geq C|n|^{-\tau}, \quad \text{for} \quad \text{any} \quad n \in \mathbb{Z}/\{0\}$ then the 2-point function decays for large distance as

$$S_2(\mathbf{x}, \mathbf{y}) \sim_{|\mathbf{x}-\mathbf{y}| \to \infty} O(|\mathbf{x} - \mathbf{y}|^{-1-\widetilde{\eta}}) \tag{31}$$

with $\widetilde{\eta} = b\lambda^2 + O(\lambda^3)$, b a positive constant, denoting an anomalous *metallic* behavior (Luttinger liquid behavior). Therefore, there is a *localization-delocalization* transition also in the presence of interaction varying the strength of the kinetic energy.

Theorem 1.1 is proven under the condition that the chemical potential is in a gap of the single particle spectrum. One may expect that Anderson localization is present also for other values of the chemical potential, and indeed in [32] in the non interacting case $\lambda = 0$, exponential coordinate decay of the two-point function in the absence of interaction was established also in correspondence of the condition

$$||\omega x \pm 2\omega\bar{x}||_1 \geq C|x|^{-\tau}, \quad \text{for} \quad \text{any} \quad 0 \neq x \in \mathbb{Z} \tag{32}$$

By an extension of the methods introduced in the present paper, one could indeed prove exponential decay of correlations in presence of interaction at finite temperatures, with a decay rate much larger than the inverse temperature; new ideas seem however necessary to consider the zero temperature case.

E. Sketch of the proof

In order to prove Theorem 1.1 we expand around the molecular limit, obtaining a series in terms of sum of product of propagators $(-ik_0 + \phi_x - \phi_{\bar{x}})^{-1}$ which, for $(\omega(x - \rho\bar{x}))_{\text{mod.1}}$ small, $\rho = \pm$, can be bonded by $C|x - \rho\bar{x}|^\tau$; such *small divisors* can accumulate so that the size of certain terms in the expansion grows as a factorial, so destroying the possibility of convergence. One has then to prove that there is no accumulation of small divisors. A similar problem arises in the Linstedt series for KAM tori [29], in which one can exploit the Diophantine condition to show the lacking of small divisors accumulation. The presence of interaction produces however an essential difference: in KAM Lindstedt series or in the non interacting case the series can be represented in terms of *tree* diagrams, while in the present case the series are expressed in terms of *diagrams with loops*. This make the small divisor problem and the structure of resonances much more involved.

We perform the analysis of the Grassmann integral (26) in an iterative way by using Renormalization Group methods. We start integrating the higher energy frequencies, see §2. After the integration of the ultraviolet scales, we have to integrate the low energy modes (infrared scales) in which one has to face a small divisor problem, as discussed in §3. The theory is *non-renormalizable* according to power counting; the scaling dimension depends on the number of vertices in any subgraph, so that one has to improve the dimensions of all possible subgraphs with any number of external fields. In order to get such improvement, we have to exploit the in incommensurability of the potential and take advantage of the diophantine condition on the frequency. One has to distinguish between two kind of terms in the effective potential, depending on whether the coordinates (measured from the Fermi coordinate) of the external fields are different (*non-resonant* terms) or equal (*resonant* terms). In the non resonant terms one uses the Diophantine condition to get good bounds, exploiting, roughly speaking, the idea that if the denominators associated to the external lines have similar small size but different coordinates, then the difference of coordinates is necessarily large (see §3.C). The result is somewhat similar to Bruno lemma as presented in [29], but new difficulties arise from the fact that the resonances have any number of external fields and not only two as in the non interacting case; in particular, one has to improve the bounds by a quantity proportional to the external lines for combinatorial reason, see §3.E . Regarding the resonances one uses that the local part of the terms with more than four external fields is vanishing. Moreover the resonances with two external fields produce a renormalization of the chemical potential or a dynamically generated a mass term implying an exponential decay in time. An important difference with respect to the non interacting case, or the Lindsted series for KAM is that in such cases one has only tree diagrams, and their number is $O(n!)$ so that a $\frac{C^n}{n!}$-bound on each diagram is sufficient for convergence. In the presence of interaction, on the contrary, one has loops so that the number of diagrams $O(n!^2)$ and a similar bound on each diagram is not sufficient; one has then to avoid graph expansion and taking into account the fact that the fermionic expectations can be represented in terms of determinants, exploiting the cancellations due to the fermionic nature of the problem, see §3.G. Finally in §3.G we study the flow of the running coupling constants, proving its boundedness as a consequence of the dynamically generated mass term, and the two point functions is analyzed in §3.H, completing the proof of the theorem.

2. THE ULTRAVIOLET INTEGRATION

A. Ultraviolet and Infrared fields

We introduce a function $\chi_h(t,k_0) \in C^\infty(\mathbb{T}\times\mathbb{R})$, such that $\chi_h(t,k_0) = \chi_h(-t,-k_0)$ and $\chi_h(t,k_0) = 1$, if $\sqrt{k_0^2+v_0^2||t||_1^2} \le a\gamma^{h-1}$ and $\chi_h(t,k_0) = 0$ if $\sqrt{k_0^2+v_0^2||t||_1^2} \le a\gamma^{h}$ with a and $\gamma > 1$ suitable constants. We choose a so that the supports of $\chi_0(\omega(x-\bar{x}),k_0)$ and $\chi_0(\omega(x+\bar{x}),k_0)$ are disjoint; note that the C^∞ function on $\mathbb{T}\times\mathbb{R}$

$$\widehat{\chi}^{u.v.}(\omega x,k_0) = 1-\chi_0(\omega(x-\bar{x}),k_0)-\chi_0(\omega(x+\bar{x}),k_0) \tag{33}$$

is equal to 0, if $\sqrt{k_0^2+|\phi_x-\phi_{\bar{x}}|^2} \le b$, with b a suitable constant. For reasons which will appear clear below, we choose $\gamma > 2^{\frac{1}{\tau}}$. We can write then

$$g(\mathbf{x},\mathbf{y}) = g^{(u.v)}(\mathbf{x},\mathbf{y}) + g^{(i.r)}(\mathbf{x},\mathbf{y}) \tag{34}$$

and

$$g^{(i.r)}(\mathbf{x},\mathbf{y}) = \sum_{\rho=\pm} g_\rho^{(\le 0)}(\mathbf{x},\mathbf{y}) \tag{35}$$

where

$$g^{(u.v.)}(\mathbf{x},\mathbf{y}) = \frac{\delta_{x,y}}{\beta}\sum_{k_0\in D_\beta}\chi(\gamma^{-M}|k_0|)\widehat{\chi}^{u.v.}(\omega x,k_0)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\phi_x-\phi_{\bar{x}}}$$

$$g_\rho^{(\le 0)}(\mathbf{x},\mathbf{y}) = \frac{\delta_{x,y}}{\beta}\sum_{k_0\in D_\beta}\chi_0(\omega(x-\rho\bar{x}),k_0)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\phi_x-\phi_{\bar{x}}} \tag{36}$$

For definiteness, we start considering the generating function (21) with $\phi = 0$. We will use the following *addition formula*; if g_1, g_2 are two propagators and $g := g_1 + g_2$, then $P_g(d\psi) = P_{g_1}(d\psi_1)P_{g_2}(d\psi_2)$, in the sense that for every polynomial f

$$\int P_g(d\psi)f(\psi) = \int P_{g_1}(d\psi_1)\int P_{g_2}(d\psi_2)f(\psi_1+\psi_2)\ . \tag{37}$$

The properties of Grassmann integrals imply that we can write

$$e^{W(0)} = \int P(d\psi)e^{-\mathcal{V}^{(M)}(\psi)} = \int P(d\psi^{(i.r.)})\int P(d\psi^{(u.v.)})e^{-\mathcal{V}^{(M)}(\psi^{(i.r.)}+\psi^{(u.v.)})} \tag{38}$$

where $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ are gaussian Grassmann integrations with propagators respectively $g^{(u.v.)}(\mathbf{x},\mathbf{y})$ and $g^{(i.r)}(\mathbf{x},\mathbf{y})$ and $\psi^{(u.v.)}$ and $\psi^{(i.r.)}$ are independent Grassmann variables. We can write

$$\int P(d\psi^{(u.v.)})e^{-\mathcal{V}^{(M)}(\psi^{(i.r.)}+\psi^{(u.v.)})} = e^{\sum_{n=0}^\infty \frac{(-1)^n}{n!}\mathcal{E}^T_{u.v.}(\mathcal{V}^{(M)};n)} \equiv e^{-\beta L E_0 - \mathcal{V}^{(0)}(\psi^{(i.r.)})} \tag{39}$$

where $\mathcal{E}^T$ is the fermionic truncated expectation with respect to $P(d\psi^{(u.v.)})$, that is, if $X(\psi+\phi)$ is a monomial

$$\mathcal{E}^T(X:n) = \frac{\partial^n}{\partial\alpha^n}\log\int P(d\psi)e^{\alpha X(\phi+\Psi)})|_{\alpha=0} \tag{40}$$

By the above definition

$$\mathcal{V}^{(0)} = \sum_{n=1}^\infty\sum_{x_1}\int dx_{0,1}....\sum_{x_n}\int dx_{0,n}W_n^{(h)}(\mathbf{x}_1,...,\mathbf{x}_n)[\prod_{i=1}^n\psi^{(\varepsilon_i)(\le 0)}_{\mathbf{x}'_i,\rho_i}] \tag{41}$$

with $\mathbf{x} = \mathbf{x}' + \rho\bar{\mathbf{x}}$, $\bar{\mathbf{x}} = (0,\bar{x})$ and E_0 is a constant; moreover

$$e^{W(0)} = e^{-\beta L E_0}\int P(d\psi^{(i.r.)})e^{-\mathcal{V}^{(0)}(\psi^{i.r.})} \tag{42}$$

Note that the kernel $W_n^{(h)}(\mathbf{x}_1,...,\mathbf{x}_n)$ will contain in general Kronecker or Dirac deltas, and we define the L_1 norm as they would be positive functions, *e.g.* if $W(\mathbf{x}_1,\mathbf{x}_2,..\mathbf{x}_n) = \delta(\sum_j\eta_j\mathbf{x}_j)\bar{W}(\mathbf{x}_1,..,\mathbf{x}_n)$ then $|W|_{L_1} = \int d\mathbf{x}_1..d\mathbf{x}_n\delta(\sum_j\eta_j\mathbf{x}_j)|\bar{W}(\mathbf{x}_1,..,\mathbf{x}_n)|$.

Lemma 2.1 *The constant* E_0 *and the kernels* $W_n^{(0)}$ *are given by power series in* $\lambda, \varepsilon, \nu$ *convergent for* $|\lambda|, |\varepsilon|, |\nu| \leq \varepsilon_0$*, for* ε_0 *small enough and independent of* β, M*. They satisfy the following bounds:*

$$|W_n^{(0)}|_{L_1} \leq L\beta C^n \varepsilon_0^{k_n} \ , \tag{43}$$

for some constant $C > 0$ *and* $k_n = \max\{1, n-1\}$*. Moreover,* $\lim_{M\to\infty} E_0$ *and* $\lim_{M\to\infty} W_n^{(0)}$ *do exist and are reached uniformly in* M*, so that, in particular, the limiting functions are analytic in the same domain.*

B. Proof of Lemma 2.1

We can write $\chi(\gamma^{-M}|k_0|) = \sum_{j=-\infty}^{M} f_j(|k_0|)$ with, for $j \leq M-1$, $f_j(|k_0|) = \chi(\gamma^{-j}|k_0|) - \chi(\gamma^{-j+1}|k_0|)$ a smooth compact support function non vanishing for $\gamma^{h-1} \leq |k_0| \leq \gamma^{h+1}$. . Therefore

$$g^{(u.v.)}(\mathbf{x}, \mathbf{y}) = \sum_{h=1}^{M} g^{(h)}(\mathbf{x}, \mathbf{y}) \ , \tag{44}$$

where

$$g^{(h)}(\mathbf{x}, \mathbf{y}) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0} \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \phi_x - \phi_{\bar{x}}} \chi^{(u.v.)}(k_0, \omega x) f_h(|k_0|) = \delta_{x,y} \bar{g}^{(h)}(x, x_0 - y_0) \tag{45}$$

where we have used that $\chi(\gamma^{-N}|k_0|) = \sum_{h=1}^{N} f_h(|k_0|)$, according to the definition after (15). By integration by parts, for any integer K

$$|\bar{g}^{(h)}(x, x_0 - y_0)| \leq \frac{C_K}{1 + [\gamma^h |x_0 - y_0|]^K} \tag{46}$$

By using (44) we can write $P(d\psi^{(u.v.)}) = \prod_{h=1}^{M} P(d\psi^{(h)})$ and the corresponding decomposition of the field $\psi_{\mathbf{x}}^{(u.v.)} = \sum_{h=1}^{M} \psi_{\mathbf{x}}^{(h)}$. Hence, we can integrate iteratively the fields $\psi^{(M)}, \psi^{(M-1)}, ..., \psi^{(h)}$ with $h \geq 1$ and, if we define $\psi^{(\leq 0)} = \psi^{i.r.}$ and $\psi^{(\leq h)} = \psi^{i.r.} + \sum_{j=1}^{h} \psi^{(j)}$, if $h \geq 0$, we get :

$$e^{W(0)} = e^{-L\beta E_h} \int P(d\psi^{\leq h})\, e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})} \tag{47}$$

Let us consider first the effective potentials on scale h, $\mathcal{V}^{(h)}(\psi^{(\leq h)})$. We want to show that they can be expressed as sums of terms, each one associated to an element of a family of labeled trees; we shall call this expansion *the tree expansion.*

The tree definition can be followed looking at Fig 1 (for a general introduction to tree expansion see for instance [32]).

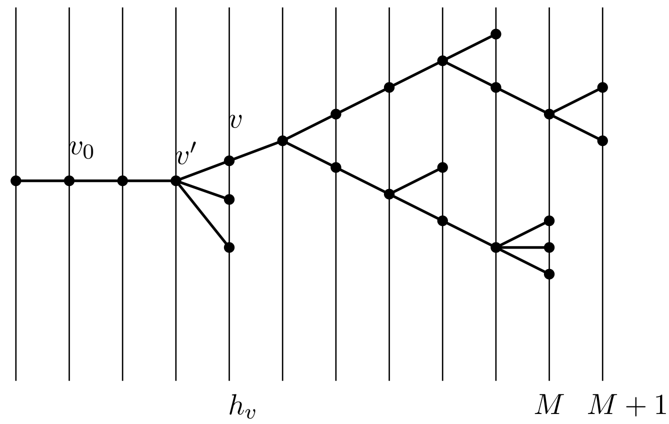

FIG. 1: A tree $\tau \in \mathcal{T}_{h,n}$ with its scale labels.

Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with $\bar{n}$ end-points is bounded by $4^{\bar{n}}$. We shall also consider the set $\mathcal{T}_{h,n,M}$ of the *labeled trees* with n endpoints (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \leq M$ with the root. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, M+1]$, and we represent any tree $\tau \in \mathcal{T}_{M,h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non trivial

vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

3) There is only one vertex immediately following the root, which will be denoted v_0; its scale is $h+1$. If v_0 is an endpoint, the tree is called the *trivial tree*; this can happen only if $n+m=1$.

4) Given a vertex v of $\tau \in \mathcal{T}_{M,h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$; the number of endpoint of these subtrees will be called n_v. If a subtree with root v contains only v and one endpoint on scale h_v+1, it will be called a *trivial subtree.*

5) Given an end-point, the vertex v preceding it is surely a non trivial vertex, if $n > 1$.

Our expansion is built by associating a value to any tree $\tau \in \mathcal{T}_{M,h,n}$ in the following way.

First of all, given a normal endpoint $v \in \tau$ with $h_v = M+1$, we associate to it one of the terms (note that to the ε interaction two terms are associated) contributing to the potential $\mathcal{V}^{(M)}(\psi)$ while, if $h_v \leq M$, we associate to it one of the terms appearing in the following expression:

$$-\mathcal{V}(\psi^{(<h_v)}) - \nu\mathcal{N}(\psi^{(<h_v)}) + \int d\mathbf{x}d\mathbf{y}\lambda v(\mathbf{x},\mathbf{y})(-\nu_C(y) + \bar{g}^{[h_v,M]}(y;0))\psi^{+(<h_v)}_{\mathbf{y}}\psi^{-(<h_v)}_{\mathbf{y}}) \quad (48)$$

We associate to the label an index to specify which term is associated to the end-point. We introduce also a *field label* f to distinguish the field variables appearing in the different terms associated to the endpoints; the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$ will denote the space-time point, the ε index of the Grassmann field variable with label f.

The previous definitions imply that, if $0 \leq h < M$, the following iterative equations are satisfied:

$$-\mathcal{V}^{(h)}(\psi^{(\leq h)}) - \beta L e_h = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{M,h,n}} \mathcal{V}^{(h)}(\tau,\psi^{(\leq h)}) \;, \quad (49)$$

where, if v_0 is the first vertex of τ and $\tau_1,\ldots,\tau_s$, $s \geq 1$, are the subtrees with root in v_0,

$$\mathcal{V}^{(h)}(\tau,\psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}\big[\bar{\mathcal{V}}^{(h+1)}(\tau_1,\psi^{(\leq h+1)});\ldots;\bar{\mathcal{V}}^{(h+1)}(\tau_s,\psi^{(\leq h+1)})\big] \;, \quad (50)$$

where $\bar{\mathcal{V}}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$ is equal to $\mathcal{V}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$ if the subtree τ_i contains more than one end-point, otherwise it is given by one of the terms contributing to the potentials in (24), if $h_v = M+1$, or one of the addends in (48), if $h_v \le M$.

Note that

$$|\lambda\nu_C(x)|, |\lambda\bar{g}^{[h_v,M]}(x,0))| \le C|\lambda| \tag{51}$$

We define

$$N_v = \sum_{i, v_i^* > v} 1 \tag{52}$$

the number of end-points following v The above definitions imply, in particular, that, if $n > 1$ and v is not an endpoint, then $N_v > 1$; in fact the vertex preceding an end-point is necessarily non trivial, if $n > 1$.

Using its inductive definition, the right hand side of (49) can be further expanded, and in order to describe the resulting expansion we need some more definitions.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v, and the set $\mathbf{x}_v$ of all space-time points associated with one of the end-points following v. The subsets P_v must satisfy various constraints. First of all, $|P_v| \ge 2$, if $v > v_0$; moreover, if v is not an endpoint and $v_1, \ldots, v_{S_v}$ are the $S_v \ge 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $S_v > 1$. Given $\tau \in \mathcal{T}_{M,h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$.

With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau,\psi^{(\le h)})$ in the r.h.s. of (49) as

$$\begin{aligned} \mathcal{V}^{(h)}(\tau,\psi^{(\le h)}) &= \sum_{\mathbf{P}\in\mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau,\mathbf{P}) \;, \\ \bar{\mathcal{V}}^{(h)}(\tau,\mathbf{P}) &= \int d\mathbf{x}_{v_0} \widetilde{\psi}^{(\le h)}(P_{v_0}) K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}) \;, \end{aligned} \tag{53}$$

where $K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v) = \frac{1}{S_v!} \prod_{i=1}^{S_v} [K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i})] \;\; \mathcal{E}_{h_v}^T[\widetilde{\psi}^{(h_v)}(P_{v_1}\setminus Q_{v_1}), \ldots, \widetilde{\psi}^{(h_v)}(P_{v_{S_v}}\setminus Q_{v_{S_v}})] \;, \tag{54}$$

Moreover, if v_i is an endpoint, $K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i})$ is equal to the kernel of one of the terms contributing to the potential in (24), if $h_{v_i} = N+1$, or one of the four terms in (48), if $h_{v_i} \le N$; if v_i is not an endpoint, $K_{v_i}^{(h_v+1)} = K_{\tau_i,\mathbf{P}_i}^{(h_v+1)}$, where $\mathbf{P}_i = \{P_w, w \in \tau_i\}$.

In order to get the final form of our expansion, we need a convenient representation for the truncated expectation in the r.h.s. of (54). Let us put $s = S_v$, $P_i := P_{v_i} \setminus Q_{v_i}$; moreover we order in an arbitrary way the sets $P_i^{\pm} := \{f \in P_i, \varepsilon(f) = \pm\}$, we call $f_{ij}^{\pm}$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f\in P_i^+}\mathbf{y}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f_{ij}^-)$, $\mathbf{y}_{ij} = \mathbf{x}(f_{ij}^+)$. Note that $\sum_{i=1}^s |P_i^-| = \sum_{i=1}^s |P_i^+| := k$, otherwise the truncated expectation vanishes. A couple $l := (f_{ij}^-, f_{i'j'}^+) := (f_l^-, f_l^+)$ will be called a line joining the fields with labels $f_{ij}^-, f_{i'j'}^+$. Then, we use the *Brydges-Battle-Federbush* formula, if $s > 1$,

$$\mathcal{E}_h^T(\widetilde{\psi}^{(h)}(P_1), \ldots, \widetilde{\psi}^{(h)}(P_s)) = \sum_T \prod_{l\in T} \left[g^{(h)}(\mathbf{x}_l - \mathbf{y}_l)\right] \int dP_T(\mathbf{t}) \det G^{h,T}(\mathbf{t}) \,, \tag{55}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{ii'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm.

$$G_{ij,i'j'}^{h,T} = t_{ii'}\delta_{x_{ij},y_{i'j'}}\left[\widetilde{g}^{(h)}(x_{ij}, x_{0,ij} - y_{0,i'j'})\right]_{\rho_{ij}^-,\rho_{i'j'}^+} \,, \tag{56}$$

with $(f_{ij}^-, f_{i'j'}^+)$ not belonging to T.

By inserting (55) in the r.h.s. of (54) we get

$$V^{(h)}(\tau, \mathbf{P}) = \sum_{T\in\mathbf{T}} \int d\mathbf{x}_{v_0} W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) \prod_{f\in P_{v_0}} \psi_{\mathbf{x}(f)}^{(\le h)\sigma(f)} \tag{57}$$

where

$$W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = \prod_{v\,\text{not e.p.}} \frac{1}{S_v!} \int dP_{T_v}(\mathbf{t}_v) det G^{h_v,T_v}(\mathbf{t}_v) \prod_{l\in T_v} \delta_{x_\ell,y_\ell} \bar{g}^{(h_v)}(x_\ell; x_{0,\ell} - y_{0,\ell} l) \tag{58}$$

$\mathbf{T}$ is the set of the tree graphs on $\mathbf{x}_{v_0}$ (which is a collection of several coordinates, defined after (52)), obtained by putting together an anchored tree graph T_v for each non trivial vertex v; $v_1^*, \ldots, v_n^*$ are the endpoints of τ, f_l^- and f_l^+ are the labels of the two fields forming the line l, "e.p." is an abbreviation of "endpoint".

Note that we can eliminate the Kronecker deltas in the propagators in the spanning tree T, so that only a single sum over the coordinate remain and the coordinate of the external

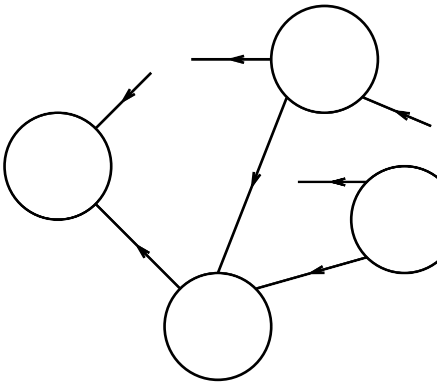

FIG. 2: A pictorial representation of one of the terms summed in the r.h.s. of (55); in the figure $s = 4$ and each of the monomial $\widetilde{\psi}^{(h)}(P)$ is represented as a set of half-lines external to a blob (not from a single point to take into account that they can have different coordinates). Some of the half-lines are contracted in propagators and their union represent the spanning tree $T \in \mathbf{T}$; the others are uncontracted and represent the fields in the determinant.

fields and of the fields in the determinants are assigned once that x, T and τ are given, as the interaction is quasi local; we can then write

$$V^{(h)}(\tau, \mathbf{P}) = \sum_{T\in\mathbf{T}} \sum_{x} \int dx_{0,v_0} H_{\tau,\mathbf{P},T}(x, x_{0,v_0}) \prod_{f\in P_{v_0}} \psi^{(\leq h)\sigma(f)}_{\widehat{\mathbf{x}}(f)} \tag{59}$$

where

$$H_{\tau,\mathbf{P},T}(x, x_{0,v_0}) = \prod_{v\,\text{not e.p.}} \frac{1}{S_v!} \int dP_{T_v}(\mathbf{t}_v) det G^{h_v,T_v}(\mathbf{t}_v) \prod_{l\in T_v} \bar{g}^{(h_v)}(\widehat{x}_\ell; x_{0,\ell} - y_{0,\ell})] \tag{60}$$

where $\widehat{\mathbf{x}}(f) = (\widehat{x}(f), x_0(f))$ and there is a field $\bar{f}$ such that $\widehat{x}(\bar{f}) = x$ and all the other coordinates $\widehat{x}(f)$ are assigned once that x, T and τ are given. We will call *resonances* the terms such that $\widehat{x}(f)$ is the same for all $f \in P_{v_0}$. Similarly $\widehat{x}_l$ is assigned, once that that x, T and τ are given.

In order to bound the above expression we introduce an Hilbert space $\mathcal{H} = \ell^2 \otimes \mathbb{R}^s \otimes L^2(\mathbb{R}^1)$ so that

$$G^{h,T}_{ij,i'j'} = \Big(\mathbf{v}_{x_{ij}} \otimes \mathbf{u}_i \otimes A(x_{0,ij}-, x_{ij}) \ , \ \mathbf{v}_{y_{i',j'}} \otimes \mathbf{u}_{i'} \otimes B(y_{0,i'j'}-, x_{ij})\Big) \ , \tag{61}$$

where $\mathbf{v} \in \mathbb{R}^L$ are unit vectors such that $(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$, $\mathbf{u} \in \mathbb{R}^s$ are unit vectors $(u_i, u_i) = t_{ii'}$, and A, B are vectors in the Hilbert space with scalar product

$$(A, B) = \int dz_0 A(x', x_0 - z_0) B^*(x', z_0 - y_0) \tag{62}$$

given by

$$\begin{aligned} A(x, x_0 - z_0) &= \frac{1}{\beta} \sum_{k_0} e^{-ik_0(x_0 - z_0)} \sqrt{\chi^{(u.v.)} f_h(|k_0|)} (k_0^2 + (\phi_x - \phi_{\bar{x}})^2)^{-1} \\ B(x, y_0 - z_0) &= \frac{1}{\beta} \sum_{k_0} e^{-ik_0(y_0 - z_0)} \sqrt{\chi^{(u.v.)} f_h(|k_0|)} (ik_0 + \phi_x - \phi_{\bar{x}}) \; . \end{aligned} \tag{63}$$

Moreover

$$||A_h||^2 = \int dz_0 |A_h(\mathbf{z})|^2 \leq C\gamma^{-3h} \; , \qquad ||B_h||^2 \leq C\gamma^{3h} \; , \tag{64}$$

for a suitable constant C.

If $\varepsilon_0 = \max\{|\lambda|, |\nu|\}$, by using (54) and (55), we get the bound

$$\begin{aligned} &\frac{1}{\beta L} \sum_{\tau \in \mathcal{T}_{M,h,n}} \sum_{T \in \mathbf{T}} \sum_{\mathbf{P} \in \mathbf{P}_\tau} \sum_{x} \int dx_{0,v_0} |H_{\tau,\mathbf{P},T}(x, x_{0,v_0})| \leq \\ &\sum_{\tau \in \mathcal{T}_{M,h,n}} \sum_{T \in \mathbf{T}} \sum_{\mathbf{P} \in \mathbf{P}_\tau} \Bigg[\prod_{v \text{ not e.p.}} \frac{1}{S_v!} \max_{\mathbf{t}_v} \left| \det G^{h_v, T_v}(\mathbf{t}_v) \right| \times \\ &\prod_{l \in T_v} \prod_{l \in T_v} \int d(x_{0,l} - y_{0,l}) | \sup_x |\bar{g}^{(h_v)}(x_l; x_{0,l} - y_{0,l})||| \Bigg] \end{aligned} \tag{65}$$

where, given the tree τ, $\mathbf{T}$ is the family of all tree graphs joining the space-time points associated to the endpoints, which are obtained by taking, for each non trivial vertex v, one of the anchored tree graph T_v appearing in (55), and by adding the lines connecting the two vertices associated to non local endpoints. Gram–Hadamard inequality (see for instance [30]), combined with (64), implies the dimensional bound:

$$|\det G^{h_v, T_v}(\mathbf{t}_v)| \leq C^{\sum_{i=1}^{S_v} |P_{v_i}| - |P_v| - 2(S_v - 1)} \; . \tag{66}$$

By the decay properties of $g^{(h)}(\mathbf{x})$ given by (46), it also follows that

$$\prod_{v \text{ not e.p.}} \frac{1}{S_v!} \prod_{l \in T_v} \int d(x_{0,l} - y_{0,l}) | \sup_x |\bar{g}^{(h_v)}(x_l; x_{0,l} - y_{0,l})|| \leq C^{n+m} \prod_{v \text{ not e.p.}} \frac{1}{S_v!} \gamma^{-h_v(S_v - 1)} \tag{67}$$

We can now perform the sum $\sum_{T \in \mathbf{T}}$, which erases the $1/S_v!$ up to a C^n factor. Then, by using the identity $\sum_{v' \geq v} (S_{v'} - 1) = n_v - 1$ and the bound $\sum_{v \geq v_0} [\sum_{i=1}^{S_v} |P_{v_i}| - |P_v| - 2(S_v - 1)] \leq$

$4n-2(n-1)$, we easily get the final bound

$$\sum_{n=1}^{\infty} C^n \varepsilon_0^n \sum_{\tau \in \mathcal{T}_{M,h,n}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}|=0}} \gamma^{-h(n-1)} \Big[\prod_{v \text{ not trivial}} \gamma^{-(h_v - h_{v'})(N_v - 1)} \Big] \tag{68}$$

where v' is the non trivial vertex immediately preceding v or v_0. This bound is suitable to control the expansion, if $n>1$, since $N_v>1$ (N_v is defined in (52)) for any non trivial vertex, as discussed below (51). If $n=1$ the allowed trees have only one endpoint of scale $h+1$.

Note that $\sum_{T \in \mathbf{T}}$ can be bounded by $\prod_v S_v! C^{\sum_{i=1}^{S_v} |P_{v_i}| - |P_v| - 2(S_v - 1)} \leq c^n \prod_v S_v!$. In order to bound the sum over τ, note that the number of unlabeled trees is $\leq 4^n$; moreover, as $N_v>1$ and, if $v>v_0$, $2 \leq |P_v| \leq 4N_v - 2(N_v-1)$, so that $N_v - 1 \geq |P_v|/6$,

$$\Big[\prod_{v \text{ not trivial}} \gamma^{-(h_v - h_{v'})(N_v - 1)} \Big] \leq \Big[\prod_{v \text{ not trivial}} \gamma^{-\frac{2}{5}(h_v - h_{v'})} \Big] \Big[\prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{10}} \Big] \tag{69}$$

The factor $\gamma^{-\frac{2}{5}(h_v - h_{v'})}$ can be used to bound the sum over the scale labels of the tree; moreover

$$\sum_{\mathbf{P} \in \mathcal{P}_\tau} \gamma^{-\frac{|P_v|}{10}} \leq C^n \tag{70}$$

Since the constant C is independent of M, β, the bounds above imply analyticity of the kernels in λ and ν, if ε_0 is small enough. It is an immediate consequence of the above bounds the proof of uniform convergence of the $M \to \infty$ limit; the proof of this is essentially identical to the one in [35] after (2.8) and it will not repeated here. ∎

3. THE INFRARED INTEGRATION AND THE SMALL DIVISOR PROBLEM

A. Multiscale analysis

In order to integrate the infrared scales we will use, in addition to (37), also the following property: if $P_g(d\psi)$ is a Grassmann integration with propagator g then

$$\frac{1}{N} \int P_g(d\psi) e^{-\nu \psi^+ \psi^-} f(\psi) = \int P_{g'}(d\psi) f(\psi) \tag{71}$$

with $g'^{-1} = g^{-1} + \nu$ and $\mathcal{N} = 1 + g\nu$; the general strategy will be to insert part of the quadratic terms of the effective potential in the fermionic integration at each iteration, so dynamically varying the propagator. We describe such a procedure inductively. Assume that we have integrated the fields $\psi^{(0)}...\psi^{(h+1)}$ obtaining

$$e^{-\beta L E_0}\int P(d\psi^{(\le 0)})e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})}=e^{-\beta L E_h}\int P(d\psi^{(\le h)})e^{-\mathcal{V}^{(h)}(\psi^{(\le h)})} \tag{72}$$

where $P(d\psi^{(\le h)})$ is the gaussian grassman integration with propagator, $\rho=\pm$

$$g^{(\le h)}_{\rho,\rho'}(\mathbf{x}',\mathbf{y}')=\delta_{x',y'}\bar g^{(\le h)}_{\rho.\rho'}(x',x_0-y_0) \tag{73}$$

with

$$g^{(\le h)}_{\rho,\rho'}(x',x_0-y'_0)=\int dk_0 e^{-ik_0(x_0-y_0)}\chi_h(\omega x',k_0)\times$$

$$\begin{pmatrix} -ik_0+v_0(\omega x')_{\mathrm{mod}.1}+r_{+,x'} & \sigma_h \\ \sigma_h & -ik_0-v_0(\omega x')_{\mathrm{mod}.1}+r_{-,x'}\end{pmatrix}^{-1}_{\rho,\rho'}\equiv \tag{74}$$

$$\int dk_0 e^{-ik_0(x_0-y_0)}\chi_h(\omega x',k_0)A^{-1}_{h,\rho,\rho'}(x',k_0) \tag{75}$$

where $\mathcal{V}^{(h)}$ can be written as sum over trees (similar to the ones for $\mathcal{V}^{(0)}$ and defined precisely below), and each tree with n end points contribute to $\mathcal{V}^{(h)}$ with a term of the form, after integrating the Koenecker deltas in the spanning tree as discussed before (59)

$$\sum_{x'}\int dx_{0,1}....\int dx_{0,n}H^{(h)}_{n;\rho_1,..,\rho_n}(x';x_{0,1},.,x_{0,n})[\prod_{i=1}^n\psi^{(\varepsilon_i)(\le h)}_{\mathbf{x}'_i,\rho_i}] \tag{76}$$

where the coordinates of the external fields x'_i are assigned once that x' and the labels of the tree are assigned. As in the previous section we call *resonant* the terms such that all the spatial components of coordinates of the external points are equal, that is $x'_i=x'_1\equiv x'$. We can split $\mathcal{V}^{(h)}$ in two parts

$$\mathcal{V}^{(h)}=\mathcal{V}^{(h)}_R+\mathcal{V}^{(h)}_{NR} \tag{77}$$

where in $\mathcal{V}^{(h)}_R$ are the resonant terms while in $\mathcal{V}^{(h)}_{NR}$ are the *non resonant terms.* We define a *localization* operation $\mathcal{L}$ as a linear operation acting on $\mathcal{V}^{(h)}$ in the following way:

1. On the non resonant part of the effective potential is defined as $\mathcal{L}\mathcal{V}^{(h)}_{NR}=0$.
2. On the resonant part of the effective potential its action consists in setting the time coordinate of the external fields equal

$$\mathcal{L}\sum_{x'}\int dx_{0,1}...\int dx_{0,n}H^{(h)}_{n,\rho_1,..,\rho_n}(x';x_{0,1},..,x_{0,n})[\prod_{i=1}^n\psi^{(\varepsilon_i)(\le h)}_{x',x_{0,i},\rho_i}]=$$

$$\sum_{x'}\int dx_{0,1}...\int dx_{0,n}H^{(h)}_{n,\rho_1,..,\rho_n}(x';x_{0,1},..,x_{0,n})[\prod_{i=1}^n\psi^{(\varepsilon_i)(\le h)}_{x',x_{0,1},\rho_i}] \tag{78}$$

From the above definitions it turns out that $\mathcal{L}\mathcal{V}^{(h)}$ is given by the following expression

$$\mathcal{L}\mathcal{V}^{(h)} = \gamma^h \nu_h F_\nu^{(h)} + F_z^{(h)} + s_h F_\sigma^{(h)} + \ F_\zeta^{(h)} + F_\lambda^{(h)} = s_h F_\sigma^{(h)} + \bar{\mathcal{L}}\mathcal{V}^{(h)} \tag{79}$$

where, if $\mathbf{x}' = (x + \rho\bar{x}, x_0)$

$$\begin{aligned}
F_\sigma^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 \psi^{+(\le h)}_{\mathbf{x}',\rho} \psi^{-(\le h)}_{\mathbf{x}',-\rho} \qquad F_\nu^{(h)} = \sum_\rho \sum_{x'} \int dx_0 \psi^{+(\le h)}_{\mathbf{x}',\rho} \psi^{-(\le h)}_{\mathbf{x}',\rho} \\
F_\zeta^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 (\omega x')_{\text{mod}.1} \zeta_{h,\rho}(x') \psi^{+(\le h)}_{\mathbf{x}',\rho} \widehat{\psi}^{-(\le h)}_{\mathbf{x}',-\rho} \\
F_z^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 (\omega x')_{\text{mod}1} z_{h,\rho}(x') \psi^{+(\le h)}_{\mathbf{x}',\rho} \widehat{\psi}^{-(\le h)}_{\mathbf{x}',\rho} \\
F_\lambda^{(h)} &= \sum_{x'} \int dx_0 \lambda_h(x') \psi^{+(\le h)}_{\mathbf{x}',+} \psi^{-(\le h)}_{\mathbf{x}',+} \psi^{+(\le h)}_{\mathbf{x}',-} \psi^{-(\le h)}_{\mathbf{x}',-}
\end{aligned} \tag{80}$$

where $\widehat{\partial}_x H^{(h)}_{2,\rho,-\rho}(x', x_0, y_0) \equiv \frac{H^{(h)}_{2,\rho,-\rho}(x', x_0, y_0) - H^{(h)}_{2,\rho,-\rho}(0, x_0, y_0)}{(\omega x')_{\text{mod}1}}$ and

$$\begin{aligned}
s_h &= \frac{1}{\beta} \int dx_0 dy_0 H^{(h)}_{2,\rho,-\rho}(0, x_0, y_0) \qquad \nu_h = \frac{1}{\beta} \int dx_0 dy_0 H^{(h)}_{2,\rho,\rho}(0, x_0, y_0) \\
\zeta_{h,\rho}(x') &= \frac{1}{\beta} \int dx_0 dy_0 \widehat{\partial}_x H^{(h)}_{2,\rho,-\rho}(x', x_0, y_0) \quad z_{h,\rho}(x') = \frac{1}{\beta} \int dx_0 dy_0 \widehat{\partial}_x H^{(h)}_{2,\rho\rho}(x', x_0, y_0) \\
\lambda_h(x') &= \frac{1}{\beta} \int dx_{0,1} ... dx_{0,4} H_4^{(h)}(x'; x_{0,1}, x_{0,2}, x_{0,3}, x_{0,4})
\end{aligned} \tag{81}$$

Note that in $\mathcal{L}\mathcal{V}^{(h)}$ there are no terms with 6 or more fields, as consequence of anticommutativity. Moreover the s_h, ν_h coefficients are *independent* from ρ and real. Note indeed that (21) is invariant under parity $x \to -x$ (in the limit $L \to \infty$), and this implies invariance under the transformation $\psi^{\pm(h)}_{x_0,x',\rho} \to \psi^{\pm(h)}_{x_0,-x',-\rho}$; therefore, if $\varepsilon = \pm$

$$\sum_{\rho,x'} \int dx_0 H^{(h)}_{2,\rho,\varepsilon\rho}(x', x_0, 0) \psi^{+(\le h)}_{x_0,x',\rho} \psi^{+(\le h)}_{x_0,x',\varepsilon\rho} = \sum_{\rho,x'} \int dx_0 H^{(h)}_{2,-\rho,-\varepsilon\rho}(-x', x_0, 0) \psi^{+(\le h)}_{x_0,x',\rho} \psi^{+(\le h)}_{x_0,x',\varepsilon\rho} \tag{82}$$

so that from (81) the independence from ρ of σ_h, ν_h follows. Moreover $g^*(k_0, x) = g(-k_0, x)$ so that $(\widehat{H}^{(h)}_{2,\rho,\varepsilon\rho}(x', k_0))^* = \widehat{H}^{(h)}_{2,\rho,\varepsilon\rho}(x', -k_0)$, and this implies reality.

We also define a *renormalization* operation as $\mathcal{R} = 1 - \mathcal{L}$ and using (71) we can rewrite (83) as

$$\int P(d\psi^{(\le h)}) e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)})} = \int P(d\psi^{(\le h)}) e^{-s_h F_\sigma^{(h)}(\psi^{(\le h)}) - \bar{\mathcal{L}}\mathcal{V}^{(h)}(\psi^{(\le h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)})} =$$

$$e^{-\beta L t_h} \int \widetilde{P}(d\psi^{(\le h)}) e^{-\bar{\mathcal{L}}\mathcal{V}^{(h)}(\psi^{(\le h)}) - \mathcal{R}\mathcal{V}^{(h)}} \tag{83}$$

with t_h coming from the normalization in (71)and $\widetilde{P}(d\psi^{(\leq h)})$ with a propagator $\widetilde{g}^{(\leq h)}$ coinciding with $g^{(\leq h)}$ with σ_h replaced by σ_{h-1} with $\sigma_{h-1}(\omega x', k_0) = \sigma_h + \chi_h^{-1}(\omega x', k_0)s_h$ and $\sigma_h \equiv \sigma_h(0,0)$; moreover $\sigma_0 = 0$.

We write then

$$\int P(d\psi^{\leq h-1}) \int P(d\psi^{(h)}) e^{-\mathcal{L}\bar{\mathcal{V}}^{(h)} - \mathcal{R}\mathcal{V}^{(h)}} = e^{-\beta L \widetilde{E}_h} \int P(d\psi^{(\leq h-1)}) e^{-\mathcal{V}^{(h-1)}(\psi^{(\leq h-1)})} \tag{84}$$

where $P(d\psi^{\leq h-1})$ have propagator $g^{(\leq h-1)}$ coinciding with (75) with $h-1$ replacing h, and $P(d\psi^{(h)}$ has propagator $g^{(h)}$ coinciding with $g^{(\leq h-1)}$ with χ_{h-1} replaced by f_h, with f_h a smooth compact support function vanishing for $c_1\gamma^{h-1} \leq \sqrt{k_0^2 + v_0^2||\omega x'||_1^2} \leq c_2\gamma^{h+1}$, for a suitable constants c_1, c_2. From the r.h.s. of (84), the procedure can be iterated. The above procedure allows to write the $W(0)$ (38) in terms of an expansion in the *running coupling constants* $\vec{v}_h = (\nu_h, \zeta_{h,\rho}, z_{h,\rho})$ with $h \leq 0$; as it is clear from the above construction, they verify a recursive equation of the form

$$\vec{v}_{h-1} = \vec{v}_h + \vec{\beta}^{(h)}(\vec{v}_h, ..\vec{v}_0) \tag{85}$$

The single scale propagator $g^{(h)}$ verify the following bound, for any integer N and a suitable constant C_N

$$|\bar{g}^{(h)}(x, x_0 - y_0)| \leq \frac{C_N}{1 + (\gamma^h |x_0 - y_0|)^N} \tag{86}$$

which can be easily obtained integrating by parts; the propagator in the infrared region then verifies the same bound than in the ultraviolet region (45). The bound (86) can however be improved at low scales. Note indeed that, in the above integration procedure, the propagator $g^{(h)}$ is "massive" due to the presence of σ_h in (75). We can then naturally define a scale h^* as

$$\gamma^{h^*} = \inf\{h : h \leq 0, \gamma^k \geq |\sigma_k| \quad \text{for} \quad \text{any} \quad k \leq h\} \tag{87}$$

and the following bound is valid

$$|\bar{g}^{(\leq h^*)}(x, x_0 - y_0)| \leq \frac{C_N}{1 + (\gamma^{h^*} |x_0 - y_0|)^N} \tag{88}$$

saying that the propagator of all the scales $\leq h^*$ verifies the same bound of the single scale propagator corresponding to a scale $h > h^*$; this fact, saying essentially that σ_h is an dynamically generated infrared cut-off, will be used to integrate all scale $\leq h^*$ in a single step.

B. Tree expansion

Also in the infrared region $\mathcal{V}^{(h)}$ can be written as sum over trees, up to the following modifications to take into account the different multiscale integration procedure.

1. The scale index now is an integer taking values in $[h, 2]$, h being the scale of the root.

2. With each vertex v of scale $h_v = +1$, which is not an endpoint, we associate one of the terms contributing to $-\mathcal{V}^{(0)}(\psi^{(\leq 0)})$, in the limit $M = \infty$. With each endpoint v of scale $h_v \leq 1$ we associate one of local terms that contribute to $\mathcal{L}\mathcal{V}^{(h_v-1)}$, and there is the constrain that $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding it or v_0; to the end-points of scale $h_v = 2$ are associated one of the terms contributing to $-\mathcal{V}$ and there is not such a constrain.

3. With each trivial or non trivial vertex $v > v_0$, which is not an endpoint, we associate the $\mathcal{R} = 1 - \mathcal{L}$ operator, acting on the corresponding kernel.

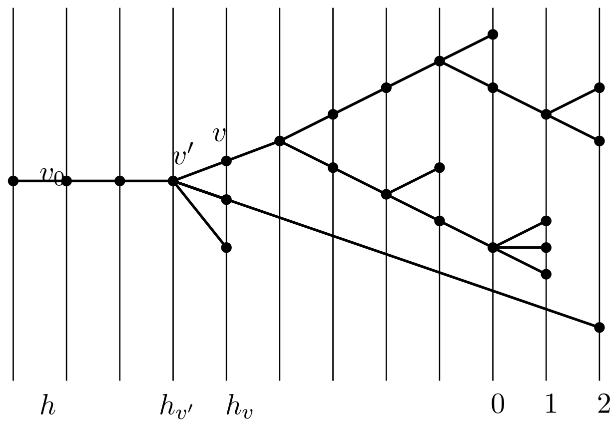

FIG. 3: A tree $\tau \in \mathcal{T}_{h,n}$ with its scale labels.

A vertex v which is not an end-point such that the spatial coordinates x' in P_v are all equal is called *resonant vertex*, while if the coordinates are different is called *non resonant vertex*; the set of resonant vertices is denoted by H and the set of non-resonant vertices is denoted by L. If $v_1, \ldots, v_{S_v}$ are the $S_v \geq 1$ vertices following the vertex v, we define

$$S_v = S_v^L + S_v^H + S_v^2 \tag{89}$$

where S_v^L is the number of *non resonant* vertices following v, S_v^H is the number of *resonant* vertices following v, while S_v^2 is the number of trivial trees with root v associated to end-points.

If $h \le -1$, the effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + L\beta\widetilde{E}_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} V^{(h)}(\tau,\psi^{(\le h)}) \tag{90}$$

where, if v_0 is the first vertex of τ and $\tau_1,..,\tau_s$ $(s = s_{v_0})$ are the subtrees of τ with root v_0, $V^{(h)}(\tau,\psi^{(\le h)})$ is defined inductively by the relation, if $s > 1$

$$V^{(h)}(\tau,\psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[\bar{V}^{(h+1)}(\tau_1,\psi^{(\le h+1)});..;\bar{V}^{(h+1)}(\tau_s,\psi^{(\le h+1)})] \tag{91}$$

where $\bar{V}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$:

1. it is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$, with $\mathcal{R}$ given by (105), if the subtree τ_i is non trivial;

2. if τ_i is trivial and $h \le -1$, it is equal to one of the terms of $\mathcal{L}\mathcal{V}^{h+1}$ or, if $h = 0$, to one of the terms in the $\mathcal{V}$.

By using (91) and the representation of the truncated expectations we get

$$V^{(h)} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{T\in\mathbf{T}}\sum_{x}\int dx_{0,v_0} H_{\tau,\mathbf{P},T}(x,x_{0,v_0})\prod_{f\in P_{v_0}}\psi^{(\le h)\sigma(f)}_{\widehat{\mathbf{x}}'(f),\rho(f)}\Big\} \tag{92}$$

where one of the spatial coordinates $\widehat{\mathbf{x}}'(f)$ of the external fields $\prod_{f\in P_{v_0}}\psi^{(\le h)\sigma(f)}_{\widehat{\mathbf{x}}'(f),\rho(f)}$ is equal to $\mathbf{x}'$ and the others are determined once that $\tau,\mathbf{T},\mathbf{P}$ are given.

Given a tree τ and $\mathbf{P}\in\mathcal{P}_\tau$, we shall define

1. The χ-vertices are the vertices v of τ , such that $\mathcal{I}_v$ (the union of of the subsets $P_{v_i}\backslash Q_{v_i}$ defined before (53), that is the set of lines contracted in v) is non empty.

2. V_χ is the family of all χ-vertices, whose number is of order n; moreover we call H_χ the resonant vertices belonging to V_χ and L_χ the non-resonant vertices belonging to V_χ.

3. $\bar{v}'$ the first vertex belonging to V_χ following v in τ.

In order to bound (92) we could proceed exactly as in §2. We define $\vec{v}_h = \varepsilon\widetilde{v}_h$ where v_h are the running coupling constants. Therefore, each contribution from the tree $\tau\in\mathcal{T}_{h,n}$ is proportional to a factor ε^n. If we neglect the $\mathcal{R}$ operation (that is, if we bound the modulus of the differences produced by $\mathcal{R}$ simply by the sum of the modulus) we get, assuming $\widetilde{v}_h$ smaller than a constant

$$\frac{1}{\beta L}\sum_x \int dx_{0,v_0} |H_{\tau,\mathbf{P},T}(x, x_{0,v_0})| \leq C^n \varepsilon^n \prod_{v \in V_\chi} \gamma^{-h_v(S_v-1)} \tag{93}$$

This bound is indeed very similar to (67) for the integration of the ultraviolet scales, the reason being being that that the bound for the single scale propagator (86) is the same both in the ultraviolet (positive scales h) or in the infrared (negative scales h) regime. However the bound (93) is unsuitable to get convergence of the sum over $\tau, \mathbf{P}$, the reason simply being that the scales h_v are *negative* and the factor $\prod_v \gamma^{-h_v(S_v-1)}$ forbids the summations over the scales (in §2 the scales were instead positive). In the Renormalization Group terminology, the infrared region correspond to a *non-renormalizable* theory which the ultraviolet region is *superrenormalizable.*

By using (113) we can write the r.h.s. of (93) as

$$\prod_{v \in V_\chi} \gamma^{-(S_v-1)h_v} = \prod_{v \in V_\chi} \gamma^{-(S_v^H + S_v^L + S_v^2 - 1)h_v} \tag{94}$$

In §3.C we will see that the contribution of the non resonant vertices L_χ can be improved taking into account certain constraint for the size of the small divisors due to the Diophantine condition. There is not such improvement for the resonant vertices, and this is why we have introduced the renormalized expansion defining the $\mathcal{R}$ operation acting on them, see §3.D.

C. Non resonant terms and the Diophantine condition

It is convenient to write in a more explicit way the relations between the coordinates $\widehat{x}'(f)$ of the external fields produced by the tree expansion. In order to do that we give the following definitions.

1. We define a tree $\bar{T}_v = \bigcup_{w \geq v} T_v$ starting from T_v and attaching to it the trees $T_{v_1}, .., T_{v_{S_v}}$ associated to the vertices $v_1, .., v_{S_v}$ following v, and repeating this operation until the end-points are reached. The tree $\bar{T}_v$ is composed by a set of lines, representing propagators with scale $\geq h_v$, connecting end-points w of the tree τ. Note that, contrary to T_v, the vertices of $\bar{T}_v$ are connected with at most four lines. To each vertex w of $\bar{T}_v$ is associated a coordinate x_w.

2. To each line ℓ of $\bar{T}_v$ we associate a label $a_\ell = 0, \pm 2\bar{x}$ respectively if the corresponding propagator $g^{(h_\ell)}_{\rho,\rho'}$ has $\rho = \rho'$, $\rho = -\rho' = 1$ or $\rho = -\rho' = -1$.

3. To each line coming in or out w is associated a factor $\delta^{i_w}_w$, where i_w is a label identifying the lines connected to w. The vertices w (which corresponds to the end-points of τ) can be of type λ, ν or λ_h, z_h, ζ_h, and a) $\delta^i_w = 0$ if w if it corresponds to a ν or ν_h, z_h end-point; b) $\delta^i_w = \pm 2\bar{x}$ if w if it correspond to a ζ_h end-point; c) $\delta^i_w = \pm 1$ it corresponds to an ε end-point; d) $\delta^i_w = (0, \pm 1)$ is a λ end-point f) $\delta^i_w = (0, \pm 2\bar{x})$ if is a λ_h end-point

Note that the value of such indices (and correspondingly the value of $\widehat{x}(f)$ is determined by the choice of $\tau, \mathbf{P}, \mathbf{T}$.

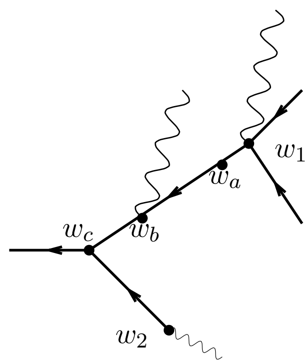

FIG. 4: A tree $\bar{T}_v$ with attached wiggly lines representing the external lines P_v; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2, representing the end-points following v in τ.

According to the above definitions, consider two vertices w_1, w_2 such that x'_{w_1} and x'_{w_2} are coordinates of the external fields, and let be c_{w_1,w_2} the path (vertices and lines) in $\bar{T}_v$ connecting w_1 with w_2 (in the example in Fig. 4 the path is composed by w_1, w_a, w_b, w_c, w_2 and the corresponding lines) ; as the path is a linear tree there is a natural orientation in the vertices, and we we call i_w the label of the line exiting fom w in c_{w_1,w_2}. We call $|c_{w_1,w_2}|$ the number of vertices in c_{w_1,w_2}. The following relation holds

$$x'_{w_1} - x'_{w_2} = (\rho_{\ell_{w_2}} - \rho_{\ell_{w_1}})\bar{x} + \sum_{w \in c_{w_1,w_2}} \delta^{i_w}_w + \sum_{\ell \in c_{w_1,w_2}} a_\ell \tag{95}$$

The Diophantine condition implies a relation between the scale h_v and the number of vertices between w_2 and w_1.

Lemma 3.1 *Given $\tau, \mathbf{P}, \mathbf{T}$, let us consider $v \in L_\chi$ and w_1, w_2 two vertices in $\bar{T}_v$ with $x'_{w_1} \neq x'_{w_2}$; then*

$$|c_{w_1,w_2}| \geq A\bar{x}^{-1}\gamma^{\frac{-h_{\bar{v}'}}{\tau}} \tag{96}$$

with a suitable constant A.

Proof. Note that $||\omega x'_{w_i}||_1 \leq cv_0^{-1}\gamma^{h_{\bar{v}'}-1}$, $i = 1,2$ by the compact support properties of the propagator; therefore by using (95) and the Diophantine condition

$$\begin{aligned} &2cv_0^{-1}\gamma^{h_{\bar{v}'}} \geq ||(\omega x'_{w_1})||_1 + ||(\omega x'_{w_2})||_1 \geq ||\omega(x'_{w_1} - x'_{w_2})||_1 = \\ &||(\rho_{\ell_{w_2}} - \rho_{\ell_{w_1}})\bar{x} + \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w} + \sum_{\ell \in c_{w_1,w_2}} a_\ell)||_1 \geq \\ &C_0|(\rho_{\ell_{w_2}} - \rho_{\ell_{w_1}})\bar{x} + \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w} + \sum_{\ell \in c_{w_1,w_2}} a_\ell)|^{-\tau} \geq C_0(4\bar{x}c_{w_2,w_1}|)^{-\tau} \end{aligned} \tag{97}$$

from which (96) follows. ∎

Lemma 3.1 says that there is a relation between the number of end-points following $v \in L_\chi$ and the scales of the external lines coming out from v.

Lemma 3.2 *If $v \in V_\chi$ and $N_v = \sum_{i, v_i^* > v} 1$ is the number of end-points following v in τ then*

$$\varepsilon^n \leq \varepsilon^{\frac{n}{2}} \prod_{v \in V_\chi} \varepsilon^{N_v 2^{h_{\bar{v}'}-1}} \tag{98}$$

Proof We can write

$$\varepsilon^{\frac{1}{2}} = \prod_{h=-\infty}^{0} \varepsilon^{2^{h-2}} \tag{99}$$

Given a tree $\tau \in \mathcal{T}_{h,n}$, we consider an end-point v^* and the path in τ from v^* to the root v_0; to each vertex $v \in V_\chi$ in such path with scale h_v we associate a factor $\varepsilon^{2^{h_v-2}}$; repeating such operation for any end-point, the vertices v followed by N_v end-points are in N_v paths, therefore we can associate to them a factor $\varepsilon^{N_v 2^{h_v-2}}$; finally we use that $\varepsilon^{2^{h_v-2}} < \varepsilon^{2^{h_{\bar{v}'}-2}}$. ∎

It is an immediate consequence of Lemma 3.1 and Lemma 3.2 the following result, ensuring that we can extract from the ε^n factor a small factor to be associated to the non resonant vertices.

Lemma 3.3 *Given $\tau, \mathbf{P}, \mathbf{T}$ the following inequality holds*

$$\varepsilon^{\frac{n}{4}} \leq \prod_{v \in L_\chi} \varepsilon^{A\bar{x}^{-1}\gamma^{\frac{-h_{\bar{v}'}}{\tau}} 2^{h_{\bar{v}'}-1}} \tag{100}$$

Proof. Note that if v is non resonant, there exists surely two external fields with coordinates x'_1, x'_2 such that $x'_1 \neq x'_2$; note that

$$N_v \geq |c_{w_1,w_2}| \geq A\bar{x}^{-1}\gamma^{\frac{-h_{\bar{v}'}}{\tau}} \tag{101}$$

therefore, by (98), (100) follows, ■.

By combing the above results we get the following final lemma which will play a crucial role in the following.

Lemma 3.4 *If $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$, given $\tau, \mathbf{P}, \mathbf{T}$ the following inequality holds*

$$[\prod_{v \in V_\chi} \gamma^{-h_v S_v^L}][\prod_{v \in L_\chi} \varepsilon^{A\bar{x}^{-1}\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}}}] \leq \bar{C}^n \tag{102}$$

with $\bar{C} = [\frac{3}{|\log \varepsilon| A\bar{x}^{-1}]}]^3 e^{-3}]$.

Proof As we assumed $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$ than, for any N

$$\varepsilon^{A\bar{x}^{-1}\gamma^{\frac{-h}{\tau}} 2^h} = e^{-|\log \varepsilon| A\bar{x}^{-1}\gamma^{-\eta h}} \leq \gamma^{N\eta h} \frac{N}{|\log \varepsilon| A\bar{x}^{-1}]^N e^N} \tag{103}$$

as $e^{-\alpha x} x^N \leq [\frac{N}{\alpha}]^N e^N$. Therefore, by choosing $N = 3$ we get

$$\prod_{v \in L_\chi} \varepsilon^{A\bar{x}^{-\tau}\gamma^{\frac{-h_{\bar{v}'}}{\tau}} 2^{h_{\bar{v}'}}} \leq \bar{C}^n \prod_{v \in V_\chi} \gamma^{3 S_v^L h_v} \tag{104}$$

■

D. Renormalization of the resonant terms

By lemma 3.4 we see that the contribution from the *non resonant* vertices $v \in V_\chi$ can be bounded by exploiting the Diophantine condition. On the other hand, the $\mathcal{R} = 1 - \mathcal{L}$-operation, with $\mathcal{L}$ defined in (78), is defined exactly to deal with the *resonant* vertices. The $\mathcal{R}$ acts on the resonant terms and its action is

$$\mathcal{R} \sum_{x'} \int dx_{0,1} .... \int dx_{0,n} H^{(h)}_{n;\rho_1,..,\rho_n}(x'; x_{0,1}, ., x_{0,n}) [\prod_{i=1}^n \psi^{(\varepsilon_i)(\leq h)}_{x',x_{0,i},\rho_i}] = \sum_{x'} \int dx_{0,1} .... \int dx_{0,n}$$

$$\{H^{(h)}_{n,\rho_1,..,\rho_n}(x'; x_{0,1}, .., x_{0,n}) [\prod_{i=1}^n \psi^{(\varepsilon_i)(\leq h)}_{x',x_{0,i},\rho_i} - \prod_{i=1}^n \psi^{(\varepsilon_i)(\leq h)}_{x',x_{0,1},\rho_i}]\} \tag{105}$$

We can write the difference $[\prod_{i=1}^n \psi^{(\varepsilon_i)(\le h)}_{x',x_{0,i},\rho_i} - \prod_{i=1}^n \psi^{(\varepsilon_i)(\le h)}_{x',x_{0,1},\rho_i}]$ as a sum of products of fields in which one $\psi^{(\varepsilon)(\le h)}_{x',x_{0,i}}$ field is replaced by a D-field defined as

$$D^{\varepsilon(\le h)}_{x',x_{0,1},x_{0,i},\rho_i} = \psi^{(\varepsilon)(\le h)}_{x',x_{0,1},\rho_i} - \psi^{(\varepsilon)(\le h)}_{x_{0,i},x',\rho_i} \tag{106}$$

This means that the effect of the $\mathcal{R}$ operation on a resonant vertex v can be expressed replacing on of the $\psi^{\varepsilon(\le h_v)}$ fields in P_v with $D^{\varepsilon(\le h_v)}$ (see for instance §3.1 of [36] for more details in a similar case). The corresponding propagator can be written as

$$g^{(h_{v'})}(x_{0,1} - z_0, x') - g^{(h_{v'})}(x_{0,i} - z_0, \bar{x}') = (x_{0,1} - x_{0,i}) \int_0^1 dt \partial g^{(h_{v'})}(\widehat{x}_{0,1i}(t) - z_0, x') \tag{107}$$

where $\widehat{x}_{0,1i}(t) = x_{0,1} + t(x_{0,i} - x_{0,1})$ is an interpolated point between $x_{0,1}$ and $x_{0,2}$. Note that, for any integer α, β

$$\begin{aligned} |(x_0 - y_0)^\alpha \partial^\beta g^{(h_v)}(x_0 - y_0)|_{L_\infty} &\le C_{\alpha,\beta} \gamma^{-\alpha h_v} \gamma^{\beta h_v} \\ |(x_0 - y_0)^\alpha \partial^\beta g^{(h_v)}(x_0 - y_0)|_{L_1} &\le C_{\alpha,\beta} \gamma^{-\alpha h_v} \gamma^{\beta h_v} \gamma^{-h} \end{aligned} \tag{108}$$

Therefore the effect of a non trivial $\mathcal{R}$ operation on a vertex v is twofold. From one side an extra factor $(x_{0,1} - x_{0,i})$ is produced, which can be written can as $(x_{0,1} - x_{0,i}) = \sum_r (x_{0,r} - x_{0,r-1})$ where $x_{0,r}$ are points in the spanning tree $\bar{T}_v$ defined above; then the factor $(x_{0,r} - x_{0,r-1})$ in the integration over the coordinates (similar to (67)) produces an extra factor γ^{-h_v} for any resonant $v \in L_\chi$. One the other hand one of the propagators associated to the external line in P_v carry an extra derivative, so that an extra factor $\gamma^{h_{\bar{v}'}}$ is obtained; therefore , with respect to the bounds in which there are no D fields, one has an extra factor in the bound (see §3.2-§3.9 for more details in a similar case)

$$\prod_{v \in V_\chi} \gamma^{(h_{\bar{v}'} - h_v)} \tag{109}$$

As we will see, the extra factors $\gamma^{h_{\bar{v}'}}$ produced by the $\mathcal{R}$ operation can be used to compensate the factors $\gamma^{-h_v S_v^H}$ in (94).

E. Bounds for vertices with a large number of external fields

In order to sum over $\mathbf{P}$ (92) we have to show that there is some gain factor also on the vertices with a large number of external fields. Let us consider the vertices $v \in V_\chi$

with $|P_v| \geq 6$ we call $\bar{\rho}, \bar{\varepsilon}$ the labels of the external fields whose number is maximal; we define this set m_v and $|m_v| \geq |P_v|/4$. We consider a tree $\bar{T}_v$ and we define a pruning operation associating to it another tree $\widehat{T}_v$ eliminating from $\bar{T}_v$ all the trivial vertices w in $\bar{T}_v$ not associated to any external line with label $\bar{\rho}, \bar{\varepsilon}$, and all the subtrees not containing any external line with label $\bar{\rho}, \bar{\varepsilon}$ (see Fig. 5 for an example), so that there is an external line associated to all end-points.

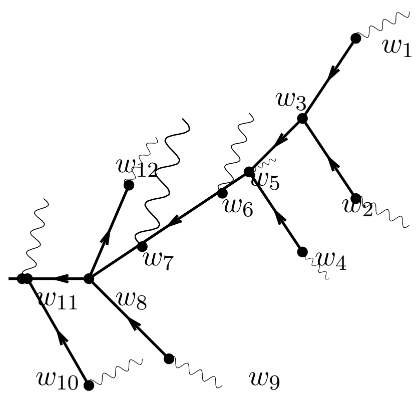

FIG. 5: In the picture the lines represent the propagators with scale $\leq h_v$ in $\widehat{T}_v$ and the wiggly lines represent the external lines P_v with label $\bar{\rho}$; note that, by definition of the pruning operation, all the end-points have associated wiggly lines, contrary to what happens in $\bar{T}_v$, see Fig. 4.

The vertices w of $\widehat{T}_v$ are then only non trivial vertices or trivial vertices with external lines $\bar{\rho}, \bar{\varepsilon}$; all the end-points have associated an external line. We define a procedure to group in two sets the fields in m_v. We start considering the end-points w_a immediately followed by vertices w_b with external lines (in the figure w_4, w_{10}), and we say that the couple of fields in w_a, w_b is of type 1 if $x'_{w_a} = x'_{w_b}$, while it is of type 2 if $x'_{w_a} \neq x'_{w_b}$. If $x'_{w_a} = x'_{w_b}$ we can replace the ψ field in w_b with a D field

$$\psi^{\varepsilon(\leq h_v-1)}_{\mathbf{x}'_{w_b},\rho}\psi^{\varepsilon(\leq h_v-1)}_{\mathbf{x}'_{w_a},\rho} = \psi^{\varepsilon(\leq h_v-1)}_{\mathbf{x}'_{w_b},\rho}(\psi^{\varepsilon(\leq h_v-1)}_{\mathbf{x}'_{w_a},\rho} - \psi^{\varepsilon(\leq h_v-1)}_{\mathbf{x}'_{w_b},\rho}) \tag{110}$$

We now prune tree $\widehat{T}_v$ canceling the end-points w already considered and the resulting subtrees with no external lines; in the resulting tree we select an end-point w_a immediately followed by vertices w_b, and again such a couple can be of type 1 or 2. We again prune the tree and we continue unless there are no end-points w followed by vertices with wiggly line. Then in the resulting tree we consider (if they are present, otherwise the tree is trivial and the procedure ends) a couple of endpoints followed by the same non trivial vertex (in the picture w_1, w_2); we call them w_a, w_b and we proceed exactly as above distinguishing the two kind of couples. We then cancel such end-points and the subtrees not containing external lines, so that the end-points are associated to external lines; we consider end-points followed by non trivial vertices with no external lines, and we proceed in the same way. If the resulting tree has again end-points with external lines followed by vertices with external lines (in the picture w_5), we prune such vertices as described above and we continue in this way so that at the end all except at most one vertex with external lines are considered. Note that by construction the paths c_{w_a,w_b} in $\bar{T}_v$ do not overlap; for instance in Fig.5 the paths can be, if the corresponding coordinates are different, $c_{w_{10},w_{11}}$, c_{w_4,w_5}, c_{w_1,w_2}, c_{w_5,w_6}, c_{w_6,w_7}, $c_{w_7,w_{12}}$, $c_{w_9,w_{11}}$.

Therefore, given a vertex v in the tree τ, we have paired all the external fields with index $\bar{\rho}, \bar{\varepsilon}$, whose number m_v is $m_v \geq |P_v|/4$, in couples both with the same x' or with different x'; we write $m_v = m_v^{(1)} + m_v^{(2)}$, where $m_v^{(1)}$ are the fields in couples with the same x' and $m_v^{(2)}$ are the fields in couples with different x'. In the couple of fields w, w' with $x'_w = x'_{w'}$ one of the fields is a D field and, by (108), this produces in the bound an extra $\gamma^{h_{\bar{v}'}-h_v}$ for each couple, so that we get an extra factor $\gamma^{-|m_v^{(1)}|(h_{\bar{v}'}-h_v)}$. For each couple w, w' with $x'_w \neq x'_{w'}$, we have $|c_{w,w''}| \geq B\gamma^{-h_{\bar{v}'}/\tau}$ by lemma 3.1 so that

$$\varepsilon^{|c_{w,w'}|2^{h_{v'}}} \leq \varepsilon^{B\gamma^{-h_{\bar{v}'}/\tau}2^{h_{\bar{v}'}}} \tag{111}$$

Moreover by Lemma 3.2 we can associate to each $v \in V_\chi$ a factor $\varepsilon^{N_v 2^{h_{\bar{v}}-1}}$ with N_v the vertices in $\bar{T}_v$; as the paths $c_{w,w'}$ are non overlapping, we get one of the factors (111) for each of the couples in m_v^2 so that

$$\varepsilon^{\frac{n}{4}} \prod_{v \in V_\chi} \gamma^{A|m_v^1|(h_{\bar{v}'}-h_v)} \leq \prod_{v \in V_\chi} \varepsilon^{\gamma^{-h_{\bar{v}'}/\tau}2^{h_{\bar{v}'}}|m_v^2|} \prod_{v \in V_\chi} \gamma^{|m_v^1|(h_{\bar{v}'}-h_v)} \leq \prod_{v \in V_\chi} \gamma^{-|P_v|/8} \tag{112}$$

Remark. It can happen that a D field emerging from a vertex v_1 is not contracted in v_2, with $v_2 = \bar{v}'_1$, but in a vertex $v_3 < v_2$; then the corresponding gain is $\gamma^{h_{\bar{v}'_2}-h_{v_1}} = \gamma^{h_{\bar{v}'_2}-h_{v_2}}\gamma^{h_{v_2}-h_{v_1}}$; therefore if in v_2 the $\mathcal{R}$ operation acts non trivially, one can simply bound the absolute value of the difference of terms due to the action of $\mathcal{R}$ in v_2 by the sum of the absolute values. Similarly if such field is in a couple w, w' belonging to $m_{v_2}^1$, there is no need to use (110) as one of the two fields is already a D-field. It is useful to avoid unnecessary renormalization as they could produce too many derivatives on a single propagators, see §3.1-§3.10 of [36] for more details in a similar case.

F. Bounds

In this section we get a bound for the kernels of the effective potential defined in (92).

Lemma 3.5 *If* $\vec{v}_h = (\widetilde{\lambda}_h, \widetilde{\nu}_h, \widetilde{z}_h, \widetilde{\zeta}_h) \equiv (\widetilde{\lambda}_h, \widetilde{\alpha}_h)$ *then*

$$\frac{1}{\beta L}\sum_{\tau\in\mathcal{T}_{M,h,n}}\sum_{T\in\mathbf{T}}\sum_{\mathbf{P}\in\mathbf{P}_\tau}\sum_x\int dx_{0,v_0}|H_{\tau,\mathbf{P},T}(x,x_{0,v_0})| \leq C^n|\varepsilon|^{\frac{n}{2}}|h|^{2n}(\gamma^{-h}|\sup_{k\geq h}|\widetilde{\lambda}_k||)^{n_\lambda}(\sup_{k\geq h}|\widetilde{\alpha}_k||)^{n_a} \tag{113}$$

where C is a suitable constant and n_λ, n_α is the number of end-points of type λ, α.

Proof The matrix $\widetilde{G}^{h,T}_{ij,i'j'}$ can be written as

$$\widetilde{G}^{h,T}_{ij,i'j'} = \Big(\mathbf{v}_{x_{ij}}\otimes\mathbf{u}_i\otimes A(x_{0,ij}-,x_{ij})\ ,\ \mathbf{v}_{y'_{i',j'}}\otimes\mathbf{u}_{i'}\otimes B(y_{0,i'j'}-,x_{ij})\Big)\ , \tag{114}$$

where $\mathbf{v}\in\mathbb{R}^L$ are unit vectors such that $(\mathbf{v}_i,\mathbf{v}_j)=\delta_{ij}$, $\mathbf{u}\in\mathbb{R}^s$ are unit vectors $(u_i,u_i)=t_{ii'}$, and A, B are vectors in the Hilbert space with scalar product

$$(A,B)=\int dz_0 A(x_0-z_0,x')B^*(z_0-y_0,x') \tag{115}$$

given by

$$\begin{aligned} A(x_0-z_0,x') &= \frac{1}{\beta}\sum_{k_0} e^{-ik_0(x_0-z_0)}\sqrt{f_h(k_0,x')}\ \mathbb{1}\ ,\\ B(y_0-z_0,x') &= \frac{1}{\beta}\sum_{k_0} e^{-ik_0(y_0-z_0)}\sqrt{f_h(k_0,y')}\Big[A_h(k_0,x')\Big]^{-1}\ . \end{aligned} \tag{116}$$

with A_h defined in (67). Therefore

$$|\det\widetilde{G}^{h_v,T_v}(\mathbf{t}_v)|\leq \bar{C}^n \tag{117}$$

By using Lemma 3.3, (109), (112) we get

$$\frac{1}{L\beta}\sum_x\int dx_{0,v_0}|H_{\tau,\mathbf{P},T,}(x,x_{0,v_0})| \leq [\prod_v\frac{1}{S_v!}][\prod_{v\in L_\chi}\varepsilon^{A\bar{x}^{-1}\gamma^{\frac{-h_{\bar{v}'}}{\tau}}2^{h_{\bar{v}'}}}][\prod_{v\in H_\chi}\gamma^{h_{\bar{v}'}-h_v}][\prod_{v\in V_\chi}\gamma^{-\alpha|P_v|}]$$
$$[\prod_{v\in V_\chi}\gamma^{-h_v(S^H_v+S^L_v-1)}][\sup_{k\geq h}|\widetilde{\alpha}_k|]^{n_\alpha}[\sup_{k\geq h}|\gamma^{-k}\widetilde{\lambda}_k|]^{n_\alpha}|\varepsilon|^{\frac{n}{2}} \tag{118}$$

Note that

$$[\prod_{v\in V_\chi}\gamma^{-h_v(S_v^H+S_v^L-1)}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}-h_v}]\le[\prod_{v\in V_\chi}\gamma^{-h_v(S_v^H+S_v^L)}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}}] \tag{119}$$

and

$$[\prod_{v\in V_\chi}\gamma^{-h_v S_v^H}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}}]=1 \tag{120}$$

so that

$$[\prod_{v\in V_\chi}\gamma^{-h_v(S_v^H+S_v^L-1)}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}-h_v}]\le[\prod_{v\in V_\chi}\gamma^{-h_v S_v^L}] \tag{121}$$

By using Lemma 3.4 $[\prod_{v\in L_\chi}\varepsilon^{A\bar x^{-1}\gamma^{\frac{-h_{\bar v'}}{\tau}}}2^{h_{\bar v'}}\prod_{v\in V_\chi}\gamma^{-h_v S_v^L}\le\bar C^n$ so that

$$\frac{1}{L\beta}\sum_x\int dx_{v_0}|H_{\tau,\mathbf{P},T}(x,\mathbf{x}_{v_0})|\le[\prod_v\frac{1}{S_v!}][\prod_{v\in V_\chi}\gamma^{-\alpha|P_v|}][\sup_{k\ge h}|\widehat\alpha_k|]^{n_\alpha}[\sup_{k\ge h}|\gamma^{-k}\widetilde\lambda_k|]^{n_\alpha}|\varepsilon|^{\frac{n}{2}} \tag{122}$$

The sum over $\mathbf{P}$ is done as in (70) using the factor $[\prod_{vnote.p.}\gamma^{-\alpha|P_v|}]$, and the sum $\sum_{\mathbf{T}}$ can be bounded by $c^n\prod_v S_v!$. The sum over the trees τ is done performing the sum of unlabelled trees and the sum over scales. The unlabeled trees can be bounded by 4^n by Caley formula. The sum over the scales is bounded by $|h|^{|V_\chi|}$ and $|V_\chi|\le 2n$; indeed given the unlabeled tree, the scales of the trivial vertices and of the end-points are determined once that the scales of the non trivial vertices are given, and their number is smaller than the number of χ-vertices; then (113) follows. ∎

G. The flow of the effective couplings

In order to sum over n in (113) we need that the running coupling constants v_k are small uniformy in h. In order to prove this we exploit the recursive equation (85). Note that the r.h.s. of (85) is expressed by a sum over trees with the constraint that over v_0, the first vertex in τ, the $\mathcal{L}$-operation acts. This immediately implies that each term verifies the same bound as the r.h.s. of (113) with an extra γ^h. The reason is that (119) is replaced by

$$[\prod_{v\in V_\chi}\gamma^{-h_v(S_v-1)}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}-h_v}]\le\gamma^h[\prod_{v\in V_\chi}\gamma^{-h_v S_v}][\prod_{v\in H_\chi}\gamma^{h_{\bar v'}}] \tag{123}$$

as $v_0\notin H_\chi$ so that $\prod_{v\in V_\chi}\gamma^{h_v}\le\gamma^{h_{v_0}}\prod_{v\neq v_0,v\in H_\chi}\gamma^{h_v}$ and $h_{v_0}=h$ as $v_0\in V_\chi$ because $\mathcal{L}\mathcal{R}=0$.

Lemma 3.6 *If $\gamma^{\bar h}\ge|\varepsilon|^{2\bar x}$ then there exists an ε_0 and a choice ν such that for $\varepsilon\le\varepsilon_0$ and $|\lambda|\le\varepsilon^{2\bar x+2}$ then there exists a suitable constant C_1 such that, for any $k\ge\bar h$*

$$|\widetilde\lambda_h|\le|\widetilde\lambda|C_1\quad|\widehat\alpha_h|\le C_1 \tag{124}$$

Proof We proceed by induction. The flow equation for ν_k is

$$\widetilde{\nu}_{k-1} = \gamma\widetilde{\nu}_h + \gamma^{-k}\int dx_0 H^{(k)}_{2,\rho\rho}(0,x_0,0) \tag{125}$$

with $\widetilde{\nu}_2 = \widetilde{\nu}$. By iteration we get

$$\widetilde{\nu}_k = \gamma^{-k+1}(\widetilde{\nu} + \sum_{k'\geq k}\int dx_0 H^{(k')}_{2,\rho\rho}(0,x_0,0)) \tag{126}$$

and by properly choosing $\widetilde{\nu}$ so that $\widetilde{\nu}_{\bar{h}} = 0$ we get

$$\widetilde{\nu}_k = -\gamma^{-k+1}\sum_{\bar{h}\leq k'\leq k}\int dx_0 H^{(k')}_{2,\rho\rho}(0,x_0,0)) \tag{127}$$

and one can show by a fixed point argument, the existence of a bounded sequence of $\widetilde{\nu}_k$ verifying (127) (the proof is identical to the one §A2.6 of [32]). Regarding the flow of $\widetilde{\zeta}_h$ assume that (124) is true for $k \geq h$. The flow equation for $\widetilde{\zeta}_h$

$$\widetilde{\zeta}_{h,\rho} = \sum_{k\geq h}\int dx_0 \widehat{\partial} H^{(k)}_{2,\rho,\rho} \tag{128}$$

where $\widetilde{z}_{1,\rho} = 0$ and $\widehat{\partial}$ is defined after (80). Using lemma 3.4 and the fact that the derivative cancels a factor γ^h we get for ε small enough

$$|\widetilde{\zeta}_h| \leq \sum_{n=2}^{\infty}\sum_{k\geq h} C^n C_1^n \varepsilon^{\frac{n}{2}}|h|^{2n} \leq |h|C_2(CC_1|h|^2\varepsilon^{\frac{1}{2}}) \leq C_1 \tag{129}$$

where we use that $|h|^2\varepsilon^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{4}}$ and $\gamma^{-k}|\widetilde{\lambda}| \leq \varepsilon$.

Similarly

$$|\widetilde{\lambda}_h| \leq |\widetilde{\lambda}_0| + \sum_{n=2}^{\infty}\sum_{n_\lambda=1}^{n}\sum_{k\geq h} C^n C_1^n \gamma^k \varepsilon^{\frac{n}{2}}|h|^{2n}(\gamma^{-k}|\widetilde{\lambda}_k|)^{n_\lambda} \leq$$

$$|\widetilde{\lambda}_0| + \sum_{n=2}^{\infty}|h|^{2n+1}\varepsilon^{\frac{n}{2}}C^n C_1^n \sum_{n_\lambda=1}^{\infty}|\lambda|(\gamma^{-\bar{h}}|\widetilde{\lambda}_k||)^{n_\lambda-1} \leq |\lambda|C_1 \tag{130}$$

■

The above lemma says the the flow is bounded up to a scale $\gamma^h \geq \varepsilon^{2\bar{x}}$. In order to integrate the smaller scales one has to use the mass term. Note indeed that if there exists two constants such that

$$c_1\varepsilon^{2\bar{x}} \leq \sigma_h \leq c_2\varepsilon^{2\bar{x}} \tag{131}$$

then there the scale h^* defined in (87) is $\bar{c}_1\varepsilon^{2\bar{x}} \leq \gamma^{h^*} \leq \bar{c}_2\varepsilon^{2\bar{x}}$. By (88) we can integrate the scale $\leq h^*$ in a single step, so that, by lemma 3.4 and 3.5 (for $h > h^* = \log \varepsilon^{2\bar{x}}$), convergence follows. It remains then to prove (131); indeed the upper bound is trivial and actually $\sigma_h, \zeta_h = O(\varepsilon^{2\bar{x}})$. In order to prove the lower bound we can write

$$\sigma_h = \sum_{k\geq h} \int dx_0 H^{(k)}_{2,\rho-\rho}(0, x_0, 0) \tag{132}$$

and

$$H^{(h)}_2 = H^{(a)(h)}_{2,\rho,-\rho} + H^{(b)(h)}_{2,\rho,-\rho} \tag{133}$$

where $H^{(a)(h)}_{2,\rho,-\rho}$ is the sum over trees with $n \leq 8\bar{x}$ and $H^{(b)(h)}_{2,\rho,-\rho}$ is the sum over trees with $n \geq 8\bar{x}+1$. By Lemma 3.5 $H^{(b)(h)}_{2,\rho,-\rho}$ is bounded by $\leq C\varepsilon^{2\bar{x}+\frac{1}{4}}$. Regarding $H^{(a)(h)}_{2,\rho,-\rho}$ we again distinguish between trees with at least a λ, λ_h end-point and the rest; the former is bounded by $C\gamma^h|\gamma^{-h}\widetilde{\lambda}| \leq C\varepsilon^{2\bar{x}+1}$. Regarding the latter, it can be represented in terms of chain graphs, and there is only one possible contribution $O(\varepsilon^{2\bar{x}})$, namely the graph with only ε-vertices and diagonal propagators; note indeed that $\widetilde{z}_h = O(\varepsilon^{2\bar{x}})$ and there are at least two vertices in each chain. In order to bound the chain graphs $O(\varepsilon^k)$ with $k \geq 2\bar{x}+1$ we note that, if x'_ℓ is the coordinate of any internal propagator with scale h and x' is the external coordinate, $x_\ell \neq x'$, c is a constant

$$c\gamma^h \geq ||\omega x'||_1 + ||\omega x'_\ell||_1 \geq ||\omega x' - \omega x'_\ell|| \geq C_0|x' - x_\ell|^{-\tau} \geq C_0|(8\bar{x})^2|^{-\tau} \tag{134}$$

Such graphs have have at most $8\bar{x}$ propagators bounded by (134) so that they are $O(\varepsilon^{2\bar{x}+1}\bar{x}!^\alpha)$. Let us then consider the chain graph with $2\bar{x}$ ε-vertices; it has only diagonal propagator and, up to higher order terms, is given by $\varepsilon^{2\bar{x}}a$ with

$$a_h = \frac{\chi_{\geq h}}{\phi_{-\bar{x}+1} - \phi_{\bar{x}}} \frac{\chi_{\geq h}}{\phi_{-\bar{x}+2} - \phi_{\bar{x}}} \cdots\cdots \frac{\chi_{\geq h}}{\phi_{\bar{x}-1} - \phi_{\bar{x}}} \tag{135}$$

where $\chi_{\geq h}$ is the cut-off function $\chi_{u,v} + \sum_{\rho=\pm}\sum_{k=h}^0 f^{(k)}(\omega(x - \rho\bar{x}))$. The terms proportional to ε^k, with $2\bar{x}+1 \leq k \leq 8\bar{x}$ have at most $8\bar{x}$ propagators bounded by (134); therefore

$$\sigma_h = \varepsilon^{2\bar{x}}(a_h + O(\varepsilon\bar{x}!^\alpha) + O(\varepsilon^{2\bar{x}+\frac{1}{4}})) \tag{136}$$

Therefore, for $\varepsilon \leq O(\bar{x}!^{-\alpha})$ then (131) follows, with $a_{-\infty} \equiv a \neq 0$.

H. The 2-point function

We have finally to get a bound for the two-point function. First of all, we note that Lemma 3.4 and Lemma 3.5 immediately imply a bound for the kernel of the effective potential with two external lines, with coordinate x and y. Indeed in the trees $\tau \in \mathcal{T}_{h,n}$ with n end-points contributing to $W_2^{(h)}$ there is necessarily a path c_{w_1,w_2} in $\widehat{T}_v$ connecting the points w_1, with $\mathbf{x}_{w_1} = \mathbf{x}$ and w_2 with $\mathbf{x}_{w_2} = \mathbf{y}$ such that by (95) $|x-y| \leq 8|c_{w_1.w_2}|\bar{x}$; moreover $|c_{w_1,w_2}| \leq n$ so that $n \geq \frac{1}{8\bar{x}}|x-y|$. Therefore no tree τ with $n < \frac{1}{8\bar{x}}|x-y|$ contribute to a kernel of the effective potential with external lines with coordinate x and y; therefore by Lemma 3.5 and Lemma 3.6 we get, for $h \geq h^*$

$$\frac{1}{\beta}\int dx_0 |W_2^{(h)}(\mathbf{x},\mathbf{y})| \leq \sum_{n \geq \frac{1}{8\bar{x}}|x-y|} C^n \gamma^h |\log\varepsilon|^n \varepsilon^{\frac{n}{2}} |h|^n \leq C\gamma^h \varepsilon^{\alpha|x-y|} \tag{137}$$

with suitable α and C.

In order to bound the 2-point function we have to consider the multiscale integration with $\phi \neq 0$, see (26); we get

$$S_2(\mathbf{x},\mathbf{y}) = \sum_{h=h^*}^{1} S_{2,h}(\mathbf{x},\mathbf{y}) \tag{138}$$

and $S_{2,h}(\mathbf{x},\mathbf{y})$ are expressed in terms of a tree expansion similar to the one for $W_2^{(h)}$, where the only difference is that two external fields are replaced by propagators $g^{(k)}(x'; x_0 - z_0)$ and $g^{(l)}(y'; x_0 - z_0)$; therefore $S_{2,h}(\mathbf{x},\mathbf{y})$ (at $\mathbf{x},\mathbf{y}$ fixed) verifies a bound similar to (137) with an extra extra factor $C_N \frac{\gamma^{-h}}{1+\gamma^{Nh}|x_0-y_0|^N}$ for any N, that is

$$|S_{2,h}(\mathbf{x},\mathbf{y})| \leq \varepsilon^{\alpha|x-y|} \frac{C_N}{1+\gamma^{Nh}|x_0-y_0|^N} \tag{139}$$

In conclusion, by (136), for any N

$$|S_2(\mathbf{x},\mathbf{y})| \leq \sum_{h=h^*}^{0} \varepsilon^{\alpha|x-y|} \frac{C_N}{1+\gamma^{Nh}|x_0-y_0|^N} \leq \widetilde{C}_N \frac{e^{-\frac{\alpha}{2}|\log\varepsilon||x-y|}}{1+[\sigma_{h^*}|x_0-y_0|]^N} \tag{140}$$

so that (29) is proved.

---

[1] P. W. Anderson Phys. Rev. 109, 1492 (1958)

[2] J. Froehlich and T. Spencer, Comm. Math. Phys. 88 (1983), 151.

[3] M. Aizenman and S. Molchanov, Comm. Math. Phys. 157 (1993), 245

[4] S. Aubry and G. Andre, Ann. Israel Phys. Soc 3, 1 (1980).

[5] E. Dinaburg, E, Y. Sinai, Funct. analysis and its app. 9, 279 (1975)

[6] L. Pastur, Usp. Mat. Nauk. 28, 3 (1973)

[7] L.H. Eliasson, Comm. Math. Phys 146, 447 (1992)

[8] J. Bellissard, R. Lima, and D. Testard, Comm. Math. Phys. 88, 207 (1983

[9] Ya. Sinai, J. Stat. Phys. 46, 861 (1987)

[10] J. Froehlich, T. Spencer, T. Wittvwer, Comm. Math. Phys. 88, 151 (1983)

[11] A. Avila, S. Jitomirskaya Ann. of. Math. 170 303 (2009)

[12] L.Fleishmann, P.W. Anderson Phys. Rev B 21, 2366 (1980)

[13] T. Giamarchi, H.J. Schulz, Europhys. Lett. 3 (1987), 1287.

[14] I. Gornyi, A. Mirlin, A., D. Polyakov, Phys. Rev. Lett. 9, 206603 (2005)

[15] J. Vidal, D. Mouhanna, T. Giamarchi, Phys, Rev. Lett. 83, 3908 (1999)

[16] D.M. Basko, IL. Alteiner , B. L. Altshuler Ann. of Physics Ann. Phys. (N. Y). 321, 1126 (2006)

[17] V. Oganesyan, D. A. Huse, Phys. Rev. B 75, 155111 (2007)

[18] M. Znidaric, T. Prosen, P. Prelovek, Phys. Rev. B 77, 064426 (2008)

[19] Pal, A., and Huse, Phys. Rev. B 82, 174411 (2010)

[20] V. Ros, M. Mueller, A. Scardicchio.Nuclear Physics B, 420 (2015)

[21] R. Nandkshore, D. Huse . Annual Review of Condensed Matter Physics, Vol. 6, 15 (2015)

[22] S. Iyer, V. Oganesyan, G. Refael, D. A. Huse Phys. Rev. B 87, 134202 (2013)

[23] A. Klein, J.F. Perez, Comm Math, Phys.,147, 241, (1992)

[24] A. Klein, J.F. Perez , Comm Math, Phys. 147,1, 99 (1990)

[25] M. Fauser, M., S Warzel. arXiv:1402.5832.

[26] J. Imbrie. arxiv1403.7837

[27] T. Spencer. Talk at a conference in Rome 2012

[28] De Roeck, W., and Huveneers, F. arXiv:1308.6263

[29] G. Gallavotti. Comm.Math. Phys 164, 1, 145 (1994)

[30] G.Gentile, V.Mastropietro. Rev. Math. Phys. 8, 3, 393 (1996).

[31] G.Benfatto, G. Gentile, V.Mastropietro. J. Stat. Phys. 89, 655 (1997)

[32] G. Gentile, V.Mastropietro. Comm. Math.Phys. 215, 69 (2000)

[33] V.Mastropietro. Comm. Math. Phys. 201, 81 (1999)

[34] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, M. Inguscio, Nature 453, 895 (2008).

[35] G.Benfatto, P.Falco V.Mastropietro. Comm. Math.Phys. 330, 153 (2014)

[36] G. Benfatto, V. Mastropietro Rev. Math. Phys. 13, 1323 (2001)