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Elliptic Systems with Nonlinearities of Arbitrary Growth

Djairo G. de Figueiredo and Bernhard Ruf

Abstract. In this paper we study the existence of nontrivial solutions for the following system of coupled semilinear Poisson equations:

$$\begin{cases} -\Delta u = v^p , & \text{in } \Omega, \\ -\Delta v = f(u) , & \text{in } \Omega, \\ u = 0 \text{ and } v = 0 , & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$. We assume that $0 < p < \frac{2}{N-2}$, and the function f is superlinear and with no growth restriction (for example $f(s) = s\,e^s$); then the system has a nontrivial (strong) solution.

1. Introduction

We consider the system of equations

$$\begin{cases} -\Delta u = g(v) \ , \ \text{in } \Omega \\ -\Delta v = f(u) \ , \ \text{in } \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases} , \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. It is known, see [5], [11], [15], that for the "model case"

$$f(s) = s^q \ , \ q > 1 \ , \quad \text{and} \qquad g(s) = s^p \ , \ p > 1 \ ,$$

(here and in what follows, $s^\alpha := \text{sgn(s)}|s|^\alpha$) the system (1) has a nontrivial solution provided that

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N} \tag{2}$$

For $N = 2$ this condition is satisfied for any $p > 1$ and $q > 1$.
For $N \geq 3$, the curve of $(p, q) \in \mathbb{R}^2$ satisfying $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}$ is the so-called "critical hyperbola": for points (p, q) on this curve one finds the typical problems of non-compactness, and non-existence of solutions, as it was proved in [23], [18], using Pohozaev type arguments.

The case N=2
As mentioned above, for $N = 2$ any pair of powers $(p, q) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfies the inequality (2). Actually, even a higher growth than polynomial is admitted: by the inequality of Trudinger-Moser, see [22], [19], [20], *subcritical growth* for a single equation is given by the condition (see [10])

$$\lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = 0 \ , \ \forall \ \alpha > 0$$

It follows from a result in de Figueiredo-do Ó-Ruf [8] that system (1) has a non-trivial solution for nonlinearities f and g with such subcritical growth (and satisfying an Ambrosetti-Rabinowitz condition, see [2]). Also existence results for certain nonlinearities with critical growth are given in [8]. In this paper we consider a different type of extension of the known results: We will show that if one nonlinearity, say g, has polynomial growth (of any order), then, to prove existence of solutions, *no growth restriction* is required on the other nonlinearity f (other than the Ambrosetti-Rabinowitz condition).

The case N=3
Note that for $N = 3$ the critical hyperbola has the asymptotes $p_\infty = 2$ and $q_\infty = 2$. In particular, if $g(s) = s^p$ with $1 < p < 2$, then the cited existence results say that there exists a solution (u, v) for system (1) with $f(s) = s^q$, for any $q > 1$. Also in this case we show that existence of solutions can be proved requiring *no growth restriction* whatsoever on the nonlinearity f (other than the Ambrosetti-Rabinowitz condition).

The case N $\geq$ 4
For $N \geq 4$ the asymptotes of the critical hyperbola are in the values $p_\infty = \frac{2}{N-2} \leq 1$ and $q_\infty = \frac{2}{N-2} \leq 1$. Note that for an exponent $p < 1$, the corresponding equation in the system is *sublinear.* i.e. we have a system with one sublinear and one superlinear equation. In this situation, the proposed approach is no longer applicable. However, in this case a reduction of the system to a single equation is possible (see Clément-Felmer-Mitidieri [6] and Felmer - Martínez [12]), which allows to prove again a result of the same form; moreover this approach also allows to extend to the whole range the cases $N = 2$ and $N = 3$, that is for $N = 2\ :\ 0 < p < +\infty$, and for $N = 3\ :\ 0 < p < 2$.

The main result of the paper is stated in the following theorem:

Theorem 1.1. *Suppose that*

1) $g(s) = s^p$, *with* $\begin{cases} 0 < p\ , & \textit{if}\ \ N = 2 \\ 0 < p < \frac{2}{N-2}\ , & \textit{if}\ \ N \geq 3 \end{cases}$

2) $f \in C(\mathbb{R})$, *and set* $F(s) = \int_0^s f(t)dt$;

- there exist constants $\theta > \begin{cases} 2\ ,\ \textit{if}\ p \geq 1 \\ 1 + \frac{1}{p}\ ,\ \textit{if}\ p < 1 \end{cases}$ *and* $s_0 \geq 0$ *such that*

$\theta F(s) \leq f(s)s\ ,\ \forall\ |s| \geq s_0$

- and for s *near* 0*:* $f(s) = \begin{cases} o(s)\ , & \textit{if}\ p \geq 1 \\ o(s^{1/p})\ , & \textit{if}\ p < 1 \end{cases}$

Then the system

$$\begin{cases} -\Delta u = v^p & in\ \ \Omega, \\ -\Delta v = f(u) & in\ \ \Omega, \\ u = 0\ \ ,\ \ v = 0 & on\ \ \partial\Omega, \end{cases} \tag{3}$$

has a nontrivial (strong) solution.

Remarks
1) It is somewhat surprising that *no growth restriction* needs to be imposed on f, since for the single equation $-\Delta u = f(u)$ growth restrictions are, in general, necessary to prove the existence of solutions; we refer to the non-existence result in [9] for $N = 2$, and to [20] for $N \geq 3$.
2) In the cases with $p > 1$, the nonlinearity $g(s) = s^p$ may be replaced by more general functions, satisfying an Ambrosetti-Prodi type condition like $f(s)$, and the growth restriction

$$|g(s)| \leq c|s|^p + d\ ,\ \ \text{for some constants}\ \ c, d > 0,\ \ \text{and}\ \begin{cases} 1 < p\ , & N = 2 \\ 1 < p < 2\ , & N = 3 \end{cases}$$

For the sake of simplicity, we restrict here to the case $g(s) = s^p$.

For completeness we also state the following theorem:

Theorem 1.2. *Suppose that*

1) (p, q) *satisfy* $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}$, *and* $\frac{2}{N-2} \leq p \leq 1$.

2) $f \in C(\mathbb{R})$, *and there exist constants* $\theta > \frac{p+1}{p}$ *and* $s_0 \geq 0$ *such that*

$$\theta F(s) := \theta \int_0^s f(t)dt \leq f(s)s \ , \quad \forall \ |s| \geq s_0 \ ,$$

and

$$|f(s)| \leq c|s|^q + d \ , \quad \textit{for some constants} \quad c, d > 0 \ .$$

Then the system

$$\begin{cases} -\Delta u &= v^p \quad in \quad \Omega, \\ -\Delta v &= f(u) \quad in \quad \Omega, \\ u = 0 \quad , & v = 0 \quad on \quad \partial\Omega, \end{cases} \tag{4}$$

has a nontrivial (strong) solution.

In the literature we have only found the cases of (p, q) below the critical hyperbola, and with the restriction that $p > 1$ and $q > 1$ (see [5], [15], [11]) and the case $0 < p \cdot q < 1$ (see Felmer-Martínez [12]). This does not cover the whole region below the critical hyperbola. The above theorem covers also the remaining cases below the critical hyperbola, namely

$$0 < p \leq 1 \quad \text{and} \quad p \cdot q \geq 1 \ ;$$

note that we need to make the restriction that the sublinear function v^p is in the form of a power, while the superlinear function $f(u)$ may be of more general form.

2. Proof: the case $p > 1$

In this section we consider the case $1 < p < \frac{2}{N-2}$, i.e. $N = 2, 3$.

2.1. The setting

A natural functional associated to system (1) is

$$J(u, v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega (F(u) + G(v))dx \ , \tag{5}$$

with $F(s) = \int_0^s f(t)dt$ and $G(s) = \int_0^s g(t)dt$. The natural space to consider this functional is the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$; however, in order to have a well-defined C^1-functional on this space, one has to impose certain *growth restrictions*:

in $N = 2$: F and G subcritical in the sense of Trudinger-Moser (see above)

in $N = 3$: $|F(s)| \leq c|s|^6 + d$, $|G(s)| \leq c|s|^6 + d$

These conditions are on the one hand too loose for $G(s) = \frac{1}{p+1}s^{p+1}$, where a more restrictive growth is given, and too strong on $F(s)$, where we do not want any growth limitation.

We therefore follow an idea of de Figueiredo-Felmer [11] and Hulshoff-vanderVorst [15], defining a related functional on suitable *fractional* Sobolev spaces.

Consider the Laplacian as the operator

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega) \ ,$$

and $\{e_i\}_{i=1}^\infty$ a corresponding system of orthogonal and L^2-normalized eigenfunctions, with eigenvalues $\{\lambda_i\}$. Then, writing

$$u = \sum_{n=1}^\infty a_n e_n \ , \quad \text{with} \quad a_n = \int_\Omega u e_n dx \ ,$$

we set

$$E^s = \{u \in L^2(\Omega) \ : \ \sum_{n=1}^{\infty} \lambda_n^s |a_n|^2 < \infty\}$$

and define a linear operator on $L^2(\Omega)$ by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n e_n \ , \ \forall \ u \in D(A^s) := E^s \ .$$

The spaces E^s are *fractional* Sobolev spaces with the inner product

$$(u,v)_s = \int_\Omega A^s u A^s v dx \ ,$$

see Lions-Magenes [16], and we have

$$\begin{aligned} &E^s = H^s(\Omega) \ \text{ if } \ 0 \le s < \tfrac{1}{2} \ , \qquad E^{1/2} \subset H^{1/2}(\Omega) \ , \\ &E^s = \{u \in H^s(\Omega) \mid u|_{\partial\Omega} = 0\} \ \text{ if } \ \tfrac{1}{2} < s \le 2 \ , \ s \ne \tfrac{3}{2} \ , \ \text{and} \\ &E^{3/2} \subset \{u \in H^{3/2}(\Omega) \mid u|_{\partial\Omega} = 0\} \end{aligned}$$

By the Sobolev imbedding theorem we therefore have continuous imbeddings

$$E^s \subset L^p(\Omega) \ , \ \text{ if } \ \frac{1}{p} \ge \frac{1}{2} - \frac{s}{N} \ ,$$

and these imebbedings are compact if $\frac{1}{p} > \frac{1}{2} - \frac{s}{N}$.

2.2. The functional

With these definitions, we now define the Hilbert space $E := E^t \times E^s$, endowed with the norm

$$\|(u,v)\|_E = (\|u\|^2_{E^t} + \|v\|^2_{E^s})^{\frac{1}{2}}$$

On the space E we consider the functional

$$\begin{aligned} &I : E \to \mathbb{R} \ , \\ &I(u,v) = \int_\Omega A^t u A^s v - \int_\Omega (\frac{1}{p+1}|v|^{p+1} + F(u))dx \end{aligned} \tag{6}$$

with s and t such that $s + t = 2$; loosely speaking, this means that we distribute the two derivatives given in the first term of the functional J, see (5), differently on the variables u and v. Of course, it is crucial to recuperate from critical points (u,v) of this functional solutions of system (3). We state this in the following

Proposition 2.1. *Suppose that $(u,v) \in E^t \times E^s$ is a critical point of the functional I, i.e. u and v are weak solutions of the system*

$$\left\{ \begin{aligned} &\int_\Omega A^t u A^s \phi = \int_\Omega v^p \phi \ , \ \forall \ \phi \in E^s \\ &\int_\Omega A^t \psi A^s v = \int_\Omega f(u)\psi \ , \ \forall \ \psi \in E^t \ . \end{aligned} \right. \tag{7}$$

Then $v \in W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$ and $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \forall \ q \ge 1$, and hence u and v are "strong" solutions of (3), i.e.

$$\left\{ \begin{aligned} &\int_\Omega (-\Delta u)\phi = \int_\Omega v^p \phi \ , \ \forall \ \phi \in C_0^\infty(\Omega) \\ &\int_\Omega (-\Delta v)\psi = \int_\Omega f(u)\psi \ , \ \forall \ \psi \in C_0^\infty(\Omega). \end{aligned} \right. \tag{8}$$

From this proposition follows by standard bootstrap arguments that u and v are classical solutions of (3) if f and Ω are smooth.

The proof of this proposition follows ideas of de Figueiredo - Felmer [11], and will be given in subsection 2.5.

In the following subsection we prove that there exist values s and t with $s+t=2$ such that the functional I is a well-defined C^1 functional, and that it has a non-trivial critical level.

2.3. The choice of the spaces E^s and E^t

We begin by proving the following Lemma:

Lemma 2.2.
Let $1 < p$ ($N=2$), or $1 < p < 2$ ($N=3$). Then there exist parameters $s > 0$ and $t > 0$ with $s+t=2$ such that the following embeddings are continuous and compact:

$$E^s(\Omega) \subset L^{p+1}(\Omega) \quad , \qquad E^t(\Omega) \subset C^0(\Omega)$$

Proof. Note that $H^s(\Omega) \subset L^{p+1}(\Omega)$ compactly, iff $\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}$.
For $N=2$, we get thus the condition

$$s > 1 - \frac{2}{p+1}$$

Choose $s < 1$ satisfying the previous condition, and set $t = 2 - s > 1$. We have a compact embedding $E^t(\Omega) \subset C^0(\Omega)$ for

$$\frac{t}{N} > \frac{1}{2} \ , \quad \text{i.e. for} \ \ t > 1 \ ;$$

and hence the Lemma holds for $N=2$.

For $N=3$, we get the condition

$$s > \frac{3}{2} - \frac{3}{p+1} \ .$$

Since

$$\sup\{\frac{3}{2} - \frac{3}{p+1} \mid 1 < p < 2\} = \frac{1}{2} \ ,$$

we can choose $s < \frac{1}{2}$, and then $t > \frac{3}{2}$, and hence $E^t(\Omega) \subset C^0(\Omega)$ compactly.

□

Thus, we now fix s and t as in Lemma 2.2, and define the functional $I(u,v)$ given by (6) on the space $E^t \times E^s =: E$.

In the next Lemma we collect a few properties of the operators A^s and the spaces E^s.

Lemma 2.3. *Let $s > 0$ and $t > 0$.*

1) $z \in E^s$ iff $A^s z \in L^2$, and $\|z\|_{E^s} = \|A^s z\|_{L^2}$

2) Let $z \in E^{s+t} = E^2 = H^2$; then $A^{s+t} z = A^s A^t z = A^t A^s z$.

Proof. 1) follows immediately from the definitions.

2) we have

$$A^{s+t} z = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{(s+t)/2} e_i = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{s/2} \lambda_i^{t/2} e_i = A^s \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{t/2} e_i = A^s A^t z$$

□

2.4. Existence of a non-trivial critical point

The functional $I(u,v) : E = E^t \times E^s$ is strongly indefinite near zero, in the sense that there exist infinite dimensional subspaces E^+ and E^- with $E^+ \oplus E^- = E$ such that the functional is (near zero) positive definite on E^+ and negative definite on E^-. Li-Willem [17] prove the following general existence theorem for such situations, which can be applied in our case:

Theorem 2.4 (Li-Willem, 1995).
Let $\Phi : E \to \mathbb{R}$ be a strongly indefinite C^1-functional satisfying
A1) Φ has a local linking at the origin, i.e. for some $r > 0$:

$$\Phi(z) \geq 0 \;\; for \;\; z \in E^+ \;,\; \|z\|_E \leq r \quad , \quad \Phi(z) \leq 0 \;,\; for \;\; z \in E^- \;,\; \|z\|_E \leq r \;.$$

A2) Φ maps bounded sets into bounded sets.
A3) Let E^+_n be any n-dimensional subspace of E^+; then $\phi(z) \to -\infty$ as $\|z\| \to \infty$, $z \in E^+_n \oplus E^-$.
A4) Φ satisfies the Palais-Smale condition (PS) (Li-Willem [17] require a weaker "(PS)-condition", however, in our case the classical (PS) condition will be satisfied).*
Then Φ has a nontrivial critical point.

We now verify that our functional satisfies the assumptions of this theorem.

First, it is clear, with the choices of s and t made above, that $I(u,v)$ is a C^1-functional on $E^s \times E^t$.

A1) Following de Figueiredo-Felmer [11] we can define the spaces

$$E^+ = \{(u, A^{t-s}u) \mid u \in E^t\} \;,\; \text{and } E^- = \{(u, -A^{t-s}u) \mid u \in E^t\}$$

which give a natural splitting $E^+ \oplus E^- = E$. It is easy to see that $I(u,v)$ has a local linking with respect to E^+ and E^- at the origin.

A2) Let $B \subset E^t \times E^s$ be a bounded set, i.e. $\|u\|_{E^t} \leq c$, $\|v\|_{E^s} \leq c$, for all $(u,v) \in B$. Then

$$\begin{aligned} |I(u,v)| &\leq \|A^t u\|_{L^2} \|A^s v\|_{L^2} + \int_\Omega |v|^{p+1} + \int_\Omega |f(u)| \\ &\leq \|u\|_{E^t} \|v\|_{E^s} + c\|v\|^{p+1}_{E^s} + \sup_{x\in\Omega} |f(u(x))| \cdot |\Omega| \leq C \end{aligned}$$

A3) Let $z_k = z_k^+ + z_k^- \in E = E^+_n \oplus E^-$ denote a sequence with $\|z_k\|_E \to \infty$. By the above, z_k may be written as

$$z_k = (u_k, A^{t-s}u_k) + (w_k, -A^{t-s}w_k) \;,\quad \text{with} \;\; u_k \in E^t_n,\; w_k \in E^t \;,$$

where E^t_n denotes an $n-$dimensional subspace of E^t. Thus, the functional $I(z_k)$ takes the form

$$\begin{aligned} I(z_k) &= \int_\Omega A^t u_k A^s A^{t-s} u_k - \int_\Omega A^t w_k A^s A^{t-s} w_k - \\ &\qquad - \frac{1}{p+1} \int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} - \int_\Omega F(u_k + w_k) \\ &= \int_\Omega |A^t u_k|^2 - \int_\Omega |A^t w_k|^2 - \frac{1}{p+1} \int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} - \int_\Omega F(u_k + w_k) \end{aligned}$$

Note that $\|z_k\| \to \infty \iff \int |A^t u_k|^2 + \int |A^t w_k|^2 = \|u_k\|^2_{E^t} + \|w_k\|^2_{E^t} \to \infty$.

Now, if

1) $\|u_k\|_{E^t} \leq c$, then $\|w_k\|_{E^t} \to \infty$, and then $I(z_k) \to -\infty$

2) $\|u_k\|_{E^t} \to \infty$, then we estimate (c, c_1 and c_2 are positive constants) using the fact that $t - s > 0$ and $p > 1$

$$\int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} \geq c \left(\int_\Omega |A^{t-s}(u_k - w_k)|^2 \right)^{\frac{p+1}{2}} \geq c_1 \|u_k - w_k\|^{p+1}_{L^2}$$

and

$$\int_\Omega F(u_k + w_k) \ge c_2 \int_\Omega |u_k + w_k|^{p+1} - d \ge c_1 \|u_k + w_k\|_{L^2}^{p+1} - d$$

and hence we obtain the estimate

$$I(z_k) \le \frac{1}{2}\|u_k\|_{E^t}^2 - c_1(\|u_k - w_k\|_{L^2}^{p+1} + \|u_k + w_k\|_{L^2}^{p+1}) + d$$

Since $\phi(t) = t^{p+1}$ is convex, we have $\frac{1}{2}(\phi(t) + \phi(s)) \ge \phi(\frac{1}{2}(s+t))$, and hence

$$\begin{aligned} I(z_k) &\le \frac{1}{2}\|u_k\|_{E^t}^2 - c_1 \frac{1}{2^p}(\|u_k - w_k\|_{L^2} + \|u_k + w_k\|_{L^2})^{p+1} + d \\ &\le \frac{1}{2}\|u_k\|_{E^t}^2 - c_1 \frac{1}{2^p}\|u_k\|_{L^2}^{p+1} + d \end{aligned}$$

Since on E_n^t the norms $\|u_k\|_{E^t}$ and $\|u_k\|_{L^2}$ are equivalent, we conclude that also in this case $J(z_k) \to -\infty$.

A4) Let $\{z_n\} \subset E$ denote a (PS)-sequence, i.e. such that

$$|I(z_n)| \to c \ , \quad \text{and} \ \ |(\Phi'(z_n), \eta)| \le \epsilon_n \|\eta\|_E \ , \ \forall \ \eta \in E, \ \ \text{and} \ \ \epsilon_n \to 0 \tag{9}$$

We first show:

Lemma 2.5. *The (PS)-sequence $\{z_n\}$ is bounded in E.*

Proof. By (9) we have for $z_n = (u_n, v_n)$

$$I(u_n, v_n) = \int_\Omega A^t u_n A^s v_n - \frac{1}{p+1}\int_\Omega v_n^{p+1} - \int_\Omega F(u_n) \to c \tag{10}$$

$$I'(u_n, v_n)(\phi, \psi) = \int_\Omega A^t u_n A^s \psi + \int_\Omega A^s v_n A^t \phi - \int_\Omega v_n^p \psi - \int_\Omega f(u_n)\phi = \epsilon_n \|(\phi, \psi)\|_E \tag{11}$$

Choosing $(\phi, \psi) = (u_n, v_n) \in E^t \times E^s$ we get by (11)

$$2\int_\Omega A^t u_n A^s v_n - \int v_n^{p+1} - \int_\Omega f(u_n) u_n = \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{12}$$

and subtracting this from $2\, I(u_n, v_n)$ we obtain, using assumption 2) of Theorem 1.1

$$(1 - \frac{2}{p+1})\int_\Omega v_n^{p+1} + (1 - \frac{2}{\theta})\int_\Omega f(u_n) u_n \le C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{13}$$

and thus

$$\int_\Omega v_n^{p+1} \le C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{14}$$

$$\int_\Omega f(u_n) u_n \le C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{15}$$

Choosing $(\phi, \psi) = (0, A^{t-s}u_n) \in E^t \times E^s$ in (11) we get

$$\int_\Omega |A^t u_n|^2 = \int_\Omega v_n^p A^{t-s} u_n + \epsilon_n \|A^{t-s} u_n\|_{E^s}$$

and hence by Hölder

$$\|u_n\|_{E^t}^2 = \|A^t u_n\|_{L^2}^2 \le (\int_\Omega |v_n|^{p+1})^{\frac{p}{p+1}} (\int_\Omega |A^{t-s} u_n|^{p+1})^{\frac{1}{p+1}} + \epsilon_n \|u_n\|_{E^t}$$

Noting that

$$(\int_\Omega |A^{t-s} u_n|^{p+1})^{\frac{1}{p+1}} \le c\|A^{t-s} u_n\|_{E^s} = c\|A^t u_n\|_{L^2} = c\|u_n\|_{E^t}$$

we obtain, using (14)

$$\|u_n\|_{E^t}^2 \le [C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s})]^{p/(p+1)} \cdot c\|u_n\|_{E^t} + \epsilon_n \|u_n\|_{E^t}$$

and thus

$$\|u_n\|_{E^t} \le C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s})^{p/(p+1)} \tag{16}$$

Similarly as above we note that $A^{s-t}v_n \in E^t$, and thus, choosing $(\phi,\psi) = (A^{s-t}v_n, 0) \in E^t \times E^s$ in (11) we get

$$\int_\Omega |A^s v_n|^2 = \int_\Omega f(u_n)A^{s-t}v_n + \epsilon_n \|A^{s-t}v_n\|_{E^t} \le \|A^{s-t}v_n\|_\infty \int_\Omega |f(u_n)| + \epsilon_n \|v_n\|_{E^s}$$

Using that $\|A^{s-t}v_n\|_{E^t} = \|A^s v_n\|_{L^2} = \|v_n\|_{E^s}$, and the fact that $E^t \subset C^0$ we then obtain, using (15)

$$\begin{aligned} \|v_n\|_{E^s} &\le c\int_\Omega |f(u_n)| + \epsilon_n = \int_{[|u_n|\le s_0]} \max_{|t|\le s_0} |f(t)| + \int_{[|u_n|>s_0]} f(u_n)u_n + \epsilon_n \\ &\le C + \epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s}) \end{aligned} \tag{17}$$

Joining (16) and (17) we finally get

$$\|u_n\|_{E^t} + \|v_n\|_{E^s} \le C + 2\epsilon_n(\|u_n\|_{E^t} + \|v_n\|_{E^s})$$

Thus, $\|u_n\|_{E^t} + \|v_n\|_{E^s}$ is bounded. $\square$

With this it is now possible to complete the proof of the (PS)-condition: since $\|u_n\|_{E^t}$ is bounded, we find a weakly convergent subsequence $u_n \rightharpoonup u$ in E^t. Since the mappings $A^t : E^t \to L^2$ and $A^{-s} : L^2 \to E^s$ are continuous isomorphisms, we get $A^t(u_n - u) \rightharpoonup 0$ in L^2 and $A^{t-s}(u_n - u) \rightharpoonup 0$ in E^s. Since $E^s \subset L^{p+1}$ compactly, we conclude that $A^{t-s}(u_n - u) \to 0$ strongly in L^{p+1}.

Similarly, we find a subsequence of $\{v_n\}$ which is weakly convergent in E^s and such that v_n^p is strongly convergent in $L^{\frac{p+1}{p}}$

Choosing $(\phi,\psi) = (0, A^{t-s}(u_n - u) \in E^t \times E^s$ in (11) we thus conclude

$$\int_\Omega A^t u_n A^t(u_n - u) = \int_\Omega v_n^p A^{t-s}(u_n - u) + \epsilon_n \|A^{t-s}(u_n - u)\|_{E^s} \tag{18}$$

By the above considerations, the righthand-side converges to 0, and thus

$$\int_\Omega |A^t u_n|^2 \to \int_\Omega |A^t u|^2$$

Thus, $u_n \to u$ strongly in E^t.

To obtain the strong convergence of $\{v_n\}$ in E^s, one proceeds similarly: as above, one finds a subsequence $\{v_n\}$ converging weakly in E^s to v, and then $A^{s-t}v_n \rightharpoonup A^{s-t}v$ weakly in A^t and $A^{s-t}v_n \to A^{s-t}v$ strongly in C^0. Choosing in (9) $(\phi,\psi) = (A^{s-t}(v_n - v), 0)$, we get

$$\int_\Omega A^s(v_n - v)A^s v_n = \int f(u_n)A^{s-t}(v_n - v) + \epsilon_n(\|A^{s-t}(v_n - v)\|) \tag{19}$$

The first term on the right is estimated by $\|A^{s-t}(v_n - v)\|_{C^0} \int_\Omega |f(u_n)| \to 0$, and thus one concludes again that

$$\int_\Omega |A^s v_n|^2 \to \int_\Omega |A^s v|^2$$

and hence also $v_n \to v$ strongly in E^s.

Thus, the conditions of Theorem 2.4 are satisfied; hence, we find a positive critical point (u, v) for the functional I, which yields a weak solution to system (3).

2.5. Strong solutions

In this section we prove Proposition 2.1.

Consider the first equation in the system (7). We can follow the arguments of [11]: If $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\int_\Omega A^t u A^s \phi = \int_\Omega u A^2 \phi = \int_\Omega u(-\Delta\phi) \tag{20}$$

On the other hand, $v^p \in L^{\frac{p+1}{p}}(\Omega)$, and hence (see [13]) there exists a unique solution

$$y \in W^{2,\frac{p+1}{p}}(\Omega) \quad \text{of} \quad -\Delta y = v^p \ .$$

By the choice of s we have $\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}$, which is equivalent to $\frac{1}{2} > \frac{p}{p+1} - \frac{s}{N}$, which in turn implies that $W^{2,\frac{p+1}{p}}(\Omega) \subset L^2(\Omega)$. Thus, we conclude that

$$\int_\Omega v^p \phi = \int_\Omega (-\Delta y)\phi = \int_\Omega y(-\Delta\phi) \ , \ \forall\ \phi \in H^2(\Omega) \cap H_0^1(\Omega) \tag{21}$$

Comparing (20) and (21) yields

$$\int_\Omega (y-u)(-\Delta\phi) = 0 \ , \ \forall\ \phi \in H^2(\Omega) \cap H_0^1(\Omega)$$

and hence $u = y$; thus $u \in W^{2,\frac{p+1}{p}}(\Omega)$.

Consider now the second equation in system (7). Again, for $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$\int_\Omega (-\Delta\psi)v = \int_\Omega A^t \psi A^s v = \int_\Omega f(u)\psi \ , \ \forall\ \psi \in E^t \ .$$

On the other hand, $E^t \subset \{u \in H^t(\Omega) \mid u|_{\partial\Omega} = 0\} \subset C^\lambda(\Omega)$, with $\lambda = t - \frac{N}{2}$.

By our choices of s and t we have

$$\begin{cases} 1 < t < 2 \ , & N = 2 \\ \frac{3}{2} < t < 2 \ , & N = 3 \end{cases}$$

and hence in both cases $u \in C^\lambda(\Omega)$ with $\lambda > 0$. This implies that $f(u) \in L^\infty(\Omega)$, and hence there exists a unique solution

$$w \in W^{2,q}(\Omega) \ , \ \forall\ q \geq 1 \ , \quad \text{of} \quad -\Delta w = f(u)$$

Note that if $f \in C^\lambda$ and $\partial\Omega$ is sufficiently smooth, then $w \in C^{2,\lambda}(\Omega)$.
We finish by concluding as above that $w = v$, and that therefore $v \in W^{2,q}, \forall\, q \geq 1$, respectively $v \in C^{2,\lambda}(\Omega)$.

3. Proof: the case $p \leq 1$

In this section we consider the cases $0 < p \leq 1$ ($N = 2, 3$) and $0 < p < \frac{2}{N-2}$ ($N \geq 4$), i.e. we consider the situation where one equation has a sublinear nonlinearity in the form of a power, and the other equation has a superlinear nonlinearity.

3.1. The functional

We consider now the system

$$\begin{cases} -\Delta u = v^p \ , & \text{with } 0 < p \leq 1 \\ -\Delta v = f(u) \end{cases} \tag{22}$$

System (22) can be written as

$$\begin{cases} (-\Delta u)^{1/p} = v \ , & \text{with } 0 < p \leq 1 \\ -\Delta v = f(u) \end{cases} \tag{23}$$

and thus we have the equivalent equation

$$\begin{cases} -\Delta(-\Delta u)^{1/p} = -\Delta v = f(u) \\ u = \Delta u = 0 \qquad \partial\Omega \end{cases} \tag{24}$$

To equation (24) we may associate the following functional

$$I(u) = \frac{p}{p+1}\int_\Omega |\Delta u|^{\frac{p+1}{p}} - \int_\Omega F(u) \ . \tag{25}$$

Indeed, the derivative of $I(u)$ in direction v yields

$$I'(u)\,v = \int_\Omega (-\Delta u)^{1/p}(-\Delta v) - \int_\Omega f(u)v \ ,$$

and thus critical points of I correspond to weak solutions of equation (23) and thus of system (22).

3.2. Existence of critical points

Note that the first term of the functional I is defined on the space $E = W^{2,\frac{p+1}{p}}(\Omega)\cap W_0^{1,\frac{p+1}{p}}(\Omega)$. Since by assumption $p < \frac{2}{N-2}$ we have $\frac{p+1}{p} > 1 + \frac{N-2}{2} > \frac{N}{2}$, and thus

$$W^{2,\frac{p+1}{p}}(\Omega) \ \subset\subset \ C(\Omega)$$

Thus, the second term of the functional I is defined if F is continuous, and no growth restriction on F is necessary. Since F is differentiable, the functional I is a well-defined C^1-functional on the space E.

We now show that the classical mountain-pass theorem of Ambrosetti-Rabinowitz may be applied to the functional I. Indeed, I has a local minimum in the origin:

$$I(u) = \frac{p}{p+1}\int_\Omega |\Delta u|^{\frac{p+1}{p}} - \int_\Omega F(u) \geq c\,\frac{p}{p+1}\ \|u\|_C^{\frac{p+1}{p}} - o(\|u\|_C^{\frac{p+1}{p}})$$

Next, let u_1 be any fixed element of E. Then

$$I(su_1) \leq \frac{p}{p+1}\ s^{\frac{p+1}{p}} \int_\Omega |\Delta u_1|^{\frac{p+1}{p}} - s^\theta \|u\|_C^\theta + d$$

with $\theta > \frac{p+1}{p}$ (by assumption), and thus $I(su_1) \to -\infty$ as $s \to \infty$.

Finally, we show that I satisfies the Palais-Smale condition (PS). Let $(u_n) \subset E$ be a (PS)-sequence, i.e.

$$|I(u_n)| \leq c \quad , \quad \text{and} \quad |I'(u_n)v| \leq \epsilon_n \|v\|_E \ , \ \epsilon_n \to 0 \ , \ \forall\ v \in E \ .$$

We have

$$\begin{aligned} c + \epsilon_n \|u_n\|_E \quad &\geq |\theta I(u_n) - I'(u_n)u_n| \\ &\geq (\theta\tfrac{p}{p+1} - 1)\textstyle\int_\Omega |\Delta u_n|^{\frac{p+1}{p}} - \theta \int_\Omega F(u_n) + \int_\Omega f(u_n)u_n \\ &\geq (\theta\tfrac{p}{p+1} - 1)\textstyle\int_\Omega |\Delta u_n|^{\frac{p+1}{p}} - c \\ &\geq \delta \|u\|_E^{\frac{p+1}{p}} - c \ , \end{aligned}$$

and thus (u_n) is bounded in E. Since E is compactly imbedded in $C(\Omega)$, we find a convergent subsequence in $C(\Omega)$, and then it is standard to conclude that u_n converges strongly also in E.

Thus, by the Mountain-Pass theorem we obtain a (non-trivial) critical point u, which gives rise to a solution to system (3).

3.3. Proof of Theorem 1.2

The proof follows the same lines as in section 3.2. We just observe that for $\frac{2}{N-2} \leq p \leq 1$

$$W^{2,\frac{p+1}{p}}(\Omega) \subset L^{\frac{N(p+1)}{Np-2(p+1)}}(\Omega) \ .$$

The exponent $\frac{N(p+1)}{Np-2(p+1)}$ satisfies

$$\frac{1}{p+1} + \frac{1}{\frac{N(p+1)}{Np-2(p+1)}} = 1 - \frac{2}{N} \ ,$$

i.e. we are on the critical hyperbola. Hence, for $q+1 < \frac{N(p+1)}{Np-2(p+1)}$ we are below the hyperbola, and we have $E \subset\subset L^{q+1}(\Omega)$ compactly. We can then proceed exactly as above, to obtain a critical point via the Mountain-Pass theorem.

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Djairo G. de Figueiredo
IMECC
Universidade Estadual de Campinas
13081-970, Campinas, SP, Brazil
e-mail: djairo@ime.unicamp.br

Bernhard Ruf
Dipartimento di Matematica
Università degli Studi
Via Saldini 50, 20133 Milano, Italy
e-mail: ruf@mat.unimi.it