Some arithmetical properties of the generating power series for the sequence $\{\zeta(2k+1)\}_{k=1}^{\infty}$

G. Molteni

Acta Math. Hungarica **90 (3)**, 133–140 (2001)

Abstract

Let $f_{\text{odd}}(z) := \sum_{k=1}^{\infty} \zeta(2k+1)z^{2k}$ be the power series with the values of the Riemann zeta function at odd integers as coefficients. This function can be analytically continued to a meromorphic function over $\mathbb{C}$. We prove that 1 and the values of f_{odd} at rational points with relatively prime denominators are linearly independent over $\overline{\mathbb{Q}}$.

Some arithmetical properties of the sequence $\{\zeta(2k+1)\}_{k=1}^{\infty}$ are deduced.

Mathematics Subject Classification (1991): 11J72, 11J99.

1 Introduction and results

Let $\zeta(s)$ be the Riemann zeta function. The arithmetical nature of its values at even integers is well known from the classical formula $\zeta(2k) = \frac{2^{2k}|B_{2k}|}{2(2k)!}\pi^{2k}$, where the Bernoulli number B_{2k} appears. Similar results hold for the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ of a totally real number field $\mathbb{F}$ when evaluated at even integers, by a theorem of Siegel-Klingen [7]. In the general context of motivic L-functions, Deligne [4] introduced the notion of critical integers (the even positive integers, in the case of $\zeta_{\mathbb{F}}(s)$) and proposed a conjecture about the values at the critical integers, implying that

$$L(m) = A(m)\Omega(m) \qquad \text{for } m \text{ critical,} \tag{1}$$

with $A(m)$ an algebraic number and $\Omega(m)$ a period, i.e., the integral over some algebraic cycle of an algebraic differential form. This conjecture is actually a theorem for the Artin L-functions.

Unlike the case of critical points, the values at non-critical integers are totally mysterious: establishing the arithmetical nature of the constants $\zeta(2k+1)$, for example, is a surprisingly very difficult problem. It is a generally shared opinion that every $\zeta(2k+1)$ is a transcendental number with an arithmetical nature different from that of $\zeta(2k)$. In particular $\zeta(2k+1)$ should not be algebraically dependent on any powers of the logarithms of algebraic numbers, but the only result about this problem is the celebrated proof of the irrationality of $\zeta(3)$ (Apéry [1], see also Beukers [3]).

Let

$$f_{\text{even}}(z) := \sum_{k=1}^{\infty} \zeta(2k) z^{2k} \quad \text{and} \quad f_{\text{odd}}(z) := \sum_{k=1}^{\infty} \zeta(2k+1) z^{2k} .$$

The aim of this paper is the study of $f_{\text{odd}}(z)$ to show that it has arithmetical properties different from that ones of $f_{\text{even}}(z)$. As a consequence, we show that a representation of the form (1) is not possible for $\{\zeta(2k+1)\}$ for some particular choices of $A(m)$ and $\Omega(m)$.

Considering the logarithmic derivative of the identity $\sin z = z \prod_n (1 - z^2/\pi^2 n^2)$, the equality

$$f_{\text{even}}(z) = \frac{1}{2} - \frac{\pi z}{2} \operatorname{cotg} \pi z \tag{2}$$

follows, a formula due to Euler.

By (2) it follows that $f_{\text{even}}(z)$ is a transcendental number for every rational z. Nevertheless, the numbers 1, $f_{\text{even}}(z_1)$ and $f_{\text{even}}(z_2)$ are linearly dependent on $\overline{\mathbb{Q}}$ for every choice of z_1, z_2 in $\mathbb{Q}$. The main result of this paper shows that this is not possible for the function $f_{\text{odd}}(z)$.

Theorem. *Let $\{r_l = a_l/b_l\}_{l=1}^{M}$ be arbitrary distinct rational numbers with $(a_l, b_l) = 1$, $0 < r_l < 1$ and suppose that for any l there exists an odd prime p_l such that $p_l | b_l$ and $p_l \nmid b_j$ when $j \neq l$. Then the numbers $\{1\} \cup \{f_{\text{odd}}(r_l)\}_{l=1}^{M}$ are linearly independent over $\overline{\mathbb{Q}}$.*

Remark 1. The hypothesis about $\{b_l\}_{l=1}^{M}$ in Theorem is a type of independence for the rational numbers r_l, and the relation $2(t-1)f_{\text{odd}}(\frac{1}{t}) - 2(t-1)f_{\text{odd}}(\frac{t-1}{t}) + t(t-2) = 0$, holding for every integer t by Proposition 1 below, shows that a some type of independence of $\{r_l\}_{l=1}^{M}$ has to be assumed in order to the claim of Theorem hold.

An obvious corollary is

Corollary 1. *Let $0 < r = a/b < 1$ be a rational number with b not a power of 2. Then $f_{\text{odd}}(r)$ is transcendental.*

A lemma giving an explicit formula for $f_{\text{odd}}(z)$ at rational points is necessary for the proof of Theorem. Since it has an independent interest, we state it explicitly.

Lemma. *Let $\frac{a}{b} \in (0,1) \cap \mathbb{Q}$, $\alpha_j := \sin^2 \frac{j\pi}{b}$ and $h_a(z) := (\sin^2 \pi a z) \ln \sin^2 \pi z$. Then*

$$f_{\text{odd}}\Big(\frac{a}{b}\Big) = b\Big(\ln 2 - \frac{1}{2a}\Big) + \sum_{j=1}^{b} \alpha_{aj} \ln \alpha_j = \sum_{j=1}^{b} h_a\Big(\frac{j}{b}\Big) - b\int_0^1 h_a(z)dz \ , \tag{3}$$

i.e., $f_{\text{odd}}(a/b)/b$ is the discrepancy between the integral of $h_a(z)$ and its Riemann approximation.

The series defining $f_{\rm odd}(z)$ converges only in the disk $|z| < 1$, but a meromorphic continuation to $\mathbb{C}$ is possible. In fact, let $\psi(z) := \Gamma'(z)/\Gamma(z)$ be the logarithmic derivative of the gamma function. The infinite product formula $\Gamma^{-1}(1+z) = e^{\gamma z}\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-z/n}$ implies that

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty}\frac{1}{n(n+z)} = -\gamma + \sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1} = \\ -\gamma + f_{\rm even}(z)/z - f_{\rm odd}(z)\ , \quad (4)$$

where γ is the Euler constant. Identities (2) and (4) provide the meromorphic continuation. Moreover, the functional equation $\Gamma(z+1) = z\Gamma(z)$ gives a functional equation for $f_{\rm odd}$ too, so that the following proposition holds.

Proposition 1. *$f_{\rm odd}(z)$ is an even meromorphic function with simple poles at $z \in \mathbb{Z}\backslash\{0\}$; it satisfies the functional equation $f_{\rm odd}(z) + \frac{1}{2z} = f_{\rm odd}(1-z) + \frac{1}{2(1-z)}$ and $\operatorname{Res}_{z=n} f_{\rm odd}(z) = -\operatorname{sgn}(n)/2$.*

Proposition 1 has simple but interesting consequences, for instance

Corollary 2. *The claims of Theorem and Corollary 1 hold without the restriction $0 < r_l < 1$ as well.*

Corollary 3. *The sequence $\{\zeta(2k+1)\}_{k=1}^{\infty}$ does not satisfy any linear recursion.*

Remark 2. By (2), identity (3) is equivalent to the formula

$$\psi\Big(\frac{a}{b}\Big) = -\gamma - \ln\frac{b}{2} - \frac{\pi}{2}\operatorname{cotg}\frac{\pi a}{b} + \sum_{0<j<b/2}\cos\frac{2\pi a j}{b}\ln\sin\frac{\pi j}{b}\ , \quad (5)$$

a celebrated relation due to Gauss. The original proof of (5) has been considerably simplified by Jensen [6], using Abel's theorem on the continuity of convergent power series on the circle of convergence, and by Lehmer [8], using a relation of $\psi(z)$ with the Euler constants for arithmetic progressions. We prove (3) independently of (5), thus providing a new and very short proof of (5).

The theorem follows from the lemma, the theory of linear forms in logarithms developed by Baker [2] and some considerations about cyclotomic fields.

2 Consequences on $\{\zeta(2k+1)\}_{k=1}^{\infty}$

The theorem and Proposition 1 give some support to the belief that the conjectural expression in (1) for the critical values probably does not hold for $\{\zeta(2k+1)\}$. Here there are some examples.

Let $\wp_\Lambda$ be the Weierstrass elliptic function for a lattice $\Lambda \subset \mathbb{C}$. $\wp_\Lambda$ has a lattice of poles, and three non-collinear poles can be found for every element of $\mathbb{L} := \mathbb{C}(z, \wp_{\Lambda_1}, \wp'_{\Lambda_1}, \cdots, \wp_{\Lambda_N}, \wp'_{\Lambda_N})$, for every choice of $\Lambda_1, \cdots, \Lambda_N$. Hence $f_{\rm odd} \notin \mathbb{L}$ and, in particular, $f_{\rm odd}(z) \neq \sum_{i=1}^{N} c_i(\wp_{\Lambda_i}(z) - 1/z^2)$, so

Proposition 2. *It does not exist* $\{\Lambda_i\}_{i=1}^N$ *lattices and* $\{c_i\}_{i=1}^N \in \mathbb{C}$ *such that*

$$\frac{\zeta(2k+1)}{2k+1} = \sum_{i=1}^{N} c_i G_{2k+2}(\Lambda_i) := \sum_{i=1}^{N} c_i (\sum_{\substack{\omega\in\Lambda_i \\ \omega\neq 0}} \frac{1}{\omega^{2k+2}}) \qquad \forall k > 0\ .$$

This is an interesting fact, in view of the series defining $\zeta(2k+1)$.

Let ω be a period for Λ. The impossibility of the equality $f_{\text{odd}}(z) = \omega\wp(\omega z) - \frac{1}{\omega z^2}$ gives

Proposition 3. *It does not exist any lattice* Λ *and any period* ω *of* Λ *such that* $\zeta(2k+1) = (2k+1)G_{2k+2}(\Lambda)\omega^{2k+1}$*, holds for every positive* k*.*

When $\wp_\Lambda$ has algebraic invariants G_4, G_6, every G_{2k} is algebraic as well by the relation $\wp'^2 = 4\wp^3 - 60G_4\wp - 140G_6$, and ω is a transcendental number by the theorem of Schneider [9]. Hence, Proposition 3 excludes a particular form of (1) for the numbers $\zeta(2k+1)$.

Let $\mathbb{K}$ be the field of meromorphic functions assuming only algebraic values at the rational points; then

Corollary 4. $f_{\text{odd}} \notin \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{C}$.

In fact, our theorem implies that arbitrarily long sets of values of f_{odd} at rational points are $\overline{\mathbb{Q}}$-linearly independent, while this is not the case for $g \in \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{C}$. In fact, g is a finite sum $\sum_{j=1}^{N} c_j f_j$, with $c_j \in \mathbb{C}$ and $f_j \in \mathbb{K}$, and the set $\{g(z_i)\}_{i=1}^{N+1}$ is always $\overline{\mathbb{Q}}$-linearly dependent for every choice of $z_i \in \mathbb{Q}$.

Remark 3. Corollary 4 states a peculiar property of f_{odd}, since $f_{\text{even}} \in \mathbb{K} \otimes_{\mathbb{Q}} \mathbb{C}$.

$\mathbb{K}$ is a too large field to deduce any direct consequences from Corollary 4; for this purpose the subfield $\mathbb{M} := \overline{\mathbb{Q}}(z, \sin\pi z) \subset \mathbb{K}$ is more convenient. Actually, every element of $\overline{\mathbb{Q}}(z, x)$ regular at $\{|x|, |z| < \delta\}$ is a series of the form $\sum_{m,n=0}^{\infty} c_{m,n} z^m x^n$ with $\{c_{m,n}\}$ satisfating the linear recursion

$$\sum_{u,v=1}^{M} a_{u,v} \quad c_{m-u,n-v} = 0 \qquad \text{for every} \quad m, n \geq 0\ ,$$

for some $\{a_{u,v}\}_1^M \in \overline{\mathbb{Q}}$.
Hence, the series $\sum_{m,n=0}^{\infty} c_{m,n} z^m \sin^m \pi z$ is an element of $\mathbb{M}$ regular for $|z| < \delta$, so that it can be written as $\sum_{h=0}^{\infty} (\sum_{l=0}^{h} d_{h,l}\pi^l) z^h$ for a suitable double sequence $\{d_{h,l}\}$ of algebraic numbers. Hence, the following proposition holds.

Proposition 4. *It does not exist N sequences* $\{a^{(i)}_{u,v,t}\}_{i=1}^N \in \overline{\mathbb{Q}}$ *and* $\{c_i\}_1^N \in \mathbb{C}$ *such that* $\{c^{(i)}_{l,m,n}\}$ *satisfy the linear recursion* $\sum_{u,v,t=1}^M a^{(i)}_{u,v,t} c^{(i)}_{l-u,m-v,n-t} = 0 \ \forall i = 1, \dots, N$, *and*

$$\zeta(2k+1) = \sum_{i=1}^N c_i \sum_{l=0}^{2k+1} d^{(i)}_{k,l} \ \pi^l \qquad \textit{for every} \quad k > 0 \ ,$$

where $d^{(i)}_{k,l}$ *are related to the coefficients* $c^{(i)}_{m,n}$ *as above.*

Again, Proposition 4 excludes a particular form of (1) for $\zeta(2k+1)$, π being the period in this case.

Corollary 3 and Propositions 2, 3, 4 are only a few implications of Theorem and Proposition 1. Other and more complicated consequences can be deduced, all showing that representations of the form (1) are impossible if some conditions are imposed on the sequence $\{A(m)\}$.

3 Proofs

3.1 The lemma ...

By the identity $\ln(4\sin^2\pi z) = \ln(1-e^{2\pi i z}) + \ln(1-e^{-2\pi i z})$ and the power series $-\ln(1-z) = \sum_{k=1}^\infty \frac{z^k}{k}$ converging uniformly in the complex region $\{|z| \le 1\} \cap \{|z-1| > \delta\}$, we obtain the Fourier series

$$\ln(2\sin\pi z) = -\sum_{k=1}^\infty \frac{\cos 2k\pi z}{k} \qquad \text{uniformly for } z \in [\delta, 1-\delta],\ \forall \delta > 0, \tag{6}$$

(see also [11], ch. 3, sec. 14). A term by term integration of (6) gives the identity $\int_0^1 h_a(z)dz = 1/2a - \ln 2$. Moreover, we remark that $h_a(z) \in \mathcal{C}^1[0,1]$, that $h_a(z) \in \mathcal{C}^\infty(0,1)$ and that $h''_a(z)$ is integrable in $[0,1]$. Hence the Euler-Maclaurin summation formula (see [10], ch. 1) gives the relation

$$\sum_{j=1}^b h_a\Big(\frac{j}{b}\Big) - b\int_0^1 h_a(z)dz = -\frac{1}{2b}\int_0^1 B_2(\{bz\}) h''_a(z)dz \ , \tag{7}$$

with $B_2(z)$ the second Bernoulli polynomial and where $\{z\}$ denotes the fractional part of z. Since $B_2(\{z\}) = \frac{1}{\pi^2}\sum_1^\infty \frac{\cos 2k\pi z}{k^2}$ uniformly for $z \in \mathbb{R}$ and

$$h''_a(z) = 2\pi^2(2a^2\cos 2\pi az \ln\sin\pi z + 2a\sin 2\pi az \frac{\cos\pi z}{\sin\pi z} - \frac{\sin^2\pi az}{\sin^2\pi z}) \ ,$$

the right side of (7) becomes

$$\frac{2a^2}{b}\sum_{k=1}^\infty\sum_{l=1}^\infty \frac{1}{k^2 l}\int_0^1 \cos 2kb\pi z \ \cos 2\pi az \ \cos 2l\pi z \ dz \ , \tag{8}$$

where we have used the $0 < a < b$ condition to cancel some zero-terms. By the orthogonality of the Fourier basis, (8) is equal to

$$\frac{a^2}{2b}\sum_{k=1}^{\infty}\Big(\frac{1}{k^2}\frac{1}{kb+a}+\frac{1}{k^2}\frac{1}{kb-a}\Big)$$

and it is easy to verify that this quantity is $f_{\text{odd}}(\frac{a}{b})$, so the lemma is proved.

3.2 . . . and the theorem

We have to prove that $\beta_l = 0\ \forall l$ is the unique solution of

$$\beta_0 + \sum_{l=1}^{M}\beta_l f_{\text{odd}}\Big(\frac{a_l}{b_l}\Big) = 0 \qquad \beta_l \in \overline{\mathbb{Q}}\ .$$

By the lemma, this linear form becomes

$$\tilde{\beta}_{-1} + \tilde{\beta}_0 \ln 2 + \sum_{l=1}^{M}\sum_{j=1}^{b_l}\tilde{\beta}_{l,j}\ln\alpha_{l,j} = 0\ , \tag{9}$$

with $\alpha_{l,j} := \sin^2\frac{j\pi}{b_l}$, $\tilde{\beta}_{-1} := \beta_0 - \frac{1}{2}\sum_{l=1}^{M}\beta_l\frac{b_l}{a_l}$, $\tilde{\beta}_0 := \sum_{l=1}^{M}\beta_l b_l$ and $\tilde{\beta}_{l,j} := \beta_l\alpha_{l,ja_l}$. Moreover, $\tilde{\beta}_{-1}$, $\tilde{\beta}_0$ and any $\tilde{\beta}_{l,j}$ belongs to $\overline{\mathbb{Q}}$, since $\alpha_{l,ja_l} \in \mathbb{Q}[b_l]_+$, the maximal real subfield of the cyclotomic field $\mathbb{Q}[b_l]$. Some values for j such that $\alpha_{l,j} = 0$ appear in (9), but in this case $\tilde{\beta}_{l,j} = 0$ too, and the total contribution is zero.

Equation (9) involves a $\overline{\mathbb{Q}}$-linear form in logarithms of algebraic numbers that can be investigated by the fundamental theorems of Baker. In fact, Theorem 1 in [2] (inhomogeneous case) shows that (9) is impossible when $\tilde{\beta}_{-1} \neq 0$, and for $\tilde{\beta}_{-1} = 0$, Theorem 2 in [2] (homogeneous case) implies that the solutions of (9) can be found in $\mathbb{Q}$. It follows that we can suppose $\beta_0 = \frac{1}{2}\sum_{l=1}^{M}\beta_l\frac{b_l}{a_l}$ and $\tilde{\beta}_0, \tilde{\beta}_{l,j} \in \mathbb{Z}$ so that (9) becomes

$$2^{\tilde{\beta}_0}\prod_{l=1}^{M}\prod_{j=1}^{b_l}\alpha_{l,j}^{\tilde{\beta}_{l,j}} = 1 \qquad \tilde{\beta}_0, \tilde{\beta}_{l,j} \in \mathbb{Z}\ . \tag{10}$$

Let $\mathbb{L}_l := \mathbb{Q}[b_l]_+$, $\mathbb{L} := \otimes_l \mathbb{L}_l$, and let $N(\cdot)$ be the norm map. It is a well known fact that (see [5], ch. VI, eq. 3.10)

$$\mathrm{N}_{\mathbb{Q}[q]_+/\mathbb{Q}}(\sin^2\frac{j\pi}{q}) = q/4^{[\mathbb{Q}[q]_+:\mathbb{Q}]} \qquad \text{for every} \quad q, \quad (j,q) = 1\ . \tag{11}$$

We apply $N_{\mathbb{L}/\mathbb{Q}} = N_{\mathbb{L}_1/\mathbb{Q}}N_{\mathbb{L}/\mathbb{L}_1}$ to (10) and we consider the terms with $l = 1$; it becomes

$$R\prod_{j=1}^{b_1}[N_{\mathbb{L}_1/\mathbb{Q}}(\alpha_{1,j})]^{\tilde{\beta}_{1,j}[\mathbb{L}\ :\mathbb{L}_1]} = 1\ , \tag{12}$$

where R is a rational that by (11) is divisible only by 2 and the primes appearing in b_l with $l > 1$. In particular $\nu_{p_1}(R) = 0$, where ν_{p_1} is the valuation at the prime p_1 that divides b_1 but not b_l for $l > 1$ by the hypothesis of our theorem, so (12) gives

$$0 = \nu_{p_1}(\prod_{j=1}^{b_1} [N_{\mathbb{L}_1/\mathbb{Q}}(\alpha_{1,j})]^{\tilde{\beta}_{1,j}}) = \sum_{j=1}^{b_1} \tilde{\beta}_{1,j}\, \nu_{p_1}(N_{\mathbb{L}_1/\mathbb{Q}}(\alpha_{1,j})) =$$

$$\beta_1 \sum_{j=1}^{b_1} \sin^2 \frac{j a_1 \pi}{b_1}\, \nu_{p_1}(N_{\mathbb{L}_1/\mathbb{Q}}(\alpha_{1,j}))\ . \quad (13)$$

By (11) again, $\nu_{p_1}(N_{\mathbb{L}_1/\mathbb{Q}}(\alpha_{1,j})) \geq 0$ and it is not zero when $(j, b_1) = 1$; moreover, $\sin^2 \frac{j a_1 \pi}{b_1} > 0$ for such a value of j, hence (13) gives $\beta_1 = 0$.

In similar way we prove that $\beta_l = 0$ for $l > 0$ and the condition $\beta_0 = \frac{1}{2}\sum_{j=1}^{M} \beta_l \frac{b_l}{a_l}$ gives $\beta_0 = 0$ too. The theorem is proved.

Acknowledgment. I thank professors Dvornicich, Perelli, Viola and Zannier for many suggestions and useful discussions about the subject of this paper.

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Giuseppe Molteni
Dipartimento di Matematica
Università di Milano
Via Saldini 50
20133 Milano
ITALY
e-mail: molteni@mat.unimi.it