Absence of interaction corrections in the optical conductivity of graphene

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The exact vanishing of the interaction corrections to the zero temperature optical conductivity of undoped graphene in the presence of weak short-range interactions is rigorously established. Our results are in agreement with measurements of graphene's ac conductivity in a range of frequencies between the temperature and the bandwidth. Even if irrelevant in the renormalization group sense, lattice effects and nonlinear bands are essential for the universality of the conductivity.

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I. INTRODUCTION

Understanding the low-temperature properties of interacting many-body systems is one of the most challenging problems in physics; even a weak interaction can radically change the behavior of the noninteracting system and produce a variety of different effects. In view of this fact, it is particularly interesting that a very small class of observables, among which is the Hall resistivity, appears to be completely independent of the interaction and of other microscopic details and that their values depend only upon fundamental constants. Even if there is agreement on the symmetries underlying this universal behavior (in the case of quantum Hall effect (QHE) it is topological invariance[1,2]), there is no first-principles derivation of this fact in any interacting many-body system.[3] The advent of graphene,[4] a two-dimensional (2D) crystal of pure carbon, finally provides such a system. Universality is not only observed in relatively accessible experiments but, as our study shows, it can also be rigorously deduced from an interacting lattice Hamiltonian.

Indeed, recent optical measurements in graphene[5] show that at half-filling and small temperatures, if the frequency is in a range between the temperature and the bandwidth, the conductivity is essentially constant and equal, up to a few percent, to $\sigma_0 = \frac{e^2}{h}\frac{\pi}{2}$, a universal value depending only on the fundamental von Klitzing constant h/e^2 and not on the material parameters, such as the Fermi velocity. Such value coincides with the theoretical prediction in a system of massless noninteracting Dirac particles,[6] a widely used effective model of half-filled graphene;[7] remarkably, the inclusion of lattice effects and nonlinear bands does not change such a value (see Ref. 9). Of course, interaction effects could produce modifications to this theoretical value, which was obtained by neglecting interactions. However, in the case of weak short-range interactions and at half-filling, we rigorously establish that this is not the case. All the interaction corrections to the zero temperature and zero frequency conductivity of the half-filled Hubbard model on the honeycomb lattice cancel out exactly as a consequence of exact lattice Ward identities (WIs) and of suitable regularity properties of the current-current response function. Note that we first perform the zero temperature and then the zero frequency limit of the conductivity, which means that we are looking at a range of frequencies that are very small (as compared to the bandwidth) but still larger than the temperature, which is precisely the range of frequencies relevant for optical measurement of the ac conductivity of undoped graphene.[5]

Besides an obvious interest for the physics of graphene, the universality phenomenon proven here appears to be closely related, as is manifest in our proof, to the universality in the QHE[10] and to the nonrenormalization of the anomalies in quantum electrodynamics.[11] Graphene provides a realization of the analog of such phenomena in a much simpler context, both from an experimental and theoretical point of view.

An important point of our analysis is that, even if irrelevant in the renormalization group (RG) sense, the effects of the underlying honeycomb lattice and the nonlinear bands are essential for the universality of conductivity in the interacting case. By using the Dirac effective description, which has been successfully used to explain several properties of graphene (see Ref. 12 for a review), one easily misses the exact cancellations necessary for universality. The problem is that the effective Dirac model has spurious ultraviolet divergences (absent in the lattice model) which need to be cured via a suitable regularization procedure, e.g., dimensional or momentum regularization. However, none of these regularization procedures have a fundamental meaning, and while they all give the same results as long as singular quantities are concerned (e.g., in the computation of critical exponents), they may fail to provide a unique answer in the computation of finite quantities, such as the corrections to the conductivity, which are sensitive to the regularization scheme. For example, in the case of Coulombic interactions, frequency-dependent corrections to the conductivity were computed in Refs. 13–15, but different values (with the same sign) were found depending on the regularization scheme; in this context, momentum regularization appears to provide more accurate results than dimensional regularization, as discussed in Ref. 15. On the other hand, in the case of short-range interactions, the momentum regularization, contrary to the dimensional one, predicts nonvanishing corrections to the optical conductivity, a fact that is believed to be a spurious effect due to the explicit breaking of gauge invariance.[14] In general, it is unclear which one of the two regularization schemes gives more accurate physical predictions.

In this paper we provide a clear answer to this problem in the simple case of a half-filled Hubbard model with weak local interactions, in the absence of disorder. As discussed in more detail below, the exact vanishing of the interaction-dependent contributions to the optical conductivity involves cancellations between contributions with momenta close to the Fermi points and those with momenta far from the cusp singularities, where the Dirac approximation fails. The paper is organized as follows: after having introduced the lattice model and the notion of conductivity, we state our main results; next we present the proof, first explaining the role of lattice Ward identities, then describing the computation of the optical conductivity (both in the noninteracting and in the interacting case), and finally sketching our RG construction of the interacting response functions.

II. THE HUBBARD MODEL ON THE HONEYCOMB LATTICE

We fix units such that $\hbar = 1$ and the lattice spacing $a = 1$. We introduce creation and annihilation fermionic operators $\psi^{\pm}_{\vec{x},\sigma} = (a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}) = L^{-2}\sum_{\vec{k}\in\mathcal{B}_\Lambda}\psi^{\pm}_{\vec{k},\sigma}e^{\pm i\vec{k}\vec{x}}$ for electrons with spin index $\sigma =\uparrow\downarrow$ sitting at the sites of the two triangular sublattices Λ_A and Λ_B of a periodic honeycomb lattice of side L. We assume that $\Lambda_A = \Lambda$ has basis vectors $\vec{l}_{1,2} = \frac{1}{2}(3, \pm\sqrt{3})$ and that $\Lambda_B = \Lambda_A + \vec{\delta}_j$, with $\vec{\delta}_1 = (1,0)$ and $\vec{\delta}_{2,3} = \frac{1}{2}(-1, \pm\sqrt{3})$ the nearest-neighbor vectors; $\mathcal{B}_\Lambda = \{\vec{k} = n_1\vec{G}_1/L + n_2\vec{G}_2/L \,:\, 0 \leqslant n_i < L\}$ with $\vec{G}_{1,2} = \frac{2\pi}{3}(1, \pm\sqrt{3})$ as the first Brillouin zone. [Note that in the thermodynamic limit $L^{-2}\sum_{\vec{k}\in\mathcal{B}_\Lambda} \to |\mathcal{B}|^{-1}\int_{\mathcal{B}} d\vec{k}$, with $|\mathcal{B}| = 8\pi^2/(3\sqrt{3})$]. The grand-canonical Hamiltonian at half-filling is $H_\Lambda = H^0_\Lambda + UV_\Lambda$, where H^0_Λ is the free Hamiltonian describing nearest-neighbor hopping (t is the hopping parameter):

$$H^0_\Lambda(t) = -t\sum_{\substack{\vec{x}\in\Lambda_A\\ j=1,2,3}}\sum_{\sigma=\uparrow\downarrow}\left(a^{+}_{\vec{x},\sigma}b^{-}_{\vec{x}+\vec{\delta}_j,\sigma} + b^{+}_{\vec{x}+\vec{\delta}_j,\sigma}a^{-}_{\vec{x},\sigma}\right),$$

and V_Λ is the local Hubbard interaction:

$$V_\Lambda = \sum_{\vec{x}\in\Lambda_A}\prod_{\sigma=\uparrow\downarrow}\left(a^{+}_{\vec{x},\sigma}a^{-}_{\vec{x},\sigma} - \frac{1}{2}\right) + \sum_{\vec{x}\in\Lambda_B}\prod_{\sigma=\uparrow\downarrow}\left(b^{+}_{\vec{x},\sigma}b^{-}_{\vec{x},\sigma} - \frac{1}{2}\right).$$

The current is defined as usual via the Peierls substitution by modifying the hopping parameter along the bond $(\vec{x},\vec{x}+\vec{\delta}_j)$ as $t \to t_{\vec{x},j}(\vec{A}) = t\, e^{ie\int_0^1 \vec{A}(\vec{x}+s\vec{\delta}_j)\cdot\vec{\delta}_j\, ds}$, where the constant e appearing at exponent is the electric charge and $\vec{A}(\vec{x}) \in \mathrm{R}^2$ is a periodic field on $\mathcal{S}_\Lambda = \{\vec{x} = L\xi_1\vec{l}_1 + L\xi_2\vec{l}_2 \,:\, \xi_i \in [0,1]\}$. If we denote by

$$H_\Lambda(\vec{A}) = H^0_\Lambda(\{t_{\vec{x},j}(\vec{A})\}) + UV_\Lambda$$

the modified Hamiltonian with the new hopping parameters, the lattice current is defined as $\vec{J}^{(A)}_{\vec{p}} = -|\mathcal{S}_\Lambda|\,\partial H_\Lambda(\vec{A})/\partial\vec{A}_{\vec{p}}$, which gives, at first order in $\vec{A}$,

$$\vec{J}^{(A)}_{\vec{p}} = \vec{J}_{\vec{p}} + \frac{1}{|\mathcal{S}_\Lambda|}\sum_{\vec{q}\in\mathcal{D}_\Lambda}\hat{\Delta}_{\vec{p},\vec{q}}\vec{A}_{\vec{q}},$$

where if $\eta^j_{\vec{p}} = \frac{1-e^{-i\vec{p}\vec{\delta}_j}}{i\vec{p}\vec{\delta}_j}$,

$$\vec{J}_{\vec{p}} = iet\sum_{\substack{\vec{x}\in\Lambda\\ \sigma,j}} e^{-i\vec{p}\vec{x}}\vec{\delta}_j\eta^j_{\vec{p}}\left(a^{+}_{\vec{x},\sigma}b^{-}_{\vec{x}+\vec{\delta}_j,\sigma} - b^{+}_{\vec{x}+\vec{\delta}_j,\sigma}a^{-}_{\vec{x},\sigma}\right)$$

is the paramagnetic current and

$$[\hat{\Delta}_{\vec{p},\vec{q}}]_{lm} = \sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}} e^{-i(\vec{p}+\vec{q})\vec{x}}(\vec{\delta}_j)_l(\vec{\delta}_j)_m\eta^j_{\vec{p}}\eta^j_{\vec{q}}\Delta_{\vec{x},j},$$

with $\Delta_{\vec{x},j} = -e^2t\sum_\sigma(a^{+}_{\vec{x},\sigma}b^{-}_{\vec{x}+\vec{\delta}_j,\sigma} + b^{+}_{\vec{x}+\vec{\delta}_j,\sigma}a^{-}_{\vec{x},\sigma})$, is the diamagnetic tensor. The two components of the paramagnetic current $\vec{J}_{\vec{p}}$ will be seen as the spatial components of a "space-time" three-component vector $\hat{J}_{\vec{p},\mu}$, $\mu = 0,1,2$, with $\hat{J}_{\vec{p},0} = e\hat{\rho}_{\vec{p}}$ and $\hat{\rho}_{\vec{p}}$ the density operator:

$$\hat{\rho}_{\vec{p}} = \sum_{\substack{\vec{x}\in\Lambda_A\\ \sigma=\uparrow\downarrow}} e^{-i\vec{p}\vec{x}}a^{+}_{\vec{x},\sigma}a^{-}_{\vec{x},\sigma} + \sum_{\substack{\vec{x}\in\Lambda_B\\ \sigma=\uparrow\downarrow}} e^{-i\vec{p}\vec{x}}b^{+}_{\vec{x},\sigma}b^{-}_{\vec{x},\sigma}. \quad (1)$$

It is also convenient to introduce the reduced current $\vec{j}_{\vec{p}}$, related to the paramagnetic current by $\vec{J}_{\vec{p}} = v_0\vec{j}_{\vec{p}}$, where $v_0 = \frac{3t}{2}$ is the free Fermi velocity. If $O_{\mathbf{x}} = e^{x_0H_\Lambda}O_{\vec{x}}e^{-x_0H_\Lambda}$, with $\mathbf{x} = (x_0,\vec{x})$, we denote by $\langle O^{(1)}_{\mathbf{x}_1}\cdots O^{(n)}_{\mathbf{x}_n}\rangle_\beta$ the thermodynamic limit of $\Xi^{-1}\mathrm{Tr}\{e^{-\beta H_\Lambda}\mathbf{T}(O^{(1)}_{\mathbf{x}_1}\cdots O^{(n)}_{\mathbf{x}_n})\}$, where $\Xi = \mathrm{Tr}\{e^{-\beta H_\Lambda}\}$, and $\mathbf{T}$ is the operator of fermionic time ordering. Moreover, we denote by $\langle O^{(1)}_{\mathbf{x}_1};\cdots;O^{(n)}_{\mathbf{x}_n}\rangle_\beta$ the corresponding truncated expectations and by $\langle O^{(1)}_{\mathbf{x}_1};\cdots;O^{(n)}_{\mathbf{x}_n}\rangle$ their zero temperature limit.

The two-point, three-point, and current-current functions, $\hat{S}^\beta(\mathbf{k})$, $\hat{G}^\beta_{2,1;\mu}(\mathbf{k},\mathbf{p})$, and $\hat{K}^\beta_{\mu\nu}(\mathbf{p})$, are defined as the 2D Fourier transforms of $\langle\psi^{-}_{\mathbf{x},\sigma}\psi^{+}_{\mathbf{y},\sigma}\rangle_\beta$, $\langle J_{\mathbf{z},\mu};\psi^{-}_{\mathbf{x},\sigma}\psi^{+}_{\mathbf{y},\sigma}\rangle_\beta$, and $\langle J_{\mathbf{x},\mu};J_{\mathbf{y},\nu}\rangle_\beta$, respectively. Finally, the conductivity is defined via the Kubo formula as[9] (here $l,m = 1,2$)

$$\sigma^\beta_{lm}(p_0) = -\frac{2}{3\sqrt{3}}\frac{1}{p_0}\left[\hat{K}^\beta_{lm}(p_0,\vec{0}) + \Delta^\beta_{lm}\right],$$

where $\Delta^\beta_{lm} = \lim_{L\to\infty}\frac{1}{L^2}\sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}}(\vec{\delta}_j)_l(\vec{\delta}_j)_m\langle\Delta_{\vec{x},j}\rangle_\beta$, and $3\sqrt{3}/2$ is the area of the hexagonal cell of the honeycomb lattice. In our notations, $\mathbf{p} = (p_0,\vec{p})$, with $p_0 \in \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})$ the Matsubara frequency.

It is known that in general the interaction modifies the values of the physical quantities. For instance, the Fermi velocity v_F, the wave-function renormalization Z, and the vertex functions are known to depend explicitly on the interaction[16] and to be analytic in U. Therefore, it is natural to expect that the interacting conductivity remains close to its free value at weak coupling; what is *a priori* unclear is whether its zero frequency limit has corrections explicitly depending upon U, or whether these cancel out exactly.

Theorem. There exists a constant $U_0 > 0$ such that, for $|U| \leqslant U_0$ and any fixed p_0, $\sigma^\beta_{lm}(p_0)$ is analytic in U uniformly in β as $\beta\to\infty$. Moreover, for $l,m = 1,2$, restoring the presence of the dimensional constant $\hbar = h/2\pi$ in the result:

$$\sigma_{lm} = \lim_{p_0\to 0^+}\lim_{\beta\to\infty}\sigma^\beta_{lm}(p_0) = \frac{e^2}{h}\frac{\pi}{2}\delta_{lm}. \quad (2)$$

Note that the definition of σ_{lm} involves a limiting procedure in which first the temperature and then the frequency are set to zero, i.e., close to the limit we have $\beta^{-1} \ll p_0 \ll t$, which corresponds to the range of frequencies investigated with optical techniques in Ref. 5. By taking the limits in the opposite order, we would get information about the dc conductivity that, in the presence of disorder, also appears to have a universal value. (See Ref. 8 for details concerning the noninteracting case in the Dirac approximation.) The rest of the paper is devoted to the proof of the theorem; for some technical aspects of the discussion, the reader is referred to Ref. 17.

III. WARD IDENTITIES

It is important to note that the two-point, three-point, and response functions are not independent. They are related to each other by WIs following from the continuity equation:

$$-ie\partial_{x_0}\rho_{(x_0,\vec{p})} + i\vec{p}\cdot\vec{J}_{(x_0,\vec{p})} = 0. \tag{3}$$

In particular, by defining $p^0 = -ip_0$, the two- and three-point functions verify[18] the WI:

$$p^\mu \hat{G}^\beta_{2,1;\mu}(\mathbf{k},\mathbf{p}) = -e\hat{S}^\beta(\mathbf{k}+\mathbf{p})\Gamma_0(\vec{p}) + e\Gamma_0(\vec{p})\hat{S}^\beta(\mathbf{k}), \tag{4}$$

where $\Gamma_0(\vec{p}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\vec{p}\vec{\delta}_1} \end{pmatrix}$. Equation (4) can be used to infer identities between the interacting Fermi velocity, wave-function renormalization, and vertex functions [see Eq. (13) below]. Similarly, the response functions verify the WIs: $p^\mu \hat{K}^\beta_{\mu 0}(\mathbf{p}) = 0$ and

$$p^\mu \hat{K}^\beta_{\mu m}(\mathbf{p}) = -\lim_{L\to\infty}\frac{1}{L^2}[\vec{p}\cdot\langle\hat{\Delta}_{\vec{p},-\vec{p}}\rangle_{\beta,L}]_m, \tag{5}$$

where $m = 1,2$ and the term in the right-hand side is known as the Schwinger term. Equation (5) can be used to conveniently rewrite the conductivity, provided that the response function is regular enough, as discussed below. The key factor is that the large distance decay of the current-current correlation can be estimated as

$$|\langle J_{\mathbf{x},\mu}; J_{\mathbf{y},\nu}\rangle| \leqslant \frac{(\text{const.})}{1+|\mathbf{x}-\mathbf{y}|^4}, \tag{6}$$

which follows from the fact that the Hubbard interaction does not change the asymptotic infrared properties of the theory. This is straightforward to check at a perturbative level (the interaction is irrelevant according to power counting), and a nonperturbative proof can be found in Refs. 16 and 17. Equation (6) implies that $\hat{K}_{\mu\nu}(\mathbf{p}) = \lim_{\beta\to\infty}\hat{K}^\beta_{\mu\nu}(\mathbf{p})$ is continuous at $\mathbf{p} = \mathbf{0}$; therefore, from the WI Eq. (5),

$$i\frac{p_0}{p_1}\hat{K}_{0m}(p_0,p_1,0) = \left[\hat{K}_{1m}(p_0,p_1,0) + \lim_{\beta,L\to\infty}\frac{1}{L^2}\langle[\hat{\Delta}_{(p_1,0),(-p_1,0)}]_{1m}\rangle_{\beta,L}\right].$$

Taking first the limit $p_0 \to 0$ and then the limit $p_1 \to 0$, we see that the left-hand side is vanishing in the limit. This implies, using the continuity of $\hat{K}_{\mu\nu}(\mathbf{p})$ at $\mathbf{p} = \mathbf{0}$, $\lim_{\mathbf{p}\to\mathbf{0}}\hat{K}_{1i}(\mathbf{p}) = -\Delta_{1i}$, with $\Delta_{lm} = \lim_{\beta\to\infty}\Delta^\beta_{lm}$; a similar argument shows that $\lim_{\mathbf{p}\to\mathbf{0}}\hat{K}_{lm}(\mathbf{p}) = -\Delta_{lm}$ for all $l,m \in \{1,2\}$. Therefore, using again the continuity at $\mathbf{p} = \mathbf{0}$ of the current-current function and the definition of conductivity, we can rewrite

$$\sigma_{lm} = -\frac{2}{3\sqrt{3}}\lim_{p_0\to0^+}\lim_{\beta\to\infty}\frac{1}{p_0}[\hat{K}_{lm}(p_0,\vec{0}) - \hat{K}_{lm}(\mathbf{0})]. \tag{7}$$

In addition, symmetry considerations immediately imply that $\hat{K}_{lm}(p_0,\vec{0})$ is even in p_0. This, together with (7), shows that there are strict relations between the regularity properties of the Fourier transform of the current-current correlations $\hat{K}_{lm}(\mathbf{p})$ and the properties of the conductivity; in particular, it says that contributions to $\hat{K}_{lm}(\mathbf{p})$ that are differentiable in $\mathbf{p}$ give zero contribution to the conductivity in the limit. This is not enough to prove that all the interaction corrections to the conductivity are vanishing in the limit, as in the perturbative expansion of $\hat{K}_{lm}(\mathbf{p})$ one can easily identify nondifferentiable terms; however, as discussed in the following, there are dramatic cancellations among such terms, which imply universality.

IV. CONDUCTIVITY IN THE NONINTERACTING CASE

Before discussing the computation of the conductivity in the interacting case, it is instructive to perform the analysis in the free gas approximation, in which case, restoring the presence of the dimensional constant $\hbar$ in the formulas,

$$\sigma_{ij}|_{U=0} = \frac{2}{3\sqrt{3}}\frac{2e^2v_0^2}{\hbar}\lim_{p_0\to0^+}\int\frac{dk_0}{2\pi}\int_{\mathcal{B}}\frac{d\vec{k}}{|\mathcal{B}|} \times \mathrm{Tr}\left\{\frac{S_0(\mathbf{k}+(p_0,\vec{0})) - S_0(\mathbf{k})}{p_0}\Gamma_i(\vec{k},\vec{0})S_0(\mathbf{k})\Gamma_j(\vec{k},\vec{0})\right\}, \tag{8}$$

where

$$\vec{\Gamma}(\vec{k},\vec{p}) = \frac{2}{3}\sum_{j=1}^{3}\vec{\delta}_j\begin{pmatrix} 0 & ie^{-i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)} \\ -ie^{+i(\vec{k}+\vec{p})(\vec{\delta}_j-\vec{\delta}_1)} & 0 \end{pmatrix},$$

$S_0(\mathbf{k})$ is the two-point function at $U = 0$,

$$S_0(\mathbf{k}) = \frac{1}{k_0^2 + v_0^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec{k}) \\ -v_0\Omega(\vec{k}) & ik_0 \end{pmatrix},$$

$v_0 = \frac{3}{2}t$ and $\Omega(\vec{k}) = \frac{2}{3}\sum_{j=1}^{3} e^{i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)}$. The complex dispersion relation $\Omega(\vec{k})$ vanishes only at the two Fermi points $\vec{p}_F^{\pm} = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$, and close to them it assumes the form of a relativistic dispersion relation $\Omega(\vec{p}_F^{\pm}+\vec{k}') \simeq ik'_1 \pm k'_2$. The integral in Eq. (8) is not absolutely convergent, so the integral and the limit $p_0 \to 0^+$ cannot be exchanged. This is well known (see Ref. 8, where a similar question is discussed in the relativistic approximation). In order to explicitly compute Eq. (8), we can proceed as follows. Let $\varepsilon > 0$ be a small number, independent of p_0, to be eventually sent to zero; we distinguish between the regions where $|\Omega(\vec{k})| \geqslant \varepsilon$ and $|\Omega(\vec{k})| \leqslant \varepsilon$. The integral associated with the region $|\Omega(\vec{k})| \geqslant \varepsilon$ is nonsingular; therefore, we can exchange the integral with the limit and check that the integral of the limit is zero (simply because the resulting integrand is odd in k_0). Next, in the integral associated with the region $|\Omega(\vec{k})| \leqslant \varepsilon$, we rewrite the propagator as the relativistic propagator plus a correction (similarly, we rewrite $\vec{\Gamma}$ as its value at the Fermi points plus a correction). The corrections are associated with absolutely convergent integrals, uniformly in p_0, and therefore, their contribution after having taken $\varepsilon \to 0$ is equal to zero. Therefore, we are left with a cut-off integral involving the Dirac propagators, in which the dependence upon v_0 disappears (by scaling). An explicit evaluation of the integral over k_0 (by residues) yields

$$\sigma_{ij}|_{U=0} = 8\delta_{ij}\frac{e^2}{h}\lim_{\varepsilon\to 0}\lim_{p_0\to 0^+}\frac{p_0}{16}\int_0^{\varepsilon} dk\frac{1}{k^2+p_0^2/4}$$
$$= \delta_{ij}\frac{e^2}{h}\lim_{\varepsilon\to 0}\lim_{p_0\to 0^+}\arctan(2\varepsilon/p_0) = \delta_{ij}\frac{e^2}{h}\frac{\pi}{2},$$

which is the desired result. This simple analysis explains why the free conductivity is universal: the integral is not absolutely convergent and only the region close to the singularity, where the Dirac approximation is valid, contributes to the result.

V. CONDUCTIVITY IN THE INTERACTING CASE

The interaction produces nontrivial renormalization of the Fermi velocity, the wave function, and the vertex function, and the universality stated in Eq. (2) appears as a delicate compensation between them, which cannot be seen in naive perturbation theory in U. By the exact RG analysis explained in Ref. 16, it follows that for small U, if $|\mathbf{k}-\mathbf{p}_F^{\omega}| \ll 1$, the two-point function is

$$S(\mathbf{k}) = \frac{1}{Z}\begin{pmatrix} -ik_0 & -v_F\Omega^*(\vec{k}) \\ -v_F\Omega(\vec{k}) & -ik_0\end{pmatrix}^{-1}[1+O(|\mathbf{k}-\mathbf{p}_F^{\omega}|^{\theta})], \quad (9)$$

where $Z = Z(U) = 1 + O(U^2)$ and $v_F = v_F(U) = \frac{3t}{2} + O(U^2)$ are analytic functions of U and $0<\theta<1$. Therefore, the effect of the interaction is simply to modify the value of the wave-function renormalization and of the Fermi velocity up to a correction $O(|\mathbf{k}-\mathbf{p}_F^{\omega}|^{\theta})$ due to the irrelevant terms in the RG sense. Moreover, if $0<|\mathbf{p}| \ll |\mathbf{k}-\mathbf{p}_F^{\omega}| \ll 1$,

$$\hat{G}_{2,1;\mu}(\mathbf{k},\mathbf{p}) = eZ_\mu S(\mathbf{k}+\mathbf{p})\Gamma_\mu(\vec{p}_F^{\pm},\vec{0})S(\mathbf{k})$$
$$\times\left[1+O\left(\left|\mathbf{k}-\mathbf{p}_F^{\omega}\right|^{\theta}\right)\right], \quad (10)$$

where $\Gamma_0(\vec{k},\vec{p}) = \left(\begin{smallmatrix}1 & 0\\ 0 & e^{-i\vec{p}\vec{\delta}_1}\end{smallmatrix}\right)$, $Z_\mu = Z_\mu(U)$ are analytic in U, and $0<\theta<1$. That is, the three-point function is identical to the free one, up to a renormalization of the vertex and to corrections $O(|\mathbf{k}-\mathbf{p}_F^{\omega}|^{\theta})$ with better infrared properties.

The RG analysis of Ref. 16 can be repeated for the current and density correlations and one gets (see following section and Ref. 17 for details)

$$\hat{K}_{lm}(\mathbf{p}) = \frac{Z_l Z_m}{Z^2}\langle \hat{j}_{\mathbf{p},l};\hat{j}_{-\mathbf{p},m}\rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p}), \quad (11)$$

where $\langle\cdot\rangle_{0,v_F}$ is the average associated with a noninteracting system with Fermi velocity $v_F(U)$, $\vec{j}_{\mathbf{p}}$ is the reduced current defined after Eq. (1), and $\hat{R}_{lm}(\mathbf{p})$ takes into account contributions from the irrelevant terms in the RG sense. This implies (see the following section and Ref. 17) that its real space counterpart has better decay properties than the bound Eq. (6), namely,

$$|R_{lm}(\mathbf{x},\mathbf{y})| \leqslant \frac{C}{1+|\mathbf{x}-\mathbf{y}|^{4+\theta}} \quad (12)$$

with $0<\theta<1$, so that $\hat{R}_{lm}(p_0,\vec{0})$ is continuous and differentiable at $p_0=0$ (and even in p_0, by symmetry).

By Eqs. (9) and (10) and the WI Eq. (4), the vertex renormalizations Z_μ are related to the wave-function renormalization Z and to the Fermi velocity v_F by simple identities[18]:

$$Z_0 = Z\,, \qquad Z_1 = Z_2 = v_F Z. \quad (13)$$

Therefore,

$$\hat{K}_{lm}(\mathbf{p}) = v_F^2\langle \hat{j}_{\mathbf{p},l};\hat{j}_{-\mathbf{p},m}\rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p}).$$

Plugging this into Eq. (7) we get

$$\sigma_{lm} = -\frac{2}{3\sqrt{3}}\lim_{p_0\to 0^+}\frac{1}{p_0}\big[[\hat{R}_{lm}(p_0,\vec{0}) - \hat{R}_{lm}(\mathbf{0})]$$
$$+\big(v_F^2\langle \hat{j}_{(p_0,\vec{0}),l};\hat{j}_{(-p_0,\vec{0}),m}\rangle_{0,v_F} - v_F^2\langle \hat{j}_{\mathbf{0},l};\hat{j}_{\mathbf{0},m}\rangle_{0,v_F}\big)\big]. \quad (14)$$

This means the conductivity can be decomposed in two class of terms, one [the first line of Eq. (14)] expressed by absolutely convergent integrals and another [the second line of Eq. (14)] by nonabsolutely convergent integrals. The first class of terms gives zero contribution to σ_{lm}, simply because $\hat{R}_{lm}(p_0,\vec{0})$ is continuous with continuous derivative at $p_0=0$ and even in p_0. The second class of terms gives a contribution that is exactly equal to the free conductivity of a system with Fermi velocity $v_F(U)$ [thanks to the WI Eq. (13)]. Since the free conductivity is independent of the Fermi velocity, this concludes the proof of Eq. (7). In the following section we explain how Eqs. (11) and (12) are derived, referring to the main technical details (in particular, the symmetries and the nonperturbative bounds) in Refs. 16 and 17.

In any case, at this point it should be clear why the universality of the conductivity is easily missed in the continuum Dirac approximation. In fact, while the free conductivity is expressed by a nonabsolutely convergent integral, whose value is dictated by the infrared singularity of the Green's function (which is the same as the Dirac propagator), in the computation of the interacting conductivity one has to distinguish between the nonabsolutely and absolutely convergent contributions. The nonabsolutely convergent integrals can be resummed and, exploiting cancellations from exact WIs, one proves universality for such terms (they sum up to the free conductivity of a system with different Fermi velocity). On the other hand, the absolutely convergent terms are associated with momenta in the whole Brillouin zone. Their cancellation requires a compensation of the contributions coming from a neighborhood of the Fermi points with those far away from them, where the Dirac approximation fails. Therefore, the Dirac approximation is intrinsically unable to predict cancellation of these convergent contributions; lattice effects and nonlinear bands, even if irrelevant in the RG sense, are essential for establishing universality of the conductivity.

VI. THE RENORMALIZATION GROUP ANALYSIS

It remains to explain how Eqs. (11) and (12) are derived; here we only sketch their proof and we refer to Refs. 16 and

17 for the technical details. The generating functional of the correlations can be written in terms of a Grassmann integral,

$$e^{W(A,\lambda)} = \int P(d\psi) e^{\mathcal{V}(\psi)+(\psi,\lambda)+B(A,\psi)}, \qquad (15)$$

where $P(d\psi)$ is the fermionic Gaussian integration for $\psi^{\pm}_{\mathbf{k},\sigma}$ with inverse propagator

$$g^{-1}(\mathbf{k}) = -Z_0 \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & ik_0 \end{pmatrix}. \qquad (16)$$

$B(A,\psi)$ is the source term describing the coupling of the Grassmann field with the external $U(1)$ gauge field A, and $(\psi,\lambda) = \int_0^\beta dx_0 \sum_{\vec{x}\in\Lambda}[\psi^+_{\mathbf{x}}\lambda^-_{\mathbf{x}} + \lambda^+_{\mathbf{x}}\psi^-_{\mathbf{x}}]$. The response function $\hat{K}_{\mu\nu}(\mathbf{p})$ can be obtained by deriving twice with respect to the gauge field A:

$$\hat{K}^{\beta}_{\mu\nu}(\mathbf{p}) = \frac{\delta^2}{\delta A_\mu(\mathbf{p})\delta A_\nu(-\mathbf{p})} W(A,0)|_{A=0}. \qquad (17)$$

Similarly, the two-point function can be obtained by deriving the generating functional twice with respect to the external field λ, and the three-point function by deriving twice with respect to λ and once with respect to A. The perturbation theory for these correlation functions is (apparently) affected by infrared divergences related to the divergence of the free propagator Eq. (16) at the Fermi points. In order to exploit cancellations in the perturbation series, it is convenient to perform the integral Eq. (15) in a multiscale fashion. We decompose the field ψ as a sum of independent Grassmann fields $\psi^{(h)}$, living on momentum scales $|\mathbf{k} - \mathbf{p}^{\pm}_F| \simeq 2^h$, with $h \leqslant 0$ a scale label. The scaling dimension of the local operator $\partial^m_{\mathbf{x}}\psi^{n_\psi}_{\mathbf{x}} A^{n_A}_{\mathbf{x}}$ turns out to be[16,17]

$$D = 3 - n_\psi - n_A - m.$$

Therefore, the only marginal terms in the RG sense are those with $n_\psi = 2, n_A = 0, m = 1$, or $n_\psi = 2, n_A = 1, m = 0$ (the terms with $n_\psi = 2, n_A = 0, m = 0$ corresponding to a possible shift of the Fermi momentum are vanishing by symmetry). All the other terms are *irrelevant*, in particular, the terms with four or more fermionic fields, corresponding to the effective multiparticle scattering terms.

After the integration of the fields with scales $\geqslant h$, we rewrite Eq. (15) (setting, for simplicity, $\lambda^{\pm} = 0$) as

$$e^{-\beta L^2 F_{\beta,L}} \int \prod_{\omega=\pm} P\left(d\psi^{(\leqslant h)}_\omega\right) e^{\mathcal{V}^{(h)}(\psi^{(\leqslant h)})+B^{(h)}(A,\psi^{(\leqslant h)})}, \qquad (18)$$

where $\psi^{\pm}_{\mathbf{k}',\omega}$ is the quasiparticle field at the Fermi point $\mathbf{p}^{\omega}_F$ (with quasimomentum $\mathbf{k}'$ relative to the Fermi point $\mathbf{p}^{\omega}_F$) and $P_{\leqslant h}(\psi_\omega)$ is a fermionic Gaussian integration with propagator

$$g^{(\leqslant h)}_\omega(\mathbf{k}') = -\frac{\chi_h(\mathbf{k}')}{Z_h} \begin{pmatrix} ik_0 & v_h\Omega^*(\vec{k}) \\ v_h\Omega(\vec{k}) & ik_0 \end{pmatrix}^{-1} (1 + O(|\mathbf{k}'|^\theta).$$

Moreover, $\chi_h(\mathbf{k}')$ is a cut-off function supported in $|\mathbf{k}'| \leqslant 2^h$, $0 < \theta < 1$, while Z_h and v_h are, respectively, the effective wave-function renormalization and Fermi velocity on scale h.

The effective potential $\mathcal{V}^{(\leqslant h)}(\psi^{(\leqslant h)})$ is a sum of monomials in $\psi^{(\leqslant h)}$ of arbitrary order, characterized at order n by kernels $W^{(h)}_{n,0}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ that are analytic in U and decay superpolynomially in the relative distances $|\mathbf{x}_i - \mathbf{x}_j|$ on scale 2^{-h}. Moreover, the effective source is given by

$$B^{(h)}(A,\psi) = \sum_{\mu=0}^{2} Z_{\mu,h} \int \frac{d\mathbf{p}}{(2\pi)^3} A_\mu(\mathbf{p}) j_\mu(\mathbf{p}) + \bar{B}^{(h)}, \qquad (19)$$

where $j_\mu(\mathbf{p}) = -ie\sum_\sigma \int \frac{d\mathbf{k}}{(2\pi)|\mathcal{B}|} \psi^+_{\mathbf{k}+\mathbf{p},\sigma}\Gamma_\mu(\vec{k},\vec{p})\psi^-_{\mathbf{k},\sigma}$, and $\bar{B}^{(h)}$ is a sum of monomials in (A,ψ) of arbitrary order, characterized at order n in ψ and m in A by kernels $W^{(h)}_{n,m}(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{y}_1,\ldots,\mathbf{y}_m)$ that are analytic in U and decay superpolynomially in the relative distances on scale 2^{-h}. In particular, for all $0 < \theta < 1$, they satisfy the bounds

$$\int \left|W^{(h)}_{n,m}\right| \leqslant (\text{const.})^{n+m}|e|^m 2^{(3-n-m)h}|U|2^{\theta h}, \qquad (20)$$

which are nonperturbative, i.e., they are based on the convergence of the expansion for the kernels $W^{(h)}$. They are obtained by exploiting the anticommutativity properties of the Grassmann variables via a determinant expansion and the use of the Gram-Hadamard inequality for determinants Refs. 16 and 17. The factor $2^{(3-n-m)h}$ corresponds to the bare scaling dimension, and the extra factor $2^{\theta h}$ is a dimensional gain due to the irrelevance of the effective electron-electron interaction. Every contribution in perturbation theory involving an effective scattering in the infrared is suppressed thanks to the irrelevance of the four-legged kernel. This dimensional gain is analogous to the one found in super-renormalizable theories such as ϕ^4_2 or ϕ^4_3, thanks to the (exponentially fast) vanishing of the effective scattering term.

The running coupling constants $Z_h, v_h, Z_{\mu,h}$ satisfy recursive equations (β-function equations) that, thanks to the bound equation (20), lead to bounded and controlled flows. The limiting values

$$Z(U) = \lim_{h\to-\infty} Z_h,\;\; Z_\mu(U) = \lim_{h\to-\infty} Z_{\mu,h},\;\; v_F(U) = \lim_{h\to-\infty} v_h$$

are analytic functions of U, analytically close to their unperturbed values $Z_0 = Z_{0,0} = 1$ and $Z_{1,0} = Z_{2,0} = v_0 = \frac{3}{2}t$; moreover, the limit is reached exponentially fast.

The A-dependent part of the generating function $W(A,\lambda)$ is given by the $h \to -\infty$ limit of the effective source Eq. (19). Therefore, the derivatives with respect to A appearing in Eq. (17) can either act on the (limit of the) first or on the (limit of the) second term in Eq. (19). In the former case we get a contribution to the first term in Eq. (11) (with $Z_h, v_h, Z_{\mu,h}$ replaced by their limiting value); in the latter, we get contributions to the remainder term $\hat{R}_{lm}(\mathbf{p})$, which satisfies improved dimensional estimates [and correspondingly improved decay properties like Eq. (12)], thanks to the extra dimensional factor $2^{\theta h}$ in Eq. (20). This concludes the (sketch of the) proof of Eqs. (11) and (12). For more technical details, concerning, in particular, the convergence of the expansion for the effective potentials and the effective sources, see Ref. 17.

VII. CONCLUSIONS

In conclusion, we rigorously proved the nonexistence of corrections to the zero temperature and zero frequency limit of undoped half-filled graphene optical conductivity due to weak short-range interactions. This solves a debated problem about the role of interactions in graphene and, remarkably, it is one of the very few examples of universality in condensed matter that can be established on firm mathematical grounds. The novelty of our approach is the use of constructive RG methods combined with exact lattice Ward identities. These are believed to play a crucial role also in the understanding of other universal phenomena, such as the QHE, which are still not accessible to a first-principles analysis.[3,10] Our proof shows the crucial role played by the lattice and by the nonlinear bands in the emergence of universality and strongly suggests that these will play an important role in the understanding of the effects of disorder or long-range interactions on the conductivity of graphene.

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