Generalised states: a multi-sorted algebraic approach to probability

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This paper is dedicated to the memory of Franco Montagna.

Abstract We introduce a generalised notion of state as an additive map from a Boolean algebra of events to an arbitrary MV-algebra. Generalised states become unary operations in two-sorted algebraic structures that we call state algebras. Since these, as we show, form an equationally defined class of algebras, universal-algebraic techniques apply. We discuss free state algebras, their geometric representation, and their connection with the theory of affine representations of lattice-groups.

Keywords probability · state · multi-sorted algebra · heterogeneous algebra · Boolean algebra · MV-algebra · Lattice-ordered Abelian group.

1 Introduction

Classically, *states* are normalised positive linear functionals on Riesz spaces with unit, or more generally, on lattice-ordered Abelian groups with unit. States provide the standard abstraction of the expected value operator. States of MV-algebras [14] are just the same thing as classical states, up to translation through the well-known theory of the functor Γ [4, Chapter 7]. However, because of the relation between MV-algebras and Boolean algebras, MV-algebraic states may also be regarded as a many-valued generalisation of finitely additive probability assignments. Here, families of (many-valued) events are no longer modelled by Boolean algebras, but rather by MV-algebras. MV-algebraic states

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have been intensively investigated in the last two decades; see [6] for a recent survey of results. On MV-algebraic matters we follow the notation and terminology of [4], to which we refer the reader for background. Given an MV-algebra $(D, \oplus, \neg, 0)$, we introduce as usual the binary operations $\odot$, $\ominus$, and the constant 1 together with the lattice supremum $\vee$ and infimum $\wedge$ on D. The lattice order of D is denoted by $\leq$.

In 2009, Flaminio and Montagna [7] introduced the notion of *internal state* as an additional unary operation $s\colon D \to D$ on an MV-algebra D. Internal states are intended to capture the basic properties of states in a setting amenable to universal-algebraic techniques. Indeed, in [7, Definition 3.1], s is required to satisfy certain *equational* laws. (See Remark 1 below for the Flaminio-Montagna axiomatisation.) Please see the references in [6, Section 8] for subsequent work on internal states. On the other hand, a *state of an MV-algebra D* is a mapping $s\colon D \to [0,1]$ satisfying $s(1) = 1$ and the following condition for every $a, b \in D$:

$$\text{if } a \odot b = 0, \text{ then } s(a \oplus b) = s(a) + s(b). \tag{1}$$

Thus:

- a state is defined on an MV-algebra and takes values in the real unit interval, and
- the form of the key axiom (1) is not equational.

Prima facie, therefore, there seems to be no hope of directly applying the theory of universal algebra to states. One way to remedy this is to make recourse to the Flaminio-Montagna theory of internal states. Unfortunately, that comes at the significant cost of renouncing the fundamental conceptual distinction between *events* — elements of the domain of a state — and *degrees of probability* — elements of the co-domain of the state.

The purpose of this note is to show that, *pace* first impressions, an equational theory of states *that preserves this conceptual distinction* is indeed feasible; more substantially, we also provide sufficient mathematical evidence to indicate that its developments may prove of considerable interest. Specifically, we provide a preliminary representation theory for free objects in the resulting equationally definable class of algebras. Our main point can already be made in the classical setting of *Boolean* algebras of events, and we shall therefore work in that setting. We do emphasise that it is straightforward to adapt our definitions to MV-algebras of many-valued events; compare Remark 1. We further alert the reader to the fact that MV-algebras already play a key rôle *in this classical setting*, as abstractions of the co-domain $[0, 1]$ of finitely additive probabilities.

Throughout this note we think of a Boolean algebra E as an abstract algebraic model of the structure of events in probability theory, an event being an element of E. The underlying order relation on a Boolean algebra is written $\subseteq$, and the binary relative complement operation is written $\setminus$. We stress for clarity that our definitions need no lattice-theoretic completeness assumption about E. As is well known, Boolean algebras are precisely those MV-algebras such that $\oplus$ is idempotent, and then $\oplus$ agrees with join and $\odot$ agrees with meet. Because we shall soon have to work with Boolean algebras and MV-algebras within a single formal framework, we need to distinguish the Boolean operations from the MV-algebraic ones. Hence we write

$$(E, \cup, \cap, ', \bot, \top)$$

to denote a Boolean algebra with its operations and their obvious meaning. Now, if $D = E$ is a Boolean algebra, states of E clearly coincide with finitely additive probability assignments of E. More precisely, a *state of a Boolean algebra E* is a mapping $s \colon E \to [0, 1]$ satisfying $s(\top) = 1$ and the following condition for every $a, b \in E$:

$$\text{if } a \cap b = \bot, \text{ then } s(a \cup b) = s(a) + s(b). \tag{2}$$

Terminology. For the rest of this note, *state* means *state of a Boolean algebra*.

If Boolean algebras are algebraic models of families of events, what is an abstract model of the set $[0, 1]$ of probability degrees? In the remainder of this note, we make the following points.

(I) The relevant structure of $[0, 1]$ as the collection of probability degrees includes its MV-algebraic structure.

(II) In light of (I), one can replace $[0, 1]$ by any MV-algebra in the definition of state (2), thereby obtaining what we call *generalised states* (see Definition 1).

(III) Furthermore, generalised states admit an *equational* definition — hence, *a fortiori*, so do states.

(IV) Items (I–III) show that generalised states, when properly interpreted, can be regarded as genuine algebraic structures, provided only one allows *multi-sorted algebras*. (We comment on multi-sorted universal algebra in Section 4.)

(V) Item (IV) thus leads to the notion of *state algebra*, a two-sorted algebraic structure equipped with a generalised state as a multi-sorted operation. In closing, we discuss *free* state algebras and prove a significant representation theorem for a class of them.

2 Generalised states

The basic operations required in elementary probability theory *on the probability degrees themselves* — not on the events — are the following.

(i) Addition of real numbers.
(ii) Involution in $[0, 1]$, that is, $x \in [0, 1] \mapsto 1 - x \in [0, 1]$.
(iii) Infima and suprema derived from the natural order of the real numbers.

Now, (i–iii) amount to considering $(\mathbb{R}, 1)$ as a *lattice-ordered Abelian group with unit* 1. It may be argued that one should add to (i–iii) the multiplicative structure of $[0, 1]$, for example to model independence of events, or conditioning. However, for the purposes of this note we do not consider multiplication.

The addition in (i) can only be restricted to $[0, 1]$ at the price of turning it into a partially defined operation. However, by the theory of the functor Γ [4, Chapter 7], the unital lattice-group $(\mathbb{R}, 1)$ [1] is uniquely determined by the MV-algebra $[0, 1]$, its unit interval. Hence, in place of (i–ii) we can equivalently consider the MV-algebraic structure of $[0, 1]$. The order in (iii) is then derivable from the MV-algebraic structure in the usual manner.

In light of the above, let (G, u) be any lattice-ordered Abelian group with strong order unit u, and let E be any Boolean algebra. We can define a generalised notion of (G, u)-valued state by considering functions

$$s \colon E \to G$$

that satisfy the normalisation condition $s(\top) = u$ and the finite additivity condition

$$\text{if } a \cap b = \bot, \text{ then } s(a \cup b) = s(a) + s(b).$$

Passing to the unit-interval MV-algebra $\Gamma(G, u) = [0, u]$, whose truncated addition is $x \oplus y := (x + y) \wedge u$ for all $x, y \in [0, u]$, such functions turn out to be in one-one correspondence with the functions $s\colon E \to [0, u]$ that satisfy

$$s(\top) = u, \text{ and} \tag{3}$$

$$\text{if } a \cap b = \bot, \text{ then } s(a \cup b) = s(a) + s(b). \tag{4}$$

We will show in Lemma 1 below how to turn the quasi-equational notion (3–4) into an equivalent equational one, thus achieving item (III) in the Introduction.

Definition 1 (Generalised state)
    A *generalised state* of a Boolean algebra E, with values in an MV-algebra D, is a mapping $s\colon E \to D$ such that for every $a, b \in E$ the following hold.

(A1) $s(a \cup b) = s(a) \oplus s(b \setminus a)$,
(A2) $s(a') = \neg s(a)$,
(A3) $s(\top) = 1$.

Remark 1 Our Definition 1 is inspired by but not identical to the one used in [7]. If the domain of s were an MV-algebra rather than a Boolean algebra, then condition (A1) would need to be replaced by

(A1′) $s(a \oplus b) = s(a) \oplus s(b \ominus (a \odot b))$

in order to prove an analogue of our Lemma 1 below. Indeed, Flaminio and Montagna defined an internal state of the MV-algebra D to be a map $s\colon D \to D$ that satisfies (A1′), (A2), and (A3), together with the additional axiom

(A4) $s(s(a) \oplus s(b)) = s(a) \oplus s(b)$

expressing the fact that the image of D under s is an MV-subalgebra of D. Further, other authors introduced and studied a concept of state for residuated lattices, allowing for values in an arbitrary residuated lattice — a generalised state in the sense of our Definition 1 is called, in that literature, an "order-preserving type I state"; see [5, Proposition 3.22].

Notation. For the rest of this note, E always denotes a Boolean algebra, D always denotes an MV-algebra, and s always denotes a function $s\colon E \to D$.

    Let us collect useful elementary facts about generalised states in

Proposition 1 *If s is a generalised state, the following hold for every $a, b \in E$.*

1. *$s(\bot) = 0$.*
2. *If $a \subseteq b$, then $s(a) \leq s(b)$.*
3. *$s(a \cup b) \leq s(a) \oplus s(b)$, and if $a \cap b = \bot$, then $s(a \cup b) = s(a) \oplus s(b)$.*

4. *$s(a \setminus b) \geq s(a) \ominus s(b)$, and if $b \subseteq a$, then $s(a \setminus b) = s(a) \ominus s(b)$.*
5. *$s(a \cap b) \geq s(a) \odot s(b)$, and if $a \cap b = \bot$, then $s(a) \odot s(b) = 0$.*
6. *$s(a \cup b) = s(a) \oplus [s(b) \ominus s(a \cap b)]$.*

Proof 1. By (A2) and (A3), $s(\bot) = \neg s(\top) = 0$.
2. Let $a \subseteq b$. Then $b = a \cup (b \setminus a)$. Since $a \cap (b \setminus a) = \bot$, we get

$$s(b) = s(a \cup (b \setminus a)) = s(a) \oplus s(b \setminus a) \geq s(a).$$

3. It follows from (A1) and item 2 that

$$s(a \cup b) = s(a) \oplus s(b \setminus (a \cap b)) \leq s(a) \oplus s(b).$$

Let $a \cap b = \bot$. Then

$$s(a \cup b) = s(a) \oplus s(b \setminus \bot) = s(a) \oplus s(b).$$

4. We get

$$\begin{aligned} s(a \setminus b) &= s((a' \cup b)') = \neg s(a' \cup b) \\ &\geq \neg(\neg s(a) \oplus s(b)) = s(a) \odot \neg s(b) = s(a) \ominus s(b). \end{aligned}$$

Let $b \subseteq a$. Then $a' \cap b = 0$ and the equality in the formula above follows from item 3.
5. We get

$$\begin{aligned} s(a \cap b) &= s(a \setminus b') \\ &\geq s(a) \ominus s(b') = s(a) \odot \neg s(b') = s(a) \odot s(b). \end{aligned}$$

Let $a \cap b = \bot$. Then

$$s(a) \odot s(b) \leq s(a \cap b) = s(\bot) = 0.$$

6. This is a consequence of (A1) and item 4.   □

We can now prove:

Lemma 1 *Let $s\colon E \to D$ satisfy $s(\top) = 1$. Then the following are equivalent.*

1. *s is a generalised state.*
2. *For every $a, b \in E$, if $a \cap b = \bot$, then $s(a \cup b) = s(a) \oplus s(b)$ and $s(a) \odot s(b) = 0$.*
3. *For every $a, b \in E$, if $a \cap b = \bot$, then $s(a \cup b) = s(a) + s(b)$, where $+$ is the addition operation of the essentially unique lattice-ordered Abelian group $(G, 1)$ such that $\Gamma(G, 1) = D$.*

Proof The implication from item 1 to 2 follows from Proposition 1. Conversely, let a mapping $s\colon E \to D$ satisfy $s(\top) = 1$ and 2. Then $a \cap a' = 0$ yields $s(a) \odot s(a') = 0$ and

$$1 = s(\top) = s(a \cup a') = s(a) \oplus s(a'), \quad a \in E.$$

Therefore $s(a') = \neg s(a)$ by [4, Lemma 1.1.3] and thus (A2) in Definition 1 holds. Finally, we prove (A1). Using

the identities $a \cap (b \setminus a) = \bot$, $a \cup b = a \cup (b \setminus a)$, we obtain

$$s(a \cup b) = s(a \cup (b \setminus a)) = s(a) \oplus s(b \setminus a), \quad a, b \in E.$$

Hence, (A1) in Definition 1 holds.

It remains to show that item 2 is equivalent to item 3. This amounts to the well-known fact that in any MV-algebra $x \oplus y$ agrees with $x + y$ of its enveloping unital Abelian lattice-group if, and only if, $x \odot y = 0$ — see [4, Lemma 2.1.3(i)].   □

As promised, Lemma 1 shows that a generalised state can be thought of as an additive map $E \to D$, that is to say, it is the same thing as a function satisfying (3–4).

Using Proposition 1 we can easily derive the following inequalities, which can be viewed as analogues of the well-known Boole–Fréchet bounds in probability theory, see e.g. [9, §3.5].

Corollary 1 (Boole–Fréchet bounds) *If s is a generalised state then the following hold for all $a, b \in E$.*

$$s(a) \vee s(b) \leq s(a \cup b) \leq s(a) \oplus s(b)$$
$$s(a) \odot s(b) \leq s(a \cap b) \leq s(a) \wedge s(b)$$

Further, these bounds are attained for appropriate choices of D, E, s, a, and b.

Proof We use Proposition 1. Both $s(a) \vee s(b) \leq s(a \cup b)$ and $s(a \cap b) \leq s(a) \wedge s(b)$ follow from monotonicity (item 2). The inequalities $s(a \cup b) \leq s(a) \oplus s(b)$ and $s(a) \odot s(b) \leq s(a \cap b)$ are item 3 and 5, respectively. To show the bounds are attained, it suffices to take $D = E$ and s the identity homomorphism, which clearly is a generalised state.
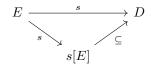
Remark 2 The notion of observable introduced by Pulmannová in [17] is formally the same as our concept of generalised state. Indeed, an *observable* was defined in that paper as a mapping x from a Boolean algebra E into an MV-algebra D satisfying $x(\top) = 1$ and condition 2 in Lemma 1. The motivation for studying observables in [17] comes from quantum probability theory.

3 Examples of generalised states

We discuss in this section two examples of generalised states that are significant for different reasons. We first point out a few easy facts.

Clearly, any state of a Boolean algebra E is a generalised state — in this case, D is the standard MV-algebra $[0, 1]$ and the rest follows from Lemma 1. Moreover, by regarding Boolean algebras as algebras of the same similarity type as MV-algebras, we can consider

the case when the function $s: E \to D$ is a homomorphism. Then Lemma 1 shows that s is a generalised state. Moreover, in this case the range $s[E]$ is in fact a Boolean algebra, because E is and s is a homomorphism that is onto its range. Hence, $s[E]$ must be contained in the Boolean skeleton $B(D)$ of D [4, Corollary 1.5.4], and $s[E]$ must be a Boolean subalgebra of $B(D)$. We thus have the following. *Any homomorphism $s: E \to D$ is a generalised state that factors as in the commutative diagram below into generalised states which are surjective and injective, respectively, and a factorisation of s with these properties is essentially unique*:

3.1 Example 1: The infinite lottery.

Let us model an experiment consisting in the random draw of a natural number $n \in \mathbb{N}$. Assume that the events to be considered ought to afford an answer at least to questions of the form '$n \in A$?', for $A \subseteq \mathbb{N}$ finite. The minimal algebraic model of observable events is then the finite-cofinite Boolean algebra $\mathscr{E}$ on $\mathbb{N}$, that is,

$$\mathscr{E} = \{A \subseteq \mathbb{N} \mid \text{either } A \text{ or } A' \text{ is finite}\}.$$

Note that there is no $[0, 1]$-valued state modelling both (i) the uniformly random character of the experiment, and (ii) the intuition that each $n \in \mathbb{N}$ may actually turn out to be the winning number. That is to say, there is no state $s: \mathscr{E} \to [0, 1]$ satisfying $p(\{n\}) = \alpha > 0$ for all $n \in \mathbb{N}$. There is, however, exactly one state $s: \mathscr{E} \to [0, 1]$ modelling (i) alone, namely, the invariant state given by

$$s(A) = \begin{cases} 0 & A \text{ finite,} \\ 1 & A \text{ cofinite,} \end{cases} \quad A \in \mathscr{E}.$$

This state s fails to account for (ii), in that according to s the probability that any given single $n \in \mathbb{N}$ be drawn is 0. The classical way out of this situation is to issue the usual *caveat* that *events of null probability may actually obtain* — probability zero and impossibility are distinct concepts. Whether one considers this solution satisfactory or not, we proceed to exhibit an alternative way out that makes use of generalised states.

We construct a generalised state for the experiment at hand that accounts for both (i) and (ii). We do so by replacing the co-domain MV-algebra $[0, 1]$ with

a non-semisimple MV-chain, the well-known *Chang's MV-algebra C* [3]. Its underlying set is

$$C = \{0, \varepsilon, 2\varepsilon, \dots, 1 - 2\varepsilon, 1 - \varepsilon, 1\},$$

where ε is a symbol interpreted as a positive infinitesimal and C is equipped with the operations:

$$x \oplus y = \begin{cases} (k+m)\varepsilon & x = k\varepsilon,\ y = m\varepsilon, \\ 1 - (m-k)\varepsilon & x = k\varepsilon,\ y = 1 - m\varepsilon,\ m > k, \\ 1 - (k-m)\varepsilon & x = 1 - k\varepsilon,\ y = m\varepsilon,\ k > m, \\ 1 & \text{otherwise,} \end{cases}$$

$$\neg x = \begin{cases} 1 - k\varepsilon & \text{if } x = k\varepsilon, \\ k\varepsilon & \text{if } x = 1 - k\varepsilon. \end{cases}$$

We define a mapping $s \colon \mathscr{E} \to C$ such that, for all $A \in \mathscr{E}$,

$$s(A) = \begin{cases} |A|\varepsilon & A \text{ finite}, \\ 1 - |A'|\varepsilon & A \text{ cofinite}, \end{cases}$$

Lemma 1 can be employed to show that s is a generalised state. First,

$$s(\mathbb{N}) = 1 - |\mathbb{N}'|\varepsilon = 1 - 0\varepsilon = 1.$$

Second, take $A, B \in \mathscr{E}$ such that $A \cap B = \emptyset$. This implies that at least one of the sets must be finite. If A and B are finite, then

$$s(A) \oplus s(B) = |A|\varepsilon \oplus |B|\varepsilon = |A \cup B|\varepsilon = s(A \cup B)$$

and, analogously, $s(A) \odot s(B) = 0$. Assume that A is finite and B is cofinite. Then the assumption $A \cap B = \emptyset$ yields $A \subseteq B'$ and

$$\begin{aligned} s(A) \oplus s(B) &= |A|\varepsilon \oplus (1 - |B'|\varepsilon) = 1 - (|B'| - |A|)\varepsilon \\ &= 1 - |A' \cap B'|\varepsilon = 1 - |(A \cup B)'|\varepsilon \\ &= s(A \cup B), \end{aligned}$$

as $A \cup B$ is cofinite. Finally,

$$s(A) \odot s(B) = |A|\varepsilon \odot (1 - |B'|\varepsilon) = \neg(\underbrace{(1 - |A|\varepsilon) \oplus |B'|\varepsilon}_{1}),$$

since $|A| \leq |B'|$. Hence, s is a generalised state $\mathscr{E} \to C$.

The state $s \colon \mathscr{E} \to C$ of this example can be viewed as a non-standard analogue of the probabilistic model for the random selection of a natural number: there is a "uniform nowhere-zero distribution" on the sample space $\mathscr{E}$ intuitively associated to s, which is the function constantly equal to ε on $\mathscr{E}$.

3.2 Example 2: Affine representations are generalised states.

In this example we apply the classical theory of affine representations of unital Riesz spaces, or more generally of unital lattice-ordered Abelian groups. See [8] for background. We formulate everything in the language of MV-algebras; this requires a straightforward translation from the group-theoretic version via the functor Γ. The Boolean algebra E comes with its associated state space. To explain this, let $\mathrm{St}\,E$ be the set of all $[0,1]$-valued states of E, and consider the set-theoretic product $[0,1]^E$ equipped with its product (Tychonoff) topology. Using the natural (injective) map $\mathrm{St}\,E \longrightarrow [0,1]^E$, we topologise $\mathrm{St}\,E$ through the subspace topology induced by $[0,1]^E$. The resulting Tychonoff space $\mathrm{St}\,E$ is the *state space* of E. It is well known that $\mathrm{St}\,E$ is a compact convex subset of the ambient locally convex topological linear space $\mathbb{R}^E$.

For any $a \in E$, let us define a function

$$\hat{a} \colon \mathrm{St}\,E \longrightarrow [0,1]$$

by setting, for every $p \in \mathrm{St}\,E$,

$$\hat{a}(p) \coloneqq p(a).$$

Then $\hat{a}$ is continuous and affine. To represent E by such affine functions, let $\mathrm{C}\,(\mathrm{St}\,E)$ be the MV-algebra of all continuous functions $\mathrm{St}\,E \to [0,1]$, with operations defined pointwise from those of the standard MV-algebra $[0,1]$. Define a function $s \colon E \to \mathrm{C}\,(\mathrm{St}\,E)$ as $s(a) = \hat{a}$, for every $a \in E$. Then s provides such a representation. The reader is cautioned that s *is not a homomorphism*, in general. However, it can be shown that s *indeed is a generalised state*. Thus, the theory of affine representations of Boolean algebras may be recast in the language of generalised states.

We shall see at the end of our note that this example is a crucially important one: states arising from affine representations enjoy the universal property of free objects, cfr. Remark 4.

4 State algebras

In the Birkhoff-Lipson approach to multi-sorted[1] algebras [2], classical universal algebra is extended to multi-sorted operations. The key to doing this is to replace the category of sets and functions by the category of *multi-sorted sets* and *multi-sorted functions*. Here we only

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[1] Called *heterogeneous* algebras in [2]. We stick to the multi-sorted terminology which seems to have become standard.

spell out the two-sorted case of interest to us. A *two-sorted set* is an ordered pair (A, B) of sets. The component sets A and B are called the *sorts* of the two-sorted set (A, B). A *two-sorted function*

$$f\colon (A_1, B_1) \longrightarrow (A_2, B_2)$$

between two-sorted sets is a pair $f := (f_1, f_2)$ of functions

$$f_1\colon A_1 \longrightarrow A_2,$$
$$f_2\colon B_1 \longrightarrow B_2.$$

Composition and identity morphisms are defined component-wise. This defines the category of two-sorted sets. One can study multi-sorted sets equipped with *multi-sorted operations* in much the same way as one studies, in general algebra, sets equipped with operations. We shall apply this idea to generalised states.

Consider, then, a generalised state $s\colon E \to D$. We can regard the pair of sets (E, D) as a two-sorted set; we call E the sort of *events*, and D the sort of *probability degrees*. Then $s\colon E \to D$ is a unary operation between the two different sorts. Hence, we are concerned with two-sorted algebras whose underlying two-sorted set is the ordered pair (E, D), and whose (multi-sorted) operations are as follows.

(T1) The operations of Boolean algebras $\cup, \cap, ', \top, \bot$ in the single sort E.
(T2) The operations of MV-algebras $\oplus, \neg, 0$ in the single sort D.
(T3) The unary operation s with sort E in the domain and D in the codomain.

Thus (T1–T3) define a *two-sorted similarity type*. Building on Definition 1, we now have:

Definition 2 (State algebra) A *state algebra* is an algebra (E, D, s) of the two-sorted similarity type (T1–T3) such that the following equational conditions hold.

(S1) $(E, \cup, \cap, ', \bot, \top)$ is a Boolean algebra.
(S2) $(D, \oplus, \neg, 0)$ is an MV-algebra.
(S3) The operation $s\colon E \to D$ is a generalised state, i.e. satisfies the equational conditions (A1–A3) in Definition 1.

When no danger of confusion arises, we write only (E, D) in place of (E, D, s).

All of the universal-algebraic constructions and fundamental theorems go through to the multi-sorted setting up to minimal modifications.[2] For example, since state

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[2] Somewhat more substantial changes are needed in the presence of algebras with underlying multi-sorted sets which are not everywhere non-empty. In our case, given that the type of MV-algebras, hence of Boolean algebras, includes constants, neither E nor D can be empty.

algebras are defined by equations, they are closed under homomorphic images, subalgebras, and products. For an important sample of such universal-algebraic multi-sorted results please see the original [2]. We now focus on free objects.

For clarity, let us first spell out the notion of homomorphism of state algebras. A *homomorphism* between state algebras (E_1, D_1) and (E_2, D_2) is a morphism

$$h\colon (E_1, D_1) \longrightarrow (E_2, D_2)$$

of two-sorted sets, which means that $h := (h_1, h_2)$ is a pair of functions $h_1\colon E_1 \to E_2$, $h_2\colon D_1 \to D_2$, such that h preserves all operations in the type (T1–T3). In turn, the latter means that h_1 is a homomorphism of Boolean algebras, h_2 is a homomorphism of MV-algebras, and h preserves the operation s — that is, the following diagram commutes:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ s\ } & D_1 \\
{\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\
E_2 & \xrightarrow{\ s\ } & D_2
\end{array}
$$

Elementwise, for all $a \in E_1$ we have

$$h_2(s(a)) = s(h_1(a)).$$

As in the classical, single-sorted case, free algebras always exist in equationally defined classes of multi-sorted algebras. We now state the defining universal property of free state algebras, which is obtained from the usual one for single-sorted algebras by replacing the category of sets with the category of two-sorted sets.

Definition 3 (Free state algebras) Let $\iota\colon (S_1, S_2) \to (E, D)$ be a two-sorted function, where (S_1, S_2) is a two-sorted set, and (E, D) is (the underlying two-sorted set of) a state algebra. We say ι is *free* (with respect to the class of all state-algebras) if for every two-sorted function $\eta\colon (S_1, S_2) \to (E', D')$, where (E', D') is (the underlying two-sorted set of) a state algebra, there is *exactly one* homomorphism of state algebras

$$h\colon (E, D) \longrightarrow (E', D')$$

that makes the following diagram commute.

$$
\begin{array}{ccc}
(S_1, S_2) & \xrightarrow{\quad\iota\quad} & (E, D) \\
 & {\scriptstyle \eta}\searrow & \downarrow{\scriptstyle h} \\
 & & (E', D')
\end{array}
$$

In standard algebraic usage, one refers to (E, D) in Definition 3 as an *algebra "freely generated by the two-sorted set"* (S_1, S_2), omitting reference to ι. It can indeed be shown that $(\iota_1[S_1], \iota_2[S_2])$, where $\iota := (\iota_1, \iota_2)$, does generate (E, D). However, care is needed with the notion

of generating system in the multi-sorted case. We will see, for instance, that even when $(\iota_1[S_1], \iota_2[S_2])$ freely generates (E, D), $\iota_2[S_2]$ *need not generate — let alone freely generate — the MV-algebra D*. This is in fact a general phenomenon about subalgebras in the two-sorted setting, and has nothing to do with freeness; we illustrate it by the simplest possible example.

Example 1 Let F_1 be the Boolean algebra freely generated by $x_1 \in F_1$, and consider the generalised state $s \colon F_1 \to F_1$ given by the identity homomorphism. Then (F_1, F_1) is a state algebra. The two-sorted subset $(\{x_1\}, \{x_1\})$ obviously generates (F_1, F_1), because x_1 generates the Boolean algebra F_1. Note, however, that also the *two-sorted subset* $(\{x_1\}, \emptyset)$ *generates* (F_1, F_1). Indeed, apply first the Boolean operations in the sort of events to obtain F_1 from x_1. Then *apply the operation s to x_1*, obtaining an element $s(x_1) \in F_1$ in the sort of probability degrees. Since s is the identity, $s(x_1) = x_1 \in F_1$; therefore, applying the MV-algebraic operations in the sort of probability degrees to x_1 yields the whole F_1.

It is easy to describe the state-algebras (E, D) freely generated by a two-sorted set $(\emptyset, S_2)$, for S_2 an arbitrary set. Indeed, let $FMV(S_2)$ denote the MV-algebra freely generated by S_2, and let $\mathbf{2}$ be the two-element Boolean algebra, and consider the state algebra $(\mathbf{2}, FMV(S_2))$ whose operation $s \colon \mathbf{2} \to FMV(S_2)$ is given by the only possible choice — $\mathbf{2}$ is the initial object in the category of MV-algebras. We leave it to the reader to verify that $(\mathbf{2}, FMV(S_2))$ is the state algebra freely generated by $(\emptyset, S_2)$.

5 Representation theorem for free state algebras

In this section we prove our main result. For each integer $n \geq 0$, we let F_n denote the Boolean algebra freely generated by the generators $x_1, \ldots, x_n$. Thus, $F_0 = \mathbf{2}$, the two-element Boolean algebra. We shall obtain a representation theorem for the state algebras freely generated by a multi-sorted set $(\{x_1, \ldots, x_n\}, \emptyset)$ in terms of F_n and an appropriate generalised state of F_n. The state algebra freely generated by $(\emptyset, \emptyset)$ is immediately seen to be $(\mathbf{2}, \mathbf{2})$ with $s \colon \mathbf{2} \to \mathbf{2}$ the only possible such generalised state, namely, the identity. Thus we assume $n \geq 1$ throughout.

In order to formulate and prove our result concisely, we will assume familiarity with basic polyhedral geometry, McNaughton functions, the geometric representation theory of finitely generated free MV-algebras, and the MV-algebraic theory of Schauder hats. The needed background can be found in [16].

In the Euclidean n-dimensional linear space $\mathbb{R}^n$, let $e_1, \ldots, e_n$ denote the standard orthonormal basis, let $[0, 1]^n \subseteq \mathbb{R}^n$ denote the unit n-cube, and let $\Delta_n := \mathrm{conv}\{e_1, \ldots, e_n\}$ denote the standard $(n-1)$-simplex, the convex hull of the standard basis. Let us further write $\pi_i \colon [0, 1]^n \to [0, 1]$ for the projection functions, $i = 1, \ldots, n$, and let $\mathrm{C}([0,1]^n)$ be the MV-algebra of all continuous $[0, 1]$-valued functions on the unit n-cube. (Cfr. Section 3.2.) By $\mathscr{M}_n$ we denote the MV-subalgebra of $\mathrm{C}([0,1]^n)$ generated by $\{\pi_i\}_{i=1}^n$. Then it is known that $\mathscr{M}_n$ is in fact freely generated by $\{\pi_i\}_{i=1}^n$, and consists precisely of the McNaughton functions, that is, the continuous piecewise-linear functions whose affine linear pieces have integer coefficients. For any subset $X \subseteq [0, 1]^n$ we write

$$\mathscr{M}_n(X)$$

for the MV-algebra obtained by restricting the elements of $\mathscr{M}_n$ to X. Writing

$$r \colon \mathscr{M}_n \longrightarrow \mathscr{M}_n(X)$$

for the restriction map, r is a surjective MV-homomorphism whose kernel $\ker r := \{f \in \mathscr{M}_n \mid r(f) = 0\}$ is the ideal of all McNaughton functions vanishing over X, or equivalently (by continuity), over the closure of X. We specifically consider the MV-algebra

$$\mathscr{S}_n := \mathscr{M}_n(\Delta_n),$$

which we call the *MV-algebra dual to the standard n-simplex*. It will be useful to have an explicit description of the kernel of

$$\rho \colon \mathscr{M}_n \longrightarrow \mathscr{S}_n. \tag{†}$$

We already used the standard fact that, in any MV-algebra M with enveloping lattice-ordered Abelian group with unit $(G, 1)$, and for any $x, y \in M$, the equality $x \oplus y = x + y$ holds if, and only if, $x \odot y = 0$. This shows that the condition $x + y = 1$ is equationally expressible in MV-algebraic language by the system of equations

$$\begin{cases} x \oplus y &= 1 \\ x \odot y &= 0. \end{cases}$$

It is an MV-algebraic exercise to extend this from two to n summands. For an explicit description of the MV-algebraic relations[3] needed, please see [15]. What matters for us is that the partition of unity condition

$$x_1 + \cdots + x_n = 1, \tag{*}$$

---

[3] Since universal algebraists reserve the name 'equation' for fully invariant identities, in the following we use 'relation' to mean an equality between two MV-algebraic terms that holds for given elements of a given MV-algebra.

for $x_1, \ldots, x_n \in M$, while *a priori* only meaningful in $(G, 1)$, is in fact expressible in the MV-algebra M by finitely many equivalent relations in the variables $x_1, \ldots, x_n$. Since it is elementary that any finite set of MV-algebraic relations is equivalent to a single one of the form $\tau = 0$, for τ an appropriate MV-algebraic term, we shall write $\sigma(x_1, \ldots, x_n)$ for a term in the variables $x_1, \ldots, x_n$ that satisfies

$$\sigma(x_1, \ldots, x_n) = 0$$

in M if, and only if, (*) holds in $(G, 1)$.

We can now consider the principal ideal

$$I_n := \langle \sigma(\pi_1, \ldots, \pi_n) \rangle$$

of $\mathscr{M}_n$ generated by the element $\sigma(\pi_1, \ldots, \pi_n) \in \mathscr{M}_n$, along with the quotient MV-algebra

$$\frac{\mathscr{M}_n}{I_n}.$$

Our discussion above of the definition of I_n shows that the zero set of I_n — the set of points in $[0,1]^n$ where the functions in I_n jointly vanish — is precisely the standard simplex $\Delta_n \subseteq [0,1]^n$. It follows immediately (by the contravariant Galois correspondence between zero sets and ideals) that the restriction map ρ in (†) factors through the natural quotient map $\mathscr{M}_n \to \frac{\mathscr{M}_n}{I_n}$. But since I_n is principal, an application of the Hay-Wójcicki Theorem yields that the second component of this factorisation — i.e., the comparison map between quotient objects — is in fact an isomorphism $\frac{\mathscr{M}_n}{I_n} \to \mathscr{S}_n$. Thus, in conclusion,

$$\ker \rho = I_n, \tag{$\sharp$}$$

which provides the promised explicit description of the kernel of ρ.

Next, let us consider F_n, the Boolean algebra freely generated by the set $\{x_1, \ldots, x_n\}$. Let us display the 2^n atoms of F_n as

$$m_1, \ldots, m_{2^n};$$

the usual convention is that m_j is the unique atom of F_n such that the binary expansion of $j-1$ has 1 in its ith digit if, and only if, m_j lies below the generator x_i in the underlying order of F_n. To each m_j there corresponds a uniquely determined vertex of the standard simplex Δ_{2^n}, namely, the unit vector e_j.

Lemma 2 *Fix an integer $n \geq 1$. Each generalised state $s \colon F_n \to D$ is uniquely determined by its action on the set $\{m_1, \ldots, m_{2^n}\}$ of atoms of F_n.*

Proof Indeed, by elementary Boolean algebra, for each $b \in F_n$ there is a unique $J_b \subseteq \{1, \ldots, 2^n\}$ such that

$$b = \bigcup_{j \in J_b} m_j.$$

Now $m_j \cap m_k = \perp$ whenever $j \neq k$. By Lemma 1, using addition of the enveloping lattice-group of D, we have

$$s(b) = \sum_{j \in J_b} s(m_j),$$

and the statement is proved. $\square$

We supplement the preceding lemma with a simple but crucial existence result:

Lemma 3 *Fix an integer $n \geq 1$. Consider a function*

$$\overline{s} \colon \{m_1, \ldots, m_{2^n}\} \longrightarrow D$$

from the set of atoms of the Boolean algebra F_n freely generated by $\{x_1, \ldots, x_n\}$ to an MV-algebra D. The following are equivalent.

1. *There is a unique extension of $\overline{s}$ to a generalised state $s \colon F_n \to D$.*
2. *In the (essentially unique) Abelian lattice-group G with unit 1 such that $\Gamma(G, 1) = D$, we have*

$$\sum_{j=1}^{2^n} \overline{s}(m_j) = 1.$$

Proof The implication from item 1 to item 2 is given by Lemma 1. For the converse implication, suppose item 2 holds. For each $b \in F_n$, set

$$s(b) := \sum_{j \in J_b} s(m_j),$$

where J_b is as in Lemma 2. Then the function $s \colon F_n \to D$ extends $\overline{s}$ because $J_{m_j} = \{m_j\}$ for each $j = 1, \ldots, 2^n$. Now, if $b_1 \cap b_2 = \perp$, it must be the case that $J_{b_1} \cap J_{b_2} = \emptyset$, and then

$$J_{b_1 \cup b_2} = J_{b_1} \cup J_{b_2}, \tag{$\sqcup$}$$

where the union is disjoint. Therefore,

$$
\begin{aligned}
s(b_1 \cup b_2) &= s\left(\bigcup_{j \in J_{b_1}} m_j \ \cup \ \bigcup_{j \in J_{b_2}} m_j \right) \\
&= s\left(\bigcup_{j \in \left(J_{b_1} \cup J_{b_2} \right)} m_j \right) \\
&= s\left(\bigcup_{j \in J_{b_1 \cup b_2}} m_j \right) && \text{(By ($\sqcup$))} \\
&= \sum_{j \in J_{b_1 \cup b_2}} s(m_j) && \text{(By def. of s.)} \\
&= \sum_{j \in \left(J_{b_1} \cup J_{b_2} \right)} s(m_j) && \text{(By ($\sqcup$))} \\
&= \sum_{j \in J_{b_1}} s(m_j) + \sum_{j \in J_{b_2}} s(m_j) \\
&= s(b_1) + s(b_2), && \text{(By def. of s)}
\end{aligned}
$$

and s is additive. By Lemma 1, it remains to show that s is normalised, i.e. satisfies $s(\top) = 1$. But since $\top = \bigcup_{j=1}^{2^n} m_j$, this is a consequence of additivity together with the assumption in item 2. □

Remark 3 Although here we only need the case of finite free algebras, it is clear that the statement in Lemma 2 holds for any finite Boolean algebra, with the same proof — atoms determine finitely additive probabilities in atomic Boolean algebras. For an extension of this basic fact to the (atomless!) free finitely generated MV-algebras of many-valued events, see [13, Lemma 7.2]. Lemma 3 also allows for other generalisations which we do not pursue here.

Next, to each vertex e_j of Δ_{2^n} there corresponds the Schauder hat H_j — the pyramidal function — with apex at e_j which is affine linear on each face of Δ_{2^n}. Of course, $\{H_j\}_{j=1}^{2^n} \subseteq \mathscr{S}_{2^n}$. Also, it is elementary that

$$
\sum_{j=1}^{2^n} H_j = 1, \tag{♭}
$$

where we use '1' to denote the function constantly equal to 1 over $\mathscr{S}_{2^n}$.

Lemma 4 *There exists exactly one generalised state $s \colon F_n \to \mathscr{S}_{2^n}$ that satisfies*

$$
s(m_j) = H_j \tag{5}
$$

for each $j = 1, \ldots, 2^n$.

Proof There is at most one such state, by Lemma 2. In light of (♭) and Lemma 3, there is at least one. □

Definition 4 For an integer $n \geq 1$ we define the *free state algebra* to be the state algebra $(F_n, \mathscr{S}_{2^n})$, where F_n is the Boolean algebra freely generated by n elements $x_1, \ldots, x_n$, $\mathscr{S}_{2^n}$ is the MV-algebra dual to the standard simplex Δ_{2^n}, and $s \colon F_n \to \mathscr{S}_{2^n}$ is the unique generalised state in Lemma 4.

Theorem 1 *For each integer $n \geq 1$, the free state algebra $(F_n, \mathscr{S}_{2^n})$ is freely generated by its multi-sorted subset $(\{x_1, \ldots, x_n\}, \emptyset)$.*

Proof Write $\iota \colon (\{x_1, \ldots, x_n\}, \emptyset) \to (F_n, \mathscr{S}_{2^n})$ for the two-sorted inclusion function $\iota := (\iota_1, \iota_2)$ such that ι_1 is the inclusion map $\{x_1, \ldots, x_n\} \subseteq F_n$, and $\iota_2 \colon \emptyset \to \mathscr{S}_{2^n}$ is the only possible function. Let (E', D') be any state algebra, and consider an arbitrary function

$$
\eta_1 \colon \{x_1, \ldots, x_n\} \longrightarrow E',
$$

which yields the obvious two-sorted function

$$
\eta \colon (\{x_1, \ldots, x_n\}, \emptyset) \longrightarrow (E', D').
$$

We define a function

$$
\overline{h}_1 \colon \{x_1, \ldots, x_n\} \longrightarrow E'
$$

by setting

$$
\overline{h}_1(x_i) := \eta_1(x_i), \ i = 1, \ldots, n. \tag{E}
$$

Since F_n is freely generated by $\{x_i\}_{i=1}^n$, there is exactly one homomorphism

$$
h_1 \colon F_n \longrightarrow E'
$$

that extends $\overline{h}_1$. We define a second function

$$
\overline{h}_2 \colon \{s(m_1), \ldots, s(m_{2^n})\} \longrightarrow D'
$$

by setting

$$
\overline{h}_2(s(m_j)) := s(h_1(m_j)), \ j = 1, \ldots, 2^n. \tag{D}
$$

(For the sake of clarity, let us stress that the symbol 's' on the right-hand side of (D) denotes the generalised state of (D', E'), whereas the one on the left-hand side is the generalised state in Lemma 4.) We show that $\overline{h}_2$, too, has a unique extension to a homomorphism. By (D) and Lemma 4 we have

$$
\overline{h}_2(H_j) = s(h_1(m_j)), \ j = 1, \ldots, 2^n.
$$

Further, let us prove

$$
\sum_{j=1}^{2^n} s(h_1(m_j)) = 1.
$$

Indeed, h_1 is a Boolean homomorphism and $\{m_j\}_{j=1}^{2^n}$ is a pairwise disjoint set whose join is $\top$, which implies

that $\{h_1(m_j)\}_{j=1}^{2^n} \subseteq E'$ is a pairwise disjoint set that joins to $\top$; by the additivity of s,

$$\sum_{j=1}^{2^n} s(h_1(m_j)) = s\left(\bigcup_{j=1}^{2^n} h_1(m_j)\right) = 1,$$

as was to be shown. Now, by the equality ($\sharp$) and the universal property of quotients, the set $\{H_j\}_{j=1}^{2^n}$ generates the MV-algebra $\mathscr{S}_{2^n}$ freely[4] subject to the relation ($\flat$). Since that relation was just shown to hold for $\{s(h_1(m_j))\}_{i=1}^{2^n}$, there is a unique extension of $\overline{h}_2$ to a homomorphism
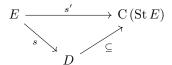
$$h_2 : D \longrightarrow D'.$$

Setting

$$h := (h_1, h_2) : (F_n, \mathscr{S}_{2^n}) \longrightarrow (E', D')$$

yields one homomorphism of state algebras that satisfies the universal property of free state algebras. The proof of the fact that any such homomorphism coincides with h is a straightforward direct verification, and is left to the reader. $\square$

Remark 4 The reader can now check the following. Let $s' : E \to \mathrm{C}(\mathrm{St}\, E)$ be the affine representation of the Boolean algebra E, as defined in Example 2 of Section 3. Let $D \subseteq \mathrm{C}(\mathrm{St}\, E)$ be the MV-subalgebra of $\mathrm{C}(\mathrm{St}\, E)$ generated by the image $s'[E]$ of E under s'. Then we have a factorisation

$$E \xrightarrow{\quad s' \quad} \mathrm{C}(\mathrm{St}\, E)$$
$$\searrow_s \qquad \nearrow_{\subseteq}$$
$$D$$

where $s : E \to D$ is a generalised state which yields a state algebra (E, D). Set $E := F_n$ in the foregoing, for $n \geq 1$ an integer. Then, as a corollary of Theorem 1: *The state algebra (E, D) is freely generated by the multi-sorted set $(\{x_1, \ldots, x_n\}, \emptyset)$.* Thus free state algebras arise from affine representations of Boolean algebras.

6 Further research

We briefly comment on a few research directions related to the algebraic framework for states presented here.

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[4] The theory of the relations satisfied by Schauder bases, which leads to the notion of abstract Schauder bases, was developed in the two papers [11], [12]. We see here a simple instance of that theory in action.

- *Fuzzy Probability Logic* FP(Ł) over infinite-valued Łukasiewicz logic was developed by Hájek in [10, Chapter 8.4]. The purpose of FP(Ł) is to formalize reasoning about properties of states. The distinctive feature of FP(Ł) is its two-level syntax: probability assessments are syntactically represented in the language by a unary modality read 'Probably', which applies to Boolean formulas only. The class of Boolean algebras and states provides a possible complete semantics for FP(Ł). It would be desirable, however, to develop an equivalent algebraic semantics in the varietal sense. State algebras seem to provide such an equivalent (multi-sorted) algebraic semantics for FP(Ł). We plan to give an account of this in a further paper.

- The traditional use of MV-algebras is to model degrees of *truth* in Łukasiewicz logic. In this note, however, MV-algebras are employed as abstract models of degrees of *probability* — for the first time, as far as we know. Like Janus, MV-algebras show one of their two faces according as they are used by way of domain or co-domain of a generalised state. This phenomenon calls for further reflection and research.

- One can apply Robinson's non-standard analysis in order to introduce infinitesimal degrees of probability. This requires fairly heavy machinery from first-order logic and model theory. In our Example 1, by contrast, *equational algebra only suffices*: MV-algebraic infinitesimals are considerably simpler than Robinson's infinitesimals. One is thus able to formally develop a theory of "infinitesimal degrees of probability" using standard algebra only. We plan to elaborate on this point in subsequent investigation.

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