VANISHING-CONCENTRATION-COMPACTNESS ALTERNATIVE FOR THE TRUDINGER-MOSER INEQUALITY IN $\mathbb{R}^N$

JOÃO MARCOS DO Ó, FEDERICA SANI, AND CRISTINA TARSI

Abstract. Let $N \geq 2$, $a > 0$ and $0 < b \leq N$. Our aim is to clarify the influence of the constraint $S_{a,b} := \{ u \in W^{1,N}(\mathbb{R}^N) \mid \|\nabla u\|_N^a + \|u\|_N^b = 1 \}$ on concentration phenomena of (spherically symmetric and non-increasing) maximizing sequences for the Trudinger-Moser supremum

$$d_N(a,b) := \sup_{u \in S_{a,b}} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx$$

where α_N is the sharp exponent of Moser, i.e. $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface measure of the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$. We obtain a vanishing-concentration-compactness alternative showing that maximizing sequences for $d_N(a,b)$ *cannot* concentrate either when $b \neq N$ or when $b = N$ and $a > 0$ is sufficiently small. From this alternative, we deduce the attainability of $d_N(a,b)$ for special values of the parameters a and b.

Keywords: Trudinger-Moser inequality, vanishing, concentration

Contents

1. Introduction

Let $a, b > 0$ and let us denote by $d_{N,\alpha}(a,b)$ the supremum corresponding to the Trudinger-Moser inequality in the whole space $\mathbb{R}^N$, $N \geq 2$, with exponent $\alpha > 0$ and constraint

$$\|\nabla u\|_N^a + \|u\|_N^b = 1 \quad \forall u \in W^{1,N}(\mathbb{R}^N) \tag{1.1}$$

More precisely,

$$d_{N,\alpha}(a,b) := \sup_{u\in W^{1,N}(\mathbb{R}^N),\ \|\nabla u\|_N^a+\|u\|_N^b=1} \int_{\mathbb{R}^N} \phi_N(\,\alpha|u|^{\frac{N}{N-1}}\,)\,dx \tag{1.2}$$

where $\alpha > 0$ and

$$\phi_N(t) := e^t - \sum_{k=0}^{N-2} \frac{t^k}{k!} \quad \forall t \geq 0$$

When $a = b = N$, we set

$$d_{N,\alpha} := d_{N,\alpha}(N,N)$$

and the above supremum $d_{N,\alpha}$ corresponds to the classical Trudinger-Moser inequality in $\mathbb{R}^N$. It is well known that

$$d_{N,\alpha_N} < +\infty \tag{1.3}$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface measure of the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$. Moreover, the exponent α_N is sharp in the sense that

$$d_{N,\alpha} = +\infty \quad \forall \alpha > \alpha_N$$

In the 2-dimensional case, the study of the attainability of the supremum $d_{2,\alpha}$ is due to B. Ruf [21] and M. Ishiwata [13]. Roughly speaking, from the delicate analysis carried out in [21] and [13], we can deduce that, given a (spherically symmetric and non-increasing) maximizing sequence $\{u_j\}_j \subset W^{1,2}(\mathbb{R}^2)$ for $d_{2,\alpha}$ with $0 < \alpha \leq \alpha_N$, the following alternative occurs: either the weak limit u in $W^{1,2}(\mathbb{R}^2)$ of the maximizing sequence $\{u_j\}_j$ is non-trivial (*compactness*) and it is a maximizer for $d_{2,\alpha}$ or $u = 0$. In the latter case, the *loss of compactness* can be caused by

- either *vanishing phenomena*, i.e.

$$\lim_{j\to+\infty} \int_{\mathbb{R}^2} |\nabla u_j|^2\,dx = 0 \quad \text{and} \quad \lim_{j\to+\infty} \int_{\mathbb{R}^2} |u_j|^2\,dx = 1$$

- or *concentration phenomena*, i.e.

$$\lim_{j\to+\infty} \int_{\mathbb{R}^2} |\nabla u_j|^2\,dx = 1\,, \quad \lim_{j\to+\infty} \int_{\mathbb{R}^2} |u_j|^2\,dx = 0 \tag{1.4}$$

and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^2\setminus B_R} |\nabla u_j|^2\,dx = 0 \quad \text{for any fixed } R > 0 \tag{1.5}$$

The proper understanding of the above alternative was a priori not obvious. However, the most valuable results obtained in [21] and [13] cannot be summarized in this way and are clearly more involved. In the *critical* case $\alpha = \alpha_2 = 4\pi$, as showed in [13], it is possible to rule out *vanishing* behaviors of maximizing sequences for $d_{2,4\pi}$ and the most hard and inspiring part of the result in [21] is to exclude *concentration* phenomena. In particular

Theorem A ([21])**.** *In the 2-dimensional case, the level of normalized concentrating sequences for the Trudinger-Moser functional is exactly $e\pi$. More precisely,*

$$\sup\Big\{ \lim_{j\to+\infty} \int_{\mathbb{R}^2} (e^{4\pi u_j^2} - 1)\,dx \,\Big|\, \{u_j\}_j \text{ is a normalized concentrating sequence} \Big\} = e\pi$$

where a normalized concentrating sequence is a (spherically symmetric and non-increasing) sequence $\{u_j\}_j \in W^{1,2}(\mathbb{R}^2)$ *such that* $\|\nabla u_j\|_2^2 + \|u_j\|_2^2 = 1$ *and satisfying* (1.4) *and* (1.5)*.*

In the *subcritical* case $0 < \alpha < \alpha_2 = 4\pi$, *concentration* cannot occur, due to the fact that one can always gain some L^p-uniform integrability with $p > 1$, and loss of compactness can be caused only by the failure of the compact embedding of $W^{1,2}(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$, i.e. by the fact that the embedding

$$W^{1,2}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$$

is continuous but not compact. Therefore *vanishing phenomena* prevail and, as enlightened in [13], provoke the non-attainability of $d_{2,\alpha}$ when $\alpha > 0$ is sufficiently small.

Theorem B ([13], Theorem 1.2)**.** *In the* 2*-dimensional case, if* $\alpha > 0$ *is sufficiently small then the Trudinger-Moser supremum* $d_{2,\alpha}$ *is not attained.*

Concerning the higher dimensional case $N \geq 3$, the study of the attainability of d_{N,α_N} is due to Y. Li and B. Ruf [18]. Even if, from [18], one can deduce that no loss of compactness of maximizing sequences occurs, a very careful blow-up analysis, as developed in [18], is needed to prove that d_{N,α_N} is attained.

Differently from the 2-dimensional case and due to the method of proof adopted in [18] which is based on blow-up analysis, one cannot deduce from [18] a precise estimate of the level of (spherically symmetric and non-increasing) concentrating sequences for the Trudinger-Moser functional in higher dimensions $N \geq 3$. This problem is heavily non-trivial and still open.

In contrast (at least apparently) with the 2-dimensional case, in the *subcritical* case $0 < \alpha < \alpha_N$ and when $N \geq 3$, the supremum $d_{N,\alpha}$ is always attained and also *vanishing phenomena* do not play any role. Actually, even in the higher dimensional case $N \geq 3$, the attainability of $d_{N,\alpha}(a,b)$ with $(a,b) \neq (N,N)$ heavily depends on the value of the exponent $0 < \alpha < \alpha_N$, as showed by M. Ishiwata and H. Wadade in [14] (see also [15] and Remark 1.2). From [14], we can deduce that the constraint (1.1) has an effect on vanishing phenomena. Then one may wonder

How does the constraint (1.1) *influence concentration phenomena?*

In a very recent paper, N. Lam, G. Lu and L. Zhang [17] proved that, when $d_{N,\alpha_N}(a,b) < +\infty$, the exponent α_N is sharp for the corresponding Trudinger-Moser inequality and it is not affected by the values of a and b

Theorem C ([17], Theorem 1.2)**.** *Let* $N \geq 2$ *and* $a,\ b > 0$*. Then* $d_{N,\alpha_N}(a,b) < +\infty$ *if and only if* $b \leq N$*. Moreover, if* $0 < b \leq N$ *then the exponent* α_N *is sharp in the sense that*

$$d_{N,\alpha}(a,b) = +\infty \quad \forall \alpha > \alpha_N$$

Remark 1.1. If $a > 0$ and $b > N$ then it is not difficult to see that for any $0 < \alpha < \alpha_N$ there exists a constant $C_\alpha > 0$ such that

$$\sup_{u \in W^{1,N}(\mathbb{R}^N),\ \|\nabla u\|_N^a + \|u\|_N^b = 1} \int_{\mathbb{R}^N} \phi_N(\, \alpha |u|^{\frac{N}{N-1}}\,)\, dx \leq C_\alpha \tag{1.6}$$

and, summarizing, we have

$$a > 0 \text{ and } b > N \quad \Rightarrow \quad d_{N,\alpha}(a,b) \begin{cases} \leq C_\alpha & \text{if } 0 < \alpha < \alpha_N \\ = +\infty & \text{if } \alpha \geq \alpha_N \end{cases}$$

Therefore, when $a > 0$ and $b > N$, we have a family of *subcritical* Trudinger-Moser type inequalities for which the exponent α_N is *not* admissible, as in the case of Adachi-Tanaka type inequalities, see [1].

Our aim is to clarify the influence of the constraint (1.1) on concentration phenomena of (spherically symmetric and non-increasing) maximizing sequences for the corresponding Trudinger-Moser inequality. Since when $a > 0$ and $b > N$, the range of the exponent α for the validity of (1.6) is an open interval, i.e. $\alpha \in (0, \alpha_N)$, it is not difficult to exclude concentration behaviors of maximizing sequences.

For this reason, from now on we will just consider the supremum $d_{N,\alpha_N}(a,b)$, with $N \geq 2$, when $a > 0$ and $0 < b \leq N$ and, to simplify notations, we will denote by $d_N(a,b)$ the Trudinger-Moser supremum $d_{N,\alpha}(a,b)$ defined by (1.2) with exponent $\alpha = \alpha_N$, i.e.

$$d_N(a,b) := d_{N,\alpha_N}(a,b) \tag{1.7}$$

Definition 1.1. *Let* $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ *be a spherically symmetric and non-increasing sequence and assume that each* u_j *satisfies the constraint* (1.1)*, i.e.*

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \geq 1$$

(I) We say that $\{u_j\}_j$ *is a* normalized vanishing sequence *if*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|u_j\|_N = 1 \tag{1.8}$$

(II) We say that $\{u_j\}_j$ *is a* normalized concentrating sequence *if*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1\,, \qquad \lim_{j\to+\infty} \|u_j\|_N = 0$$

and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N \setminus B_R} |\nabla u_j|^N \, dx = 0 \quad \text{for any fixed } R > 0$$

Our main result is a vanishing-concentration-compactness alternative for normalized (spherically symmetric and non-increasing) sequences in $W^{1,N}(\mathbb{R}^N)$, that we will state in Section 2 (see Lemma 2.2). In particular, this alternative entails the following precise estimates of the energy level of normalized vanishing and concentrating sequences.

Theorem 1.1. *Let* $N \geq 2$*,* $a > 0$ *and* $0 < b \leq N$*. Then any normalized vanishing sequence* $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ *satisfies*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!} \tag{1.9}$$

If we assume in addition that $b \neq N$, i.e. $a > 0$ and $0 < b < N$, then any normalized concentrating sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ satisfies

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0$$

We call the attention to the fact that, when $a > 0$ and $0 < b < N$, the energy level of normalized concentrating sequences is *zero*, see also Remark 2.4. Therefore, the corresponding problem of the attainability of the Trudinger-Moser supremum $d_N(a,b)$, with this range of the parameters a and b defining the constraint (1.1), becomes much easier: if $a > 0$ and $0 < b < N$ then maximizing sequences for $d_N(a,b)$ *cannot* concentrate. In other words, concentration phenomena may occur only when $b = N$, while when

$$a > 0 \quad \text{and} \quad 0 < b < N\,,$$

if we restrict our attention to maximizing sequences for $d_N(a,b)$ then the vanishing-concentration-compactness alternative expressed by Lemma 2.2 reduces to a vanishing-compactness alternative, see Section 6 and more precisely Lemma 6.2.

Exploiting Lemma 6.2, and in particular the energy level of vanishing sequences (1.9), we will prove the following attainability result

Theorem 1.2. *Let $N \geq 2$, $a > \frac{N}{N-1}$ and $0 < b < N$ be fixed. Then the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7) is attained.*

Note that, the additional condition

$$a > \frac{N}{N-1}$$

is meant to exclude possible vanishing behaviors of maximizing sequences for $d_N(a,b)$, see Section 7.

Remark 1.2. The case $0 < a \leq \frac{N}{N-1}$ is beyond our aims, since it is significant for vanishing phenomena, as one can deduce from the interesting analysis carried out in [14]. We also refer the reader to the new result by M. Ishiwata and H. Wadade [15], where the authors address explicitly the problem of the attainability of $d_{N,\alpha}(\gamma,\gamma)$, with *subcritical* exponent $0 < \alpha < \alpha_N$ and $\gamma > 0$, showing that vanishing phenomena may prevent the *subcritical* supremum $d_{N,\alpha}(\gamma,\gamma)$ to be attained.

2. A Vanishing-Concentration-Compactness alternative

In our analysis, the following improved version of the Adachi-Tanaka inequality [1] will be crucial

Theorem 2.1 ([6] and [17])**.** *Let $N \geq 2$. Then there exists a constant $C_N > 0$ such that for any $\gamma \in (0,1)$ we have*

$$\int_{\mathbb{R}^N} \phi_N(\, \alpha_N\, \gamma\, |u|^{\frac{N}{N-1}}\,)\, dx \leq \frac{C_N}{1-\gamma^{N-1}}\, \|u\|_N^N \quad \forall u \in W^{1,N}(\mathbb{R}^N) \;\; \textit{with} \;\; \|\nabla u\|_N \leq 1 \tag{2.1}$$

We recall that (2.1), in the 2-dimensional case, was deduced in [6, Theorem 1.2] as a direct consequence of the *critical* Trudinger-Moser inequality in $\mathbb{R}^2$, i.e. (1.3) with $N = 2$. N. Lam, G. Lu and L. Zhang in [17, Theorem 1.1] obtained the generalization to the higher dimensional case $N \geq 3$, without assuming a priori the validity of (1.3).

Remark 2.1. In this framework, it is important to mention also a Lions-type result [12, Theorem 1.1] in the whole space $\mathbb{R}^N$. This result tells us that, if a sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$, satisfying the constraint (1.1) with $a = b = N$, i.e.

$$\|\nabla u_j\|_N^N + \|u_j\|_N^N = 1 \quad \forall j \geq 1\,,$$

weakly converges to a *non*-trivial function $u \neq 0$ then an inequality of Trudinger-Moser type holds along the sequence with an exponent larger than α_N. More precisely,

$$\sup_j \int_{\mathbb{R}^N} \phi_N(\,\alpha_N p\,|u_j|^{\frac{N}{N-1}}\,)\,dx < +\infty \qquad \text{for any } \; 0 < p < \left[\,1 - (\|\nabla u\|_N^N + \|u\|_N^N)\,\right]^{-\frac{1}{N-1}} \tag{2.2}$$

The scale invariant inequality (2.1) implies only a weaker version of (2.2) but it is more flexible to treat the case of normalized sequences with respect to the constraint (1.1). Moreover, inequality (2.1) will enable us to describe the effect of the constraint (1.1) on concentration phenomena.

Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. We begin considering a (spherically symmetric and non-increasing) maximizing sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ for the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7), i.e. $u_j \geq 0$ a.e. in $\mathbb{R}^N$ for any $j \geq 1$,

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \geq 1$$

and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx = d_N(a,b)$$

Remark 2.2. By Schwarz symmetrization, it is well know that given a maximizing sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ for $d_N(a,b)$, one may always assume that each u_j is non-negative, spherically symmetric and non-increasing with respect to the radial variable.

We will set $\theta_j := \|u_j\|_N^b \in (0,1)$, so that $\|\nabla u_j\|_N^a = 1 - \theta_j$. Since $\{\theta_j\}_j \in (0,1)$, without loss of generality, we may assume

$$\lim_{j\to+\infty} \theta_j = \overline{\theta} \in [0,1]$$

and, it becomes natural to distinguish three cases according to $\overline{\theta} = 1$, $\overline{\theta} = 0$ and $\overline{\theta} \in (0,1)$. Intuitively, in terms of the maximizing sequence $\{u_j\}_j$ for $d_N(a,b)$, this suggests the following alternative:

(I) (*vanishing*) if $\overline{\theta} = 1$ then $\{u_j\}_j$ is a *vanishing maximizing sequence* for $d_N(a,b)$;

(II) (*concentration*) if $\overline{\theta} = 0$ then $\{u_j\}_j$ is a *concentrating maximizing sequence* for $d_N(a,b)$;

(III) (*compactness*) If $\overline{\theta} \in (0,1)$ then

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = (1-\overline{\theta})^{\frac{1}{a}} \in (0,1) \quad \text{and} \quad \lim_{j\to+\infty} \|u_j\|_N = \overline{\theta}^{\frac{1}{b}} \in (0,1)$$

and $\{u_j\}_j$ weakly converges in $W^{1,N}(\mathbb{R}^N)$ to a maximizer of $d_N(a,b)$. In other words, $d_N(a,b)$ is attained.

In fact, in Section 6, we will show that this intuition can be derived from the following vanishing-concentration-compactness alternative

Lemma 2.2. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. We consider a (spherically symmetric and non-increasing) sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ satisfying the constraint* (1.1)*, i.e.*

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \geq 1$$

and we assume that $u_j \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$.

Then

(I) either $\{u_j\}_j$ is a normalized vanishing sequence *and*

$$\lim_{j\to+\infty} \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \textit{for any fixed } R > 0$$

More precisely,

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}$$

(II) or $\{u_j\}_j$ is such that

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1 \quad \textit{and} \quad \lim_{j\to+\infty} \|u_j\|_N = 0$$

In this case there is a striking difference between the range $0 < b < N$ and $b = N$. If $0 < b < N$, independently of $a > 0$, we have

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0$$

While, if $b = N$

- *either $\{u_j\}_j$ is a* normalized concentrating sequence *and*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N \setminus B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \textit{for any fixed } R > 0$$

- *or*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0$$

(III) Finally, if both (I) and (II) do not occur then

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx + \frac{\alpha_N^{N-1}}{(N-1)!}\left(\overline{\theta}^{\frac{N}{b}} - \|u\|_N^N\right)$$

where, up to subsequences,

$$\overline{\theta}^{\frac{N}{b}} := \lim_{j\to+\infty} \|u_j\|_N^N$$

In particular, if $u_j \to u$ in $L^N(\mathbb{R}^N)$ then

$$\lim_{j\to+\infty}\int_{\mathbb{R}^N}\phi_N(\,\alpha_N|u_j|^{\frac{N}{N-1}}\,)\,dx = \int_{\mathbb{R}^N}\phi_N(\,\alpha_N|u|^{\frac{N}{N-1}}\,)\,dx$$

It is important to recall that, in the pioneering work [19], P.-L. Lions developed a version of his Concentration-Compactness Principle for the limiting case of the Sobolev embedding theorem, i.e. for the Trudinger-Moser case. The result of P.-L. Lions concerns *bounded* domains and has been sharpened by R. Černý, A. Cianchi and S. Hencl [8]. The approach introduced in [8] is different from Lions' technique and yields to deal, not only with functions vanishing on the boundary but, with functions with unrestricted boundary values on a fixed bounded domain. The case of *unbounded* domains has already been considered in [3], [12] (see Remark 2.1) and [7]. L. Battaglia and G. Mancini in [3] focused their attention to the 2-dimensional case and in particular to the planar strip $\mathbb{R}\times(-1,1)\subset\mathbb{R}^2$. R. Černý in [7] obtained a version of the Concetration-Compactness Principle for the Trudinger-Moser functional on the whole space $\mathbb{R}^N$, i.e.

$$u\in W^{1,N}(\mathbb{R}^N)\quad\mapsto\quad\int_{\mathbb{R}^N}\phi_N(\,\alpha_N|u|^{\frac{N}{N-1}}\,)\,dx$$

with respect to the constraint

$$S_M := \{\,u\in W^{1,N}(\mathbb{R}^N)\quad|\quad\|\nabla u\|_N\le 1\quad\text{and}\quad\|u\|_N\le M\,\}\,,\quad M>0\,,$$

which is different from (1.1).

We mention that Lions' Concentration-Compactness Principle [19] inspired Adimurthi and O. Druet [2] to study a valuable improvement of the Trudinger-Moser inequality on bounded domains of $\mathbb{R}^2$. The improved inequality by Adimurthi and O. Druet [2] has been extended to the higher dimensional case by Y. Yang [23, 24]. Related partial results in the whole space $\mathbb{R}^N$ have been approached in [9] and [10]. A new interpretation and a further generalization of [2] has been introduced by C. Tintarev [22], see also Y. Yang [25] for a study of the corresponding problem of attainability.

Remark 2.3. Definition 1.1 of vanishing sequences is apparently different from the notion of *vanishing* introduced by M. Ishiwata [13, Definition 2.1]. Recall that, a (spherically symmetric and non-increasing) sequence $\{u_j\}_j\subset W^{1,N}(\mathbb{R}^N)$ is *vanishing according to Ishiwata* [13] if each u_j satisfies the constraint (1.1) and

$$\lim_{R\to+\infty}\lim_{j\to+\infty}\int_{B_R}\phi_N(\,\alpha_N|u_j|^{\frac{N}{N-1}}\,)\,dx = 0 \tag{2.3}$$

Nevertheless, as a consequence of (I) of Lemma 2.2, condition (1.8) implies (2.3). In other words, any *normalized vanishing sequence* is also a vanishing sequence according to Ishiwata.

Let $\{u_j\}_j\subset W^{1,N}(\mathbb{R}^N)$ be a (spherically symmetric and non-increasing) sequence satisfying the constraint (1.1) and suppose

$$u_j \rightharpoonup u \text{ in } W^{1,N}(\mathbb{R}^N)$$

As mentioned above, if we set $\theta_j := \|u_j\|_N^b \in (0,1)$, so that $\|\nabla u_j\|_N^a = 1 - \theta_j$, then we may assume, without loss of generality,

$$\lim_{j\to+\infty} \theta_j = \overline{\theta} \in [0,1] \tag{2.4}$$

In the next Sections, we will prove Lemma 2.2 through the following steps:

- if $\overline{\theta} = 1$ then (I) occurs (see Section 3);
- if $\overline{\theta} = 0$ then (II) holds (see Section 4). More precisely in this case, either
$$\lim_{j\to+\infty} \int_{\mathbb{R}^N \setminus B_R} |\nabla u_j|^N \, dx = 0 \quad \text{for any fixed } R > 0$$
or
$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \tag{2.5}$$
In particular, if $0 < b < N$ then (2.5) holds also for *normalized concentrating sequences*.
- Finally, in Section 4, we will show that if $\overline{\theta} \in (0,1)$ then we have the convergence result expressed by (III), but we do not know a priori whether or not $u_j \to u$ in $L^N(\mathbb{R}^N)$.

Remark 2.4. Looking at case (I) and case (II) of Lemma 2.2, we can say that the constraint (1.1) has not effects on the energy level of normalized *vanishing* sequences while, in contrast, the energy level corresponding to normalized *concentrating* sequences is heavily influenced by the constraint (1.1). On one hand, in view of (I) of Lemma 2.2, the level of normalized *vanishing* sequences is always

$$\frac{\alpha_N^{N-1}}{(N-1)!}$$

independently of the parameters $a > 0$ and $0 < b \le N$ defining the constraint (1.1). On the other hand, taking into consideration (II) of Lemma 2.2, if $0 < b < N$ then the level of normalized *concentrating* sequences is *zero*, independently of $a > 0$. The same is *not* true when $b = N$, at least in general. This cannot be deduced from (II) of Lemma 2.2 but, as mentioned in the Introduction (see Theorem A), it is well known that in the 2-dimensional case and when $a = b = 2$ then

$$\sup\left\{ \lim_{j\to+\infty} \int_{\mathbb{R}^2} (e^{4\pi u_j^2} - 1)\, dx \;\middle|\; \{u_j\}_j \text{ is a normalized concentrating sequence} \right\} = e\pi$$

It is important to point out that, even if the value of the parameters $a > 0$ and $0 < b \le N$ does not influence the energy level of normalized *vanishing* sequences, we *cannot* deduce that the constraint (1.1) has not effects on *vanishing phenomena*.

3. Alternative (I) – Vanishing

In this Section, we consider the case of *normalized vanishing sequences*, i.e. (spherically symmetric and non-increasing) sequences $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ such that each u_j satisfies the constraint (1.1), namely

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \ge 1\,,$$

and

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|u_j\|_N = 1$$

First, we will show that the energy of any *normalized vanishing sequence* $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ can be localized in the exterior of any fixed ball of radius $R > 0$; more precisely,

$$\lim_{j\to+\infty} \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \text{for any fixed } R > 0$$

To this aim, we will use the classical Trudinger-Moser inequality on balls $B_R \subset \mathbb{R}^N$ of radius $R > 0$ and centered at the origin, i.e.

Theorem 3.1 ([20])**.** *There exists a constant $C_N > 0$ such that for any $R > 0$*

$$\int_{B_R} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \le C_N\, R^N\, \|\nabla u\|_N^N \quad \forall u \in W_0^{1,N}(B_R) \setminus \{0\} \text{ with } \|\nabla u\|_N \le 1 \tag{3.1}$$

We point out that the local estimate expressed by (3.1) is not the original version of Moser's inequality [20], but it can be deduced directly from the famous inequality in [20] with the aid of the rescaled function $\tilde{u} := u/\|\nabla u\|_N$, see for instance [26, Lemma 2.1].

Lemma 3.2. *Let $N \ge 2$, $a > 0$ and $0 < b \le N$ be fixed. If $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) sequence satisfying, for some $\overline{\theta} > 0$,*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 0 \quad \textit{and} \quad \sup_j \|u_j\|_N \le \overline{\theta}$$

then

$$\lim_{j\to+\infty} \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \textit{for any fixed } R > 0$$

Proof. Let $R > 0$ be arbitrarily fixed. In order to apply Theorem 3.1, the idea is to reconstruct zero-Dirichlet boundary conditions on the boundary of B_R letting

$$w_j := u_j - u_j(R) \quad \text{on } B_R$$

By construction $w_j \in W_0^{1,N}(B_R)$ and

$$\|\nabla w_j\|_{L^N(B_R)} = \|\nabla u_j\|_{L^N(B_R)} \to 0 \quad \text{as} \quad j \to +\infty$$

Note that, for any fixed $\alpha > 0$, if we set

$$\tilde{w}_j := \left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N}} w_j \quad \text{on } B_R$$

then there exists $\overline{j} \ge 1$ such that

$$\|\nabla \tilde{w}_j\|_N^N = \left(\frac{\alpha}{\alpha_N} \right)^{N-1} \|\nabla w_j\|_N^N \le 1 \quad \forall j \ge \overline{j}$$

Therefore in view of Lemma 3.1, for any fixed $\alpha > 0$, there exists $\overline{j} \ge 1$ such that

$$\int_{B_R} \phi_N(\, \alpha w_j^{\frac{N}{N-1}}\,)\, dx = \int_{B_R} \phi_N(\, \alpha_N \tilde{w}_j^{\frac{N}{N-1}}\,)\, dx \le C_N \left(\frac{\alpha}{\alpha_N} \right)^{N-1} R^N\, \|\nabla w_j\|_N^N \quad \forall j \ge \overline{j} \tag{3.2}$$

Next, applying the one-dimensional inequality

$$(1+x)^p \leq (1+\varepsilon)x^p + \left(1 - \frac{1}{(1+\varepsilon)^{1/(p-1)}} \right)^{1-p} \quad x \geq 0,\ p > 1,\ \varepsilon > 0 \tag{3.3}$$

we get

$$\alpha_N |u_j|^{\frac{N}{N-1}} = \alpha_N \big(w_j + u_j(R) \big)^{\frac{N}{N-1}} \leq C_1 u_j^{\frac{N}{N-1}}(R) + C_2 w_j^{\frac{N}{N-1}} \quad \text{on } B_R$$

for some constants $C_1,\ C_2 > 0$ depending on N. We do not need to explicitly write the value of the constant $C_2 > 0$ and the reason for that is essentially (3.2).

Summarizing,

$$\int_{B_R} \phi_N\big(\alpha_N |u_j|^{\frac{N}{N-1}} \big)\, dx \leq \int_{B_R} \phi_N\big(C_1 u_j^{\frac{N}{N-1}}(R) + C_2 w_j^{\frac{N}{N-1}} \big)\, dx$$

and, if we show that

$$\begin{aligned} \lim_{j\to+\infty} \int_{B_R} & \phi_N\big(C_1 u_j^{\frac{N}{N-1}}(R) + C_2 w_j^{\frac{N}{N-1}} \big)\, dx \\ & \leq \lim_{j\to+\infty} \exp\big(C_1 u_j^{\frac{N}{N-1}}(R) \big) \int_{B_R} \phi_N\big(C_2 w_j^{\frac{N}{N-1}} \big)\, dx \end{aligned} \tag{3.4}$$

then the proof is complete.

On one hand, by means of the following *Radial Lemma*, which holds for any spherically symmetric and non-increasing function $\varphi \in W^{1,N}(\mathbb{R}^N)$

$$\varphi^N(|x|) \leq \frac{N}{\omega_{N-1}|x|^{N-1}} \, \|\varphi\|_N^{N-1} \|\nabla\varphi\|_N \quad \text{whenever } |x| \neq 0 \tag{3.5}$$

we may estimate

$$\exp\big(C_1 u_j^{\frac{N}{N-1}}(R) \big) \leq \exp\left(\tilde{C}_1 \, \overline{\theta} \, \frac{\|\nabla u_j\|_N^{\frac{1}{N-1}}}{R} \right)$$

On the other hand, by means of (3.2) with $\alpha = C_2 > 0$, we get

$$\int_{B_R} \phi_N\big(C_2 w_j^{\frac{N}{N-1}} \big)\, dx \leq \tilde{C}_N \, R^N \, \|\nabla w_j\|_N^N \quad \forall j \geq \overline{j}$$

Hence,

$$\begin{aligned} \lim_{j\to+\infty} \int_{B_R} \phi_N\big(\alpha_N |u_j|^{\frac{N}{N-1}} \big)\, dx & \leq \lim_{j\to+\infty} \exp\big(C_1 u_j^{\frac{N}{N-1}}(R) \big) \int_{B_R} \phi_N\big(C_2 w_j^{\frac{N}{N-1}} \big)\, dx \\ & \leq \lim_{j\to+\infty} \tilde{C}_N \, R^N \, \|\nabla w_j\|_N^N \, \exp\left(\tilde{C}_1 \, \overline{\theta} \, \frac{\|\nabla u_j\|_N^{\frac{1}{N-1}}}{R} \right) = 0 \end{aligned}$$

To complete the proof, it remains to show that (3.4) holds. To this aim, we begin with an elementary one-dimensional estimate. For any $s,\ t \geq 0$, we have

$$\phi_N(s+t) = e^{s+t} \pm e^s \sum_{k=0}^{N-2} \frac{t^k}{k!} - \sum_{k=0}^{N-2} \frac{(s+t)^k}{k!} \leq e^s \phi_N(t) + e^s \sum_{k=1}^{N-2} \frac{t^k}{k!} + (e^s - 1)$$

In particular, when $N = 2$,

$$e^{s+t} - 1 = e^s(e^t - 1) + (e^s - 1)$$

If we set

$$A_j(2) = 0 \quad \text{and } A_j(N) := \exp(\, C_1 u_j^{\frac{N}{N-1}}(R)\,) \sum_{k=1}^{N-2} \frac{C_2^k}{k!} \int_{B_R} w_j^{\frac{N}{N-1}\,k}\, dx \quad \text{when } N \geq 3$$

and

$$B_j(N) := \Big[\, \exp(\, C_1 u_j^{\frac{N}{N-1}}(R)\,) - 1 \,\Big] |B_R|$$

then

$$\int_{B_R} \phi_N(\, C_1 u_j^{\frac{N}{N-1}}(R) + C_2 w_j^{\frac{N}{N-1}}\,)\, dx \leq \exp(\, C_1 u_j^{\frac{N}{N-1}}(R)\,) \int_{B_R} \phi_N(\, C_2 w_j^{\frac{N}{N-1}}\,)\, dx + A_j(N) + B_j(N)$$

Therefore, it suffices to prove that

$$\lim_{j\to+\infty} A_j(N) = 0 \tag{3.6}$$

and

$$\lim_{j\to+\infty} B_j(N) = 0 \tag{3.7}$$

From the *Radial Lemma* (3.5), we deduce (3.7) and

$$A_j(N) \leq \exp\!\left(\tilde{C}_1\, \overline{\theta}\, \frac{\|\nabla u_j\|_N^{\frac{1}{N-1}}}{R} \right) \sum_{k=1}^{N-2} \frac{C_2^k}{k!} \int_{B_R} w_j^{\frac{N}{N-1}\,k}\, dx$$

Moreover, $w_j \rightharpoonup 0$ in $W_0^{1,N}(B_R)$ and the embedding

$$W_0^{1,N}(B_R) \hookrightarrow L^{\frac{N}{N-1}\,k}(B_R)$$

is compact for any $1 \leq k \leq N-2$ with $N \geq 3$. Hence, also (3.6) holds and the proof is complete. □

We complete this Section with a precise estimate of the level of *vanishing sequences.*

Lemma 3.3. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. If $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) sequence satisfying, for some $\overline{\theta} > 0$,*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|u_j\|_N = \overline{\theta}$$

then

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}\, \overline{\theta}^N$$

Proof. Since

$$\phi_N(t) \geq \frac{t^{N-1}}{(N-1)!} \quad \forall t \geq 0$$

it is easy to see that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx \geq \lim_{j\to+\infty} \frac{\alpha_N^{N-1}}{(N-1)!} \|u_j\|_N^N = \frac{\alpha_N^{N-1}}{(N-1)!} \,\overline{\theta}^N$$

Therefore, the proof of Lemma 3.3 is complete if we show that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx \leq \frac{\alpha_N^{N-1}}{(N-1)!} \,\overline{\theta}^N$$

To obtain the preceding estimate from above, let us fix $R > 0$ and let us rewrite

$$\int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N \setminus B_R} + \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx$$

To this aim, it is clear from Lemma 3.2 that, it suffices to show that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N \setminus B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx \leq \frac{\alpha_N^{N-1}}{(N-1)!} \,\overline{\theta}^N \tag{3.8}$$

Using the elementary inequality

$$\phi_N(t) \leq \frac{t^{N-1}}{(N-1)!}\, e^t \quad \forall t \geq 0 \tag{3.9}$$

and the *Radial Lemma* (3.5), we get

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx &\leq \frac{\alpha_N^{N-1}}{(N-1)!} \int_{\mathbb{R}^N \setminus B_R} |u_j|^N\, e^{\alpha_N |u_j|^{\frac{N}{N-1}}}\, dx \\ &\leq \frac{\alpha_N^{N-1}}{(N-1)!} \,\overline{\theta}^N \exp\Big(\frac{1}{R}\, N^{\frac{N}{N-1}} \|\nabla u_j\|_N^{\frac{1}{N-1}} \Big)\end{aligned}$$

which gives (3.8). □

Note that, the case of *normalized vanishing sequence* is included in Lemma 3.2 and Lemma 3.3 with $\overline{\theta} = 1$ and hence

Corollary 3.4. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. If $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) normalized vanishing sequence then*

$$\lim_{j\to+\infty} \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \textit{for any fixed } R > 0$$

Moreover,

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}$$

4. Alternative (II) – Concentration

Let $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ be a (spherically symmetric and non-increasing) sequence satisfying the constraint (1.1), i.e.

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \geq 1\,,$$

and

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1 \quad \text{ and } \quad \lim_{j\to+\infty} \|u_j\|_N = 0$$

which is the case of normalized *concentrating* sequences. In fact, we have

Lemma 4.1. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. If $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) sequence satisfying the constraint* (1.1) *and*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1 \quad \textit{ and } \quad \lim_{j\to+\infty} \|u_j\|_N = 0$$

then

- *either $\{u_j\}_j$ is a normalized concentrating sequence and*
$$\lim_{j\to+\infty} \int_{\mathbb{R}^N\setminus B_R} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx = 0 \quad \textit{ for any fixed } R > 0 \tag{4.1}$$
- *or*
$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx = 0 \tag{4.2}$$

Proof. First, recalling the elementary inequality (3.9) and the *Radial Lemma* (3.5), we may estimate for any fixed $R > 0$

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx &\leq \frac{\alpha_N^{N-1}}{(N-1)!}\int_{\mathbb{R}^N\setminus B_R} |u_j| e^{\alpha_N |u_j|^{\frac{N}{N-1}}}\,dx \\ &\leq \frac{\alpha_N^{N-1}}{(N-1)!}\, e^{C(N)/R}\, \|u_j\|_N^N \to 0 \quad \text{ as } j \to +\infty\end{aligned} \tag{4.3}$$

and this gives (4.1).

Next, we consider the case when the sequence $\{u_j\}_j$ is *not* a normalized concentrating sequence. More precisely, we assume the existence of $\overline{R} > 0$, $\delta \in (0,1)$ and $\overline{j} \geq 1$ such that

$$\int_{\mathbb{R}^N\setminus B_{\overline{R}}} |\nabla u_j|^N\,dx \geq \delta$$

Under this assumption, if we show that (4.2) holds then the proof is complete. Even if the arguments are standard, we give a sketch for the convenience of the reader.

Note that, from (4.3), we deduce

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx = \lim_{j\to+\infty} \int_{B_R} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx$$

for any fixed $R > 0$.

To obtain a uniform estimate of the integral on balls of fixed radius $0 < R \le \overline{R}$, we argue as in the proof of Lemma 3.2 and we set

$$w_j := u_j - u_j(R) \in W_0^{1,N}(B_R) \quad \forall j \ge 1$$

Applying the one-dimensional inequality (3.3), with $p = \frac{N}{N-1} >$ and $\varepsilon = \frac{\delta}{2} > 0$, and the *Radial Lemma* (3.5), we may estimate

$$\begin{aligned}\int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx &\le \int_{B_R} \exp(\, \alpha_N u_j^{\frac{N}{N-1}}\,)\, dx \\ &\le \exp\Big(\, C(\delta) u_j^{\frac{N}{N-1}}(R)\,\Big) \int_{B_R} \exp\bigg(\, \alpha_N \Big(1 + \frac{\delta}{2}\Big) w_j^{\frac{N}{N-1}}\,\bigg) dx \\ &\le \exp\Big(\, C(\delta, N)\, \frac{1}{R}\, \|u_j\|_N\,\Big) \int_{B_R} \exp\bigg(\, \alpha_N \Big(1 + \frac{\delta}{2}\Big) w_j^{\frac{N}{N-1}}\,\bigg) dx\end{aligned}$$

If $0 < R \le \overline{R}$ then

$$\begin{aligned}\|\nabla w_j\|_{L^N(B_R)}^N = \|\nabla w_j\|_{L^N(B_R)}^N \le \|\nabla u_j\|_{L^N(B_{\overline{R}})}^N &= \big(\, 1 - \|u_j\|_N^b\,\big)^{\frac{N}{a}} - \|\nabla u_j\|_{L^N(\mathbb{R}^N \setminus B_{\overline{R}})}^N \\ &\le \big(\, 1 - \|u_j\|_N^b\,\big)^{\frac{N}{a}} - \delta\end{aligned}$$

where we also used the constraint (1.1), and

$$\lim_{j \to +\infty} \|\nabla w_j\|_{L^N(B_R)}^N \le 1 - \delta$$

Therefore, there exists $\overline{j} \ge 1$ such that

$$\|\nabla w_j\|_{L^N(B_R)}^N \le 1 - \frac{\delta}{2} \quad \forall j \ge \overline{j}$$

and, from the classical Trudinger-Moser inequality on bounded domains, we deduce

$$\sup_{j \ge \overline{j}} \int_{B_R} \exp\bigg(\, \alpha_N \Big(1 + \frac{\delta}{2}\Big) w_j^{\frac{N}{N-1}}\,\bigg) dx \le \sup_{v \in W_0^{1,N}(B_R), \|\nabla v\|_N = 1} \int_{B_R} \exp(\, \alpha_N |v|^{\frac{N}{N-1}}\,)\, dx \le C_N R^N$$

In conclusion, for any fixed $0 < R \le \overline{R}$,

$$\lim_{j \to +\infty} \int_{B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx \le \lim_{j \to +\infty} C_N R^N \, \exp\Big(\, C(\delta, N)\, \frac{1}{R}\, \|u_j\|_N\,\Big) = C_N R^N$$

and

$$\lim_{j \to +\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx \le C_N R^N \qquad \text{for any fixed } \ 0 < R \le \overline{R}$$

from which we deduce (4.2), letting $R \to 0$. □

We will see that the alternative expressed by Lemma 4.1 is meaningful only when $b = N$. In fact, we are going to show that if $0 < b < N$ then the level of normalized concentrating sequences is zero; more precisely, if $0 < b < N$ then (4.2) holds also for normalized concentrating sequences.

Let $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ be a (spherically symmetric and non-increasing) sequence satisfying the assumptions of Lemma 4.1. If we set $\theta_j := \|u_j\|_N^b \in (0,1)$ then, by assumptions,

$$\lim_{j\to+\infty} \theta_j = 0$$

and $\|\nabla u_j\|_N^a = 1-\theta_j$. In order to apply the improved version of the Adachi-Tanaka inequality (2.1), it turns out to be convenient to introduce the normalized sequence with respect to the Dirichlet norm,

$$v_j := \frac{u_j}{\|\nabla u_j\|_N}$$

so that

$$\int_{\mathbb{R}^N} \phi_N\Big(\alpha_N \, (1-\theta_j)^{\frac{N}{N-1}\,\frac{1}{a}} \, |v_j|^{\frac{N}{N-1}} \Big)\, dx \le \frac{C_N}{1-(1-\theta_j)^{\frac{N}{a}}} \, \|v_j\|_N^N = \frac{C_N}{1-(1-\theta_j)^{\frac{N}{a}}} \, \frac{\theta_j^{\frac{N}{b}}}{(1-\theta_j)^{\frac{N}{a}}}$$

Note that

$$\lim_{j\to+\infty} \frac{C_N}{1-(1-\theta_j)^{\frac{N}{a}}} \, \frac{\theta_j^{\frac{N}{b}}}{(1-\theta_j)^{\frac{N}{a}}} = \lim_{j\to+\infty} \frac{aC_N}{N} \, \theta_j^{\frac{N}{b}-1}$$

Hence,

$$\begin{aligned} \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\alpha_N |u_j|^{\frac{N}{N-1}})\, dx &= \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N\Big(\alpha_N \, (1-\theta_j)^{\frac{N}{N-1}\,\frac{1}{a}} \, |v_j|^{\frac{N}{N-1}} \Big)\, dx \\ &\le \lim_{j\to+\infty} \frac{aC_N}{N} \, \theta_j^{\frac{N}{b}-1} \end{aligned} \tag{4.4}$$

In the case $0 < b < N$, the above estimate yields

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\alpha_N |u_j|^{\frac{N}{N-1}})\, dx = 0$$

Therefore, if $b \neq N$ we have

Lemma 4.2. *Let $N \ge 2$, $a > 0$ and $0 < b < N$ be fixed. If $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) sequence satisfying the constraint* (1.1) *and*

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1 \qquad \textit{and} \qquad \lim_{j\to+\infty} \|u_j\|_N = 0$$

then

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\alpha_N |u_j|^{\frac{N}{N-1}})\, dx = 0$$

Remark 4.1. If $b = N$ then (4.4) is not useful to obtain a precise estimate of the level of normalized concentrating sequences. However, from (4.4), we can deduce that for any fixed $\delta > 0$ there exists $\overline{a} = \overline{a}(\delta) > 0$ such that if $b = N$ and $0 < a \le \overline{a}$ then

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\alpha_N |u_j|^{\frac{N}{N-1}})\, dx < \delta$$

5. Alternative (III) – Compactness

We will use the following convergence result, which holds for sequences which are neither *vanishing* nor *concentrating*, i.e. in particular $\overline{\theta} \neq 1$ and $\overline{\theta} \neq 0$, see (2.4).

Lemma 5.1. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. We consider a (spherically symmetric and non-increasing) sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ satisfying the constraint* (1.1) *with $\theta_j := \|u_j\|_N^b$, so that $\|\nabla u_j\|_N^a = 1-\theta_j$. Assume, up to subsequences, $u_j \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$ and*

$$\lim_{j\to+\infty} \theta_j = \overline{\theta} \in (0,1)$$

Then

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,)\,dx = \int_{\mathbb{R}^N} \phi_N(\,\alpha_N |u|^{\frac{N}{N-1}}\,)\,dx + \frac{\alpha_N^{N-1}}{(N-1)!}\left(\overline{\theta}^{\frac{N}{b}} - \|u\|_N^N\right)$$

Proof. It is enough to show that

$$\begin{aligned} &\lim_{j\to+\infty} \int_{\mathbb{R}^N} \left[\phi_N(\,\alpha_N |u_j|^{\frac{N}{N-1}}\,) - \frac{\alpha_N^{N-1}}{(N-1)!}\,|u_j|^N\right] dx \\ &= \int_{\mathbb{R}^N} \left[\phi_N(\,\alpha_N |u|^{\frac{N}{N-1}}\,) - \frac{\alpha_N^{N-1}}{(N-1)!}\,|u|^N\right] dx \end{aligned} \tag{5.1}$$

In view of the improved version of the Adachi-Tanaka inequality (2.1), we will obtain (5.1) simply by means of Strauss' Lemma (see [4, Theorem A.I])

Since $\overline{\theta} \in (0,1)$, we have also that $(1-\overline{\theta})^{\frac{N}{N-1}\frac{1}{a}} \in (0,1)$ and there exists $\varepsilon > 0$ and $\overline{j} \geq 1$ such that

$$\alpha_N (1-\overline{\theta})^{\frac{N}{N-1}\frac{1}{a}} \leq \alpha_N (1-\varepsilon) \quad \forall j \geq \overline{j}$$

Let

$$P_N(t) := \phi_N\Big(\,|t|^{\frac{N}{N-1}}\,\Big) - \frac{t^N}{(N-1)!} \quad \text{ and } \quad Q_N(t) := \phi_N\Big(\,(1+\varepsilon)|t|^{\frac{N}{N-1}}\,\Big)$$

then

$$\lim_{|t|\to+\infty} \frac{P_N(t)}{Q_N(t)} = 0 \quad \text{ and } \quad \lim_{t\to 0} \frac{P_N(t)}{Q_N(t)} = 0$$

Next, we introduce the normalized sequence with respect to the Dirichlet norm

$$v_j := \frac{u_j}{\|\nabla u_j\|_N} = \frac{u_j}{(1-\theta_j)^{\frac{1}{a}}} \quad \forall j \geq 1$$

By construction

$$\int_{\mathbb{R}^N} Q_N\Big(\,\alpha_N^{\frac{N-1}{N}}\,u_j\,\Big) = \int_{\mathbb{R}^N} Q_N\Big(\,\alpha_N^{\frac{N-1}{N}}\,(1-\theta_j)^{\frac{1}{a}}\,v_j\,\Big)\,dx$$

Applying the improved version of the Adachi-Tanaka inequality (2.1) to the Dirichlet-normalized sequence $\{v_j\}_j$, we get for any $j \geq \overline{j}$

$$\int_{\mathbb{R}^N} Q_N\Big(\alpha_N^{\frac{N-1}{N}} (1-\theta_j)^{\frac{1}{a}} v_j \Big)\, dx \leq \int_{\mathbb{R}^N} \phi_N\Big(\alpha_N (1-\varepsilon^2) v_j^{\frac{N}{N-1}} \Big)\, dx$$

$$\leq \frac{C_N}{1-(1-\varepsilon^2)^{N-1}} \|v_j\|_N^N = \frac{C_N}{1-(1-\varepsilon^2)^{N-1}} \frac{\theta_j^{\frac{N}{b}}}{(1-\theta_j)^{\frac{N}{a}}}$$

Hence,

$$\sup_j \int_{\mathbb{R}^N} Q_N\Big(\alpha_N^{\frac{N-1}{N}} u_j \Big)\, dx \leq \frac{C_N}{1-(1-\varepsilon^2)^{N-1}} \sup_j \frac{\theta_j^{\frac{N}{b}}}{(1-\theta_j)^{\frac{N}{a}}} < +\infty$$

Recalling that the sequence $\{u_j\}_j$ is spherically symmetric, we can apply Strauss' Lemma (see [4, Theorem A.I]) obtaining

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} P_N\Big(\alpha_N^{\frac{N-1}{N}} u_j \Big)\, dx = \int_{\mathbb{R}^N} P_N\Big(\alpha_N^{\frac{N-1}{N}} u \Big)\, dx$$

that is (5.1). □

To complete the proof of Lemma 2.2, in view of the analysis carried out in Section 3 and Section 4, we just need to consider the case for which the assumptions of the above convergence result are fulfilled. More precisely, we consider a (spherically symmetric and non-increasing) sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ satisfying the constraint (1.1) and we assume, up to subsequences, that $u_j \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$ and

$$\lim_{j\to+\infty} \|u_j\|_N^N = \overline{\theta}^{\frac{N}{b}} \in (0,1)$$

In this case, we can apply Lemma 5.1, to conclude that alternative (III) holds. Moreover, if $u_j \to u$ in $L^N(\mathbb{R}^N)$ then

$$\lim_{j\to+\infty} \|u_j\|_N^N = \|u\|_N^N$$

and, from Lemma 5.1, we deduce

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx$$

6. The case of maximizing sequences for the Trudinger-Moser inequality

Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. We begin this Section deducing a useful lower bound for the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7).

Recalling that

$$\phi_N(t) \geq \frac{t^{N-1}}{(N-1)!} + \frac{t^N}{N!} \quad \forall t \geq 0\,,$$

for any $u \in W^{1,N}(\mathbb{R}^N)$ satisfying the constraint (1.1), i.e.

$$\|\nabla u\|_N^a + \|u\|_N^b = 1$$

we may estimate

$$d_N(a,b) \geq \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \geq \frac{\alpha_N^{N-1}}{(N-1)!} \|u\|_N^N + \frac{\alpha_N^N}{N!} \|u\|_{N^2/(N-1)}^{N^2/(N-1)}$$
$$= \frac{\alpha_N^{N-1}}{(N-1)!} \left(\|u\|_N^N + \frac{\alpha_N}{N} \|u\|_{N^2/(N-1)}^{N^2/(N-1)} \right)$$

If we consider the supremum

$$D_N(a,b) := \sup_{u \in W^{1,N}(\mathbb{R}^N),\, \|\nabla u\|_N^a + \|u\|_N^b = 1} \left(\|u\|_N^N + \frac{\alpha_N}{N} \|u\|_{N^2/(N-1)}^{N^2/(N-1)} \right)$$

then it is clear that

$$d_N(a,b) \geq \frac{\alpha_N^{N-1}}{(N-1)!} D_N(a,b) \tag{6.1}$$

Remark 6.1. When $a = b$, we set

$$D_N(\gamma) := D_N(\gamma,\gamma) \quad \text{ with } \; 0 < \gamma \leq N$$

This is a particular case of the more general maximizing problem considered by M. Ishiwata and H. Wadade in [14]. As pointed out in [14], the attainability of the supremum $d_N(\gamma,\gamma)$ associated with the Trudiger-Moser inequality is closely related to the behavior of $D_N(\gamma)$. In fact, we can observe that the constant appearing on the right hand side of (6.1)

$$\frac{\alpha_N^{N-1}}{(N-1)!}$$

corresponds to the level of normalized *vanishing* sequences (see (I) of Lemma 2.2). Intuitively, when we look at maximizing sequences for $d_N(\gamma,\gamma)$ then the behavior of $D_N(\gamma)$ could be crucial to exclude possible vanishing phenomena. More precisely, if $0 < \gamma \leq N$ is such that

$$D_N(\gamma) > 1$$

then maximing sequences for $d_N(\gamma,\gamma)$ *cannot* vanish, i.e. *cannot* be normalized *vanishing* sequences. The careful study developed in [14] shows that both the behavior of $D_N(\gamma)$ and its attainability are intimately related to the value of γ in the range $(0, N]$.

We mention that the attainability of $D_N(\gamma)$ is not only interesting in the limiting case of the Sobolev embedding theorem but also in the classical Sobolev case. We refer the reader to [16], where the authors approach the study of the existence of maximizers for

$$\sup_{u \in W^{1,p}(\mathbb{R}^N),\, \|\nabla u\|_p^\gamma + \|u\|_p^\gamma = 1} \left(\|u\|_p^p + \alpha\, \|u\|_q^q \right)$$

with $N \geq 2$, $1 < p < N$, $p < q < \frac{Np}{N-p}$ and $\alpha,\, \gamma > 0$.

Following the arguments introduced by M. Ishiwata and H. Wadade [14], it is not difficult to show that

Lemma 6.1. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. Then*

$$D_N(a,b) \geq 1 \tag{6.2}$$

and hence, the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7) satisfies

$$d_N(a,b) \geq \frac{\alpha_N^{N-1}}{(N-1)!} \tag{6.3}$$

Proof. The proof of (6.2) can be deduced arguing exactly as in [14]; for the convenience of the reader, we briefly sketch it.

As showed in [14, Section 2.2], given any $u \in W^{1,N}(\mathbb{R}^N)$ satisfying the constraint (1.1), i.e.

$$\|\nabla u\|_N^a + \|u\|_N^b = 1$$

the family of comparison functions $w_t \in W^{1,N}(\mathbb{R}^N)$ depending on the parameter $t \in (0,1)$ and defined by

$$w_t(x) := \frac{(1-t)^{\frac{1}{a}}}{\|\nabla u\|_N} u(\lambda_t x) \quad \lambda_t := \frac{(1-t)^{\frac{1}{a}}}{t^{\frac{1}{b}}} \frac{\|u\|_N}{\|\nabla u\|_N} \tag{6.4}$$

still satisfies the constraint (1.1). In fact,

$$\|\nabla w_t\|_N = (1-t)^{\frac{1}{a}}$$

and

$$\|w_t\|_N = \frac{(1-t)^{\frac{1}{a}}}{\|\nabla u\|_N} \frac{1}{\lambda_t} \|u\|_N = t^{\frac{1}{b}}$$

Therefore, we may estimate

$$D_N(a,b) \geq \|w_t\|_N^N + \frac{\alpha_N}{N} \|w_t\|_{N^2/(N-1)}^{N^2/(N-1)} \geq \|w_t\|_N^N = t^{\frac{N}{b}} \quad \forall t \in (0,1) \tag{6.5}$$

and (6.2) follows. □

Remark 6.2. We mention that (6.3) can also be directly deduced from [17, Theorem 1.2]. In fact, denoting by

$$AT_N(\gamma) := \sup_{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}} \frac{1}{\|u\|_N^N} \int_{\mathbb{R}^N} \phi_N(\alpha_N \gamma |u|^{\frac{N}{N-1}})\, dx \quad \gamma \in (0,1)\,,$$

N. Lam, G. Lu and L. Zhang in [17] obtained the following more precise version of (2.1)

$$d_N(a,b) = \sup_{\gamma \in (0,1)} \frac{\left(1 - \gamma^{\frac{N-1}{N}a}\right)^{\frac{N}{b}}}{\gamma^{N-1}} AT_N(\gamma) \quad a > 0,\ 0 < b \leq N \tag{6.6}$$

Now, a simple scaling argument shows that

$$AT_N(\gamma) = \sup_{u \in W^{1,N}(\mathbb{R}^N),\ \|\nabla u\|_N = \|u\|_N = 1} \int_{\mathbb{R}^N} \phi_N(\alpha_N \gamma |u|^{\frac{N}{N-1}})\, dx \tag{6.7}$$

Note that the supremum on the right hand side of the above identity corresponds to the inequality first studied in [11]. Combining (6.6) with (6.7), it is easy to see that (6.3) follows.

Next, we consider a (spherically symmetric and non-increasing) maximizing sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ for the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7), i.e. $u_j \geq 0$ a.e. in $\mathbb{R}^N$ for any $j \geq 1$,

$$\|\nabla u_j\|_N^a + \|u_j\|_N^b = 1 \quad \forall j \geq 1$$

and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = d_N(a,b)$$

Dealing with a *maximizing* sequence for $d_N(a,b)$, the alternative expressed by Lemma 2.2 becomes simpler.

Case (I) – If $\{u_j\}_j$ is a *vanishing* maximizing sequence then

$$d_N(a,b) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}$$

Case (II) – If $0 < b < N$ then the following conditions

$$\lim_{j\to+\infty} \|\nabla u_j\|_N = 1 \quad \text{and} \quad \lim_{j\to+\infty} \|u_j\|_N = 0 \tag{6.8}$$

cannot hold, since otherwise

$$d_N(a,b) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0$$

which would contradict Lemma 6.1. While when $b = N$, if (6.8) holds then $\{u_j\}_j$ must be a *concentrating* maximizing sequence and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N \setminus B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \quad \text{for any fixed } R > 0$$

Moreover, in the latter case, combining Remark 4.1 with the estimate from below of $d_N(a,N)$ (i.e. Lemma 6.1), we deduce the existence of $\overline{a} > 0$ such that if $0 < a < \overline{a}$ and (6.8) holds then

$$\frac{\alpha_N^{N-1}}{(N-1)!} \leq d_N(a,N) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx < \frac{\alpha_N^{N-1}}{(N-1)!}$$

which is a contradiction. Consequently, if $b = N$ and $a > 0$ is sufficiently small then concentration *cannot* occur.

Case (III) – Finally, let $\theta_j := \|u_j\|_N^b \in (0,1)$ and let us consider a subsequence still denoted by $\{\theta_j\}_j$ such that

$$\lim_{j\to+\infty} \theta_j = \overline{\theta}$$

Since we already discussed the cases $\overline{\theta} = 1$ and $\overline{\theta} = 0$, without loss of generality, we may assume $\overline{\theta} \in (0,1)$. We may also assume, up to subsequences,

$$u_j \rightharpoonup u \quad \text{in } W^{1,N}(\mathbb{R}^N)$$

From Lemma 2.2, we deduce that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx + \frac{\alpha_N^{N-1}}{(N-1)!} \left(\overline{\theta}^{\,\frac{N}{b}} - \|u\|_N^N \right) \tag{6.9}$$

In particular, this implies that $u \neq 0$. In fact, if not then

$$d_N(a,b) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}\, \overline{\theta}^{\frac{N}{b}} < \frac{\alpha_N^{N-1}}{(N-1)!}$$

contradicting Lemma 6.1. Therefore, we can define

$$\tau := \frac{\overline{\theta}^{\frac{1}{b}}}{\|u\|_N} = \frac{1}{\|u\|_N} \lim_{j\to+\infty} \|u_j\|_N \geq 1$$

Note that, in view of Brezis-Lieb Lemma [5], if we show that $\tau = 1$ then we can conclude that $u_j \to u$ in $L^N(\mathbb{R}^N)$ and

$$d_N(a,b) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx$$

from which we deduce that u is a maximizer for $d_N(a,b)$.

Let

$$u_\tau(x) := u\Big(\frac{x}{\tau}\Big) \quad x \in \mathbb{R}^N$$

so that $\|\nabla u_\tau\|_N = \|\nabla u\|_N$, $\|u_\tau\|_N = \tau \|u\|_N = \overline{\theta}^{\frac{1}{b}}$ and

$$\|\nabla u_\tau\|_N^a + \|u_\tau\|_N^b = \|\nabla u\|_N^a + \overline{\theta} \leq \lim_{j\to+\infty} (\, \|\nabla u_j\|_N^a + \|u_j\|_N^b\,) = 1$$

Therefore, we may estimate

$$\begin{aligned} d_N(a,b) &\geq \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_\tau|^{\frac{N}{N-1}}\,)\, dx = \tau^N \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \\ &= \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx + (\tau^N - 1) \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \end{aligned}$$

and, using (6.9), we get

$$\begin{aligned} d_N(a,b) &\geq d_N(a,b) - \frac{\alpha_N^{N-1}}{(N-1)!} \left(\overline{\theta}^{\frac{N}{b}} - \|u\|_N^N \right) + (\tau^N - 1) \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \\ &= d_N(a,b) - \frac{\alpha_N^{N-1}}{(N-1)!} \|u\|_N^N (\tau^N - 1) + (\tau^N - 1) \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \\ &= d_N(a,b) + (\tau^N - 1) \int_{\mathbb{R}^N} \phi_{N+1}(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx \end{aligned} \tag{6.10}$$

Since $u \neq 0$, we have

$$\int_{\mathbb{R}^N} \phi_{N+1}(\, \alpha_N |u|^{\frac{N}{N-1}}\,)\, dx = \int_{\mathbb{R}^N} \Big(e^{\alpha_N |u|^{\frac{N}{N-1}}} - \sum_{k=0}^{N-1} \frac{\alpha_N^k}{k!} |u|^{\frac{Nk}{N-1}} \Big)\, dx > 0$$

Consequently, $\tau = 1$. In fact, if not then $\tau > 1$ and (6.10) gives a contradiction.

Summarizing,

Lemma 6.2. *Let $N \geq 2$, $a > 0$ and $0 < b \leq N$ be fixed. We consider a (spherically symmetric and non-increasing) maximizing sequence $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ for the Trudinger-Moser supremum defined by* (1.7) *and we assume that $u_j \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$.*
If $b = N$ then one of the following alternatives occurs:

(I) either $\{u_j\}_j$ is a vanishing *maximizing sequence and*

$$d_N(a,b) = \lim_{j\to+\infty} \int_{\mathbb{R}^N} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = \frac{\alpha_N^{N-1}}{(N-1)!}$$

(II) or $\{u_j\}_j$ is a concentrating *maximizing sequence and*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N\setminus B_R} \phi_N(\, \alpha_N |u_j|^{\frac{N}{N-1}}\,)\, dx = 0 \qquad \text{for any fixed } R > 0$$

(III) or the weak limit u is non-trivial and it is a maximizer for $d_N(a,b)$.
If either

$$0 < b < N \quad \text{and} \quad a > 0$$

or

$$b = N \quad \text{and} \quad 0 < a << 1$$

then maximizing sequences for $d_N(a,b)$ cannot *concentrate and we have just two alternatives: either vanishing (I) or attainability (III) occurs.*

When either

$$0 < b < N \quad \text{and} \quad a > 0$$

or

$$b = N \quad \text{and} \quad 0 < a << 1$$

since *concentration* cannot occur, the lack of compactness of maximizing sequences for $d_N(a,b)$ can be only caused by *vanishing* phenomena. This possible lack of compactness may prevent the supremum $d_N(a,b)$ to be attained. In this respect, the analysis carried out in [14] plays a crucial role, see also [15] and Remark 6.1.

7. Proof of Theorem 1.2 – Attainability of the supremum

Let $N \geq 2$, $a > \frac{N}{N-1}$ and $0 < b < N$. To prove the attainability of the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7), we will follow the arguments introduced by M. Ishiwata and H. Wadade in [14] (see also [15]). In fact, since

$$0 < b < N$$

Lemma 6.2 expresses a vanishing-compactness alternative for (spherically symmetric and non-increasing) maximizing sequences of $d_N(a,b)$. More precisely, if $0 < b < N$ and $\{u_j\}_j \subset W^{1,N}(\mathbb{R}^N)$ is a (spherically symmetric and non-increasing) maximizing sequence for $d_N(a,b)$ then $\{u_j\}_j$ *cannot* concentrate and if it would be possible to exclude vanishing phenomena then $d_N(a,b)$ would be attained. In this respect, the restriction to the case

$$a > \frac{N}{N-1}$$

plays a crucial role. If $a > \frac{N}{N-1}$ then it is possible to improve the lower bound of $d_N(a,b)$ expressed by Lemma 6.1 showing that (I) of Lemma 6.2 *cannot* occur.

Proposition 7.1. *Let $N \geq 2$, $a > \frac{N}{N-1}$ and $0 < b \leq N$. Then the Trudinger-Moser supremum $d_N(a,b)$ defined by (1.7) satisfies*

$$d_N(a,b) > \frac{\alpha_N^{N-1}}{(N-1)!} \tag{7.1}$$

If in addition $b \neq N$ then $d_N(a,b)$ is attained.

Proof. As already mentioned, in view of Lemma 6.2, if $0 < b < N$ then the validity of (7.1) would enable to conclude the attainability of $d_N(a,b)$. Therefore, we just need to prove (7.1) and for this, we have to restrict the range of the parameter $a > 0$ defining the constraint (1.1).

Following [14, Section 2.2], we consider a suitable family of comparison functions $\{w_t\}_{t\in(0,1)} \subset W^{1,N}(\mathbb{R}^N)$, generated by a fixed function $u \in W^{1,N}(\mathbb{R}^N)$ satisfying the constraint (1.1). We used the same argument in the proof of Lemma 6.1 and we refer to (6.4) for the definition of w_t with $t \in (0,1)$.

Note that

$$\|w_t\|_p^p = t^{\frac{N}{b}} (1-t)^{\frac{p-N}{a}} \frac{\|u\|_p^p}{\|\nabla u\|_N^{p-N} \|u\|_N^N} \qquad \forall p \geq N$$

Let

$$B(u) := \frac{\|u\|_{N^2/(N-1)}^{N^2/(N-1)}}{\|\nabla u\|_N^{\frac{N}{N-1}} \|u\|_N^N} > 0$$

and

$$f(t) = f_{N,a,b}(t) := \|w_t\|_N^N + \frac{\alpha_N}{N} \|w_t\|_{N^2/(N-1)}^{N^2/(N-1)} = t^{\frac{N}{b}} \left(1 + \frac{\alpha_N}{N} (1-t)^{\frac{N}{N-1}\frac{1}{a}} B(u) \right)$$

Combining (6.1) with (6.5), we get

$$d_N(a,b) \geq \frac{\alpha_N^{N-1}}{(N-1)!} D_N(a,b) \geq \frac{\alpha_N^{N-1}}{(N-1)!} f(t) \quad \forall t \in (0,1)$$

If

$$f(t) > 1 \quad \text{ for some } t \in (0,1) \tag{7.2}$$

then it would be possible to conclude that (7.1) holds. Note that $f(1) = 1$, therefore (7.2) would follow if

$$f'(t) < 0 \quad \text{ for some } t \in (0,1) \text{ sufficiently close to } 1$$

We can compute

$$f'(t) = \frac{N}{b} t^{\frac{N}{b}-1} \left(1 + \frac{\alpha_N}{N} (1-t)^{\frac{N}{N-1}\frac{1}{a}} B(u) \right) - \frac{\alpha_N}{N-1} \frac{1}{a} t^{\frac{N}{b}} (1-t)^{\frac{N}{N-1}\frac{1}{a}-1} B(u)$$

and, if

$$\frac{N}{N-1} \frac{1}{a} - 1 < 0, \quad \text{i.e. } a > \frac{N}{N-1},$$

we have

$$\lim_{t\to 1^-} f'(t) = -\infty$$

□

References

[1] S. Adachi, K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents.* Proc. Amer. Math. Soc. **128** (2000), no. 7, 2051–2057

[2] Adimurthi, O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality.* Comm. Partial Differential Equations **29** (2004), no. 1-2, 295–322

[3] L. Battaglia, G. Mancini, *Remarks on the Moser-Trudinger inequality.* Adv. Nonlinear Anal. **2** (2013), no. 4, 389–425

[4] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state.* Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345

[5] H. Brezis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals.* Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490

[6] D. Cassani, F. Sani, C. Tarsi, *Equivalent Moser type inequalities in $\mathbb{R}^2$ and the zero mass case.* J. Funct. Anal. **267** (2014), no. 11, 4236–4263

[7] R. Černý, *Concentration-compactness principle for embedding into multiple exponential spaces on unbounded domains.* Czechoslovak Math. J. **65** (140) (2015), no. 2, 493–516

[8] R. Černý, A. Cianchi, S. Hencl, *Concentration-compactness principles for Moser-Trudinger inequalities: new results and proofs.* Ann. Mat. Pura Appl. (4) **192** (2013), no. 2, 225–243

[9] M. de Souza, J. M. do Ó, *A sharp Trudinger-Moser type inequality in* $\mathbb{R}^2$. Trans. Amer. Math. Soc. **366** (2014), no. 9, 4513–4549

[10] M. de Souza, J. M. do Ó, *A sharp inequality of Trudinger-Moser type and extremal functions in* $H^{1,n}(\mathbb{R}^n)$. J. Differential Equations **258** (2015), no. 11, 4062–4101

[11] J. M. do Ó, *N-Laplacian equations in $\mathbb{R}^N$ with critical growth.* Abstr. Appl. Anal. **2** (1997), no. 3-4, 301–315

[12] J. M. do Ó, M. de Souza, E. de Medeiros, U. Severo, *An improvement for the Trudinger-Moser inequality and applications.* J. Differential Equations **256** (2014), no. 4, 1317–1349

[13] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in* $\mathbb{R}^N$. Math. Ann. **351** (2011), no. 4, 781–804

[14] M. Ishiwata, H. Wadade, *On the effect of equivalent constraints on a maximizing problem associated with the Sobolev type embeddings in* $\mathbb{R}^N$. Math. Ann. to appear

[15] M. Ishiwata, H. Wadade, *Existence and non-existence of maximizers for the Moser-Trudinger inequalities of the inhomogeneous type.* Preprint 2015

[16] M. Ishiwata, H. Wadade, *On a maximizing problem associated with the Sobolev type embeddings in $\mathbb{R}^N$ under equivalent constraints.* Preprint 2015

[17] N. Lam, G. Lu, L. Zhang, *Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities.* Preprint 2015

[18] Y. Li, B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^n$. Indiana Univ. Math. J. **57** (2008), no. 1, 451–480

[19] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I.* Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201

[20] J. Moser, *A sharp form of an inequality by N. Trudinger.* Indiana Univ. Math. J. **20** (1970/1971), 1077–1092

[21] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$. J. Funct. Anal. **219** (2005), no. 2, 340–367

[22] C. Tintarev, *Trudinger-Moser inequality with remainder terms*. J. Funct. Anal. **266** (2014), no. 1, 55–66

[23] Y. Yang, *A sharp form of Moser-Trudinger inequality in high dimension*. J. Funct. Anal. **239** (2006), no. 1, 100–126

[24] Y. Yang, *Corrigendum to: "A sharp form of Moser-Trudinger inequality in high dimension"*. J. Funct. Anal. **242** (2007), no. 2, 669–671

[25] Y. Yang, *Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two*. J. Differential Equations **258** (2015), no. 9, 3161–3193

[26] Y. Yang, X. Zhu, *A new proof of subcritical Trudinger-Moser inequalities on the whole Euclidean space*. J. Partial Differ. Equ. **26** (2013), no. 4, 300–304

Universidade Federal da Paraíba, 58051-900, João Pessoa, PB, Brazil
E-mail address: jmbo@pq.cnpq.br

Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy
E-mail address: federica.sani@unimi.it

Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy
E-mail address: cristina.tarsi@unimi.it