Measure of the non-Gaussian character of a quantum state

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We address the issue of quantifying the non-Gaussian character of a bosonic quantum state and introduce a non-Gaussianity measure based on the Hilbert-Schmidt distance between the state under examination and a reference Gaussian state. We analyze in detail the properties of the proposed measure and exploit it to evaluate the non-Gaussianity of some relevant single-mode and multimode quantum states. The evolution of non-Gaussianity is also analyzed for quantum states undergoing the processes of Gaussification by loss and de-Gaussification by photon-subtraction. The suggested measure is easily computable for any state of a bosonic system and allows one to define a corresponding measure for the non-Gaussian character of a quantum operation.

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I. INTRODUCTION

Gaussian states play a crucial role in quantum information processing with continuous variables. This is especially true for quantum optical implementations since radiation at thermal equilibrium, including the vacuum state, is itself a Gaussian state and most of the Hamiltonians achievable within the current technology are at most bilinear in the field operators—i.e., preserve the Gaussian character [1–3]. As a matter of fact, using single-mode and entangled Gaussian states, linear optical circuits, and Gaussian operations, like homodyne detection, several quantum information protocols have been implemented, including teleportation, dense coding, and quantum cloning [4].

On the other hand, quantum information protocols required for long-distance communication, such as, for example, entanglement distillation and entanglement swapping, rely on non-Gaussian operations. In addition, it has been demonstrated that teleportation [5–7] and cloning [8] of quantum states may be improved by using non-Gaussian states and non-Gaussian operations. Indeed, de-Gaussification protocols for single-mode and two-mode states have been proposed [5–7] and realized [9]. It should be also noticed that any strongly superadditive function is minimized, at fixed covariance matrix, by Gaussian states. This is crucial to prove the extremality of Gaussian states and Gaussian operations [10,11] for what concerns various quantities such as channel capacities [12], multipartite entanglement measures [13], and distillable secret keys in quantum key distribution protocols. Since in most cases these quantities can be computed only for Gaussian states, a non-Gaussianity measure may serve as a guideline to quantify them for the class of non-Gaussian states. Overall, non-Gaussianity is revealing itself as a resource for continuous variable quantum information, and thus we urge a measure able to quantify the non-Gaussian character of a quantum state.

In this paper we introduce a quantity, the non-Gaussianity $\delta[\varrho]$ of a quantum state, which quantifies how much a state fails to be Gaussian. Our measure, which is based on the Hilbert-Schmidt distance between the state itself and a reference Gaussian state, can be easily computed for any state, either single-mode or multimode.

The paper is structured as follows. In the next section we introduce notation and review the basic properties of Gaussian states. Then, in Sec. III we introduce the formal definition of $\delta[\varrho]$ and study its properties in details. In Sec. IV we evaluate the non-Gaussianity of relevant quantum states, whereas in Sec. V we analyze the evolution of non-Gaussianity for known Gaussification and de-Gaussification maps. Section VI closes the paper with some concluding remarks.

II. GAUSSIAN STATES

For concreteness, we will use here the quantum optical terminology of modes carrying photons, but our theory applies to general bosonic systems. Let us consider a system of n modes described by mode operators a_k, $k=1,\ldots,n$, satisfying the commutation relations $[a_k,a_j^\dagger]=\delta_{kj}$. A quantum state ϱ of n modes is fully described by its characteristic function [14]

$$\chi[\varrho](\boldsymbol{\lambda}) = \mathrm{Tr}[\varrho D(\boldsymbol{\lambda})],$$

where $D(\boldsymbol{\lambda})=\otimes_{k=1}^{n} D_k(\lambda_k)$ is the n-mode displacement operator, with $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)^T$, $\lambda_k\in\mathbb{C}$, and where

$$D_k(\lambda_k) = \exp\{\lambda_k a_k^\dagger - \lambda_k^* a_k\}$$

is the single-mode displacement operator. The canonical operators are given by

$$q_k = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger),$$

$$p_k = \frac{1}{i\sqrt{2}}(a_k - a_k^\dagger),$$

with commutation relations given by $[q_j,p_k]=i\delta_{jk}$. Upon introducing the real vector $\boldsymbol{R}=(q_1,p_1,\ldots,q_n,p_n)^T$, the commutation relations can be rewritten as

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$$[R_k,R_j]=i\Omega_{kj},$$

where Ω_{kj} are the elements of the symplectic matrix $\mathbf{\Omega}=i\oplus_{k=1}^{n}\sigma_2$, σ_2 being the y Pauli matrix. The covariance matrix $\boldsymbol{\sigma}\equiv\boldsymbol{\sigma}[\varrho]$ and the vector of mean values $\boldsymbol{X}\equiv\boldsymbol{X}[\varrho]$ of a quantum state ϱ are defined as

$$X_j=\langle R_j\rangle,$$

$$\sigma_{kj}=\frac{1}{2}\langle\{R_k,R_j\}\rangle-\langle R_j\rangle\langle R_k\rangle, \qquad (1)$$

where $\{A,B\}=AB+BA$ denotes the anticommutator and $\langle O\rangle=\mathrm{Tr}[\varrho O]$ is the expectation value of the operator O.

A quantum state ϱ_G is referred to as a Gaussian state if its characteristic function has the Gaussian form

$$\chi[\varrho_G](\mathbf{\Lambda})=\exp\left\{-\frac{1}{2}\mathbf{\Lambda}^T\boldsymbol{\sigma}\mathbf{\Lambda}+\boldsymbol{X}^T\mathbf{\Omega}\mathbf{\Lambda}\right\},$$

where $\mathbf{\Lambda}$ is the real vector $\mathbf{\Lambda}=(\mathrm{Re}\,\lambda_1,\mathrm{Im}\,\lambda_1,\ldots,\mathrm{Re}\,\lambda_n,\mathrm{Im}\,\lambda_n)^T$. Of course, once the covariance matrix and the vector of mean values are given, a Gaussian state is fully determined. For a single-mode system the most general Gaussian state can be written as

$$\varrho_G=D(\alpha)S(\zeta)\nu(n_t)S^\dagger(\zeta)D^\dagger(\alpha),$$

$D(\alpha)$ being the displacement operator, $S(\zeta)=\exp[\frac{1}{2}\zeta(a^\dagger)^2-\frac{1}{2}\zeta^*a^2]$ the squeezing operator, $\alpha,\zeta\in\mathbb{C}$, and $\nu(n_t)=(1+n_t)^{-1}[n_t/(1+n_t)]^{a^\dagger a}$ a thermal state with an n_t average number of photons.

III. MEASURE OF THE NON-GAUSSIAN CHARACTER OF A QUANTUM STATE

In order to quantify the non-Gaussian character of a quantum state ϱ we use a quantity based on the distance between ϱ and a reference Gaussian state τ, which itself depends on ϱ. Specifically, we define the non-Gaussianity $\delta[\varrho]$ of the state ϱ as

$$\delta[\varrho]=\frac{D_{HS}^2[\varrho,\tau]}{\mu[\varrho]}, \qquad (2)$$

where $D_{HS}[\varrho,\tau]$ denotes the Hilbert-Schmidt distance between ϱ and τ,

$$D_{HS}^2[\varrho,\tau]=\frac{1}{2}\mathrm{Tr}[(\varrho-\tau)^2]=\frac{\mu[\varrho]+\mu[\tau]-2\kappa[\varrho,\tau]}{2}, \qquad (3)$$

with $\mu[\varrho]=\mathrm{Tr}[\varrho^2]$ and $\kappa[\varrho,\tau]=\mathrm{Tr}[\varrho\tau]$ denoting the purity of ϱ and the overlap between ϱ and τ, respectively. The Gaussian reference τ is the Gaussian state such that

$$\boldsymbol{X}[\varrho]=\boldsymbol{X}[\tau], \quad \boldsymbol{\sigma}[\varrho]=\boldsymbol{\sigma}[\tau];$$

i.e., τ is the Gaussian state with the same covariance matrix $\boldsymbol{\sigma}$ and the same vector $\boldsymbol{X}$ of the state ϱ.

The relevant properties of $\delta[\varrho]$, which confirm that it represents a good measure of the non-Gaussian character of ϱ, are summarized by the following lemmas.

Lemma 1. $\delta[\varrho]=0$ if and only if ϱ is a Gaussian state.

Proof. If $\delta[\varrho]=0$, then $\varrho=\tau$ and thus it is a Gaussian state. If ϱ is a Gaussian state, then it is uniquely identified by its first and second moments and thus the reference Gaussian state τ is given by $\tau=\varrho$, which, in turn, leads to $D_{HS}[\varrho,\tau]=0$ and thus to $\delta[\varrho]=0$.

Lemma 2. If U is a unitary map corresponding to a symplectic transformation in phase space—i.e., if $U=\exp\{-iH\}$ with Hermitian H that is at most bilinear in the field operators—then $\delta[U\varrho U^\dagger]=\delta[\varrho]$. This property ensures that displacement and squeezing operations do not change the Gaussian character of a quantum state.

Proof. Let us consider $\varrho'=U\varrho U^\dagger$. Then the covariance matrix transforms as $\boldsymbol{\sigma}[\varrho']=\Sigma\boldsymbol{\sigma}[\varrho]\Sigma^T$, Σ being the symplectic transformation associated to U. At the same time the vector of mean values simply translates to $\boldsymbol{X}'=\boldsymbol{X}+\boldsymbol{X}_0$, where $\boldsymbol{X}_0$ is the displacement generated by U. Since any Gaussian state is fully characterized by its first and second moments, the reference state must necessarily transform as $\tau'=U\tau U^\dagger$—i.e., with the same unitary transformation U. Since the Hilbert-Schmidt distance and the purity of a quantum state are invariant under unitary transformations, the lemma is proved.

Lemma 3. $\delta[\varrho]$ is proportional to the squared $L^2(\mathbb{C}^n)$ distance between the characteristic functions of ϱ and of the reference Gaussian state τ. In the formula,

$$\delta[\varrho]\propto\int d^{2n}\boldsymbol{\lambda}\{\chi[\varrho](\boldsymbol{\lambda})-\chi[\tau](\boldsymbol{\lambda})\}^2. \qquad (4)$$

Since the notion of Gaussianity of a quantum state is defined through the shape of its characteristic function and since the characteristic function of a quantum state belongs to the $L^2(\mathbb{C}^n)$ space [14], we address $L^2(\mathbb{C})$ distance to as a good indicator of the non Gaussian character of ϱ.

Proof. Since characteristic functions of self-adjoint operators are even functions of λ and by means of the identity

$$\mathrm{Tr}[O_1O_2]=\int\frac{d^{2n}\boldsymbol{\lambda}}{\pi^n}\chi[O_1](\boldsymbol{\lambda})\chi[O_2](-\boldsymbol{\lambda}),$$

we obtain

$$D_{HS}^2[\varrho,\tau]=\frac{1}{2}\int\frac{d^{2n}\boldsymbol{\lambda}}{\pi^n}\{\chi[\varrho](\boldsymbol{\lambda})-\chi[\tau](\boldsymbol{\lambda})\}^2.$$

Lemma 4. Consider a bipartite state $\varrho=\varrho_A\otimes\varrho_G$. If ϱ_G is a Gaussian state, then $\delta[\varrho]=\delta[\varrho_A]$.

Proof. We have

$$\mu[\varrho]=\mu[\varrho_A]\mu[\varrho_G],$$

$$\mu[\tau]=\mu[\tau_A]\mu[\tau_G],$$

$$\kappa[\varrho,\tau]=\kappa[\varrho_A,\tau_A]\kappa[\varrho_G,\varrho_G].$$

Therefore, since $\kappa[\varrho_G,\varrho_G]=\mu[\varrho_G]$, we arrive at

$$\delta[\varrho] = \frac{\mu[\varrho_A]\mu[\varrho_G] + \mu[\tau_A]\mu[\varrho_G] - 2\kappa[\varrho_A,\tau_A]\kappa[\varrho_G,\varrho_G]}{2\mu[\varrho_A]\mu[\varrho_G]} = \delta[\varrho_A]. \quad (5)$$

The four properties illustrated by the above lemmas are the natural properties required for a good measure of the non-Gaussian character of a quantum state. Notice that by using the trace distance $D_T[\varrho,\tau]=\frac{1}{2}\mathrm{Tr}|\varrho-\tau|$ instead of the Hilbert-Schmidt distance, we would lose lemmas 3 and 4 and that the invariance expressed by lemma 4 holds thanks to the *renormalization* of the Hilbert-Schmidt distance through the purity $\mu[\varrho]$. We stress the fact that our measure of non-Gaussianity is a computable one: It may be evaluated for any quantum state of n modes by the calculation of the first two moments of the state, followed by the evaluation of the overlap with the corresponding Gaussian state.

Notice that $\delta[\varrho]$ is not additive (nor multiplicative) with respect to the tensor product. If we consider a (separable) multipartite quantum state in the product form $\varrho=\otimes_{k=1}^{n}\varrho_k$, the non-Gaussianity is given by

$$\delta[\varrho] = \frac{\prod_{k=1}^{n}\mu[\varrho_k] + \prod_{k=1}^{n}\mu[\tau_k] - 2\prod_{k=1}^{n}\kappa[\varrho_k,\tau_k]}{2\prod_{k=1}^{n}\mu[\varrho_k]}, \quad (6)$$

where τ_k is the Gaussian state with the same moments of ϱ_k. In fact, since the state ϱ is factorizable, we have that the corresponding Gaussian τ is a factorizable state too.

IV. NON-GAUSSIANITY OF RELEVANT QUANTUM STATES

Let us now exploit the definition (2) to evaluate the non-Gaussianity of some relevant quantum states. At first we consider Fock number states $|p\rangle$ of a single mode as well as multimode factorizable states $|p\rangle^{\otimes n}$ made of n copies of a number state. The reference Gaussian states are a thermal state $\tau_p=\nu(p)$ with average photon number p and a factorizable thermal state $\tau_N=[\nu(p)]^{\otimes n}$ with average photon number p in each mode [15]. Non-Gaussianity may be analytically evaluated, leading to

$$\delta[|p\rangle\langle p|] = \frac{1}{2}\left(1+\frac{1}{2p+1}\right) - \frac{1}{p+1}\left(\frac{p}{p+1}\right)^{p},$$

$$\delta[(|p\rangle\langle p|)^{\otimes n}] = \frac{1}{2}\left[1+\left(\frac{1}{2p+1}\right)^{n}\right] - \left[\frac{1}{p+1}\left(\frac{p}{p+1}\right)^{p}\right]^{n}.$$

In the multimode case of $|p\rangle^{\otimes n}$, we seek the number of copies that maximizes the non-Gaussianity. In Fig. 1 we show both $\delta_p \equiv \delta[|p\rangle\langle p|]$ and $\bar{\delta}_p=\max_n \delta[(|p\rangle\langle p|)^{\otimes n}]$ as a function of p. As is apparent from the plot, the non-Gaussianity of the Fock states $|p\rangle$ increases monotonically with the number of photons, p, with the limiting value $\delta_p=1/2$ obtained for $p\to\infty$. Upon considering multimode copies of Fock states we obtain a larger value of non-Gaussianity: $\bar{\delta}_p$ is a decreasing function of p, approaching $\bar{\delta}=1/2$ from above. The value of $\bar{\delta}_p$ corresponds to $n=3$ for $p<26$ and to $n=2$ for $27\leq p \lesssim 250$.

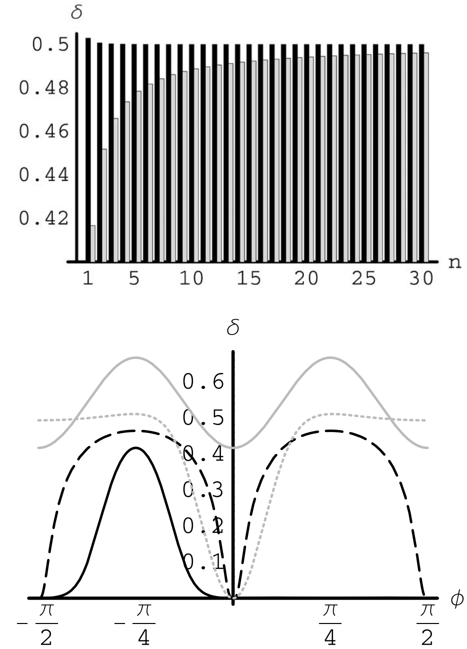

FIG. 1. (Top) Non-Gaussianity of single mode Fock states (gray line) $|p\rangle$ and of multimode Fock states $|p\rangle^{\otimes n}$ (black line) as a function of p. Non-Gaussianity for multimode states has been maximized over the number of copies, n. (Bottom) Non-Gaussianity, as a function of the parameter ϕ, for the two-mode superpositions $|\Phi\rangle$ (dashed gray line), $|\Psi\rangle$ (solid gray line), and for the single-mode superposition of coherent states, $|\psi_S\rangle$, for $\alpha=0.5$ (solid black line) and $\alpha=5$ (dashed black line).

Another example is the superposition of coherent states:

$$|\psi_S\rangle = \mathcal{N}^{-1/2}(\cos\phi|\alpha\rangle + \sin\phi|-\alpha\rangle), \quad (7)$$

with normalization $\mathcal{N}=1+\sin(2\phi)\exp\{-2\alpha^2\}$, which for $\phi=\pm\pi/4$ reduces to the so-called Schrödinger cat states and whose reference Gaussian state is a displaced squeezed thermal state $\tau_S=D(C)S(r)\nu(N)S^{\dagger}(r)D^{\dagger}(C)$, where the real parameters C, r, and N are analytical functions of ϕ and α. Finally, we evaluate the non-Gaussianity of the two-mode Bell-like superpositions of Fock states:

$$|\Phi\rangle = \cos\phi|0,0\rangle + \sin\phi|1,1\rangle,$$

$$|\Psi\rangle = \cos\phi|0,1\rangle + \sin\phi|1,0\rangle,$$

which for $\phi=\pm\pi/4$ reduces to the Bell states $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$. The corresponding reference Gaussian states are, respectively, a two-mode squeezed thermal state $\tau_\Phi=S_2(\xi)[\nu(N)\otimes\nu(N)]S_2^{\dagger}(\xi)$, where $S_2(\xi)=\exp(\xi a_1^{\dagger}a_2^{\dagger}-\xi^{*}ab)$ denotes the two-mode squeezing operator and $\tau_\Psi=R(\theta)[\nu(N_1)\otimes\nu(N_2)]R^{\dagger}(\theta)$—namely, the correlated two-mode state obtained by mixing two thermal states at a beam splitter of transmissivity $\cos^2\theta$, i.e., $R(\theta)=\exp[i\theta(a_1^{\dagger}a_2+a_2^{\dagger}a_1)]$. All the parameters involved in these reference Gaussian states are analytical functions of the superposition parameter ϕ. Non-Gaussianities are thus evaluated by means of (2) and are reported in Fig. 1 as a function of the parameter ϕ. As is apparent from the plot, the non-Gaussianity of single-mode states does not surpass the value $\delta=1/2$, and this fact is confirmed by other examples not reported here.

As concerns the Schrödinger-cat-like states, we notice that for small values of α the non-Gaussianity of the superposition $|\psi_S\rangle$ shows a different behavior for positive and negative values of the parameter ϕ: for $\phi>0$ and $\alpha=0.5$ we have almost zero δ, while higher values are achieved for $\phi<0$. For higher values of α ($\alpha=5$ in Fig. 1), non-Gaussianity becomes an even function of ϕ. This different behavior can be understood by looking at the Wigner functions of even and odd Schrödinger cat states for different values of α: for small values of α the even cat's Wigner function is similar to a Gaussian function, while the odd cat's Wigner function shows a non-Gaussian hole in the origin of phase space; increasing the value of α, the Wigner functions of the two kind of states become similar and deviate from a Gaussian function.

We have also done a numerical analysis of the non-Gaussianity of single-mode quantum states represented by a finite superposition of Fock states:

$$\varrho_d=\sum_{n,k=0}^{d}\varrho_{nk}|n\rangle\langle k|. \tag{8}$$

To this aim we generate random quantum states in a finite-dimensional subspace, $\dim(H)\equiv d+1\leq 21$, following the algorithm proposed by Życzkowski *et al.* [16,17]—i.e., by generating a random diagonal state (i.e., a point on the simplex) and a random unitary matrix according to the Haar measure. In Fig. 2 we report the distribution of non-Gaussianity $\delta[\varrho_d]$, as evaluated for 10^5 random quantum states, for three different values of the maximum number of photons, d. As is apparent from the plots, the distribution of $\delta[\varrho_d]$ becomes Gaussian-like for increasing d. In the fourth panel of Fig. 2 we thus report the mean values and variances of the distributions as a function of the maximum number of photons, d. The mean value increases with the dimension, whereas the variance is a monotonically decreasing function of d.

Also for finite superpositions simulations we did not observe non-Gaussianity higher than 1/2. Therefore, although we have no proof, we conjecture that $\delta=1/2$ is a limiting value for the non-Gaussianity of a single-mode state. Higher values are achievable for two-mode or multimode quantum states (e.g., $\delta=2/3$ for the Bell states $|\Psi^{\pm}\rangle$).

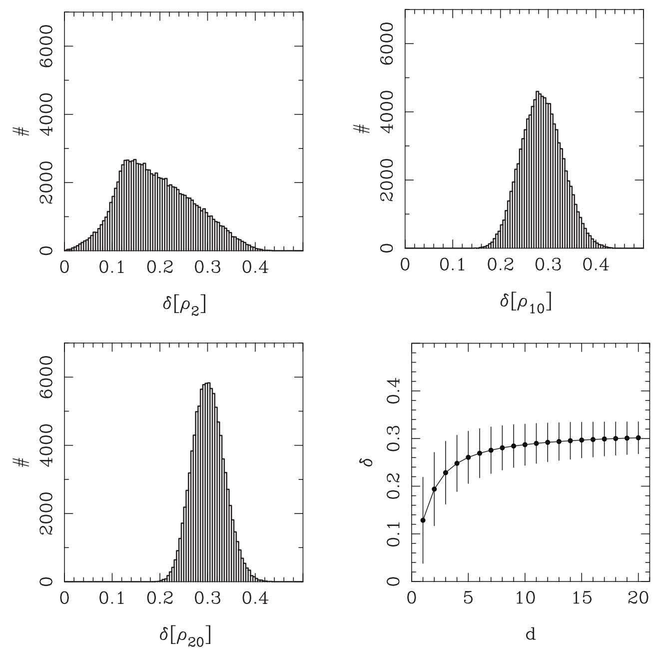

FIG. 2. Distribution of non-Gaussianity $\delta[\varrho_d]$ as evaluated for 10^5 random quantum states, for three different value of the maximum number of photons, d. Top: $d=2$ (left), $d=10$ (right). Bottom: $d=20$ (left). (Bottom right) Mean values and variances of the non-Gaussianities evaluated for 10^5 random quantum states, as a function of the maximum number of photons, d.

V. GAUSSIFICATION AND DE-GAUSSIFICATION PROCESSES

We have also studied the evolution of non-Gaussianity of quantum states undergoing either Gaussification or de-Gaussification processes. First, we have considered the Gaussification of Fock states due do the interaction of the system with a bath of oscillators at zero temperature. This is perhaps the simplest example of a Gaussification protocol. In fact, the interaction drives asymptotically any quantum state to the vacuum state of the harmonic system, which, in turn, is a Gaussian state. The evolution of the system is governed by the Lindblad master equation $\dot{\varrho}=\frac{\gamma}{2}\mathcal{L}[a]\varrho$, where $\dot{\varrho}$ denotes time derivative, γ is the damping factor, and the Lindblad superoperator acts as follows: $\mathcal{L}[a]\varrho=2a^{\dagger}\varrho a-a^{\dagger}a\varrho-\varrho a^{\dagger}a$. Upon writing $\eta=e^{-\gamma t}$ the solution of the master equation can be written as

$$\varrho(\eta)=\sum_m V_m\varrho V_m^{\dagger},$$

$$V_m=[(1-\eta)^m/m!]^{1/2}a^m\eta^{1/2(a^{\dagger}a-m)}, \tag{9}$$

where ϱ is the initial state. In particular, for the system initially prepared in a Fock state, $\varrho_p=|p\rangle\langle p|$, we obtain, after evolution, the mixed state

$$\varrho_p(\eta)=\sum_m V_m\varrho_p V_m^{\dagger}=\sum_{l=0}^{p}\alpha_{l,p}(\eta)|l\rangle\langle l|, \tag{10}$$

with $\alpha_{l,p}(\eta)=\binom{p}{l}(1-\eta)^{p-l}\eta^l$. The reference Gaussian state corresponding to $\varrho_p(\eta)$ is a thermal state $\tau_p(\eta)=\nu(p\eta)$ with average photon number $p\eta$. Non-Gaussianity of $\varrho_p(\eta)$ can be evaluated analytically; we have

$$\delta_{p\eta}\equiv\delta[\varrho_p(\eta)]=\frac{1}{2(1-\eta)^{2m}{}_2F_1\left(-m,-m,1;\frac{\eta^2}{(\eta-1)^2}\right)}\times\left\{(1-\eta)^{2m}{}_2F_1\left(-m,-m,1;\frac{\eta^2}{(\eta-1)^2}\right)+(1+2m\eta)^{-1}-\frac{2[1+(m-1)\eta]^m}{(1+m\eta)^{m+1}}\right\}, \tag{11}$$

${}_2F_1(a,b,c;x)$ being a hypergeometric function. We show the behavior of $\delta_{p\eta}$ in Fig. 3 as a function of $1-\eta$ for different values of p. As is apparent from the plot, $\delta_{p\eta}$ is a monotonically decreasing function of $1-\eta$ as well as a monotonically increasing function of p. That is, at fixed time t, the higher the initial photon number p is, the larger the resulting non-Gaussianity.

Let us now consider the de-Gaussification protocol obtained by the process of photon subtraction. Inconclusive photon subtraction (IPS) has been introduced for single-mode and two-mode states in [6,7,18] and experimentally realized in [9]. In the IPS protocol an input state $\varrho^{(in)}$ is mixed with the vacuum at a beam splitter (BS) with transmissivity T and then, on and off photodetection with quantum efficiency ϵ is performed on the reflected beam. The process can be thus characterized by two parameters: the transmissivity T and the detector efficiency ϵ. Since the detector can only discriminate the presence from the absence of light, this measurement is inconclusive; namely it does not resolve the number of detected photons. When the detector clicks, an unknown number of photons is subtracted from the initial state and we obtain the conditional IPS state ϱ_{IPS}. The conditional map induced by the measurement is non-Gaussian [7], and the output state is de-Gaussified. Upon applying the IPS protocol to the (Gaussian) single-mode squeezed vacuum $S(r)|0\rangle$ $(r\in\mathbb{R})$, where $S(r)$ is the real squeezing operation, we obtain [18] the conditional state ϱ_{IPS}, whose characteristic function $\chi[\varrho_{IPS}](\lambda)$ is a sum of two Gaussian functions and therefore is no longer Gaussian. The corresponding Gaussian reference state is a squeezed thermal state $\tau_{IPS}=S(\xi_{IPS})\nu(N_{IPS})S^{\dagger}(\xi_{IPS})$ where the parameters ξ_{IPS} and N_{IPS} are analytic functions of r, T, and ϵ. Non-Gaussianity $\delta_{IPS}=\delta_{IPS}(T,\epsilon,r)$ has been evaluated, and in Fig. 3 (bottom) we report δ_{IPS} for $r=0.5$ as a function of the transmittivity T for different values of the quantum efficiency ϵ. As is apparent from the plot, the IPS protocol indeed de-Gaussifies the input state; i.e., nonzero values of the non-Gaussianity are obtained. We found that δ_{IPS} is an increasing function of the transmissivity T which is the relevant parameter, while the quantum efficiency ϵ only slightly affects the non-Gaussian character of the output state. The highest value of non-Gaussianity is achieved in the limit of unit transmissivity and unit quantum efficiency:

$$\lim_{T,\eta\to 1}\delta_{IPS}=\delta[|1\rangle\langle 1|]=\delta[S(r)|1\rangle\langle 1|S^{\dagger}(r)],$$

where the last equality is derived from lemma 2. This result is in agreement with the fact that a squeezed vacuum state undergoing the IPS protocol is driven toward the target state $S(r)|1\rangle$ in the limit of $T,\epsilon\to 1$ [18]. Finally, we notice that for $T,\epsilon\neq 1$ and for $r\to\infty$ the non-Gaussianity vanishes. In turn, this corresponds to the fact that one of the coefficients of the two Gaussians of $\chi[\varrho_{IPS}](\lambda)$ vanishes; i.e., the output state is again a Gaussian one.

VI. CONCLUSION AND OUTLOOK

Having at our disposal a good measure of non-Gaussianity for the quantum state allows us to define a measure of the non-Gaussian character of a quantum operation. Let us denote by $\mathcal{G}$ the whole set of Gaussian states. A convenient definition for the non-Gaussianity of a map $\mathcal{E}$ reads as follows: $\delta[\mathcal{E}]=\max_{\varrho\in\mathcal{G}}\delta[\mathcal{E}(\varrho)]$, where $\mathcal{E}(\varrho)$ denotes the quantum state obtained after the evolution imposed by the map. Indeed, for a Gaussian map $\mathcal{E}_g$, which transforms any input Gaussian state into a Gaussian state, we have $\delta[\mathcal{E}_g]=0$. Work along this line is in progress, and results will be reported elsewhere.

In conclusion, we have proposed a measure of the non-Gaussian character of a *CV* quantum state. We have shown that our measure satisfies the natural properties expected from a good measure of non-Gaussianity and have evaluated the non-Gaussianity of some relevant states, in particular of states undergoing Gaussification and de-Gaussification protocols. Using our measure, an analog non-Gaussianity measure for quantum operations may be introduced.

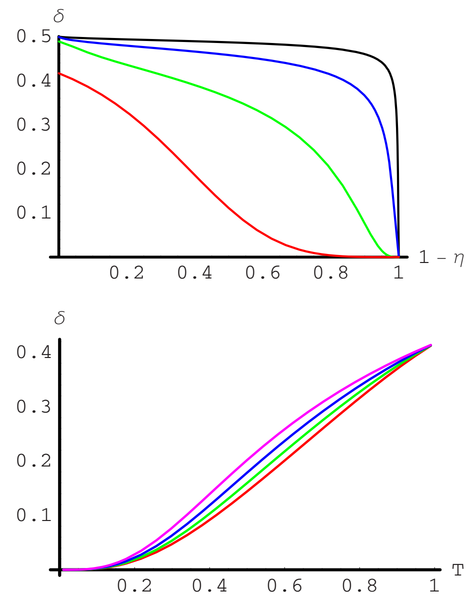

FIG. 3. (Color online) (Top) Non-Gaussianity of Fock states $|p\rangle$ undergoing Gaussification by the loss mechanism due to the interaction with a bath of oscillators at zero temperature. We show $\delta_{\eta p}$ as a function of $1-\eta$ for different values of p: from bottom to top, $p=1,10,100,1000$. (Bottom) Non-Gaussianity of ϱ_{IPS} as a function of T for $r=0.5$ and for different values of $\epsilon=0.2,0.4,0.6,0.8$ (from bottom to top). δ_{IPS} results to be a monotonous increasing function of T, while ϵ only slightly changes the non-Gaussian character of the state.

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APPENDIX: GAUSSIAN REFERENCE WITH UNCONSTRAINED MEAN VALUE

As we have seen from the above examples, $\delta[\varrho]$ of Eq. (2) represents a good measure of the non-Gaussian character

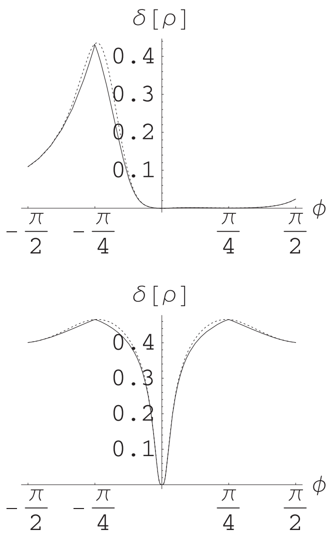

FIG. 4. Non-Gaussianity of a Schrödinger-cat-like state as a function of the superposition parameter ϕ, with either C obtained by numerical minimization (solid line) or with $C=\mathrm{Tr}[a\varrho]$ (dotted line). (Left) $\alpha=0.5$. (Right) $\alpha=5$.

of a quantum state. A question arises at to whether different choices for the reference Gaussian state τ may lead to alternative, valid, definitions. For example (for single-mode states), we may define

$$\delta'[\varrho] = \min_{\tau} \mathrm{D}^2_{HS}[\varrho,\tau]/\mu[\varrho], \tag{A1}$$

where $\tau=D(C)S(\xi)\nu(N)S^{\dagger}(\xi)D^{\dagger}(C)$ is a Gaussian state with the same covariance matrix of ϱ and unconstrained vector of mean values $\boldsymbol{X}=(\mathrm{Re}\, C, \mathrm{Im}\, C)$ used to minimize the Hilbert-Schmidt distance. Here we report a few examples of the comparison between the results already obtained using (2) with that coming from (A1). As we will see, either the two definitions coincide or δ' and δ are monotone functions of each other. Since the definition (2) corresponds to an easily computable measure, we conclude that it represents the most convenient choice.

Let us first consider the Fock state $\varrho=|p\rangle\langle p|$. According to (A1), the reference Gaussian state is given by a displaced thermal state $\tau'=D(C)\nu_p D^{\dagger}(C)$. The overlap between ϱ and τ' is given by

$$\kappa[|p\rangle\langle p|,\tau'] = \frac{1}{1+p}\exp\left\{-\frac{C^2}{1+p}\right\}\left(\frac{p}{1+p}\right)^p L_p\left(-\frac{C^2}{p(1+p)}\right). \tag{A2}$$

The maximum of (A2) is achieved for $C=0$, which coincides with the assumptions $C=\mathrm{Tr}[a|p\rangle\langle p|]$.

Let us consider the quantum state (10) obtained as the solution of the loss master equation for an initial Fock state $|p\rangle\langle p|$. The unconstrained Gaussian reference is again a displaced thermal state $\tau'=D(C)\nu_{p\eta}D^{\dagger}(C)$, and the overlap is given by

$$\kappa[\varrho_p(\eta),\tau'] = \mathrm{Tr}[\tau\varrho_p(\eta)] = \frac{[1+\eta(p-1)]^p}{(1+p\eta)^{p+1}} L_p\left(\frac{\eta|C|^2}{(1+p\eta)[\eta(1-p)-1]}\right) \times e^{-|C|^2/1+p\eta}.$$

Again, since the overlap is maximum for $C=\mathrm{Tr}[a\varrho_p(\eta)]=0$, both definitions give the same results for the non-Gaussianity.

Let us now consider the Schrödinger-cat-like states of (7). The reference Gaussian state is a displaced squeezed thermal state, with squeezing and thermal photons as calculated before. The optimization over the free parameter C may be done numerically. In Fig. 4 we show the non-Gaussianitiy, both as resulting from (A1) and by choosing $C=\mathrm{Tr}[a\varrho_S]$ as in (2), as a function of ϵ. The two curves are almost the same, with no qualitative differences.

[1] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Napoli, 2005).
[2] J. Eisert and M. B. Plenio, Int. J. Quantum Inf. **1**, 479 (2003).
[3] F. Dell'Anno *et al.*, Phys. Rep. **428**, 53 (2006).
[4] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[5] T. Opatrny, G. Kurizki, and D. G. Welsch, Phys. Rev. A **61**, 032302 (2000).
[6] P. T. Cochrane, T. C. Ralph, and G. J. Milburn, Phys. Rev. A **65**, 062306 (2002).
[7] S. Olivares, M. G. A. Paris, and R. Bonifacio, Phys. Rev. A **67**, 032314 (2003); S. Olivares and M. G. A. Paris, Laser Phys. **16**, 1533 (2006).
[8] N. J. Cerf, O. Kruger, P. Navez, R. F. Werner, and M. M. Wolf, Phys. Rev. Lett. **95**, 070501 (2005).
[9] J. Wenger, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **92**, 153601 (2004); A. Ourjoumtsev *et al.*, Science **312**, 83 (2006).
[10] M. M. Wolf, G. Giedke, and J. I. Cirac, Phys. Rev. Lett. **96**, 080502 (2006).
[11] M. M. Wolf, D. Perez-Garcia, and G. Giedke, Phys. Rev. Lett. **98**, 130501 (2007).
[12] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).
[13] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 4002 (2000); R. F. Werner and M. M. Wolf, *ibid.* **86**, 3658 (2001).
[14] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1882 (1969).
[15] P. Marian and T. A. Marian, Phys. Rev. A **47**, 4474 (1993).
[16] K. Życzkowski and M. Kús, J. Phys. A **27**, 4235 (1994).
[17] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).
[18] S. Olivares and M. G. A. Paris, J. Opt. B: Quantum Semiclassical Opt. **7**, S392 (2005).