On the algebraic independence in the Selberg class

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Abstract

We prove that a function F of the Selberg class $\mathcal{S}$ is a b-th power in $\mathcal{S}$, i.e., $F = H^b$ for some $H \in \mathcal{S}$, if and only if b divides the order of every zero of F and of every p-component F_p. This implies that the equation $F^a = G^b$ with $(a, b) = 1$ has the unique solution $F = H^b$ and $G = H^a$ in $\mathcal{S}$. As a consequence, we prove that if F and G are distinct primitive elements of $\mathcal{S}$, then the transcendence degree of $\mathbb{C}[F, G]$ over $\mathbb{C}$ is two.

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Statement of the results

The Selberg class $\mathcal{S}$ (see the survey [3] for the basic definitions, conjectures and properties) has a natural structure of semigroup, i.e., $FG \in \mathcal{S}$ if $F, G \in \mathcal{S}$. The additivity of the degree d_F, the existence of a unique element $F = 1$ with zero-degree and the non-existence of functions with $0 < d_F < 1$ imply that every $F \in \mathcal{S}$ has a factorization into primitive elements.

In this paper we consider the following problem: given $b \in \mathbb{N}^*$, when an element of $\mathcal{S}$ is a b-th power? In other words, when the equation

$$x^b = F \tag{1}$$

has a solution in $\mathcal{S}$? It is known that under Selberg's orthonormality conjecture the factorization is actually unique, so that (1) has a solution if and only if the multiplicity of every primitive factor of F is divisible by b; nevertheless, there are no unconditional results about this subject, thus our problem is interesting and non-trivial.

It is not difficult to show that (1) admits always one solution at most: in fact, if G and H are both solutions in $\mathcal{S}$, then $(G(s)/H(s))^b = 1$ identically, but G and H are meromorphic functions (with a pole at $s = 1$ at most), therefore $G(s)/H(s)$ is entire (from $(G(s)/H(s))^b = 1$) and hence it is constant (from $(G(s)/H(s))^b = 1$ again). Being elements of $\mathcal{S}$, both G and H have a representation as Dirichlet series, convergent for $\sigma > 1$, so $G(\sigma), H(\sigma) \to 1$ for $\sigma \to +\infty$, therefore the constant is 1 and $G = H$.

Our approach to (1) is the following: every function F of $\mathcal{S}$ has a unique representation as Euler product, $F(s) = \prod_p F_p(p^{-s})$ for $\sigma > 1$, moreover by the Ramanujan conjecture about the growth of the coefficients, every p-component $F_p(p^{-s})$ is a holomorphic function on $\sigma > 0$. Therefore, let $m_f(\rho)$ be the order of ρ as zero of a meromorphic function f (with $m_f(\rho) < 0$ when ρ is a pole of f), then

$$\text{(1) has a solution} \quad \Longrightarrow \quad \begin{cases} b|m_F(\rho) & \text{for every } \rho \in \mathbb{C}, \\ b|m_{F_p}(\rho) & \text{for every } p, \text{ for every } \Re\rho > 0. \end{cases}$$

The following theorem states the principal result of this paper.

Theorem. *Let $F \in \mathcal{S}$, then*

$$x^b = F \textit{ has a solution } x \in \mathcal{S} \quad \Longleftrightarrow \quad \begin{cases} b|m_F(\rho) & \textit{for every } \rho \in \mathbb{C}, \\ b|m_{F_p}(\rho) & \textit{for every } p, \textit{ for every } \Re\rho > 0. \end{cases}$$

Remark 1. The gamma factor γ_F giving the functional equation of F can be described as the component F_∞ corresponding to the archimedean valuation of $\mathbb{Q}$. In $\mathcal{S}$ the gamma factor is a product of exponentials and Γ-functions, therefore it is a meromorphic function without zeros, moreover, by the functional equation, its poles are located at the trivial zeros of F so that $b|m_{F_\infty}(\rho)$ when $b|m_F(\rho)$. We expect that conditions of "local type" or of "global type", alone, are sufficient for the existence of a solution of (1) when are assumed on the entire complex plain, i.e., that the following equivalences hold:

$$x^b = F \quad \Longleftrightarrow \quad b|m_{F_p}(\rho) \;\; \forall\, \rho \in \mathbb{C}, \; \forall\; p\text{-adic valuation, } \infty \text{ included} \quad \Longleftrightarrow \quad b|m_F(\rho) \;\; \forall\, \rho \in \mathbb{C}.$$

Our Theorem is weaker in this respect, since its hypotheses are both of local and of global type. The obstruction to a proof of this conjecture, also when a meromorphic continuation to all $\mathbb{C}$ is assumed for every p-component F_p, is the lack of a relation between $m_{F_p}(\rho)$ and $m_F(\rho)$ into the critical strip $0 \le \Re\rho \le 1$. At last, we remark that it is conjectured that every function of $\mathcal{S}$ is of polynomial type, i.e., that $F_p^{-1}(z)$ is polynomial for every (finite) p: in this case $F_p(p^{-s})$ has no zeros and the condition $b|m_{F_p}(\rho)$, for $\Re\rho > 0$, becomes trivial.

Our Theorem has interesting consequences about a second problem. When the equation $F^a = G^b$ has solutions in $\mathcal{S}$? In some cases the analysis of this equation is very simple; for example, when the two functions have the same degrees $d_F = d_G$ (in this case the additivity of the degree gives $ad_F = d_{F^a} = d_{G^b} = bd_G$ so that $a = b$ and the equation becomes $(F(s)/G(s))^a = 1$, so that $F = G$ is the unique solution) or when there exists a zero for F which is not a zero for G (then the only possibility is for $a = b = 0$). The last case is particularly important since many results support the conjecture that primitive functions always have distinct zeros; if this is true then not only the equation $F^a = G^b$ has the unique solution $F = G$, $a = b$, but the unique factorization would hold. For the moment it is only known that distinct primitive functions have zeros of different multiplicity (see [1], for example); this is a too weak result to deduce some consequence about the problem of the unicity of the factorization.

Remark 2. When F and G satisfy functional equations containing different Γ-factors, there exists a zero of F (a trivial one) which is not a zero for G; this is the only case where we are able to prove this fact. Our present knowledge of the distribution of the zeros of these functions does not leave out the possibility that two functions F and G satisfying the same functional equation have all their zeros located at the same points (but with different multiplicity, of course), nevertheless, nobody believes this can happen.

As a consequence of our theorem we obtain a complete description of the solutions of the equation $F^a = G^b$.

Corollary 1. *Let $F, G \in \mathcal{S}$; if $F^a = G^b$ for some $(a,b) = 1$, then $F = H^b$ and $G = H^a$ for some $H \in \mathcal{S}$.*

Proof. In fact, $am_F(s) = m_{F^a}(s) = m_{G^b}(s) = bm_G(s)$ and $am_{F_p}(s) = m_{F_p^a}(s) = m_{G^b}(s) = bm_{G_p}(s)$ for every prime p, by the Euler product. Since $(a,b) = 1$, it follows that $b|m_F$ and $b|m_{F_p}$; by our Theorem there exists $H \in \mathcal{S}$ such that $F = H^b$. Therefore, $G^b = F^a = H^{ab}$, hence $(G/H^a)^b = 1$ and as usual this implies $G = H^a$. □

In [2] it has been proved that distinct elements of $\mathcal{S}$ are linearly independent over the ring $\mathcal{F}$ of the *p-finite* Dirichlet series, i.e., the ring of the Dirichlet series $\sum_{n=1}^{\infty} c_n n^{-s}$ absolutely converging somewhere and whose coefficients are supported on a finite set of primes. In the same paper it has been pointed out that this fact implies the equivalence of the algebraic independence in $\mathcal{S}$ to the unique factorization in $\mathcal{S}$; thus, it is not surprising that every result about the factorization has a consequence about the algebraic independence. In fact, the mere linear independence implies

Corollary 2. *Let F be a non-trivial function of the Selberg class, then F is a transcendental element over the ring $\mathcal{F}$.*

Proof. Suppose that there exists a non-trivial polynomial $P(x) \in \mathcal{F}[x]$ such that $P(F(s)) = 0$, identically. By the linear independence over $\mathcal{F}$ of different elements of $\mathcal{S}$ it follows that $F^a = F^b$ for some $0 \le a < b$, and therefore $F^{b-a} = 1$: a contradiction. □

Corollary 1 strengthens Corollary 2 in the following way.

Corollary 3. *Let F, G, both powers of distinct primitive element of $\mathcal{S}$, then the transcendence degree of the ring $\mathcal{F}[F, G]$ over $\mathcal{F}$ is two.*

Proof. We can restrict the proof to the case of F and G both primitive; moreover, by Corollary 2 it is sufficient to prove that F and G are algebraically independent over $\mathcal{F}$. Assume that a polynomial equation of type $P(F(s), G(s)) = 0$ holds identically, for some non-trivial element $P \in \mathcal{F}[x, y]$. By the linear independence already proved in [2] it follows that $F^{\tilde{a}} G^{\tilde{b}} = F^{\tilde{c}} G^{\tilde{d}}$ for some $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in $\mathbb{N}$. By cancelling the common factors we can obtain $F^a G^b = 1$ or $F^a = G^b$ for some $a, b \in \mathbb{N}$, not both equal to 0. The first identity is obviously impossible; for the second one, let m be a common factor of a and b, then $(F^{a/m}/G^{b/m})^m = 1$ and $F^{a/m} = G^{b/m}$ follows so that we can assume $(a,b) = 1$. By Corollary 1 we get $F = H^b$ and $G = H^a$ for some $H \in \mathcal{S}$ but F and G are both primitive by hypothesis, hence $a = b = 1$ and $G = F$: a contradiction. □

Proof of the theorem

By hypothesis $b|m_F(\rho)$ for every $\rho \in \mathbb{C}$, hence there exists a unique meromorphic function H such that $F = H^b$ and verifying the condition $\lim_{\sigma \to +\infty} H(\sigma) = 1$. Moreover, H has not zeros for $\sigma > 1$, it is of finite order and there exists an integer m such that $(s-1)^m H(s)$ is entire since $H^b = F$ has these properties. The proof of the theorem will be completed by proving that $H \in \mathcal{S}$. it will be useful to split the proof in some lemmas.

Representations and Ramanujan conjecture

Lemma 1. (Dirichlet series and Euler product) *Let F and H be as before, then H has a representation as Dirichlet series $H(s) = \sum_{n=1}^{\infty} h_n n^{-s}$ and as Euler product $H(s) = \prod_p (1 + \sum_{m=1}^{\infty} h_{p^m} p^{-s})$, both absolutely converging for $\sigma > 1$.*

Proof. Let $F(s) = \sum_{n=1} a_n n^{-s}$ be the Dirichlet representation of F, then the Ramanujan hypothesis $a_n \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$ gives

$$|a_{p^m}| \leq c(\epsilon) p^{m\epsilon} \quad \forall \epsilon > 0, \text{ for some constant } c(\epsilon) > 0 \text{ independent of } p, \tag{2}$$

therefore $F_p(p^{-s})$ is holomorphic when $\sigma > 0$; moreover, $b | m_{F_p}(\rho)$ for every $\Re\rho > 0$ by hypothesis so that there exists a unique $H_p(p^{-s})$, holomorphic for $\sigma > 0$, such that $H_p^b(p^{-s}) = F_p(p^{-s})$ and $\lim_{\sigma \to +\infty} H_p(p^{-\sigma}) = 1$. Let $H_p(p^{-s}) = 1 + \sum_{m=1}^{\infty} h_{p^m} p^{-ms}$ its representation as p^{-s}-power series, holding for $\sigma > 0$. By (2), into the disk $|z| \leq p^{-\epsilon}$ we have

$$|F_p(z)| \leq \Big| \sum_{m=0}^{\infty} a_{p^m} z^m \Big| \leq \sum_{m=0}^{\infty} c(\epsilon/2) p^{m\epsilon/2} p^{-m\epsilon} = \frac{c(\epsilon/2)}{1 - p^{-\epsilon/2}} \leq c_1(\epsilon) \text{ independent of } p,$$

so that into the same disk

$$|H_p(z)| \leq |F_p(z)|^{1/b} \leq c_2(\epsilon) \text{ independent of } p.$$

By this upper-bound and the Cauchy's estimate about the coefficients of a power series of an holomorphic function we get

$$|h_{p^m}| \leq \sup_{|z| = p^{-\epsilon}} |H_p(z)| p^{m\epsilon} \leq c_2(\epsilon) p^{m\epsilon}, \quad \text{for every } \epsilon > 0, p \text{ and } m. \tag{3}$$

From (3) it follows that $\prod_p (1 + \sum_{m=1}^{\infty} |h_{p^m}| p^{-m\sigma})$ converges for $\sigma > 1$, so that if h_n is the multiplicative sequence defined on the p-th powers as h_{p^m}, then both $\sum_{n=1}^{\infty} h_n n^{-s}$ and $\prod_p (1 + \sum_{m=1}^{\infty} h_{p^m} p^{-ms})$ are absolutely convergent for $\sigma > 1$ and represent the same function $K(s)$. Besides, $\lim_{\sigma \to +\infty} K(\sigma) = 1$ since $K(s)$ is an Euler product and

$$K^b(s) = \prod_p (1 + \sum_{m=1}^{\infty} h_{p^m} p^{-ms})^b = \prod_p H_p^b(p^{-s}) = \prod_p F_p(p^{-s}) = F(s),$$

again from the absolute convergence for $\sigma > 1$, therefore $H = K$ and the claim is proved. □

Lemma 2. (Ramanujan conjecture) *Let F and H be as before, then $\ln H(s) = \sum_{n=1}^{\infty} \tilde{h}_n n^{-s}$ with $|\tilde{h}_{p^m}| \leq p^{\theta m}$ for some $0 < \theta < 1/2$; moreover, $H(s) = \sum_{n=1}^{\infty} h_n n^{-s}$ with $h_n \ll_\epsilon n^\epsilon$, for every $\epsilon > 0$.*

Proof. The first claim is trivial since the same statement holds for $\ln F(s)$ and $\ln H(s) = \frac{1}{b} \ln F(s)$. For the second one, let $\nu(n) = \sum_{p|n} 1$, from (3) we get

$$|h_n| \leq c_2^{\nu(n)}(\epsilon) n^\epsilon,$$

and from the density of primes $\nu(n) \ll \ln n / \ln\ln n$, hence $c_2^{\nu(n)}(\epsilon) \ll_\epsilon n^\epsilon$ and $h_n \ll_\epsilon n^{2\epsilon}$ follows. □

Functional equation

The functional equation of the elements in the Selberg class is ruled by a product of functions of the type $Q^s\Gamma(\alpha s+\beta)$, said γ-factors in this context. Clearly, there is a relation between the values of $H(s)$ and $\bar{H}(1-s)=\overline{H(1-\bar{s})}$ coming from the functional equation of $F=H^b$; we will be able to show that this relation is actually given by a functional equation of standard type as consequence of the following lemma about γ-factors.

Lemma 3. (Roots of γ-factors) *Let $\mathcal{T}$ be the semigroup generated by the functions $aQ^s\Gamma(\alpha s+\beta)$, where $a\in\mathbb{C}^*$, $Q,\alpha>0$ and $\Re\beta\geq 0$, with the usual product. Then, the equation*

$$x^b=\mathcal{F} \qquad \text{with} \qquad b\in\mathbb{N}^*,\ \mathcal{F}\in\mathcal{T}, \tag{4}$$

has a solution in $\mathcal{T}$ if and only if b divides the order of every pole of $\mathcal{F}$.

Proof. The necessity of the required condition is evident. Let

$$\mathcal{F}(s)=aQ^s\prod_j\Gamma(\alpha_j s+\beta_j) \tag{5}$$

and assume that b divides the order of every pole of $\mathcal{F}$. As in [4], we say that a Γ-factor $\Gamma(\alpha s+\beta)$ is $\mathbb{Q}$-equivalent to $\Gamma(\tilde{\alpha}s+\tilde{\beta})$ when $\alpha/\tilde{\alpha}\in\mathbb{Q}$. We split the Γ-factors appearing in (5) into families $l=1,\dots,h$ belonging to different $\mathbb{Q}$-classes

$$\mathcal{F}(s)=aQ^s\prod_{l=1}^h\prod_j\Gamma(\alpha_{l,j}s+\beta_{l,j});$$

by the definition of $\mathbb{Q}$-equivalence, there exist h real numbers $\tilde{\alpha}_l$ such that $\alpha_{l,j}=n_{l,j}\tilde{\alpha}_l$ with $n_{l,j}\in\mathbb{N}^*$, therefore

$$\mathcal{F}(s)=aQ^s\prod_{l=1}^h\prod_j\Gamma\big(n_{l,j}(\tilde{\alpha}_l s+\tfrac{\beta_{l,j}}{n_{l,j}})\big), \tag{6}$$

and applying the Legendre-Gauss formula

$$\Gamma(s)=m^{s-\frac{1}{2}}(2\pi)^{\frac{1-m}{2}}\prod_{k=0}^{m-1}\Gamma\Big(\frac{s+k}{m}\Big) \qquad m=2,3,\dots,$$

to every Γ-factor of (6) we obtain

$$\mathcal{F}(s)=\check{a}\check{Q}^s\prod_{l=1}^h\prod_j\Gamma(\tilde{\alpha}_l s+\tilde{\beta}_{l,j}) \tag{7}$$

for some $\check{a}\in\mathbb{C}^*$, $\check{Q}>0$ and suitable $\tilde{\beta}_{l,j}$ with $\Re(\tilde{\beta}_{l,j})\geq 0$. Now we introduce a second relation between the Γ-factors belonging to the same $\mathbb{Q}$-class: we say that $\tilde{\beta}_{l,j_1}$ is $\mathbb{Z}$-equivalent to $\tilde{\beta}_{l,j_2}$ when $\tilde{\beta}_{l,j_1}-\tilde{\beta}_{l,j_2}\in\mathbb{Z}$. Let $\check{\beta}_{l,j}$ be the smallest element in every $\mathbb{Z}$-class appearing in (7). Every Γ-factor belonging to the $\mathbb{Z}$-class of $\check{\beta}_{l,j}$ can be written as

$$(\tilde{\alpha}_l s+\check{\beta}_{l,j})(\tilde{\alpha}_l s+\check{\beta}_{l,j}+1)\cdots(\tilde{\alpha}_l s+\check{\beta}_{l,j}+r_{l,j})\Gamma(\tilde{\alpha}_l s+\check{\beta}_{l,j})$$

by repeated applications of the factorial identity $\Gamma(s+1) = s\Gamma(s)$, therefore we can introduce the polynomials

$$p_l(s) = \prod_j (\tilde{\alpha}_l s + \check{\beta}_{l,j})(\tilde{\alpha}_l s + \check{\beta}_{l,j} + 1) \cdots (\tilde{\alpha}_l s + \check{\beta}_{l,j} + r_{l,j}) \tag{8}$$

and the partial γ-factors

$$\gamma_l(s) = \prod_j \Gamma^{k_{l,j}}(\tilde{\alpha}_l s + \check{\beta}_{l,j}), \qquad k_{l,j} \in \mathbb{N}$$

in such a way that $\check{\beta}_{l,j}$ be distinct and

$$\phi_l(s) = p_l(s)\gamma_l(s), \qquad F(s) = \check{a}\check{Q}^s \prod_{l=1}^h \phi_l(s).$$

We remark that two Γ-factors $\Gamma(\tilde{\alpha}_1 s + \check{\beta}_{1,j})$ and $\Gamma(\tilde{\alpha}_2 s + \check{\beta}_{2,k})$ belonging to different $\mathbb{Q}$-classes can have a common pole at most, hence there is an infinite set S_1 of poles of $\gamma_1(s)$ that are neither poles of any $\phi_l(s)$ with $l \neq 1$ or zeros of $p_1(s)$. Moreover, two elements $\Gamma(\tilde{\alpha}_1 s + \check{\beta}_{1,j})$ and $\Gamma(\tilde{\alpha}_1 s + \check{\beta}_{1,k})$ in the same $\mathbb{Q}$-class but different $\mathbb{Z}$-class cannot have any common pole, so that we conclude that there exists at least a pole $\rho \in S_1$ such that $k_{1,1} = m_{\Gamma^{k_{1,1}}(\tilde{\alpha}_1 s + \check{\beta}_{1,1})}(\rho) = m_F(\rho)$. By hypothesis b divides $m_F(\rho)$, therefore $b|k_{1,1}$; in a similar way we prove that $b|k_{l,j}$ for every l, j. Let $k_{l,j,-1} = k_{l,j}/b$, then

$$F(s) = \check{a}\check{Q}^s \Big(\prod_{l=1}^h p_l(s)\Big) \Big(\prod_{l=1}^h \prod_j \Gamma^{k_{l,j,-1}}(\tilde{\alpha}_l s + \check{\beta}_{l,j})\Big)^b. \tag{9}$$

Again, by hypothesis b divides the order of every pole of $F(s)$ and b divides obviously the order of every pole of the γ-factor to the right-side of (9), therefore b divides the order of every zero of the polynomial $\prod_{l=1}^h p_l(s)$, too. By the form (8) of every $p_l(s)$, every zero of $\prod_{l=1}^h p_l(s)$ is $-(\check{\beta}_{l,j} + n)/\tilde{\alpha}_l$ for some index l, j and some non-negative integer n; by (8) again in this case the product

$$(\tilde{\alpha}_l s + \check{\beta}_{l,j})(\tilde{\alpha}_l s + \check{\beta}_{l,j} + 1) \cdots (\tilde{\alpha}_l s + \check{\beta}_{l,j} + n)$$

divides $\prod_{l=1}^h p_l(s)$, therefore we get that

$$\prod_{l=1}^h p_l(s) = \prod_{l=1}^h \Big[\prod_j (\tilde{\alpha}_l s + \check{\beta}_{l,j})^{k_{l,j,0}} (\tilde{\alpha}_l s + \check{\beta}_{l,j} + 1)^{k_{l,j,1}} \cdots (\tilde{\alpha}_l s + \check{\beta}_{l,j} + r_{l,j})^{k_{l,j,r_{l,j}}}\Big]^b$$

for some positive integers

$$k_{l,j,-1} \geq k_{l,j,0} \geq k_{l,j,1} \geq \cdots \geq k_{l,j,r_{l,j}}, \tag{10}$$

so that

$$F(s) = \check{a}\check{Q}^s \prod_{l=1}^h \Big[\prod_j (\tilde{\alpha}_l s + \check{\beta}_{l,j})^{k_{l,j,0}} (\tilde{\alpha}_l s + \check{\beta}_{l,j} + 1)^{k_{l,j,1}} \cdots (\tilde{\alpha}_l s + \check{\beta}_{l,j} + r_{l,j})^{k_{l,j,r_{l,j}}} \Gamma^{k_{l,j,-1}}(\tilde{\alpha}_l s + \check{\beta}_{l,j})\Big]^b. \tag{11}$$

The inequalities (10) show that the polynomial terms appearing in (11) can be completely absorbed into the Γ-factors by the factorial identity; taking a b-th root of $\check{a}$ and the positive b-th root of $\check{Q}$ we obtain the claim. □

Lemma 4. (Functional equation) *Let F and H be as before, then H satisfies a functional equation of the type required in the Selberg class.*

Proof. Let

$$\Gamma_F(s) = \prod_{j=1}^{l} \Gamma(\alpha_j s + \beta_j),$$
$$\phi(s) = Q^s \Gamma_F(s) F(s),$$
$$\phi(s) = \omega \bar{\phi}(1-s),$$

with $Q, \alpha_j > 0$, $\Re\beta_j \geq 0$, and $|\omega| = 1$, be the functional equation satisfied by F.

We have already remarked that the poles of Γ_F are located at the trivial zeros of F, therefore b divides the order of every pole of $\mathcal{F}(s) = \phi(s)/F(s) = Q^s\Gamma_F(s) \in \mathcal{T}$: by Lemma 3 there exists an element $\tilde{\mathcal{F}} \in \mathcal{T}$ such that $\mathcal{F} = \tilde{\mathcal{F}}^b$. Let $\psi(s) = \tilde{\mathcal{F}}(s)H(s)$; then $\phi = \psi^b$ and from the functional equation of F we get

$$\Big(\frac{\psi(s)}{\bar{\psi}(1-s)}\Big)^b = \frac{\phi(s)}{\bar{\phi}(1-s)} = \omega;$$

but $\psi(s)$ is a meromorphic function of $\mathbb{C}$, hence there exists a b-th root $\tilde{\omega}$ of ω, such that

$$\psi(s) = \tilde{\omega}\bar{\psi}(1-s) \qquad \forall s \in \mathbb{C}.$$

□

Theorem follows by Lemma 1, 2 and 4.

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