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On Elliptic Equations and Systems with Critical Growth in Dimension Two

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Dedicated to Professor S. Nikol'skii
on the occasion of his 100th birthday

Abstract—We consider nonlinear elliptic equations of the form $-\Delta u = g(u)$ in Ω, $u = 0$ on $\partial\Omega$, and Hamiltonian-type systems of the form $-\Delta u = g(v)$ in Ω, $-\Delta v = f(u)$ in Ω, $u = 0$ and $v = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^2$ and $f, g \in C(\mathbb{R})$ are superlinear nonlinearities. In two dimensions the maximal growth (= critical growth) of f and g (such that the problem can be treated variationally) is of exponential type, given by Pohozaev–Trudinger-type inequalities. We discuss existence and nonexistence results related to the critical growth for the equation and the system. A natural framework for such equations and systems is given by Sobolev spaces, which provide in most cases an adequate answer concerning the maximal growth involved. However, we will see that for the system in dimension 2, the Sobolev embeddings are not sufficiently fine to capture the true maximal growths. We will show that working in *Lorentz spaces* gives better results.

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1. INTRODUCTION

In this paper we consider nonlinear elliptic equations of the form

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$

and Hamiltonian-type systems of the form

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = 0 \ \text{ and } \ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

where Ω is a bounded domain in $\mathbb{R}^2$. Here, f and g are continuous and superlinear functions.

Consider first equation (E): its solutions correspond to the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(u)\,dx,$$

where $G(t) = \int_0^t g(s)\,ds$ denotes the primitive of $g(t)$. Considering this functional on the Sobolev space $H_0^1(\Omega)$, one obtains, by the well-known inequality of Pohozaev [20] and Trudinger [25], a natural growth restriction (critical growth) on $G(t)$. This inequality says that if u is a $H_0^1(\Omega)$ function, then the integral $\int_\Omega e^{u^2}dx$ is finite. Furthermore, by the sharpened form of this inequality

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due to J. Moser [19], one has

$$\sup_{\|u\|_{H_0^1}\le 1} \int_\Omega e^{\alpha u^2}\,dx \begin{cases} <+\infty & \text{if } \alpha \le 4\pi, \\ =+\infty & \text{if } \alpha > 4\pi. \end{cases} \tag{1}$$

The problems that arise naturally are

(i) existence of solutions if g has a subcritical growth,

(ii) noncompactness and related phenomena if g has a critical growth.

Turning to system (S), we first note that the natural functional associated with (S) has the form

$$J(u,v) = \int_\Omega \nabla u \nabla v\,dx - \int_\Omega F(u)\,dx - \int_\Omega G(v)\,dx,$$

where F and G are again the primitives of f and g, respectively. The functional $J(u,v)$ can again be defined on the space $H_0^1(\Omega)\times H_0^1(\Omega)$, and then one finds as before a maximal growth of type

$$f(t)\sim e^{t^2}, \qquad g(t)\sim e^{t^2}.$$

Again, the same questions (i) and (ii) as for equation (E) arise. We note that the functional $J(u,v)$ is considerably more difficult to handle than $I(u)$, since $J(u,v)$ is strongly indefinite (i.e., is positive and negative definite on ∞-dimensional subspaces). On the other hand, we note that the choice of the space $H_0^1\times H_0^1$ is not as compelling as in the case of $I(u)$. Indeed, the term $\int_\Omega \nabla u\nabla v\,dx$ in $J(u,v)$ is also defined on $W_0^{1,\alpha}(\Omega)\times W_0^{1,\beta}(\Omega)$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, in view of the estimate

$$\left|\int_\Omega \nabla u\nabla v\,dx\right| \le \|\nabla u\|_\alpha \|\nabla v\|_\beta.$$

We note that by Sobolev's embedding theorem, the choice $0<\alpha<2$ implies

$$W_0^{1,\alpha}(\Omega)\subset L^{\frac{N\alpha}{N-\alpha}}(\Omega),$$

and furthermore, since $\beta=\frac{\alpha}{\alpha-1}>2$, we get

$$W_0^{1,\beta}(\Omega)\subset L^\infty(\Omega).$$

Thus, we have a maximal growth of polynomial type for F,

$$|F(s)|\le |s|^{\frac{N\alpha}{N-\alpha}},$$

and *no growth restriction* on the nonlinearity G. So this choice of spaces brings us immediately outside the range of exponential nonlinearities.

However, a refined choice of spaces is possible, namely, the so-called *Lorentz spaces* or, more precisely, *Sobolev–Lorentz spaces*, which are interpolation spaces between the usual Sobolev spaces. This approach allows us to derive an "exponential critical curve," in complete analogy to the so-called "critical hyperbola" in the case of $N\ge 3$. Again, the same questions (i) and (ii) may now be asked with respect to this new critical growth.

2. EQUATION (E)

In this section we consider equation (E). We say that $g(t)$ has a *subcritical growth* if

$$\lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = 0 \qquad \text{for every} \quad \alpha > 0, \tag{2}$$

and $g(t)$ has a *critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = 0 \quad \text{if} \quad \alpha > \alpha_0 \qquad \text{and} \qquad \lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = +\infty \quad \text{if} \quad 0 < \alpha < \alpha_0. \tag{3}$$

We first consider

2.1. Subcritical growth. Concerning the subcritical growth, one has the following existence result:

Theorem 2.1 (see [4, 14]). *Assume that* $f\colon \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the following conditions*:

(H1) *there exist constants* $t_0 > 0$ *and* $M > 0$ *such that*

$$0 < F(x,t) = \int_0^t f(x,s)\,ds \le M|f(x,t)| \qquad \forall\, |t| \ge t_0, \quad \forall\, x \in \Omega;$$

(H2) $0 < F(x,t) \le \frac{1}{2} f(x,t)t \ \ \forall\, t \in \mathbb{R}\setminus\{0\}$, $\forall\, x \in \Omega$.

Then problem (E) *has a nontrivial solution. Moreover, if* $f(x,t)$ *is an odd function of* t, *then equation* (E) *has infinitely many solutions.*

Proof. The proof follows (by now) standard lines: the assumptions guarantee that the functional has a mountain pass structure around the origin (cf. [3, 22]). The subcritical growth yields compactness (through the Pohozaev–Trudinger inequality, see also P.L. Lions [18]), and hence the critical level is attained. $\square$

2.2. Critical growth. We consider now equation (E) with a critical growth in the sense specified above. For reference, let us first briefly recall the situation for $N \ge 3$, which has been studied extensively and is by now very well understood. Here, a critical growth is given by $g(s) = |s|^{2^*-2}s$, where $2^* = \frac{2N}{N-2}$ is the limiting Sobolev exponent, and so the corresponding equation is

$$-\Delta u = |u|^{2^*-2}u. \tag{4}$$

One knows that

- equation (4) has, on all of $\mathbb{R}^N$, the family of solutions $u_\lambda(x) = c\big(\frac{\lambda}{\lambda^2+|x-x_0|^2}\big)^{\frac{N-2}{2}}$, which "concentrate" as $\lambda \to 0$ (see G. Talenti [24]);
- equation (4) has no nontrivial solutions on bounded and star-shaped domains: this is due to the seminal *Pohozaev identity* [21];
- the nonexistence may also be understood as a result of the loss of compactness due to the concentrating family of solutions on $\mathbb{R}^N$;
- the existence of solutions can be recovered if (4) is perturbed by suitable lower order terms. This is the groundbreaking result by Brezis–Nirenberg [6]. The argument goes as follows: due to the concentrating sequence, the noncompactness of the perturbed functional occurs at a specific (and explicitly known) level. The (suitable) perturbations of the functional produce a minimax level *below* the noncompactness level; thus, there is compactness at this level, and hence one may prove the existence of a critical point at this level.

In the case $N = 2$ the situation is more complicated and the known results are less complete. The difficulties start already with the fact that there is no natural "model problem" for the critical case. Thus, let us write the "critical" equation (with $\alpha_0 = 4\pi$, see (3)) in the form

$$-\Delta u = h(u)e^{4\pi u^2},$$

where $h \in C(\mathbb{R})$ is subcritical, i.e., satisfies condition (2). The question is now whether there exist optimal (= sharp) conditions on $h(t)$ such that we have again the situation of nonexistence and existence of solutions.

Related to the study of this question is the behavior of the supremum in (1). Indeed, it came as a surprise when L. Carleson and A. Chang [8] proved in 1986 that the supremum in (1) is attained on the unit ball in $\mathbb{R}^2$ (this result was extended to arbitrary domains in $\mathbb{R}^2$ by M. Flucher [15]). Carleson and Chang prove their result by the following steps:

- the supremum is radial and is thus characterized by an ODE (the radial equation);
- if the supremum is not attained, then there exists a maximizing sequence that tends weakly to 0 and concentrates at the origin;
- determine an explicit upper bound, namely $\int_{B_1(0)} e^{4\pi u_n^2} \le (1+e)\pi$, for any normalized concentrating sequence $(u_n) \in H_0^1(B_1)$;
- provide an explicit normalized function $w \in H_0^1(B_1)$ with $\int_{B_1} e^{4\pi w^2} > (1+e)\pi$.

Clearly, $(1+e)\pi$ takes the role of the (highest) noncompactness level, analogous to the situation for $N \ge 3$ described above, and since the supremum lies above this noncompactness level, it is attained.

In the recent paper by de Figueiredo, do Ó, and the author [10], the following explicit normalized concentrating and *maximizing* sequence for $(1+e)\pi$ was constructed:

For $n \in \mathbb{N}$ set $\delta_n = \frac{2\log n}{n}$ and $A_n = \frac{1}{en^2} + O\big(\frac{1}{n^4}\big)$; then define

$$y_n(|x|) = \begin{cases} \left(\dfrac{1-\delta_n}{n}\right)^{1/2} \log \dfrac{1}{|x|}, & \dfrac{1}{n} \le |x| \le 1, \\ \dfrac{1}{(n(1-\delta_n))^{1/2}} \log \dfrac{A_n+1}{A_n+n|x|} + (n(1-\delta_n))^{1/2}, & 0 \le |x| \le \dfrac{1}{n}. \end{cases} \tag{5}$$

This sequence allows one to give a *new proof* of the last step in the argument of Carleson–Chang, as well as a slight generalization of their result. In view of this and the above remarks, it is of interest to consider

$$\sup_{u \in H_0^1(\Omega),\ \|u\|=1} \int\limits_\Omega h(u)e^{4\pi u^2}\,dx = S$$

and give optimal (= sharp) conditions on the subcritical function $h(t)$ such that the supremum S is *attained* or *not attained*, respectively.

2.3. Critical growth: existence. For the corresponding equation

$$\begin{cases} -\Delta u = h(u)e^{4\pi u^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

some progress has recently been made concerning the determination of an optimal subcritical function $h(t)$. We remark that concerning nonexistence results, a fundamental difference from the case $N \ge 3$ is that (up to now) no suitable identity of Pohozaev type is known for the case $N = 2$.

In [14] the following theorem was proved by de Figueiredo, Miyagaki, and the author (see also Adimurthi [2]):

Theorem 2.2. *Assume that $h \in C(\mathbb{R})$ and let $f(s) = h(s)e^{4\pi s^2}$. Assume that*

(H1) $f(0) = 0$;

(H2) *there exist constants $s_0 > 0$ and $M > 0$ such that*

$$0 < F(s) = \int_0^s f(r)\,dr \le M|f(s)| \qquad \forall\,|s| \ge s_0;$$

(H3) $0 < F(s) \le \frac{1}{2}f(s)s \;\; \forall\, s \in \mathbb{R} \setminus \{0\}$.

Furthermore, let d denote the inner radius of Ω, i.e.,

$$d := \text{radius of the largest ball} \subset \Omega.$$

Then equation (6) *has a nontrivial solution provided that*

(H4) $\lim_{|s|\to\infty} h(s)s = \beta > \frac{1}{2\pi d^2}$.

The proof of this theorem follows closely the scheme by Brezis–Nirenberg mentioned above; that is,

- determine (explicitly) the level of noncompactness;
- use an explicit concentrating sequence and the hypothesis on $h(t)$ to show that the minimax level is below this noncompactness level;
- thus, the compactness is recovered and the existence of a solution follows.

The concentrating sequence used in the proof of this theorem is the so-called *Moser sequence* given by

$$w_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{1}{(\log n)^{1/2}} \log \dfrac{1}{|x|}, & \dfrac{1}{n} \le |x| \le 1, \\ (\log n)^{1/2}, & 0 \le |x| \le \dfrac{1}{n}. \end{cases}$$

We remark that this sequence is *not* an optimal concentrating sequence; in fact, one easily calculates that

$$\lim_{n\to\infty} \int_{B_1} e^{4\pi w_n^2} dx = 2\pi < (1+e)\pi.$$

We remark that condition (H4) in Theorem 2.2 may be slightly improved to

$$\beta > \frac{1}{e\pi d^2}$$

by using the optimal maximizing sequence (5) mentioned above.

2.4. Critical growth: nonexistence. Concerning the nonexistence, only a partial result is known; in the following theorem, the nonexistence of a positive radial solution on $\Omega = B_1(0)$ is proved under conditions comparable to those of Theorem 2.2.

Theorem 2.3 (de Figueiredo–Ruf [13])**.** *Let $\Omega = B_1(0)$. Suppose that $h \in C^2(\mathbb{R})$ and that there exist constants $r_1 > 0$ and $\sigma > 0$ such that the following relations hold for some constants $K > 0$ and $c > 0$:*

(i) $h(r) = \frac{K}{r}$ *for* $r \ge r_1$,

(ii) $0 \le h(r) \le cKr^{1+\sigma}$ *for* $0 \le r \le r_1$.

Then there exists a constant $K_0 > 0$ *such that, for* $K < K_0$, *the equation*

$$\begin{cases} -\Delta u = h(u)e^{4\pi u^2} & \text{in } B_1(0) \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{7}$$

has no nontrivial radial solution.

We remark that by Gidas–Ni–Nirenberg [16] any positive solution of equation (7) is *radial*, and hence Theorem 2.3 says that equation (7) has *no positive solution.*

The proof of Theorem 2.3 uses techniques of the theory of ordinary differential equations, in particular the *shooting method.* More precisely, considering only the radial solutions on $\Omega = B_1(0)$, one can reduce equation (7) to the radial equation

$$\begin{cases} u_r r + \dfrac{1}{r} u_r + h(u)e^{4\pi u^2} = 0 & \text{in } (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{8}$$

Using the transformation $t = -2\log\frac{r}{2}$ and setting $y(t) = u(r)$, we obtain

$$\begin{cases} -y'' = h(y)e^{4\pi y^2 - t} & \text{for } t > 2\log 2, \\ y(2\log 2) = 0,\ y'(+\infty) = 0. \end{cases} \tag{9}$$

That is, we have transformed equation (8), which has a singularity at 0, to equation (9) on $(2\log 2, +\infty)$, thus transporting the singularity at 0 to $+\infty$. The shooting method consists now in considering solutions $y(t)$ of (9) with $y'(+\infty) = \gamma$; i.e., one shoots horizontally from infinity and tries to land at the point $2\log 2$. The estimates to achieve this are delicate and lengthy and are a refinement of the work of Atkinson–Peletier [3].

We summarize: if we assume that the asymptotic condition in the existence Theorem 2.2 is optimal (at least on the unit ball B_1), then the major *open problem* may be stated as follows:

Find a good model equation (i.e., properties of $h(u)$*) under which one may prove the existence of a nontrivial solution for* $\lim_{t\to\infty} h(t)t > \frac{1}{e\pi}$ *and the nonexistence of a solution for* $\lim_{t\to\infty} h(t)t < \frac{1}{e\pi}$.

As already mentioned, what seems to be missing is a kind of *Pohozaev identity* to obtain a sharp nonexistence result.

3. SYSTEM (S)

We now consider the system of equations (S), where $\Omega \subset \mathbb{R}^2$ is a bounded domain and the nonlinearities f and g are again continuous functions with exponential growth.

As mentioned in the Introduction, the natural functional associated with system (S) is

$$J(u,v) = \int\limits_\Omega \nabla u \nabla v\, dx - \int\limits_\Omega F(u)\, dx - \int\limits_\Omega G(v)\, dx. \tag{10}$$

Considering this functional on the product of Sobolev spaces $H_0^1(\Omega) \times H_0^1(\Omega)$, we obtain $F(t) \sim e^{t^2}$ and $G(t) \sim e^{t^2}$ as the maximal growth for both nonlinearities.

We recall that in dimension $N \geq 3$ different maximal growths can be obtained by working either with fractional Sobolev spaces $H^s(\Omega) \times H^t(\Omega)$ (here $H^t(\Omega)$ is the space of functions whose derivative of order $t \in \mathbb{R}^+$ lies in $L^2(\Omega)$; these spaces can be obtained via Fourier series or interpolation, see Adams–Fournier [1]) or with Banach spaces of type $W^{1,\alpha}(\Omega) \times W^{1,\beta}(\Omega)$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The first approach was used by Hulshof–van der Vorst [17] and de Figueiredo–Felmer [9], and the second, by de Figueiredo–do Ó–Ruf [11]. In both cases one obtains the so-called critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}; \tag{11}$$

that is, the critical growth $F(t) \sim |t|^{2^*}$ with $2^* = \frac{2N}{N-2}$ of the scalar equation is replaced by $F(t) = |t|^{p+1}$ and $G(t) = |t|^{q+1}$ with $p+1$ and $q+1$ satisfying (11).

As already mentioned in the Introduction, in the case of systems in two dimensions Sobolev spaces do not seem suitable to extend the notion of criticality; we thus turn now to Lorentz spaces.

3.1. Lorentz spaces. We begin by recalling the definition of a Lorentz space: For a measurable function $\phi\colon \Omega \to \mathbb{R}$, we denote by

$$\mu_\phi(t) = |\{x \in \Omega\colon \phi(x) > t\}|, \qquad t \ge 0,$$

its *distribution function.* The *decreasing rearrangement* $\phi^*(s)$ of ϕ is defined by

$$\phi^*(s) = \sup\{t > 0\colon \mu_\phi(t) > s\}, \qquad 0 \le s \le |\Omega|.$$

The Lorentz space $L(p,q)$ is given as follows: $\phi \in L(p,q)$, $1 < p < \infty$, $1 \le q < \infty$, if

$$\|\phi\|_{p,q} = \left(\int_0^\infty \left[\phi^*(t) t^{1/p}\right]^q \frac{dt}{t} \right)^{1/q} < +\infty.$$

We recall the following properties of Lorentz spaces (see Adams–Fournier [1]):

1. $L(p,p) = L^p$, $1 < p < +\infty$.
2. The following inclusions hold for $1 < q < p < r < \infty$:

$$L^r \subset L(p,1) \subset L(p,q) \subset L(p,p) = L^p \subset L(p,r) \subset L^q.$$

3. The Hölder inequality

$$\left| \int_\Omega f g \, dx \right| \le \|f\|_{p,q} \|g\|_{p',q'}, \qquad \text{where} \quad p' = \frac{p}{p-1}, \quad q' = \frac{q}{q-1}.$$

Furthermore, we recall the following embedding results:

Theorem A (see [1])**.** *Suppose that $1 \le p < N$ and $\nabla u \in L(p,q)$; then $u \in L(p^*,q)$, where $p^* = \frac{Np}{N-p}$ and $1 \le q < \infty$.*

For the next theorem, see H. Brezis [5].

Theorem B. *Suppose that $u \in W^{j,p}$ with $p < \frac{N}{j}$; then $u \in L(p^*,p)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{j}{N}$.*

Note that this result improves Sobolev's theorem, which gives $u \in L^{p^*} = L(p^*,p^*)$, which is a larger space than $L(p^*,p)$.

The following refinement of Trudinger's result (see H. Brezis and S. Wainger [7], H. Brezis [5], and R.S. Strichartz [23]) is of particular importance for our considerations:

Theorem C. *Assume that $\nabla u \in L(N,q)$ for some $1 < q < \infty$. Then $e^{|u|^{\frac{q}{q-1}}} \in L^1$.*

We make the following

Definition. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that $1 < p < \infty$ and $1 < q < \infty$ and set

$$W_0^1 L(p,q)(\Omega) = \mathrm{cl}\bigl\{u \in C_0^\infty(\Omega)\colon \|\nabla u\|_{p,q} < \infty\bigr\}.$$

On $W_0^1 L(p,q)$ we have the following norm:

$$\|u\|_{1;\,p,q} := \|\nabla u\|_{p,q},$$

with which $W_0^1 L(p,q)$ becomes a reflexive Banach space.

3.2. An exponential "critical hyperbola." We consider again system (S). We look for an analogue of the critical hyperbola. Considering the functional on the space $H_0^1 \times H_0^1$, one sees that the nonlinearities $G(v) \sim e^{|v|^2}$ and $F(u) \sim e^{|u|^2}$ lie on this "critical curve." We assume that $F(t)$ and $G(t)$ have "exponential polynomial growth," i.e.,

$$F(t) \sim e^{|t|^p} \quad \text{and} \quad G(t) \sim e^{|t|^q} \qquad \text{for some} \quad 1 < p \quad \text{and} \quad q < +\infty.$$

We prove

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then we have an "exponential critical curve" given by*

$$(F(s), G(s)) = \bigl(e^{|s|^p}, e^{|s|^q}\bigr), \qquad \textit{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. We consider the functional (10). We want to consider the term $\int_\Omega \nabla u \nabla v\,dx$ on a product of Lorentz spaces; i.e., we want to estimate this term using the Hölder inequality on Lorentz spaces:

$$\left|\int\limits_\Omega \nabla u \nabla v\,dx\right| \le \|\nabla u\|_{L(2,q)} \|\nabla v\|_{L(2,p)}, \qquad \frac{1}{q} + \frac{1}{p} = 1.$$

By Theorem C we have

$$e^{|u|^{\frac{q}{q-1}}} \in L^1 \qquad \text{and} \qquad e^{|v|^{\frac{p}{p-1}}} \in L^1,$$

and thus the maximal growth allowed for $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$ is given by

$$F(u) \sim e^{|u|^p}, \quad p = q' = \frac{q}{q-1}, \qquad \text{and} \qquad G(v) \sim e^{|v|^q}, \quad q = p' = \frac{p}{p-1}. \qquad \square$$

In other words, Theorem 3.1 says that the maximal growth for system (S) is obtained by the embeddings

$$W_0^1 L(2,p) \times W_0^1 L(2,q) \subset L_{e^{|s|^q}} \times L_{e^{|s|^p}}, \qquad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where L_ϕ denotes the Orlicz space with growth function (N-function) ϕ.

3.3. Subcritical growth. In this section we consider system (S) for nonlinearities F and G with exponential and subcritical growth, i.e.,

$$F(t) \sim e^{|t|^p} \quad \text{and} \quad G(t) \sim e^{|t|^q}, \qquad \text{with} \quad \frac{1}{p} + \frac{1}{q} > 1.$$

As mentioned, we consider functional (10) on the space

$$E = W_0^1 L(2,p) \times W_0^1 L(2,q), \qquad \text{with} \quad q = p'.$$

We make the following assumptions on f and g (our aim is to give simple assumptions, at the expense of greater generality):

(A1) f and g are continuous functions, with $f(s) = o(s)$ and $g(s) = o(s)$ near the origin.

(A2) There exist constants $\mu > 2$, $\nu > 2$, and $s_0 > 0$ such that

$$0 < \mu F(s) \le s f(s) \quad \text{and} \quad 0 < \nu G(s) \le s g(s) \qquad \forall\, |s| \ge s_0.$$

(A3) F and G have at most critical growth; i.e., there exist constants a_1, a_2 and b_1, b_2 such that

$$F(s) \le a_1 + a_2 e^{|s|^{q'}} \qquad \text{and} \qquad G(s) \le b_1 + b_2 e^{|s|^q}.$$

(A4) Either F or G is subcritical; i.e.,

$$\lim_{|s|\to\infty} \frac{F(s)}{\Phi(s)} = 0 \qquad \text{or} \qquad \lim_{|s|\to\infty} \frac{G(s)}{\Psi(s)} = 0,$$

where $\Phi(s) = e^{|s|^{q'}}$ and $\Psi(s) = e^{|s|^q}$.

Example. $F(s) = e^{|s|^\alpha} - 1 - |s|^\alpha$ and $G(s) = e^{|s|^\beta} - 1 - |s|^\beta$, with either $1 < \alpha < p$ and $1 < \beta \le q$ or $\alpha = p$ and $1 < \beta < q$.

Theorem 3.2. *Under assumptions* (A1)–(A4), *system* (S) *has a nontrivial positive* (*weak*) *solution* $(u, v) \in E$.

Proof. The proof follows the ideas from [11]. The main problems one has to deal with are the following:

- the functional J given in (10) is strongly indefinite, being unbounded from above and below on infinite-dimensional subspaces of E;
- E is a Banach space rather than (as usual) a Hilbert space. $\square$

3.4. Critical growth. Little is known about system (S) with critical growth in the sense of Theorem 3.1. Only in the particular case when both $F(s)$ and $G(s)$ have the same growth e^{t^2}, i.e., when one can work on the space $H_0^1(\Omega) \times H_0^1(\Omega)$, one has an existence result (see de Figueiredo–do Ó–Ruf [11]):

Theorem 3.3. *Assume that*

(H1) *f and g are continuous functions and both f and g have critical growth with* $\alpha_0 = 4\pi$;

(H2) $f(s) = o(s)$ *and* $g(t) = o(t)$ *near the origin*;

(H3) *there exist constants $\theta > 2$ and $t_0 > 0$ such that, for all $t \ge t_0$,*

$$0 < \theta F(t) \le t f(t) \qquad \textit{and} \qquad 0 < \theta G(t) \le t g(t);$$

(H4) *there exist $M > 0$ and $t_0 > 0$ such that, for all $t \ge t_0$,*

$$0 < F(t) \le M f(t) \qquad \textit{and} \qquad 0 < G(t) \le M g(t);$$

(H5) *d denote the inner radius of Ω and*

$$\lim_{|t|\to\infty} \frac{t f(t)}{e^{4\pi t^2}} > \frac{1}{\pi d^2} \qquad \textit{and} \qquad \lim_{|t|\to\infty} \frac{t g(t)}{e^{4\pi t^2}} > \frac{1}{\pi d^2}.$$

Then system (S) *has a nontrivial weak solution* $(u, v) \in E$.

The proof combines the techniques of Theorem 2.2 with the methods of the critical case for systems in dimension $N \ge 3$ (in particular, the problems arising from the indefiniteness of the functional; see, e.g., [12]).

We conclude by noting that for nonlinearities with critical growth in the sense of Theorem 3.1 that are different from those considered in Theorem 3.3, many questions remain open. In particular, there are

- no known concentration phenomena,
- no existence results and no nonexistence results.

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