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Probing deformed quantum commutators

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Several quantum gravity theories predict a minimal length at the order of magnitude of the Planck length, under which the concepts of space and time lose their physical meaning. In quantum mechanics, the insurgence of such a minimal length can be described by introducing a modified position-momentum commutator, which in turn yields a generalized uncertainty principle, where the uncertainty on position measurements has a lower bound. The value of the minimal length is not predicted by theories and must be estimated experimentally. In this paper, we address the quantum bound to the estimability of the minimal uncertainty length by performing measurements on a harmonic oscillator, which is analytically solvable in the deformed algebra induced by the deformed commutation relations.

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I. INTRODUCTION

The existence of a minimal length is a general feature of many quantum gravity theories (see Refs. [1,2] and references therein). According to these theories, the Planck length

$$\ell_{\mathrm{P}} = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-35}\ \mathrm{m} \tag{1}$$

sets an order of magnitude under which the concepts of space and time lose their physical meaning. In turn, this corresponds to the existence of a minimal uncertainty in position measurements, which sets a limit to the localizability of an object.

The uncertainty principle derived from the standard commutation relations between position and momentum does not predict the existence of any inferior bound to the position uncertainty, as the latter may be arbitrarily small, provided that the momentum uncertainty gets bigger. From this fact originates the idea of modifying the commutation relation between position and momentum, in order to obtain the prediction of a minimal position uncertainty [3–7].

In one dimension, let us consider the minimal deformation

$$[\boldsymbol{x},\boldsymbol{p}] = i\hbar\left[1+\beta_0\left(\frac{\ell_{\mathrm{P}} p}{\hbar}\right)^2\right], \tag{2}$$

with β_0 being a positive dimensionless parameter. It is easy to see that the following generalized uncertainty principle holds:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1+\beta_0\left(\frac{\ell_{\mathrm{P}}\Delta p}{\hbar}\right)^2\right]. \tag{3}$$

Equation (3) does indeed predict an inferior bound to position uncertainty, given by $\Delta x_0 = \ell_{\mathrm{P}}\sqrt{\beta_0}$.

The introduction of a deformed commutator as in Eq. (2) modifies the algebra of the Hilbert space and alters the spectral decomposition of the Hamiltonian operator of many quantum systems of theoretical and experimental interest. Among them, the harmonic oscillator is of paramount theoretical importance, and several studies have been focused on it in the context of deformed commutators [5,8,9]. The energy eigenvalues can be found analytically in an arbitrary number of dimensions and the eigenstates in the momentum basis can be obtained [5,9].

The value of β_0 in Eqs. (2) and (3), usually assumed to be around unity [10], has to be found experimentally since theoretical predictions are still lacking. Recently, beside proposed tests with high-energy or neutrino experiments [11,12], an optomechanical experimental scheme has been proposed [13], and an upper bound to the value of β_0 has been set in Ref. [14], using micro- and nanomechanical harmonic oscillators. Since β_0 does not correspond to a proper quantum observable, its value should be inferred through some indirect measurements, which causes an additional error in its estimation. In particular, if this extra uncertainty is too big compared to the value of the

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parameter, it may be intrinsically inestimable, and no experiment may be able to observe its presence.

The purpose of this work is to analyze the ultimate limits to precision in the estimation of β_0, exploiting tools from local quantum estimation theory (QET) [15–18], and presenting the results for a harmonic oscillator prepared in various initial states. Estimation theory provides a rigorous framework to determine the bound to the precision achievable in an estimation procedure of experimental data. This bound, known as the Cramér-Rao (CR) inequality [19], is connected to the Fisher information of the probability distribution. QET is a generalization to quantum systems: the ultimate bound to precision is found by optimizing the Fisher information over all the quantum measurements that can be made on a system. By providing the tools to find the optimal measurement and state preparation, QET allows one to go beyond standard classical limits in precision and has been successfully applied to a wide range of metrological problems [20,21], in particular in quantum interferometry and quantum optics [22], and in experiments with photons [23,24] and trapped ions [25,26].

Remarkably, the study of the modified algebra of the Hilbert space induced by the deformed commutators has highlighted a shortcoming of standard QET, that in turn has led us to a critical revision and generalization of the standard Cramér-Rao bounds [27], which we will discuss in the following. We also notice that a deformation of position commutators also occurs in other models, e.g., due to spin-induced uncertainty [28], and the corresponding effects may be observable at different length scales.

The paper is structured as follows. In Sec. II we report the solution of the eigenvalue problem for the harmonic oscillator in the modified algebra, reporting explicit expressions for the energy spectrum and for the eigenfunctions. In Sec. III we review some results of local QET, reporting the expression for the Fisher information (FI), quantum Fisher information (QFI), and estimability of a parameter. In Sec. IV we present the main results of our work. We discuss the modifications to QET required for this problem, and we show the ultimate bounds on precision in the measure of the parameter, calculating also the performance of the momentum operator. Analytical expansions for small values of β_0 are derived for FI and QFI relative to pure states. We also analyze the QFI and FI for mixed states and the thermal state. Finally, we analyze the dependence of the results on the mass and frequency of the oscillator, in order to find the best experimental configurations. Section V closes the paper with some concluding remarks.

II. HARMONIC OSCILLATOR

In this section we consider the linear harmonic oscillator in the algebra generated by $\boldsymbol{x}$ and $\boldsymbol{p}$ obeying the commutation relation

$$[\boldsymbol{x},\boldsymbol{p}] = i\hbar(1+\beta p^2), \tag{4}$$

with $\beta = \ell_P^2/\hbar^2\beta_0$, which has the units of inverse square momentum.

The action of position and momentum as differential operators in the momentum representation is given by

$$\boldsymbol{p}\psi(p) = p\psi(p), \tag{5}$$

$$\boldsymbol{x}\psi(p) = i\hbar(1+\beta p^2)\partial_p\psi(p). \tag{6}$$

For the operators $\boldsymbol{x}$ and $\boldsymbol{p}$ to be symmetric, and thus represent physical observables, the scalar product of the Hilbert space must be modified:

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} dp\mu_\beta(p)\psi^*(p)\phi(p) \tag{7}$$

$$1 = \int_{-\infty}^{+\infty} dp\mu_\beta(p)|p\rangle\langle p|, \tag{8}$$

where

$$\mu_\beta(p) = \frac{1}{(1+\beta p^2)}. \tag{9}$$

The presence of the nontrivial integration measure $\mu_\beta(p)$ has a remarkable impact on the estimatibility of β, as we will explain in the following section.

The Hamiltonian of the harmonic oscillator,

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + m\omega^2\frac{\boldsymbol{x}^2}{2}, \tag{10}$$

leads to the following stationary Schrödinger equation in the momentum representation:

$$\left[-\frac{\hbar^2 k}{2}\left((1+\beta p^2)\frac{\partial}{\partial p}\right)^2 + \frac{p^2}{2m}\right]\psi(p) = E\psi(p), \tag{11}$$

where $k = m\omega^2$.

The solution of Eq. (11) has been addressed in Ref. [5] and, in a different way, in Ref. [9]. In the former, the solutions were found using the general theory of totally Fuchsian equations, in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$, while in the latter it was given in terms of the Gegenbauer polynomials $C_n^{(\lambda)}(s)$. The solutions of Refs. [5] and [9] in the momentum basis are, respectively,

$$\psi_n(p) = \mathcal{N}_n(1+\beta^2)^{-\frac{1}{2}(n+\lambda)}{}_2F_1\left(-n, 1-n-2\lambda; 1-n-\lambda; \frac{1}{2}(1+ip\sqrt{\beta})\right) \tag{12}$$

$$=\frac{\sqrt{4}\beta 2^{\lambda-\frac{1}{2}}}{\sqrt{\pi}}\Gamma(\lambda)\sqrt{\frac{n!(\lambda+n)}{\Gamma(n+2\lambda)}}(1+\beta p^2)^{-\frac{\lambda}{2}}C_n^{(\lambda)}\left(p\sqrt{\frac{\beta}{1+\beta p^2}}\right), \tag{13}$$

where $\lambda=\frac{1}{2}\{1+\sqrt{1+4/[(\hbar m\omega)^2\beta^2]}\}$ and $\mathcal{N}_n$ is a normalization constant. The relation between these two solutions involves transformation formulas for the hypergeometric functions. Besides, in Ref. [5] the normalization constant $\mathcal{N}_n$ of Eq. (12) was not derived explicitly. The two solutions are compared in the Appendix, where the normalization constant is found to be

$$\mathcal{N}_n=\frac{(-i)^n\sqrt{\pi}\sqrt{4}\beta 2^{\lambda+n-\frac{1}{2}}}{\sin(\pi\lambda)\Gamma(1-n-\lambda)}\sqrt{\frac{\lambda+n}{n!\Gamma(n+2\lambda)}}. \tag{14}$$

The energy eigenvalues, according to Refs. [5,8,9], are

$$E_n=\frac{k}{2}\left[\left(n+\frac{1}{2}\right)(\Delta x_0^2+\sqrt{\Delta x_0^4+4a^4})+\Delta x_0^2 n^2\right], \tag{15}$$

with $\Delta x_0=\hbar\sqrt{\beta}$ and $a=\sqrt{\frac{\hbar}{m\omega}}$.

III. LOCAL QUANTUM ESTIMATION THEORY

The parameter β introduced in the commutator (4) does not correspond to a proper quantum observable and it cannot be measured directly. In order to get information about β, we have to resort to indirect measurements, inferring its value by the measurements of a different observable or a set of observables; that is, we have a parameter estimation problem.

QET provides tools to find the optimal measurement according to some given criterion. In this context we exploit local QET which looks for the quantum measurement that maximizes the so-called *Fisher information*, i.e., minimizing the variance of the estimator at a fixed value of the parameter. Our aim is to evaluate the ultimate bound on precision, i.e., the smallest value of the parameter that can be discriminated, and to determine the optimal measurement achieving these bounds.

In the following, we briefly review the main concepts of local QET and set the notation for the rest of the paper. We refer the reader to Ref. [18] for a more detailed review of the subject. In the following section we also discuss the generalization of standard QET that is required in the problem at hand, in which the geometry of the Hilbert space is affected by the minimal length, i.e., by the parameter to be estimated.

In order to solve an estimation problem we have to find an estimator, i.e., a map from the set of measurements $x_1,x_2,\ldots,x_n$ into the space of parameters β:

$$\hat{\beta}=\hat{\beta}(x_1,x_2,\ldots,x_n). \tag{16}$$

Optimal estimators are those saturating the Cramér-Rao inequality [19]

$$\mathrm{Var}(\beta)\geq\frac{1}{MF(\beta)}, \tag{17}$$

which sets a lower bound on the variance $\mathrm{Var}(\beta)=E_\beta[(\hat{\beta}(x)-\beta)^2]$ of any estimator. M is the number of measurements and $F(\beta)$ is the Fisher information, defined by

$$F(\beta)=\int dxP(x|\beta)(\partial_\beta\ln P(x|\beta))^2, \tag{18}$$

where $P(x|\beta)$ is the probability of obtaining the value x when the parameter has the value β, and ∂_β is a shorthand for $\frac{\partial}{\partial\beta}$.

In quantum mechanics, we consider a *quantum statistical model*, i.e., a family of quantum states ρ_β defined on a Hilbert space H and labeled by the parameter β which in our problem is real and positive. We want to estimate its value through the measurement of some observable on the state ρ_β. A quantum estimator for the parameter β is a pair, consisting of a positive-operator-valued measurement (POVM) and a classical estimator that accounts for the post-processing of the sampled data. The choice of the quantum measurement is the central problem of QET, since different choices in general lead to different attainable precisions.

In quantum mechanics the probability of a certain outcome is given by the Born rule $P(x|\beta)=\mathrm{Tr}[\Pi_x\rho_\beta]$, where Π_x, are the elements of the POVM we measure and satisfy $\int dx\Pi_x=1$. The FI is then written as

$$F(\beta)=\int dx\frac{[\partial_\beta\mathrm{Tr}(\Pi_x\rho_\beta)]^2}{\mathrm{Tr}(\Pi_x\rho_\beta)}. \tag{19}$$

Upon defining the symmetric logarithmic derivative (SLD) L_β as the self-adjoint operator satisfying the equation

$$\frac{L_\beta\rho_\beta+\rho_\beta L_\beta}{2}=\frac{\partial\rho_\beta}{\partial\beta}, \tag{20}$$

we have that the FI $F(\beta)$ of any POVM is bounded [17] by the so-called quantum Fisher information $H(\beta)$:

$$F(\beta)\leq H(\beta)\equiv\mathrm{Tr}[\rho_\beta {L_\beta}^2]=\mathrm{Tr}[\partial_\beta\rho_\beta L_\beta]. \tag{21}$$

The Cramér-Rao inequality now takes the form

$$\mathrm{Var}(\beta) \geq \frac{1}{MH(\beta)}, \tag{22}$$

which gives the ultimate bound to precision for any unbiased estimator of β.

Equation (20) is a Lyapunov matrix equation and a general solution exists. An explicit form for the symmetric logarithmic derivative can be given in the basis in which the density operator is diagonal. Upon writing

$$\rho_\beta = \sum_n p_n(\beta)|\psi_n(\beta)\rangle\langle\psi_n(\beta)|, \tag{23}$$

where $\{|\psi_n\rangle\}$ is a complete set in the Hilbert space, we have [18]

$$L_\beta = 2\sum_{nm} \frac{\langle\psi_m|\partial_\beta\rho_\beta|\psi_n\rangle}{p_n + p_m}|\psi_m\rangle\langle\psi_n|, \tag{24}$$

where it is understood that the sum is on the indices for which $p_n + p_m \neq 0$. From Eq. (24) follows the explicit formula for the QFI,

$$H(\beta) = 2\sum_{nm} \frac{|\langle\psi_m|\partial_\beta\rho_\beta|\psi_n\rangle|^2}{p_n + p_m}. \tag{25}$$

The expression of the QFI gets simpler when we consider a family of pure states described by the wave function ψ_β. In standard quantum mechanics it is straightforward to find that the SLD is $L_\beta = 2\partial_\beta\rho_\beta$ by noticing that $\partial_\beta\rho_\beta = \partial_\beta(\rho_\beta^2) = \partial_\beta\rho_\beta\rho_\beta + \rho_\beta\partial_\beta\rho_\beta$, where ρ_β is a projector onto the pure state [18]. This yields

$$H(\beta) = 4(\langle\partial_\beta\psi|\partial_\beta\psi\rangle + \langle\partial_\beta\psi|\psi\rangle^2). \tag{26}$$

From a geometrical perspective, the precision in the estimation of the parameter β is related to the distinguishability of the corresponding state ρ_β from its neighbors. If we have to discriminate between the two values β and $\beta + d\beta$, with $d\beta$ infinitesimal, a larger "distance" between ρ_β and $\rho_{\beta+d\beta}$ generally corresponds to an easier discrimination by quantum-limited measurement on the system. Among the different definitions of distance that can be made on the manifold of quantum states, the one that turns out to capture the notion of estimation measure is the Bures distance [29,30], defined as

$$D_B(\rho_1, \rho_2) = \sqrt{2[1 - F(\rho_1, \rho_2)]}, \tag{27}$$

where $F(\rho_1, \rho_2) = \mathrm{Tr}[(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{1/2}]$ is the quantum fidelity between the states ρ_1 and ρ_2 [31]. By evaluating the infinitesimal Bures distance explicitly, one finds that the Bures metric is indeed proportional to the QFI [32].

In order to quantify the performance of an estimator and thus the estimability of a certain parameter, a relevant figure of merit is the signal-to-noise ratio (SNR)

$$R_\beta \equiv \beta^2 F(\beta) \geq \frac{\beta^2}{\mathrm{Var}(\beta)}, \tag{28}$$

which is larger for a better estimator. We can easily derive an upper bound for this ratio using the Cramér-Rao inequality, obtaining

$$R_\beta \leq Q_\beta \equiv \beta^2 H(\beta), \tag{29}$$

which we refer to as the quantum signal-to-noise ratio (QSNR). The larger the quantities $R(\beta)$ and $Q(\beta)$ the smaller the relative error in the estimation of the parameter β.

IV. QUANTUM LIMITS TO PRECISION IN PROBING DEFORMED COMMUTATORS

We investigate the value of the QFI and the performance of a momentum measurement through the calculation of the FI as functions of β for different states of the harmonic oscillator. In this way we find the estimability and the precision available through a momentum measurement as a function of the value of β, clarifying what values of β could allow better estimation through experiments. In the following, we take $\hbar = 1$ and $k_B = 1$. The parameters characterizing the harmonic oscillator, i.e., its mass m and its pulsation ω are initially taken equal to 1. We discuss the dependence of the QFI and FI on these parameters in Sec. IV D.

In the last section we discussed the tools of QET. In the problem at hand, however, standard QET has proven to be inaccurate, due to the particular geometry of the Hilbert space induced by the deformed commutators (2). Indeed the scalar product has a nontrivial measure $\mu_\beta(p)$ [Eq. (9)] that depends on the parameter β. This in turn introduces a β-dependent measure in the sample space on which the probability $P(p|\beta)$ is defined, thus making the Cramér-Rao surpassable. This situation has been addressed recently in Ref. [27], where an additional contribution to the FI was introduced. Let us redefine the FI as

$$\mathcal{F}(\beta) = F(\beta) + \mathcal{I}_\mu(\beta), \tag{30}$$

where

$$\mathcal{I}_\mu(\beta) = \int dp\mu_\beta(p)P(p|\beta)[\partial_\beta \log \mu_\beta(p)]^2. \tag{31}$$

Correspondingly, we redefine the SNR $\mathcal{R}(\beta) \equiv \beta^2\mathcal{F}(\beta)$. As $\mathcal{I}_\mu$ is a positive quantity, it follows that Eq. (22) does not give the ultimate bound to the variance of any estimator of β. It is not known whether $\mathcal{F}$ in Eq. (30) can be optimized over all possible quantum measurements so that a new quantum Cramér-Rao bound can be found.

A. Pure states

We first consider the estimation of β from a measurement on the harmonic oscillator prepared in a pure state $|\psi_\beta\rangle$. Equation (26), derived in Sec. III, does not hold here because $\partial_\beta(\rho_\beta^2) \neq \partial_\beta\rho_\beta\rho_\beta + \rho_\beta\partial_\beta\rho_\beta$. Nevertheless, we can obtain a simplified expression for the QFI starting from Eq. (25). We write $\rho_\beta = \sum_n p_n|\phi_n\rangle\langle\phi_n|$, where $|\phi_0\rangle \equiv |\psi_\beta\rangle$, $p_n = \delta_{n0}$, and $\{|\phi_n\rangle\}_{n\neq0}$ form a basis of the subspace orthogonal to $|\psi_\beta\rangle$. We obtain

$$\begin{aligned}H(\beta) &= 2\sum_{\substack{n,m\\ \delta_{n0}+\delta_{m0}\neq0}} \frac{|\delta_{m0}\langle\partial_\beta\phi_0|\phi_n\rangle + \delta_{n0}\langle\phi_n|\partial_\beta\phi_0\rangle|^2}{\delta_{n0}+\delta_{m0}} \\ &= |\langle\partial_\beta\phi_0|\phi_0\rangle + \langle\phi_0|\partial_\beta\phi_0\rangle|^2 + 4\sum_{n=1}^{\infty}|\langle\phi_n|\partial_\beta\phi_0\rangle|^2 \\ &= 4\langle\partial_\beta\psi_\beta|\partial_\beta\psi_\beta\rangle - 4\mathrm{Im}(\langle\psi_\beta|\partial_\beta\psi_\beta\rangle)^2. \end{aligned} \tag{32}$$

Consider now a momentum measurement on the state described by the wave function $\psi_\beta(p)$. The probability of getting p as an outcome is given by $P(p|\beta) = |\psi_\beta(p)|^2$, so the corresponding FI (30) is

$$\mathcal{F}(\beta) = \int dp\left\{\mu_\beta\frac{[\partial_\beta|\psi_\beta|^2]^2}{|\psi_\beta|^2} + |\psi_\beta|^2\frac{[\partial_\beta\mu_\beta]^2}{\mu_\beta}\right\}. \tag{33}$$

Notice that if the wave function $\psi_\beta(p)$ is real, the first term of Eq. (33), corresponding to $F(\beta)$, is equal to the QFI (32). Thus the FI for the momentum measurement is greater than the QFI and the standard Cramér-Rao bound is violated.

Using Eq. (32) and performing numerical integration of the scalar product, we calculate the QFI $H(\beta)$ for the first eigenstates of the harmonic oscillator. In all cases $H(\beta)$ is a decreasing function of β, but looking at the estimability $Q(\beta)$, which is the relevant quantity to consider, we have an increasing function of the parameter. If we consider eigenstates of higher energy, the QFI increases as can be checked numerically.

Since the value of β is believed to be much smaller than one, the wave functions in Eqs. (12) and (13) and the QFI (32) can be expanded around $\beta = 0$ in order to get analytic solutions which confirm the consistency of the numerical integrations. We obtain the following polynomial expressions:

$$H_{\psi_0}(\beta) = \frac{9}{8} - \frac{53}{8}\beta + \frac{803}{32}\beta^2 + O(\beta^3), \tag{34}$$

$$H_{\psi_1}(\beta) = \frac{45}{8} - \frac{351}{8}\beta + \frac{7633}{32}\beta^2 + O(\beta^3), \tag{35}$$

$$H_{\psi_2}(\beta) = \frac{123}{8} - \frac{1255}{8}\beta + \frac{36401}{32}\beta^2 + O(\beta^3). \tag{36}$$

Figure 1 compares the analytical results with the numerical findings at various values of β. For $\beta \lesssim 0.01$, i.e., the expected range of values for β [13], the approximation is very good with a relative error of at most 10^{-3}.

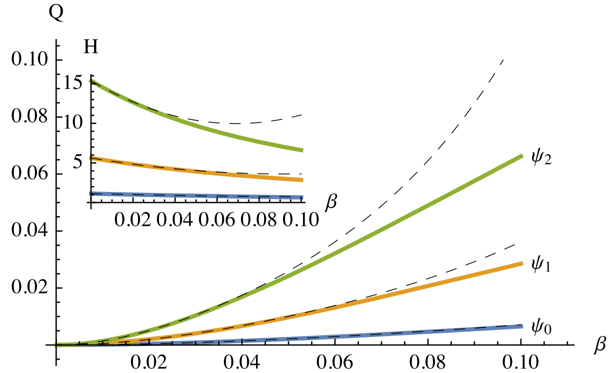

FIG. 1. From bottom to top, the estimability $Q(\beta)$ for the ground state (blue) and the first (orange) and second excited state (green) as obtained by numerical integration of the scalar product in Eq. (32). In the inset, we show the QFI $H(\beta)$ for the same states. The dashed lines are obtained from the Taylor expansions of the QFI [Eqs. (34)–(36)]. The estimability increases as we employ more excited states. The QFI decreases with β. In the region $\beta \lesssim 0.01$ the Taylor expansion provides a very good approximation.

The term $\mathcal{I}_\mu(\beta)$, for small β, reads

$$\mathcal{I}_{\mu,\psi_0}(\beta) = \frac{3}{4} - 3\beta + 9\beta^2 + O(\beta^3), \tag{37}$$

$$\mathcal{I}_{\mu,\psi_1}(\beta) = \frac{15}{4} - \frac{45}{2}\beta + \frac{405}{4}\beta^2 + O(\beta^3), \tag{38}$$

$$\mathcal{I}_{\mu,\psi_2}(\beta) = \frac{39}{4} - \frac{165}{2}\beta + \frac{2043}{4}\beta^2 + O(\beta^3). \tag{39}$$

Notice that $\mathcal{I}_{\mu,\psi_n}(\beta) \simeq 2/3H_{\psi_n}(\beta) = F_{\psi_n}(\beta)$: the integration-measure term of $\mathcal{F}(\beta)$ gives a relevant contribution to the estimability of β through a momentum measurement.

We also studied the behavior of the QFI of the generic superposition of the ground and first excited state, to determine if the best estimability is attained by choosing the first excited state. The system is thus described by

$$|\psi\rangle = \cos(\phi)|\psi_0\rangle + \sin(\phi)|\psi_1\rangle \tag{40}$$

and the QFI is a function of the parameters β and ϕ. The QFI has been calculated through numerical integration and it is shown in Fig. 2 (left): the maximal values of the function are obtained for $\phi \to \pi/2$ and $\phi \to 3/2\pi$, i.e., the first excited state is the optimal state among those of Eq. (40). This can be seen numerically for arbitrary β and analytically for small β, when the following expression holds:

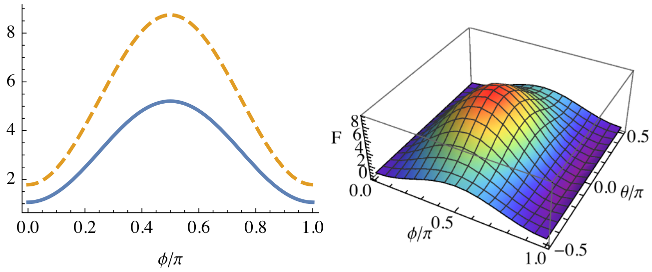

FIG. 2. Left: QFI (solid blue) and FI (dashed orange) relative to the state $|\psi\rangle = \cos(\phi)|\psi_0\rangle + \sin(\phi)|\psi_1\rangle$ as functions of ϕ, with $\beta = 0.01$. The maximal values are reached when $\phi \to \pi/2$: among the superpositions of $|\psi_0\rangle$ and $|\psi_1\rangle$ the optimal state is the first excited state. Right: FI as a function of the angles θ and ϕ for a superposition of the first three eigenstates, cf. Eq. (42), for $\beta = 10^{-2}$. We can see that the maximal QFI is attained when $\theta = 0$ and $\phi \simeq 0.43\pi$, i.e., when the system is in a superposition of the states $|\psi_0\rangle$ and $|\psi_2\rangle$.

$$H(\beta) = H_{\psi_0}(\beta) + [H_{\psi_1}(\beta) - H_{\psi_0}(\beta)]\sin^2\phi. \quad (41)$$

We now consider a superposition of the first three eigenstates of the harmonic oscillator:

$$|\psi\rangle = \cos\phi|\psi_0\rangle + \sin\phi\sin\theta|\psi_1\rangle + \sin\phi\cos\theta|\psi_2\rangle. \quad (42)$$

In this case, the optimal state is not $|\psi_2\rangle$ as one would expect, given the previous result. The right panel of Fig. 2 shows the QFI for the superposition of the form of Eq. (42) as a function of θ and ϕ. $|\psi_2\rangle$ is given by $\theta = 0$ and $\phi = \pi/2$ but the maximum is for $\theta = 0$ and $\phi \simeq 0.43\pi$. Thus, in general, the eigenstates of the harmonic oscillator are not the states that give the best estimability.

B. Mixed states

When the system is prepared in a mixed state $\rho_\beta = \sum_m p_m|\psi_m\rangle\langle\psi_n|$, by expanding $\partial_\beta\rho$ in Eq. (25) we obtain the following formula for the QFI:

$$H(\beta) = 2\sum_{nm}\frac{1}{p_n + p_m}|\partial_\beta p_m\delta_{mn} + p_n\langle\psi_m|\partial_\beta\psi_n\rangle + p_m\langle\partial_\beta\psi_m|\psi_n\rangle|^2. \quad (43)$$

The FI $\mathcal{F}(\beta)$ for the momentum measurement (30), on the other hand, is given by the two contributions

$$F(\beta) = \sum_n p_n\int dp\mu_\beta(p)|\psi_n(p)|^2\partial_\beta\ln|\psi_n(p)|^2 \quad (44)$$

and

$$\mathcal{I}_\mu = \sum_n p_n\int dp\mu_\beta(p)|\psi_n(p)|^2\partial_\beta\ln\mu_\beta(p). \quad (45)$$

As an example, we consider the estimation of β from a measurement on the harmonic oscillator prepared in a generic statistical mixture of the ground and the first excited state. The system is thus described by the statistical operator

$$|\psi\rangle\langle\psi| = \cos(\theta)^2|\psi_0\rangle\langle\psi_0| + \sin(\theta)^2|\psi_1\rangle\langle\psi_1|. \quad (46)$$

We performed numerical integration of Eqs. (43) and (44) and the results are shown in Fig. 3. The FI is much higher than the QFI due to the contribution of the term $\mathcal{I}_\mu$. While for $\theta \to 0$ and $\theta \to \frac{\Pi}{2}$ (i.e., when the state is pure) $F(\beta) = H(\beta)$, for intermediate values of θ, $F(\beta)$ does not saturate the QFI, as we see in Fig. 3. Thus, while in general the momentum measurement is not optimal for mixed states, the FI is much greater than the QFI due to the dependence of the geometry of the Hilbert space on β.

C. Thermal state

In a typical experimental setup it is generally challenging to prepare the oscillator in a pure state. Due to the interaction with the environment, the system will most likely be in a thermal state characterized by a temperature T. The density operator describing the state is then

$$\rho_T = Z^{-1}\sum_n e^{-E_n(\beta)/T}|\psi_n\rangle\langle\psi_n|, \quad (47)$$

where $Z = \sum_n e^{-E_n(\beta)/T}$ is the partition function of the thermal distribution. What is the maximum precision achievable if the oscillator is in the thermal state ρ_T? We focus on states with temperatures close to zero (compared to the ground-state energy) so that only the lower eigenstates have significant populations. Indeed, the scalar products of the form $\langle\partial_\beta\psi_n|\psi_m\rangle$ that appear in Eq. (43),

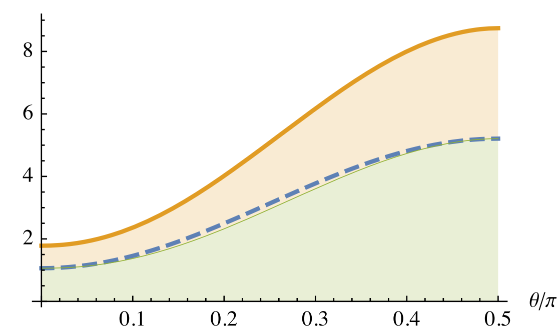

FIG. 3. Comparison of QFI (dashed blue) and FI $\mathcal{F}$ (solid orange) for the statistical mixture of the ground and first excited state [Eq. (46)] as a function of θ, with $\beta = 0.01$. The two shaded regions represent the contributions to the FI coming from $F(\beta)$ (bottom, green) and $\mathcal{I}_\mu$ (top, orange), cf. Eq. (30). The FI is much greater than the QFI due to the relevant contribution of the integration-measure term $\mathcal{I}_\mu$. For $\theta = 0$ and $\theta = \Pi/2$ (i.e., for pure states) $F(\beta)$ is equal to the QFI, while for intermediate values of θ it is slightly lower, which means that the momentum measurement is not the optimal one (in the sense of the standard QET).

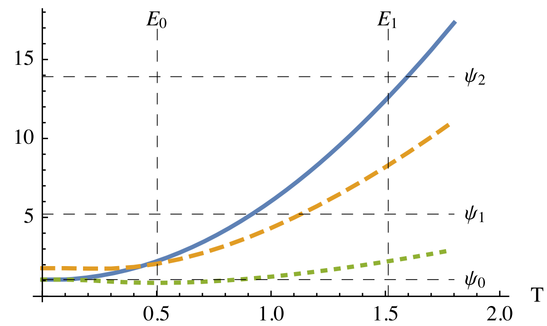

FIG. 4. QFI $H(\beta)$ (solid blue), FI $\mathcal{F}(\beta)$ (dashed orange), and $F(\beta)$ (dotted green) as functions of T for $\beta = 0.01$ (with $\hbar = k_B = 1$). The FI and QFI increase with temperature, because higher eigenstates of the oscillator are populated, but the performance of the momentum measurement gets worse as temperature increases. For T close to zero $\mathcal{F}(\beta)$ violates the CR bound, but at a temperature comparable with $E_0(\beta) \simeq \frac{1}{2} + \frac{\beta}{4}$, $\mathcal{F}(\beta)$ gets lower than the QFI.

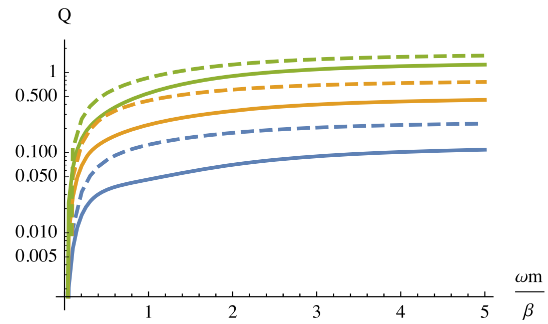

FIG. 5. Log-plot of the quantum estimability $Q(\beta)$ (solid lines) and estimability $\mathcal{R}(\beta)$ for the momentum measurement (dashed lines) as functions of $\omega m/\beta$ for the pure states (from bottom to top) ψ_0, ψ_1, and ψ_2. The plots do not depend on $\beta.Q(\beta)$ and $\mathcal{R}(\beta)$ increase with the product ωm and reach the limits reported in Eq. (49).

for high m and n, involve highly oscillating functions and are thus hard to compute numerically to an acceptable accuracy.

As can be seen in Fig. 4, the QFI and FI are increasing functions of T. This is due to the fact that the population of higher eigenstates increases with T and the QFI and FI increase with the energy of the eigenstate. When $T \lesssim E_0$, $\mathcal{F}(\beta)$ is greater than $H(\beta)$, violating the quantum Cramér-Rao bound; on the other hand, when the temperature increases, the momentum measurement is not optimal anymore.

D. Dependence on m and ω

In the previous section we have shown the behavior of the QFI as a function of β assuming $\omega = 1$ and $m = 1$. In this section we show how the QFI depends on the mass and frequency of the harmonic oscillator.

By looking at Eqs. (12) and (13), we notice that the eigenstates of the harmonic oscillator depend on m and ω only through the product $\omega m \beta$ in the term λ.

As we see in Fig. 5, in the example of the ground state, $H(\beta)$ is an increasing function of ωm. We can analytically obtain the limits for $\omega m \to 0$,

$$H(\beta) \underset{\omega m \to 0}{\to} 0, \tag{48}$$

and $\omega m \gg \beta$,

$$Q_{\psi_0}(\beta) \sim \frac{1}{8}, \qquad Q_{\psi_1}(\beta) \sim \frac{1}{2}, \qquad Q_{\psi_2}(\beta) \sim \frac{11}{8}. \tag{49}$$

As for the FI, we find that for large ωm the SNR is twice the QSNR: $\mathcal{R}_{\psi_n}(\beta) \sim 2Q_{\psi_n}(\beta)$. Equation (49) shows that the SNR and QSNR of β do not depend on its value for large enough ωm.

V. CONCLUSIONS

Although a minimal length at the Planck scale is predicted by many theories of quantum gravity, due to the lack of theoretical predictions about its value and the formidable technological challenges required, experimental tests have been so far inconclusive. The aim of this paper is to provide theoretical tools to asses the best achievable precision in the estimation of the deformation of the canonical commutation relations induced by the minimal length. We focused on measurements on a harmonic oscillator, a relevant testbed both from a theoretical point of view, as it is analytically solvable, and from an experimental point of view, since experiments can and have been made with nanomechanical and optomechanical oscillators.

We have shown that a measurement of the momentum is optimal if the oscillator is in a pure state and the achievable precision goes beyond the bounds of standard quantum estimation theory. This is a relevant result, due to the altered geometry of the Hilbert space, and shows the necessity of redefining the quantities of QET in a more general way [27].

Our results indicate that the estimability improves by preparing the oscillator in a higher-energy eigenstate. Moreover, increasing the mass and frequency of the oscillator allows for better precision and the temperature is not detrimental for the probing, although the momentum measurement ceases to be the optimal measurement as the temperature increases above the energy of the ground state.

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APPENDIX: RELATION BETWEEN THE SOLUTIONS OF THE HARMONIC OSCILLATOR IN THE MOMENTUM BASIS

Here we show the relation between the two solutions of the harmonic oscillator. We also find the normalization constant $\mathcal{N}_n$ for the solution (12), involving the hypergeometric function.

The solution of Ref. [9] is normalized. Let us start from Eq. (12) and show that it can be cast into the form of Eq. (13). We assume that n is even, i.e., we set $n = 2\nu$, with $\nu \in N$. The case with odd n is analogous. The argument of ${}_2F_1$ in Eq. (12) is complex, but we can apply Kummers' quadratic transformation (15.8.18) from Ref. [33] to obtain

$$\psi_n(p) = \mathcal{N}_n(1+\beta^2)^{-\frac{\lambda}{2}-\nu}{}_2F_1\left(-\nu, \frac{1}{2}-\lambda-\nu; 1-\lambda-2\nu; 1+\beta p^2\right). \tag{A1}$$

Next, we apply Eq. (15.8.6) of Ref. [33] to invert the argument of ${}_2F_1$: we end up with

$$\psi_n(p) = \frac{\mathcal{N}_n\sqrt{\pi}(-1)^\nu 4^{-\lambda-\nu}}{(1+\beta p^2)^{\frac{\lambda}{2}}}\frac{\sec(\pi\lambda)\Gamma(-\lambda-2\nu+1)}{\Gamma(\lambda+\frac{1}{2})+\Gamma(-2\lambda-2\nu+1)}{}_2F_1\left(-\nu, \lambda+\nu; \lambda+\frac{1}{2}; \frac{1}{1+\beta p^2}\right). \tag{A2}$$

By applying Eq. (20) of Ref. [34] and by plugging n back in, we finally reach the functional form of Eq. (13):

$$\psi_n(p) = \mathcal{N}_n\frac{i^n n!\sin(\pi\lambda)\Gamma(\lambda)\Gamma(1-n-\lambda)}{(-2)^n\pi}(1+\beta p^2)^{-\frac{\lambda}{2}}C_n^{(\lambda)}\left(\sqrt{\frac{\beta p^2}{1+\beta p^2}}\right). \tag{A3}$$

The same result can be obtained for odd n by applying Eq. (21) of Ref. [34].

By comparing Eqs. (A3) and (12) we obtain an expression for the normalization constant,

$$\mathcal{N}_n = \frac{(-i)^n\sqrt{\pi}\sqrt{4\beta}2^{\lambda+n-\frac{1}{2}}}{\sin(\pi\lambda)\Gamma(1-n-\lambda)}\sqrt{\frac{\lambda+n}{n!\Gamma(n+2\lambda)}}. \tag{A4}$$

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