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# Basic principles of *hp* virtual elements on quasiuniform meshes

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In the present paper we initiate the study of *hp* Virtual Elements. We focus on the case with uniform polynomial degree across the mesh and derive theoretical convergence estimates that are explicit both in the mesh size $h$ and in the polynomial degree $p$ in the case of finite Sobolev regularity. Exponential convergence is proved in the case of analytic

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solutions. The theoretical convergence results are validated in numerical experiments. Finally, an initial study on the possible choice of local basis functions is included.

*Keywords*: Virtual elements; polygonal methods; *hp* error bounds.

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## 1. Introduction

The Virtual Element Method (VEM) is a very recent generalization of the Finite Element Method, introduced in Ref. 8, that responds to the increasing interest in using general polyhedral and polygonal meshes, also including non-convex elements and hanging nodes. The main idea of VEM is to use richer local approximation spaces that include (but are typically not restricted to) polynomial functions and, most importantly, avoid the explicit integration of the associated shape functions. Indeed, the operators and matrices appearing in the problem are evaluated by introducing an innovative construction that only requires an implicit knowledge of the local shape functions. By following such developments, the VEM acquires very interesting properties and advantages with respect to more standard Galerkin methods, yet still keeping the same implementation complexity. For instance, in addition to allowing for polygonal and polyhedral meshes, it can handle approximation spaces of arbitrary $C^k$ global regularity on unstructured meshes.

Although the Virtual Element Method has been applied to a large range of problems (a non-exhaustive list being Refs. 3, 4, 8–10, 12, 13, 17, 18, 22, 23 and 26), all the present works on VEM are focused, both theoretically and numerically, on the $h$-behaviour of the method. In other words, the convergence properties of the schemes are investigated assuming that the polynomial degree $p$ is fixed and only the mesh is refined. On the other hand, looking at the Finite Element literature, a very successful approach in applications is to allow for a variable value of $p$ and to focus not only in the accuracy that can be obtained by reducing the mesh size $h$, but also by increasing $p$. As in the FEM literature, we here refer to such approach as *hp* analysis; we mention for instance Refs. 6, 7, 19, 25 and 27 as very short list of papers and books among the very large literature of *hp* FEM.

The aim of the present paper is to initiate the study of *hp* Virtual Element Methods. The first motivation of such study is to show that the powerful *hp* methodology can be adopted also in the framework of Virtual Elements. The second, but not secondary, motivation is that we believe that combining the huge mesh flexibility of VEM with the advantages of a full (possibly adaptive) *hp* method can yield a very efficient and competitive methodology.

The present contribution focuses on the initial foundations of such ambitious plan, mainly in terms of convergence estimates. We restrict our attention to a two-dimensional scalar elliptic model problem (as in Ref. 8) and assume a polynomial degree $p$ that is the same on all elements of the (quasi-uniform) mesh. First of all, we prove fundamental convergence results (and the associated interpolation estimates) that is explicit in both $h$ and $p$. As a second result, we show also that

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Virtual Elements can attain exponential convergence when the target solution is analytic on a suitable (small) extension of the domain. We then explore numerically the behaviour of Virtual Elements in terms of $p$, both in the case of solutions with finite Sobolev regularity and for analytic solutions, and the stability bounds of the virtual bilinear form, always in terms of $p$. In the Appendix, we start to explore another interesting issue of *hp* elements, that is the choice of the basis and the condition number of the associated stiffness matrix. Note that, since in this work we focus on scalar problems in a planar two-dimensional domain, direct solvers can generally be used and the condition number issue is not of primary importance. Indeed, it is mainly the stability of the solver that determines the best attainable accuracy, as we show in the numerical tests. Nevertheless, in order to answer to some natural questions (such as: how do Legendre bases cope on general polygons?) we decided to include an initial study related to the choice of the basis.

The paper is organized as follows. After presenting the continuous model problem, in Sec. 3 we make a brief review of the Virtual Element Method. Afterwards, in Sec. 4 we present theoretical error estimates, whereas in Sec. 5 we show analogous error estimates leading to exponential convergence of the $p$ method if the solution of the Poisson problem is analytic. Successively, in Sec. 6 we develop the associated numerical tests validating the convergence results on the errors; we also give numerical bounds dealing with the stabilization of the method. Finally, the Appendix follows.

## 2. The Model Problem

Let $\Omega$ be a simply connected polygonal domain and let $\Gamma$ be its boundary. Let $H^l(\omega)$, with $l \in \mathbb{N}_0$ and $\omega$ open measurable set, denote the usual Sobolev space with square integrable weak derivatives of order $l$; let $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$ denote the associated norm and seminorm, respectively (see Ref. 1). Let $f \in L^2(\Omega)$. We consider the two-dimensional Poisson problem with homogeneous Dirichlet boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma. \tag{2.1}$$

We set $V := H_0^1(\Omega)$ and we consider the weak formulation of problem (2.1):

$$\text{find } u \in V \text{ such that } a(u,v) = (f,v)_{0,\Omega}, \quad \forall\, v \in V, \tag{2.2}$$

where $(\cdot,\cdot)_{0,\Omega}$ is the $L^2$-scalar product on $\Omega$ and $a(\cdot,\cdot) := (\nabla\cdot, \nabla\cdot)_{0,\Omega}$.

It is well known that problem (2.2) is well-posed (see for instance Ref. 16) since the bilinear form $a$ is continuous and coercive (i.e. $a(v,v) \geq \alpha\|v\|_{1,\Omega}^2$, where $\alpha > 0$) thanks to the Poincaré inequality. Throughout this paper, $C$ denotes a positive constant whose dependence on certain parameters will be made explicit where necessary.

## 3. Virtual Elements for the Poisson Problem

In the present section, we introduce a Virtual Element Method for the Poisson problem (2.2) based on polygonal meshes. Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions

of $\Omega$ into non-overlapping polygonal elements $K$ of diameter $h_K = \mathrm{diam}(K) := \sup\{|\mathbf{x}-\mathbf{y}| : \mathbf{x},\mathbf{y} \in K\}$. The characteristic mesh size is denoted by $h := \max\{h_K : K \in \mathcal{T}_h\}$. Let $\mathcal{V}_h$ and $\mathcal{E}_h$ be the sets of all vertices and edges in the mesh $\mathcal{T}_h$ respectively. Moreover, we denote by $\mathcal{V}_h^b := \mathcal{V}_h \cap \partial\Omega$ the set of all boundary vertices and by $\mathcal{E}_h^K$ the set of edges $e$ of an element $K \in \mathcal{T}_h$.

Henceforth, we assume that there exist two positive real numbers $\gamma$ and $\widetilde{\gamma}$ such that the sequence of decompositions satisfies the following:

**(D0)** the decomposition $\mathcal{T}_h$ is made of a finite number of simple polygons of diameter $h_K$,
**(D1)** for all $K \in \mathcal{T}_h$, $K$ is star-shaped with respect to a ball of radius $\geq h_K\gamma$,
**(D2)** for all $K \in \mathcal{T}_h$, the distance between any two vertices of $K$ is $\geq h_K\widetilde{\gamma}$.

To every edge $e \in \mathcal{E}_h$ we associate a tangential vector $\boldsymbol{\tau}_e$ and a normal unit vector $\mathbf{n}_e$ obtained by a counter-clockwise rotation of $\boldsymbol{\tau}_e$.

We split the bilinear form $a$ as a sum of local contributions

$$a(u,v) := \sum_{K\in\mathcal{T}_h} a^K(u,v), \quad \forall\, u,\ v \in V,$$

with $a^K(u,v) := (\nabla u, \nabla v)_{0,K}$.

It was shown in Ref. 8 that it is possible to build:

- $V_h(K)$, a finite-dimensional subspace of $H_0^1(\Omega)|_K$;
- symmetric local bilinear forms $a_h^K : V_h(K) \times V_h(K) \to \mathbb{R}$;
- $V_h$ a finite-dimensional subspace of $H_0^1(\Omega)$ such that $V_h|_K = V_h(K)$;
- a symmetric bilinear form $a_h : V_h \times V_h \to \mathbb{R}$, of the form $a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} a_h^K(u_h, v_h)$, $\forall\, u_h, v_h \in V_h$;
- an element $f_h \in V_h'$ and a duality pairing $\langle\cdot;\cdot\rangle_h$;

in such a way that the resulting discrete problem

$$\text{find } u_h \in V_h \text{ such that } a_h(u_h, v_h) = \langle f_h; v_h\rangle_h, \quad \forall\, v_h \in V_h \tag{3.1}$$

has a unique solution $u_h \in V_h$ which is close to the solution $u$ of the original problem (2.2). More precisely, when $u \in H^k(\Omega)$ the error in the energy norm admits the upper bound

$$\text{if } u \in H^k(\Omega), \quad k \geq 1, \quad |u - u_h|_{1,\Omega} \leq Ch^{k-1}|u|_{k,\Omega}, \tag{3.2}$$

where the constant $C = C(p)$ depends implicitly on the (fixed) polynomial degree $p$ but not on the characteristic mesh size $h$. We now briefly review the local Virtual Spaces introduced in Ref. 8. Let $K \in \mathcal{T}_h$ and let $p \in \mathbb{N}$, $p \geq 1$. Let $\mathbb{P}_p(e)$ and $\mathbb{P}_{p-2}(K)$ be respectively the set of polynomials of degree $p$ over the edge $e$ and of degree $p-2$ over the polygon $K$, with the convention $\mathbb{P}_{-1}(K) = \{0\}$. With the space of continuous piecewise polynomials over the boundary of each element $K$:

$$\mathbb{B}_p(\partial K) := \{v_h \in \mathcal{C}^0(\partial K) | v_h|_e \in \mathbb{P}_p(e), \forall\, e \in \mathcal{E}_h^K\}, \tag{3.3}$$

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we define the local Virtual Element spaces

$$V_h(K) := \{v_h \in H^1(K) \,|\, \Delta v_h \in \mathbb{P}_{p-2}(K), v_h|_{\partial K} \in \mathbb{B}_p(\partial K)\}. \tag{3.4}$$

Observe that $\mathbb{P}_p(K) \subseteq V_h(K)$ for any $K \in \mathcal{T}_h$. For any fixed function $v \in V_h(K)$ we identify the following set of local degrees of freedom:

- the values of $v$ at vertices of $K$;
- the values of $v$ at $(p-1)$ internal nodes of each edge $e \in \mathcal{E}_h^K$ (for instance at the internal Gauß–Lobatto nodes, as done in Refs. 8 and 10);
- the internal moments $\frac{1}{|K|}\int_K q_{\boldsymbol{\alpha}} v_h dx$, where $\{q_{\boldsymbol{\alpha}} : 0 \leq |\boldsymbol{\alpha}| \leq p(p-1)/2\}$ is a basis for $\mathbb{P}_{p-2}(K)$. For instance, Beirão da Veiga *et al.*[8,10] employed a basis of shifted and scaled monomials: let $\mathbf{x_K}$ and $h_K$ be the barycenter and the diameter of $K$ respectively, then $q_{\boldsymbol{\alpha}}(\mathbf{x}) := (\frac{\mathbf{x}-\mathbf{x_K}}{h_K})^{\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ such that $|\alpha| := \alpha_1 + \alpha_2 \leq p-2$.

The global Virtual Space is obtained by the continuous matching of the local spaces over the element boundaries

$$V_h := \{v_h \in C^0(\overline{\Omega}) : v_h|_K \in V_h(K), v_h|_{\partial\Omega} = 0\} \subset H_0^1(\Omega)$$

with the natural definition of the global degrees of freedom from the local ones. It was shown in Ref. 8 that, given $K \in \mathcal{T}_h$, the bilinear forms $a_h^K$ must satisfy the two following assumptions:

**(A1) *p*-consistency:** $\forall\, K \in \mathcal{T}_h$ it holds that

$$a^K(q, v_h) = a_h^K(q, v_h), \quad \forall\, v_h \in V_h(K), \ \forall\, q \in \mathbb{P}_p(K); \tag{3.5}$$

**(A2) stability:** $\forall\, K \in \mathcal{T}_h$ there exist two constants $0 < \alpha_* < \alpha^* < \infty$ such that

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h), \quad \forall\, v_h \in V_h(K). \tag{3.6}$$

In the following study we will assume that the constants $\alpha_*, \alpha^*$ are independent of $h, p$, see also Remark 4.5. Let $\varphi$ be a sufficiently regular function, e.g. $\varphi \in H^1(K)$. We introduce the local averaging operator:

$$\overline{\varphi} := \begin{cases} \dfrac{1}{|\partial K|}\displaystyle\int_{\partial K} \varphi(x)dx & \text{if } p = 1, \\[2ex] \dfrac{1}{|K|}\displaystyle\int_K \varphi(x)dx & \text{if } p > 1. \end{cases} \tag{3.7}$$

Having this, we introduce the projection operator $\Pi_p^\nabla : V_h(K) \to \mathbb{P}_p(K)$ as follows: for any $v_h \in V_h(K)$ its projection $\Pi_p^\nabla v_h \in \mathbb{P}_p(K)$ is the unique polynomial satisfying,

$$\begin{aligned} a^K(\Pi_p^\nabla v_h - v_h, q) &= 0 \quad \forall\, q \in \mathbb{P}_p(K), \\ \overline{\Pi_p^\nabla v_h - v_h} &= 0 \quad \text{where the averaging operator is defined in (3.7).} \end{aligned} \tag{3.8}$$

Then, a candidate bilinear form $a_h$ satisfying **(A1)** and **(A2)** can be sought in the form:

$$a_h^K(u_h, v_h) = a^K(\Pi_p^\nabla u_h, \Pi_p^\nabla v_h) + S^K(u_h - \Pi_p^\nabla u_h, v_h - \Pi_p^\nabla v_h), \quad \forall\, u_h, v_h \in V_h(K),$$

where $S^K$ is a positive definite bilinear form satisfying

$$c_0 a^K(v_h, v_h) \le S^K(v_h, v_h) \le c_1 a^K(v_h, v_h),$$
$$\forall\, v_h \in V_h(K), \quad \text{such that} \quad \Pi_p^\nabla v_h = 0, \tag{3.9}$$

for some positive constants $c_0$ and $c_1$ independent on $h$, $p$ and $K$.

A possible choice for $S^K$ can be found in (6.1). The global discrete bilinear form reads

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h), \quad \forall\, u_h, v_h \in V_h.$$

Finally, we recall from Ref. 8 a possible choice for the loading term. Let $P_{p-2}^{0,K}$ and $P_0^{0,K}$ be the $L^2$-projector on polynomials of degree $p-2$ and 0 respectively over the polygon $K$ and let the averaging operator be defined in (3.7). Then, we may define

$$\langle f_h, v_h \rangle_h := \begin{cases} \displaystyle\sum_{K \in \mathcal{T}_h} \int_K \left[ P_{p-2}^{0,K} f \right] v_h dx, & \forall\, v_h \in V_h \quad \text{if } p \ge 2, \\ \displaystyle\sum_{K \in \mathcal{T}_h} \int_K \left[ P_0^{0,K} f \right] \overline{v_h} dx, & \forall\, v_h \in V_h \quad \text{if } p = 1. \end{cases} \tag{3.10}$$

Under the above choices for $V_h$, $a_h$ and $f_h$, the paper by Beirão da Veiga *et al.*[8] guarantees well-posedness and $h$-convergence (3.2).

**Remark 3.1.** As shown in Ref. 8, the projection operator $\Pi_p^\nabla$ in (3.8) is computable using the degree of freedom values, without the need of any further information on the virtual shape functions. We finally note that the definition in (3.7) is not the only possible one; other (computable) choices could be used instead.

## 4. Approximation Results

In this section, we give a convergence result for the error of the Virtual Element Method measured in the energy norm in terms of both $h$ and $p$.

### 4.1. *Auxiliary approximation results*

Let $u$ and $u_h$ be the solutions of (2.2) and (3.1) respectively, and denote by $S_h^{p,-1}(\mathcal{T}_h)$ the space of the piecewise discontinuous polynomials of degree $p$ over the decomposition $\mathcal{T}_h$. Given $u \in H^1(K)$, $\forall\, K \in \mathcal{T}_h$, we define the broken $H^1$-seminorm as

$$|u|_{h,1,\Omega} = \sum_{K \in \mathcal{T}_h} (|u|_{1,K}^2)^{\frac{1}{2}}. \tag{4.1}$$

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Let $\mathcal{F}_h$ be the smallest constant satisfying

$$(f, v_h)_{0,\Omega} - \langle f_h, v_h \rangle_h \leq \mathcal{F}_h |v_h|_{1,\Omega}, \quad \forall\, v_h \in V_h. \tag{4.2}$$

Then the following best approximation estimate holds (see Theorem 3.1 in Ref. 8):

$$|u - u_h|_{1,\Omega} \leq C \left( \inf_{u_\pi \in S_h^{p,-1}(\mathcal{T}_h)} |u - u_\pi|_{1,h,\Omega} + \inf_{u_I \in V_h} |u - u_I|_{1,\Omega} + \mathcal{F}_h \right), \tag{4.3}$$

where $C$ is a constant depending only on $\alpha_*$ and $\alpha^*$ from assumption **(A2)**. In what follows, we shall derive estimates for the three terms in (4.3) that are explicit in both $h$ and $p$.

### 4.1.1. *Polynomial approximation term*

We start by bounding the term $|u - u_\pi|_{h,1,\Omega}$. In order to derive the bound, we need to prove a generalized-polygonal version of a classic result, namely Lemma 4.1 in Ref. 6. In this lemma, the existence of a sequence of polynomials which approximate $H^k$ functions over the triangular and square reference elements was shown. We extend this result for generic polygons having the unit diameter. Thus, we are ready to show the following lemma.

**Lemma 4.1.** *Let $\widehat{K} \subseteq \mathbb{R}^2$ be a polygon with* $\operatorname{diam}(\widehat{K}) = 1$. *Moreover, assume that $\widehat{K}$ is star-shaped with respect to a ball of radius $\geq\gamma$ and the distance between any two vertices of $\widehat{K}$ is $\geq\widetilde{\gamma}$, $\gamma$ and $\widetilde{\gamma}$ being the constants introduced in assumptions* **(D1)** *and* **(D2)** *of Sec.* 3. *Then, there exists a family of projection operators $\{\widehat{\Pi}_{\widehat{K},p}\}$, $p = 1, 2, \ldots$ with $\widehat{\Pi}_{\widehat{K},p} : H^{k+1}(\widehat{K}) \to \mathbb{P}_p(\widehat{K})$ such that, for any $0 \leq \ell \leq k+1$, $\widehat{u} \in H^{k+1}(\widehat{K})$, $k \in \mathbb{N}$, it holds*

$$\|\widehat{u} - \widehat{\Pi}_{\widehat{K},p}\widehat{u}\|_{\ell,\widehat{K}} \leq C p^{-(k+1-\ell)} \|\widehat{u}\|_{k+1,\widehat{K}} \tag{4.4}$$

*with $C$ a constant independent on $u$ and $p$.*

**Proof.** We assume without loss of generality that $\mathbf{x}_{\widehat{\mathbf{K}}}$, the barycenter of $\widehat{K}$, coincides with the origin $\mathbf{0}$. For a given $r > 0$, we define

$$R(r) := \{(x, y) \in \mathbb{R}^2 \,|\, |x| < r, |y| < r\}. \tag{4.5}$$

Thanks to the fact that $\operatorname{diam}(\widehat{K}) = 1$ and $\mathbf{x}_{\widehat{K}} = \mathbf{0}$, we have $R(1) \supset \overline{\widehat{K}}$.

Let $r_0 > 1$. Then, it obviously holds $\overline{\widehat{K}} \subset R(r_0)$. We note that $\partial\widehat{K}$ is Lipschitz; consequently, using Ref. 28, there exists $E : H^{k+1}(\widehat{K}) \to H^{k+1}(R(2r_0))$ extension operator such that $E\widehat{u} = 0$ on $R(2r_0)\backslash R(\frac{3}{2}r_0)$ and $\|E\widehat{u}\|_{k+1,R(2r_0)} \leq C\|\widehat{u}\|_{k+1,\widehat{K}}$. A careful inspection of Theorem 5 in Chap. VI of Ref. 28 shows that the constant $C$ depends only on $k$ and on the "worst angle" value

$$\theta_{\widehat{K}} = \min_{\theta \in \mathcal{A}_{\widehat{K}}} \min\{\theta, 2\pi - \theta\},$$

where $\mathcal{A}_{\widehat{K}}$ denotes the set of the (amplitude of) internal angles of $\widehat{K}$. In particular, the constant $C$ may explode when $\theta_{\widehat{K}} \to 0$. It is possible to check that, under the regularity hypotheses on $K$, the angle parameter $\theta_{\widehat{K}}$ is bounded from below by a constant depending only on $\gamma, \widetilde{\gamma}$. Thus the constant $C$ can be bounded in terms of $k$ and $\gamma, \widetilde{\gamma}$. Therefore, it holds $\|E\widehat{u}\|_{k+1,R(2r_0)} \leq C(k,\gamma,\widetilde{\gamma})\|\widehat{u}\|_{k+1,\widehat{K}}$. The remaining part of the proof, that is based on the approximation of the extended function $E\widehat{u}$, follows exactly the same steps as in Lemma 4.1 of Ref. 6 and is therefore not shown. □

Using this result, we are able to give a generalized-polygonal version of Lemma 4.5 of Ref. 6, which will play the role of local *hp* estimate result on $|u - u_\pi|_{1,K}$, where $K$ is a polygon of the decomposition $\mathcal{T}_h$.

**Lemma 4.2.** *Let $K \in \mathcal{T}_h$ satisfying assumptions* **(D1)** *and* **(D2)** *and $u \in H^{k+1}(K)$. Then there exists a sequence of projection operators $\{\Pi^h_{K,p}\}$, $p = 1, 2, \ldots,$ with $\Pi^h_{K,p} : H^{k+1}(K) \to \mathbb{P}_p(K)$ such that for any $0 \leq \ell \leq k+1$, $k \in \mathbb{N}$:*

$$|u - \Pi^h_{K,p} u|_{\ell,K} \leq C \frac{h_K^{\mu+1-\ell}}{p^{k+1-\ell}} \|u\|_{k+1,K}, \tag{4.6}$$

*where $\mu = \min(p,k)$ and $C$ is independent on $u$, $h$ and $p$.*

**Proof.** We consider the mapping $F(\mathbf{x}) = \frac{1}{h_K}(\mathbf{x} - \mathbf{x_K})$. Let $\widehat{K} = F(K)$, where $h_K$ denotes the barycenter of $K$. Obviously $\mathrm{diam}(\widehat{K}) = 1$ and the barycenter of $\widehat{K}$ is in the origin, $\mathbf{x}_{\widehat{\mathbf{K}}} = \mathbf{0}$. Let $\widehat{\Pi}_{\widehat{K},p}\widehat{u}$ be the sequence of approximating polynomials of degree $p$, introduced in Lemma 4.1. We let $\Pi^h_{K,p} u$ be the push forward of the above sequence with respect to the transformation $F$, i.e. $\Pi^h_{K,p} u = (\widehat{\Pi}_{\widehat{K},p}(\widehat{u})) \circ F$, where $\widehat{\varphi} = \varphi \circ F^{-1}$ for a sufficiently regular function $\varphi$. Then, it is possible to check, by a simple change of variables argument, that

$$|u - \Pi^h_{K,p} u|_{\ell,K} \leq C h_K^{1-\ell} |\widehat{u} - \widehat{\Pi}_{\widehat{K},p}\widehat{u}|_{\ell,\widehat{K}},$$

where $C$ is a constant independent on $K$ (hence on $\widehat{K}$), $h$, $u$ and $p$; besides, $C$ is independent also on $\ell$, $\gamma$ and $\widetilde{\gamma}$, thanks to the fact that $F$ is the composition of a translation with a dilatation.

We apply Lemma 4.1 and we obtain, by adding and subtracting any $\widehat{q} \in \mathbb{P}_p(\widehat{K})$,

$$\begin{aligned} |u - \Pi^h_{K,p} u|_{\ell,K} &\leq C h_K^{1-\ell} \|(\widehat{u} - \widehat{q}) - \widehat{\Pi}_{\widehat{K},p}(\widehat{u} - \widehat{q})\|_{\ell,\widehat{K}} \\ &\leq C \frac{h_K^{1-\ell}}{p^{k+1-\ell}} \|\widehat{u} - \widehat{q}\|_{k+1,\widehat{K}}, \quad \forall\, \widehat{q} \in \mathbb{P}_p(\widehat{K}), \end{aligned} \tag{4.7}$$

where $C$ in the right-hand side of (4.7) is a constant depending on $k$. Using the classical Scott–Dupont theory (see e.g. Ref. 16 or Ref. 21) and a scaling argument,

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bound (4.7) yields

$$|u - \Pi^h_{K,p} u|_{\ell,K} \le C \frac{h_K^{1-\ell}}{p^{k+1-\ell}} \left( \sum_{i=\mu+1}^{k+1} |\widehat{u}|^2_{i,\widehat{K}} \right)^{\frac{1}{2}}$$
$$\le C \frac{h_K^{\mu+1-\ell}}{p^{k+1-\ell}} \|u\|_{k+1,K}, \quad \mu = \min(p,k), \tag{4.8}$$

where $C$ is independent on $u$, $p$ and $h$. □

**Remark 4.1.** We note that if $k \le p$ then the classical Bramble–Hilbert lemma allows to take the seminorm in the right-hand side of (4.6), yielding

$$|u - \Pi^h_{K,p} u|_{\ell,K} \le C \frac{h_K^{k+1-\ell}}{p^{k+1-\ell}} |u|_{k+1,K},$$

where $C$ is a constant independent on $h$, $p$ and $u$.

We are now able to give a global estimate on $|u - u_\pi|_{h,1,\Omega}$ in (4.3), where $u_\pi \in S_h^{p,-1}(\mathcal{T}_h)$, $S_h^{p,-1}(\mathcal{T}_h)$ being defined at the beginning of Sec. 4.1. In fact, by choosing $u_\pi|_K = \Pi^h_{K,p} u$ for all $K \in \mathcal{T}_h$ and recalling the shape regularity properties **(D1)**–**(D2)**, we obtain:

$$\begin{aligned} |u - u_\pi|_{h,1,\Omega} &\le C_1 \frac{h^\mu}{p^k} \|u\|_{k+1,\Omega}, \quad \mu = \min(p,k), \\ |u - u_\pi|_{h,1,\Omega} &\le C_2 \frac{h^k}{p^k} |u|_{k+1,\Omega}, \quad \text{for } p \ge k, \end{aligned} \tag{4.9}$$

where $C_1$ and $C_2$ are two constants independent on $u$, $p$ and $h$.

### 4.1.2. *Virtual interpolation term*

We turn now to the term $|u - u_I|_{1,\Omega}$ in (4.3). Preliminarily, we observe that **(D1)** and **(D2)**, defined in Sec. 3, imply that there exists $\widetilde{\mathcal{T}}_h$, an auxiliary conformal triangular mesh that refines $\mathcal{T}_h$, obtained by connecting, for all $K \in \mathcal{T}_h$, the $N_K$ vertices to the center of the ball that realizes assumption **(D1)** for $K$. Moreover, it is possible to check that each triangle $T \in \widetilde{\mathcal{T}}_h$ is uniformly shape regular.

Let $\widetilde{S}_h^{p,0}(\widetilde{\mathcal{T}_h})$ be the set of continuous piecewise polynomials of degree $p$ over the auxiliary triangular decomposition introduced above. It is well known (see Theorem 4.6 in Ref. 6) that there exists $\varphi_p^h \in \widetilde{S}_h^{p,0}(\widetilde{\mathcal{T}_h})$ such that for any $u \in H^{k+1}(\Omega)$, $k \in \mathbb{R}$:

$$\begin{aligned} \|u - \varphi_p^h\|_{1,\Omega} &\le C_1 \frac{h^\mu}{p^k} \|u\|_{k+1,\Omega}, \quad k > \frac{1}{2}, \\ |u - \varphi_p^h|_{1,\Omega} &\le C_2 \frac{h^k}{p^k} |u|_{k+1,\Omega}, \quad k > \frac{1}{2}, \ p \ge k, \end{aligned} \tag{4.10}$$

where $C_1$ and $C_2$ are two constants independent on $u$, $p$ and $h$ and where $\mu = \min(p,k)$.

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Now, we use $\varphi_p^h$ in (4.10) to construct an approximant $u_I \in V_h$ of $u$. For this purpose, we modify a particular technique introduced in Ref. 26.

**Lemma 4.3.** *Under* **(D1)** *and* **(D2)**, *for all* $u \in H^{k+1}(\Omega)$, $k \in \mathbb{N}$, *there exists* $u_I \in V_h$ *such that*

$$|u - u_I|_{1,\Omega} \leq C\frac{h^\mu}{p^k}\|u\|_{k+1,\Omega}, \quad \mu = \min(p, k), \tag{4.11}$$

*where* $C$ *is independent on* $h$, $p$ *and* $u$.

**Proof.** Let $u_\pi$ be the function defined in (4.9). Let $\varphi_p^h$ be the function described in (4.10). For each $K \in \mathcal{T}_h$, we define $u_I|_K$ the solution of the following problem:

$$\begin{cases} -\Delta u_I = -\Delta u_\pi & \text{in } K, \\ u_I = \varphi_p^h & \text{on } \partial K. \end{cases} \tag{4.12}$$

It is possible to check that $u_I|_K \in V_h|_K$. Moreover, since $u_I \in H^1(\Omega)$, it holds that $u_I \in V_h$.

Using (4.12), we can write

$$\begin{cases} -\Delta(u_I - u_\pi) = 0 & \text{in } K, \\ u_I - u_\pi = \varphi_p^h - u_\pi & \text{on } \partial K. \end{cases}$$

Therefore, since $(u_I - u_\pi)$ is harmonic it holds

$$\begin{aligned} |u_I - u_\pi|_{1,K} &= \inf\{|z|_{1,K}, z \in H^1(K) \,|\, z = \varphi_p^h - u_\pi \text{ on } \partial K\} \\ &\leq |\varphi_p^h - u_\pi|_{1,K}. \end{aligned} \tag{4.13}$$

Finally by (4.13) we obtain

$$\begin{aligned} |u - u_I|_{1,K} &\leq |u - u_\pi|_{1,K} + |u_\pi - u_I|_{1,K} \leq |u - u_\pi|_{1,K} + |u_\pi - \varphi_p^h|_{1,K} \\ &\leq 2|u - u_\pi|_{1,K} + |u - \varphi_p^h|_{1,K}. \end{aligned} \tag{4.14}$$

The proof is completed by summing on all the elements in (4.14) and using (4.9), (4.10). □

**Remark 4.2.** We point out that if $k \leq p$ and under the hypothesis of Lemma 4.3, the following holds:

$$|u - u_I|_{1,\Omega} \leq C\frac{h^k}{p^k}|u|_{k+1,\Omega},$$

where $C$ is a constant independent on $h$, $p$ and $u$.

### 4.1.3. *Loading approximation term*

It remains to estimate the term $\mathcal{F}_h$ in (4.3). We have the following result.

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**Lemma 4.4.** *Under assumptions* **(D1)** *and* **(D2)**, *let* $f \in H^{\widetilde{k}+1}(K)$ *be the loading term for all* $K \in \mathcal{T}_h$, $\widetilde{k} \in \mathbb{N}$. *Then it holds*

$$\mathcal{F}_h \leq C \frac{h^{\widetilde{\mu}}}{p^{\widetilde{k}+2}} \left( \sum_{K \in \mathcal{T}_h} \|f\|^2_{\widetilde{k}+1,K} \right)^{\frac{1}{2}}, \quad \widetilde{\mu} = \min(p, \widetilde{k}+2), \tag{4.15}$$

*where* $C$ *is a constant independent on* $h$, $p$ *and* $u$.

**Proof.** Since the case $p = 1$ has been already analyzed in Ref. 8, we only consider the case $p \geq 2$. Let $v_h \in V_h$. Let $P^{0,K}_{p-2}$ be the $L^2$-projector on polynomials of degree $p-2$ over the polygon $K$, for all $K \in \mathcal{T}_h$. We get by (3.10):

$$\begin{aligned}
(f, v_h)_{0,\Omega} - \langle f_h, v_h \rangle_h &= \sum_{K \in \mathcal{T}_h} (f - P^{0,K}_{p-2} f, v_h)_{0,K} \\
&= \sum_{K \in \mathcal{T}_h} (f - P^{0,K}_{p-2} f, v_h - P^{0,K}_{p-2} v_h)_{0,K} \\
&\leq \sum_{K \in \mathcal{T}_h} \|f - P^{0,K}_{p-2} f\|_{0,K} \|v_h - P^{0,K}_{p-2} v_h\|_{0,K} \\
&\leq \sum_{K \in \mathcal{T}_h} \|f - f^{\pi}_{p-2}|_K\|_{0,K} \|v_h - v^{\pi}_{p-2}|_K\|_{0,K},
\end{aligned}$$

where $f^{\pi}_{p-2}|_K$ and $v^{\pi}_{p-2}|_K$ are the piecewise polynomial functions of degree $p-2$ that realize the bound (4.8) with $\ell = 0$ on each $K \in \mathcal{T}_h$. An adaptation of Lemma 4.1 (and so also of Lemma 4.1 in Ref. 6 or Lemma 3.1 in Ref. 7) implies that, given $\widetilde{p} = \max(1, p-2)$,

$$\begin{aligned}
(f, v_h)_{0,\Omega} - \langle f_h, v_h \rangle &\leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{\min((p-2)+1, \widetilde{k}+1)}}{\widetilde{p}^{\widetilde{k}+1}} \|f\|_{\widetilde{k}+1,K} \frac{h_K}{\widetilde{p}} |v_h|_{1,K} \\
&\leq C \frac{h^{\min(p, \widetilde{k}+2)}}{\widetilde{p}^{\widetilde{k}+2}} \left( \sum_{K \in \mathcal{T}_h} \|f\|^2_{\widetilde{k}+1,K} \right)^{\frac{1}{2}} |v_h|_{1,K}.
\end{aligned}$$

The final result follows by the definition of $\mathcal{F}_h$ in (4.2) and substituting $\widetilde{p}$ with $p$, up to a change of the constant $C$. □

By observing that, if the solution $u$ of (2.2) is in $H^{k+1}(\Omega)$ then $f \in H^{k-1}(\Omega)$, Lemma 4.4 immediately gives also the following corollary.

**Corollary 4.1.** *Under assumptions* **(D1)** *and* **(D2)**, *let the solution* $u$ *of* (2.2) *be in* $H^{k+1}(\Omega)$, $k \in \mathbb{N}$. *Then it holds*

$$\mathcal{F}_h \leq C(k, \gamma, \widetilde{\gamma}) \frac{h^{\mu}}{p^k} \|u\|_{k+1,\Omega}, \quad \mu = \min(p, k), \tag{4.16}$$

*where* $C$ *is a constant independent on* $h$, $p$ *and* $u$.

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Finally, we note that an analogous observation as in Remark 4.1 and Remark 4.2 holds also for Corollary 4.1, yielding

$$\mathcal{F}_h \leq C(k,\gamma,\widetilde{\gamma})\frac{h^k}{p^k}|u|_{k+1,\Omega}, \quad 1 \leq k+1 \leq p+1, \tag{4.17}$$

where $C$ is a constant independent on $h$, $p$ and $u$.

**Remark 4.3.** We stress the fact that using the same enhancing strategy introduced in Ref. 3 it is possible to obtain a more accurate load approximation. Nevertheless, the global order of convergence of the method does not change due to the presence of the other terms in the error estimates.

### 4.2. *hp estimate in the energy norm*

Finally, we are able to show the following convergence result.

**Theorem 4.1.** *Let $k \in \mathbb{N}$, $k > \frac{1}{2}$ and let the mesh assumptions* **(D1)** *and* **(D2)** *hold. Let $u$ and $u_h$ be respectively the solution of problems* (2.2) *and* (3.1), *with $u \in H^{k+1}(\Omega)$. Then, the following hp estimates hold*:

$$|u-u_h|_{1,\Omega} \leq C_1\frac{h^\mu}{p^k}\|u\|_{k+1,\Omega}, \quad \mu = \min(p,k), \tag{4.18}$$

$$|u-u_h|_{1,\Omega} \leq C_2\frac{h^k}{p^k}|u|_{k+1,\Omega}, \quad \text{if } k \leq p, \tag{4.19}$$

*where $C_1$ and $C_2$ are two constants independent on $h$, $p$ and $u$.*

**Proof.** It suffices to combine (4.3), (4.2), (4.9), (4.11) and (4.16). □

**Remark 4.4.** Let the domain $\Omega$ be convex. Following the argument shown in Ref. 9 (and, if $p = 1, 2$ suitably changing the definition of the discrete loading term (3.10)) and applying approximation results similar to those shown above, one can also easily derive $L^2$-estimates of the form:

$$\|u-u_h\|_{0,\Omega} \leq C(k,\gamma,\widetilde{\gamma})\frac{h^{\mu+1}}{p^{k+1}}\|u\|_{k+1,\Omega}, \quad \mu = \min(p,k), \tag{4.20}$$

where $C$ is a constant independent on $h$, $p$ and $u$, with the usual modification for the case $k \leq p$ and where $C$ is a constant independent on $h$ and $p$.

**Remark 4.5.** In the present analysis we made the assumption that the constants $\alpha_*, \alpha^*$ of the stability condition (3.6) are independent of $h, p$. For the independence in $h$, that is easy to achieve under the current mesh assumptions, see Ref. 8. For the independence in $p$, see Remark 6.1 in the numerical tests section. As a final note, we observe that it is immediate to check that the dependence of the constants $C_1, C_2$ in Theorem 4.1 is linear in the ratio $\alpha^*/\alpha_*$; the same holds for the convergence results of the next section.

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## 5. Exponential Convergence for Analytic Solutions

In this section, we derive an exponential convergence result for analytic solutions, under a further regularity assumption on the decomposition. We recall that we are given a polygonal decomposition $\mathcal{T}_h$ and a triangular auxiliary subdecomposition $\widetilde{\mathcal{T}}_h$ (described in Sec. 4.1.2). Given $K$ polygon in $\mathcal{T}_h$, we define $Q = Q(K)$ as any of the smallest square containing $K$; besides, given $\widetilde{K}$ triangle in $\widetilde{\mathcal{T}}_h$, we define $\widetilde{Q} = \widetilde{Q}(\widetilde{K})$ the parallelogram given by $\overline{\widetilde{Q}} = \overline{\widetilde{K} \cup \widetilde{K}^*}$, where $\widetilde{K}^*$ is the reflection of $\widetilde{K}$ with respect to a midpoint of anyone of its edges. We point up that there are three possible $\widetilde{Q}(\widetilde{K})$; we fix arbitrarily one of them. Next, we define:

$$\Omega_{\text{ext}} = \Omega_{\text{ext}}(h) := \Omega \cup \left( \bigcup_{K \in \mathcal{T}_h} Q(K) \right) \cup \left( \bigcup_{\widetilde{K} \in \widetilde{\mathcal{T}}_h} \widetilde{Q}(\widetilde{K}) \right). \tag{5.1}$$

We observe that $\text{dist}(x, \Omega) \leq d(h)$, $\forall\, x \in \Omega_{\text{ext}}$, $d(\cdot)$ being a non-decreasing function in $h$. Therefore, $\forall\, h \leq \overline{h}$ one has $d(h) \leq d(\overline{h})$ and thus $\Omega_{\text{ext}}$ is a uniformly bounded domain in terms of $h$, if $h$ is bounded. We demand for the following regularity assumption on the mesh:

**(D3)** there exists $N \in \mathbb{N}$ independent on $h$ such that there are at most $N$ overlapping squares in the collection $\{Q(K)\}$ and $N$ parallelograms in the collection $\{\widetilde{Q}(\widetilde{K})\}$, i.e. for all $Q(K)$ in $\{Q(K)\}$ and for all $\widetilde{Q}(\widetilde{K})$ in $\{\widetilde{Q}(\widetilde{K})\}$, given $I_{K'} := \{Q(K) \,|\, Q(K) \cap Q(K') \neq \emptyset\}$ and $\widetilde{I}_{\widetilde{K}'} := \{\widetilde{Q}(\widetilde{K}) \,|\, \widetilde{Q}(\widetilde{K}) \cap \widetilde{Q}(\widetilde{K}') \neq \emptyset\}$, it holds that $\text{card}(I_{K'})$, $\text{card}(\widetilde{I}_{\widetilde{K}'}) \leq N$, $\forall\, K \in \mathcal{T}_h$ and $\forall\, \widetilde{K} \in \widetilde{\mathcal{T}}_h$.

We note that, given $u \in H^{k+1}(\Omega_{\text{ext}})$, $k \in \mathbb{N}$, under assumption **(D3)**, the following expressions hold:

$$\sum_{K \in \mathcal{T}_h} \|u\|^2_{k+1,Q(K)} \leq N \|u\|^2_{k+1,\Omega_{\text{ext}}}, \qquad \sum_{\widetilde{K} \in \widetilde{\mathcal{T}}_h} \|u\|^2_{k+1,\widetilde{Q}(\widetilde{K})} \leq N \|u\|_{k+1,\Omega_{\text{ext}}},$$

with $\Omega_{\text{ext}}$ defined in (5.1). In order to obtain exponential convergence estimates for analytic functions, we must show bounds analogous to (4.18) and (4.19) by making explicit the dependence of the constants $C_1$ and $C_2$ on $k$, i.e. on the Sobolev regularity of the solution $u$. For this reason, we split this section in two parts. In Sec. 5.1, we derive estimates of types (4.18) and (4.19) with the dependence on $k$ explicated; in Sec. 5.2, we derive an exponential convergence estimate.

We stress that in the following we will assume that $u$, the solution of (2.2), is in fact the restriction of a regular function on the set $\Omega_{\text{ext}}$, $\Omega_{\text{ext}}$ being defined in (5.1); with an abuse of notation we will call again $u$ such a regular function.

### 5.1. *hp estimate using an overlapping square method*

In this section, we use an overlapping square technique which allows us, under assumption **(D3)**, to explicit the dependence on the Sobolev regularity in the estimate proven in Lemma 4.1 (consequently also on Lemmas 4.2 and 4.4) and on

Lemma 4.3. Finally, we restate Theorem 4.1 on a proper extended domain, under assumption **(D3)**. We note that the polynomial approximation which allows to have an estimate in $p$ will be different from that discussed in Sec. 4; such a polynomial, introduced by Babuška and Suri in Refs. 6 and 7, is a Fourier-type approximation. We decide to use here a different choice, by choosing an approximant of Legendre type whose properties are studied for instance in Ref. 27. The reason for this change is discussed in Remark 5.1.

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### 5.1.1. *A first local estimate*

Here, we give an explicit representation of the constant $C$ in (4.4) in terms of $k$, $k$ being the Sobolev regularity of the target function. We start by showing the counterpart of Lemma 4.1. As a minor note, we point out that the estimate of Lemma 5.1 does not require explicitly a shape regularity condition on the polygons, differently from Lemma 4.1.

**Lemma 5.1.** *Let $\widehat{Q}$ be the square $[-1,1]^2$. Let $\widehat{K} \subseteq \widehat{Q}$ be a polygon with barycenter $\mathbf{x}_{\widehat{K}} = \mathbf{0}$. Moreover, assume that $p \geq 2k$, with $k \in \mathbb{N}$. Then, there exists a family of projection operators $\{\widehat{\Pi}_{\widehat{Q},p}\}$, $p = 1, 2, \ldots$ with $\widehat{\Pi}_{\widehat{Q},p} : H^2(\widehat{Q}) \to \mathbb{P}_p(\widehat{Q})$ such that, for any $\widehat{u} \in H^{k+1}(\widehat{Q})$, it holds*

$$|\widehat{u} - \widehat{\Pi}_{\widehat{Q},p}\widehat{u}|_{1,\widehat{K}} \leq C 2^k e^k p^{-k} |\widehat{u}|_{k+1,\widehat{Q}}, \tag{5.2}$$

*with $C$ a constant independent on $u$, $k$ and $p$.*

**Proof.** Let $\widehat{Q} = [-1,1]^2$. Let $\{V_i\}_{i=1}^4$ be the set of vertices of $\widehat{Q}$. Let $\widehat{u} \in H^{k+1}(\widehat{Q})$. Let $\mathbb{Q}_p(\Theta)$ be the set of polynomials of maximum degree $p$ in each variable over a domain $\Theta \in \mathbb{R}^2$. As a consequence of Lemma 4.67 in Ref. 27, it is possible to show the existence of $\widehat{\varphi}_p \in \mathbb{Q}_p(\widehat{Q})$ such that:

$$\widehat{\varphi}_p(V_i) = \widehat{u}(V_i), \quad \forall\, i = 1, \ldots, 4 \tag{5.3}$$

and

$$|\widehat{u} - \widehat{\varphi}_p|^2_{1,\widehat{Q}} \leq 2 \left\{ \frac{(p-k)!}{(p+k)!} + \frac{1}{p(p+1)} \cdot \frac{(p-k+1)!}{(p+k-1)!} \right\} |\widehat{u}|^2_{k+1,\widehat{Q}}. \tag{5.4}$$

Since $p \geq k$, it is possible to show that (5.4) leads to the following simpler bound:

$$|\widehat{u} - \widehat{\varphi}_p|_{1,\widehat{Q}} \leq C e^k p^{-k} |\widehat{u}|_{k+1,\widehat{Q}}, \quad \text{with } C = \sqrt{e}. \tag{5.5}$$

In order to show this, we perform the computations only on the first term in the right-hand side of (5.4) since the treatment of the other one is analogous. Using Stirling formula:

$$\frac{(p-k)!}{(p+k)!} = \frac{(p-k)^{(p-k)} \cdot e^{-(p-k)} \cdot \sqrt{2\pi(p-k)} \cdot e^{\theta_{p-k}}}{(p+k)^{(p+k)} \cdot e^{-(p+k)} \cdot \sqrt{2\pi(p+k)} \cdot e^{\theta_{p+k}}},$$

with

$$\frac{1}{12n+1} \leq \theta_n \leq \frac{1}{12n}, \quad \forall\, n \in \mathbb{N}.$$

Then:

$$\frac{(p-k)!}{(p+k)!} \leq p^{-2k} \cdot e^{2k} \cdot e^{\theta_{p-k}} \leq C e^{2k} p^{-2k}, \quad \text{with } C = e. \tag{5.6}$$

At this point, we observe that $\mathbb{Q}_p(\widehat{Q}) \subseteq \mathbb{P}_{2p}(\widehat{Q})$. This fact and (5.5) immediately imply that there exists $\widehat{\varphi}_p \in \mathbb{P}_p(\widehat{Q})$ which interpolates $\widehat{u}$ at the vertices of $\widehat{Q}$ as in (5.3) and which satisfies

$$|\widehat{u} - \widehat{\varphi}_p|_{1,\widehat{Q}} \leq C 2^k e^k p^{-k} |\widehat{u}|_{k+1,\widehat{Q}},$$

provided that $p \geq 2k$. We note that, owing to the fact that $\widehat{K} \subseteq \widehat{Q}$, it holds

$$|\widehat{u} - \widehat{\varphi}_p|_{1,\widehat{K}} \leq |\widehat{u} - \widehat{\varphi}_p|_{1,\widehat{Q}} \leq C 2^k e^k p^{-k} |\widehat{u}|_{k+1,\widehat{Q}}.$$

In order to conclude, it suffices to define $\widehat{\Pi}_{\widehat{Q},p}\widehat{u} := \widehat{\varphi}_p$. □

The counterpart of Lemma 4.2 follows.

**Lemma 5.2.** *Let $K \in \mathcal{T}_h$. Let $Q = Q(K)$ be the smallest square containing $K$ and let $u \in H^{k+1}(Q)$. Let $p \geq 2k$. Then, there exists a sequence of projection operators $\Pi^h_{Q,p}$, $p = 1, 2, \ldots$ with $\Pi^h_{Q,p} : H^2(Q) \to \mathbb{P}_p(Q)$ such that for any $k \in \mathbb{N}$:*

$$|u - \Pi^h_{Q,p} u|_{1,K} \leq C M^k \frac{h_K^\mu}{p^k} \|u\|_{k+1,Q}, \quad \mu = \min(p, k),$$

*where $C$ and $M$ are two constants independent on $k$, $h$, $p$ and $u$.*

**Proof.** It suffices to apply Lemma 5.1 and a classical scaling argument. The mapping $F$ between $Q$ and $\widehat{Q}$ is the composition of a rotation, a translation and a dilatation in $\mathbb{R}^2$. The polygon $\widehat{K} \in \widehat{Q}$ and the operator $\Pi^h_{Q,p} u$ will be simply given by $\widehat{K} = F(K)$ and $\Pi^h_{Q,p} u = (\Pi^h_{Q,p}(u \circ F^{-1})) \circ F$ respectively. □

As done in Sec. 4.1.1, we define $u_\pi \in S_h^{p,-1}(\mathcal{T}_h)$, $S_h^{p,-1}(\mathcal{T}_h)$ being introduced at the beginning of Sec. 4.1, as

$$u_\pi|_K = (\Pi^h_{Q,p} u)|_K, \quad \text{with } Q = Q(K), \ \forall\, K \in \mathcal{T}_h.$$

Owing to assumption **(D3)** and Lemma 5.2, we are able to give the following global estimate:

$$|u - u_\pi|_{h,1,\Omega} \leq C A^k \frac{h^\mu}{p^k} \|u\|_{k+1,\Omega_{\text{ext}}}, \quad \mu = \min(p, k), \tag{5.7}$$

where $\Omega_{\text{ext}}$ is defined in (5.1) and $C$ and $A$ are two constants independent on $h$, $p$, $k$, $\gamma$, $\widetilde{\gamma}$ and $u$ ($A$ is independent also on $N$).

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5.1.2. *A second local estimate*

In the present section, we give an explicit representation of the constant $C$ in (4.11) in terms of $k$. We point out that here the shape regularity assumption is needed; in fact, the usual scaling arguments used herein are based on affine mappings of shape regular triangles into the master triangle.

**Lemma 5.3.** *Let $u$ be the solution of* (2.2) *with $u \in H^{k+1}(\Omega_{\text{ext}})$, $\Omega_{\text{ext}}$ being defined in* (5.1)*. Under assumptions* **(D1)**, **(D2)** *and* **(D3)**, *provided that $p \geq 2k$, there exists $u_I \in V_h$ such that*

$$|u - u_I|_{1,\Omega} \leq C \cdot B^k \frac{h^k}{p^k} |u|_{k+1,\Omega_{\text{ext}}}, \tag{5.8}$$

*where $C$ and $B$ are two constants independent on $k$, $p$, $h$ and $u$ ($B$ is independent also on $N$).*

**Proof.** The proof of this lemma is a combination of the arguments used in Lemma 5.1 and the construction of Lemma 4.3. Therefore, we only give the sketch of the proof. We start by considering a triangle $\widetilde{K}$ in the subtriangular decomposition $\widetilde{\mathcal{T}}_h$, we map it into the master triangle $\widehat{T}$ (i.e. the triangle obtained halving the square $[-1,1]^2$ through its diagonal), we use a Legendre-type approximant in order to derive an estimate in $p$ as in Lemma 5.1, we go back to the triangle $\widetilde{K}$. Let $\widetilde{Q}$ be the parallelogram $\widetilde{Q} = \widetilde{Q}(\widetilde{K})$ (see assumption **(D3)**) and let $\{\widetilde{V}_i\}_{i=1}^3$ be the set of the vertices of $\widetilde{K}$. Therefore, it is possible to show the existence of a $\varphi_p^h \in \mathbb{P}_p(\widetilde{K})$ such that $\varphi_p^h(\widetilde{V}_i) = u(\widetilde{V}_i)$, $\forall\, i = 1,2,3$ and such that

$$|u - \varphi_p^h|_{1,\widetilde{K}} \leq C\widetilde{B}^k \frac{h^k}{p^k} |u|_{k+1,\widetilde{Q}}, \tag{5.9}$$

where $C$ and $\widetilde{B}$ are two constants independent on $p$, $h$, $k$ and $u$ ($\widetilde{B}$ is also independent on $N$, $\gamma$ and $\widetilde{\gamma}$). We point out that this estimate holds for all the triangles in the triangular subdecomposition $\widetilde{\mathcal{T}}_h$. We denote, with a little abuse of notation, by $\varphi_p^h : \Omega \to \mathbb{R}$ the global piecewise polynomial function whose restriction on each triangle $\widetilde{K}$ satisfies (5.9).

So far, we have obtained discontinuous piecewise polynomials. We set:

$$E = E(\widetilde{K}) := \left( \bigcup_{\{\widetilde{\widetilde{K}} \in \widetilde{\mathcal{T}}_h \mid \widetilde{K} \cap \widetilde{\widetilde{K}} = e\}} \widetilde{Q}(\widetilde{\widetilde{K}}) \right) \cup \widetilde{Q}(\widetilde{K}), \quad e \in \mathcal{E}_{\widetilde{K}},$$

where we recall that $\mathcal{E}_{\widetilde{K}}$ is the set of the edges of $\widetilde{K}$ and $\widetilde{Q}(\widetilde{\widetilde{K}})$ is defined in assumption **(D3)**. We need to modify $\varphi_p^h$ in order to get a continuous piecewise polynomial over $\widetilde{\mathcal{T}}_h$ without changing the approximation property (5.9). This can be done following the same approach as in Theorem 4.6 and Lemma 4.7 of Ref. 6, i.e. by correcting $\varphi_p^h$ with suitable polynomial extensions of its edge jumps. It is possible to check that such step does not introduce constants depending on $k$.

With another little abuse of notation, we have obtained a $\varphi_p^h \in H_0^1(\Omega)$ piecewise continuous polynomial of degree $p$ over the subtriangular decomposition $\widetilde{\mathcal{T}}_h$, such that an analogous of (5.9) holds for all $\widetilde{K} \in \widetilde{\mathcal{T}}_h$:

$$|u - \varphi_p^h|_{1,\widetilde{K}} \leq c(\gamma, \widetilde{\gamma})\widetilde{\widetilde{B}}^k \frac{h^k}{p^k}|u|_{k+1,E}.$$

Using assumption **(D3)** and the arguments described in Lemma 4.3, one can conclude the proof. □

The counterpart of Lemma 4.4 follows easily from Lemma 4.4 and Lemma 5.2. In particular the following lemma holds.

**Lemma 5.4.** *Under assumptions* **(D1)**, **(D2)** *and* **(D3)**, *let* $\Omega_{\text{ext}}$ *be defined in* (5.1), *let the loading term* $f \in H^{\widetilde{k}+1}(\Omega_{\text{ext}})$. *Then it holds*

$$\mathcal{F}_h \leq CD^k \frac{h^{\widetilde{\mu}}}{p^{\widetilde{k}+2}}\|f\|_{\widetilde{k}+1,\Omega_{\text{ext}}}, \quad \widetilde{\mu} = \min(p, \widetilde{k}+2), \tag{5.10}$$

*where* $C$ *and* $D$ *are two constants independent on* $k$, $h$, $p$ *and* $u$ ($D$ *is also independent on* $N$).

5.1.3. *A global estimate result*

Combining bounds (5.7), (5.8), (5.10) and (4.3) yields the following result.

**Theorem 5.1.** *Let* $k \in \mathbb{N}$, $k > \frac{1}{2}$. *Let the mesh assumptions* **(D1)**, **(D2)** *and* **(D3)** *hold. Let* $u$ *and* $u_h$ *be respectively the solution of problems* (2.2) *and* (3.1). *Let* $\Omega_{\text{ext}}$ *be defined as in* (5.1). *Let* $u \in H^{k+1}(\Omega_{\text{ext}})$. *Let* $\gamma$, $\widetilde{\gamma}$ *and* $N$ *be the constants introduced in assumptions* **(D1)**, **(D2)** *and* **(D3)**. *Assume also* $p \geq 2k$. *Then, the following hp estimate holds*:

$$|u - u_h|_{1,\Omega} \leq C\widetilde{A}^k \frac{h^k}{p^k}|u|_{k+1,\Omega_{\text{ext}}}, \tag{5.11}$$

*where* $C$ *and* $\widetilde{A}$ *are two constants independent on* $h$, $p$, $k$ *and* $u$ ($\widetilde{A}$ *is also independent on* $N$).

As done in Remark 4.4, we point out that if the domain $\Omega$ is convex it is possible to derive easily, owing to the approximation properties of Legendre polynomials, $L^2$-estimates of the form:

$$\|u - u_h\|_{0,\Omega} \leq C\widetilde{A}^k \frac{h^{k+1}}{p^{k+1}}|u|_{k+1,\Omega_{\text{ext}}}, \tag{5.12}$$

where $C$ and $\widetilde{A}$ are two constants independent on $h$, $p$, $k$ and $u$ ($\widetilde{A}$ is also independent on $N$).

**Remark 5.1.** We point out that in order to obtain the *hp* estimates of Theorem 4.1 and of Theorem 5.1, we used two different approximant polynomials. Throughout

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Sec. 4, we decided to follow the Babuška–Suri construction (see Refs. 6 and 7) which is based on a Fourier series expansion on a proper domain. Nevertheless this construction obliges, also in the case of the overlapping square technique introduced at the beginning of Sec. 5, to use some extension operator (for instance the one described in Ref. 28 for Lipschitz domains). Thus, to give an explicit representation of the dependence of the involved constant on the Sobolev regularity $k$ is not a trivial work. On the other hand, throughout Sec. 5, we made use of Legendre-type approximant (as done for instance in Ref. 27). In this case, owing to Legendre polynomials properties, we are able to obtain exponential estimates (see Sec. 5.2), since the dependence in the constant with respect to the Sobolev regularity $k$ can be derived. We stress that the Legendre approach could be used also in Sec. 4; the choice of Fourier-type approximation, which we recall is not applicable in Sec. 5, is essentially a matter of taste and has the merit of avoiding to use bi-polynomial functions; furthermore, the latter approach is much easier than the former and this is why we show the details of both.

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### 5.2. *Exponential convergence*

We have the following exponential convergence result for analytic solutions $u$ over the extended domain $\Omega_{\text{ext}}$ (see (5.1)).

**Theorem 5.2.** *Let the mesh assumptions* **(D1)**, **(D2)** *and* **(D3)** *hold. Let $u$ and $u_h$ be respectively the solution of problems* (2.2) *and* (3.1), *with $u \in \mathcal{A}(\overline{\Omega_{\text{ext}}})$, $\mathcal{A}(\overline{\Omega_{\text{ext}}})$ being the set of analytic function over the closure of $\Omega_{\text{ext}}$ defined in* (5.1). *Then, the following exponential convergence estimate holds*:

$$\|u - u_h\|_{1,\Omega} \leq Ce^{-bp}, \tag{5.13}$$

*for some positive constants $C$ and $b$ independent on $p$.*

**Proof.** We recall (see for instance Ref. 15) that an analytic function in the closure of a domain $\Theta \in \mathbb{R}^2$ is characterized by the following bound:

$$\|D^{\boldsymbol{\alpha}} u\|_{\infty,\overline{\Theta}} \leq CA^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!, \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in {\mathbb{N}_0}^2, \tag{5.14}$$

where $\boldsymbol{\alpha}! = \alpha_1!\alpha_2!$ and where $C$ and $A$ are constants independent on the multi-index $\boldsymbol{\alpha}$; nevertheless, $C$ and $A$ depend on $u$ and on $\overline{\Theta}$. Recalling (5.11), we have

$$|u - u_h|_{1,\Omega} \leq C(\gamma, \widetilde{\gamma}, N)\widetilde{A}(\gamma, \widetilde{\gamma})^k \frac{h^k}{p^k}|u|_{k+1,\Omega_{\text{ext}}},$$

if $p \geq 2k$. Using standard results from space interpolation theory,[14,29] from the above bound one can easily derive

$$|u - u_h|_{1,\Omega} \leq C(\gamma, \widetilde{\gamma}, N)\widetilde{A}(\gamma, \widetilde{\gamma})^s \frac{h^s}{p^s}|u|_{s+1,\Omega_{\text{ext}}} \tag{5.15}$$

for all $s \in \mathbb{R}$ with $2 \leq 2s \leq p$. The combination of (5.15) and (5.14) yields

$$|u - u_h|_{1,\Omega} \leq C \left( \widetilde{A} \frac{h}{p} \right)^s A^{s+1} (s+1)!.$$

By means of Stirling formula, we obtain:

$$|u - u_h|_{1,\Omega} \leq C \left( \frac{hA\widetilde{A}}{p} \right)^s \left( \frac{s+1}{e} \right)^{s+1} \sqrt{2\pi} (s+1)^{\frac{1}{2}},$$

easily yielding

$$|u - u_h|_{1,\Omega} \leq C \left( \frac{hA\widetilde{A}}{ep} s \right)^s s^{\frac{3}{2}}.$$

By denoting $\delta = \frac{hA\widetilde{A}}{e}$ we can write:

$$|u - u_h|_{1,\Omega} \leq C \left( \frac{s}{p} \delta \right)^s s^{\frac{3}{2}}.$$

Since this last inequality holds true for all $s$ such that $2 \leq 2s \leq p$, we may choose $s = \frac{p}{2(\delta+1)}$. Hence:

$$|u - u_h|_{1,\Omega} \leq C \left( \frac{\delta}{2(\delta+1)} \right)^{\frac{p}{2(\delta+1)}} p^{\frac{3}{2}} = Ce^{-bp} p^{\frac{3}{2}}, \quad \text{with } b = \frac{\log(\frac{\delta}{2(\delta+1)})}{2(\delta+1)}. \tag{5.16}$$

The multiplier $p^{\frac{3}{2}}$ can be absorbed by $e^{-bp}$ by making $b$ a little bit smaller and increasing $C$; therefore, (5.16) immediately yields

$$|u - u_h|_{1,\Omega} \leq Ce^{-bp}, \tag{5.17}$$

for some constants $C$ and $b$ independent on $p$. The result follows by the Poincaré inequality. □

## 6. Numerical Results

In this section, we present numerical results experimentally validating the error estimates (4.18), (4.19), (4.20) and (5.17). We consider four types of meshes (see Fig. 1) on the domain $\Omega = [0,1]^2$, namely an unstructured triangular mesh, a regular square mesh, a regular hexagonal mesh and a Voronoi–Lloyd mesh (see Ref. 20). The basis $\{q_{\boldsymbol{\alpha}}\}$ of the space $\mathbb{P}_{p-2}(K)$ introduced in Sec. 3, $\forall\, K \in \mathcal{T}_h$, is taken to be the same as that introduced for instance in Ref. 8 or Ref. 10. Different choices are investigated in the Appendix. Moreover, we fix a possible choice for the stabilizing term $S^K$ introduced in (3.9) (see for instance Ref. 8) as:

$$S^K(u_h, v_h) = \sum_{r=1}^{\dim(V_h(K))} \chi_r(u_h)\chi_r(v_h), \quad \forall\, u_h,\, v_h \in V_h(K),\, K \in \mathcal{T}_h, \tag{6.1}$$

where $\chi_r$, $\forall\, r = 1, \ldots, \dim(V_h(K))$ is the operator which associates to each function in the local space $V_h(K)$ its $r$th local degree of freedom.

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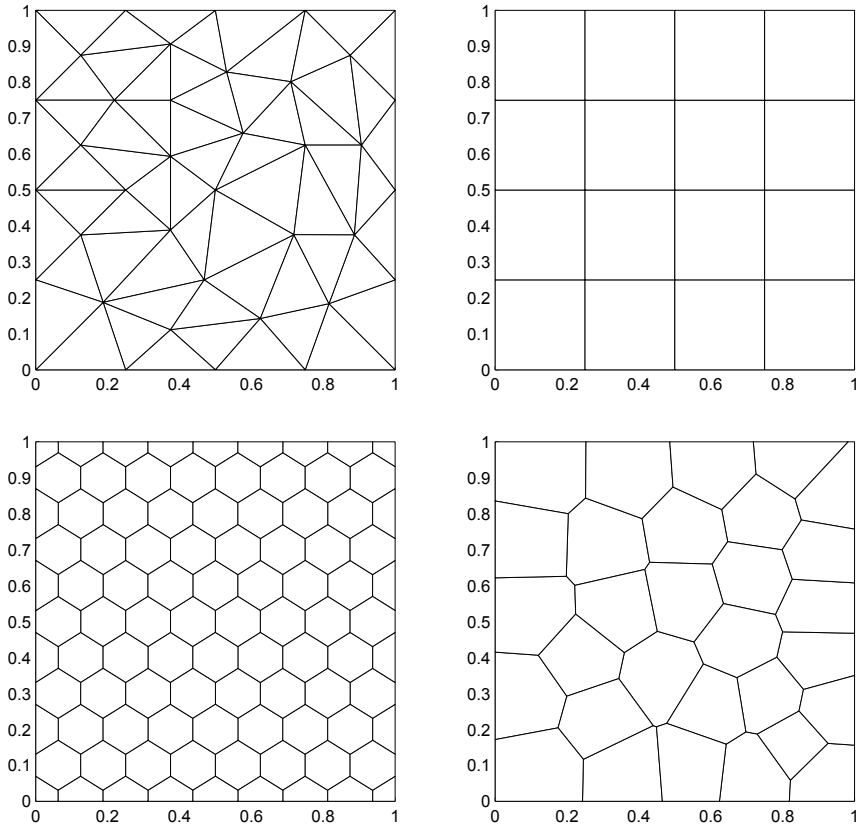


Fig. 1. From left to right: unstructured triangular mesh, regular square mesh, regular hexagonal mesh, Voronoi–Lloyd mesh.

**Remark 6.1.** The stabilizing bilinear form (6.1) does not guarantee, in principle, that the stability property (3.9) is uniform in $p$. A theoretical analysis of the dependence in $p$ of the constants appearing in (3.9), possibly for different choices of the stabilizing term, is a more specific topic that is beyond the scope of the present (general) work and will be investigated in future communications. Nevertheless, in Sec. 6.4, we show numerically the dependence on $p$ of the stability bounds in (3.9) for the stabilizing term (6.1) for different sample polygons.

In order to estimate the error introduced by the MATLAB algebraic sparse solver, we have solved a problem whose exact solution is the polynomial $u(x, y) = x^2+y^2$. Since the VEM passes the patch test, in this case, for $k \geq 2$, the approximate solution $u_h$ coincides with $u$ and the error that we measure is only due to the algebraic solver (it should be zero in exact arithmetic). Hence, together with the standard error in the $H^1$-norm and in the $L^2$-norm with respect to the solutions (6.2) and (6.3), in our convergence figures we also plot this algebraic error. When the error curve comes close to the algebraic error curve, the convergence error and

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the error introduced by the MATLAB algebraic solver are of the same order and the expected theoretical behaviour does not hold anymore.

### 6.1. *Convergence in p for an analytic function*

We consider problem (2.2) with loading term $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$. The exact solution is given by:

$$u(x,y) = \sin(\pi x)\sin(\pi y). \tag{6.2}$$

In this test, the mesh is kept fixed (see Fig. 1) and the polynomial degree is raised. In Figs. 2 and 3 we report the errors among the discrete and exact solutions. Since we are dealing with a virtual element solution $u_h$ (that is unknown inside elements) we cannot directly compute the error $\|u - u_h\| s, \Omega$, $s = 0, 1$. Therefore, as is standard in VEM, we plot instead $\|u - \Pi_p^\nabla u_h\|_{0,\Omega}$ and $|u - \Pi_p^\nabla u_h|_{h,1,\Omega}$, that are good representatives of the above errors (see (3.8) for the definition of the operator $\Pi_p^\nabla$ and (4.1) for the definition of the $H^1$ broken Sobolev seminorm).

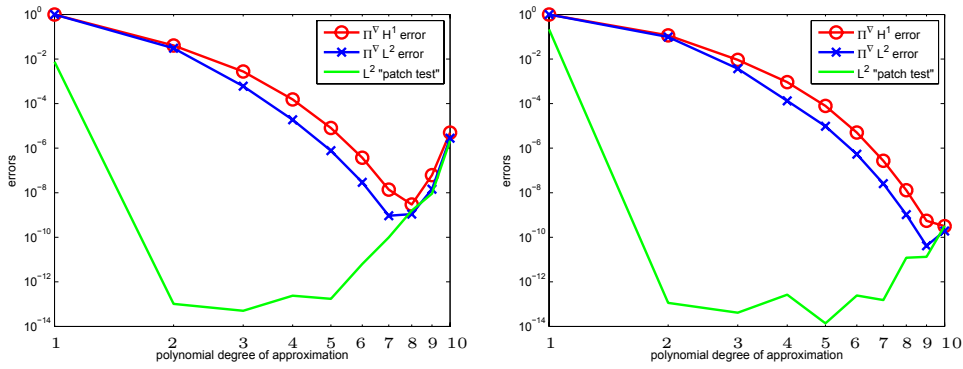


Fig. 2. $u(x,y) = \sin(\pi x)\sin(\pi y)$; unstructured triangle mesh (left); regular square mesh (right).

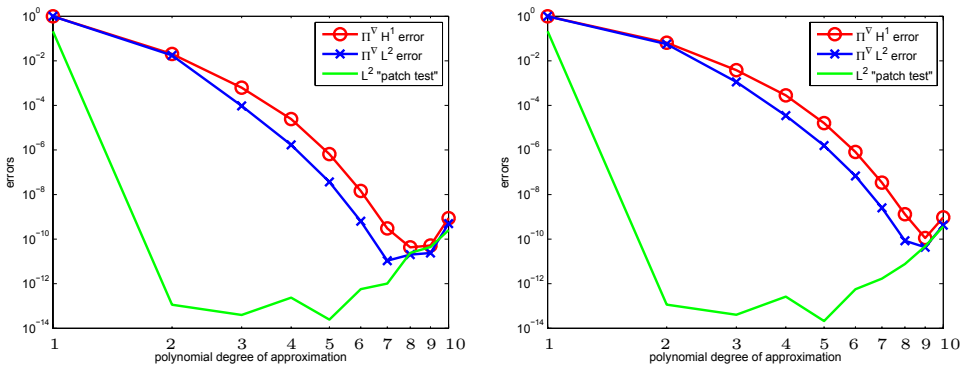


Fig. 3. $u(x,y) = \sin(\pi x)\sin(\pi y)$; regular hexagonal mesh (left); 4: Voronoi–Lloyd mesh (right).

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In accordance with Theorem 5.2, the exponential convergence is evident from the decreasing slope in the error graphs. Moreover, from Figs. 2 and 3 we can observe that the lower line is a good marker for the indication of the machine algebra error.

### 6.2. *Convergence in p for a function with finite Sobolev regularity*

Secondly, we present a similar behaviour test for the case of a problem with solution

$$u(r,\theta) = r^{2.5}\sin(2.5\theta), \tag{6.3}$$

where $(r,\theta)$ are the polar coordinates with respect to the origin. Since the function is harmonic, the loading term $f = 0$ and the Dirichlet boundary conditions are set in accordance with $u|_{\partial\Omega}$. We note that $u \in H^{3.5-\varepsilon}(\Omega)$, $\forall\,\varepsilon > 0$. In Figs. 4 and 5, the segmented line represents a line of slope $5 = 2 \cdot 2.5$. Owing to Theorem 4.1, we should have an estimate in $p$ of the type $p^{-a}$, $a = 2.5$. Anyhow, for this type of corner singularity, one could extend the technical result of Refs. 6 and 7 obtaining error estimate in $p$ of the type $p^{-2a}$ also in our VEM framework. Thus, we expect

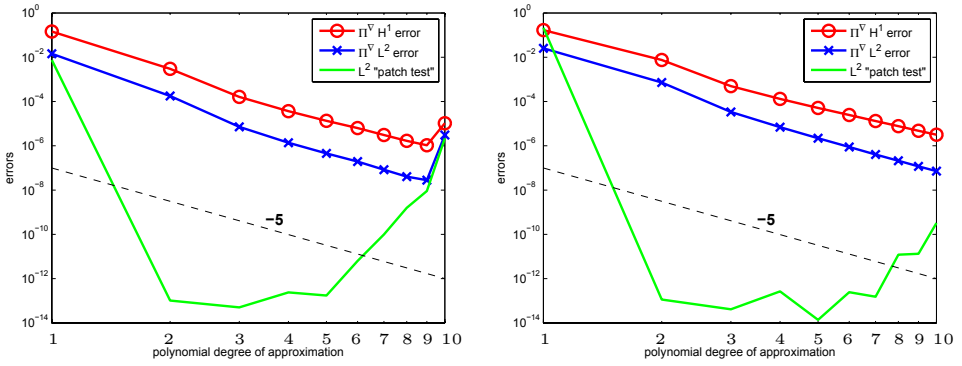


Fig. 4. $u = r^{2.5}\sin(2.5\theta)$; unstructured triangle mesh (left); 2: regular square mesh (right).

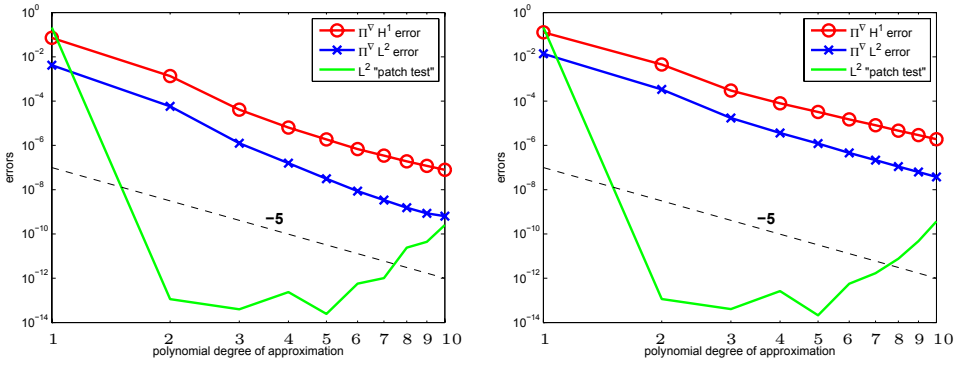


Fig. 5. $u = r^{2.5}\sin(2.5\theta)$; regular hexagonal mesh (left); 4: Voronoi–Lloyd mesh (right).

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a slope for the $H^1$ error of the type $p^{-5}$, which is represented with the dashed line in Figs. 4 and 5. Figures 4 and 5 are in agreement with such observation.

### 6.3. *Convergence in h*

In this subsection, we show the convergence rate when the polynomial degree is kept fixed and the mesh size goes to zero. We consider a sequence of hexagonal and Voronoi–Lloyd meshes and we study the same harmonic test case as in Sec. 6.1. In particular, we examine the case $p = 3$ and $p = 5$. We observe that the slope of the errors are in accordance with Theorem 4.1 and with estimate (4.20). The same considerations about Fig. 6 are still valid for Fig. 7. We only point out that the strange $L^2$-error behaviour for the final step is due to the machine precision error.

### 6.4. *Numerical tests for the stability bounds*

As already observed in Remark 6.1, the stabilizing bilinear form (6.1) does not guarantee, in principle, that the stability property (3.9) is uniform in $p$. Indeed,

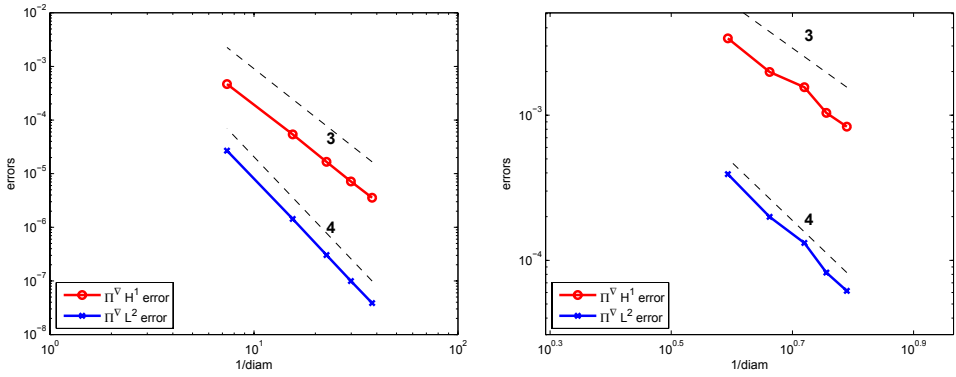


Fig. 6. Regular hexagonal mesh (left); Voronoi–Lloyd mesh (right); $p = 3$.

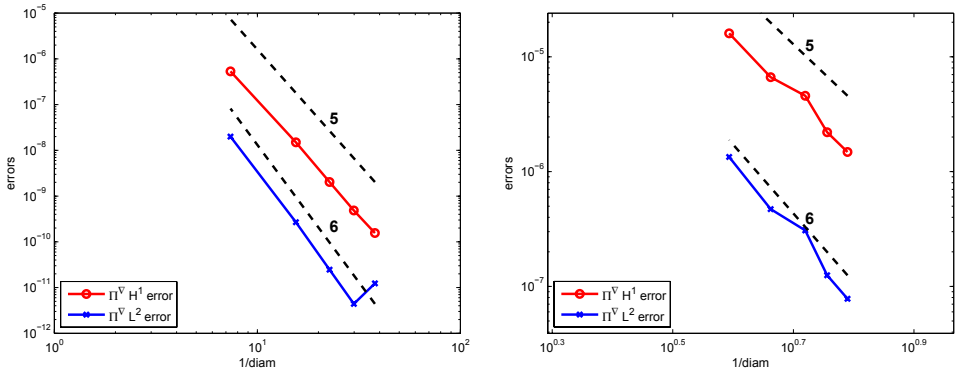


Fig. 7. Regular hexagonal mesh (left); Voronoi–Lloyd mesh (right); $p = 5$.

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while it is quite easy to show that these two constants are independent on $h$ (at least under our current geometric assumptions on the elements), the behaviour in $p$ needs a different approach. A theoretical analysis of the dependence in $p$ of the constants appearing in (3.9) (or equivalently (3.6)), possibly for different choices of the stabilizing term, will be the topic of future communications. Nevertheless, in the present section we show numerically the dependence on $p$ of the stability bounds in (3.6) for the stabilizing term (6.1) for different sample polygons.

Given $K \in \mathcal{T}_h$, we want to investigate the behaviour in $p$ of the constants $\alpha_*, \alpha^*$ in the stability bound (3.6):

$$\alpha_* |v_h|_{1,K} \leq a_h^K(v_h, v_h) \leq \alpha^* |v_h|_{1,K}. \tag{6.4}$$

We note that finding $\alpha_*$ and $\alpha^*$ in (6.4) is equivalent to find the minimum and the maximum eigenvalue, say $\lambda_{\min}$ and $\lambda_{\max}$, of the generalized eigenvalue problem:

$$\mathbf{A}_h^K \mathbf{v} = \lambda \mathbf{A}^K \mathbf{v}, \tag{6.5}$$

where $\mathbf{A}_h^K$, $\mathbf{A}^K \in \mathbb{R}^{\dim(V_h(K)) \times \dim(V_h(K))}$ and

$$(\mathbf{A}_h^K)_{i,j} = a_h^K(\varphi_i, \varphi_j); \quad (\mathbf{A}^K)_{i,j} = a^K(\varphi_i, \varphi_j),$$

$\{\varphi_i\}_{i=1}^{\dim(V_h(K))}$ being the usual canonical virtual basis. Since both matrices (that are symmetric and positive semi-definite) have a kernel given by the vectors representing constant functions, without loss of generality we restrict the analysis to the zero-average functions in $V_h(K)$. Since matrix $\mathbf{A}^K$ is not computable exactly, we approximate its entries by solving numerically the associated diffusion problem, using a very fine triangular mesh on the polygon and *hp* FEM.

We stress that we are in particular interested in the behaviour in terms of $p$ of:

$$\frac{\alpha^*}{\alpha_*}, \quad \text{i.e. } \frac{\lambda_{\max}}{\lambda_{\min}}, \tag{6.6}$$

since such quantity is the one that could affect the convergence results discussed in Secs. 4 and 5, see Theorem 3.1 of Ref. 8. In Table 1 we present tests on three regular sample polygons: a triangle, a square and a hexagon.

Table 1. Minimum and maximum eigenvalues of the generalized eigenvalue problem (6.5) on: tr = a triangle; sq = a square; he = a hexagon.

| $p$ | tr. $\lambda_{\min}$ | tr. $\lambda_{\max}$ | sq. $\lambda_{\min}$ | sq. $\lambda_{\max}$ | he. $\lambda_{\min}$ | he. $\lambda_{\max}$ |
|---|---|---|---|---|---|---|
| 2 | 1.0000e+00 | 1.0825e+00 | 1.0000e+00 | 1.1225e+00 | 4.6394e−01 | 1.2079e+00 |
| 3 | 6.1649e−01 | 1.0000e+00 | 3.7333e−01 | 1.1978e+00 | 4.4218e−01 | 1.2234e+00 |
| 4 | 2.7391e−01 | 1.0000e+00 | 3.0515e−01 | 1.0365e+00 | 4.5071e−01 | 1.3473e+00 |
| 5 | 6.1234e−02 | 1.0000e+00 | 3.0203e−01 | 1.2361e+00 | 4.2189e−01 | 1.3208e+00 |
| 6 | 4.4998e−02 | 1.0000e+00 | 2.0408e−01 | 1.0580e+00 | 4.2771e−01 | 1.2256e+00 |
| 7 | 2.3220e−02 | 1.0000e+00 | 2.0026e−01 | 1.1509e+00 | 4.0537e−01 | 1.2696e+00 |
| 8 | 7.9998e−03 | 1.0000e+00 | 1.3968e−01 | 1.0433e+00 | 4.0755e−01 | 1.2439e+00 |
| 9 | 5.2406e−03 | 1.0000e+00 | 1.3176e−01 | 1.1310e+00 | 4.0183e−01 | 1.2364e+00 |
| 10 | 2.6406e−03 | 1.0000e+00 | 8.9375e−02 | 1.0390e+00 | 3.8648e−01 | 1.2514e+00 |

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First of all, we note that $\lambda = 1$ is always an eigenvalue since, due to the consistency condition **(A1)**, for all vectors $\mathbf{v}$ associated to polynomial functions the two operators above give the same result. Therefore $\lambda_{\max}$ is always bigger or equal than one and $\lambda_{\min}$ always smaller or equal than one. We moreover observe that the maximum eigenvalue is almost constant in all the three cases. On the other hand, the minimum eigenvalue behaves differently, and depends on the polygon. In the case of the hexagon, it is still almost constant. For the square, we notice a very slow decay. Finally, for the triangle we have instead a considerable decay in terms of $p$. Nonetheless, looking at the numerical experiments discussed in Sec. 6.2, we do not notice an equivalent loss in terms of $p$-convergence rates. A possible *heuristic* reason (among others) is the following. At the theoretical level, the bounds in (6.4) are not applied to the complete local space $V_h(K)$ but only (see Theorem 3.1 of Ref. 8) to polynomials and to $u_h - u_I$, $u_h$ being the solution of the discrete problem (3.1) and $u_I$ being the virtual interpolation term; therefore, looking at the "worst rate" (6.6) on the whole space $V_h(K)$ could be pessimistic in many situations.

## Appendix

In this Appendix, we study the behaviour of the condition number of the global stiffness matrix of problem (3.1) and we explore some alternatives in the choice of the local VEM basis. We note that, as it happens in Finite Elements (see Refs. 2 and 5), the main responsible for the growth of the condition number (when $p$ increases) are the internal "bubble" basis functions. In Appendix A.1, we numerically investigate the behaviour of the condition number by changing the polynomial basis $\{q_{\boldsymbol{\alpha}}\}$ of $\mathbb{P}_{p-2}(K)$ (for all $K \in \mathcal{T}_h$), introduced in Sec. 3 for the definition of the local internal degrees of freedom. In Appendix A.2, we discuss an orthogonalization technique that strongly reduces the condition number, but is unstable with respect to machine precision.

### A.1. *Three explicit bases*

In this subsection, we consider three explicit bases:

- $\{q^1_{\boldsymbol{\alpha}}\}$, the same basis introduced for instance in Refs. 8 and 10, that is to say:
$$q^1_{\boldsymbol{\alpha}} = \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K}\right)^{\boldsymbol{\alpha}}, \quad \forall\, \boldsymbol{\alpha} \in \mathbb{N}_0^2, \ \ |\boldsymbol{\alpha}| \le p-2, \tag{A.1}$$
where $\mathbf{x}_K$ and $h_K$ are respectively the barycenter and the diameter of the polygon $K$.
- $\{q^2_{\boldsymbol{\alpha}}\}$, which is defined by
$$q^2_{\boldsymbol{\alpha}} = \frac{q^1_{\boldsymbol{\alpha}}}{\|q^1_{\boldsymbol{\alpha}}\|_{0,K}}, \quad \forall\, \boldsymbol{\alpha} \in \mathbb{N}_0^2, \ \ |\boldsymbol{\alpha}| \le p-2. \tag{A.2}$$
- $\{q^3_{\boldsymbol{\alpha}}\}$, which is a Legendre-type basis. In order to define it, we recall that $N_K$ is the number of vertices of $K$ and $\{V_i\}_{i=1}^{N_K}$ is the set of vertices of $K$; moreover,

we set

$$\widetilde{\mathbf{x}}_{\mathbf{K}} = (\widetilde{x}_K, \widetilde{y}_K) = \left( \frac{x_{V,\min} + x_{V,\min}}{2}; \frac{y_{V,\min} + y_{V,\min}}{2} \right),$$
$$h_K^x = |x_{V,\min} - x_{V,\min}|, \quad h_K^y = |y_{V,\min} - y_{V,\min}|,$$

where $x_{V,\min} = \max_{i=1}^{N_K} x_i$, $x_{V,\min} = \min_{i=1}^{N_K} x_i$, $y_{V,\min} = \max_{i=1}^{N_K} y_i$, $y_{V,\min} = \min_{i=1}^{N_K} y_i$. Besides, let $L_s(\cdot)$ be the Legendre polynomial of degree $s$ on the segment $[-1, 1]$ (see e.g. Ref. 24 for the properties of Legendre polynomials). Then, we are able to define the basis:

$$q_{\boldsymbol{\alpha}}^3 = L_{\alpha_1}\left(2\frac{x - \widetilde{x}_K}{h_K^x}\right) L_{\alpha_2}\left(2\frac{y - \widetilde{y}_K}{h_K^y}\right), \quad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^2, \ |\boldsymbol{\alpha}| \le p - 2. \tag{A.3}$$

We observe that the third choice should be, at least at a first glance, better than the other two, thanks to orthogonality properties of Legendre polynomials. We will see that instead this is not the case in general. Indeed, the orthogonality properties of the Legendre basis are quickly lost when the considered domain is not rectangular. In our tests, we consider the same meshes already used in Sec. 6; see Fig. 1. In Figs. A.1 and A.2, we compare the behaviour of the condition number, given by the MATLAB command *cond*, for the three choices of the bases mentioned above and the four meshes. We stress the fact that the Legendre-type basis performs better in the case of the square mesh; this is believable thanks to the orthogonality properties of the Legendre polynomials. On the contrary, one can see that more general meshes, such as the hexagonal, unstructured triangular and Voronoi–Lloyd ones, the best result is obtained with the $L^2$-scaled basis. Finally, we present in Figs. A.3–A.6 some "comparison tests" in which we report the error $|u - \Pi_p^\nabla u_h|_{1,\Omega}$ (see (3.8)), by using the four different meshes in Fig. 1 and the three different bases. We consider the same two test cases of Sec. 6. From Figs. A.3 and A.4, one can see that the Legendre-type basis is a good choice for the square case, while on

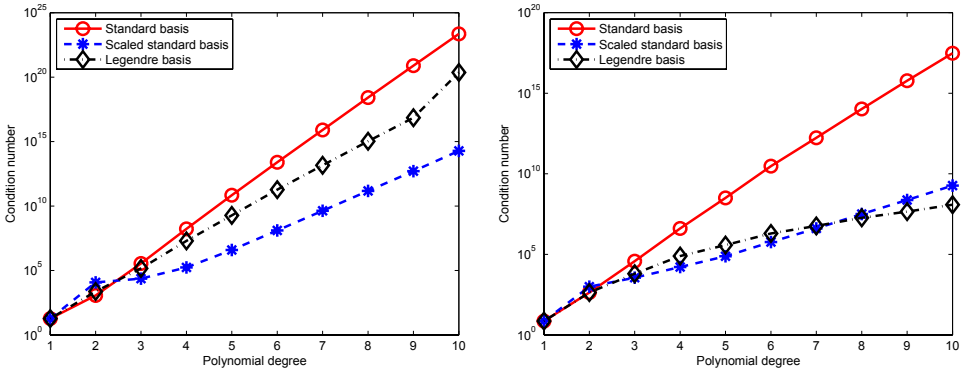


Fig. A.1. Unstructured triangle mesh (left); regular square mesh (right).

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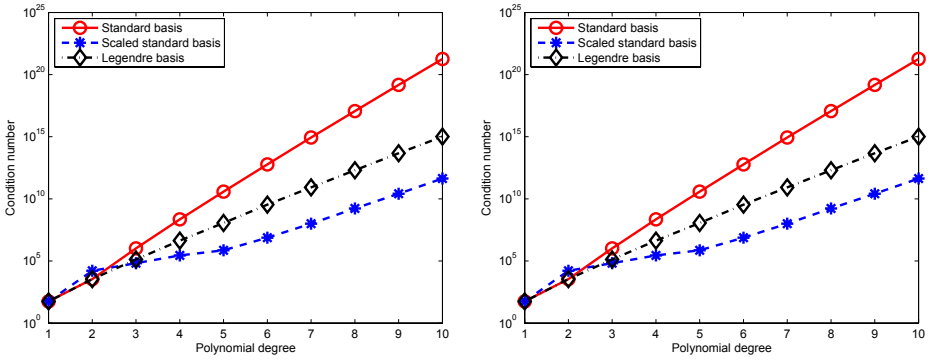


Fig. A.2. Regular hexagonal mesh (left); Voronoi–Lloyd mesh (right).

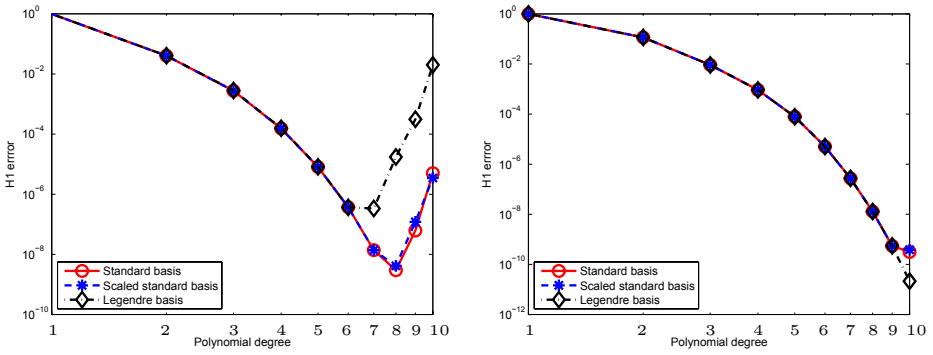


Fig. A.3. $u(x, y) = \sin(\pi x)\sin(\pi y)$; unstructured triangle mesh (left); regular square mesh (right).

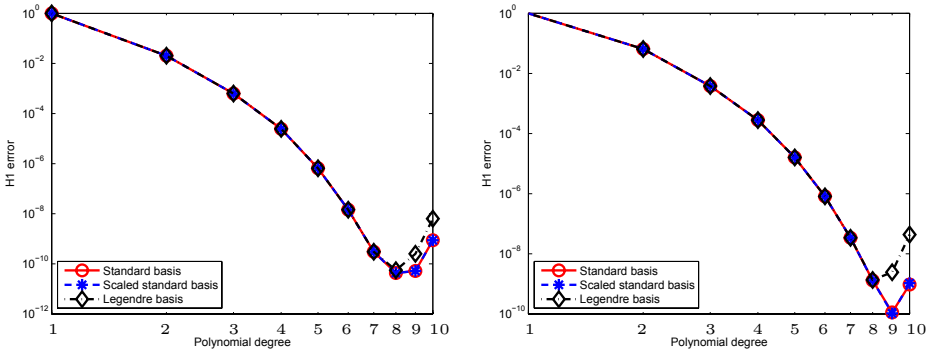


Fig. A.4. $u(x, y) = \sin(\pi x)\sin(\pi y)$; regular hexagonal mesh (left); Voronoi–Lloyd mesh (right).

triangles it is very unstable; besides, for general meshes, it seems that the slope of the error with the other two bases is almost the same and performs better than the Legendre-type basis. The same considerations for Figs. A.3 and A.4 hold for Figs. A.5 and A.6.

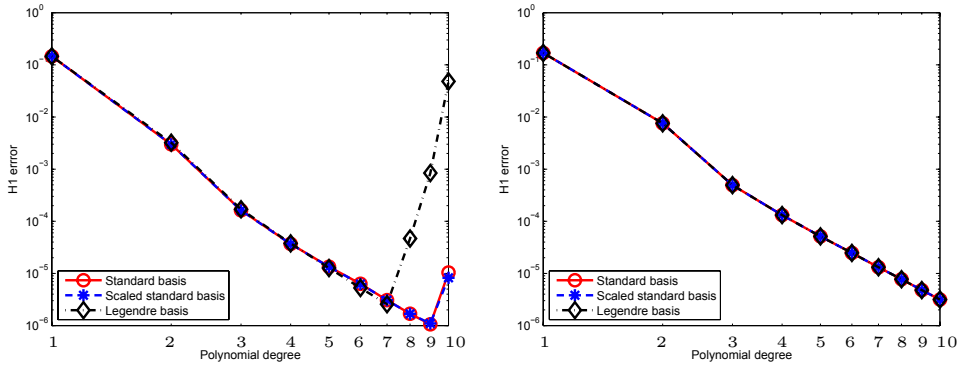


Fig. A.5. $u(r,\theta) = r^{2.5}\sin(2.5\theta)$; unstructured triangle mesh (left); regular square mesh (right).

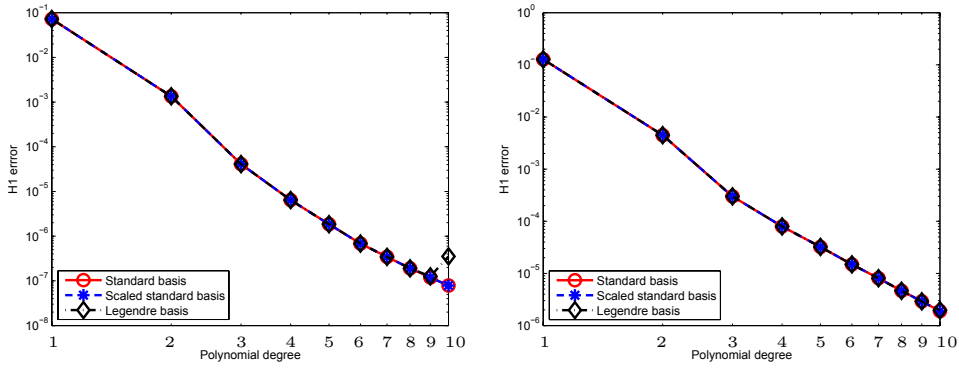


Fig. A.6. $u(r,\theta) = r^{2.5}\sin(2.5\theta)$; regular hexagonal mesh (left); Voronoi–Lloyd mesh (right).

We point out that in our numerical tests we have used a direct solver in order to work out the global system arising from the discrete problem (3.1). A consequence of this fact is that the condition number of the global matrix does not affect the resolution of the linear system as it would do if we used an iterative solver. This explains why the behaviour of the errors with the choice of the classical basis $\{q^1_{\boldsymbol{\alpha}}\}$ and the scaled basis $\{q^2_{\boldsymbol{\alpha}}\}$ is almost the same, notwithstanding the large difference in the condition number of the global matrix as shown in Fig. A.1 and in Fig. A.2.

### A.2. *A "virtual" Gram–Schmidt process*

From the previous subsection, it is clear that by suitably changing the basis of the space one obtains better condition numbers for the global stiffness matrix. Despite this, from Figs. A.1 and A.2 we note that the condition numbers are still large and, although in our codes we use a direct solver, it would be preferable to reduce such numbers. Therefore, in this subsection we consider an extension of the Gram–Schmidt technique that considerably reduces the condition number of the global stiffness matrix, but at the price of an unstable propagation of the machine error

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precision (as better discussed later). The idea behind this procedure consists in orthonormalizing the internal virtual basis functions with respect to the discrete bilinear form $a^h$, the additional difficulty being that such shape functions are not known explicitly. Nevertheless, in order to derive an orthonormalized basis, we can use the fact that the discrete scalar product between two discrete virtual functions is computable (following for instance Ref. 10). The detailed description of such a process can be found in Ref. 11.

We present now some numerical experiments about the behaviour of the condition number. We consider the four different meshes in Fig. 1 and we compare the condition number of the $L^2$-scaled basis introduced in Sec. A.1 and the new Gram–Schmidt basis (see Ref. 11). We remind that, from the previous numerical experiments, the $L^2$-scaled basis seems to be the most well-conditioned among the choices of Appendix A.1. From the results in Figs. A.7 and A.8, it follows that the Gram–Schmidt basis performs much better, at least for what concerns the condition number.

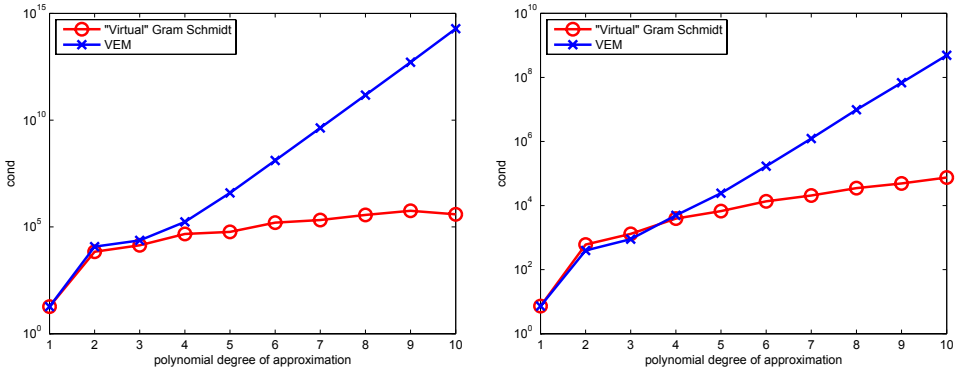


Fig. A.7. Unstructured triangle mesh (left); regular square mesh (right).

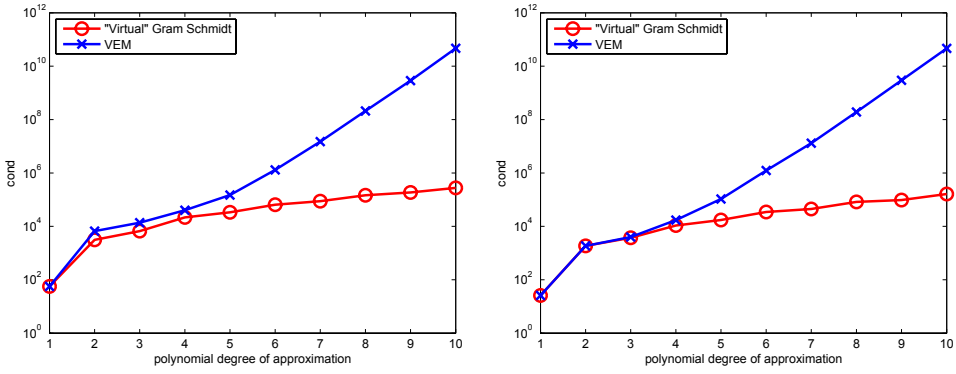


Fig. A.8. Regular hexagonal mesh (left); Voronoi–Lloyd mesh (right).

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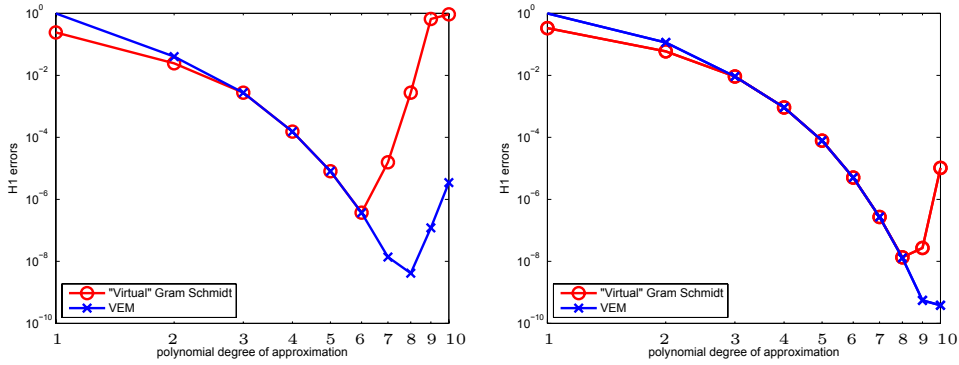


Fig. A.9. Unstructured triangle mesh (left); regular square mesh (right).

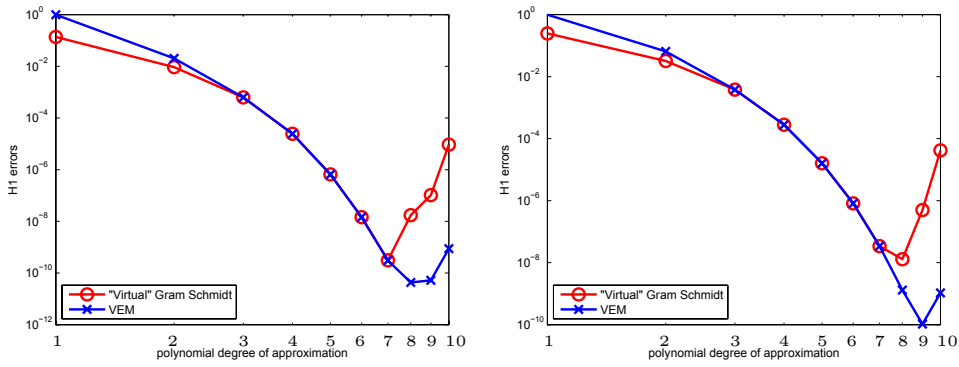


Fig. A.10. Regular hexagonal mesh (left); Voronoi–Lloyd mesh (right).

In Figs. A.9 and A.10, we compare the behaviour of the error $|u - \Pi_p^\nabla u_h|_{1,\Omega}$ using the two bases above on the usual test case $u(x, y) = \sin(\pi x)\sin(\pi y)$. We observe that, although the method described in this subsection improves the condition number of the global stiffness matrix, it is numerically unstable. Therefore, in practice, the proposed Gram–Schmidt method may be preferable to the simple basis choice in (A.2) only for mid-low values of $p$.

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