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Post-Newtonian cosmology in the Eulerian and Lagrangian frames

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Cosmologists are often in error but never in doubt

Lev Davidovich Landau

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Contents

Introduction

Motivation

The large-scale structure of the Universe is the result of the evolution of cosmological perturbations generated during Inflation: quantum fluctuations of the inflaton field set the seeds of primordial curvature perturbations in the very early Universe. These perturbations manifest themselves as inhomogeneities in the matter-energy fields and are observed as temperature and polarisation anisotropies in the Cosmic Microwave Background. Subsequent evolution via gravitational instability significantly enhances these initial perturbations, ultimately leading to the formation of the cosmic structures. The theoretical study of structure formation connects the early homogeneous Universe with that observed today and is carried out with the use of different approximations, depending on the specific range of applicability. The standard approach is the following. Relativistic perturbation theory around a homogeneous background is used at large scales (of the order of the Hubble horizon), where the grow of structures is in the linear regime. Even if second-order perturbations are considered, e.g. to compute non-Gaussianity, the matter density perturbations must be small. At small scales, well inside the Hubble horizon, General Relativity (GR) is replaced by Newtonian gravity, and non-linear gravitational instabilities are treated with Newtonian N-body simulations. However, there are several issues concerning this approach:

- The Newtonian approximation of GR, which in the Eulerian picture contains the scalar gravitational potential in the time-time component of the space-time metric, fails to produce accurate description of the photons trajectories. On the other hand, if all the calculations of the cosmological observables are performed within the relativistic perturbation theory, distances are computed assuming a homogeneous background distance-redshift relation, thus missing the true non-linear effects.

- Many investigations have been carried out in trying to understand how the output of non-linear N-body simulations can be used in a relativistic perturbative framework and even if it can be done at all, given that the simulations are run using a completely Newtonian approach. Moreover, the existing analyses of relativistic correction to matter power spectrum are often restricted to the linear dynamics.

- The upcoming galaxy surveys (such as Euclid) will probe scales approaching the Hubble horizon, where Newtonian approximation is no longer valid. Therefore, on the one hand a relativistic treatment is mandatory to study the grow of structures. On the other hand, the observations are performed along our past light cone and it is necessary to take into account light cone and gauge relativistic effects for the correct interpretation of all the data.

In order to deal with the above problems, we should seek for a relativistic and non-perturbative approach, capable to disentangle the Newtonian from GR contributions. An alternative approximation scheme is well-suited for this purpose: the Newtonian analysis can be improved with the post-Newtonian (PN) approximation of GR, which provides the first relativistic corrections for a system of slowly moving particles bound together by gravitational forces and thus it can be used to account for the moderately non-linear gravitational field generated during the highly non-linear stage of the evolution of matter fluctuations on intermediate scales. It is a crucial improvement of both the aforementioned approximations, as it could bridge the gap between relativistic perturbation theory and Newtonian structure formation, providing a unified approximation scheme able to describe the evolution of cosmic inhomogeneities from the largest observable scales to small ones, including also the intermediate range, where the relativistic effects cannot be ignored and non-linearity starts to be relevant. Nonetheless, few attempts have been made so far to go beyond the Newtonian approximation on non-linear scales. A relevant difficulty of this scheme is that in the PN framework the background is not merely the FRW metric but a Newtonian metric yet describing non-linearities. This very fact has so far prevented from proceeding in this direction because of the computational complexity, except for symmetric situations. The PN approximation has by construction a direct correspondence with Newtonian quantities: the PN expressions are sourced only by the non-linear Newtonian terms which can be extracted e.g. from N-body simulations (or by approximate analytical expressions obtained for example by via the Zel'dovich approximation). Such a correspondence becomes increasingly important especially when studying frame-dependent quantities.

The PN approximation is also suitable to address the so-called dark energy problem: the lack of a theoretical explanation for dark energy, what is it made of and, in some sense, even if it really exists are yet open questions. It has been proposed that small scale non-linear inhomogeneities play a dynamical role in the expansion history and could give a contribution to the accelerated expansion. The basic idea is the following: GR is a non-linear theory thus, in principle, non-linear effects on small scales can lead to unexpected non-perturbative behaviour on large scales. If it is the case, the back-reaction mechanism would link the recent onset of the accelerated expansion to the beginning of non-linear structure formation, thus providing also a natural solution to the coincidence problem. The present controversy about the cosmological back-reaction concerns mostly how to construct an adequate smoothing procedure. Nevertheless, it is widely believed that back-reaction is a purely GR and non-linear effect, thus its quantitative estimation should be performed at least in the PN approach.

Main results

Driven by the motivations described above, the present thesis is devoted to the cosmological dynamics in the PN approximation of GR. The main results are summarised below.

The PN Zel'dovich solution: we obtained the analytical solution of the Einstein field equations describing the non-linear cosmological dynamics of plane-parallel perturbations in the synchronous-comoving gauge. The solution is non-perturbative in the standard sense, exact up to PN order and extends the Zel'dovich approximation which, in turn, is exact for non-linear plane-parallel dynamics in Newtonian gravity. This work has been completed in collaboration with D. Maino and S. Matarrese and has been published in *JCAP*, Villa et al. 2011.

The cosmological back-reaction of plane-parallel perturbations: within the averaging prescription proposed in Buchert 2000, we performed for the first time a PN calculation for the average expansion rate whose result implies that the average dynamics of plane-parallel perturbations is the same as the Friedmann-Robertson-Walker background with negligible PN contributions. Nevertheless, the analytical solution obtained allowed a careful analysis of the late stage of plane-parallel dynamics: it was explicitly showed that for the pancake singularity the matter density and the space-time curvature diverge but their averages are well-defined, Villa et al. 2011.

PN gauge transformations: we analysed the procedure for passing from GR to Newtonian theory in both the Eulerian and the Lagrangian approaches to gravitational dynamics and also the connection between the two. In the GR framework we provided the transformation equations from the Poisson gauge to the synchronous and comoving gauge for the metric in the Newtonian approximation, being completely non-linear in the standard perturbation theory. We fully recovered equations and solution for the metric both in the relativistic perturbation theory up to second order and in the Newtonian limit at any order. This work has been carried out in collaboration with D. Maino and S. Matarrese and was submitted to *JCAP* and is currently under review, Villa et al. 2014. We then extend our transformation to the PN approximation and our results are consistent with GR up to second order in perturbation theory. This work has been carried out in collaboration with D. Maino and S. Matarrese and is currently in preparation, Villa et al. 2014.

Thesis overview

Apart from two introductory chapters, the present thesis consists of two parts, for a total of eight chapters. Part I spans from chapter 3 to chapter 6. In chapters 3 and 4 we review the description of the cosmological dynamics in the Newtonian and in the PN approximation of GR, and in relativistic standard perturbation theory, up to second order. Chapters 5 and 6 deal with the transformation from the Eulerian to the Lagrangian frame in the Newtonian and in the PN approximation of GR. Part II, which actually corresponds solely to chapter 7, is dedicated to the PN cosmological dynamics of plane-parallel perturbation and to a PN estimation of the kinematical back-reaction. Chapter 8 deals with conclusions and future work.

Chapter 1 - Basic cosmology: this first introductory chapter consists of three sections. In the first we review briefly the characterisation of the FRW models for the Universe. In the last two we give the initial conditions from Inflation and the

assumptions we use to study the gravitational dynamics of large-scale structures, respectively.
Chapter 2 - Gauge fixing and gauge transformations in GR: in this second introductory chapter we discuss the gauge fixing in perturbation theory and in the $3+1$ approach to GR. In this thesis we are interested in the connection between GR and the Newtonian treatment of cosmological perturbation. We will therefore consider two particular gauges, which we will define more precisely in the following: the Poisson gauge, which in the Newtonian limit reduce to the Eulerian frame, and the synchronous-comoving gauge, which is intrinsically a Lagrangian frame. We also present the basics of the theory of gauge transformations, both in the active and in the passive approaches, up to second order in perturbation theory.

Chapter 3 and 6 - Cosmological dynamics in the Eulerian and in the Lagrangian frames: we discuss the Newtonian limit of GR in the Eulerian and Lagrangian frame and we present the equations for the cosmological dynamics in the Newtonian limit and the PN approximation in the Poisson gauge and in the synchronous-comoving gauge. The GR solutions up to second order in perturbation theory in both frames are reported in appendix B.

Chapters 5 and 6 - From the Eulerian to the Lagrangian frame: Newtonian and PN approximations: these two chapters are the core of the thesis. In chapter 5 we provide the transformation from the Eulerian to the Lagrangian frame in the Newtonian limit of GR. In chapter 6 we extend this analysis to the PN approximation. The approach adopted for our transformation is fully non-perturbative, which implies that if our quantities are expanded according to the rules of standard perturbation theory, all terms are exactly recovered at any order in perturbation theory, only provided they are Newtonian and PN. We explicitly show this result in both frames.

Chapter 7: PN cosmological dynamics for plane-parallel perturbations - This chapter deals with the PN extension of the Newtonian Zel'dovich solution of the plane-parallel dynamics and its application in the context of the cosmological backreaction proposal.

CHAPTER 1

Basic cosmology

This introductory chapter consists of three sections. In the first section we present the characterisation of the homogeneous space-time in the language of the $3+1$ formalism for GR, Misner et al. 1973, since we will use this approach in the following. In the second section we give the initial conditions for cosmological dynamics. We consider here the simplest inflationary model, namely the standard single scalar model. In the last section we recall the fundamental assumptions and the validity of the pressure-less fluid approximation for the dark matter. The homogeneous background considered in this thesis is the flat Einstein-de Sitter model.

1.1 Homogeneity and isotropy: a geometrical perspective

Cosmology deals with the large-scale properties of the Universe. In the past the attempts to describe our Universe started from some "philosophical" assumptions: for example, since the time of Copernicus, it has generally been assumed that we do not occupy a privileged position. Nowadays we take advantage of the modern observations which show that the distribution of cosmic structures at small scales has highly inhomogeneous clustering properties, whereas the Universe appears homogeneous on large scales. A stronger confirmation of the symmetry properties of the whole Universe comes from the isotropy of the Cosmic Microwave Background (CMB) radiation ($\frac{\Delta T}{T} \simeq 10^{-5}$). This observational evidence suggests that we can assume that the large-scale structure of Universe is isotropic. The Copernican Principle is formulated as follows: it is assumed that there exist a special family of observers to which the Universe seems the same in all directions, independently of their location. The observed isotropy supplied by the Copernican Principle implies the Cosmological Principle which states that the Universe is homogeneous and isotropic on a sufficiently large scale. If one assumes the Cosmological Principle as a valid tool to describe the Universe, tight constraints on the geometry of space-time ensue. In order to clarify this statement, we give the precise mathematical definitions of homogeneity and isotropy following the textbooks Wald 1984 and Misner et al. 1973.

A space-time is said to be (spatially) homogeneous if there exists a one-parameter family of (spacelike) hypersurfaces Σ_t foliating the space-time such that for each $t \in \mathbb{R}$ and for any points $p, q \in \Sigma_t$ there exists an isometry of space-time metric g_{ab} which takes p into q. In other words, this means that at any given time t, each point of the space looks like any other point. This peculiar foliation of the space-time is made of the so-called hypersurfaces of homogeneity.

With regard to isotropy, it must be kept in mind that it is strictly connected with the physical properties of the motion of the observers: for example, any observer in motion

relative to the matter filling the Universe sees an anisotropic distribution of velocity. The following definition selects a peculiar family of observers and gives a mathematical characterisation of isotropy. A space-time is said to be (spatially) isotropic at each point if there exists a congruence of timelike curves (i.e. observers), with tangent denoted u^a, filling the space-time and satisfying the following property: given any point p and any two spatial unit tangent vectors s_1^a and s_2^a at p orthogonal to u^a, there exists an isometry of space-time metric g_{ab} which leaves p and u^a at p fixed but rotates s_1^a into s_2^a. This means that in an isotropic Universe it is impossible to construct a geometrically preferred vector orthogonal to u^a.
In the case of a space-time simultaneously homogeneous and isotropic the hypersurfaces of homogeneity must be orthogonal to the world lines of isotropic observers: the failure of the tangent subspace orthogonal to u^a to coincide with Σ_t would make possible to construct a geometrically preferred vector orthogonal to u^a, namely the velocity of the observer on Σ_t, in violation of isotropy, since this would actually provide the observer with a way to distinguish one space direction in his rest frame from all others. On the contrary, the isotropic observers see the hypersurfaces of homogeneity invariant under spatial translations (because of homogeneity itself) and under spatial rotations. Each Σ_t is said to be maximally symmetric.
It can be showed (Wald 1984) that the requirement of isotropy forces the Riemann tensor of the hypersurfaces of homogeneity to have the peculiar expression

$$ {}^{(3)}R_{abcd} = \mathcal{K} h_{c[a} h_{b]d} \,, \tag{1.1} $$

where h_{ab} is the spatial metric and $\mathcal{K}$ is the scalar curvature parameter of Σ_t. Moreover, the homogeneity implies that $\mathcal{K}$ can not vary from point to point on the hypersurfaces Σ_t. From the Riemann tensor one can compute the Ricci tensor and the scalar curvature of such a space. They read

$$ {}^{(3)}R_{ab} = 2\mathcal{K} h_{ab} \qquad \text{and} \qquad {}^{(3)}R = 6\mathcal{K}. \tag{1.2} $$

1.1.1 Gauge choice and metric tensor

The natural choice of the observers in the homogeneous and isotropic space-time is that of isotropic observers. They are comoving with the matter, otherwise they would see an anisotropic distribution of velocity. The above symmetry assumptions also assure that their world-line u^a are orthogonal to the hypersurfaces of homogeneity and then irrotational. With regard to the matter content, we consider an irrotational dust Universe, thus the stress-energy tensor has the form

$$ T_{ab} = \varrho u_a u_b, \tag{1.3} $$

where ϱ is the energy density measured by u_a and the isotropic observers, comoving with the matter, move along the geodesics. Homogeneity also requires the energy density ϱ to be function of the time t only.

The choice of isotropic observers determines a line element of the form

$$ ds^2 = -dt^2 + h_{\alpha\beta}(t, \mathbf{q}) dq^\alpha dq^\beta \,, \tag{1.4} $$

where, in the language of the $3 + 1$ formalism, we say that the time slicing is chosen to be geodesic: the time coordinate t, constant on every hypersurface of homogeneity, coincides with the proper time along isotropic observes.

The assumptions of homogeneity and isotropy turn out also to determine the form of the metric $h_{\alpha\beta}(t, \mathbf{q})$. This can be easily seen as follows (Misner et al. 1973).
Firstly, one constructs the isotropic coordinate frame: among the hypersurfaces of homogeneity $\{\Sigma_t\}$ we choose one slice to represent the initial slice Σ_{in} on which we set $t = 0$ and suppose that there exists an arbitrary spatial coordinate system $\{q^\alpha\}$. Each curve of the isotropic congruence is permanently labelled by the coordinate values $\{q^\alpha\}$ of the point from which it propagates from Σ_{in}. The spatial metric of the initial hypersurface of homogeneity is completely determined by the above form of its Riemann tensor. For example, if we choose quasi-Cartesian coordinates, the components of the metric of a space with constant curvature are given by (see e.g. Misner et al. 1973)[1]

$$\gamma_{\alpha\beta}(\mathbf{q}) = \left(1 + \frac{1}{4}\mathcal{K}\delta_{\mu\nu}q^\mu q^\nu\right)^{-2} \delta_{\alpha\beta}. \tag{1.5}$$

Thus one can set

$$h_{\alpha\beta,in} = \gamma_{\alpha\beta} \tag{1.6}$$

and then carries this spatial geometry to each of the other hypersurfaces of homogeneity $\{\Sigma_t\}$ by means of the isotropic observers. More precisely, once one has chosen the metric of Σ_{in}, the spatial separation of the world line of two isotropic observers

$$d\sigma_{in} := h_{\alpha\beta,in} dq^\alpha dq^\beta \tag{1.7}$$

$$= \left(\gamma_{\alpha\beta} dq^\alpha dq^\beta\right)^{\frac{1}{2}} \tag{1.8}$$

is known. At some later time they will be separated by some other distance $d\sigma_t$. Homogeneity guarantees that the ratio $\frac{d\sigma_t}{d\sigma_{in}}$ will be independent of the spatial position of the two observers on the slices and, because of isotropy, it will be independent also of the direction connecting them. Then one can define the dimensionless time function

$$a(t) := \frac{d\sigma_t}{d\sigma_{in}} \tag{1.9}$$

describing how the separation of the world line of the congruence of the observers change with time onto the orthogonal hypersurfaces of homogeneity. It is called the scale-factor. By combining the above results, one obtains the spatial separation at time t as

$$d\sigma_t = h_{\alpha\beta,t} dq^\alpha dq^\beta \tag{1.10}$$

$$= a(t) d\sigma_{in} \tag{1.11}$$

$$= a(t) \left(\gamma_{\alpha\beta} dq^\alpha dq^\beta\right)^{\frac{1}{2}}. \tag{1.12}$$

Therefore the components of the spatial metric are given by

$$h_{\alpha\beta}(t, \mathbf{q}) = a(t)\gamma_{\alpha\beta}(\mathbf{q}) \tag{1.13}$$

and the space-time line element (1.4) reads

$$ds^2 = -dt^2 + a^2(t)\gamma_{\alpha\beta} dq^\alpha dq^\beta \tag{1.14}$$

[1]In the Einstein-de Sitter model the space is flat ($\mathcal{K} = 0$) and, if we choose Cartesian coordinates, $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$

which is the well-known Friedmann-Robertson-Walker (FRW) line element: q^α are the spatial, time-independent coordinates comoving with the matter and t labels the hypersurfaces of homogeneity.
We will consider the spatially flat Einstein-de Sitter model and use conformal time η defined by $d\eta = dt/a(t)$. The background metric is given by

$$ds^2 = a^2(\eta)\left[-dt^2 + \delta_{\alpha\beta}dq^\alpha dq^\beta\right]. \tag{1.15}$$

The Einstein equations describe the evolution of the scale factor and the mass density and read

$$\begin{aligned} 3\mathcal{H}^2 &= a^2 8\pi G\varrho_b & (1.16)\\ \varrho_b' &= -3\mathcal{H}\varrho_b & (1.17)\end{aligned}$$

where a prime stands for differentiation with respect to conformal time, and $\mathcal{H} = a'/a$.

1.2 Inflation and initial conditions

In the standard approach to cosmology the inflationary paradigm provides the initial conditions for structure formation and generally for cosmological perturbation theory. Inflation is an extensive topic, dealing with the early epoch of the Universe. It constitutes a wide field of research, with a large number of different inflationary models, see e.g. Lyth & Riotto 1999. Here we just recall some of the basics of the inflationary theory and give initial condition for relativistic perturbation theory which we will use for our calculations throughout this thesis.

Our current understanding of the origin of structures in the Universe is that once the Universe became matter dominated (z $\sim$ 3200) primeval density inhomogeneities ($\delta\varrho/\varrho \sim 10^{-5}$) were amplified by gravity and grew in the structure we see today. The primary success of inflation is to give a model for the origin of the primordial density perturbations from vacuum fluctuations during a period of accelerated expansion at very early times. In the standard single field scenario, zero-point vacuum fluctuations of a light, weakly coupled scalar field were stretched up to super-Hubble scales during the inflationary accelerated expansion. The stress-energy tensor of the inflaton field was dominating and its perturbations were coupled with the perturbations of the space-time metric through Einstein equations and generated the first seed of density fluctuations.
The components of a spatially flat FRW metric perturbed up to second order can be written in any gauge as

$$\begin{aligned} g_{00} &= -a^2(\eta)\left(1 + 2\phi^{(1)} + \phi^{(2)}\right)\\ g_{0i} &= a^2(\eta)\left(\hat{\omega}_i^{(1)} + \frac{1}{2}\hat{\omega}_i^{(2)}\right)\\ g_{ij} &= a^2(\eta)\left[(1 - 2\psi^{(1)} - \psi^{(2)})\delta_{ij} + \left(\hat{\chi}_{ij}^{(1)} + \frac{1}{2}\hat{\chi}_{ij}^{(2)}\right)\right]. & (1.18)\end{aligned}$$

The functions $\phi^{(r)}, \hat{\omega}_i^{(r)}, \psi^{(r)}$ and $\hat{\chi}_{ij}^{(r)}$, where $(r) = (1), (2)$, stand for the rth-order perturbations of the metric. Notice that such an expansion could a priori include terms of arbitrary order, Matarrese et al. 1998, but for our purposes the first and second-order terms are sufficient. It is standard use to split the perturbations into the so-called scalar, vector and tensor parts according to their transformation properties with respect to the three-dimensional space with background metric δ_{ij}, where scalar parts are related to a scalar potential, vector parts to transverse (divergence-free) vectors and tensor parts to transverse trace-free tensors. That is

$$\hat{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)} \tag{1.19}$$

$$\hat{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)}\,, \tag{1.20}$$

where ω_i and χ_i are transverse vectors, *i.e.* $\partial^i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0$, $\chi_{ij}^{(r)}$ is a symmetric transverse and trace-free tensor, *i.e.* $\partial^i \chi_{ij}^{(r)} = 0$, $\chi_{\ i}^{i(r)} = 0$ and $D_{ij} = \partial_i \partial_j - (1/3)\,\delta_{ij}\,\nabla^2$ is a trace-free operator. [2] The reason why such a splitting has been introduced, Bardeen 1980 and Kodama & Sasaki 1984, is that in linear theory, these different modes are decoupled from each other in the perturbed evolution equations, so that they can be studied separately. This property does not hold anymore beyond the linear regime where second-order perturbations are sourced by first-order perturbations.

Thanks to the inflationary accelerated expansion, the gauge invariant curvature perturbation of uniform density hypersurfaces ζ becomes constant on super-horizon scales after is has been generated at a primordial epoch. The conserved value of the curvature perturbation ζ allows to set the initial conditions for the metric and matter perturbations accounting for the primordial contributions. We conveniently fix the initial conditions at the time when cosmological perturbations relevant for large-scale structures are well outside the Hubble radius deeply in matter dominated era. In order to follow the evolution on super-horizon scales of the density fluctuations coming from the various mechanisms, we expand the gauge invariant curvature perturbation up to second order $\zeta = \zeta_1 + 1/2\zeta_2$, where

$$\zeta_1 = -\psi_1 - \mathcal{H}\frac{\delta_1 \varrho}{\varrho_b'} \tag{1.21}$$

and the expression for ζ_2 is given by Bartolo et al. 2004

$$\zeta_2 = -\psi_2 - \mathcal{H}\frac{\delta_2 \varrho}{\varrho'} + \Delta\zeta_2\,, \tag{1.22}$$

with

$$\Delta\zeta_2 = 2\mathcal{H}\frac{\delta_1 \varrho'}{\varrho'}\frac{\delta_1 \varrho}{\varrho'} + 2\frac{\delta_1 \varrho}{\varrho'}(\psi_1' + 2\mathcal{H}\psi_1) - \left(\frac{\delta_1 \varrho}{\varrho'}\right)^2 \left(\mathcal{H}\frac{\varrho''}{\varrho}\right)\,. \tag{1.23}$$

In particular, ζ_2 provides the necessary information about the level of non-Gaussianity of primordial perturbations. Different inflationary scenarios are characterized by different values of ζ_2. For example, in the standard single-field model $\zeta_2 = 2\left(\zeta_1\right)^2 + \mathcal{O}\left(\epsilon, \eta\right)$, where ϵ and η are the standard slow-roll parameters, Lyth & Riotto 1999. In general, we may parametrize the primordial non-Gaussianity level in terms of the conserved curvature perturbation as in Bartolo et al. 2004

$$\zeta_2 = 2a_{\rm nl}\left(\zeta_1\right)^2\,, \tag{1.24}$$

where the parameter $a_{\rm nl}$ depends on the physics of a given scenario. At linear order during the matter-dominated epoch and on large scales $\zeta_1 = -5\varphi_{in}/3$, where φ_{in} is the

[2] Latin indices are raised and lowered using δ^{ij} and δ_{ij}, respectively.

gravitational potential during matter domination, when the cosmological constant was still negligible. Thus we can write

$$\zeta_2 = \frac{50}{9}\, a_{\rm nl}\, \varphi_{in}^2 = \frac{50}{9}\, a_{\rm nl}\, \varphi_{in}^2\,. \tag{1.25}$$

In addition, since we consider the standard single field inflation, the metric (1.18) can be simplified. The fact that first-order vector perturbations have decreasing amplitudes and that are not generated in the presence of scalar fields, allows us to conclude that they can be safely disregarded. Moreover, the first-order tensor part gives a negligible contribution to second-order perturbations. Thus, in the following we will neglect $\omega_i^{(1)}$, $\chi_i^{(1)}$ and $\chi_{ij}^{(1)}$. However the same reasoning does not apply to second-order perturbations: since in the non-linear case scalar, vector and tensor modes are dynamically coupled, the second-order vector and tensor contributions are generated by first-order scalar perturbations even if they were initially zero, Matarrese et al. 1998.
Therefore the spatially flat FRW metric perturbed up to second order reads in any gauge

$$\begin{aligned}
g_{00} &= -a^2(\tau)\left(1+2\phi^{(1)}+\phi^{(2)}\right)\\
g_{0i} &= a^2(\tau)\left(\partial_i\omega^{(1)}+\frac{1}{2}\,\partial_i\omega^{(2)}+\frac{1}{2}\,\omega_i^{(2)}\right)\\
g_{ij} &= a^2(\tau)\left[\left(1-2\psi^{(1)}-\psi^{(2)}\right)\delta_{ij}+D_{ij}\left(\chi^{(1)}+\frac{1}{2}\chi^{(2)}\right)\right.\\
&+ \left.\frac{1}{2}\left(\partial_i\chi_j^{(2)}+\partial_j\chi_i^{(2)}+\chi_{ij}^{(2)}\right)\right]\,.
\end{aligned} \tag{1.26}$$

1.3 The irrotational and pressure-less fluid approximation

In this section we recall briefly the assumptions which are usually adopted for analytic techniques for studying the gravitational dynamics of the inhomogeneities of the large-scale structure. According to the ΛCDM model, matter decouples from the primordial plasma at the time of early matter domination, Komatsu et al. 2011. Modelling the large-scale structure thus means to describe the gravitational evolution during the matter domination era. In this phase the non-collisional fluid of cold dark matter is believed to dominate the other components, until dark energy becomes relevant. In the study of the dynamics of dark matter the single–stream approximation is commonly assumed, i.e. we neglect the velocity dispersion. We also assume zero vorticity because, according to Kelvin's circulation theorem and in absence of dissipation, vorticity is conserved: a fluid with vanishing initial vorticity will forever remain irrotational. The vanishing of initial vorticity is in turn guaranteed by the initial conditions consistent with the inflationary paradigm, as we recall in 1.2. All these assumptions are valid as long as we restrict our analysis to suitable large scales, where the cosmological dynamics is only governed by gravitational self–interaction of the dark matter fluid of dust. At small scales the irrotational and pressure-less fluid approximation breaks down: a number of highly non-linear phenomena, such as vorticity generation by multistreaming, merging, tidal disruption and fragmentation, occur and therefore pressure gradients and viscosity become important and affect the dynamics significantly.

CHAPTER 2

Gauge fixing and gauge transformations in GR

2.1 The gauge problem and the gauge transformation rules in perturbation theory

In this section we present the basics of the theory of the gauge transformations in perturbation theory, up to second order. The subject is rather complicated and here we want to give a concise but precise description of the computational techniques. The present discussion is based on Bruni et al. 1997 and Matarrese et al. 1998[1].

2.1.1 The gauge freedom in GR: equivalence between active and passive approach

GR describes nature in terms of the space-time manifold $\mathcal{M}$ and a collection of tensor fields, the metric tensor g and other tensor fields $T^{(i)}$, e.g. the stress-energy tensor.

Consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ mapped onto each other by a diffeomorphism ϕ, namely a map which is C^∞, i.e. infinitely continuously differentiable in the advanced calculus sense, one-to-one, onto and its inverse ϕ^{-1} is also C^∞. Such properties assure that the two manifold have identical structure and allow us to move fields back and forth from $\mathcal{M}$ to $\mathcal{N}$ and viceversa. For example, consider a scalar field $f : \mathcal{N} \to \mathbb{R}$. The map ϕ

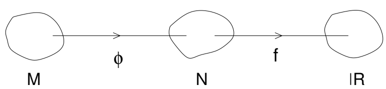


Figure 2.1: Pull-back of a scalar field.

naturally pulls back the scalar field f on $\mathcal{N}$ in the scalar field $f \circ \phi : \mathcal{M} \to \mathbb{R}$ on $\mathcal{M}$ such that

$$f(Q) = f(\phi(P)) \in \mathbb{R} \qquad \forall P \in \mathcal{M} \text{ and } \forall Q \in \mathcal{N}$$

Similarly, we can can push forward on $\mathcal{N}$ tensor fields defined on $\mathcal{M}$

$$\phi_* : \underset{P\in M}{\mathcal{T}} \binom{k}{l} \to \underset{\phi(P)\in N}{\mathcal{T}} \binom{k}{l} \tag{2.1}$$

such that

$$\phi_*(T(P)) = (\phi_* T)(\phi(P)) \in \mathbb{R} \tag{2.2}$$

and also pull back on $\mathcal{M}$ tensor fields defined on $\mathcal{N}$

$$\phi^* : \underset{\phi(P)\in N}{\mathcal{T}} \binom{k}{l} \to \underset{P\in M}{\mathcal{T}} \binom{k}{l} \tag{2.3}$$

[1]Indices notation for this section: μ, ν for spacetime indices; $_{,\mu}$ for partial derivative with respect to x^μ

such that

$$\phi^*(T(\phi(P))) = (\phi^* T)(P) \in \mathbb{R}\,. \tag{2.4}$$

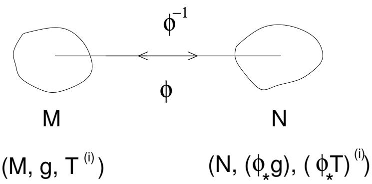


Figure 2.2: Gauge freedom of GR: active point of view.

This means that the collections $(\mathcal{M}, g, T^{(i)})$ and $(\mathcal{N}, (\phi_* g), (\phi_* T)^{(i)})$ represent the same space-time and describe the same physics. This very fact is the gauge freedom of GR: we can use a diffeomorphism ϕ to construct a representation in terms of an identification between points on $\mathcal{M}$ and $\mathcal{N}$ and in terms of the push-forward on $\mathcal{N}$ of the tensor fields defined on $\mathcal{M}$[2]. All these representations are fully equivalent: any physically meaningful statement about $(\mathcal{M}, g, T^{(i)})$ will hold with equal validity for $(\mathcal{N}, (\phi_* g), (\phi_* T)^{(i)})$ and therefore we are free to choose the representation which is more convenient for our purpose. This point of view is called active, because it deals with manifolds and tensor fields. We can reformulate the gauge freedom of GR in the passive point of view, referring to the components of tensor fields in some coordinate system. Suppose that we have two coordinate systems: x^μ on $\mathcal{M}$ and x'^μ on $\mathcal{N}$. We can use the

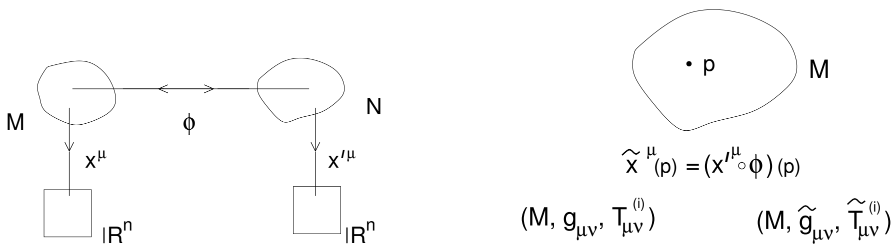


Figure 2.3: Gauge freedom in GR: passive point of view.

diffeomorphism ϕ between the two manifolds to construct a new coordinate system $\tilde{x}^\mu$ on $\mathcal{M}$ by simply pulling back the coordinate system x'^μ defined on $\mathcal{N}$: $\tilde{x}^\mu = x'^\mu \circ \phi$. The effect of ϕ is that of a standard coordinate transformation, i.e. it changes components and basis, leaving the tensors unchanged. We use the same manifold $\mathcal{M}$ and the same tensors g and $T^{(i)}$: we just write them in two different coordinates systems, x^μ and $\tilde{x}^\mu$.

The passive point of view on diffeomorphisms is, philosophically, drastically different from the active one but in practice they are really equivalent since the components of the push forward tensors $(\phi_* g)$ and $(\phi_* T)^{(i)}$ in the coordinate system x'^μ on $\mathcal{N}$ in the active viewpoint are precisely the components of the tensors g and $T^{(i)}$ in the coordinate system $\tilde{x}^\mu$ on $\mathcal{M}$ in the passive viewpoint, Wald 1984.

[2] Of course we can change the roles of $\mathcal{M}$ and $\mathcal{N}$ and consider the pull-back on $\mathcal{M}$ of the tensor fields defined on $\mathcal{N}$

2.1.2 The gauge problem in perturbation theory

The idea underlying the theory of space-time perturbations is the same that we have in any perturbative formalism: we try to find approximate solutions of the field equations, regarding them as small deviations from a known background solution. However, the application of perturbation methods in GR brings in a new problem, since among the physical quantities to be perturbed is the space-time itself. In perturbation theory we have to deal with two collections of space-time manifold and tensor fields: the first is $\left(\mathcal{M}_{ph}, g_{ph}, T^{(i)}_{ph}\right)$, where $\mathcal{M}_{ph}$ is the physical perturbed space-time and $g_{ph}, T^{(i)}_{ph}$ are the unknown solution of the Einstein equations and the second is $\left(\mathcal{M}_b, g_b, T^{(i)}_b\right)$, where $\mathcal{M}_b$ is some background space-time and $g_b, T^{(i)}_b$ are the known solutions of the Einstein equations. In perturbation theory we write the unknown solution of any relevant quantity, say, represented by a tensor field T_{ph} as $T_{ph} = T_b + \Delta T$, where ΔT is a small perturbation. The first problem in GR is actually the correct definition of the perturbation: it should be defined as the difference between T_{ph} and T_b. The problem is that, since T_{ph} and T_b are defined in different space-times, they can be compared only after a prescription for identifying points between these space-times is given. A gauge choice in perturbation theory is precisely this, i.e. a one-to-one correspondence between the background and the physical space-time, by means of a diffeomorphism, which allow us to compare the perturbed and the background tensors. We start on the physical space-time $\mathcal{M}_{ph}$ and make a gauge choice, i.e. we consider a diffeomorphism ϕ identifying a point R on $\mathcal{M}_{ph}$ with a point P on $\mathcal{M}_b$. This gives us two tensors at P: T_b itself and the pull-back of T_{ph}, $\phi^* T_{ph}$, which can be directly compared. Now it is natural to define the perturbation of T_b as the difference

$$\Delta^\phi T = \phi^* T_{ph} - T_b \,. \tag{2.5}$$

Let us make a different gauge choice i.e. consider a different diffeomorphism ψ. We start at the same point R on $\mathcal{M}_{ph}$ and end up in a different point Q on $\mathcal{M}_b$. The perturbation is defined as before

$$\Delta^\psi T = \psi^* T_{ph} - T_b \,, \tag{2.6}$$

this time using the map ψ. It is clear that both the representation of T_{ph} in the background space-time and the perturbation are completely dependent on which map is chosen, and the freedom we have in choosing it gives rise to an arbitrariness in the value of the perturbation. This is the essence of the so-called gauge problem in perturbation theory.

2.1.3 The gauge transformation rules in perturbation theory: active and passive approaches

A gauge transformation is a change in the diffeomorphism between the background and the physical space-times and we want to calculate how the two representations on $\mathcal{M}_b$ of T_{ph}, - $\phi^* T_{ph}$ in P and $\psi^* T_{ph}$ in Q - change accordingly. We are interested in perturbation theory, so now we introduce a small parameter ϵ and we perform the calculation up to second order in perturbation theory. We choose a coordinate system x^μ on $\mathcal{M}_b$ and consider a parametric curve in the manifold on which we can take the point P to correspond to $\epsilon = 0$. A gauge transformation is an infinitesimal point transformation from P to Q

on $\mathcal{M}_b$ which at second order has the form[3]

$$x^\mu(Q) = x^\mu(P) + \epsilon\xi^\mu(x^\alpha(P)) + \frac{\epsilon^2}{2}\left(\xi^\mu_{,\nu}\xi^\nu + \zeta^\mu\right)(x^\alpha(P)) + \mathcal{O}(\epsilon^3) \tag{2.7}$$

and gives the $x^\mu(Q)$, namely the coordinates of a second point Q along the curve.
In order to find how the two representations of T_{ph} on $\mathcal{M}_b$ change, we have to transform the pull-back under the point transformation (2.7). For a mixed tensor field (with covariant and controvariant indices) we also need the inverse transformation which is given by

$$x^\mu(P) = x^\mu(Q) - \epsilon\xi^\mu(x^\alpha(P)) - \frac{\epsilon^2}{2}\left(\xi^\mu_{,\nu}\xi^\nu + \zeta^\mu\right)(x^\alpha(P)) + \mathcal{O}(\epsilon^3) \tag{2.8}$$

but we want the r.h.s in terms of $x(Q)$ expanded up to second order. At first order the inverse reads

$$x^\mu(P) = x^\mu(Q) - \epsilon\xi^\mu(x^\alpha(Q)) + \mathcal{O}(\epsilon^2)\,, \tag{2.9}$$

since for the first-order term at first order $x(P) \sim x(Q)$. For the inverse at second order we have to change the first-order term up to second order using the inverse at first order obtaining

$$\begin{aligned}
-\epsilon\xi^\mu(x^\alpha(P)) &= -\epsilon\xi^\mu\left(x^\alpha(Q) - \epsilon\xi^\alpha(x^\beta(Q))\right) \\
&= -\epsilon\left[\xi^\mu(x^\alpha(Q)) - \epsilon\xi^\mu_{,\nu}\xi^\nu(x^\alpha(Q))\right] \\
&= -\epsilon\xi^\mu(x^\alpha(Q)) + \epsilon^2\xi^\mu_{,\nu}\xi^\nu(x^\alpha(Q)) + \mathcal{O}(\epsilon^3)\,,
\end{aligned} \tag{2.10}$$

whereas for the second-order term at second order $x(P) \sim x(Q)$. Then the inverse at second order is

$$x^\mu(P) = x^\mu(Q) - \epsilon\xi^\mu(x(Q)) + \frac{\epsilon^2}{2}\left(\xi^\mu_{,\nu}\xi^\nu - \zeta^\mu\right)(x(Q)) + \mathcal{O}(\epsilon^3)\,. \tag{2.11}$$

Now we can apply the standard rules for the components of the pull-back tensors $\phi^* T_{ph}$ in the coordinates x^μ under the action the gauge transformation (2.7), regardless the perturbative order of the quantity itself, for the moment. Let us start with a scalar. The pull back of a scalar in P - $F(x(P))$ - and in Q - $F(x(Q))$ - are related simply changing by the transformation of their argument according to (2.7). At second order we have

$$F(x(P)) = F(x(Q)) + \epsilon F_{,\nu}\xi^\nu(x(Q)) + \frac{\epsilon^2}{2}\left(F_{,\omega\sigma}\xi^\omega\xi^\sigma + F_{,\omega}\xi^\omega_{,\nu}\xi^\nu + F_{,\sigma}\zeta^\sigma\right)(x(Q)) + \mathcal{O}(\epsilon^3) \tag{2.12}$$

namely

$$F(x(P)) = F(x(Q)) + \epsilon\mathcal{L}_\xi F + \frac{\epsilon^2}{2}\left(\mathcal{L}^2_\xi F + \mathcal{L}_\zeta F\right) + \mathcal{O}(\epsilon^3) \tag{2.13}$$

where $\mathcal{L}_\xi$ is the Lie derivative of a scalar along the vector field ξ^μ.
Then we consider a one-index object. A 1-form transform as

$$\left(\phi^*\omega\right)_\mu(x(P)) = \left(\frac{\partial x^\alpha(Q)}{\partial x^\mu(P)}\right)\omega_\alpha(x(Q))|_{x(Q)\to x(P)} \tag{2.14}$$

[3]This is known as a "knight diffeomorphism". Its particular form is due to the fact that in going from P to Q on $\mathcal{M}_b$ we actually move from P to R on $\mathcal{M}_{ph}$ using ϕ^{-1} and finally from R to Q on $\mathcal{M}_b$ using ψ. The Taylor expansion of ϕ and ψ gives the form in (2.7) for the composition up to second order. See Bruni et al. 1997 for the proof. The vectors ξ^μ and ζ^μ are the first and second-order generators of the knight diffeomorphism.

and a vector transform as

$$(\phi^* v)^\mu (x(P)) = \left(\frac{\partial x^\mu(P)}{\partial x^\alpha(Q)} \right) \Big|_{x(Q)\to x(P)} v^\alpha(x(Q))|_{x(Q)\to x(P)} \tag{2.15}$$

where we have to expand every term around $x(P)$ to get the l.h.s at second order. After properly collecting the various terms and omitting the point dependence for brevity, the result is

$$(\phi^* \omega)_\mu = \omega_\mu + \epsilon \mathcal{L}_\xi \omega_\mu + \frac{\epsilon^2}{2} \left(\mathcal{L}_\xi^2 \omega_\mu + \mathcal{L}_\zeta \omega_\mu \right) + \mathcal{O}(\epsilon^3) \tag{2.16}$$

for the 1-form and

$$(\phi^* v)^\mu = v^\mu + \epsilon \mathcal{L}_\xi v^\mu + \frac{\epsilon^2}{2} \left(\mathcal{L}_\xi^2 v^\mu + \mathcal{L}_\zeta v^\mu \right) + \mathcal{O}(\epsilon^3) \tag{2.17}$$

for the vector, where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ^μ. For vectors

$$\mathcal{L}_\xi v^\alpha = v^\alpha_{,\nu} \xi^\nu - \xi^\alpha_{,\nu} v^\nu \tag{2.18}$$

$$\mathcal{L}_\xi^2 v^\alpha = v^\alpha_{,\nu\rho} \xi^\nu \xi^\rho + v^\alpha_{,\nu} \xi^\nu_{,\rho} \xi^\rho - 2\xi^\alpha_{,\rho} v^\rho_{,\nu} \xi^\nu - \xi^\alpha_{,\nu\rho} \xi^\nu v^\rho + \xi^\alpha_{,\rho} \xi^\rho_{,\nu} v^\nu \tag{2.19}$$

and for 1-forms

$$\mathcal{L}_\xi \omega_\alpha = \omega_{\alpha,\nu} \xi^\nu + \xi^\nu_{,\alpha} \omega_\nu \tag{2.20}$$

$$\mathcal{L}_\xi^2 \omega_\alpha = \omega_{\alpha,\nu\rho} \xi^\nu \xi^\rho + \omega_{\alpha,\nu} \xi^\nu_{,\rho} \xi^\rho + 2\xi^\rho_{,\alpha} \omega_{\rho,\nu} \xi^\nu + \xi^\rho_{,\nu\alpha} \xi^\nu \omega_\rho + \xi^\nu_{,\rho} \xi^\rho_{,\alpha} \omega_\nu \,. \tag{2.21}$$

The metric tensor transforms as

$$(\phi^* g)_{\mu\nu} (x(P)) = g_{\mu\nu} + \epsilon \mathcal{L}_\xi g_{\mu\nu} + \frac{\epsilon^2}{2} \left(\mathcal{L}_\xi^2 g_{\mu\nu} + \mathcal{L}_\zeta g_{\mu\nu} \right) + \mathcal{O}(\epsilon^3) \,, \tag{2.22}$$

where substituting the Lie derivative for a 2 covariant indices object gives

$$\begin{aligned}(\phi^* g)_{\mu\nu} = {} & g_{\mu\nu} + \epsilon \left(g_{\mu\nu,\sigma} \xi^\sigma + \xi^\sigma_{,\nu} g_{\mu\sigma} + \xi^\sigma_{,\mu} g_{\nu\sigma} \right) + \\ & + \frac{\epsilon^2}{2} \big(2\xi^\omega_{,\mu} g_{\omega\nu,\sigma} \xi^\sigma + \xi^\omega_{,\mu} \xi^\sigma_{,\omega} g_{\sigma\nu} + 2\xi^\omega_\mu \xi^\sigma_{,\nu} g_{\omega\sigma} + 2\xi^\omega_{,\nu} g_{\omega\mu,\sigma} \xi^\sigma + \\ & + \xi^\omega_{,\nu} \xi^\sigma_{,\omega} g_{\mu\sigma} + g_{\mu\nu,\sigma\omega} \xi^\sigma \xi^\omega + \xi^\sigma_{,\mu\omega} g_{\sigma\nu} \xi^\omega + \xi^\sigma_{,\nu\omega} g_{\mu\sigma} \xi^\omega + \\ & + g_{\mu\nu,\sigma} \zeta^\sigma + \zeta^\sigma_{,\nu} g_{\mu\sigma} + \zeta^\sigma_{,\mu} g_{\nu\sigma} \big) + \mathcal{O}(\epsilon^3) \,. \end{aligned} \tag{2.23}$$

As the gauge freedom in GR, also the theory of gauge transformations can be formulated in the active and in the passive approach. The discussion above is in the active point of view. The passive point of view is obvious, once we recall that every diffeomorphism introduces a new coordinate system defined by

$$x^\mu(Q) = \tilde{x}^\mu(P) \,. \tag{2.24}$$

The gauge transformations written as infinitesimal point transformations above, eq. (2.7) with inverse (2.11), can be equivalently written as the infinitesimal coordinate transformation at the same point

$$\tilde{x}^\mu(P) = x^\mu(P) + \epsilon \xi^\mu(x^\alpha(P)) + \frac{\epsilon^2}{2} \left(\xi^\mu_{,\nu} \xi^\nu + \zeta^\mu \right) (x^\alpha(P)) + \mathcal{O}(\epsilon^3) \tag{2.25}$$

with inverse

$$x^\mu(P) = \tilde{x}^\mu(P) - \epsilon\xi^\mu(\tilde{x}^\alpha(P)) + \frac{\epsilon^2}{2}\left(\xi^\mu_{,\nu}\xi^\nu - \zeta^\mu\right)(\tilde{x}^\alpha(P)) + \mathcal{O}(\epsilon^3) \qquad (2.26)$$

and we already know that the components of the pull-back of tensor fields in the coordinate system x^μ in different points are precisely the components of the original tensors in the new coordinate system $\tilde{x}^\mu$ at the same physical point. In the formulas above we simply have the standard Jacobian matrix and the standard transformation rules, expanded with respect to the infinitesimal coordinate transformation (2.25). This equivalence is crucial: once it is established we will adopt in the following the passive point of view and refer to a gauge transformation in perturbation theory as a coordinate transformation in the perturbed space-time. Note that at the zeroth order the coordinates coincide: that is why in cosmological perturbation theory we say that a gauge transformation does not change the background coordinates, Bardeen 1980.

Summarising, so far we have found that any given quantity U, - a scalar, a vector etc.. - regardless its perturbative order, transforms under a gauge transformation as

$$\tilde{U} = U + \epsilon\mathcal{L}_\xi U + \frac{\epsilon^2}{2}\left(\mathcal{L}^2_\xi U + \mathcal{L}_\zeta U\right) + \mathcal{O}(\epsilon^3) \qquad (2.27)$$

but we have not jet obtained how an expanded quantity transforms, i.e. how the perturbations transform. Then, consider now the expansion up to second-order of same quantity in two different gauges[4]:

$$T = T_0 + \epsilon\delta T + \frac{\epsilon^2}{2}\delta^2 T + \mathcal{O}(\epsilon^3) \qquad \text{and} \qquad \tilde{T} = T_0 + \epsilon\delta\tilde{T} + \frac{\epsilon^2}{2}\delta^2\tilde{T} + \mathcal{O}(\epsilon^3) \qquad (2.28)$$

The transformation rule for the perturbations is easily obtained by plugging (2.28) in (2.27) and collecting terms of the same order. It reads

$$\begin{aligned} T_0 + \epsilon\delta\tilde{T} + \frac{\epsilon^2}{2}\delta^2\tilde{T} + \mathcal{O}(\epsilon^3) &= T_0 + \epsilon\left(\delta T + \mathcal{L}_\xi T_0\right) + \\ &+ \frac{\epsilon^2}{2}\left(\delta^2 T + 2\mathcal{L}_\xi\delta T + \mathcal{L}^2_\xi T_0 + \mathcal{L}_\zeta T_0\right) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \qquad (2.29)$$

Therefore this formula tells us how to obtain the perturbations at first order $\delta\tilde{T}$ and at second order $\delta^2\tilde{T}$ in one gauge from the corresponding perturbations δT and $\delta^2 T$ in the other gauge.

2.2 Gauge fixing in the $3+1$ approach to GR

GR makes no fundamental distinction between time and space, although we do. In order to compare Einstein equations with the dynamical equations of Newtonian gravity we will use the $3+1$ (or ADM) approach to GR, Misner et al. 1973 and Smarr & York 1978a. The space-time $\mathcal{M}$ is assumed to be globally hyperbolic, i.e. sliced into a family of spacelike hypersurfaces $\{\Sigma\}$ parametrized by a global time function t constant on each slice. This description is formulated in terms of the construction of the slices and the choice of the congruence of the world line of the observers. This allows us to present the

[4]From (2.5) the perturbation is defined as the Taylor expansion of the pull back of the tensor field T_{ph}, i.e. as the Lie derivative of the tensor field along the generator of the diffeomorphism.

characterisation of the two specific frames we are interested in, namely the Eulerian and Lagrangian frames. The connection between the gauge choices in the ADM approach and the standard gauge fixing in cosmological perturbation theory from the point of view of gauge invariant variables was initially point out in Bardeen 1980.

2.2.1 Basics of the $3+1$ formalism

A foliation $\{\Sigma\}$ is a family of spacelike hypersurfaces embedded in the spacetime and is mathematically described by a closed one-form Ω_a satisfying[5]

$$\nabla_{[a}\Omega_{b]} = 0 \tag{2.30}$$

Because of its closure, this form is locally exact, i.e. there exists a scalar function t such that $\Omega_a = \nabla_a t$. Therefore each slice arises as the level surface of the function t and can be labelled by the relation $t =$ constant. It can be shown (see Wald 1984) that the function t is monotonically increasing, playing the role of the global time function of the space-time, consistently with its causal structure. Given a space-time metric g_{ab}, the norm of Ω_a defines a strictly positive scalar field, the lapse function $\mathcal{N}$, by

$$g^{ab}\Omega_a\Omega_b = -\frac{1}{\mathcal{N}^2} \tag{2.31}$$

where the minus sign is for spacelike hypersurfaces. Hence every foliation specifies a normalized one-form ω_a given by

$$\omega_a = \mathcal{N}\Omega_a = \mathcal{N}\nabla_a t. \tag{2.32}$$

The unit normal vector field of the slices is defined to be hypersurface-orthogonal by

$$n^a = -g^{ab}\omega_b = -g^{ab}\mathcal{N}\Omega_b = -g^{ab}\mathcal{N}\nabla_b t \tag{2.33}$$

where the minus sign is chosen so that n^a points in the direction of increasing t. The above definition is equivalent to say that (see e.g. Wald 1984)

$$n_{[a}\nabla_b n_{c]} = 0 \tag{2.34}$$

which is the necessary and sufficient condition that n^a define a family of hypersurfaces building up a foliation. The vector field n^a determines the orthogonal projector tensor on the slices

$$h_{ab}(n) = g_{ab} + n_a n_b \tag{2.35}$$

which is the induced spatial metric on Σ_t.

The Eulerian observers This most natural choice of observers is suggested by the geometrical specification of the foliation itself: observers moving along the unit normal to the slices are by definition at rest in the slices and are called Eulerian, in analogy with hydrodynamics.

As a consequence of the above definitions, we have this relation between the normalised one-form and the unit normal associated with a specific foliation

$$\mathcal{N}n^a\nabla_a t = 1 \tag{2.36}$$

[5] In this section a, b, ... are apace-time indices, i,j, ... are spatial indices and ∇_a denotes the covariant derivative.

which makes clear the physical role of the lapse function: from (2.36) one has

$$\mathcal{N} = (n^a \nabla_a t)^{-1} = \left(\frac{dt(x^\mu(\lambda))}{d\lambda} \right)^{-1} = \frac{d\lambda}{dt} \,, \tag{2.37}$$

namely $d\lambda = \mathcal{N} dt$, which means that $\mathcal{N}$ measures the rate of flow of proper time λ with respect to the global time t as one moves along the normal congruence; it is equivalent to say that $\mathcal{N} dt$ is the orthogonal proper time between the slices Σ_t and Σ_{t+dt}. In general the non-vanishing acceleration of Eulerian observers is given by

$$a^a = n^b \nabla_b n^a = h^{ab} h^c_b \nabla_c (\ln \mathcal{N}) \tag{2.38}$$

where the last equality follows from (2.33).

Every other choice of observers, which we call coordinate observers, is represented by the timelike vector field t^a, decomposed into its parts normal and tangential to Σ_t and written as

$$t^a = \mathcal{N} n^a + \mathcal{N}^a \qquad \text{with} \qquad \mathcal{N}^a n_a = 0 \,. \tag{2.39}$$

The vector t^a determines another orthogonal projector tensor

$$h_{ab}(t) = g_{ab} + t_a t_b \tag{2.40}$$

on the hypersurfaces orthogonal to t^a, i.e. on the rest frame of the coordinate observers. This is different from the hyperurface of constant coordinate time t. The tensor $h_{ab}(t)$ is actually the spatial metric of the rest frame of the observers moving along t^a, but the collection of these three dimensional spaces may not form a foliation made of Σ labelled by some parameter representing the time coordinate. This is the case if the rotation of t^a is non vanishing, i.e. $t_{[a} \nabla_b t_{c]} \neq 0$ implying that the hypersufaces may cross itself or others. [6]

The line element in the $3+1$ formalism is written as

$$ds^2 = a^2 \left\{ (-\mathcal{N}^2 + \mathcal{N}^i \mathcal{N}_i) d\eta^2 + 2\mathcal{N}_i d\eta dx^i + h_{ij}(n) dx^i dx^j \right\} , \tag{2.41}$$

where the time coordinate labelling the slices is the conformal time η. The inverse metric has components

$$g^{00} = -\frac{1}{a^2 \mathcal{N}^2} \qquad g^{0i} = \frac{\mathcal{N}^i}{a^2 \mathcal{N}^2} \qquad g^{ij} = \frac{1}{a^2} \left(h^{ij} - \frac{\mathcal{N}^i \mathcal{N}^j}{\mathcal{N}^2} \right) . \tag{2.42}$$

Here the spatial coordinates are constant along t^a and the time coordinate η is constant on each hypersurface. This line element is completely general, i.e. we can still fix the gauge. The lapse function $\mathcal{N}$, as shown above, represent the ratio of the proper time distance to the coordinate-time distance between two neighbour constant time hypersurfaces as we move along the normal congruence. The shift vector $\mathcal{N}^i$ represents the rate of deviation of the constant space coordinate line t^a from the normal to the constant

[6] The $3+1$ approach corresponds to the slicing of the space-time into a series of spatial hypersurfaces, each labelled by a coordinate time t. The complementary approach is the $1+3$ approach, corresponding to the threading of the space-time by the world line of the observers t^a. In the $1+3$ approach it is not required that the rest frame of the observers define a time slicing of space-time. In the fluid approach of Ellis 1971 the dynamics is described by the threading point of view using the four velocity of the fluid.

time hypersurfaces or, in other words, describe how spatial coordinates are propagated from one hypersurface to the next.

Instead of defining a gauge in a coordinate based approach by means of the value assumed by some gauge invariant variables, Kodama & Sasaki 1984, Malik & Wands 2009, in the $3+1$ approach we define a gauge in terms of the slicing and of the properties of the world line of the observers such that, once we have chosen the observers, we know that spatial coordinates are constant along their world lines and the relation between the coordinate time t constant on each Σ_t and the proper time of our observers is represented by the lapse function, Smarr & York 1978a. This turns out to be very convenient for our purpose, since we are interested in characterising the Lagrangian and the Eulerian frames. Indeed, although the use of the coordinate observers t^a has no particular physical meaning in the fluid description and the choice of t^a rather than n^a is apparently more complicated, with appropriate choice of a non vanishing shift vector the coordinate observers define a frame comoving with the matter, as we discuss in the following.

The relation between Eulerian and coordinate observers, eq. (2.39), is specified by the shift vector $\mathcal{N}^a$, which is the velocity in η time of the spatial triad dragged along t^a relative to the normal direction. Thus the transformation between the two frame can be written as a boost. Indeed from (2.39) the unit coordinate velocity $\hat{t}^a\hat{t}_a = -1$ is given by

$$\hat{t}^a = \frac{1}{(\mathcal{N}^2 - \mathcal{N}^a\mathcal{N}_a)^{1/2}} t^a \,. \tag{2.43}$$

Thus we have from (2.39)

$$\hat{t}^a = \frac{1}{\mathcal{N}(1 - v^a v_a)^{1/2}} \left(\mathcal{N} n^a + \mathcal{N} v^a\right) = \Gamma\left(n^a + v^a\right) \tag{2.44}$$

where $\Gamma = \left(1 - v^2\right)^{-1/2}$ is the Lorentz factor. The boost is between the normalised velocity of the coordinate observers $\hat{t}^a$ and the unit normal n^a and is defined by the velocity $v^a = \mathcal{N}^a/\mathcal{N}$, which is the relative velocity between the two frames as measured in the proper time of the Eulerian observers.

Many choices are possible, see Smarr & York 1978b and Smarr & York 1978a. In the following we will discuss only those relevant for our aim.

2.2.2 Zero shift vector condition

If the shift vector is vanishing, the spatial coordinates are fixed along the normal congruence. We can call it the normal frame. This frame is not comoving with the matter. But, since the shift vector is vanishing, the coordinate velocity of the matter and the velocity measured by the Eulerian observers moving along the normal congruence coincide.

2.2.3 The comoving condition

If the matter content is a fluid, we can use the fluid four-velocity u^a itself as the vector field for the coordinate observers. The relation between u^a and n^a is given by

$$u^a = \Gamma(n^a + w^a) \tag{2.45}$$

with

$$u^a u_a = -1 \qquad \text{and} \qquad n^a w_a = 0\,, \tag{2.46}$$

where $\Gamma = \left(1 - w^2\right)^{-1/2}$ is the Lorentz factor and w^a is the three-velocity of the fluid relative to the Eulerian observers. This suggests one possible physically motivated way of choosing the shift $\mathcal{N}^a$: we can correlate the fluid motion and the coordinate observers frame by demanding that $w^a = \mathcal{N}^a$, which is called the comoving condition. In this case the coordinate observers are comoving with the fluid, i.e. a given fluid element has fixed spatial coordinates. Of course, even when $w^a = \mathcal{N}^a$ the time η on the slices does not coincide with the proper time defined in the rest frame of the fluid element.

If the matter is spatially rotating the comoving choice would cause the spatial coordinate grid eventually to wind up into a mess. This choice is therefore well posed in our case of irrotational dust and until caustic formation.

2.3 The Eulerian frame

The Poisson gauge is defined at any order in perturbation theory, by demanding that, recall (1.18) and see Malik & Wands 2009, Bertschinger 1996 and Matarrese et al. 1998

$$\partial_i \hat{\omega}^{i(r)} = 0 \qquad \text{and} \qquad \partial^i \hat{\chi}^{(r)}_{ij}\,. \tag{2.47}$$

This eliminates one scalar degree of freedom from the shift g_{0i} and one scalar and two vector degrees of freedom from g_{ij}, completely fixing the gauge.

Consider the line element in the Poisson gauge, up to the PN order

$$ds^2 = a^2 \left[-\left(1 + U + \frac{V}{c^2}\right) d\eta^2 + 2\frac{P_i}{c^2} d\eta dx^i + \left(1 - \frac{U}{c^2}\right) \delta_{ij} dx^i dx^i \right]\,, \tag{2.48}$$

where $\partial_i P^i = 0$ is one condition of the Poisson gauge, which eliminate the scalar part of g_{0i}. The metric variables in the PN expansion are not expanded in standard perturbation theory. From the initial condition from Inflation we know that the Newtonian first-order vector perturbation in the shift $\hat{\omega}^{i(1)}$ is zero. In the Poisson gauge the scalar $\hat{\omega}^{(1)}$ is also vanishing because of the gauge condition and we know that the second-order vector perturbation is PN. Therefore, at first order in perturbation theory and in the Newtonian limit at any order the Poisson gauge has zero shift vector. The line element reads

$$ds^2 = a^2 \left[-\left(1 + U\right) d\eta^2 + \delta_{ij} dx^i dx^i \right]\,. \tag{2.49}$$

We identify the Eulerian frame with the Poisson gauge. In the Newtonian approximation the components of the metric are those required by the Newtonian equation of motion at any order in perturbation theory, not just at linear order. The lapse perturbation is found from the Newtonian limit of the Einstein equations and the result is $U = 2\varphi_g$, φ_g being the Newtonian gravitational potential, see section 3.1. The velocity in the equation of Newtonian gravity is measured by the Eulerian observers along unit normal n^a, which has components $n^a = \frac{1}{a}\left(1 - \varphi_g, 0, 0, 0\right)$. From GR perturbation theory we know that second-order vector (shift) and tensor perturbation are generated by the scalar first-order perturbations and we are force to include them if we want to study the non-linear regime of gravitational instability. The natural extension of our Newtonian Eulerian frame is the Poisson gauge.

2.4 The Lagrangian frame

Consider the case of irrotational and pressure-less matter flow. We can fix the four velocity of the observers t^a equal to the velocity of matter u^a (comoving condition) with a vanishing shift vector, since the matter world lines have no rotation: thus we have $u^a = \mathcal{N} n^a$ and this is called the comoving-orthogonal condition. The cosmic time labelling each slice of the foliation, which coincide with the family of the matter rest spaces (which are also surfaces of simultaneity for all the observers) can be also normalized to measure the proper time along each world line: since the fluid moves along normal geodesics, we can set $\mathcal{N} = 1$. With this choice, the normal vector coincides with the four velocity of the fluid, $n^a = u^a$. The spatial coordinates are constant along the geodesics of the matter, namely the spatial coordinates are Lagrangian, and the proper time along the fluid world lines is the time coordinate of the slicing. Note that in this gauge the $3 + 1$ and the $1 + 3$ approach to Einstein equations are fully equivalent and the projector in the rest frame of the observers comoving with the fluid is the induced spatial metric of the hypersurfaces of constant proper time. In the following we identify our Lagrangian frame with this gauge and will call it the synchronous and comoving gauge. This is a well motivated and non-perturbative gauge choice with a clear physical interpretation and is valid beyond the linear regime within our assumptions, see (1.3), until caustic formation. The line element in the Lagrangian frame reads:

$$ds^2 = a^2(\tau) \left[-d\tau^2 + \gamma_{\alpha\beta}(\tau, \mathbf{q}) dq^\alpha dq^\beta \right] \ , \tag{2.50}$$

where obviously in a perturbed space-time the spatial metric depends on the coordinates. The spatial metric is the solution of the ADM Einstein equation or, equivalently, of the equations in the $1 + 3$ approach written in this gauge, see Matarrese & Terranova 1996.

Part I

Cosmological dynamics from the Eulerian to the Lagrangian frame

CHAPTER 3

Cosmological dynamics in the Eulerian frame

It is believed that the distribution of matter in the early Universe was very smooth, the best indication being the tiny fluctuations in the Cosmic Microwave Background. However, the distribution of matter in the Universe at present time is inhomogeneous on scales below about 100 h^{-1} Mpc (h being the Hubble constant in units of 100 km s^{-1} Mpc^{-1}) and the gravitational dynamics is non-linear below about 10 h^{-1} Mpc. The theoretical study of structure formation connects the early Universe with that observed today. The gravitational instability is governed by the equations provided by GR but, for most purposes, we can use the Newtonian approximation, namely the weak-field and slow-motion limit of GR. These conditions are verified at small scales, well inside the Hubble radius, where the peculiar gravitational potential φ_g, divided by the square of the speed of light to obtain a dimensionless quantity, remains much less than unity, while the peculiar velocity never become relativistic. The peculiar gravitational potential is related to the matter-density fluctuation δ via the cosmological Poisson equation

$$\nabla^2 \varphi_g = 4\pi G a^2 \rho_b \delta \tag{3.1}$$

where ρ_b is the Freidmann-Robertson-Walker (FRW) background matter density, $\delta = (\rho - \rho_b)/\rho_b$ is the density contrast and $a(t)$ is the scale-factor, which obeys the Friedmann equation. This implies that for a fluctuation of proper scale λ,

$$\frac{\varphi_g}{c^2} \sim \delta \left(\frac{\lambda}{r_H}\right)^2 , \tag{3.2}$$

where $r_H = cH^{-1}$ is the Hubble radius. This very fact tells us that the weak-field approximation does not necessarily imply small density fluctuations, rather it depends on the ratio of the perturbation scale λ to the Hubble radius. That is why Newtonian gravity is widely used to study structure formation at small scales, also in the non-linear regime. The evolution of perturbations is dealt with analytically, within perturbation theory, or numerically, by means of N-body simulations.

However, there are situations, even within the Hubble horizon, where the Newtonian treatment is not well suited. The Newtonian approximation of GR in the Eulerian approach consists in perturbing the time-time component of the space-time metric by an amount $2\varphi_g/c^2$. The calculation of the photon geodesics of this metric fails to produce an accurate description of photon trajectories: it is well know that the Newtonian estimate of the Rees-Sciama effect and of gravitational lensing is incorrect by a factor of 2. The correct calculation involves the weak-field limit of GR, which is valid for slow motions of the sources of the gravitational field, but allows test particles to be relativistic. The related metric is perturbed also in the space-space component of the space-time metric by an amount $-2\varphi_g/c^2$ in the orthogonal zero-shear gauge defined in ..... However, as we recalled in 1.2, the zero-shear condition is applicable only to scalar perturbations and we simply cannot make this choice to study non-linear dynamics in the GR framework: we know that, even if vector and tensor modes in the space-time metric vanish initially, they are dynamically generated by scalar modes. Therefore the appropriate gauge for the Eulerian approach in GR is the Poisson gauge defined in ...

This chapter consists of three sections which deal with the Eulerian cosmological dynamics in the Newtonian limit and in the PN approximation, respectively. Here we just recall the basic procedure for the calculation and the results. We refer to the references cited in the text for the details.

3.1 The Newtonian limit

In the Eulerian picture the matter dynamics is described with respect to a system of coordinates not comoving with the the matter. In a uniformly expanding Universe, all physical separations scale in proportion with a cosmic scale factor $a(t)$. Even though the expansion is not perfectly uniform, it is perfectly reasonable to factor out the Hubble expansion and we do this by using FRW comoving spatial coordinates and conformal time defined as

$$x = \frac{r}{a(t)}\,, \qquad d\eta = \frac{dt}{a(t)}\,, \tag{3.3}$$

where r is proper spatial coordinates, t is the cosmic time and $a(t)$ the scale-factor. With this choice all physical quantities appearing in the equations are measured by observers comoving with the FRW background expansion. The coordinate velocity is

$$v^A = \frac{dx^A}{d\eta} = \frac{dr^A}{dt} - Hr^A\,, \tag{3.4}$$

where $H = \partial_t a/a$ is the Hubble parameter. It is the physical velocity of the matter minus the Hubble expansion, i.e. the peculiar velocity.

The Newtonian equations in the Eulerian picture read

$$0 = \frac{\partial \delta_{\mathcal{E}}}{\partial \eta} + v^C \partial_C \delta_{\mathcal{E}} + \partial_A v^A (1 + \delta_{\mathcal{E}}) \tag{3.5}$$

$$0 = \frac{\partial v_K}{\partial \eta} + v^C \partial_C v_K + \mathcal{H} v_K + \partial_K \varphi_g^{\mathcal{E}} \tag{3.6}$$

$$\nabla^2 \varphi_g^{\mathcal{E}} = 4\pi G a^2 \rho_b \delta_{\mathcal{E}}\,, \tag{3.7}$$

where $\mathcal{H} = \partial_\eta a/a$. For the irrotational dust considered here we have also the condition

$$\epsilon^{ABC} \partial_A v_B = 0\,, \tag{3.8}$$

where ϵ^{ABC} is the totally antisymmetric Levi-Civita tensor relative to the Euclidean spatial metric, such that $\epsilon^{ABC} = 1$, etc...

The fundamental variables in the Eulerian picture are the velocity and density field, evaluated at the Eulerian coordinates x^A. The same equations can be obtained in the GR framework: the continuity and Euler equations are the lowst-order equations in the $1/c^2$

expansion of $\nabla^a T_{a0}$ and $\nabla^a T_{aK}$ respectively. The time-time component of the Einstein equations

$$R^0_0 = \frac{8\pi G}{c^4}\left(T^0_0 - \frac{1}{2}T\right) \tag{3.9}$$

reduces to the Poisson equation and implies that the lapse perturbation in the time-time component of the metric tensor coincides with the Newtonian gravitational potential.

The line-element in the Newtonian approximation of GR in the Eulerian picture then reads

$$ds^2 = a^2\left[-\left(1 + 2\frac{\varphi_g^{\mathcal{E}}}{c^2}\right)c^2 d\eta^2 + \delta_{AB}dx^A dx^B\right]. \tag{3.10}$$

The point of view illustrated above is *purely Newtonian*: the matter is viewed as responsive to the gravitational field given by φ_g and the Newtonian order in the metric is established considering just the equations of motion for the fluid, eqs. (3.6) and (3.5), where only $\gamma^{\mathcal{E}}_{00} = -\left(1 + 2\varphi^{\mathcal{E}}_g/c^2\right)$ is required, with $\varphi^{\mathcal{E}}_g$ satisfying the Poisson equation (3.7). The line element in eq. (3.10) is our lowest, i.e. Newtonian, order approximation in the Poisson gauge.[1]. Accordingly, in our approach the PN line element in the Poisson gauge contains a divergence-less vector contribution in the space-time component and a scalar contribution in space-space component of the metric:

$$ds^2 = a^2\left[-\left(1 + \frac{2\varphi_g^{\mathcal{E}}}{c^2} + \frac{V^{\mathcal{E}}}{c^4}\right)c^2 d\eta^2 + \frac{P^{\mathcal{E}}_A}{c^3}cd\eta dx^A + \left(1 - \frac{2\varphi_g^{\mathcal{E}}}{c^2}\right)\delta_{AB}dx^A dx^B\right]. \tag{3.11}$$

The PN terms here are given by the lowest-order of the $1/c^2$ expansion of all the Einstein equations. In this gauge the transverse and traceless tensor modes, generated by the non-linear growth of scalar perturbations, appear at PPN level ($\mathcal{O}\left(1/c^4\right)$) only.

3.1.1 Second-order results in perturbation theory

In this subsection we report the results for the dynamics of irrotational dust in the Eulerian picture up to second order in perturbation theory. The density contrast and the peculiar velocity are expanded about the background solution $\delta = 0$ and $\mathbf{v} = 0$. Then the differential equations are solved order by order. For the complete calculation, we refer to e.g. Catelan et al. 1995. The second-order expressions for the peculiar gravitational potential, the peculiar velocity and the density contrast are

$$\varphi_g^{\mathcal{E}} = \phi - \frac{5}{21}\eta^2\Psi_{\mathcal{E}} + \frac{\eta^2}{12}\partial^K\phi\partial_K\phi \tag{3.12}$$

$$\mathbf{v}_{\mathcal{E}} = \nabla\left(-\frac{\eta}{3}\phi - \frac{\eta^3}{36}\partial^K\phi\partial_K\phi + \frac{\eta^3}{21}\Psi_{\mathcal{E}}\right) \tag{3.13}$$

$$\delta_{\mathcal{E}} = \frac{\eta^2}{6}\partial^K\partial_K\phi + \frac{5}{252}\eta^4\left(\nabla_x\phi\right)^2 + \frac{\eta^4}{126}\partial^K\partial^N\phi\partial_K\partial_N\phi + \frac{\eta^4}{36}\partial^K\phi\partial^K\nabla_x\phi\,, \tag{3.14}$$

where ϕ is the peculiar gravitational potential evaluated at the initial time η_{in} and the potential $\Psi_{\mathcal{E}}$ is given by

$$\nabla^2_x\Psi_{\mathcal{E}} = -\frac{1}{2}\left[\left(\nabla^2_x\phi\right)^2 - \partial^N\partial^K\phi\partial_N\partial_K\phi\right]. \tag{3.15}$$

[1]We remark that the line element in eq. (3.10) is not referred to the so-called longitudinal gauge, where vector and tensor modes are set to zero by hand at all orders and only the scalar mode in the spatial metric is present. Actually, it is not even a gauge, since only one among the six physical degrees of freedom in the metric are allowed.

These expressions coincide with the time-time perturbation of the metric, the matter peculiar velocity and the density contrast in the Poisson gauge, at second order in GR perturbation theory, retaining the Newtonian terms only, see Bartolo et al. 2010.

3.2 The PN approximation

The PN approximation in the cosmological framework has been studied in Eulerian (or more generally non-comoving) coordinates in Futamase 1988, Futamase 1989, Tomita 1991, Shibata & Asada 1995. Here we refer to Carbone & Matarrese 2005, where in the Poisson gauge a hybrid approximation scheme is proposed which upgrades the weak-field limit of Einstein's field equations to account for post-Newtonian scalar and vector metric perturbations and for leading-order source terms of gravitational waves, while including also the first and second-order perturbative approximations.

The PN approximation is obtained by expanding the Einstein equations in powers of $1/c$. Up to the PN order the space-time metric is expanded as

$$ds^2 = a^2 \left[-\left(1 + U + \frac{V}{c^2}\right) d\eta^2 + 2\frac{P_A}{c^2} d\eta dx^A + \left(1 - \frac{U}{c^2}\right) \delta_{AB} dx^A dx^B \right] \tag{3.16}$$

where $U = 2\varphi_g^{\mathcal{E}}$ for the Newtonian term and $\partial_A P^A = 0$ for the Poisson gauge. We report the equations for the time-time component V and the time-space component P_A of the metric. For a Universe filled of irrotational dust and after the subtraction of the Einstein-de Sitter background we have

$$\begin{aligned} \nabla_x^2 \nabla_x^2 V \;=\; & \frac{7}{2} \nabla_x^2 \left(\partial_F \varphi_g^{\mathcal{E}} \partial^F \varphi_g^{\mathcal{E}} \right) - 3\partial^C \partial_F \left(\partial_C \varphi_g^{\mathcal{E}} \partial^F \varphi_g^{\mathcal{E}} \right) + \\ & + \frac{3}{2} \mathcal{H}^2 \left\{ 2\nabla_x^2 \left[(1+\delta_{\mathcal{E}}) v^2 \right] - 3\mathcal{H} \partial^A \left[(1+\delta_{\mathcal{E}}) v_A \right] - 3\partial^C \partial_F \left[(1+\delta_{\mathcal{E}}) v_C v^F \right] \right\} \end{aligned} \tag{3.17}$$

and

$$\nabla_x^2 \nabla_x^2 P_A = 6\mathcal{H}^2 \partial^K \left[v_A \partial_K [(1+\delta_{\mathcal{E}}) - v_K \partial_A [(1+\delta_{\mathcal{E}}) \right] \tag{3.18}$$

where $v^A = \partial^A \Phi_v$ for irrotational dust, $\delta_{\mathcal{E}}$ and $\varphi_g^{\mathcal{E}}$ are given by the Newtonian equations (3.5), (3.6), (3.7).

At this order the time-space component is not dynamical. Using the well known vector calculus identity $\nabla_x \times \nabla_x \times \mathbf{P} = \nabla_x \left(\nabla_x \cdot \mathbf{P} - \nabla_x^2 \mathbf{P} \right)$ and the condition for P_A in the Poisson gauge, $\nabla_x \cdot \mathbf{P} = 0$, this equation can be re-written as, Bruni et al. 2014b

$$\nabla_x \times \nabla_x^2 \mathbf{P} = 6\mathcal{H}^2 \nabla_x \times [(1+\delta_{\mathcal{E}})\mathbf{v}] \;, \tag{3.19}$$

which shows clearly that the PN vector potential is sourced by the curl of the mass-energy current. This is of course the case also for irrotational dust, since the curl of the quantity $[(1+\delta_{\mathcal{E}})\mathbf{v}]$ is not vanishing, even if $\mathbf{v} = \nabla_x \Phi$. The solutions of the second-order expansion in perturbation theory are reported in appendix B.

CHAPTER 4

Cosmological dynamics in the Lagrangian frame

In the Lagrangian picture the dynamics is described with respect to a system of coordinates attached to the matter: at some arbitrarily chosen initial time we label the fluid elements by spatial coordinates q^α; at all later times, the same element is labelled by the same coordinate value. The perturbation theory in the Lagrangian picture is more powerful than the Eulerian one. We concentrate on the Newtonian approximation in the following since it offers the easiest understanding. The spatial Lagrangian position vector of the fluid elements is given by the curve q^α = const., implying that the velocity of matter vanishes in Lagrangian coordinates. On the other hand, the Eulerian, time-dependent position vector could be expressed in terms of Lagrangian spatial coordinates as

$$\mathbf{x}(\mathbf{q},\tau) = \mathbf{q} + \mathcal{S}(\mathbf{q},\tau) \tag{4.1}$$

and the peculiar velocity is

$$\mathbf{v} = \frac{d\mathcal{S}}{d\tau} \,. \tag{4.2}$$

Similarly to the Eulerian trajectory, also the peculiar velocity of the matter $d\mathbf{x}/d\tau$, can be expressed in terms of the initial labels of the fluid element q^α through the mapping in eq. (4.1). Note that the relation above refers to an inhomogeneous universe and is fully non-perturbative. For a perfectly uniform expansion, the comoving position vector $\mathbf{x}$ is fixed in time and coincides with its initial, i.e. Lagrangian, coordinate value $\mathbf{q}$. On the contrary, in a perturbed universe it changes with time as irregularities grow, in the way described by the relation above. Therefore, all the information about the evolution of the perturbations is contained in the mapping relation (4.1) and the displacement vector $\mathcal{S}$ is the only fundamental field: one can equivalently write the equations of motion of the fluid in terms of either the displacement vector $\mathcal{S}$, as in ref. Catelan 1995, or in terms of the Jacobian matrix of the map

$$\mathcal{J}^A_\alpha = \frac{\partial x^A}{\partial q^\alpha} = \delta^A_\alpha + \frac{\partial S^A}{\partial q^\alpha} \,, \tag{4.3}$$

as in Matarrese & Terranova 1996, where $\mathcal{S}^A_\alpha = \partial S^A/\partial q^\alpha$ is called the deformation tensor. Given that, as in the Eulerian case, it is impossible to work out the general solution $\mathcal{S}$, a perturbative approach is again introduced. The key novelty with respect to the Eulerian approach is that one searches for solutions of perturbed trajectories about the homogeneous expansion instead of the perturbed density and velocity fields, which are the perturbed quantities in the Eulerian approach. The important point is that a slight perturbation of the Lagrangian particle paths carries a large amount of non–linear information about the corresponding Eulerian evolved observables, since the Lagrangian picture does not rely on the smallness of the density and velocity fields, but on the smallness of the deviation of the trajectory field, in a coordinate system that moves with the fluid and is intrinsically non-linear in the density field.

This chapter consists of two sections which deal with the Lagrangian cosmological dynamics in GR and in the Newtonian limit. The powers of the speed of light appear explicitly, being essential for the PN expansion performed in the following.

4.1 The relativistic Lagrangian dynamics

In this section the dynamics of the inhomogeneous irrotational dust model of Universe is analysed in the framework of general relativity in the Lagrangian frame. In GR, for a pressureless fluid, the time coordinate τ can be defined in such a way that it satisfies the following two conditions: any hypersurface τ = const. is orthogonal to the world-line of the fluid elements in any point and the variation of τ along each world-line coincide with the proper time variation along it. In the synchronous and comoving gauge, comoving hypersufaces are orthogonal to the matter flow ($g_{0\beta} = 0$) and coincide with the synchronous ones, ($g_{00} = -1$), orthogonal to geodesics. This is the coordinate frame geometrically characterized by the lapse $\mathcal{N} = 1$ and the shift vector $\mathcal{N}^a = 0$, see section 2.4. The line element is then

$$ds^2 = -dt^2 + h_{\alpha\beta}(t, \mathbf{q})dq^\alpha dq^\beta. \tag{4.4}$$

The fluid four velocity in Lagrangian coordinates has components $u^\alpha = (1, 0, 0, 0)$. It is worth to stress that the possibility of making the synchronous gauge choice and comoving gauge choice simultaneously is a peculiarity of fluid with vanishing acceleration, as indeed the dust, and holds also beyond the linear regime. It is essential to recall that in this gauge the velocity-gradient tensor $\nabla_\beta u_\alpha$ coincides with (minus) the extrinsic curvature $\mathcal{K}_{\alpha\beta}$ and fully characterises the geometry of the hypersufaces[1]:

$$\nabla_\beta u_\alpha = \Theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} = -\mathcal{K}_{\alpha\beta} = \frac{1}{2}\dot{h}_{\alpha\beta}. \tag{4.5}$$

This means that if one considers an irrotational dust model and uses the Lagrangian picture to describe it, the extrinsic curvature of the hypersurfaces of constant t is of purely kinematical nature. This is clearly expressed through the ADM equations, which encode the dynamical description of the space-time and can be written in terms of the gradient-velocity tensor. Here the equations for the mixed tensor are written for the components

$$\Theta^\alpha_\beta = \nabla_\beta u^\alpha = \frac{1}{2}h^{\alpha\gamma}\dot{h}_{\gamma\beta}. \tag{4.6}$$

The energy and momentum constraints read

$$\Theta^2 - \Theta^\alpha_\beta\Theta^\beta_\alpha + c^2\mathcal{R} = 16\pi G\varrho \tag{4.7}$$

$$\mathcal{D}_\alpha\Theta^\alpha_\beta = \mathcal{D}_\beta\Theta \tag{4.8}$$

and the evolution equations are

$$\dot{\Theta}^\alpha_\beta + \Theta\Theta^\alpha_\beta + c^2\mathcal{R}^\alpha_\beta = 4\pi G\varrho\delta^\alpha_\beta. \tag{4.9}$$

[1] A dot denotes partial differentiation with respect to time t.

Taking the trace of the last equation and combining it with the energy constraint one obtain the Raychaudhuri equation

$$\dot{\Theta} + 2\sigma^2 + \frac{1}{3}\Theta^2 + 4\pi G\varrho = 0. \tag{4.10}$$

Finally, since here $T_{\alpha\beta} = \varrho u_\alpha u_\beta$, the ADM conservation equation gives the continuity equation

$$\dot{\varrho} = -\Theta\varrho \tag{4.11}$$

which can be solved exactly in this gauge giving

$$\varrho(\mathbf{q}, t) = \varrho_{in}(\mathbf{q})\sqrt{\frac{h_{in}(\mathbf{q})}{h(\mathbf{q}, t)}} \tag{4.12}$$

where h is the determinant of the spatial metric.

It is convenient to factor out the Einstein-de Sitter background solution of the above equations. To this aim, it is firstly performed a conformal rescaling of the metric with conformal factor $a(\tau)$, the scale factor of the Einstein-de Sitter model. This procedure involves the transformation of the time coordinate t to conformal time τ via

$$d\tau = \frac{dt}{a(t)} \tag{4.13}$$

which corresponds the conformal transformation of the whole metric

$$\tilde{g}_{\alpha\beta} = \frac{1}{a^2} g_{\alpha\beta}. \tag{4.14}$$

The line element is then written in the form

$$ds^2 = a^2(\tau)\left[-c^2 d\tau^2 + \gamma_{\alpha\beta}(\tau, \mathbf{q}) dq^\alpha dq^\beta\right] , \tag{4.15}$$

where $\gamma_{\alpha\beta}(\tau, \mathbf{q}) = \frac{h_{\alpha\beta}(t(\tau), \mathbf{q})}{a^2(t(\tau))}$ and $a(\tau)$ is the solution of the Friedmann equations for the Einstein-de Sitter model.

The conformal gradient velocity tensor is

$$\tilde{\Theta}^\alpha_\beta = \frac{1}{c}\frac{a'}{a^2}\delta^\alpha_\beta + \frac{1}{2c}\frac{1}{a}\gamma^{\alpha\sigma}\gamma'_{\sigma\beta}. \tag{4.16}$$

In order to factor out the Einstein de-Sitter background, the isotropic Hubble flow with gradient velocity tensor $\frac{a'}{a}\delta^\alpha_\beta$ is subtracted from (4.16). The result is a tensor which describes the gradient of peculiar velocity only and coincide with the extrinsic curvature of hypersurfaces of constant τ:

$$\vartheta^\alpha_\beta = ac\tilde{\Theta}^\alpha_\beta - \frac{a'}{a}\delta^\alpha_\beta = \frac{1}{2}\gamma^{\alpha\sigma}\gamma'_{\sigma\beta}. \tag{4.17}$$

The matter content is also written in terms of the density contrast, defined as the adimensional deviation of the matter density from that of Einstein de-Sitter background

$$\delta(\tau, \mathbf{q}) := \frac{\varrho(\tau, \mathbf{q}) - \varrho_b(\tau)}{\varrho(\tau)}. \tag{4.18}$$

The ADM equations are then recast in a more convenient form as the energy constraint

$$\vartheta^2 - \vartheta^\mu{}_\nu \vartheta^\nu{}_\mu + 4\mathcal{H}\vartheta - 16\pi G a^2 \rho_b \delta = -c^2\, {}^{(3)}\!\mathcal{R}\,, \tag{4.19}$$

the momentum constraint

$$\mathcal{D}_\alpha \vartheta^\alpha{}_\beta = \partial_\beta \vartheta\,, \tag{4.20}$$

and the evolution equation

$$\vartheta^{\alpha\prime}_\beta + 2\mathcal{H}\vartheta^\alpha_\beta + \vartheta\vartheta^\alpha_\beta + \mathcal{H}\vartheta\delta^\alpha_\beta - \frac{3}{2}\mathcal{H}^2\delta\delta^\alpha_\beta = -c^2\, {}^{(3)}\!\mathcal{R}^\alpha_\beta\,. \tag{4.21}$$

Here ${}^{(3)}\!\mathcal{R}^\alpha_\beta$ is the conformal Ricci tensor of the three-dimensional space, $\mathcal{D}_\alpha$ is the covariant derivative corresponding to the metric $\gamma^{\mathcal{L}}_{\alpha\beta}$ and primes denote differentiation with respect to the coordinate time τ. After replacing the density from the energy constraint, the evolution equation can be rewritten as

$$\vartheta^{\alpha\prime}_\beta + 2\mathcal{H}\vartheta^\alpha_\beta + \vartheta\vartheta^\alpha_\beta + \frac{1}{4}\left(\vartheta^\mu_\nu\vartheta^\nu_\mu - \vartheta^2\right)\delta^\alpha_\beta = -\frac{c^2}{4}\left[4\, {}^{(3)}\!\mathcal{R}^\alpha_\beta - {}^{(3)}\!\mathcal{R}\delta^\alpha_\beta\right]. \tag{4.22}$$

The trace part of the evolution equation combined with the energy constraint to eliminate ${}^{(3)}\!\mathcal{R}$ gives the Raychaudhuri equation, which describes the evolution of the peculiar volume expansion scalar and reads

$$\vartheta' + \mathcal{H}\vartheta + \frac{1}{3}\vartheta^2 + \sigma^\lambda{}_\rho \sigma^\rho{}_\lambda + 4\pi G a^2 \rho_b \delta = 0\,, \tag{4.23}$$

where $\sigma^\alpha{}_\beta \equiv \vartheta^\alpha{}_\beta - \frac{1}{3}\delta^\alpha{}_\beta\vartheta$ is the shear tensor, i.e. the trace-free part of the velocity-gradient tensor.

Finally, mass conservation implies

$$\delta' + (1+\delta)\,\vartheta = 0\,, \tag{4.24}$$

which in this gauge can be solved exactly, by virtue of eq. (4.17). The solution is

$$\delta(\mathbf{q},\tau) = (1 + \delta_{in}(\mathbf{q}))\,[\gamma(\mathbf{q},\tau)/\gamma_{in}(\mathbf{q})]^{-1/2} - 1\,, \tag{4.25}$$

where γ is the determinant of the conformal spatial metric $\gamma_{\alpha\beta}$.

The main advantage of this formalism is that there is only one dimensionless (tensor) variable in the equations, namely the spatial metric tensor $\gamma_{\alpha\beta}$, which is present with its partial time derivatives through ϑ^α_β and with its spatial gradients through the spatial Ricci tensor $\mathcal{R}^\alpha_\beta$. A relevant advantage of having a single tensorial variable, for the following PN expansion, is that there can be no extra powers of c hidden in the definition of different quantities.

4.2 The Newtonian limit

The Newtonian approximation is obtained in the $c \to \infty$ limit: the energy constraint and the evolution equation require that the spatial Ricci tensor is zero, see Ellis 1971, Matarrese & Terranova 1996 and Buchert & Ostermann 2012 for a derivation in the tetrad formalism. This in turns implies that $\overline{\gamma}^{\mathcal{L}}_{\alpha\beta}$ can be transformed to the Euclidean metric δ_{AB} globally. In other words, at each time τ there exist global Eulerian observers comoving with the Hubble flow for which the components of the metric are δ_{AB}. This means that, according to the tensor transformation law, we can write the spatial metric as[2]

$$\overline{\gamma}^{\mathcal{L}}_{\alpha\beta} = \mathcal{J}^A_\alpha \mathcal{J}^A_\beta \delta_{AB} \,, \tag{4.26}$$

where $\mathcal{J}^A_\alpha$ is the Newtonian Jacobian matrix given by eq. (4.3). We can also find the transformation of the Christoffel symbols from the usual rule

$$\overline{\Gamma}^M_{LK} w^K_{\mathcal{E}} = \mathcal{J}^M_\nu \mathcal{J}^\rho_L \overline{\Gamma}^\nu_{\rho\sigma} w^\sigma_{\mathcal{L}} - \mathcal{J}^\rho_L \partial_\sigma \mathcal{J}^M_\rho w^\sigma_{\mathcal{L}} \tag{4.27}$$

Since $\overline{\Gamma}^A_{BC} = 0$ in Eulerian coordinates, we find the Christoffel symbols in Lagrangian coordinates

$$\overline{\Gamma}^\alpha_{\beta\sigma} = \mathcal{J}^\alpha_M \partial_\sigma \mathcal{J}^M_\beta \,. \tag{4.28}$$

We can therefore reformulate the Newtonian limit in this gauge, referring to the metric which results from the $c \to \infty$ limit of the Einstein equations: eq. (4.19) and eq. (4.22) tell us that we can write the spatial metric in the form of eq. (4.26). The Ricci tensor calculated from (4.26) is zero but the Christoffel symbols involved in spatial covariant derivatives do not vanish. On the other hand, the vanishing of the spatial curvature implies that these covariant derivatives always commute. The resulting geometry in Lagrangian coordinates reproduces the properties of the Eulerian velocity and density fields, which come from the mapping (4.1): all it is needed is the Jacobian matrix, eq. (4.3), which is actually found by solving the remaining equations, the Raychaudhuri equation and the momentum constraint. They contain no explicit power of c, preserving their form in the Newtonian limit, and no curvature terms, which would involve higher (PN) terms of the metric.

Now, we finally obtain the Newtonian expression of eq. (4.20) and eq. (4.23), where by Newtonian expression here we mean the expression that comes from (4.26). Then, we rewrite the peculiar velocity-gradient tensor as

$$\overline{\vartheta}^\alpha_\beta = \mathcal{J}^\alpha_B \mathcal{J}^{B\prime}_\beta \,. \tag{4.29}$$

The Raychaudhuri equation is therefore given by

$$\mathcal{J}^\alpha_B \mathcal{J}^{B\prime\prime}_\alpha + \mathcal{H}\frac{\mathcal{J}'}{\mathcal{J}} = \frac{3}{2}\mathcal{H}^2 \left(1 - \frac{1}{\mathcal{J}}\right), \tag{4.30}$$

where $\mathcal{J}$ is the determinant of the Jacobian matrix and $\left(1 - \frac{1}{\mathcal{J}}\right)$ is the solution for the density contrast from eq. (4.25).[3] The momentum constraint reads

$$\overline{\mathcal{D}}_\alpha \left(\mathcal{J}^\alpha_B \mathcal{J}^{B\prime}_\beta\right) = \partial_\beta \left(\frac{\mathcal{J}'}{\mathcal{J}}\right), \tag{4.31}$$

where $\overline{\mathcal{D}}_\alpha$ is the covariant derivative related to the Newtonian metric, eq. (4.26).
On the other hand, eq. (4.29) together with $\overline{\vartheta}^\alpha_\beta = (1/2)\,\overline{\gamma}^{\alpha\sigma}\,\overline{\gamma}'_{\sigma\beta}$ gives

$$\mathcal{J}_{A\alpha}{}' \mathcal{J}^A{}_\beta = \mathcal{J}_{A\alpha} \mathcal{J}^A{}_\beta{}' \tag{4.32}$$

[2]Note that Eulerian indices $A, B, \ldots$ are raised/lowered by the flat Euclidean metric.

[3]We assumed for simplicity $\delta_{in} = 0$ and used the residual gauge freedom of the synchronous and comoving gauge to set $\mathcal{J}_{in} = 1$ in the Newtonian limit, as in ref. Matarrese & Terranova (1996).

which is identical to the standard Newtonian form of the irrotational condition in Lagrangian space

$$\overline{\vartheta}_{[\alpha\beta]} = 0 \tag{4.33}$$

This equation, together with the relation $\partial_\beta \mathcal{J}^A_\alpha = \partial_\alpha \mathcal{J}^A_\beta$, which follows from the symmetry of the Newtonian Christoffel symbols, reduce the momentum constraint to an identity.

From the Newtonian limit of Einstein's equations in the synchronous and comoving gauge we then find eq.(4.30) and eq.(4.32). These are identical to the well-known Lagrangian equations of Newtonian gravity, see e.g. Catelan 1995, Buchert 1989, andBouchet et al. 1992.
A final remark about the energy constraint and the evolution equation: it would be wrong to take a Newtonian version of eq. (4.19), and eq. (4.21) or eq. (4.22), by setting the l.h.s to zero. They simply imply that the Newtonian spatial curvature vanishes. On the contrary, they must be thought perturbatively: as a consequence of our gauge choice no odd powers of c appear in the equations, so the expansion parameter is $1/c^2$. The spatial metric is then expanded up to PN order in the form

$$\gamma_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + \frac{1}{c^2} w_{\alpha\beta} \tag{4.34}$$

Therefore the Newtonian l.h.s of eq. (4.21), or of eq. (4.19) and eq. (4.22), determine the spatial PN Ricci tensor, as shown in Matarrese & Terranova 1996.

4.2.1 The Zel'dovich approximation

The linear result in Lagrangian perturbation theory is the well-known Zel'dovich approximation, Zel'dovich 1970. The peculiarity of this treatment, at any order, is that, while the displacement vector is calculated from the equations at the required perturbative order, all the other dynamical variables, such as the mass density, are calculated exactly from their non-perturbative definition. Since the equations in Lagrangian coordinates are intrinsically non-linear in the density, what comes out is a fully non-linear description of the system, which, though not being generally exact, "mimics" the true non-linear behaviour. This perturbation treatment basically exploits the advantages of the Lagrangian picture, leading, in particular, to a more accurate description of high density regions. Its limitations are generally set by the emerging of caustic singularities.

The Zel'dovich approximation is obtained by expanding equations (4.30) and (4.31) to the first order in the displacement vector. The result is

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} + D(\tau)\nabla\Phi(\mathbf{q}), \tag{4.35}$$

where $D(\tau) \propto a(\tau) \propto \tau^2$ is the growing mode solution for the Einstein-de Sitter model and $\Phi(\mathbf{q})$ is related to the initial peculiar gravitational potential φ by the cosmological Poisson equation, yielding

$$\Phi_{\rm in} = -\frac{\phi}{4\pi G a^2_{\rm in} \varrho_{b,{\rm in}}}. \tag{4.36}$$

Starting from the displacement vector at first order, we calculate all other quantities exactly. The Zel'dovich metric is found from (4.26) and reads

$$\gamma^{ZEL}_{\alpha\beta}(\mathbf{q}, \tau) = \delta_{\sigma\omega}\left[\delta^\sigma_\alpha + D(\tau)\partial^\sigma\partial_\alpha\Phi_{\rm in}(\mathbf{q})\right]\left[\delta^\omega_\beta + D(\tau)\partial^\omega\partial_\beta\Phi_{\rm in}(\mathbf{q})\right], \tag{4.37}$$

where we used the fact that at first order in the displacement vector covariant and partial derivatives with respect to the coordinates q_α coincide, since the Newtonian Christoffel symbols are second-order quantities.
One can of course diagonalize this expression by going to the principal axes of the deformation tensor. Calling λ_α the eigenvalues of the matrix $\partial^\alpha \partial_\beta \Phi_{\rm in}$, one finds

$$\gamma_\alpha^{ZEL}(\mathbf{q}, \tau) = \left[1 + D(\tau)\lambda_\alpha(\mathbf{q})\right]^2 . \tag{4.38}$$

Note that, contrary to what has been commonly done sometime in the literature, the metric tensor must be evaluated at second order in the displacement vector, in order to obtain back the correct Zel'dovich expressions for the dynamical variables (density, shear, etc ...).

The above diagonal form of the metric allows a straightforward calculation of all the relevant quantities. The well-known expression for the mass density is consistently recovered,

$$\varrho^{ZEL} = \varrho_b \prod_\alpha \left(1 + D\lambda_\alpha\right)^{-1} . \tag{4.39}$$

The peculiar velocity-gradient tensor has the same eigenframe of the metric; its eigenvalues read

$$\vartheta_\alpha^{ZEL} = \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} . \tag{4.40}$$

By summing over α the latter expression we can obtain the peculiar volume-expansion scalar

$$\vartheta^{ZEL} = \sum_\alpha \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} \tag{4.41}$$

and then the shear eigenvalues

$$\sigma_\alpha^{ZEL} = \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} - \frac{1}{3}\sum_\alpha \frac{D'\lambda_\alpha}{1 + D\lambda_\alpha} . \tag{4.42}$$

The electric tide comes out just proportional to the shear. Its eigenvalues read [4]

$$\mathcal{E}_\alpha^{ZEL} = -4\pi G a^2 \varrho_b \frac{D}{D'}\, \sigma_\alpha^{ZEL} . \tag{4.43}$$

The expressions above provide non–perturbative formulas not only for the matter density but also for the metric and extrinsic curvature (the velocity-gradient tensor) which can be used to extrapolate far into the non–linear regime and thus beyond the capacity of perturbation theory.

[4]Note that hese expressions for the shear and the tide completely agree with those obtained by Kofman & Pogosyan 1995 and Hui & Bertschinger 1996.

CHAPTER 5

From the Eulerian to the Lagrangian frame I - Newtonian approximation

In this chapter we consider the non-linear dynamics of cosmological perturbations of an irrotational collisionless fluid, the FRW background being the Einstein-de Sitter model. We discuss the connection between GR and Newtonian gravity in the Eulerian and in the Lagrangian picture and we provide the transformation rule between the Eulerian and the Lagrangian frames: it is fully four-dimensional and non-perturbative and clarifies the role of the transformation of the time and spatial coordinates. Our approach here is different from the standard perturbative gauge transformation in GR, see Matarrese et al. 1998, Bruni et al. 1997 and Bardeen 1980, and from the spatial coordinate transformation of Newtonian theory.

Most derivations of the Newtonian limit of GR are coordinate-dependent, thus a precise understanding of the Newtonian correspondence between the Eulerian and the Lagrangian frame has to be considered as the starting point e.g. for studying the gauge dependence when we want to add GR corrections in a perturbed space-time from a non-perturbative perspective.

This chapter is based on our paper Villa et al. 2014, which was submitted to *JCAP* and is currently under review.

5.1 The Newtonian transformation from the Eulerian to the Lagrangian frame

In this section we provide the coordinate transformation for passing from the Poisson gauge to the synchronous and comoving gauge in the Newtonian limit.

5.1.1 The transformation of the spatial coordinates

The Poisson gauge is defined in Bertschinger 1996 starting from the perturbed Einstein-de Sitter line element, the background spatial metric being δ_{ij}. In comoving spatial Cartesian coordinates and conformal time the line element can be written in any gauge as

$$ds^2 = a^2(\eta)\left\{-\left(1+2\psi\right)c^2 d\eta^2 + 2w_i c d\eta dx^i + \left[\left(1-2\phi\right)\delta_{ij} + 2h_{ij}\right]dx^i dx^j\right\} \tag{5.1}$$

where the tensor perturbation is trace-less and we have written explicitly the c factor in the time coordinate. The four gauge modes are eliminating by setting

$$\partial_i w^i = 0\,, \qquad \partial_i h^i_j = 0\,, \tag{5.2}$$

which fixes the Poisson gauge, including all the six physical degrees of freedom present in the metric. In particular, the Poisson gauge has no residual gauge ambiguity, since it can be shown that a coordinate transformation from an arbitrary gauge completely fixes this gauge. It is important to stress that in our approach all the degrees of freedom in the metric should be understood as *a priori* containing perturbations at any order in standard perturbation theory. The scalar potentials ψ, ϕ and the tensor h_{ij} contain even powers of the speed of light, starting from $1/c^2$ and $1/c^4$, respectively. The vector w_i contains odd powers of the speed of light, starting from $1/c^3$. We are free to change the time coordinate from $c\eta$ to η, since it just represents a change in the units of time, obtaining

$$ds^2 = a^2(\eta)\left\{-\left(c^2+2\psi\right)d\eta^2 + 2w_i d\eta dx^i + \left[\left(1-2\phi\right)\delta_{ij} + 2h_{ij}\right]dx^i dx^j\right\} \tag{5.3}$$

The Newtonian limit in this gauge is obtained by retaining in the metric the only potential required in the Newtonian equations of motion, i.e. $\psi = \varphi_g$, as already explained in section3.1. The Newtonian line-element in then

$$ds^2 = a^2\left[-\left(c^2+2\varphi_g\right)d\eta^2 + \delta_{ij}dx^i dx^j\right] \tag{5.4}$$

For our purposes, it is useful to reinterpret this line element in the language of the $3+1$ splitting of space-time, see Smarr & York 1978a, where the chosen coordinate system, i.e. the gauge, is related to the observers. In this formalism, the space-time is split in a family of three-dimensional hypersurfaces, the "space", plus the "the time direction", in strict analogy with the Newtonian treatment. On every three-dimensional hypersurface, the chosen time coordinate is constant, thus every hypersuface corresponds to the rest frame of the chosen observers. This perspective will of help here, since we are dealing with the Eulerian and Lagrangian frames.

Eulerian observers are represented by a set of curves with unit four-velocity n^a always orthogonal to the constant-time slices. Orthogonality implies that Eulerian observers are at rest on each slice and that there exists a scalar function $\mathcal{N}$, called the lapse function, such that

$$n_a = -\mathcal{N}\nabla_a\eta\ . \tag{5.5}$$

It represent the rate of change of the proper time along n^a with respect to the time η. Given arbitrary three-dimensional coordinates on the initial slice, $\{q^\alpha\}$, one can construct a non-normal congruence threading the slices with tangent vectors t^a. Each curve is permanently labelled by the coordinate values it acquires on the initial slice. The vector basis related to the $\{q^\alpha\}$ is dragged along these curves, and not along the curves of Eulerian observers. The relation between the two four-velocities is given by

$$t^a = \mathcal{N}n^a + \mathcal{N}^a\ , \qquad n^a\mathcal{N}_a = 0\ , \tag{5.6}$$

where $\mathcal{N}^a$ is the projection of the velocity shift between the two frames on the slices. We can fix the shift vector such that t^a coincides with the matter four-velocity u^a, namely we can choose the well-known comoving condition: in this case, a given element of the fluid has fixed spatial coordinates and the $\{q^\alpha\}$ are called Lagrangian. Of course, the time η does not coincide with the proper (conformal) time defined in the rest frame of the fluid. The shift vector projected on the slices, i.e. on the rest frame of Eulerian observers, measures in η-time the spatial velocity of the matter with respect to the Eulerian observers. The line-element is that of the ADM formalism

$$ds^2 = a^2\left[-\mathcal{N}^2 d\eta^2 + g_{\alpha\beta}\left(dq^\alpha + \mathcal{N}^\alpha d\eta\right)\left(dq^\beta + \mathcal{N}^\beta d\eta\right)\right]. \tag{5.7}$$

Instead of the basis related to the coordinates $\{q^\alpha\}$, we can alternatively choose an orthonormal basis on the spatial slices, whereas the time coordinate η remains unchanged. The spatial coordinates are related in the usual way : $dx^A = \mathcal{J}^A_\alpha dq^\alpha$. The orthonormal basis is also dragged along the world-line of the corresponding fluid-element: the parallel transport condition of the tetrad $\mathcal{J}^A_\alpha$ along u^a reads

$$\mathcal{J}^A_{\alpha;a} u^a = 0 \, , \tag{5.8}$$

where the semicolon indicates the four-dimensional covariant derivative. Of course, the orthonormal basis dragged along u^a is not at rest with respect to the Eulerian observers: its relative velocity coincides with the peculiar velocity of the matter. Using the orthonormal basis the line-element is

$$ds^2 = a^2 \left[-\mathcal{N}^2 d\eta^2 + \delta_{AB} \left(\mathcal{J}^A_\alpha dq^\alpha + \mathcal{N}^A d\eta \right) \left(\mathcal{J}^B_\alpha dq^\beta + \mathcal{N}^B d\eta \right) \right] . \tag{5.9}$$

In the Newtonian limit, the shift vector is the Newtonian peculiar velocity of the matter measured by Eulerian observers comoving with the Hubble flow, $\mathcal{N}^A = v^A$, and the lapse function is found to be $\mathcal{N} = 1 - 2\varphi_g^{\mathcal{E}}$ from the time-time component of Einstein equations. The Newtonian line-element is then

$$ds^2 = a^2 \left[-\left(c^2 + 2\varphi_g^{\mathcal{E}} \right) d\eta^2 + \delta_{AB} \left(\mathcal{J}^A_\alpha dq^\alpha + v^A d\eta \right) \left(\mathcal{J}^B_\alpha dq^\beta + v^B d\eta \right) \right] \tag{5.10}$$

where the matrix $\mathcal{J}^A_\alpha$ is the Jacobian matrix of the map

$$\mathbf{x}(\mathbf{q}, \eta) = \mathbf{q} + \mathcal{S}(\mathbf{q}, \eta) \, . \tag{5.11}$$

The coordinates x^A on the slices representing the rest frame of Eulerian observers are the usual Eulerian coordinates of Newtonian gravity. This is our starting point for the transformation to the synchronous and comoving gauge. We transform from the Eulerian spatial coordinates x^A to the Lagrangian ones q^α in the line-element (5.10), obtaining

$$ds^2 = a^2 \left[-\left(c^2 + 2\varphi_g^{\mathcal{E}} - v^C v^D \delta_{CD} \right) d\eta^2 + 2\delta_{AB} \mathcal{J}^A_\sigma v^B dq^\sigma d\eta + \delta_{AB} \mathcal{J}^A_\sigma \mathcal{J}^B_\lambda dq^\sigma dq^\lambda \right] . \tag{5.12}$$

Note that at this step we have not changed the time coordinate, according to a purely Newtonian treatment where the time is absolute. The slices $\eta =$ const. still set the rest frame of the Eulerian observers, not yet the rest frame of the matter. On the $\eta =$ const. slices, we have performed a spatial transformation from the spatial orthonormal basis to the Lagrangian one, the two bases moving with relative velocity v^A with respect to each other. This transformation can be alternatively viewed as a boost with spatially varying relative velocity v^A between the Lagrangian and the Eulerian frame:

$$u^a = \Gamma \left(n^a + v^a \right) \, , \tag{5.13}$$

where $\Gamma = \left(1 - v^2/c^2 \right)^{-1/2}$ is the Lorentz factor. In the Newtonian limit, the unit normal n^a has no spatial component and the coordinate three-velocity v^A is that measured by the Eulerian observers in η time and we have $\Gamma \simeq 1$. The boost is supplemented by a spatial coordinate transformation from the orthonormal coordinate basis related to the x^A to that related to the q^α. In matrix form, our resulting transformation reads

$$\mathcal{A}^a_b = \begin{pmatrix} 1 & 0 \\ & \\ v^A & \mathcal{J}^A_\alpha \, . \end{pmatrix} \tag{5.14}$$

Now, we need a second step to arrive at the synchronous and comoving gauge: we have to change from the Eulerian observers rest frame to the fluid rest frame.

5.1.2 The transformation of the time coordinate

We write the time transformation required to obtain the correct quantities in the Lagrangian frame as

$$\tau = \eta - \frac{1}{c^2}\xi^0_{\mathcal{E}}(x^A, \eta) \tag{5.15}$$

and its inverse

$$\eta = \tau + \frac{1}{c^2}\xi^0_{\mathcal{E}}(x^A, \eta)\ , \tag{5.16}$$

where we want the function $\xi^0_{\mathcal{E}}$ in terms of the q^a. Note that the functions $\xi^0_{\mathcal{E}}$ and $\xi^0_{\mathcal{L}}$ are functions of the Eulerian and Lagrangian coordinates, respectively; the (conformal) time dependence at the required order is the same, $\eta = \tau$, and the spatial coordinates on the slices change as $dx^A = \mathcal{J}^A_\alpha dq^\alpha$. We have

$$\xi^0_{\mathcal{E}}(x^a(q^b)) = \mathcal{J}^0_s \xi^s_{\mathcal{L}}(q^b) = \xi^0_{\mathcal{L}}(q^b)\,, \tag{5.17}$$

thus ξ^0 transforms simply as a scalar under the change in spatial coordinates, i.e.

$$\xi^0_{\mathcal{E}}(x^A(q^\beta), \eta) = \xi^0_{\mathcal{L}}(q^\alpha, \tau). \tag{5.18}$$

Now, we go back to the line-element of eq. (5.12). In order to get

$$ds^2 = a^2\left[-c^2 d\tau^2 + \gamma^{\mathcal{L}}_{\alpha\beta} dq^\alpha dq^\beta\right] \tag{5.19}$$

we substitute

$$\tau = \eta - \frac{1}{c^2}\xi^0_{\mathcal{L}}(\eta, q^\alpha) \tag{5.20}$$

in $-c^2a^2(\tau)d\tau^2$, obtaining

$$-c^2a^2(\tau)d\tau^2 = -c^2\left(a^2(\eta) - \frac{2}{c^2}a^2(\eta)\mathcal{H}\xi^0_{\mathcal{L}}\right)\left(d\eta - \frac{1}{c^2}d\xi^0_{\mathcal{L}}\right)^2\ , \tag{5.21}$$

which, at $1/c^2$ order reduces to

$$-c^2a^2(\tau)d\tau^2 = -c^2a^2(\eta)d\eta^2 + a^2(\eta)\left(2\mathcal{H}\xi^0_{\mathcal{L}} + 2\frac{\partial\xi^0_{\mathcal{L}}}{\partial\eta}\right)d\eta^2 + 2a^2(\eta)\frac{\partial\xi^0_{\mathcal{L}}}{\partial q^\sigma}dq^\sigma d\eta\,. \tag{5.22}$$

Comparing with eq. (5.12)

$$ds^2 = a^2(\eta)\left[-\left(c^2 + 2\varphi^{\mathcal{L}}_g - v^C v^D\delta_{CD}\right)d\eta^2 + 2\delta_{AB}\mathcal{J}^A_\sigma v^B dq^\sigma d\eta + \mathcal{J}^A_\sigma\mathcal{J}^B_\rho\delta_{AB}dq^\sigma dq^\rho\right] \tag{5.23}$$

we finally get the equations for $\xi^0_{\mathcal{L}}$

$$2\mathcal{H}\xi^0_{\mathcal{L}} + 2\frac{\partial\xi^0_{\mathcal{L}}}{\partial\eta} = -2\varphi_g + v^A v^B\delta_{AB} \tag{5.24}$$

$$v^K = \frac{\partial\xi^0_{\mathcal{L}}}{\partial q^\lambda}\mathcal{J}^\lambda_F\delta^{FK}\,. \tag{5.25}$$

Integrating eq. (5.24) gives

$$\xi^0_{\mathcal{L}} = \frac{1}{a}\int_{\eta_{in}}^{\eta} a\left(-\varphi^{\mathcal{L}}_g + \frac{1}{2}v^A v^B\delta_{AB}\right)d\tilde{\eta} + \frac{C(q^\alpha)}{a}\,, \tag{5.26}$$

where $C(q^\alpha)$ is an integration constant which we will fix using (5.25): we re-write this equation as

$$\frac{\partial \xi^0_{\mathcal{L}}}{\partial q^\sigma} = v^K \mathcal{J}^F_\sigma \delta_{FK} \,, \tag{5.27}$$

then $C(q^\alpha)$ is found from

$$v^K \mathcal{J}^F_\sigma \delta_{FK} - \frac{1}{a}\frac{\partial}{\partial q^\sigma}\left[\int_{\eta_{in}}^{\eta} a\left(-\varphi^{\mathcal{L}}_g + \frac{1}{2}v^A v^B \delta_{AB}\right) d\tilde{\eta}\right] = \frac{1}{a}\frac{\partial C}{\partial q^\sigma} \,. \tag{5.28}$$

We obtain the same expression for ξ^0 also following the procedure outlined in Kolb et al. 2005, which exploits directly the fact that the time coordinate of the synchronous and comoving gauge is the proper time along the world-line of the fluid: in Kolb et al. 2005 the calculation was performed at second order in standard perturbation theory, whereas ours is not restricted to any perturbative order. We will show in the next section that our solution for ξ^0, when expanded at second order, coincides with that of Kolb et al. (2005).

5.1.3 Four-dimensional gauge transformation of the metric

The transformation between the Eulerian and Lagrangian frame is

$$\mathcal{A}^a_b = \begin{pmatrix} 1 & 0 \\ v^A & \mathcal{J}^A_\alpha \end{pmatrix} , \tag{5.29}$$

and the time transformation is

$$\mathcal{B}^a_b = \begin{pmatrix} 1 + \frac{1}{c^2}\frac{\partial \xi^0}{\partial \tau} & \frac{1}{c^2}\frac{\partial \xi^0}{\partial q^\beta} \\ 0 & \delta^A_\alpha \end{pmatrix} . \tag{5.30}$$

Their product gives the transformation of the space-time coordinates. The result is the four dimensional coordinate transformation:

$$\mathcal{J}^a_b = \begin{pmatrix} 1 + \frac{1}{c^2}\frac{\partial \xi^0}{\partial \tau} & \frac{1}{c^2}\frac{\partial \xi^0}{\partial q^\beta} \\ v^A & \mathcal{J}^A_\beta \end{pmatrix} . \tag{5.31}$$

We now calculate how the metric tensor transforms under the transformation $x^a \to q^a$: from the standard rule we have

$$g^{\mathcal{L}}_{ab}(q^c) = \frac{\partial x^d}{\partial q^a}\frac{\partial x^f}{\partial q^b} g^{\mathcal{E}}_{df}(x^d(q^c)) \,, \tag{5.32}$$

We also need the Newtonian Eulerian metric $g^{\mathcal{E}}_{ab}$

$$g^{\mathcal{E}}_{ab} = a^2 \begin{pmatrix} -\left(c^2 + 2\varphi^{\mathcal{E}}_g\right) & 0 \\ 0 & \delta_{AB} \end{pmatrix} . \tag{5.33}$$

Applying the synchronous and comoving gauge conditions to the time-time and space-time components on the l.h.s., i.e. $g^{\mathcal{L}}_{00} = -1$ and $g^{\mathcal{L}}_{0\alpha} = 0$, and expanding up to $1/c^2$ we find the same equations for $\xi^0_{\mathcal{L}}$ that we found in section 5.1.2: the equation for $g^{\mathcal{L}}_{00}$ is

$$2\mathcal{H}\xi^0_{\mathcal{L}} + 2\frac{\partial \xi^0_{\mathcal{L}}}{\partial \eta} = -2\varphi^{\mathcal{E}}_g + v^A v^B \delta_{AB} \tag{5.34}$$

with solution

$$\xi^0_{\mathcal{L}} = \frac{1}{a}\int_{\eta_{in}}^{\eta} a\left(-\varphi^{\mathcal{L}}_g + \frac{1}{2}v^A v^B \delta_{AB}\right) d\tilde{\eta} + \frac{C(q^\alpha)}{a}\,, \tag{5.35}$$

where $C(q^\alpha)$ has to be fixed from the equation for $g^{\mathcal{L}}_{0\alpha}$

$$v^K = \frac{\partial \xi^0_{\mathcal{L}}}{\partial q^\lambda}\mathcal{J}^\lambda_F \delta^{FK}\,. \tag{5.36}$$

For the spatial components we get

$$\overline{\gamma}^{\mathcal{L}}_{\alpha\beta} = \mathcal{J}^A_\alpha \mathcal{J}^B_\beta \delta_{AB}\,. \tag{5.37}$$

The second equation for ξ^0 obtained from our transformation, eq. (5.25), is very important: it shows that $\xi^0_{\mathcal{L}}$ can be thought as the velocity potential in Lagrangian space[1]. From the irrotationality condition in Eulerian space we know that $v^A = \partial^A \Phi_v$ and the peculiar velocity-gradient tensor is $\vartheta^A_B = \partial^A\partial_B\Phi_v$. By changing the spatial derivative in Lagrangian coordinates according to $\partial_A = \mathcal{J}^\sigma_A \partial_\sigma$ and using eq. (5.25), it is easy to show that the peculiar velocity-gradient tensor in Eulerian space transforms to[2]

$$\overline{\vartheta}^\alpha_\beta = \overline{\mathcal{D}}^\alpha \overline{\mathcal{D}}_\beta \xi^0_{\mathcal{L}} \tag{5.38}$$

in Lagrangian space. The Lagrangian velocity-gradient tensor appears as the spatial covariant derivative of a scalar in the Lagrangian space, but when expanded at second order in perturbation theory it acquires a true tensorial part, owing the expression of the Christoffel symbol in Lagrangian space.

Substituting eq. (5.25) in eq. (5.34) we find a relation linking the gravitational potential and the velocity potential $\xi^0_{\mathcal{L}}$, i.e. a Lagrangian version of the Bernoulli equation, which reads

$$\mathcal{H}\xi^0_{\mathcal{L}} + \frac{\partial \xi^0_{\mathcal{L}}}{\partial \tau} + \varphi^{\mathcal{L}}_g - \frac{1}{2}\partial_\alpha \xi^0_{\mathcal{L}} \partial_\beta \xi^0_{\mathcal{L}} \overline{\gamma}^{\alpha\beta} = 0\,. \tag{5.39}$$

We will exploit this equation in the PN coordinate transformation.

5.2 Consistency up to second order in perturbation theory

The aim of this section is to calculate the spatial metric $g^{\mathcal{L}}_{\alpha\beta}$ and the function of the time transformation $\xi^0_{\mathcal{L}}$, starting from the Eulerian field at second order in perturbation theory given by eqs. (3.12), (3.13), and (3.14) in section 3.1.1.

For the second-order expansion we have to be careful: every term is a function $F_{\mathcal{E}}(x^a(q^b))$ which has to be expanded itself together with its argument. In order for all

[1]Remember that although the three-velocity vanishes in the Lagrangian space, the velocity-gradient tensor is well defined.

[2]See eq. (C.11).

the expressions to be Newtonian, all terms are functions of the coordinates $x^A = q^A + \mathcal{S}^A$, with absolute time $\eta = \tau$. Therefore, we have this two-step Taylor expansion: the second-order expansion of an Eulerian function is given by

$$F_{\mathcal{E}}(x^a) = F_{\mathcal{E}}^{(0)}(x^a) + F_{\mathcal{E}}^{(1)}(x^a) + \frac{1}{2}F_{\mathcal{E}}^{(2)}(x^a) \,, \tag{5.40}$$

where we have to expand the argument with respect to the Newtonian perturbative transformation for the x^a:

$$x^A = q^A + S^{A(1)} + \frac{1}{2}S^{A(2)} \,, \qquad \eta = \tau \,. \tag{5.41}$$

By collecting all terms of the same order, we find the second-order expansion of the Lagrangian function. The result is

$$\begin{aligned} F_{\mathcal{L}}(q^a) \quad = \quad & F_{\mathcal{E}}^{(0)}(q^a) + \left(\left.\frac{\partial F_{\mathcal{E}}^{(0)}}{\partial x^A}\right|_{x=q} S^{A(1)} + F_{\mathcal{E}}^{(1)}(q^a) \right) + \frac{1}{2}\left(2\, \left.\frac{\partial F_{\mathcal{E}}^{(1)}}{\partial x^A}\right|_{x=q} S^{A(1)} + \right. \\ & \left. + F_{\mathcal{E}}^{(2)}(q^a) + \left.\frac{\partial^2 F_{\mathcal{E}}^{(0)}}{\partial x^A \partial x^B}\right|_{x=q} S^{A(1)}S^{B(1)} + \left.\frac{\partial F_{\mathcal{E}}^{(0)}}{\partial x^A}\right|_{x=q} S^{A(2)} \right) \,. \end{aligned} \tag{5.42}$$

The FRW background, in particularly the Einstein-de Sitter background considered here, keeps its form in any of the two gauges considered

$$ds^2 = a^2(\eta)\eta_{ab}dx^a dx^b \,, \tag{5.43}$$

where η is the conformal time. The lapse function is simply the scale factor and the shift vector vanishes, so that the Eulerian and Lagrangian spatial coordinates coincide. In standard perturbation theory, the background spatial transformation is simply $dx^A = \delta^A_\alpha dq^\alpha$. So, in the following we can set the Latin indices equal to the Greek ones in the spatial derivatives of second-order quantities, whereas the spatial derivatives and the arguments of first-order quantities change with the Jacobian matrix of the first-order transformation.

We then follow an iterative procedure, starting from the Eulerian first-order peculiar velocity: the time integration gives the first-order spatial transformation

$$\mathbf{x} = \mathbf{q} - \frac{\tau^2}{6}\nabla_q \phi \tag{5.44}$$

with Jacobian $\mathcal{J}^\alpha_\beta = \delta^\alpha_\beta - \tau^2/6\, \partial^\alpha\partial_\beta\phi$ and inverse $\mathcal{J}^\alpha_\beta = \delta^\alpha_\beta + \tau^2/6\, \partial^\alpha\partial_\beta\phi$, where $A = \alpha$ at first order. Then we use the Eulerian velocity at second order to find the second-order spatial transformation: after changing coordinates in the first-order term and integrating over time we find

$$x^\alpha = q^\alpha - \frac{\tau^2}{6}\partial^\alpha\phi + \frac{\tau^4}{84}\partial^\alpha\Psi \,. \tag{5.45}$$

We can now find the solution for $\xi^0_{\mathcal{L}}$, eq. (5.35):

$$\xi^0_{\mathcal{L}} = \frac{1}{a}\int_{\eta_{in}}^{\eta} a\left(-\varphi^{\mathcal{L}}_g + \frac{1}{2}v^A v^B \delta_{AB}\right) d\tilde{\eta} + \frac{C(q^\alpha)}{a} \,. \tag{5.46}$$

In the integral, we have to transform the scalar $\varphi_g^{\mathcal{E}}$ with the Taylor expansion of the form of equation (5.42). The result is

$$\varphi_g^{\mathcal{L}} = \phi - \frac{10}{21}\tau^2\Psi \tag{5.47}$$

where the potential $\Psi_{\mathcal{L}}$ is given by

$$\nabla_q^2\Psi = -\frac{1}{2}\left[\left(\nabla_q^2\phi\right)^2 - \partial^\sigma\partial^\lambda\phi\partial_\sigma\partial_\lambda\phi\right]. \tag{5.48}$$

The integration gives the solution $\xi_{\mathcal{L}}^0$ up to second order

$$\xi_{\mathcal{L}}^0 = -\frac{1}{3}\tau\phi + \frac{1}{21}\tau^3\Psi + \frac{1}{36}\tau^3\partial^\sigma\phi\partial_\sigma\phi + \frac{C(q^\alpha)}{a}. \tag{5.49}$$

We use the second-order Eulerian velocity and remaining equation obtained from our transformation, eq. (5.36), to fix the constant $C(q^\alpha)$. It turns out to vanish at second-order, thus it is at least a third-order quantity.
Our final expressions

$$\mathbf{x} = \mathbf{q} - \frac{\tau^2}{6}\nabla_q\phi + \frac{\tau^4}{84}\nabla_q\Psi \tag{5.50}$$

$$\eta = \tau - \frac{1}{3}\tau\phi + \frac{\tau^3}{36}\partial^\sigma\phi\partial_\sigma\phi + \frac{\tau^3}{21}\Psi \tag{5.51}$$

coincide with the second-order gauge transformation from the Poisson gauge to the synchronous and comoving gauge obtained from a fully relativistic calculation, see Matarrese et al. 1998,[3] retaining the Newtonian terms only. In particular, the time transformation also coincides with the result in Kolb et al. 2005, where a different procedure was adopted.

Finally, our result for the Newtonian Lagrangian metric up to second order from eq. (5.37) and eq. (5.50) is

$$g_{\alpha\beta}^{\mathcal{L}} = \delta_{\alpha\beta} - \frac{\tau^2}{3}\partial_\alpha\partial_\beta\phi + \frac{\tau^4}{36}\partial_\sigma\partial_\alpha\phi\partial^\sigma\partial_\beta\phi + \frac{\tau^4}{42}\partial_\alpha\partial_\beta\Psi. \tag{5.52}$$

This is exactly the same expression obtained from the second-order solution of the Einstein equations in the synchronous and comoving gauge, considering only the Newtonian terms in the metric, see Matarrese et al. 1998. It is important to note that this expression includes the contribution of second-order Newtonian tensor modes generated by scalar initial perturbations. It is well known that tensor modes appear already at the Newtonian and PN level in the metric in this gauge, whereas they are only PPN in the Poisson gauge.

5.3 Discussion

In this chapter we examined the Newtonian approximation to the non-linear gravitational dynamics of cosmological perturbations in two frames.

[3]Note that in Matarrese et al. 1998 the inverse gauge transformation is performed, from the synchronous gauge to the Poisson gauge. We found our inverse transformation, using the procedure explained in 2.1.3, and the result agrees with Matarrese et al. 1998.

Our starting point was the Newtonian line-element in the Poisson gauge, given by

$$ds^2 = a^2 \left[- \left(1 + 2\frac{\varphi_g^{\mathcal{E}}}{c^2} \right) c^2 d\eta^2 + \delta_{AB} dx^A dx^B \right] . \quad (5.53)$$

We then transformed this metric to the Lagrangian frame and obtained the Newtonian line-element in the synchronous and comoving gauge as in ref. Matarrese & Terranova (1996), which is given by

$$ds^2 = a^2 \left[-c^2 d\tau^2 + \delta_{AB} \mathcal{J}^A_\alpha \mathcal{J}^B_\beta dq^\alpha dq^\beta \right] . \quad (5.54)$$

As we said, our starting point was the metric of eq. (5.53) in the Poisson gauge, which we dubbed Newtonian, as the metric variables appearing there are just those needed for the Newtonian equations of motion. With our transformation we arrived at a Newtonian metric in the synchronous and comoving gauge, where, once again, we have just the variables needed for the Newtonian Lagrangian equations of motion, namely the spatial Jacobian matrix. However, the Newtonian three-dimensional space has vanishing spatial curvature, therefore, the Newtonian Lagrangian metric, eq. (5.54), can be transformed globally to the Einstein-de Sitter background metric, the transformation being just the spatial transformation to Eulerian coordinates $dx^A = \mathcal{J}^a_\alpha dq^\alpha$, without changing the time coordinate. As we will see below, this inconsistency can be easily overcome by requiring that the Ricci four-dimensional curvature scalar is preserved by the transformation, as it should.

As we have shown, the Lagrangian frame can be transformed to the locally flat inertial frame by means of the transformation to coordinates $\left(\tau + 1/c^2\, \xi^0, x^A\right)$:

$$ds^2 = a^2 \left[-c^2 \left(d\tau + \frac{1}{c^2} d\xi^0 \right)^2 + \delta_{AB} \mathcal{J}^A_\alpha \mathcal{J}^B_\beta dq^\alpha dq^\beta \right] . \quad (5.55)$$

When our solution ξ^0,

$$\xi^0_{\mathcal{L}} = \frac{1}{a} \int_{\eta_{in}}^{\eta} a \left(-\varphi_g^{\mathcal{L}} + \frac{1}{2} v^A v^B \delta_{AB} \right) d\tilde{\eta} + \frac{C(q^\alpha)}{a} \quad (5.56)$$

with $C(q^\alpha)$ fixed from

$$v^K = \frac{\partial \xi^0_{\mathcal{L}}}{\partial q^\lambda} \mathcal{J}^\lambda_F \delta^{FK} , \quad (5.57)$$

is expanded in $1/c^2$, at lowest order this line-element reproduces the Newtonian one in the Eulerian frame, eq. (5.53).

On the other hand, the spatial scalar curvature ${}^{(3)}\mathcal{R}$ vanishes at lowest order in both frames. The conformal four-dimensional scalar curvature[4], ${}^{(4)}\mathcal{R} \equiv \mathcal{R}$ of the metric (5.53) is given by

$$\mathcal{R} = -2\partial^A \partial_A \varphi_g^{\mathcal{E}} \quad (5.58)$$

On the other hand, the conformal four-dimensional curvature in the synchronous and comoving gauge is given by

$$\mathcal{R} = 2\vartheta' + \vartheta^2 + \vartheta^\mu_\nu \vartheta^\nu_\mu + c^2\, {}^{(3)}\mathcal{R} \quad (5.59)$$

[4]We only give the conformal curvature here because, apart from the a^{-2} factor, the extra term in the physical curvature is simply the Einstein-de Sitter scalar curvature in both gauges

and, at lowest order in our $1/c^2$ expansion, the PN spatial curvature contributes to the four-dimensional $\mathcal{R}$

$$\mathcal{R} = 2\overline{\vartheta}' + \overline{\vartheta}^2 + \overline{\vartheta}^\mu_\nu \overline{\vartheta}^\nu_\mu + {}^{(3)}\mathcal{R}^{PN} \,. \tag{5.60}$$

In other words, at lowest order in our $1/c^2$ expansion, in the Eulerian frame, only the perturbation of the time-time component of the metric contributes to the scalar curvature $\mathcal{R}$, whereas in the Lagrangian frame we need also the spatial PN term coming from the spatial PN metric: setting ${}^{(3)}\mathcal{R} = 0$ in eq. (5.59) at the lowest order would be incorrect. This very fact is clear from the PN expansion which actually shows that the metric contributes at different orders to the four-dimensional curvature.
In order to find the correct four-dimensional curvature in the Lagrangian frame, one cannot use the metric in eq. (5.54), since the required PN part is missing. This means that, in order to obtain the same expression for the scalar curvature $\mathcal{R}$, at lowest order after the change of frame, we need to start from the weak-field metric in the Poisson gauge:

$$ds^2 = a^2 \left[-\left(1 + 2\frac{\varphi_g^{\mathcal{E}}}{c^2}\right) c^2 d\eta^2 + \left(1 - 2\frac{\varphi_g^{\mathcal{E}}}{c^2}\right) \delta_{AB} dx^A dx^B \right] , \tag{5.61}$$

where only the scalar PN mode is considered in the metric, since vector and tensor modes give higher-order contributions to $\mathcal{R}$, once transformed to the Lagrangian frame. The transformation to the Lagrangian frame finally leads to

$$ds^2 = a^2 \left[-c^2 d\tau^2 + \left(1 + \frac{\chi}{c^2}\right) \mathcal{J}^A_\alpha \mathcal{J}^B_\beta \delta_{AB} dq^\alpha dq^\beta \right] , \tag{5.62}$$

where, see Matarrese & Terranova 1996 and the next chapter,

$$\chi = 2\mathcal{H}\xi^0_{\mathcal{L}} - 2\varphi^{\mathcal{L}}_g - \Upsilon^{\mathcal{L}}. \tag{5.63}$$

In the latter expression, the potential $\Upsilon^{\mathcal{L}}$ is given by

$$\overline{\mathcal{D}}^\sigma \overline{\mathcal{D}}_\sigma \Upsilon^{\mathcal{L}} = -\frac{1}{2}\left(\overline{\vartheta}^2 - \overline{\vartheta}^\mu_\nu \overline{\vartheta}^\nu_\mu\right) , \tag{5.64}$$

and the peculiar velocity-gradient tensor is written in terms of our solution $\xi^0_{\mathcal{L}}$, eq. (5.56), as $\overline{\vartheta}^\mu_\nu = \overline{\mathcal{D}}^\mu \overline{\mathcal{D}}_\nu \xi^0_{\mathcal{L}}$. The PN scalar mode χ comes from the transformation of the time coordinate, keeping only the scalar contributions in the PN spatial metric, see the next chapter.

Following this procedure, starting from the scalar curvature $\mathcal{R} = -2\partial^A\partial_A\varphi^{\mathcal{E}}_g$ in the Eulerian frame we arrive at the same expression $\mathcal{R} = -2\overline{\mathcal{D}}^\alpha \overline{\mathcal{D}}_\alpha \varphi^{\mathcal{L}}_g$ in the Lagrangian frame.

Let us now compare our approach with that of Matarrese et al. 1993 and Matarrese et al. 1994a. Both papers deal with relativistic dynamics and consider the parallel transport condition, eq. (5.8), which in the Lagrangian approach of the synchronous and comoving gauge becomes

$$\mathcal{J}^{A'}_\alpha = \vartheta^\sigma_\alpha \mathcal{J}^A_\sigma \,. \tag{5.65}$$

Then they solve the Einstein equations and find the velocity-gradient tensor[5] and the Jacobian matrix. Finally the Eulerian trajectories and peculiar velocity are obtained from

[5]In ref. Matarrese et al. (1993) the Einstein equations are integrated numerically in the special case of the so-called "silent Universe", see Matarrese et al. 1994b and Bruni et al. 1995, and solved analytically for the plane-parallel dynamics. In Matarrese et al. 1994a the calculation is performed at second order in relativistic perturbation theory.

$$dx^A = \mathcal{J}^A_\sigma dq^\sigma \; . \tag{5.66}$$

In this paper instead we have reconstructed the Lagrangian dynamics from the Eulerian fields in the Newtonian limit, thus our procedure actually goes in the opposite direction.

To conclude, let us emphasise once again that our transformation is different from the standard perturbation theory one. In standard perturbation theory, the background spatial transformation is simply $dx^A = \delta^A_\alpha dq^\alpha$; in a relativistic calculation, the same background transformation from the Eulerian to Lagrangian frame is simply the standard Lorentz transformation, with boost velocity equal to the first-order peculiar velocity, see Bertschinger & Hamilton 1994. Instead, in the Newtonian limit, we have performed a non-trivial spatial transformation from the Eulerian to the Lagrangian frame - the two frames moving with relative velocity v^A - and we have subsequently changed the parametrisation of the hyper-surfaces from the rest frame of Eulerian observers to the rest frame of the matter. This transformation is fully non-perturbative and will provide the background transformation for the same procedure in the PN approach.

CHAPTER 6

From the Eulerian to the Lagrangian frame II - PN approximation

In this chapter we obtain the PN metric in the Lagrangian frame by transforming from the Eulerian frame, the background transformation being the Newtonian one of the previous chapter.
A PN expansion in the synchronous and comoving gauge was performed in Matarrese & Terranova 1996. Our aim here is to obtain the metric by the aforementioned transformation and we will compare our result with that of Matarrese & Terranova 1996.

This work is currently in preparation for submission to *JCAP*, Villa et al. 2014.

6.1 The PN transformation from the Eulerian to the Lagrangian frame

In this section we provide the transformation from the Eulerian frame to the Lagrangian frame in the PN approximation.

First, we explain how our transformation is different from the gauge transformation in perturbation theory. The key point is that the gauge choice of the background spacetime do precede the decomposition of three-dimensional quantities into scalars, vectors, and tensors. This very fact is true when we deal with non-perturbative quantities as well as with perturbations with respect to some background, e.g. the standard perturbation theory or the PN expansion. To be more clear, we need a little step back. It is customary to decompose the perturbed metric tensor into its scalar, vector and tensor part. This denomination is due to the transformation behaviour of these quantities under a spatial coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^j)$ which leaves the time coordinate unchanged, $\tilde{t} = t$, just as our background transformation. The transformation rules are very well known: any scalar F is invariant

$$\tilde{F}(\tilde{x}^i, t) = F(x^i, t) \tag{6.1}$$

where F is the function evaluated in the coordinate system x^i and $\tilde{F}$ the function evaluated in coordinate system $\tilde{x}^i$ and e.g. the metric tensor transform as

$$\tilde{g}_{ab}\left(\tilde{x}^i, \tilde{t}\right) = \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}\left(x^i\left(\tilde{x}^j\right), t\right), \tag{6.2}$$

with $\tilde{t} = t$.

Moreover, we decompose any vector into a scalar - the curl-free or longitudinal part - and a divergence-less vector or transverse part and we decompose any tensor into scalars, vector - the divergence free part - and tensor - the divergence-free and traceless parts -, see equations (1.19) and (1.20). The properties "divergence-free" and "traceless" involve a metric and a derivative operator. In perturbation theory they are the background spatial metric and its covariant derivative.

Our background is the Newtonian limit in two different frames. The time coordinate is the same at the lowest order - the absolute Newtonian time - but the spatial coordinates are different. Moreover, in the Newtonian limit the three dimensional space is flat but the covariant derivative of the two spatial metrics δ_{AB} and $\overline{\gamma}_{\alpha\beta} = \mathcal{J}^A_\alpha \mathcal{J}^B_\beta \delta_{AB}$ is different. Therefore the decomposition of the PN perturbation of the metric in scalar, vector and tensor modes is different in the two frames[1]. This is also crucial for the form of the transformation of the time and spatial coordinates in the PN approximation. In our approach, scalars behave like scalars under the change in the spatial coordinates, i.e. they do not transform - we have $F_{\mathcal{L}}(q^\alpha, \tau) = F_{\mathcal{E}}(x^A(q^\beta), \tau)$ as in (6.1) - but transform along the generators of the time coordinate according to the gauge transformation rule for the scalars, eq. (2.13). Obviously, if we expand our transformation in perturbation theory, i.e. with respect to the same background coordinates, the results have to coincide. We have shown that this is verified for the Newtonian transformation in section 5.2 and will explicitly prove this up to second-order also for the PN transformation in section 6.3.

6.1.1 The transformation of the spatial coordinates

We write the spatial PN transformation as

$$x^A = q^A + S^A(q^\alpha, \tau) + \frac{1}{c^2}\xi^A_{\mathcal{E}}(q^A + S^A, \tau)\,. \tag{6.3}$$

The PN displacement vector ξ^A is a three dimensional vector in the hypersurface of constant absolute time η, (with $\eta = \tau$) which is the rest frame of the Newtonian Eulerian observers and it is a function of the Eulerian coordinates $q^A + S^A$. Consequently, it transforms as a vector when we change from Eulerian to Lagrangian spatial coordinates, i.e. $\xi^A_{\mathcal{E}} = \mathcal{J}^A_\alpha \xi^\alpha_{\mathcal{L}}$. Therefore the spatial transformation becomes

$$x^A(q^\alpha, \tau) = q^A + S^A(q^\alpha, \tau) + \frac{1}{c^2}\mathcal{J}^A_\alpha \xi^\alpha_{\mathcal{L}}(q)(q^\alpha, \tau)\,, \tag{6.4}$$

where all the functions depend on the Lagrangian coordinates q^α and on the absolute Newtonian time $\eta = \tau$.
We also need in the following the decomposition of the vector $\xi^A_{\mathcal{E}}$ in its longitudinal and transverse part. In the Eulerian frame it reads

$$\xi^A_{\mathcal{E}} = \partial^A \xi_{\mathcal{E}} + \xi^{A\perp}_{\mathcal{E}} \tag{6.5}$$

with $\partial_A \xi^{A\perp}_{\mathcal{E}} = 0$ and in the Lagrangian frame the decomposition becomes

$$\xi^\alpha_{\mathcal{L}} = \partial^\alpha \xi_{\mathcal{L}} + \xi^{\alpha\perp}_{\mathcal{L}} \tag{6.6}$$

with $\partial_\alpha \xi^{\alpha\perp}_{\mathcal{L}} = 0$.

[1]For the same reason, when we expand a second spatial derivative of a scalar in the Lagrangian frame, $\mathcal{D}_\alpha \mathcal{D}_\beta f$ at second order in perturbation theory we find a mode mixing, as noted above.

6.1.2 The transformation of the time coordinate

The background time is the same Newtonian absolute time in both frames, but the functions in the transformation depend on the spatial coordinate also. It turns out that to obtain the spatial metric in synchronous and comoving gauge up to $1/c^2$ we need the time transformation up to $1/c^4$, which is given by

$$\eta(q^\alpha,\tau) = \tau + \frac{1}{c^2}\xi^0_{\mathcal{E}}(x^A,\tau) + \frac{1}{2c^4}\left(\partial_c\xi^0_{\mathcal{E}}\xi^c_{\mathcal{E}} + \zeta^0_{\mathcal{E}}\right)(q^\alpha,\tau) \tag{6.7}$$

as a second-order gauge transformation, where we have to change the spatial coordinates. The functions $\xi^0_{\mathcal{E}}$ and $\zeta^0_{\mathcal{E}}$ transform as scalars, for the spatial derivative we have $\partial_A = \mathcal{J}^\alpha_A\partial_\alpha$ and $\xi^A_{\mathcal{E}} = \mathcal{J}^A_\alpha\xi^\alpha_{\mathcal{L}}$ for the vector. The result is

$$\begin{aligned}\eta(q^\alpha,\tau) &= \tau + \frac{1}{c^2}\xi^0_{\mathcal{L}}(q^\alpha,\tau) + \frac{1}{2c^4}\left(\partial_c\xi^0_{\mathcal{L}}\xi^c_{\mathcal{L}} + \zeta^0_{\mathcal{L}}\right)(q^\alpha,\tau)\\ &= \tau + \frac{1}{c^2}\xi^0_{\mathcal{L}}(q^\alpha,\tau) + \frac{1}{2c^4}\Psi^0_{\mathcal{L}}(q^\alpha,\tau)\,,\end{aligned} \tag{6.8}$$

where here and in the following $\partial_c = (\partial_\tau,\partial_\alpha)$. All the functions depend on the Lagrangian coordinates q^α and on the absolute Newtonian time $\eta = \tau$.

6.2 Four-dimensional frame transformation of the metric

We want to obtain the spatial PN metric in synchronous and comoving gauge from the metric in Poisson gauge. The metric transforms as[2]

$$g^{\mathcal{L}}_{ab}(q^c) = \frac{\partial x^d}{\partial q^a}\frac{\partial x^f}{\partial q^b}g^{\mathcal{E}}_{df}(x^d(q^c))\,. \tag{6.9}$$

The quantities at the r.h.s are the four dimensional Jacobian matrix and the metric in the Poisson gauge and they have to be expanded up to the correct order to obtain the spatial l.h.s up to $1/c^2$.
We write the four dimensional coordinate transformation as

$$\begin{aligned}x^a(q^\alpha,\tau) &= x^a + S^a + \frac{1}{c^2}\mathcal{A}^a_b\xi^b_{\mathcal{L}}(q^\alpha,\tau) + \frac{1}{2c^4}\left(\partial_c\left(\mathcal{A}^a_b\xi^b_{\mathcal{L}}\right)\xi^c_{\mathcal{L}} + \mathcal{A}^a_b\zeta^b_{\mathcal{L}}\right)(q^\alpha,\tau)\\ &= \overline{x}^a + \frac{1}{c^2}\mathcal{A}^a_b\xi^b_{\mathcal{L}}(q^\alpha,\tau) + \frac{1}{2c^4}\Psi^a_{\mathcal{L}}(q^\alpha,\tau)\,,\end{aligned} \tag{6.10}$$

which for $a=0$ and $a=A$ reduces to eq. (6.8) and eq. (6.4) retaining only the PN term, respectively. With $\overline{x}^a$ we denote the space-time Newtonian Eulerian coordinates $\left(x^A,\tau\right)$ and $\mathcal{A}^a_b$ indicates our background transformation, (5.29)

$$\mathcal{A}^a_b = \begin{pmatrix} 1 & 0 \\ & \\ v^A & \mathcal{J}^A_\alpha \end{pmatrix}. \tag{6.11}$$

The four-dimensional Jacobian matrix of the transformation (6.10) is, up to $1/c^4$, given by

$$\frac{\partial x^a}{\partial q^b} = \mathcal{A}^a_b + \frac{1}{c^2}\frac{\partial\left(\mathcal{A}^a_c\xi^c_{\mathcal{L}}\right)}{\partial q^b} + \frac{1}{2c^4}\left[\frac{\partial\left(\partial_c\left(\mathcal{A}^a_f\xi^f_{\mathcal{L}}\right)\xi^c_{\mathcal{L}} + \mathcal{A}^a_f\zeta^f_{\mathcal{L}}\right)}{\partial q^b}\right]. \tag{6.12}$$

[2]Note that we are adopting the passive point of view and both sides of this equation are evaluated at the same physical point.

For the transformation of the metric tensor we need the product $\frac{\partial x^d}{\partial q^a}\frac{\partial x^f}{\partial q^b}$ which up to $1/c^4$ reads

$$
\begin{aligned}
\frac{\partial x^c}{\partial q^a}\frac{\partial x^d}{\partial q^b} &= \mathcal{A}^c_a\mathcal{A}^d_b + \frac{1}{c^2}\left[\frac{\partial\left(\mathcal{A}^c_f\xi^f_{\mathcal{L}}\right)}{\partial q^a}\mathcal{A}^d_b + \mathcal{A}^c_a\frac{\partial\left(\mathcal{A}^d_f\xi^f_{\mathcal{L}}\right)}{\partial q^b}\right] + \\
&+ \frac{1}{2c^4}\left\{2\frac{\partial\left(\mathcal{A}^c_f\xi^f_{\mathcal{L}}\right)}{\partial q^a}\frac{\partial\left(\mathcal{A}^d_f\xi^f_{\mathcal{L}}\right)}{\partial q^b} + \mathcal{A}^c_a\left[\frac{\partial\left(\partial_n\left(\mathcal{A}^d_f\xi^f_{\mathcal{L}}\right)\xi^n_{\mathcal{L}} + \mathcal{A}^d_f\zeta^f_{\mathcal{L}}\right)}{\partial q^b}\right] + \right.\\
&+ \left.\left[\frac{\partial\left(\partial_n\left(\mathcal{J}^c_f\xi^f_{\mathcal{L}}\right)\xi^n_{\mathcal{L}} + \mathcal{J}^c_f\zeta^f_{\mathcal{L}}\right)}{\partial q^a}\right]\mathcal{A}^d_b\right\},
\end{aligned}
\tag{6.13}
$$

in compact form

$$
\frac{\partial x^c}{\partial q^a}\frac{\partial x^d}{\partial q^b} = \mathcal{A}^c_a\mathcal{A}^d_b + \frac{1}{c^2}\mathcal{E}^{cd}_{ab} + \frac{1}{2c^4}\mathcal{G}^{cd}_{ab}\,. \tag{6.14}
$$

The last step at the r.h.s of eq. (6.9) is to expand the components $g^{\mathcal{E}}_{df}(x^d(q^c))$: we have to expand the components $g^{\mathcal{E}}_{df}$ up to $1/c^4$ and the argument $(x^d(q^c))$ up to $1/c^4$ separately and finally collect all the terms $\mathcal{O}\left(1/c^2\right)$ and $\mathcal{O}\left(1/c^4\right)$. Then first consider a scalar function $F(x^d(q^c))$ and expand its argument with respect to

$$
x^a(q^\alpha,\tau) = x^a + S^a + \frac{1}{c^2}\mathcal{A}^a_b\xi^b_{\mathcal{L}}(q^\alpha,\tau) + \frac{1}{2c^4}\left(\partial_c\left(\mathcal{A}^a_b\xi^b_{\mathcal{L}}\right)\xi^c_{\mathcal{L}} + \mathcal{A}^a_b\zeta^b_{\mathcal{L}}\right)(q^\alpha,\tau)\,. \tag{6.15}
$$

The result is

$$
\begin{aligned}
F(x^a) &= F(\overline{x}^a) + \frac{1}{c^2}\left(\left.\frac{\partial F}{\partial x^c}\right|_{x=\overline{x}}\mathcal{A}^c_b\xi^b_{\mathcal{L}}\right) + \\
&+ \frac{1}{2c^4}\left[\left.\frac{\partial^2 F}{\partial x^e\partial x^f}\right|_{x=\overline{x}}\mathcal{A}^e_n\xi^n_{\mathcal{L}}\mathcal{A}^f_p\xi^p_{\mathcal{L}} + \left.\frac{\partial F}{\partial x^f}\right|_{x=\overline{x}}\left(\partial_c\left(\mathcal{A}^f_b\xi^b_{\mathcal{L}}\right)\xi^c_{\mathcal{L}} + \mathcal{A}^f_b\zeta^b_{\mathcal{L}}\right)\right].
\end{aligned}
\tag{6.16}
$$

Now expanding the function itself

$$
F(x^a) = F^N(x^a) + \frac{1}{c^2}F^{PN}(x^a) + \frac{1}{2c^4}F^{PPN}(x^a) \tag{6.17}
$$

and applying (6.16) to each term we finally obtain

$$
\begin{aligned}
F(\overline{x}^a) &= F^N(\overline{x}^a) + \frac{1}{c^2}\left(\left.\frac{\partial F^N}{\partial x^c}\right|_{x=\overline{x}}\mathcal{A}^c_b\xi^b_{\mathcal{L}} + F^{PN}(\overline{x}^a)\right) + \frac{1}{2c^4}\left[2\left.\frac{\partial F^N}{\partial x^c}\right|_{x=\overline{x}}\mathcal{A}^c_b\xi^b_{\mathcal{L}} + \right.\\
&\left. + F^{PPN}(\overline{x}^a) + \left.\frac{\partial^2 F^N}{\partial x^e\partial x^f}\right|_{x=\overline{x}}\mathcal{A}^e_n\xi^n_{\mathcal{L}}\mathcal{A}^f_p\xi^p_{\mathcal{L}} + \left.\frac{\partial F^N}{\partial x^f}\right|_{x=\overline{x}}\left(\partial_c\left(\mathcal{A}^f_b\xi^b_{\mathcal{L}}\right)\xi^c_{\mathcal{L}} + \mathcal{A}^f_b\zeta^b_{\mathcal{L}}\right)\right].
\end{aligned}
\tag{6.18}
$$

Collecting all the terms at the required order gives the transformation equations in order to have the synchronous and comoving gauge spatial metric up to PN order. They are

$$
\begin{aligned}
g_{00}^{\mathcal{L}} &= -a^2c^2 + a^2\Big\{-2\mathcal{H}\xi^0_{\mathcal{L}} - 2\partial_\tau\xi^0_{\mathcal{L}} - 2\varphi_g + v^A v^B\delta_{AB} + \frac{1}{c^2}\Big[-\mathcal{H}\Psi^0_{\mathcal{L}} - \partial_\tau\Psi^0_{\mathcal{L}} - V^{\mathcal{L}} + \\
&\quad +2P^{\mathcal{L}}_B v^B - 4\mathcal{H}\xi^0_{\mathcal{L}}\varphi_g - 2\Big(\partial_A\varphi_g\left(\mathcal{J}^B_\omega\xi^\omega_{\mathcal{L}}\right) + \partial_\tau\varphi_g\xi^0_{\mathcal{L}}\Big) + v^A\partial_\tau\xi^B_{\mathcal{L}}\delta_{AB} - \Big(\partial_\tau\xi^0_{\mathcal{L}}\Big)^2 + \\
&\quad -\frac{3}{2}\mathcal{H}^2\Big(\xi^0_{\mathcal{L}}\Big)^2 + v^A v^B\delta_{AB}\Big(-2\varphi_g + 2\mathcal{H}\xi^0_{\mathcal{L}}\Big) + 2\partial_\tau\xi^0_{\mathcal{L}}\Big(-2\varphi_g - 2\mathcal{H}\xi^0_{\mathcal{L}}\Big)\Big]\Big\} \qquad (6.19)
\end{aligned}
$$

$$
\begin{aligned}
g_{0\alpha}^{\mathcal{L}} &= v^A\mathcal{J}^K_\alpha\delta_{AK} - \partial_\alpha\xi^0_{\mathcal{L}} + \frac{1}{c^2}\Big[\mathcal{J}^K_\alpha P^{\mathcal{L}}_K - \partial_\tau\xi^0_{\mathcal{L}}\partial_\alpha\xi^0_{\mathcal{L}} - 4\varphi_g\partial_\alpha\xi^0_{\mathcal{L}} - \frac{1}{2}\partial_\alpha\Psi^0_{\mathcal{L}} + \\
&\quad +\Big(v^A\partial_\alpha\left(\mathcal{J}^B_\omega\xi^\omega_{\mathcal{L}}\right) + \mathcal{J}^A_\alpha\partial_\tau\left(\mathcal{J}^B_\omega\xi^\omega_{\mathcal{L}}\right)\Big)\delta_{AB}\Big] \qquad (6.20)
\end{aligned}
$$

$$
\begin{aligned}
g_{\alpha\beta}^{\mathcal{L}} &= a^2\Big(\overline{\gamma}_{\alpha\beta} + \frac{1}{c^2}w_{\alpha\beta}\Big) = a^2\Big\{\mathcal{J}^A_\alpha\mathcal{J}^B_\beta\delta_{AB} + \frac{1}{c^2}\Big[\Big(-2\varphi_g + 2\mathcal{H}\xi^0_{\mathcal{L}}\Big)\mathcal{J}^A_\alpha\mathcal{J}^B_\beta\delta_{AB} + \\
&\quad -\partial_\alpha\xi^0_{\mathcal{L}}\partial_\beta\xi^0_{\mathcal{L}} + \Big(\mathcal{J}^A_\alpha\partial_\beta\left(\mathcal{J}^B_\omega\xi^\omega_{\mathcal{L}}\right) + \partial_\alpha\left(\mathcal{J}^A_\omega\xi^\omega_{\mathcal{L}}\right)\mathcal{J}^B_\beta\Big)\delta_{AB}\Big]\Big\} \qquad (6.21)
\end{aligned}
$$

In these expressions, $\tau = \eta$ is the Newtonian absolute time, V is the perturbation in g_{00} of the order of $1/c^2$ and P_K is the perturbation in g_{0K} of the order of $1/c^2$ in the Poisson gauge. Now we impose the synchronous-comoving gauge conditions at the l.h.s, $g^{\mathcal{L}}_{00} = -a^2c^2$. and $g^{\mathcal{L}}_{0\alpha} = 0$. At the Newtonian order, we find the equations of the previous chapter.
At the PN order, we use (5.39) and (5.25) to simplify our equations for the time-time and time-space components of the metric. We find

$$
\begin{aligned}
\frac{1}{a}\partial_\tau\Big(a\Psi^0_{\mathcal{L}}\Big) &= -2\partial_\sigma\varphi^{\mathcal{L}}_g\xi^\sigma_{\mathcal{L}} - V^{\mathcal{L}} + 2P_Av^A + \frac{3}{2}\mathcal{H}^2\xi^{0\,2}_{\mathcal{L}} + 3\Big(\varphi^{\mathcal{L}}_g\Big)^2 \qquad (6.22)\\
&\quad +2\mathcal{H}\xi^0_{\mathcal{L}}\varphi^{\mathcal{L}}_g - 3\varphi^{\mathcal{L}}_g v^2 + \frac{1}{4}v^4 + \mathcal{H}\xi^0_{\mathcal{L}}v^2 + \partial_\omega\xi^0_{\mathcal{L}}\overline{\mathcal{D}}_0\xi^\omega_{\mathcal{L}}
\end{aligned}
$$

and

$$
\overline{\mathcal{D}}_0\xi_{\alpha\,\mathcal{L}} + \partial_\sigma\xi^0_{\mathcal{L}}\overline{\mathcal{D}}_\alpha\xi^\sigma_{\mathcal{L}} + \mathcal{J}^K_\alpha P_K - \partial_\alpha\xi^0_{\mathcal{L}}\partial_\tau\xi^0_{\mathcal{L}} - \frac{1}{2}\partial_\alpha\Psi^0_{\mathcal{L}} - 4\varphi^{\mathcal{L}}_g\partial_\alpha\xi^0_{\mathcal{L}} = 0\,, \qquad (6.23)
$$

where $\overline{\mathcal{D}}_0$ and $\overline{\mathcal{D}}_\alpha$ are the Newtonian covariant derivatives in Lagrangian space, i.e.

$$
\overline{\mathcal{D}}_0\xi^\omega_{\mathcal{L}} = \partial_\tau\xi^\omega_{\mathcal{L}} + \overline{\Gamma}^\omega_{\sigma 0}\xi^\sigma_{\mathcal{L}} = \partial_\tau\xi^\omega_{\mathcal{L}} + \overline{\vartheta}^\omega_\sigma\xi^\sigma_{\mathcal{L}}
$$

and

$$
\overline{\mathcal{D}}_\alpha\xi^\omega_{\mathcal{L}} = \partial_\alpha\xi^\omega_{\mathcal{L}} + \overline{\Gamma}^\omega_{\alpha\sigma}\xi^\sigma_{\mathcal{L}} = \partial_\alpha\xi^\omega_{\mathcal{L}} + \partial_\alpha\mathcal{J}^K_\sigma\mathcal{J}^\omega_K\xi^\sigma_{\mathcal{L}}\,.
$$

Finally, the equation for the PN perturbation of the spatial metric in synchronous and comoving gauge is

$$
w_{\alpha\beta} = \overline{\gamma}_{\alpha\beta}\left(-2\varphi^{\mathcal{L}}_g + 2\mathcal{H}\xi^0_{\mathcal{L}}\right) + 2\overline{D}_\alpha\overline{D}_\beta\xi_{\mathcal{L}} + \overline{D}_\alpha\xi^\perp_\beta + \overline{D}_\beta\xi^\perp_\alpha - \partial_\alpha\xi^0_{\mathcal{L}}\partial_\beta\xi^0_{\mathcal{L}}\,, \qquad (6.24)
$$

where we have split the PN Lagrangian displacement vector ξ_α into its scalar part $\xi_{\mathcal{L}}$ and vector part $\xi^{\perp}_{\alpha}$.

Equations (6.22)-(6.24) determine the PN spatial Lagrangian metric $w_{\alpha\beta}$ and the functions $\Psi^0_{\mathcal{L}}$ and $\xi^\alpha_{\mathcal{L}}$ in the coordinate transformation: in order to find $w_{\alpha\beta}$ we need $\xi^0_{\mathcal{L}}$, which we found from the Newtonian transformation, and $\xi^\alpha_{\mathcal{L}}$, which is found solving the system of the four coupled equations (6.22)-(6.24) for $\Psi^0_{\mathcal{L}}$ and $\xi^\alpha_{\mathcal{L}}$.

In the equation for the spatial components (6.24) we want to identify the scalar, vector and tensor modes of the metric. In synchronous and comoving gauge the decomposition of the PN perturbation is

$$w_{\alpha\beta} = \chi\overline{\gamma}_{\alpha\beta} + \overline{\mathcal{D}}_\alpha\overline{\mathcal{D}}_\beta\zeta + \frac{1}{2}\left(\overline{\mathcal{D}}_\alpha\lambda_\beta + \overline{\mathcal{D}}_\beta\lambda_\alpha\right) + \pi_{\alpha\beta} \tag{6.25}$$

where the vector is divergence-less ($\overline{\mathcal{D}}^\alpha\lambda_\alpha = 0$) and the tensors are divergence-less and trace-free ($\overline{\mathcal{D}}^\alpha\pi_{\alpha\beta} = 0$ and $\pi^\alpha_\alpha = 0$). We begin taking the trace of the equations (6.24) and (6.25). This gives

$$3\chi + \overline{\mathcal{D}}^2\zeta = -6\varphi^{\mathcal{L}}_g + 6\mathcal{H}\xi^0_{\mathcal{L}} + 2\overline{\mathcal{D}}^2\xi_{\mathcal{L}} - \partial_\alpha\xi^0_{\mathcal{L}}\partial_\beta\xi^0_{\mathcal{L}}\overline{\gamma}^{\alpha\beta}\,. \tag{6.26}$$

Then we apply the operator $\mathcal{D}^\alpha\mathcal{D}^\beta$ to equations (6.24) and (6.25) and using eq. (5.38) we obtain this second equation

$$\overline{\mathcal{D}}^2\chi + \overline{\mathcal{D}}^2\overline{\mathcal{D}}^2\zeta = -2\overline{\mathcal{D}}^2\varphi^{\mathcal{L}}_g + 2\mathcal{H}\overline{\mathcal{D}}^2\xi^0_{\mathcal{L}} + 2\overline{\mathcal{D}}^2\overline{\mathcal{D}}^2\xi_{\mathcal{L}} - \overline{\vartheta}^2 - \overline{\vartheta}^\mu_\nu\overline{\vartheta}^\nu_\mu - 2\partial_\rho\overline{\vartheta}\overline{\gamma}^{\sigma\rho}\partial_\sigma\xi^0_{\mathcal{L}}\,. \tag{6.27}$$

By solving the two above equation we find that the scalars χ and ζ are given by

$$\chi = 2\mathcal{H}\xi^0_{\mathcal{L}} - 2\varphi^{\mathcal{L}}_g - \Upsilon_{\mathcal{L}} \tag{6.28}$$

and

$$\zeta = 2\xi_{\mathcal{L}} + 3\mathcal{Z}_{\mathcal{L}} \tag{6.29}$$

where

$$\overline{\mathcal{D}}^2\Upsilon_{\mathcal{L}} = -\frac{1}{2}\left(\overline{\vartheta}^2 - \overline{\vartheta}^\mu_\nu\overline{\vartheta}^\nu_\mu\right), \tag{6.30}$$

and

$$\overline{\mathcal{D}}^2\mathcal{Z}_{\mathcal{L}} = \Upsilon_{\mathcal{L}} - \frac{1}{3}\left(\nabla_q\xi^0_{\mathcal{L}}\right)^2\,. \tag{6.31}$$

In order to identify the vector modes, we take the divergence of the equations (6.24) and (6.25) and substitute the expressions for the scalars. We find

$$\overline{\mathcal{D}}^2\lambda_\beta = 2\overline{\mathcal{D}}^2\xi^\perp_\beta - 4\partial_\beta\Upsilon_{\mathcal{L}} - 2\partial_\beta\xi^0_{\mathcal{L}}\overline{\vartheta} + 2\partial_\sigma\xi^0_{\mathcal{L}}\overline{\vartheta}^\sigma_\beta\,. \tag{6.32}$$

Finally, applying the operator $\overline{\mathcal{D}}^2$ to equations (6.24) and (6.25) and substituting the expressions for the scalars and vector gives the equation for the tensor mode

$$\overline{\mathcal{D}}^2\pi_{\alpha\beta} = \overline{\mathcal{D}}_\alpha\overline{\mathcal{D}}_\beta\Upsilon_{\mathcal{L}} + \overline{\mathcal{D}}^2\Upsilon_{\mathcal{L}}\overline{\gamma}_{\alpha\beta} + 2\left(\overline{\vartheta}\overline{\vartheta}_{\alpha\beta} - \overline{\vartheta}_{\alpha\mu}\overline{\vartheta}^\mu_\beta\right)\,. \tag{6.33}$$

Equations (6.28) and (6.33) are identical to the equations found in Matarrese & Terranova 1996 from the PN approximation of the energy constraint and the trace-less evolution equation in synchronous and comoving gauge. In Matarrese & Terranova 1996 the equations for the remaining scalar mode and for the vector mode are obtained from the expansion of the Raychaudhuri equation and the momentum constraint and are very complicated. On the other hand, our expressions, eq. (6.29) and eq. (6.32), are quite simple and, most importantly, disentangle the contributions from the Newtonian dynamics (the terms involving the peculiar velocity-gradient tensor $\overline{\vartheta}^\alpha_\beta$) from the contributions from the spatial PN displacement vector.

6.3 Consistency up to second order in perturbation theory

In this section we show that our PN transformation, once expanded at second order in standard perturbation theory, reproduces the correct result for the metric in synchronous and comoving gauge. The procedure which we will follow is slightly different from that section 5.2, where we expanded the exact (with respect to perturbation theory) solution for $\xi^0_{\mathcal{L}}$ in eq. (5.35). The difference is that, up to now, we are not able to find the exact solutions of (6.22) and (6.23). Therefore here we expand these equations at second order in perturbation theory, then the solution is trivial since they are decoupled at second order. Finally we substitute the solutions in eq. (6.24) obtaining our expression for the PN perturbation of the spatial metric at second order. As expected, our expression turns out to be identical to the PN terms of the second order metric in synchronous and comoving gauge, see Matarrese et al. 1998 and Bartolo et al. 2005, retaining only PN terms.
We expand the coordinate transformation at second order as[3]

$$\begin{aligned} \eta &= \tau + \frac{1}{c^2}\left(\xi^{0(1)} + \xi^{0(2)}\right) + \frac{1}{2c^4}\Psi^{0(2)} && (6.34)\\ &= \tau + \frac{1}{c^2}\left(\xi^{0(1)} + \xi^{0(2)}\right) + \frac{1}{2c^4}\left(\partial_\tau \xi^{0(1)}\xi^{0(1)} + \zeta^{0(2)}\right) && (6.35)\end{aligned}$$

for the time transformation and

$$\begin{aligned} x^\alpha &= q^\alpha + \left(S^{\alpha(1)} + S^{\alpha(2)}\right) + \frac{1}{c^2}\xi^{\alpha(2)} && (6.36)\\ &= q^\alpha + \left(S^{\alpha(1)} + S^{\alpha(2)}\right) + \frac{1}{c^2}\left(\partial^\alpha \xi^{(2)} + \xi_\perp^{\alpha(2)}\right) && (6.37)\end{aligned}$$

for the spatial transformation. Note that both Ψ^0 and ξ^α start from second order, since the first order term in the time and spatial transformation is Newtonian only. In particular, for the PN displacement vector at second order we have $\xi^A = \delta^A_\alpha \xi^\alpha$, partial derivative with respect to q^α and covariant derivative $\overline{\mathcal{D}}_\alpha$ coincide and we can safely set $A = \alpha$.

6.3.1 The transformation of the time coordinate

Equation (6.22) at second-order is an equation for $\Psi^{0(2)}$ only and reads

$$\begin{aligned} \frac{1}{a}\partial_\tau\left(\Psi^{0(2)}\right) &= -V^{(2)} + 3\phi^2 + \frac{3}{2}\mathcal{H}\left(\xi^{0(1)}\right)^2 + 2\mathcal{H}\phi\xi^{0(1)} && (6.38)\\ &= \frac{1}{3}\phi^2 + \frac{10}{3}\left(a_{\rm nl} - 1\right)\phi^2 - 12\Theta \end{aligned}$$

where ϕ is the initial peculiar gravitational potential, Θ is defined in (B.5) and in the second line we have substituted the PN part of the second-order perturbation in the time-time component of the metric in the Poisson gauge, see eq. (B.26). The time integration gives

$$\Psi^{0(2)} = \frac{\tau}{9}\phi^2 + \frac{10}{9}\tau\left(a_{\rm nl} - 1\right)\phi^2 - 4\tau\Theta + C_1(q^\alpha)\,, \qquad (6.39)$$

where $C(q^\alpha)$ is the integration constant and the term $\frac{\tau}{9}\phi^2$ is the product $\partial_\tau\xi^{0(1)}\xi^{0(1)}$ in (6.35). We fix the constant $C_1(q^\alpha)$ exploiting the results of Matarrese et al. 1998: in this

[3] In this section we omit the superscript/subscript $\mathcal{L}$ for simplicity.

paper the coordinate transformation from the synchronous-comoving gauge to the Poisson gauge is found at second order in perturbation theory, with $a_{\rm nl} = 1$ in the initial conditions. After inverting our transformation, as explained in section 2.1.3, their results are consistent with ours for $a_{\rm nl} = 1$ and for $C_1(q^\alpha) = 0$. We therefore conclude that the integration constant vanishes at second order[4]. Our final expression for the transformation of the time coordinate is

$$\eta = \tau - \frac{1}{3}\tau\phi + \frac{\tau^3}{36}\partial^\sigma\phi\partial_\sigma\phi + \frac{\tau^3}{21}\Psi + \frac{5}{9}\tau^2\left(a_{\rm nl}-1\right)\phi^2 + \frac{\tau^2}{18}\phi^2 - 2\tau\Theta\,. \tag{6.40}$$

6.3.2 The transformation of the spatial coordinates

At second order equation (6.23) and reads

$$\partial_\tau\xi_\alpha = -P_K^{(2)}\delta_\alpha^K + \partial_\tau\xi^{0(1)}\partial_\alpha\xi^{0(1)} + \frac{1}{2}\partial_\alpha\Psi^{0(2)} + 4\phi\partial_\alpha\xi^{0(1)}\,, \tag{6.41}$$

where $P_K^{(2)}$ in the PN second-order perturbation in the time-space component in the Poisson gauge, see eq. (B.27). We now split this equation into its scalar and vector parts in the usual way. Taking the divergence gives for the scalar part

$$\begin{aligned}\partial_\tau\left(\nabla_q^2\xi^{(2)}\right) &= \partial^\alpha\left(\partial_\tau\xi^{0(1)}\partial_\alpha\xi^{0(1)}\right) + \frac{1}{2}\nabla_q^2\Psi^{0(2)} + 4\partial^\alpha\left(\phi\partial_\alpha\xi^{0(1)}\right) \\ &= \frac{10}{18}\tau\left(a_{\rm nl}-1\right)\nabla_q^2\left(\phi^2\right) - 2\tau\nabla_q^2\Theta - \frac{10}{18}\tau\nabla_q^2\left(\phi^2\right)\,.\end{aligned} \tag{6.42}$$

Time integration gives

$$\xi^{(2)} = \frac{5}{18}\tau^2\left(a_{\rm nl}-1\right)\phi^2 - \tau^2\Theta - \frac{10}{18}\tau^2\phi^2 + C_2(q^\alpha) \tag{6.43}$$

We can then subtract the scalar part from eq. (6.41) and are left with the equation for the vector part

$$\partial_\tau\xi_\alpha^{\perp(2)} = -P_\alpha^{(2)}\,, \tag{6.44}$$

whose solution is

$$\xi_\alpha^{\perp(2)} = \frac{2}{3}\tau^2\nabla_q^{-2}\left(2\partial_\alpha\Psi + \partial_\alpha\phi\nabla^2\phi - \partial^\sigma\partial_\alpha\phi\partial_\sigma\phi\right) + C_3(q^\alpha)\,, \tag{6.45}$$

where Ψ is defined (B.4). We compare again with Matarrese et al. 1998 and find that our result is consistent for $C_2(q^\alpha) = C_3(q^\alpha) = 0$ and $a_{\rm nl} = 1$. Our final expression for the spatial transformation up to second order is

$$x^\alpha = q^\alpha - \frac{\tau^2}{6}\partial^\alpha\phi + \frac{\tau^4}{84}\partial^\alpha\Psi_{\mathcal{L}} - \frac{5}{9}\phi\partial^\alpha\phi - \tau^2\partial^\alpha\Theta + \frac{2}{3}\tau^2\nabla^{-2}\left(2\partial^\alpha\Psi + \partial^\alpha\phi\nabla^2\phi - \partial^\sigma\partial^\alpha\phi\partial_\sigma\phi\right) \tag{6.46}$$

To conclude this two subsections, let us note that the second-order gauge transformation obtained here, equations (6.40) and (6.46) represent a little improvement of the analogous expressions in Matarrese et al. 1998, since here we show the dependence on $a_{\rm nl}$, which parametrises the primordial non-Gaussianity.

[4]Our expression is also consistent with the approach adopted in Kolb et al. 2005 with $a_{\rm nl} = 1$ and $C_1(q^\alpha) = 0$.

6.3.3 The PN transformed metric up to second order in synchronous-comoving gauge

Now we have calculated all the quantities that we need to find the PN perturbation in the spatial metric synchronous and comoving gauge up to second order. Here we just report our final result and we refer to appendix B for the second-order expansion of the PN scalar, vector and tensor modes. We find

$$w_{\alpha\beta} = -\frac{10}{3}\phi + \frac{5}{18}\tau^2 \left(\nabla\phi\right)^2 \delta_{ij} + \frac{10}{9}\tau^2 \left(a_{\rm nl} - 1\right) \left(\phi\partial_i\partial_j\phi + \partial_i\phi\partial_j\phi\right) - \frac{5}{9}\tau^2\partial_i\phi\partial_j\phi + \Delta_{ij} \,, \tag{6.47}$$

where Δ_{ij} are PN second-order tensors generated by first-order scalar and are given by

$$\Delta_{ij} = \frac{2}{3}\tau^2\Psi\delta_{ij} + \frac{2}{3}\tau^2\nabla^{-2}\left(\partial_i\partial_j\Psi + 2\partial_i\partial_j\phi\nabla^2\phi - 2\partial^k\partial_i\phi\partial_k\partial_j\phi\right) \,. \tag{6.48}$$

This expression for the metric is the same as that in Matarrese et al. 1998 and Bartolo et al. 2005[5], retaining the PN terms only. Let us remark that the metric in Matarrese et al. 1998 and Bartolo et al. 2005 is the second-order solution of the Einstein equations in synchronous and comoving gauge whereas our expression for the metric come out from the PN transformation from the Poisson gauge, successively expanded up to second order. The very fact that we have been able to recover the same expression should be considered as a further confirmation of our method.

[5]We consider the expression for the metric in Bartolo et al. 2005 which includes the dependence on $a_{\rm nl}$, supplemented by the explicit expression for the tensor modes given in Matarrese et al. 1998.

Part II

PN cosmological dynamics for plane-parallel perturbations and back-reaction

CHAPTER 7

PN cosmological dynamics for plane-parallel perturbations and back-reaction

We specialize the PN Lagrangian dynamics of chapter 4 to globally plane-parallel configurations, i.e. to the case where the initial perturbation field depends on a single coordinate. The leading order of our expansion, corresponding to the Newtonian background, is the Zel'dovich approximation, which, for plane-parallel perturbations in the Newtonian limit, represents an exact solution. This allows us to find the exact analytical form for the PN metric, thereby providing the PN extension of the Zel'dovich solution: this accounts for some relativistic effects, such as the non-Gaussianity of primordial perturbations. Our approach here is very different from those of Kasai 1995 and Russ et al. 1996, who proposed relativistic generalizations of the Zel'dovich approximation. Ref. Kasai 1995 introduced a relativistic, tetrad-based, perturbative approach, which is then solved to linear order and used to obtain non-perturbative expressions for the velocity-gradient tensor and mass density. The solution of Russ et al. 1996 is instead equivalent to a relativistic second-order perturbation theory treatment in the synchronous and comoving gauge, in which all quantities (metric, velocity-gradient tensor and mass density) are calculated at second order, thereby partially missing the non-perturbative character of the Zel'dovich approximation. Our approach instead aims at obtaining a non-perturbative description of both metric and fluid properties (velocity-gradient tensor and mass density), within the post-Newtonian approximation of General Relativity: our expansion in inverse powers of the speed of light is fully non-perturbative from the point of view of standard perturbation theory; thus our results contain all second and higher-order terms of standard perturbation theory calculations, as long as they are post-Newtonian and one deals with the plane-parallel dynamics. In our approximation scheme the Zel'dovich solution represents the Newtonian background over which PN corrections can be computed as small perturbations.

An application of our solution in the context of the back-reaction proposal is eventually given, providing a PN estimation of kinematical back-reaction, mean spatial curvature, average scale-factor and expansion rate.

The main results of this chapter are the subject of our paper, Villa et al. 2011.

7.1 Characterization of the Newtonian background

The starting point of our PN expansion is the Newtonian background described by the Zel'dovich approximation, with the peculiar gravitational potential depending on the conformal time and on the Lagrangian coordinate q_1 only. As it is well known, in the particular case of planar perturbations the Zel'dovich approximation yields an exact solution of the Newtonian equations. The *Zel'dovich solution* is given by

$$\begin{aligned} x_1 &= q_1 + D(\tau)\partial_1 \Phi_{\rm in}(q_1) \\ x_2 &= q_2 \\ x_3 &= q_3 \end{aligned} \tag{7.1}$$

where $\partial_1 = \partial/\partial q_1$, $D(\tau) \propto a(\tau) \propto \tau^2$ is the growing mode solution for the Einstein-de Sitter model and the potential $\Phi_{\rm in}$ is defined so that $\partial_1^2 \Phi_{\rm in} = -\delta_{\rm in}/D_{\rm in}$ and is related to the initial peculiar gravitational potential φ by the cosmological Poisson equation, yielding

$$\Phi_{\rm in} = -\frac{\varphi}{4\pi G a_{\rm in}^2 \varrho_{b,{\rm in}}} \,. \tag{7.2}$$

The Jacobian matrix is

$$\mathcal{J}^\alpha_\beta = \begin{pmatrix} 1 - \tau^2 \partial_1^2 \varphi/6 & & \\ & 1 & \\ & & 1 \end{pmatrix} . \tag{7.3}$$

For the following calculations, it is useful to define the function f

$$f \equiv D(\tau)\partial_1^2 \Phi_{\rm in} = -\frac{\tau^2}{6}\partial_1^2 \varphi \tag{7.4}$$

and the function $\eta \equiv \ln(1+f)$. Hereafter, the peculiar gravitational potential is meant to be evaluated at the initial time $\tau_{\rm in}$ and the subscript "in" is dropped for notational convenience.

The components of the metric in Lagrangian coordinates are given by (recall eq. (4.26))

$$\overline{\gamma}_{\alpha\beta} = \delta_{\sigma\omega}\left[\delta^\sigma_\alpha + D(\tau)\partial^\sigma \partial_\alpha \Phi\right]\left[\delta^\omega_\beta + D(\tau)\partial^\omega \partial_\beta \Phi\right] , \tag{7.5}$$

where we used the fact that at first order in the displacement vector covariant and partial derivatives with respect to the coordinates q_α coincide, since the Newtonian Christoffel symbols are second-order quantities. In our case $\Phi_{\rm in} = \Phi_{\rm in}(q_1)$ and for the Zel'dovich metric we find

$$\overline{\gamma}_{\alpha\beta} = \begin{pmatrix} (1 - \tau^2 \partial_1^2 \varphi/6)^2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{7.6}$$

or in more compact form

$$\overline{\gamma}_{\alpha\beta} = \begin{pmatrix} (1+f)^2 & & \\ & 1 & \\ & & 1 \end{pmatrix} . \tag{7.7}$$

It is important to keep in mind that this Newtonian metric is non-linear with respect to the peculiar gravitational potential, thus it characterizes the mildly non-linear stage of the gravitational instability. Starting from this metric at first order in the displacement vector, all the other dynamical variables are calculated exactly. The only non-vanishing component of the peculiar velocity-gradient tensor is $\overline{\vartheta}^1_1 = \eta'$ and for the shear tensor we have $\overline{\sigma}^1_1 = 2\eta'/3$ and $\overline{\sigma}^2_2 = \overline{\sigma}^3_3 = -\overline{\sigma}^1_1/2$.
Finally the density contrast in equation (4.25) takes the form

$$\overline{\delta} = \frac{1}{1+f} - 1 \,. \tag{7.8}$$

7.2 The PN expansion

For the PN expansion, we write the metric in the form

$$\gamma_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + \frac{1}{c^2} w_{\alpha\beta} \,. \tag{7.9}$$

For consistency with our Newtonian background solution, in which the peculiar gravitational potential depends only on q_1, the PN perturbation $w_{\alpha\beta}$ can be assumed to depend on the conformal time and on the Lagrangian coordinate q_1 only.

A clue for the form of the perturbation $w_{\alpha\beta}$ follows from the initial conditions of cosmological perturbations. Even though our case of globally planar dynamics is purely a toy-model we prefer to assume that our initial perturbation seed is consistent with having been generated during Inflation in the early Universe, so that our analytical results can give us a hint of what happens in the real Universe, where the initial perturbation seed is a random field that depends on all the spatial coordinates. We then set our initial conditions at the end of inflation, effectively coinciding with $\tau_{\rm in} = 0$. Considering only scalar perturbations from the Einstein-de Sitter Universe, we have at early times

$$\gamma_{\alpha\beta} = \left(1 - \frac{10}{3c^2}\varphi\right)\delta_{\alpha\beta} - \frac{\tau^2}{3}\partial_\alpha\partial_\beta\varphi \,. \tag{7.10}$$

We can use the residual gauge freedom of the synchronous and comoving gauge to set the Newtonian perturbation to zero, as in Matarrese & Terranova 1996. The initial metric perturbation is therefore given by a diagonal PN part. Thus, for the PN perturbation at initial time we have

$$w^{\alpha}_{\beta \rm in} = -\frac{10}{3}\varphi\delta^{\alpha}_{\beta} \,. \tag{7.11}$$

Starting from these initial conditions, we can assume that the evolution does not switch on the off-diagonal components of the PN metric, i.e. that w^α_β with $\alpha \neq \beta$ vanish at any time. This assumption derives from the physical picture of our one-dimensional dynamics. The mass distribution whose self-gravity generates a one-dimensional potential is made of parallel sheets of matter. For every point $\mathbf{q}$ the peculiar velocity has the same direction as the spatial derivative of the peculiar potential, thus the matter moves only in the direction perpendicular to the sheets. The collapse of this structure cannot involve tensor perturbations, which would lead to the emission of gravitational waves, because it cannot undergo any alteration of its shape. However, there is surely a scalar trace part in the PN metric, arising from our inflationary initial conditions. In addition, because of the asymmetry in the q_1 spatial direction, the function w^1_1 is assumed to differ from w^2_2 and w^3_3, whereas the latter functions can only be equal.

Therefore, the PN expansion is performed according to the following ansatz for the metric[1]

$$\gamma_{11} = (1+f)^2 + \frac{1}{c^2}(1+f)^2 g \tag{7.12}$$

$$\gamma_{22} = 1 + \frac{1}{c^2} h \tag{7.13}$$

$$\gamma_{33} = 1 + \frac{1}{c^2} h \,, \tag{7.14}$$

[1] The indices of the perturbation are lowered (raised) with the background metric $\overline{\gamma}_{\alpha\beta}$ ($\overline{\gamma}^{\alpha\beta}$).

with initial conditions

$$g_{\rm in} = h_{\rm in} = -\frac{10}{3}\varphi\,. \tag{7.15}$$

The PN expansion of the momentum constraint and of the Raychaudhuri equation gives

$$\begin{aligned} \eta'\partial_1 h &= (\partial_1 h)' && (7.16)\\ g'' + 2h'' + \frac{2}{\tau}(g'+2h') + 2\eta' g' &= -\frac{6}{\tau^2}\frac{(g_{\rm in}+2h_{\rm in})-(g+2h)}{1+f}\,. && (7.17)\end{aligned}$$

We also have the energy constraint and evolution equation, connecting the PN scalar curvature with Newtonian kinematical quantities:

$$\begin{aligned} \frac{8}{\tau}\eta' + \frac{1}{(1+f)^2}\left(-2\partial_1^2 h + 2\partial_1\eta\partial_1 h\right) &= \frac{24}{\tau^2}\left(\frac{1}{1+f}-1\right) && (7.18)\\ \eta'' + \frac{4}{\tau}\eta' + \frac{1}{4}\frac{1}{(1+f)^2}\left(-2\partial_1^2 h + 2\partial_1\eta\partial_1 h\right) &= 0\,. && (7.19)\end{aligned}$$

The momentum constraint is an equation for the spatial derivative of the function h. Setting $m \equiv \partial_1 h$, it reads

$$\eta' m = m'\,, \tag{7.20}$$

with initial condition $m_{\rm in} = (-10/3)\partial_1\varphi$. The solution reads

$$m = -\frac{10}{3}\partial_1\varphi\left(1-\frac{\tau^2}{6}\partial_1^2\varphi\right). \tag{7.21}$$

Then, by spatial integration we obtain

$$h = -\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2 + C_0(\tau)\,, \tag{7.22}$$

where the homogeneous mode $C_0(\tau)$ is a time-dependent constant of integration (w.r.t. q_1). We can use the Newtonian evolution equation and energy constraint to check the consistency of this solution: substitutions of (7.22) in (7.18) and (7.19) leads to the identity.

Note that the initial condition for the function h sets $C_0(\tau_{\rm in}) = 0$. In addition, the function $C_0(\tau)$ is an additive term in the perturbation h which would modify the background dynamics even in the absence of any initial perturbation (i.e. for $\varphi = 0$). Therefore, for consistency, we set $C_0(\tau) = 0$ for all times.

The Raychaudhuri equation becomes an equation for the function g only, whose solution reads

$$g = -\frac{10}{3}\varphi + \frac{\tau^2 C_2}{5\left(-6+\tau^2\partial_1^2\varphi\right)} + \frac{C_1}{t^3\left(-6+\tau^2\partial_1^2\varphi\right)} - \frac{5\tau^4\partial_1^2\varphi(\partial_1\varphi)^2}{21\left(-6+\tau^2\partial_1^2\varphi\right)}\,, \tag{7.23}$$

where C_1 and C_2 are integration constants. Consistency with our initial conditions, eq. (7.15), requires $C_1 = 0$.

In conclusion, the PN metric reads

$$
\begin{aligned}
\gamma_{11} &= \left(1-\frac{\tau^2}{6}\partial_1^2\varphi\right)^2 + \\
&+ \frac{1}{c^2}\left(-6+\tau^2\partial_1^2\varphi\right)\left(\frac{21\tau^2 C_2 - 25\tau^4\partial_1^2\varphi(\partial_1\varphi)^2 - 350\varphi\left(-6+\tau^2\partial_1^2\varphi\right)}{3780}\right) \\
\gamma_{22} &= 1+\frac{1}{c^2}\left(-\frac{10}{3}\varphi+\frac{5}{18}\tau^2(\partial_1\varphi)^2\right) \\
\gamma_{33} &= 1+\frac{1}{c^2}\left(-\frac{10}{3}\varphi+\frac{5}{18}\tau^2(\partial_1\varphi)^2\right). \qquad (7.24)
\end{aligned}
$$

Determination of the integration constant C_2

In the metric (7.24) the initial condition $C_2(q_1)$ is still undetermined. To determine it we take advantage of the results obtained in Refs. Bartolo et al. 2010. The authors consider the primordial non-Gaussianity set at inflationary epochs on super-Hubble scales. At later times, cosmological perturbations re-enter the Hubble radius. They show how the information on the primordial non-Gaussianity, set on super-Hubble scales, flows into smaller scale using a general relativistic computation. Their calculations, which are performed in the synchronous and comoving gauge, show how the primordial non-Gaussianity affects the PN part of the density contrast at second order. Once again the use of inflationary initial conditions in our case of globally planar dynamics is justified by our ultimate goal of having a hint on what happens in the fully three-dimensional dynamics.

First of all, we consider our fully non-linear PN expression for the density contrast. The PN contribution is given by $\delta^{PN} = (1/2)\left(1+\overline{\delta}\right)(w_{\rm in}-w)$, where w is the trace of the PN perturbation w^α_β of the metric

$$
w = \frac{-3150\varphi\left(\tau^2\partial_1^2\varphi-6\right)+\tau^2\left[63C_2+50(\partial_1\varphi)^2\left(2\tau^2\partial_1^2\varphi-21\right)\right]}{315\left(\tau^2\partial_1^2\varphi-6\right)}\,. \qquad (7.25)
$$

For a comparison with the result of Bartolo et al. 2010, our expression for the density contrast

$$
\delta = \frac{\tau^2\partial_1^2\varphi}{6-\tau^2\partial_1^2\varphi}+\frac{1}{c^2}\left(\frac{21C_2\tau^2-25\tau^4\partial_1^2\varphi(\partial_1\varphi)^2}{35\left(6-\tau^2\partial_1^2\varphi\right)^2}-\frac{5\tau^2(\partial_1\varphi)^2}{3\left(6-\tau^2\partial_1^2\varphi\right)}\right) \qquad (7.26)
$$

must be expanded up to second order with respect to the peculiar gravitational potential. As usual, we split the density contrast into a first and second order part $\delta = \delta^{(1)} + (1/2)\delta^{(2)}$, finding

$$
\delta = \frac{1}{6}\tau^2\partial_1^2\varphi+\frac{C_2}{60c^2}\tau^2+\frac{C_2}{180c^2}\tau^4\partial_1^2\varphi-\frac{5}{18c^2}\tau^2(\partial_1\varphi)^2+\frac{C_2}{720c^2}\tau^6(\partial_1^2\varphi)^2+\frac{1}{36}\tau^4(\partial_1^2\varphi)^2\,. \qquad (7.27)
$$

Actually, in this expression one can only be sure about the order of the terms that do not contain C_2, since the latter implicitly depends on the initial peculiar gravitational potential, as it will be shown. The first-order term, i.e. $\tau^2\partial_1^2\varphi/6$, obviously coincides with the result of linear perturbation theory in the synchronous and comoving gauge and it is a Newtonian term, as it is well known. The remaining terms are at least of second order. They read

$$\delta^{(2)} = \frac{C_2}{30c^2}\tau^2 + \frac{C_2}{90c^2}\tau^4\partial_1^2\varphi - \frac{5}{9c^2}\tau^2(\partial_1\varphi)^2 + \frac{C_2}{360c^2}\tau^6(\partial_1^2\varphi)^2 + \frac{1}{18}\tau^4(\partial_1^2\varphi)^2 \,. \tag{7.28}$$

This expression can be compared with eq. (45) of Bartolo et al. 2010, by specializing the latter to an Einstein-de Sitter background model and to globally planar perturbations[2]. It reads (in $c = 1$ units)

$$\delta^{(2)} = \frac{10}{9}\left(\frac{3}{4} - a_{\rm nl}\right)\tau^2(\partial_1\varphi)^2 + \frac{10}{9}\left(2 - a_{\rm nl}\right)\tau^2\varphi(\partial_1^2\varphi) + \frac{1}{18}\tau^4(\partial_1^2\varphi)^2 \,, \tag{7.29}$$

where the deviation of the parameter $a_{\rm nl}$ from unity measures the strength of the initial (i.e. inflationary) non-Gaussianity (see Bartolo et al. 2010 for more details). Looking at these expressions, we first note that the second-order Newtonian term, i.e. $(1/18)\tau^4(\partial_1^2\varphi)^2$, is the same in both expression, as it should be. For what concerns the form of $C_2(q_1)$, it should be recalled that it is the initial condition of the PN growing mode $\propto \tau^2$ in the solution (7.23). Thus, we already know that it must be (at least) a second-order term, since the analogous first-order term is Newtonian. The next step is to recognize the PN terms in eq. (7.29). Although the explicit powers of c are not shown, one knows from dimensional analysis that the second order, i.e. $\propto \varphi^2$, PN terms should be $\propto \tau^2$ and contain two spatial derivatives, or they should be $\propto \tau^4$ with four spatial derivatives and so on, in order to have the correct powers of c and to be dimensionless, second-order quantities. The PN terms in eq. (7.29) are then

$$\frac{10}{9}\left(\frac{3}{4} - a_{\rm nl}\right)\tau^2(\partial_1\varphi)^2 \tag{7.30}$$

and

$$\frac{10}{9}\left(2 - a_{\rm nl}\right)\tau^2\varphi(\partial_1^2\varphi) \,. \tag{7.31}$$

Notice that it is precisely the PN terms which bring all the relevant information about (quadratic) primordial non-Gaussianity. Note also that PN terms $\propto \tau^4$ with four spatial derivatives and PPN terms are absent (indeed, they would appear at third order in perturbation theory). Now, in the expression (7.28), it is explicitly shown that $\left(C_2/30c^2\right)\tau^2$ is a PN term. Therefore, C_2 must contain two spatial derivatives. This fact completely determines the form of C_2: the most general expression that can be constructed is

$$C_2 = A(\partial_1\varphi)^2 + B\varphi(\partial_1^2\varphi) \,. \tag{7.32}$$

At this point, notice that the PN terms in eq. (7.28)

$$\frac{C_2}{90c^2}\tau^4\partial_1^2\varphi \tag{7.33}$$

and

$$\frac{C_2}{360c^2}\tau^6(\partial_1^2\varphi)^2 \tag{7.34}$$

[2]For the Einstein-de Sitter background in (45) of Bartolo et al. 2010 we set: $\Omega_{0m} = 1$, $f(\Omega_{0m}) = 1$, $\mathcal{H}_0 = 2/\tau_0$, where the subscript "0" denotes the present time, and $D_+(\tau) = \tau^2/\tau_0^2$ is the linear growing mode solution.

are actually third and fourth-order terms, respectively. Substitution of eq. (7.32) in eq. (7.28) leads to

$$\left(\frac{A}{30c^2} - \frac{5}{9c^2}\right)\tau^2(\partial_1\varphi)^2 + \frac{B}{30c^2}\tau^2\varphi(\partial_1^2\varphi) \tag{7.35}$$

for our second-order PN terms. By comparison with eq. (7.29), we find A and B in terms of a_{nl} we finally obtain

$$C_2 = \frac{25}{3}\left[\left(1 - 4\left(a_{\mathrm{nl}} - 1\right)\right)(\partial_1\varphi)^2 + \left(4 - 4\left(a_{\mathrm{nl}} - 1\right)\right)\varphi(\partial_1^2\varphi)\right]. \tag{7.36}$$

The final expression of our PN metric reads

$$\begin{aligned}
\gamma_{11} &= \left(1 - \frac{\tau^2}{6}\partial_1^2\varphi\right)^2 + \frac{1}{c^2}\left\{\left[\frac{5}{108}\tau^2\left(\left(4\left(a_{\mathrm{nl}} - 1\right) - 1\right)(\partial_1\varphi)^2 + \right.\right.\right. \\
&+ \left.\left.\left(4\left(a_{\mathrm{nl}} - 1\right) - 4\right)\varphi\partial_1^2\varphi\right) + \frac{5}{576}\tau^4\partial_1^2\varphi\,(\partial_1\varphi)^2\right]\left(6 - \tau^2\partial_1^2\varphi\right) - \left.\frac{5}{54}\varphi\left(6 - \tau^2\partial_1^2\varphi\right)^2\right\} \\
\gamma_{22} &= 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right) \\
\gamma_{33} &= 1 + \frac{1}{c^2}\left(-\frac{10}{3}\varphi + \frac{5}{18}\tau^2(\partial_1\varphi)^2\right).
\end{aligned} \tag{7.37}$$

These expressions for the metric represent the main result of this paper: they provide the post-Newtonian extension of the well-known Zel'dovich solution for plane-parallel cosmological dynamics in Newtonian gravity.

Convergence of the perturbation series

The actual convergence of the perturbative series requires that the PN metric is much smaller than the background Newtonian one. To estimate the order of magnitude of the different contributions, one should keep in mind that, on sub-Hubble scales, the peculiar gravitational potential is suppressed with respect to the matter density contrast by the square of the ratio of the proper scale of the perturbation λ_{proper} to the Hubble radius $r_H = cH^{-1}$. Indeed, from the cosmological Poisson equation,

$$\frac{\varphi}{c^2} \sim \left(\frac{\lambda_{proper}}{cH^{-1}}\right)^2 \delta\,, \tag{7.38}$$

which makes it clear that the gravitational potential divided by the square of the speed of light can remain small even on scales characterized by a large density contrast (only provided $|\delta| \ll (cH^{-1}/\lambda_{\mathrm{proper}})^2$), which is indeed at the basis of the well-known validity of the Newtonian approach to cosmological structure formation. We must also recall that, in the Newtonian limit, the square of the peculiar velocity is of the same order as the peculiar gravitational potential and both remain small even in the non-linear regime of structure formation.
Let us then consider the various terms in the metric of eq. (7.24). For γ_{11} we find

$$\mathcal{O}\left(\frac{1}{\left(1+\bar{\delta}\right)^2}\right) + \mathcal{O}\left(\frac{\varphi/c^2}{1+\bar{\delta}}\right) + \mathcal{O}\left(\frac{\varphi/c^2}{\left(1+\bar{\delta}\right)^2}\right), \tag{7.39}$$

where the first term belongs to the Newtonian part. Similarly, for $\gamma_{22} = \gamma_{33}$ we have

$$\mathcal{O}(1) + \mathcal{O}\left(\frac{\varphi}{c^2}\right). \tag{7.40}$$

It is clear that the PN terms are sub-leading.

7.2.1 Comparison with the Szekeres solution

In this paper we have considered the evolution of an irrotational and collisionless fluid in General Relativity in the synchronous and comoving gauge. Following the fluid-flow approach, Ellis 1971, it is possible to alternatively describe our system in terms of the fluid properties of irrotational dust, i.e. mass density, volume-expansion and shear tensor, and the electric and magnetic parts of the Weyl tensor. In addition, in the special case of plane-parallel dynamics considered here, the magnetic Weyl tensor vanishes identically, thus leading to the so-called *silent universe* case, see Matarrese et al. 1993, Matarrese et al. 1994b, Bruni et al. 1995, described by the set of equations

$$\begin{aligned}
\dot{\varrho} &= -\Theta\varrho && (7.41)\\
\dot{\Theta} &= -6\Sigma^2 - \frac{1}{3}\Theta^2 - 4\pi G\varrho && (7.42)\\
\dot{\Sigma} &= \Sigma^2 - \frac{2}{3}\Theta\Sigma - E && (7.43)\\
\dot{E} &= -3E\Sigma - \Theta E - 4\pi G\varrho\Sigma\,, && (7.44)
\end{aligned}$$

where Θ is the trace of velocity-gradient tensor, Σ and E are the eigenvalues of the shear tensor and of the electric Weyl tensor

$$E^\alpha_\beta = \frac{1}{3}\delta^\alpha_\beta\left(\Theta^\mu_\nu\Theta^\nu_\mu - \Theta^2\right) + \Theta\Theta^\alpha_\beta - \Theta^\alpha_\gamma\Theta^\gamma_\beta + c^2\left({}^{(3)}R^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta\,{}^{(3)}R\right) \tag{7.45}$$

in the directions q_2 and q_3. In this framework Croudace *et al.* Croudace et al. 1994 obtain what they refere to as *relativistic Zel'dovich solution*, a sub-case of the exact solutions by Szekeres, Szekeres 1975. This solution is made more appealing by the recent growing interest on the Szekeres metric in the general framework of studying inhomogeneous cosmologies as possible alternatives to FRW (see, e.g. Meures & Bruni 2011, Meures & Bruni 2012 and refs. therein). In Croudace et al. 1994 the authors consider the relativistic evolution equations of silent universes. In the special case of local planar symmetry, i.e. $\lambda_2 = \lambda_3 = 0$, where λ_i are the eigenvalues of the matrix $\partial_\alpha\partial_\beta\varphi$, the velocity-gradient tensor reads Matarrese et al. 1993

$$\Theta^\alpha_\beta = \frac{\dot{a}}{a}\begin{pmatrix} 1 - \frac{a\lambda_1}{1-a\lambda_1} & & \\ & 1 & \\ & & 1 \end{pmatrix}, \tag{7.46}$$

which corresponds to the Newtonian Zel'dovich solution. Croudace *et el.* then compute the associated metric from

$$\Theta^\alpha_\beta = \frac{1}{2}h^{\alpha\sigma}\dot{h}_{\beta\sigma} \tag{7.47}$$

via the time-time component of Einstein's equations, i.e. the energy constraint, that closes the relativistic fluid-flow equations, completely fixing the spatial dependence of the metric. They show that the resulting metric coincides with the Szekeres form

$$h_{22} = h_{33} = 1 \tag{7.48}$$
$$h_{11} = (d(q_\alpha) - a(t)c(q_1))^2 \tag{7.49}$$

with

$$d(q_\alpha) = d_{\rm in}(q_1) - \frac{5}{9}c(q_1)((q_2)^2 + (q_3)^2) - a(t)c(q_1)\,. \tag{7.50}$$

This solution leaves the FRW expansion unperturbed in the directions q_2 and q_3. In fact, following Matarrese et al. 1993, Croudace *et al.* discard the sub-leading PN trace part of the initial conditions (7.10), thereby considering perturbations in the direction q_1 only. On the contrary, we kept the one-dimensional initial seed $\phi(q_1)$ in all directions, thereby allowing for perturbations in the component h_{22} and h_{33} of the metric *ab initio*. As a consequence, our solution for h_{22} and h_{33} changes with time, showing a non-linear dependence on the gravitational potential ϕ. Hence our (approximate) solution *cannot* be recast into the Szekeres form.

7.3 A PN estimation of cosmological back-reaction

7.3.1 The cosmological back-reaction proposal

Traditionally, cosmological models have been based on the assumption that, on a sufficiently large scale, the Universe is described by isotropic and homogeneous solutions of the Einstein equations. This approach to cosmology is based both on observational facts, such as the near-perfect isotropy of the CMB radiation, and on an *a priori* assumption, the Cosmological Principle. It allows us to circumvent our inability to obtain information about the Universe outside our past light-cone by assuming that a symmetry principle is valid everywhere.

Observations tell us that the Universe is far from homogeneous and isotropic on small scales: homogeneity in the galaxy distribution is only achieved over some large smoothing scale. When we refer to homogeneity and isotropy of the Universe, we tacitly assume that spatial smoothing over some suitably large filtering scale has been applied, so that fine-grained details can be ignored. In other words, by the assumption that the same background model can be used to describe the properties of nearby and very distant objects in the Universe, the smoothing process is implicit in the way one fits a FRW model to observations, Kolb et al. 2010. Cosmological parameters like the Hubble expansion rate or the energy density of the various cosmic components are to be considered as volume averaged quantities: only these can be compared with observations. From this point of view, assuming that the isotropic and homogeneous ΛCDM model is a good observational fit to the real inhomogeneous Universe, this does not imply that a primary source of dark energy exists, but only that it exists effectively as far as the observational fit is concerned.

To describe the time evolution of the inhomogeneities in a patch of the Universe as large as our local Hubble radius, we would construct the effective dynamics from which observable average properties can be inferred. Of course, this implies a scale-dependent description of inhomogeneities: one can say that smoothing out inhomogeneities renormalizes the description of the Universe.

There is, however, a technical difficulty inherent in any smoothing procedure. While the smoothing of the matter content is somewhat straightforward (*e.g* in the fluid description), the smoothing of the space-time metric is more complex and immediately leads to an important and unexpected feature, pointed out in Ellis 1984. Let's look at figure 7.1 as a starting point: it compares models of the same region of the Universe on three scales showing different amounts of detail.

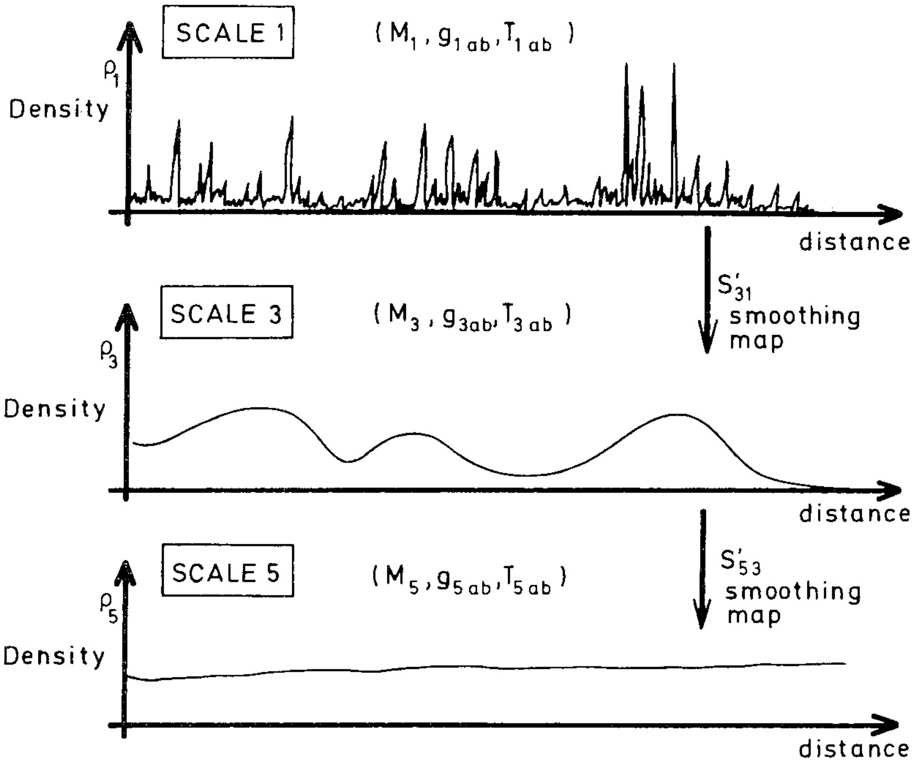
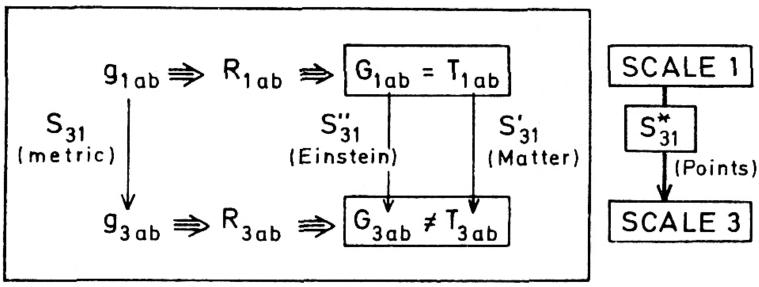


Figure 7.1: Comparison of models of the same region of Universe on three scales showing different amounts of detail. Scale 1 represents all details down to very small scales. Scale 3 represents intermediate scales. Scale 5 represents only large scale features. From Ellis 1984.

The three different matter tensors and the related metric tensors are intended to describe the same physical system and the same space-time, but at different scales of description. General Relativity tests confirm that Einstein's equations hold on some suitably small scale where the Universe is highly inhomogeneous and anisotropic, corresponding to Scale 1. It is the starting point in the flow chart below.



The chart summarizes tensors correspondence from Scale 1 to Scale 3. The two models are related by:

- Map S^*, which determines which points in the different underlying manifolds are related to each other by the smoothing procedure.
- Map S, which determines the metric tensor of the smoothed representation from the more detailed one.
- Map S', which determines the matter tensor.
- Map S'', which determines the Einstein tensor.

Cosmology deals with Scale 5, where the same procedure applies. Because of the non-linear nature of the fields equations, in general, the operations of smoothing will not commute with going to the field equations. It means that the Einstein tensor computed from the smoothed metric would generally differ from that computed from the smoothed stress–energy tensor, that is

$$S'' \neq S' \qquad \text{or} \qquad \langle G_{cd}(g_{ab})\rangle \neq G_{cd}(\langle g_{ab}\rangle), \tag{7.51}$$

where $\langle \quad \rangle$ indicates some averaging prescription. One can better set the problem defining a tensor P_{ab} representing the difference between the Einstein tensor $G_{ab}(\langle g_{ab}\rangle)$ defined from the smoothed metric, and that of related to the smoothed matter tensor T_{ab}^{fluid} via the Einstein equations, $\langle G_{ab}(g_{ab})\rangle = 8\pi T_{ab}^{fluid}$. This correction will take care of the change of scale of description:

$$G_{ab}(\langle g_{ab}\rangle) = 8\pi T_{ab}^{fluid} + P_{ab}\,. \tag{7.52}$$

The tensor P_{ab} represents the effects of small-scale inhomogeneities in the Universe on the dynamical behaviour at the smoothed-out scale. In the standard approach we take

$$G_{ab}(\langle g_{ab}\rangle) = G_{ab}(g_{ab}^{FRW}) = 8\pi\left(T_{ab}^{fluid} + \rho_{DE}\, g_{ab}\right). \tag{7.53}$$

We assumed *a priori* that the smoothed metric on this scale is that of FRW and the smoothed stress energy tensor is that of a fluid with $\varrho = \varrho(t)$. One then simply takes the P_{ab} correction term as a primary, physical source in the stress-energy tensor, the dark energy, and is also forced by observations to take $w = -1$ in the equation of state of this fluid, violating the usual energy conditions. On the contrary, the extra term P_{ab}, interpreted as an effective contribution emerging after smoothing, need not satisfy the usual energy conditions, even if the original matter stress-energy tensor does, as clearly pointed out in Ellis 1984. There would be no conceptual problems in having $w_P < -1$, for example. Therefore, it is natural to ask whether this extra term in the dynamical smoothed equations is connected with the accelerated expansion of the Universe: the possibility that modification in the average expansion rate due to structure formation would explain the late-time observations of faster expansion and longer distances is called the back-reaction conjecture, Räsänen 2006, Kolb et al. 2005.

The present controversy about the back-reaction mechanism concerns mostly how to construct an adequate smoothing procedure in GR. Nevertheless, it is well-known that both standard perturbative treatments even at higher than linear order (see e.g. the discussion in Kolb et al. 2006) and the Newtonian approximation (see in this respect the analysis of Buchert & Ehlers 1997) are totally inadequate to correctly evaluate the relevance of back-reaction terms in the average Einstein's equations (see also Räsänen 2010).

The issue has also been studied in Green & Wald 2013 with a non-standard perturbative formalism proposed in Green & Wald 2011.

There is no fully realistic calculation yet, and the amplitude of back-reaction in the real Universe remains an open question: the issue of the quantitative relevance of back-reaction effects is a largely controversial one, see Räsänen 2011 and Buchert & Räsänen 2012 for some recent review. Given the truly non-Newtonian e non-linear features of the back-reaction, it is tempting to evaluate how PN terms can affect the average expansion rate.

7.3.2 A set of effective Friedmann equations

In this section we present the smoothing procedure proposed in Buchert & Ehlers 1997 in the context of Newtonian gravity and in Buchert 2000 in GR. In both papers the Universe consists of irrotational dust only and its dynamics is described in the Lagrangian approach.
In GR, the smoothing is based on a spatial average for a scalar Ψ defined as

$$\langle\Psi\rangle_{\mathcal{D}} = \frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}} \Psi\sqrt{h}\, d^3q\,, \tag{7.54}$$

where h is the determinant of the spatial metric $h_{\alpha\beta}$ in synchronous and comoving gauge and $\mathcal{V}_{\mathcal{D}}$ is the volume of the coarse-graining comoving domain $\mathcal{D}$

$$\mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}} \sqrt{h}\, d^3q\,. \tag{7.55}$$

By smoothing the scalar Einstein equations, the energy constrain and the Raychaudhuri equation, the following effective Friedmann equations for the average scale factor $a_{\mathcal{D}} = (\mathcal{V}_{\mathcal{D}}/\mathcal{V}_{\mathcal{D}_0})^{1/3}$ are obtained:

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = \frac{8}{3}\pi G\varrho_{\mathcal{D}}^{eff} \tag{7.56}$$

$$\left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) = -\frac{4}{3}\pi G\left(\varrho_{\mathcal{D}}^{eff} + \frac{3P_{\mathcal{D}}^{eff}}{c^2}\right), \tag{7.57}$$

where the source can be viewed as a perfect fluid with effective energy density and pressure terms given by

$$\varrho_{\mathcal{D}}^{eff} = \langle\varrho\rangle_{\mathcal{D}} - \frac{\mathcal{Q}_{\mathcal{D}}}{16\pi G} - \frac{c^2\langle{}^{(3)}R\rangle_{\mathcal{D}}}{16\pi G} \tag{7.58}$$

$$P_{\mathcal{D}}^{eff} = -\frac{c^2\mathcal{Q}_{\mathcal{D}}}{16\pi G} + \frac{c^4\langle{}^{(3)}R\rangle_{\mathcal{D}}}{48\pi G} \tag{7.59}$$

obeying the continuity equation

$$\dot{\varrho}_{\mathcal{D}}^{eff} + 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\left(\varrho_{\mathcal{D}}^{eff} + \frac{P_{\mathcal{D}}^{eff}}{c^2}\right) = 0\,, \tag{7.60}$$

where

$$\langle{}^{(3)}R\rangle_{\mathcal{D}} = \frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}} {}^{(3)}R\sqrt{h}\, d^3q \tag{7.61}$$

is the average spatial curvature and we have introduced the kinematical back-reaction

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3}\langle(\Theta - \langle\Theta\rangle_{\mathcal{D}})^2\rangle_{\mathcal{D}} - 2\langle\Sigma^2\rangle_{\mathcal{D}}\,. \tag{7.62}$$

By performing the conformal rescaling of the metric $h_{\alpha\beta}$, the velocity-gradient tensor splits into a FRW and a peculiar term

$$\Theta^\alpha_\beta = \frac{1}{a}\left(\vartheta^\alpha_\beta + \frac{2}{\tau}\delta^\alpha_\beta\right), \tag{7.63}$$

where a is the scale factor of the underlying FRW background and we use here the conformal time coordinate. In terms of the peculiar quantities, the kinematical back-reaction becomes

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3a^2}\left(\langle\vartheta^2\rangle_{\mathcal{D}} - \langle\vartheta\rangle^2_{\mathcal{D}}\right) - \frac{2}{a^2}\langle\sigma^2\rangle_{\mathcal{D}}\,. \tag{7.64}$$

This term arises from the inhomogeneities of the dust which cause the congruence of its geodesics to have a non vanishing fluctuations of the expansion rate and a non vanishing averaged shear. These features describe a coarse grained picture of the large scale dust Universe in contrast to the homogeneous one, in which the geodesics have an expansion rate space-independent and a vanishing shear.
Consistency of eq. (7.60) with mass conservation, which can be re-written as

$$\dot{\varrho}_{\mathcal{D}} + 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\varrho_{\mathcal{D}} = 0\,, \tag{7.65}$$

requires that the kinematical back-reaction and the mean spatial curvature satisfy the integrability condition

$$\left(a^6_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}}\right)^{\boldsymbol{\cdot}} + c^2 a^4_{\mathcal{D}}\left(a^2_{\mathcal{D}}\langle{}^{(3)}R\rangle_{\mathcal{D}}\right)^{\boldsymbol{\cdot}} = 0\,. \tag{7.66}$$

We remark here that such a condition is a genuinely general relativistic effect, which has no analogue in Newtonian gravity, since the curvature ${}^{(3)}R$ of comoving hypersurfaces vanishes identically in the Newtonian limit. Indeed, in Newtonian gravity, $\mathcal{Q}_{\mathcal{D}}$ exactly (i.e. at any order in perturbation theory) reduces to the volume integral of a total-derivative term in Eulerian coordinates (see Buchert & Ehlers 1997)

$$\mathcal{Q}^{Newt}_{\mathcal{D}} = \frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}} \nabla\cdot(u\nabla\cdot u - u\cdot\nabla u)\,d^3x \tag{7.67}$$

where u is the peculiar velocity of the dust. In this expressions, the averaging is meant to be performed over a volume of the order of the Hubble volume. We know that inhomogeneities only exist on scales much smaller than the Hubble radius and peculiar velocities are small on the boundary of our Hubble patch. Thus, by the Gauss theorem, the volume integral can be transformed into a negligible boundary term and the standard FRW matter-dominated model can be applied without any substantial correction from the back-reaction. This result demonstrates that in order to deal with the backreaction, going beyond the Newtonian approximation is mandatory.

7.3.3 Kinematical back-reaction for plane-parallel perturbations

In this section we use our non-linear PN metric to estimate the back-reaction term in eq. (7.64) in the PN approximation. First, we recover the Newtonian result. Explicitly, for the plane-parallel dynamics we have $\overline{\sigma}^2 = \overline{\vartheta}^2/3$, thus eq. (7.64) becomes

$$\overline{\mathcal{Q}}_{\mathcal{D}} = -\frac{2}{3a^2}\langle\overline{\vartheta}\rangle^2_{\mathcal{D}} \,, \tag{7.68}$$

where a is our Einstein-de Sitter scale-factor, $\overline{\vartheta} = -\tau\partial_1^2\varphi/(3(1-\tau^2\partial_1^2\varphi/6))$ and $\langle\ldots\rangle_{\mathcal{D}}$ is our Newtonian average

$$\langle\ldots\rangle_{\mathcal{D}} = \frac{a^3}{\overline{\mathcal{V}}_{\mathcal{D}}}\int_{\mathcal{D}}\ldots\left(1-\tau^2\partial_1^2\varphi/6\right)d^3q \,, \tag{7.69}$$

where $\overline{\mathcal{V}}_{\mathcal{D}} = a^3\int_{\mathcal{D}}\left(1-\tau^2\partial_1^2\varphi/6\right)d^3q$. Recalling that

$$\frac{\overline{\vartheta}}{a} = \partial_{r_1}\overline{v}_1 \,, \tag{7.70}$$

where r_α are proper Eulerian coordinates $r_\alpha = ax_\alpha$, $\overline{v}_1 \equiv -\tau\partial_1\varphi/3$ is the Newtonian peculiar velocity in the direction r_1, we can re-write eq. (7.68), as we already know, as a total derivative:

$$\overline{\mathcal{Q}}_{\mathcal{D}} = -\frac{2}{3}\left(\langle\partial_{r_1}\overline{v}_1\rangle^E_{\mathcal{D}}\right)^2 = 0 \,, \tag{7.71}$$

where we introduced the Eulerian average $\langle\ldots\rangle^E_{\mathcal{D}} \equiv (1/\int_{\mathcal{D}}d^3r)\int_{\mathcal{D}}\ldots d^3r$.

In order to calculate the PN kinematical back-reaction $\mathcal{Q}^{PN}_{\mathcal{D}}$, we consider the kinematical quantities related to the PN peculiar velocity-gradient tensor

$$\vartheta^\alpha_\beta = \overline{\vartheta}^\alpha_\beta + \frac{1}{2c^2}w^{\alpha\,\prime}_\beta \,. \tag{7.72}$$

In full generality, without any assumption about the Newtonian background, the PN expansion of the average integrals in eq. (7.64) leads to (see appendix D)

$$\begin{aligned}\mathcal{Q}^{PN}_{\mathcal{D}} \;=\;& \frac{1}{3a^2}\langle\overline{\vartheta}^2 w\rangle_{\mathcal{D}} + \frac{2}{3a^2}\langle\overline{\vartheta}w'\rangle_{\mathcal{D}} - \frac{1}{3a^2}\langle\overline{\vartheta}^2\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}} - \frac{2}{3a^2}\langle\overline{\vartheta}\rangle_{\mathcal{D}}\langle w\rangle'_{\mathcal{D}} + \\ & -\frac{1}{a^2}\langle\overline{\sigma}^2 w\rangle_{\mathcal{D}} - \frac{1}{a^2}\langle\left(\overline{\vartheta}^\alpha_\beta w^{\beta\,\prime}_\alpha - \frac{1}{3}\overline{\vartheta}w'\right)\rangle_{\mathcal{D}} + \frac{1}{a^2}\langle\overline{\sigma}^2\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}} \,,\end{aligned} \tag{7.73}$$

where with $\langle\ldots\rangle_{\mathcal{D}}$ we indicate the Newtonian average (PN corrections in the averaging procedure would yield higher-order terms) and w is the trace of the PN perturbation of the metric.

Recalling that in our case $\overline{\sigma}^2 = \overline{\vartheta}^2/3$, equation (7.73) reduces to

$$\begin{aligned}\mathcal{Q}^{PN}_{\mathcal{D}} &= \frac{2}{a^2}\langle\overline{\vartheta}w^{2\,\prime}_2\rangle_{\mathcal{D}} - \frac{2}{3a^2}\langle\overline{\vartheta}\rangle_{\mathcal{D}}\langle w\rangle'_{\mathcal{D}} \\ &= \frac{2}{a^2}\langle\overline{\vartheta}w^{2\,\prime}_2\rangle_{\mathcal{D}} \,,\end{aligned} \tag{7.74}$$

where we used the commutation rule for w

$$\langle w\rangle'_{\mathcal{D}} - \langle w'\rangle_{\mathcal{D}} = \langle w\overline{\vartheta}\rangle_{\mathcal{D}} - \langle w\rangle_{\mathcal{D}}\langle\overline{\vartheta}\rangle_{\mathcal{D}} \tag{7.75}$$

and the last equality in eq. (7.74) follows from the previous Newtonian calculation, having neglected the last term, which involves $\langle \overline{\vartheta} \rangle_{\mathcal{D}}$. Finally, using our Newtonian average, eq. (7.69) and our solution for $w_2^{2'}$, we find

$$\mathcal{Q}_{\mathcal{D}}^{PN} = -\frac{10 a \tau^2}{81 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \partial_1 (\partial_1 \varphi)^3 d^3 q \, . \tag{7.76}$$

We can alternatively express this result in terms of Newtonian peculiar velocity $\overline{v}_1$. We find

$$\mathcal{Q}_{\mathcal{D}}^{PN} = \frac{10}{3 a \tau} \langle \partial_{r_1} \overline{v}_1^3 \rangle_{\mathcal{D}}^E \, , \tag{7.77}$$

which indeed indicates a negligible PN correction.
Finally, using the PN spatial curvature

$${}^{(3)}R^{PN} = \frac{(20/3) \partial_1^2 \varphi}{a^2 \left(1 - \tau^2 \partial_1^2 \varphi / 6\right)} \tag{7.78}$$

the PN expansion of eq. (7.61) leads to

$$\langle {}^{(3)}R \rangle_{\mathcal{D}}^{PN} = \frac{20 a}{3 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \partial_1^2 \varphi d^3 q \, . \tag{7.79}$$

It is then clear that, in the case of planar dynamics, both the kinematical back-reaction and the mean spatial curvature reduce to purely boundary terms [3] and, as a consequence, they cannot lead to acceleration. Also interesting is the fact that $\langle {}^{(3)}R \rangle_{\mathcal{D}}^{PN} \propto a^{-2}$, which provides a self-consistent solution of the integrability condition eq. (7.66), for vanishing $\mathcal{Q}_{\mathcal{D}}$.

7.3.4 PN corrections to the average scale-factor and expansion rate

An alternative approach to back-reaction is that of computing directly the PN contribution to the average scale-factor and expansion rate. Indeed, using the definition $a_{\mathcal{D}} = \left(\mathcal{V}_{\mathcal{D}} / \mathcal{V}_{\mathcal{D}_0}\right)^{1/3}$, we can calculate the PN expansion of the average scale-factor, written as

$$a_{\mathcal{D}} = \overline{a}_{\mathcal{D}} + \frac{1}{c^2} a_{\mathcal{D}}^{PN} \, . \tag{7.80}$$

For the Newtonian term, we have for the volume

$$\overline{\mathcal{V}}_{\mathcal{D}} = \int_{\mathcal{D}} a^3 \left(1 - \frac{\tau^2}{6} \partial_1^2 \varphi\right) d^3 q \, , \tag{7.81}$$

thus, neglecting the boundary term, we obtain

$$\overline{a}_{\mathcal{D}} = \frac{\tau^2}{\tau_0^2} \, , \tag{7.82}$$

which is just the Einstein de Sitter scale factor $a(\tau)$.

[3]Note that these quantities can be seen as boundary terms both in Lagrangian and Eulerian coordinates, since – at the Newtonian level – $(dF(q_1)/dq^1) dq^1 = (dF(q_1(x_1, \tau)/dx) dx$, for any function $F(q_1)$.

For the PN term we find

$$a_{\mathcal{D}}^{PN} = \frac{1}{6} a \left(\langle w \rangle_{\mathcal{D}} - \langle w_0 \rangle_{\mathcal{D}_0} \right) , \tag{7.83}$$

where $\langle \ldots \rangle_{\mathcal{D}}$ is the Newtonian average and our solution gives

$$w = \frac{-\frac{20}{63}\tau^4 \partial_1^2 \varphi \left(\partial_1 \varphi\right)^2 + \frac{5}{3}\tau^2 \left[\left(4\left(a_{\rm nl} - 1\right) + 1\right) \left(\partial_1 \varphi\right)^2 + \left(4\left(a_{\rm nl} - 1\right) + 2\right) \varphi \partial_1^2 \varphi \right] - 60\varphi}{6 - \tau^2 \partial_1^2 \varphi} . \tag{7.84}$$

Explicitly, we have

$$\begin{aligned} a_{\mathcal{D}}^{PN} &= \frac{a^4}{6\overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \left\{ -\frac{10}{189} \partial_1^2 \varphi (\partial_1 \varphi)^2 \tau^4 + \frac{5}{18}\tau^2 \left[\left(4\left(a_{\rm nl} - 1\right) + 1\right) \left(\partial_1 \varphi\right)^2 + \right. \right. \\ &+ \left. \left. \left(4\left(a_{\rm nl} - 1\right) + 2\right) \varphi \partial_1^2 \varphi \right] \right\} d^3 q \, , \end{aligned} \tag{7.85}$$

which can be written as

$$a_{\mathcal{D}}^{PN} = \frac{a^4}{6\overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \left(-\frac{5}{18}\tau^2 \left(\partial_1 \varphi\right)^2 \right) d^3 q \tag{7.86}$$

up to negligible boundary terms. The average scale-factor then becomes

$$a_{\mathcal{D}} = a \left(1 - \frac{5}{12c^2} \langle \left(1 + \overline{\delta}\right) \overline{v}_1^2 \rangle_{\mathcal{D}} \right) , \tag{7.87}$$

which indicates a negligible PN correction.
The PN contribution to the average expansion rate $\langle \Theta \rangle_{\mathcal{D}} = (3/a) a_{\mathcal{D}}' / a_{\mathcal{D}}$ is obtained by the expansion of its very definition, leading to

$$\langle \Theta \rangle_{\mathcal{D}} = 3H + \frac{1}{2ac^2} \langle w \rangle_{\mathcal{D}}' \, , \tag{7.88}$$

where $H(t) = 2/(a\tau)$ is the Hubble expansion rate of our Einstein-de Sitter background. Neglecting boundary terms, it is straightforward to obtain [4]

$$\langle \Theta \rangle_{\mathcal{D}} = 3H \left(1 - \frac{5}{12c^2} \langle \left(1 + \overline{\delta}\right) \overline{v}_1^2 \rangle_{\mathcal{D}} \right) . \tag{7.89}$$

This result again implies that the PN correction is fully negligible for plane parallel perturbations.

7.3.5 Discussion

Our results here bring both good and bad news for the back-reaction of cosmic inhomogeneities to represent a potentially viable alternative to dark energy at a fundamental level. Let us start with the bad news: as we have shown the kinematical back-reaction

[4] If we expand eq. (7.89) up to second-order in perturbation theory, we find quantitative agreement with eq. (41) in Kolb et al. 2005. We checked that the different numerical factor is simply due to the different definition of $\langle \delta\theta \rangle_{\mathcal{D}}$ used in the two calculations.

scalar remains a negligible boundary term also at the post-Newtonian level, so that no relevant back-reaction effect is implied by our PN solution. We should stress, however, that such a result is clearly a consequence of our very specific (but analytically solvable) model, which relies on one-dimensional dynamics. It is a very reasonable hypothesis that such a result will be modified by a (more complex!) full 3D calculation. The good news is the very fact that the Lagrangian approach allows to obtain a quantitative estimate of back-reaction and that our analytical expression for the PN metric also allow us to study the final stages of plane-parallel collapse, which obviously leads to a shell-crossing, pancake-like singularity. Caustic formation is considered the main limitation of the Lagrangian description, both in the Newtonian approximation and in GR. As is well-known, caustics arise because several fluid elements coming from different positions may converge to the same Eulerian position, thus forming infinite density regions. Such a pathological behaviour occurs in our case when $f = -1$, i.e. when $\tau^2 \partial_1^2 \varphi / 6 = 1$: at this time the determinants of both the Newtonian, eq. (7.6) and PN metric, eq. (7.37) go to zero, while the density contrast at both the Newtonian and PN level becomes infinite,

$$\delta = \frac{\tau^2 \partial_1^2 \varphi}{6 - \tau^2 \partial_1^2 \varphi} + \frac{5\tau^2 \left[(\partial_1 \varphi)^2 \left(5\partial_1^2 \varphi \tau^2 + 42(3 - 4a_{\rm nl}) \right) - 168(a_{\rm nl} - 2)\varphi \partial_1^2 \varphi \right]}{42 c^2 \left(\tau^2 \partial_1^2 \varphi - 6 \right)^2} \tag{7.90}$$

as does the PN spatial curvature,

$$^{(3)}R = \frac{(20/3)\partial_1^2 \varphi}{c^2 a^2 \left(1 - \tau^2 \partial_1^2 \varphi / 6 \right)}. \tag{7.91}$$

The appearance of shell-crossing singularities can be understood as indicating the breakdown of the dust approximation, rather than the occurrence of a true physical singularity of the gravitational collapse. The formation of caustics appears as an artefact of the extrapolation of the pressure-less fluid approximation beyond the point at which pressure has become important. In addition, we have explicitly shown that the appearance of shell-crossing singularities in the final stage of the gravitational collapse does not lead to divergences in the average dynamics of inhomogeneous dust Universes. An important result of our analysis is that the divergence of the PN spatial curvature at caustic formation is completely eliminated by the spatial smoothing procedure. Indeed $\mathcal{R}^{PN}$ diverges like $(1 + \overline{\delta})$, which is exactly compensated by the square root of the spatial metric determinant $\propto 1/(1 + \overline{\delta})$:

$$\langle {}^{(3)}R \rangle_{\mathcal{D}} = \frac{20a}{3c^2 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \partial_1^2 \varphi \, d^3q. \tag{7.92}$$

The averaged matter density shows the same behaviour:

$$\langle \rho \rangle_{\mathcal{D}} = \varrho_{EdS} \left(1 + \frac{5a^3}{36 c^2 \overline{\mathcal{V}}_{\mathcal{D}}} \int_{\mathcal{D}} \tau^2 \left(\partial_1 \varphi \right) \, d^3 q \right) = \varrho_{EdS} \left(1 + \frac{5}{4c^2} \langle \left(1 + \overline{\delta} \right) \overline{v}_1^2 \rangle_{\mathcal{D}} \right). \tag{7.93}$$

This very fact confirms that the instability found in Kolb et al. 2006, using a gradient expansion technique, and in Notari 2006, using a different approximation scheme, cannot be interpreted as a consequence of a shell-crossing singularity, but really arises from the back-reaction of sub-Hubble modes.

CHAPTER 8

Conclusions and future work

Roughly speaking, cosmologists study linear and non-linear scales in different ways. On the largest scales cosmological perturbation theory or one of its extensions is used. These approximation schemes have the advantage of being fully relativistic but can only be used on scales where the matter perturbations are small. To deal with non-linearities at small scales, Newtonian N-body simulations are used but they obviously do not include GR effects. This distinction however breaks down in at least two, and maybe three cases. The first case concerns the addition of relativistic fields in a context where non-linearities are important. The second case is due to the ongoing revolution in observational cosmology where surveys are now reaching unprecedented sizes, mapping out a significant fraction of the observable Universe. On large scales and at large distances it becomes necessary to take into account relativistic effects. Although small, the impact of the perturbations, e.g. on distance measurements, is not negligible and can be used as an additional probe of cosmology. If we would be able to follow the relativistic evolution of the Universe, we could also test the third (and more speculative) case for a general relativistic framework for cosmological simulations in the non-linear regime: GR is a non-linear theory, and in principle non-linear effects on small scales can leak to larger scales and lead to unexpected non-perturbative behaviour.

The PN approximation of GR could be the key ingredient in all these three cases, being a non-linear approximation scheme to study relativistic structure formation in the Universe on all scales, including scales where the density contrast is large. The key novelty is that the PN approximation has by construction a direct correspondence with Newtonian quantities: the PN expressions are sourced only by Newtonian terms, which can be extracted from N-body simulations. In this thesis we have analysed such a correspondence in the Eulerian and Lagrangian frame.

We have performed a transformation between the Eulerian and the Lagrangian frame: our starting point is the Eulerian frame, i.e. the Newtonian limit of the Poisson gauge, where spatial coordinate are comoving with the FRW background observers and the hypersurfaces of constant coordinate conformal time η represent their rest frame. We have change the spatial coordinate and boost the spatial triad such that the spatial coordinates are constant along the matter world line, obtaining the comoving line element, where the shift vector satisfies the comoving condition and the lapse function represent the relation between the coordinate time and the proper time of observers comoving with the matter. We finally have change the coordinate time η and set it equal to the proper time along the geodesic of the matter, arriving in the synchronous-comoving gauge. We have then extended this transformation to the PN approximation, starting from the PN metric in the Poisson gauge. Our transformation is fully non-perturbative and it is consistent with the results of the Newtonian approximation in both frames at any order and with the standard GR perturbation theory up to the second-order, including Newtonian and

PN contributions. Our main result are the equations for the PN metric in synchronous and comoving gauge.

This, will allow us to focus on several interesting developments. In particular, the following separate yet related issues will be addressed.

PN and post-Friedmann approaches: In Bruni et al. 2014b a new approximation scheme for the gravitational dynamics at all scales is developed in the Poisson gauge. It is named post-Friedmann, (PF), and it extends the standard PN approach by including, at leading order in the Newtonian approximation, the gravitational scalar potential in the spatial part of the metric and also a frame-dragging vector potential in the time-space component: both additions are crucial for the full consistency of the approximate Einstein equations, as well as to obtain the correct photon trajectories. Firstly, we will fully clarify the difference in the two approaches (standard PN and PF) in the limit where GR reduces to Newtonian theory, adapting the calculations in Villa et al. 2014 in the Eulerian and Lagrangian frames. Secondly, the procedure we explain in chapter 6 will be exploited to provide the corresponding PF metric in synchronous-comoving gauge. Our aim is to obtain and compare the PN and PF space-time metrics. This is part of an ongoing work in collaboration with Dr Bruni at the Institute of Cosmology and Gravitation in Portsmouth, United Kingdom.

PN extension of the Zel'dovich approximation: The Zel'dovich approximation arises in Newtonian theory and one might wonder about its extension within GR. In most papers it is restricted to the second order in perturbation theory. On the other hand, we can obtain a PN, non-perturbative extension of the Zel'dovich approximation, simply starting from the Zel'dovich Newtonian metric taken as the background and then express the PN Zel'dovich metric in terms of the PN displacement field using the equations provided by our transformation. This space-time metric would describe the dynamics beyond the second-order and would have many interesting applications.

PN correction to initial conditions: The GR extension of the Zel'dovich approximation is particularly relevant since the Zel'dovich approximation and its second order extension provide the most convenient way of setting up the initial conditions of a cosmological N-body simulation, Scoccimarro 1998 and Crocce et al. 2006. For large-scale simulations approaching the size of the Hubble horizon we have to estimate the contribution of relativistic corrections: a full understanding of the Newtonian and GR contributions to the non-linear matter density field in the synchronous gauge is crucial to correctly set up initial conditions consistent with general relativity in N-body simulation, see Bruni et al. 2014a for a second-order analysis taking into account large-scale relativistic effects and non-Gaussianity. A more careful analysis of the connection between Newtonian N-body simulations and GR is also of great interest today: in Chisari & Zaldarriaga 2011 the authors provide a dictionary for how to interpret the outputs of numerical simulation, run using Newtonian dynamics, with respect to GR at first order. In Adamek et al. 2013 a formalism for GR N-body simulations which go beyond standard perturbation theory is provided in the weak-field limit of GR, which is then used in Adamek et al. 2014 to compute the distance-redshift relation for a plane-symmetric universe. For the interpretation of N-Body simulations and the connection to the Lagrangian and Eulerian perspectives see Rigopoulos & Valkenburg 2013, where the relativistic corrections to Newtonian cosmology are analysed with the gradient expansion method on scales close to the Hubble scale.

A fully second-order implementations of GR corrections to N-body simulations is still missing and would certainly be useful, although second-order perturbation theory should not to be considered as an exhaustive study for non-linear dynamics. We will investigate these issues within the PN approximation.

Galaxy bias and matter power spectrum: Recently, a GR analysis of galaxy clustering, Bartolo et al. 2005, Verde & Matarrese 2009, Yoo et al. 2009, Baldauf et al. 2011, Bonvin & Durrer 2011, Jeong et al. 2012 and refs. therein, has drawn attention to other relativistic effects in the observations, such as gauge effects, which become sizeable near the Hubble horizon, see Bruni et al. 2012 and Wands & Slosar 2009. In addition, the large-scale structure measurements are affected by the lack of knowledge of the precise underlying biasing scheme, i.e. the local ratio between visible and dark matter. Galaxy bias is naturally defined in synchronous-comoving gauge. We will analyse the Eulerian and Lagrangian approaches to the galaxy bias and compute PN-type corrections to the matter power spectrum. In particular, the PN approach would allow us to trace non-Gaussianity of the primordial perturbations.

Kinematical back-reaction: We are interested in a more realistic fully 3D estimation for perturbations with no symmetry, with special attention to the impact of singularities in the average dynamics. We will calculate the PN average expansion rate, using the PN extension of the Zel'dovich Newtonian metric and the PN formalism for back-reaction developed in Villa et al. 2011.

Optical back-reaction: A true dynamical back-reaction modifying the expansion history would be the most radical effect of the presence of cosmic structures. However, even if it is negligible, being the smoothed dynamics FRW, an optical back-reaction is surely relevant: the presence of inhomogeneities affects redshifts and distances and generates gravitational lensing, affecting all cosmological observables. The first step is to analyse the properties of photon geodesics in a space-time with a fully non-linear density field, comparing the standard PN approach with the ongoing work on PF lensing by Dr Bruni and collaborators. This would lead to non-perturbative expressions for: distance-redshift relation, luminosity distance, redshift space distortions and gravitational lensing, including that of the CMB. The goal of this part of our project will be to build up a numerical implementation of the obtained equations in order to compute ray tracing and lensing in N-body simulation incorporating relativistic PN corrections.

APPENDIX A

Notation and symbols

Table A.1: Notation and symbols

$\mathcal{M}$	space-time manifold with a Lorentz metric with signature (-+++)
g_{ab}	space-time metric
g	determinant of the space-time metric
h_{ij}	spatial metric
h	determinant of the spatial metric
γ_{ij}	conformal spatial metric
γ	determinant of the conformal spatial metric
a, b, c	space-time indices in any gauge
i, j, k	space indices in any gauge
α, β, γ	indices for spatial components in Lagrangian coordinates
A, B, C	indices for spatial components in Eulerian coordinates
∂_α	partial derivative with respect to Lagrangian coordinates
∂_A	partial derivative with respect to Eulerian coordinates
$_{;a}$	covariant derivative with respect to g_{ab}
$\overline{\mathcal{D}}_\alpha$	Newtonian covariant derivative with respect to $\overline{\gamma}_{\alpha\beta}$
$\overline{\mathcal{D}}^2$	Laplacian in Lagrangian coordinates $\overline{\mathcal{D}}_\alpha \overline{\mathcal{D}}^\alpha$
D_{ij}	trace-free operator $\partial_i \partial_i - \frac{1}{3}\nabla^2 \delta_{ij}$
$\dot{f}$	partial differentiation with respect to cosmic time t
f'	partial differentiation with respect to conformal time τ
H	Hubble rate in cosmic time $\frac{\dot{a}}{a}$
$\mathcal{H}$	Hubble rate in conformal time $\frac{a'}{a}$
ϱ_b	matter density in the Einstein-de Sitter background

APPENDIX B

PN and second-order expansion of the space-time metric

Notation

In a generic system of coordinates, in conformal time $\tilde{y}^0 = ct$ and spatial coordinates y^i the line element is

$$ds^2 = a^2\left(\tilde{g}_{00}d\tilde{y}^{0\,2} + 2\tilde{g}_{0i}d\tilde{y}^0 dy^i + \tilde{g}_{ij}dy^i dy^j\right) \tag{B.1}$$

We consider as our time coordinate $y^0 = \tilde{y}^0/c$. In this way we do not have hidden powers of c in the coordinates. Our line element is therefore

$$ds^2 = a^2\left(g_{00}dy^{0\,2} + 2g_{0i}dy^0 dy^i + \tilde{g}_{ij}dy^i dy^j\right) \tag{B.2}$$

where

$$g_{00} = c^2\tilde{g}_{00} \qquad g_{0i} = c\tilde{g}_{0i} \qquad \tilde{g}_{ij} = \tilde{g}_{ij}\,. \tag{B.3}$$

The conformal metric is $\gamma_{ab} = \dfrac{g_{ab}}{a^2}$.

Synchronous and comoving gauge

The following expressions are obtained from Matarrese et al. 1998 and from Bartolo et al. 2005.

Second-order expansion

Second-order scalar potentials

$$\nabla^2\Psi = -\frac{1}{2}\left[\left(\nabla^2\phi\right)^2 + \partial^k\partial^l\phi\partial_k\partial_l\phi\right] \tag{B.4}$$

$$\nabla^2\Theta = \Psi - \frac{1}{3}\left(\nabla\phi\right)^2 \tag{B.5}$$

Second-order spatial metric

$$\begin{aligned} g_{ij} \;=\;& a^2\left\{\delta_{ij} - \frac{\tau^2}{3}\partial_i\partial_j\phi + \frac{\tau^4}{252}\left[-12\partial_i\partial_j\phi\nabla^2\phi - 6\nabla^2\Psi + 19\partial^k\partial_i\phi\partial_k\partial_j\phi\right] + \overline{\Delta}_{ij} + \right. \\ & \left. + \frac{1}{c^2}\left[-\frac{10}{3}\phi + \frac{5}{18}\tau^2\left(\nabla\phi\right)^2\delta_{ij} + \frac{10}{9}\tau^2\left(a_{\rm nl} - 1\right)\left(\phi\partial_i\partial_j\phi + \partial_i\phi\partial_j\phi\right) - \frac{5}{9}\tau^2\partial_i\phi\partial_j\phi + \Delta_{ij}\right]\right\} \\ & + \frac{1}{c^4}\frac{50}{9}a_{\rm nl}\phi^2 \end{aligned} \tag{B.6}$$

where $\overline{\Delta}_{ij}$ and Δ_{ij} are the Newtonian and PN second-order tensors generated by first-order scalar and are given by

$$\overline{\Delta}_{ij} = \frac{\tau^4}{42}\left(\partial_i\partial_j\Psi + \nabla^2\Psi\delta_{ij}\right) + \frac{\tau^4}{21}\left(\partial_i\partial_j\phi\nabla^2\phi - \partial^k\partial_i\phi\partial_k\partial_j\phi\right) \tag{B.7}$$

and

$$\Delta_{ij} = \frac{2}{3}\tau^2\Psi\delta_{ij} + \frac{2}{3}\tau^2\nabla^{-2}\left(\partial_i\partial_j\Psi + 2\partial_i\partial_j\phi\nabla^2\phi - 2\partial^k\partial_i\phi\partial_k\partial_j\phi\right) \tag{B.8}$$

PN expansion

The following expressions are obtained from Matarrese & Terranova 1996 and from chapter 6.

PN scalar potentials

$$\overline{\mathcal{D}}^2\Upsilon_{\mathcal{L}} = -\frac{1}{2}\left(\overline{\vartheta}^2 - \overline{\vartheta}^\mu_\nu\overline{\vartheta}^\nu_\mu\right), \tag{B.9}$$

$$\overline{\mathcal{D}}^2\mathcal{Z}_{\mathcal{L}} = \Upsilon_{\mathcal{L}} - \frac{1}{3}\left(\nabla_q\xi^0_{\mathcal{L}}\right)^2 \tag{B.10}$$

Second-order expansion

$$\Upsilon^{(2)}_{\mathcal{L}} = \frac{\tau^2}{9}\Psi \tag{B.11}$$

$$\mathcal{Z}^{(2)}_{\mathcal{L}} = \frac{\tau^2}{9}\Theta \tag{B.12}$$

PN spatial metric

$$ds^2 = a^2\left\{-c^2d\tau^2 + \left[\overline{\gamma}_{\alpha\beta} + \frac{1}{c^2}w_{\alpha\beta}\right]dq^\alpha dq^\beta\right\}, \tag{B.13}$$

where

$$\overline{\gamma}_{\alpha\beta} = \mathcal{J}^A_\alpha\mathcal{J}^B_\beta\delta_{AB} \tag{B.14}$$

and

$$w_{\alpha\beta} = \chi\overline{\gamma}_{\alpha\beta} + \overline{\mathcal{D}}_\alpha\overline{\mathcal{D}}_\beta\zeta + \frac{1}{2}\left(\overline{\mathcal{D}}_\alpha\lambda_\beta + \overline{\mathcal{D}}_\beta\lambda_\alpha\right) + \pi_{\alpha\beta} \tag{B.15}$$

where the vector is divergence-less ($\overline{\mathcal{D}}^\alpha\lambda_\alpha = 0$) and the tensors are divergence-less and trace-free ($\overline{\mathcal{D}}^\alpha\pi_{\alpha\beta} = 0$ and $\pi^\alpha_\alpha = 0$).

PN spatial conformal curvature

$$^{(3)}\mathcal{R}^\alpha_\beta = -\frac{1}{2}\overline{\mathcal{D}}^\alpha\overline{\mathcal{D}}_\beta\chi - \frac{1}{2}\overline{\mathcal{D}}^2\chi\delta^\alpha_\beta - \frac{1}{2}\overline{\mathcal{D}}^2\pi^\alpha_\beta \tag{B.16}$$

$$^{(3)}\mathcal{R} = -2\overline{\mathcal{D}}^2\chi \tag{B.17}$$

PN scalars

$$\chi = 2\mathcal{H}\xi^0_{\mathcal{L}} - 2\varphi^{\mathcal{L}}_g - \Upsilon_{\mathcal{L}} \tag{B.18}$$

$$\zeta = 2\xi_{\mathcal{L}} + 3\mathcal{Z}_{\mathcal{L}} \tag{B.19}$$

Second-order expansion

$$\chi^{(1)} + \frac{1}{2}\chi^{(2)} = -\frac{10}{3}\phi + \frac{2}{3}\tau^2\Psi + \frac{5}{18}\tau^2\left(\nabla\phi\right)^2 - \frac{\tau^2}{9}\Psi \tag{B.20}$$

$$\zeta^{(2)} = \frac{5}{9}\tau^2\left(a_{\mathrm{nl}} - 1\right)\phi^2 - \frac{5}{9}\tau^2\phi^2 - \frac{5}{3}\tau^2\Theta \tag{B.21}$$

PN vector

$$\overline{\mathcal{D}}^2\lambda_\alpha = 2\overline{\mathcal{D}}^2\xi^{\perp}_\alpha - 4\overline{\mathcal{D}}_\alpha\Upsilon_{\mathcal{L}} - 2\partial_\alpha\xi^0_{\mathcal{L}}\overline{\vartheta} - 2\partial_\sigma\xi^0_{\mathcal{L}}\overline{\vartheta}^\sigma_\alpha \tag{B.22}$$

Second-order expansion

$$\nabla^2\lambda^{(2)}_\alpha = \frac{20}{9}\tau^2\partial_\alpha\Psi + \frac{10}{9}\tau^2\partial_\alpha\phi\nabla^2\phi - \frac{10}{9}\tau^2\partial_\sigma\phi\partial_\alpha\partial^\sigma\phi \tag{B.23}$$

PN tensors

$$\overline{\mathcal{D}}^2\pi_{\alpha\beta} = \overline{\mathcal{D}}_\alpha\overline{\mathcal{D}}_\beta\Upsilon_{\mathcal{L}} + \overline{\mathcal{D}}^2\Upsilon_{\mathcal{L}}\overline{\gamma}_{\alpha\beta} + 2\left(\overline{\vartheta\vartheta}_{\alpha\beta} - \overline{\vartheta}_{\alpha\sigma}\overline{\vartheta}^\sigma_\beta\right) \tag{B.24}$$

Second-order expansion

$$\nabla^2\pi^{(2)}_{\alpha\beta} = \frac{1}{9}\tau^2\partial_\alpha\partial_\beta\Psi + \frac{1}{9}\tau^2\nabla^2\Psi\delta_{\alpha\beta} + \frac{2}{9}\tau^2\partial_\alpha\partial_\beta\phi\nabla^2\phi - \frac{2}{9}\tau^2\partial_\alpha\partial_\sigma\phi\partial_\beta\partial^\sigma\phi \tag{B.25}$$

Poisson gauge

Second-order metric

The following expressions are obtained from Bartolo et al. 2010 and from Matarrese et al. 1998

$$g_{00} - a^2\left[1 + 2\phi - \frac{10}{21}\eta^2\Psi + \frac{\eta^2}{6}\left(\nabla\phi\right)^2 + \frac{1}{c^2}\left(2\phi^2 - \frac{10}{3}\left(a_{\mathrm{nl}} - 1\right)\phi^2 + 12\Theta\right)\right] \tag{B.26}$$

$$g_{0i} = a^2\left[-\frac{8}{3c^2}\tau\nabla^{-2}\left(2\partial_i\Psi + \partial_i\phi\nabla^2\phi - \partial^k\partial_i\phi\partial_k\phi\right)\right] \tag{B.27}$$

$$g_{ij} = a^2\left[1 + \frac{1}{c^2}\left(-2\phi + \frac{10}{21}\eta^2\Psi - \frac{\eta^2}{6}\left(\nabla\phi\right)^2\right) + \frac{1}{c^4}\left(2\phi^2 + \frac{10}{3}\left(a_{\mathrm{nl}} - 1\right)\phi^2 + 8\Theta\right)\right] \tag{B.28}$$

PN metric

$$ds^2 = a^2 \left[-\left(1 + U + \frac{V}{c^2}\right) d\eta^2 + 2\frac{P_A}{c^2} d\eta dx^A + (1 - U)\, \delta_{AB} dx^A dx^B \right] \tag{B.29}$$

where $U = 2\varphi_g^{\mathcal{E}}$ for the Newtonian term and $\partial_A P^A = 0$

PN gauge transformation from the Poisson gauge to the synchronous-comoving up to second order

$$\eta = \tau - \frac{1}{3}\tau\phi + \frac{\tau^3}{36}\partial^\sigma\phi\partial_\sigma\phi + \frac{\tau^3}{21}\Psi + \frac{5}{9}\tau^2 \left(a_{\rm nl} - 1\right)\phi^2 + \frac{\tau^2}{18}\phi^2 - 2\tau\Theta \tag{B.30}$$

$$x^\alpha = q^\alpha - \frac{\tau^2}{6}\partial^\alpha\phi + \frac{\tau^4}{84}\partial^\alpha\Psi - \frac{5}{9}\phi\partial^\alpha\phi - \tau^2\partial^\alpha\Theta + \frac{2}{3}\tau^2\nabla^{-2}\left(2\partial^\alpha\Psi + \partial^\alpha\phi\nabla^2\phi - \partial^\sigma\partial^\alpha\phi\partial_\sigma\phi\right) \tag{B.31}$$

APPENDIX C

Spatial transformation rules

The Newtonian metric tensors in the Eulerian and in the Lagrangian frame are related by

$$\overline{\gamma}_{\alpha\beta} = \mathcal{J}^A_\alpha \mathcal{J}^B_\beta \delta_{AB} \tag{C.1}$$

where $\mathcal{J}^A_\alpha$ is the Newtonian spatial Jacobian matrix with the inverse is given by

$$\mathcal{J}^\mu_A = \delta_{AB}\overline{\gamma}^{\mu\nu}\mathcal{J}^B_\nu \tag{C.2}$$

and satisfies

$$\delta^\alpha_\beta = \mathcal{J}^\alpha_B \mathcal{J}^B_\beta \tag{C.3}$$

and

$$\delta^A_B = \mathcal{J}^A_\sigma \mathcal{J}^\sigma_B \, . \tag{C.4}$$

The relation between the basis vector is

$$e_A = \mathcal{J}^\mu_A e_\mu \qquad \text{and} \qquad e_\mu = \mathcal{J}^B_\mu e_B \, , \tag{C.5}$$

where $e_A = \partial_A$ and $e_\mu = \partial_\mu$.

The relation between the basis 1-form is

$$\theta^A = \mathcal{J}^A_\alpha \theta^\alpha \qquad \text{and} \qquad \theta^\alpha = \mathcal{J}^\alpha_B \theta^B \, , \tag{C.6}$$

where $\theta^A = dx^A$ and $\theta^\alpha = dq^\alpha$.

Vector Consider a vector written in the two basis, $w = w^\alpha e_\alpha$ and $w = w^A e_A$. The set of components are related by

$$w^A = \mathcal{J}^A_\sigma w_\sigma \qquad \text{and} \qquad w^\alpha = \mathcal{J}^\alpha_B w^B \, . \tag{C.7}$$

1-form Consider a 1-form written in the two basis, $\omega = w_\alpha \theta^\alpha$ and $\omega = w_A \theta^A$. The set of components are related by

$$\omega_A = \mathcal{J}^\sigma_A \omega_\sigma \qquad \text{and} \qquad \omega_\alpha = \mathcal{J}^B_\alpha \omega_B \, . \tag{C.8}$$

Tensors Analogous rules hold for the components of tensors, *e.g.* we have

$$T^A_B = \mathcal{J}^A_\sigma T_B = \mathcal{J}^\nu_B T^A_\nu = \mathcal{J}^A_\sigma \mathcal{J}^\nu_B T^\sigma_\nu \tag{C.9}$$

Second spatial derivative of a scalar Consider the second spatial derivative of a scalar written in the Eulerian basis, $\partial_A \partial_B f_{\mathcal{E}} dx^A dx^B$. Its component in the basis dq^α are found from

$$\begin{aligned}
\partial_A \partial_B f_{\mathcal{E}} dx^A \otimes dx^B &= \mathcal{J}_A^\sigma \partial_\sigma \left(\mathcal{J}_B^\rho \partial_\rho\right) f_{\mathcal{L}} \mathcal{J}_\mu^A dq^\mu dq^\nu \otimes \partial_A \mathcal{J}_\lambda^B dq^\lambda \\
&= \left(\mathcal{J}_A^\sigma \partial_\sigma \mathcal{J}_B^\rho \partial_\rho f_{\mathcal{L}} + \mathcal{J}_A^\sigma \mathcal{J}_B^\rho \partial_\sigma \partial_\rho f_{\mathcal{L}}\right) \mathcal{J}_\mu^A dq^\mu \otimes \mathcal{J}_\lambda^B dq^\lambda \\
&= \left(\delta_\mu^\sigma \mathcal{J}_\lambda^B \partial_\sigma \mathcal{J}_B^\rho \partial_\rho f_{\mathcal{L}} + \delta_\mu^\sigma \delta_\lambda^\rho \partial_\sigma \partial_\rho f_{\mathcal{L}}\right) dq^\mu \otimes dq^\lambda \\
&= \left(\overline{\Gamma}_{\mu\lambda}^\rho \partial_\rho f_{\mathcal{L}} + \partial_\mu \partial_\lambda f_{\mathcal{L}}\right) dq^\mu \otimes dq^\lambda \\
&= \overline{D}_\mu \overline{D}_\nu f_{\mathcal{L}} dq^\mu \otimes dq^\lambda
\end{aligned} \tag{C.10}$$

Laplancian of a scalar From the previous relation, the components of the Eulerian Laplacian of a scalar transform as

$$\begin{aligned}
\partial_A \partial^A f_{\mathcal{E}} &= \partial_A \delta^{AS} \partial_S f_{\mathcal{E}} \\
&= \mathcal{J}_A^\beta \partial_\beta \left(\delta^{AS} \mathcal{J}_S^\mu \partial_\mu f_{\mathcal{L}}\right) \\
&= \mathcal{J}_A^\beta \partial_\beta \left(\delta^{AS} \delta_{SB} \overline{\gamma}^{\mu\nu} \mathcal{J}_\nu^B \partial_\mu f_{\mathcal{L}}\right) \\
&= \mathcal{J}_A^\beta \partial_\beta \left(\overline{\gamma}^{\mu\nu} \mathcal{J}_\nu^A \partial_\mu f_{\mathcal{L}}\right) \\
&= \mathcal{J}_A^\beta \partial_\beta \left(\mathcal{J}_\nu^A \partial^\nu f_{\mathcal{L}}\right) \\
&= \mathcal{J}_A^\beta \partial_\beta \mathcal{J}_\nu^A \partial^\nu f_{\mathcal{L}} + \mathcal{J}_A^\beta \mathcal{J}_\nu^A \partial_\beta \partial^\nu f_{\mathcal{L}} \\
&= \overline{\Gamma}_{\beta\nu}^\beta \partial^\nu f_{\mathcal{L}} + \partial_\nu \partial^\nu f_{\mathcal{L}} \\
&= \overline{\mathcal{D}}^2 f_{\mathcal{L}}
\end{aligned} \tag{C.11}$$

Object in a mixed basis An object in a mixed basis, like *e.g.* $\partial_\nu \xi^A dq^\nu \otimes \partial_A$ which we have in the PN coordinate transformation, transforms as

$$\begin{aligned}
\partial_\nu \xi^A dq^\nu \otimes \partial_A &= \partial_\nu \left(\mathcal{J}_\sigma^A \xi^\sigma\right) dq^\nu \otimes \mathcal{J}_A^\beta \partial_\beta \\
&= \left(\partial_\nu \mathcal{J}_\sigma^A \xi^\sigma \mathcal{J}_A^\beta + \mathcal{J}_\sigma^A \mathcal{J}_A^\beta \partial_\nu \xi^\sigma\right) dq^\nu \otimes \partial_\beta \\
&= \left(\overline{\Gamma}_{\nu\rho}^\beta \xi^\rho + \partial_\nu \xi^\beta\right) dq^\nu \otimes \partial_\beta \\
&= \overline{D}_\nu \xi^\beta dq^\nu \otimes \partial_\beta
\end{aligned} \tag{C.12}$$

APPENDIX D

PN expansion of average kinematical quantities

In this appendix the perturbative expansions of kinematical quantities are given with respect to the decomposition of the spatial metric

$$\gamma_{\alpha\beta} = \overline{\gamma}_{\alpha\beta} + \frac{1}{c^2} w_{\alpha\beta}. \tag{D.1}$$

The Newtonian background quantities are indicated with a bar, the averaged perturbed quantities are indicated with $\langle \quad \rangle_{\mathcal{D}}^{PN}$, so that for a generic scalar Ψ one has

$$\langle \Psi \rangle_{\mathcal{D}}^{PN} = \langle \overline{\Psi} \rangle_{\mathcal{D}} + \frac{1}{c^2} \delta \langle \Psi \rangle_{\mathcal{D}}. \tag{D.2}$$

With $\langle \quad \rangle_{\mathcal{D}}$ is indicated the averaging using the background spatial metric $\overline{\gamma}_{\alpha\beta}$. A dot denotes partial differentation with respect to time. The trace of the perturbation $w_{\alpha\beta}$ is denoted with w.

Volume

$$\mathcal{V}_{\mathcal{D}} := \int_{\mathcal{D}} \sqrt{\gamma} d^3 q$$
$$\mathcal{V}_{\mathcal{D}}^{PN} = \int_{\mathcal{D}} \sqrt{\overline{\gamma}} d^3 q + \frac{1}{2c^2} \int_{\mathcal{D}} \sqrt{\overline{\gamma}} w d^3 q$$

Trace of gradient-velocity tensor

$$\langle \vartheta \rangle_{\mathcal{D}} := \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \vartheta \sqrt{\gamma} d^3 q$$
$$\langle \vartheta \rangle_{\mathcal{D}}^{PN} = \langle \overline{\vartheta} \rangle_{\mathcal{D}} + \frac{1}{2c^2} \left(\langle \overline{\vartheta} w \rangle_{\mathcal{D}} + \langle \dot{w} \rangle_{\mathcal{D}} - \langle \overline{\vartheta} \rangle_{\mathcal{D}} \langle w \rangle_{\mathcal{D}} \right)$$

Square of trace of gradient-velocity tensor

$$\langle \vartheta^2 \rangle_{\mathcal{D}} := \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \vartheta^2 \sqrt{\gamma} d^3 q$$
$$\langle \vartheta^2 \rangle_{\mathcal{D}}^{PN} = \langle \overline{\vartheta}^2 \rangle_{\mathcal{D}} + \frac{1}{c^2} \left(\frac{1}{2} \langle \overline{\vartheta}^2 w \rangle_{\mathcal{D}} + \langle \dot{w} \rangle_{\mathcal{D}} - \frac{1}{2} \langle \overline{\vartheta}^2 \rangle_{\mathcal{D}} \langle w \rangle_{\mathcal{D}} \right)$$

Shear magnitude

$$\langle\sigma^2\rangle_{\mathcal{D}} := \frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}}\frac{1}{2}\sigma^{\alpha}_{\beta}\sigma^{\beta}_{\alpha}\sqrt{\gamma}d^3q$$
$$\langle\sigma^2\rangle^{PN}_{\mathcal{D}} = \langle\overline{\sigma}^2\rangle_{\mathcal{D}} + \frac{1}{2c^2}\left(\langle\overline{\sigma}^2 w\rangle_{\mathcal{D}} + \langle\overline{\vartheta}^{\alpha}_{\beta}\dot{w}^{\beta}_{\alpha} - \frac{1}{3}\overline{\vartheta}\dot{w}\rangle_{\mathcal{D}} - \langle\overline{\sigma}^2\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}}\right)$$

Square of averaged trace of gradient-velocity tensor

$$\langle\vartheta\rangle^2_{\mathcal{D}} := \left(\frac{1}{\mathcal{V}_{\mathcal{D}}}\int_{\mathcal{D}}\vartheta\sqrt{\gamma}d^3q\right)^2$$
$$\langle\vartheta\rangle^{2PN}_{\mathcal{D}} = \langle\overline{\vartheta}\rangle^2_{\mathcal{D}} + \frac{1}{c^2}\langle\overline{\vartheta}\rangle_{\mathcal{D}}\left(\langle\overline{\vartheta}w\rangle_{\mathcal{D}} + \langle\dot{w}\rangle_{\mathcal{D}} - \langle\overline{\vartheta}\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}}\right)$$

These quantities are used for the calculation of the PN kinematical backreaction

Kinematical backreaction

$$\mathcal{Q}_{\mathcal{D}} := 2\langle\frac{1}{3}\vartheta^2 - \sigma^2\rangle_{\mathcal{D}} - \frac{2}{3}\langle\vartheta\rangle^2_{\mathcal{D}}$$

Recalling that $\overline{\mathcal{Q}_{\mathcal{D}}} = 0$, one finds

$$\begin{aligned}\mathcal{Q}^{PN}_{\mathcal{D}} = \frac{1}{c^2}\Bigg(&\frac{1}{3}\langle\overline{\vartheta}^2 w\rangle_{\mathcal{D}} + \frac{2}{3}\langle\dot{w}\rangle_{\mathcal{D}} - \frac{1}{3}\langle\overline{\vartheta}^2\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}} - \frac{2}{3}\langle\overline{\vartheta}\rangle_{\mathcal{D}}\langle\overline{\vartheta}w\rangle_{\mathcal{D}} + \\ &-\frac{1}{3}\langle\overline{\vartheta}\rangle_{\mathcal{D}}\langle\dot{w}\rangle_{\mathcal{D}} + \frac{2}{3}\langle\overline{\vartheta}\rangle^2_{\mathcal{D}}\langle w\rangle_{\mathcal{D}} - \langle\overline{\sigma}^2 w\rangle_{\mathcal{D}} - \langle\overline{\vartheta}^{\alpha}_{\beta}\dot{w}^{\beta}_{\alpha}\rangle_{\mathcal{D}} + \frac{1}{3}\langle\overline{\vartheta}\dot{w}\rangle_{\mathcal{D}} + \langle\overline{\sigma}^2\rangle_{\mathcal{D}}\langle w\rangle_{\mathcal{D}}\Bigg).\end{aligned}$$

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