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On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type

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ABSTRACT

For linear and semilinear equations of Tricomi type, existence, uniqueness and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem on characteristic triangles will be established. For the linear problem, a robust L^2-based theory will be developed, including well-posedness, elements of a spectral theory, partial regularity results and maximum and comparison principles. For the nonlinear problem, existence of weak solutions with nonlinearities of unlimited polynomial growth at infinity will be proven by combining standard topological methods of nonlinear analysis with the linear theory developed here. For homogeneous *supercritical* nonlinearities, the uniqueness of the trivial solution in the class of weak solutions will be established by combining suitable Pohožaev-type identities with well tailored mollifying procedures. For the linear problem, the weak existence theory presented here will also be connected to known explicit representation formulas for sufficiently regular solutions with the aid of the partial regularity results. For the nonlinear problem, the question what constitutes critical growth for the problem will be clarified and differences with equations of mixed elliptic–hyperbolic type will be exhibited.

1. Introduction

In this work, we will study the existence and uniqueness of *weak solutions u* for the semilinear Goursat[1] problem

$$\begin{cases} Tu = f(x,y,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases} \tag{1.1}$$

where $T \equiv -y\partial_x^2 - \partial_y^2$ is the Tricomi operator on $\mathbb{R}^2$ with cartesian coordinates (x, y), f is a nonlinearity to be specified and $\Omega = ABC$ is a *characteristic triangle*; that is, a simply connected region in the plane whose boundary consists of the segment

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[1] Often called the (first) Darboux problem in the Russian literature.

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AB of the x-axis and the two characteristics arcs AC and BC (of negative and positive slope respectively) that issue from A and B and intersect at C. The solutions will be found in the subspace of the standard Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$ of elements having zero trace on Γ. For the linear problem, where $f(x, y, s) = f(x, y) + \lambda s$ with $\lambda \in \mathbb{R}$, we will develop a robust L^2-based theory whose compact solution operator provides some spectral information and is compatible with weak maximum and comparison principles, which are obtained with the aid of some regularity theory. Then, using standard topological tools of nonlinear analysis and carefully constructed mollifying procedures, we will establish results on existence and uniqueness for the nonlinear problem under suitable hypotheses on the nonlinearity f.

Our primary interest in the nonlinear version (1.1) of the well studied linear Goursat problem is purely mathematical as a companion to our study of analogous questions on mixed type domains (i.e. Ω intersects also the region where $y > 0$). The existence of *weak solutions* with Tricomi boundary conditions has been treated in [1] and with Dirichlet conditions in [2–4] while the uniqueness of *classical solutions* has been treated for various class of domains and boundary conditions [5–8]. More precisely, we will seek to clarify the interaction between the form of the boundary conditions (i.e. Dirichlet conditions on the entire boundary or on a suitable proper subset of the boundary), the geometry of the domain at the parabolic boundary points (i.e. A and B, where the operator degenerates), the regularity of the solutions for $f \in L^2(\Omega)$ (i.e. the presence or not of a weight in the $H^1(\Omega)$ norm of the weak solutions), the resulting barriers to p-summability coming from the Sobolev imbedding theorem and its relation to critical exponent phenomena.

We will see that the geometry of having corners in A and B combined with placing the Dirichlet conditions only on the proper subset $\Gamma = AB \cup AC$ allows for weak solutions in $H^1(\Omega)$ and hence no barrier to immersion in $L^p(\Omega)$ for each $p \in [1, +\infty)$. For the mixed type Tricomi problem in *angular domains* (with corners in A and B), the weak solutions also lie in $H^1(\Omega)$, as was shown in [9,1]. On the other hand, both for Tricomi problem in *normal domains* (where the elliptic boundary is orthogonal to the x-axis in A and B) and for the Dirichlet problem on suitable domain, the weak solutions carry the weight $|y|$ on the first derivative in x, as was shown in [10,2]. As a result, one has a critical exponent in the Sobolev imbedding which is $2^*(1, 1) = 10$, as noted in [5]. Moreover, the Goursat problem will be shown to admit maximum and comparison principles for weak solutions such as those in the mixed type setting of the Tricomi problem in normal domains [10]; however, for weak solutions with weights as noted above in the mixed type case. The better regularity of the solutions in the Goursat case allows us to apply monotone methods (upper and lower solutions) with no limit on the polynomial growth in s for the nonlinearity $f(x, y, s)$ in contrast to the strong restrictions on growth required in the mixed type setting, as one knows from [1].

In addition, a nonhomogeneous dilation invariance in the Tricomi operator T is known to yield a Pohožaev-type result on the nonexistence of nontrivial solutions u with homogeneous boundary conditions $u = 0$ placed on a large enough portion of the boundary of a suitably star-shaped domains; that is, if $f(x, y, s) = s|s|^{p-2}$ with $p > 10$ and then the only $C^2(\overline{\Omega})$ solutions must vanish identically, as shown in [5,6]. By exploiting the special geometry of the Goursat domain through well-tailored mollifying operators and by exploiting the absence of weights in the weak solutions, we will close the regularity gap between $C^2(\overline{\Omega})$ (where one had uniqueness) and $H^1(\Omega)$ (where one has existence results). Closing this regularity gap was the original motivation for studying the nonlinear Goursat problem (1.1), but much more has come out of the investigation. In particular, with respect to what may constitute critical growth for the problem (1.1), the following situation emerges. There is no polynomial growth barrier for the purposes of existence and no polynomial critical growth exponent for the Sobolev imbedding for the weak solutions with no weights. This dissimilarity with elliptic problems should be perhaps explained by the fact that the Goursat problem is not variational. On the other hand, if one were to impose the boundary condition also on the characteristic BC, the problem becomes variational but the extra boundary data forces the weak solutions to carry weights (as in the mixed type Dirichlet problem [2]), which in turn yields the critical exponent for the Sobolev immersion and a probable barrier to existence for weak solutions at supercritical growth. Moreover, the Dirichlet problem loses the maximum principle and hence also the possibility for monotone methods which are used here to solve the superlinear problems.

The plan of the paper is as follows. In Section 2, we recall the necessary machinery and develop the linear solvability and spectral theory. In Section 3, we examine the question of regularity of the weak solutions and maximum/comparison principles compatible with the solvability theory. An important byproduct will be the bridging of a possible gap between *generalized solutions* as given by explicit integral representations involving hypergeometric functions and our notion of weak solutions whose existence follows from suitable *a priori* estimates (see Theorem 3.2 and the discussion in Step 1 of the proof). In Section 4, we prove results on existence of weak solutions. We exploit the contraction mapping principle and Leray–Schauder principles for sublinear and asymptotically linear nonlinearities and monotone methods for superlinear nonlinearities. In Section 5, we prove the aforementioned extension of the uniqueness of the trivial solution for weak solutions. In addition there are two appendices which give the proofs of two technical lemmas concerning the compactness on $C^0(\overline{\Omega})$ of the linear solution operator and the weak maximum principle for regular solutions.

We conclude this introduction with a few additional remarks on the problems considered herein. Some of the results continue to hold for operators of Tricomi type where the coefficient y in the Tricomi operator T is replaced by $K(y)$ which has the sign of y; for example, results on solvability and maximum principles for regular solutions. On the other hand, the maximum principle for weak solutions (and hence the monotone methods of Section 5) makes use of the regularity result of Nakhushev [11] which holds for $K(y) = (-y)^m$ for $m < 2$. We have treated problems in only two dimensions. In part, this is due to the importance of the Tricomi equation in the context of two dimensional transonic potential flow (see the modern survey of Morawetz [12]), but it should be noted that the analogous boundary value problem in higher dimensions (the so-called *Protter problem*) has a solvability theory which is much more delicate (see [13] for example). On the other hand, questions of nonexistence for weak solutions to nonlinear degenerate hyperbolic Cauchy problems in general dimensions has been well studied by Mitidieri and Pohožaev (see [14] and the references therein). Using similar techniques, Laptev [15] has treated nonexistence for weak solutions to nonlinear hyperbolic Cauchy problems in cones.

2. Linear theory: weak solvability and spectral theory

In this section, we will analyze the question of existence of weak solutions the problem:

$$\begin{cases} Tu = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AB \cup AC, \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega)$. We will also consider the same problem with $T - \lambda I$ in place of T and with a nonhomogeneous boundary condition $u = \gamma$ on Γ with $\gamma \in \mathbb{R}$ in order to derive some elements of a spectral theory as well as maximum and comparison principles which will be used in order to establish existence results for the nonlinear problems in Section 4. We will first recall some basic notions.

2.1. Notations and background

Since the operator T is invariant with respect to translations in x, we may assume that the domain $\Omega = ABC$ is symmetric with respect to the y-axis.[2] In particular we will denote by A, B and C the points $(-x_0, 0)$, $(x_0, 0)$ and $(0, y_C)$ where $x_0 > 0$ and $y_C = -(3x_0/2)^{2/3}$. The characteristics are then given by

$$AC : x + x_0 - \frac{2}{3}(-y)^{3/2} = 0 \quad \text{and} \quad BC : x - x_0 + \frac{2}{3}(-y)^{3/2} = 0, \tag{2.2}$$

while the parabolic segment is $AB = \{(x,0) : |x| < x_0\}$. The *conjugate boundary* is the set $\Gamma^* = AB \cup BC$ and the *adjoint problem* is

$$\begin{cases} Tu = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma^* = BC \cup AB, \end{cases} \tag{2.3}$$

where one should note that T is *formally self-adjoint*, that is, $T = T^t$ where T^t is the formal adjoint defined by

$$T^t u = -D_x^2(yu) - D_y^2(u) = Tu. \tag{2.4}$$

Weak solutions to (2.1) will belong to the space $H^1_\Gamma(\Omega)$ which is the completion in the norm

$$\|\psi\|_{H^1(\Omega)} = \left(\int_\Omega (\psi_x^2 + \psi_y^2 + \psi^2)\, dxdy \right)^{1/2}$$

of the space

$$C^\infty_\Gamma(\overline{\Omega}) = \left\{ \psi \in C^\infty(\overline{\Omega}) : \psi \equiv 0 \text{ on } N_\epsilon(\Gamma) \text{ for some } \epsilon > 0 \right\},$$

where $N_\epsilon(\Gamma)$ is an ϵ neighborhood of $\Gamma = AB \cup AC$. Since $\partial\Omega$ is Lipschitz, there is a well defined linear and continuous trace operator (see Section 4.3 of [16])

$$\mathrm{tr}_\Gamma : H^1(\Omega) \to L^2(\Gamma) \tag{2.5}$$

and clearly $\psi \in H^1(\Omega)$ lies in $H^1_\Gamma(\Omega)$ if and only if $\mathrm{tr}_\Gamma(\psi) = 0$ in $L^2(\Gamma)$. Since ψ has zero trace on a sufficiently large part of the boundary, one has a *Poincaré inequality*

$$\|\psi\|_{L^2(\Omega)} \le C_P \left(\int_\Omega (\psi_x^2 + \psi_y^2)\, dxdy \right)^{1/2}$$

for some constant $C_P > 0$ and hence an equivalent norm

$$\|\psi\|_{H^1_\Gamma(\Omega)} = \left(\int_\Omega (\psi_x^2 + \psi_y^2)\, dxdy \right)^{1/2}$$

on $H^1_\Gamma(\Omega)$. Similar considerations hold for the conjugate boundary Γ^*. We will denote by $H^{-1}_\Gamma(\Omega)$ the dual space to H^1_Γ equipped with its negative norm in the sense of Lax [17]. The spaces $H^1_{\Gamma^*}(\Omega)$ and $H^{-1}_{\Gamma^*}(\Omega)$ are defined analogously. One easily verifies the following estimates: there exist constants $C_1, C_2 > 0$ such that

$$\|Tu\|_{H^{-1}_{\Gamma^*}(\Omega)} \le C_1 \|u\|_{H^1_\Gamma(\Omega)}, \quad u \in C^\infty_\Gamma(\overline{\Omega}) \tag{2.6}$$

[2] For the same reason, we may assume instead that A (or B) is the origin without loss of generality.

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and

$$\|Tv\|_{H^{-1}_{\Gamma}(\Omega)} \le C_2 \|v\|_{H^1_{\Gamma^*}(\Omega)}, \quad v \in C^\infty_{\Gamma^*}(\overline{\Omega}). \tag{2.7}$$

Hence one has continuous extensions of the Tricomi operator T (defined on dense subspaces of smooth functions)

$$T_\Gamma : H^1_\Gamma(\Omega) \to H^{-1}_{\Gamma^*}(\Omega) \quad \text{and} \quad T_{\Gamma^*} : H^1_{\Gamma^*}(\Omega) \to H^{-1}_{\Gamma}(\Omega). \tag{2.8}$$

We recall that the placement of the boundary conditions on only a portion of the boundary implies that the problem (2.1) is not self-adjoint. In fact, one checks easily that the continuous extensions (2.8) satisfy $T_{\Gamma^*} = (T_\Gamma)^*$. We will find weak solutions u to the linear problem (2.1) in the following sense.

Definition 2.1. Given $f \in L^2(\Omega)$ one says that $u \in H^1_\Gamma(\Omega)$ is a weak solution of (2.1) if one of the following equivalent conditions hold:

(i) There exists a sequence $\{u_j\} \subset C^\infty_\Gamma(\overline{\Omega})$ such that

$$\lim_{j\to\infty} \|u_j - u\|_{H^1_\Gamma(\Omega)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|Tu_j - f\|_{H^{-1}_{\Gamma^*}(\Omega)} = 0. \tag{2.9}$$

(ii) One has the relation

$$\mathcal{B}(u,v) := \int_\Omega \left(yu_x v_x + u_y v_y\right)\, dxdy = \int_\Omega f v\, dxdy, \quad \forall\, v \in C^\infty_{\Gamma^*}(\overline{\Omega}), \tag{2.10}$$

where the bilinear form $\mathcal{B}$ in (2.10) is clearly continuous on $H^1_\Gamma(\Omega) \times H^1_{\Gamma^*}(\Omega)$.

Weak solutions of (2.3) are defined in the analogous way. As shown by Didenko [18], the conditions (2.9) and (2.10) are equivalent. Moreover, a necessary and sufficient condition to have the generalized solvability for the problems (2.1) and (2.3) for each $f, g \in L^2(\Omega)$ is to have the continuity estimates (2.6) and (2.7) as well as the following a priori estimates: there exist positive constants C_3 and C_4 such that

$$\|\varphi\|_{L^2(\Omega)} \le C_3 \|T\varphi\|_{H^{-1}_{\Gamma^*}(\Omega)}, \quad \varphi \in C^\infty_\Gamma(\overline{\Omega}) \tag{2.11}$$

$$\|\psi\|_{L^2(\Omega)} \le C_4 \|T\psi\|_{H^{-1}_{\Gamma}(\Omega)}, \quad \psi \in C^\infty_{\Gamma^*}(\overline{\Omega}). \tag{2.12}$$

Notice that for a general second order operator, one should use the formal adjoint T^t in the estimate (2.11), but here T is formally self-adjoint as noted in (2.4).

2.2. Solvability theory

The first result is the following theorem.

Theorem 2.2. *For every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H^1_\Gamma(\Omega)$ in the sense of Definition 2.1 to the problem (2.1). Moreover, the solution operator*

$$S_\Gamma : L^2(\Omega) \to H^1_\Gamma(\Omega) \tag{2.13}$$

which assigns to $f \in L^2(\Omega)$ the unique weak solution $u \in H^1_\Gamma(\Omega)$ of the problem (2.1) is linear and continuous. Analogous statements hold for the adjoint problem (2.3).

Proof. For the solvability results, it is enough to establish the a priori estimates (2.11) and (2.12). By the symmetry of the problem it is clear that it suffices to show (2.11). One can do this by estimating from above and below the expression

$$\mathcal{I} = \int_\Omega \psi T\varphi\, dxdy \tag{2.14}$$

with $\varphi \in C^\infty_\Gamma(\overline{\Omega})$ fixed but arbitrary and ψ the solution to the auxiliary Cauchy problem

$$\begin{cases} M\psi = b\psi_x + c\psi_y = \varphi & \text{in } \Omega \\ \psi = 0 & \text{on } \Gamma^* = BC \cup AB, \end{cases}$$

where the coefficients (b, c) of M can be taken as

$$b = -(1+\varepsilon x) \quad \text{and} \quad c = -h(1+\varepsilon x) \tag{2.15}$$

with

$$0 < \varepsilon < 1/x_0 \quad \text{and} \quad 0 < h < (3x_0/2)^{-1/3}. \tag{2.16}$$

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This has been done in Proposition 2.2 of [9] for Ω a Tricomi domain where the parabolic segment AB is replaced by a suitable arc σ in the elliptic region (where $y > 0$) and endpoints at A and B. One easily checks that everything goes through as before. See the appendix of [9] for details. The linearity of S_Γ is obvious and the continuity of the solution operator uses a standard argument by contradiction and the relation (2.10). □

Remark 2.3. Since Ω is a bounded Lipschitz domain, one has the compactness of the imbedding into Lebesgue spaces (cf. Section 4.6 of [16]): for each $p \in [1, \infty)$ one has

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \tag{2.17}$$

where there is no upper limit on $p \in \mathbb{R}$ since $\Omega \subset \mathbb{R}^2$. In particular, the solution operator S_Γ defined in (2.13) yields a compact operator on $L^2(\Omega)$ which is an injective but non surjective map. This has obvious consequences for the spectral theory and a Fredholm alternative for the problem (2.1).

In order to discuss the spectral theory and as preparation for the use of monotone methods for the existence of solutions to (1.1) with superlinear nonlinearities, we will be interested in weak solutions to the following generalization of the problem (2.1):

$$\begin{cases} Tu - \lambda u = f & \text{in } \Omega \\ u = \gamma & \text{on } \Gamma, \end{cases} \tag{2.18}$$

where $\lambda, \gamma \in \mathbb{R}$ and $f \in L^2(\Omega)$. The notion of weak solution is the obvious one.

Definition 2.4. An element $u \in H^1(\Omega)$ will be called a weak solution of (2.18) if $u - \gamma \in H^1_\Gamma(\Omega)$ and the following analog of (2.10) holds:

$$\mathcal{B}_\lambda(u, v) := \int_\Omega \left(y u_x v_x + u_y v_y - \lambda u v\right)\, dxdy = \int_\Omega f v\, dxdy, \quad \forall\, v \in C^\infty_{\Gamma^*}(\overline{\Omega}).$$

Theorem 2.5. *Let $f \in L^2(\Omega), \gamma \in \mathbb{R}$ and $\lambda \leq 0$. The problem (2.18) admits a unique weak solution in the sense of Definition 2.4 and the solution operator*

$$S^{\lambda,\gamma}_\Gamma : L^2(\Omega) \to H^1(\Omega)$$

which assigns to $f \in L^2(\Omega)$ the unique weak solution $u \in H^1(\Omega)$ of the problem (2.18) is linear and continuous. Analogous statements hold for the adjoint problem with the boundary condition $u = \gamma$ on Γ^*.

Proof. We first consider the case of homogeneous boundary conditions $\gamma = 0$. When $\lambda = 0$, this is just Theorem 2.2 and $S^{0,0}_\Gamma$ is just the solution operator S_Γ. For $\lambda < 0$, one repeats the argument used in the proof of Theorem 2.2 with $T - \lambda I$ in place of T. The new term corresponding to λ in the expression (2.14) satisfies

$$-\lambda \int_\Omega \psi\varphi\, dxdy = -\frac{\lambda}{2}\left[\int_{AC} \psi^2(b, c) \cdot \nu\, ds + \varepsilon \int_\Omega \psi^2\, dxdy\right] \geq 0,$$

as $(b, c) \cdot \nu \geq 0$ on AC if (2.15)–(2.16) hold. A solution operator $S^{\lambda,0}_\Gamma : L^2(\Omega) \to H^1_\Gamma(\Omega)$ is thus well defined, linear and continuous.

If $\gamma \neq 0$ and $\lambda \leq 0$, we look for $u = w + \gamma$ where $w \in H^1_\Gamma(\Omega)$ is a weak solution of

$$\begin{cases} Tw - \lambda w = f + \lambda\gamma & \text{in } \Omega \\ w = 0 & \text{on } \Gamma, \end{cases}$$

Since Ω is bounded, $f + \lambda\gamma \in L^2(\Omega)$ and since $\lambda \leq 0$, one has that w exists and is unique by the previous step. Thus a weak solution $u \in H^1(\Omega)$ to (2.18) exists and is clearly unique. The resulting solution operator $S^{\lambda,\gamma}_\Gamma : L^2(\Omega) \to H^1(\Omega)$ defined by

$$S^{\lambda,\gamma}_\Gamma(f) = \gamma + S^{\lambda,0}_\Gamma(f + \lambda\gamma)$$

is linear, continuous and satisfies $\mathrm{tr}_\Gamma(S^{\lambda,\gamma}_\Gamma(f)) = \gamma$ where tr_Γ is the trace operator (2.5). □

2.3. Spectral theory

Given that the operators T_Γ and T_{Γ^*} defined in (2.8) do not have self-adjoint realizations on $L^2(\Omega)$, their spectra will be in general complex. For the applications to existence for the nonlinear problem (1.1), we will be interested in the *real spectrum* of T_Γ; that is, the description of $(\lambda, u) \in \mathbb{R} \times H^1_\Gamma$ such that $u \neq 0$ is a weak solution of $Tu = \lambda u$. We will denote by

$\Sigma(T_\Gamma)$ the set of such real λ for which a nontrivial u exists. Composing the solution operator S_Γ defined in (2.13) with the compact imbedding (2.17) with $p = 2$ gives rise to a compact solution operator

$$S_\Gamma : L^2(\Omega) \to H^1_\Gamma(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

whose spectrum $\sigma(S_\Gamma) \subset \mathbb{C}$ consists of $\{0\}$ and eigenvalues of finite multiplicity. Hence $\lambda \in \Sigma(T_\Gamma)$ if and only if $0 \neq \mu := 1/\lambda \in \sigma(S_\Gamma)$ and is real. Similar considerations hold for T_{Γ^*}.

Remark 2.6. Since the operator norm $M_0 := \|S_\Gamma\|_{\text{op}}$ of the compact operator S_Γ equals the spectral radius of S_Γ one has

$$\lambda \in \Sigma(T_\Gamma) \Rightarrow |\lambda| \geq M_0^{-1}.$$

Moreover, the estimate (2.11) shows that the solution operator satisfies

$$\|S_\Gamma f\|_{L^2(\Omega)} \leq C_3 \|f\|_{H_\Gamma^{-1}(\Omega)} \leq C_3 \|f\|_{L^2(\Omega)}$$

and hence $M_0 \leq C_3$ can be estimated from above by C_3 and one has a lower bound C_3^{-1} for the absolute value of $\lambda \in \Sigma(T_\Gamma)$. Optimizing the constant C_3 in the a priori estimate refines the spectral bound (see Example 2.7 in [1] in the mixed type setting).

Combining these considerations with the solvability established in Theorem 2.5 yields the following result.

Theorem 2.7. *One has $\Sigma(T_\Gamma) \cap (-\infty, M_0^{-1}) = \emptyset$, with $M_0 = \|S_\Gamma\|_{\text{op}}$ as above.*

Proof. By Remark 2.6, one has $\Sigma(T_\Gamma) \cap (-M_0^{-1}, M_0^{-1}) = \emptyset$ so it is enough to show that $\lambda \notin \Sigma(T_\Gamma)$ for each $\lambda < 0$. But this is a direct consequence of Theorem 2.5 in the case $\lambda < 0$ and $\gamma = 0$. □

We conclude by noting that the comparison principle Theorem 3.4 suggests that T_Γ might admit a *principal eigenvalue*; that is a real (and positive) eigenvalue of minimum modulus with an associated positive eigenfunction. This has been done in [19] for the mixed type Tricomi problem and involves an application of Krein–Rutman theory and the strong maximum principle which is valid in the elliptic region. Here in the degenerate hyperbolic case, we have no such strong maximum principle.

3. Linear theory: regularity and maximum principles

In this section, we analyze some partial regularity results for solutions to the linear problem (2.18) on $\Omega = ABC$ if $f \in C^0(\overline{\Omega})$ or $f \in C_0^\infty(\Omega)$ and then examine the validity of maximum and comparison principles for regular and weak solutions. The main point is that a combination of the solvability result (Theorem 2.5), some regularity theory and a refinement of the maximum principle of Agmon, Nirenberg and Protter [20] yields a comparison principle for the problem (2.18) which is compatible with the solvability theory. This has been done in the case of the mixed type Tricomi problem (see Theorem 3.1 of [10]). The main difference is that in place of the C^0 solvability result of Agmon [21] for the Tricomi problem, we will use a regularity result of Nakhushev [11] for the Goursat problem with homogeneous boundary data and $\lambda = 0$ prove an analogous C^0 solvability result for the problem (2.18) (see the remark after the proof of Theorem 3.2).

3.1. *Interior regularity and continuity up to the boundary*

We begin with the following interior regularity result.

Theorem 3.1. *Let $\gamma \in \mathbb{R}, \lambda \leq 0$ and $f \in C_0^\infty(\Omega)$. Then the unique weak solution $u \in H^1(\Omega)$ to problem (2.18) belongs to $C^\infty(\Omega)$; that is, there exists $u^* \in C^\infty(\Omega)$ such that $u = u^*$ a.e. in Ω.*

Proof. This result has been proven for the mixed elliptic–hyperbolic Tricomi problem when $\lambda = \gamma = 0$ (see Lemma 3.1 of [10]) by using the estimates of Kim [22] in the hyperbolic region, which corresponds to Ω here. A simple analysis of the proof of this lemma shows that the argument carries over if $\gamma \neq 0$ and $\lambda < 0$. □

Next we consider continuity up to the boundary.

Theorem 3.2. *Let $\gamma \in \mathbb{R}, \lambda \leq 0$ and $f \in C^0(\overline{\Omega})$. Then the unique weak solution $u \in H^1(\Omega)$ to problem (2.18) is continuous up to the boundary; that is, there exists a unique $u_* \in C^0(\overline{\Omega})$ such that $u = u_*$ a.e. in Ω. Moreover $u_* = \gamma + v$ where v is the unique $C^0(\overline{\Omega}) \cap H^1_\Gamma(\Omega)$ solution to the equation*

$$v = S_0(f + \lambda\gamma + \lambda v) \tag{3.1}$$

where $S_0 : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is a bounded linear integral operator whose kernel is explicitly determined in terms of the Riemann–Hadamard function for the Goursat problem (2.1) with homogeneous boundary data and $\lambda = 0$. Moreover,

$$S_0 : C^0(\overline{\Omega}) \to C^0(\overline{\Omega}) \text{ is a compact operator} \tag{3.2}$$

and for each $f \in C^0(\overline{\Omega})$

$$S_0(f) = S_\Gamma^{0,0}(f) \quad \text{in } L^2(\Omega), \tag{3.3}$$

where $S_\Gamma^{0,0} = S_\Gamma$ is the weak solution operator of Theorem 2.5. Hence S_0 yields a continuous representative of the unique $H^1_\Gamma(\Omega)$ solution to the problem (2.18) in the case $\lambda = \gamma = 0$.

As a corollary, we obtain that for each $f \in C^0(\overline{\Omega})$ there exists unique weak solution $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ and that there is a representation formula for its continuous representative:

$$u_* = S_0(f + \lambda u_*) + \gamma. \tag{3.4}$$

Proof. We will define explicitly the solution operator and then study its properties.

Step 1. (*Definition of the integral operator S_0*). The form of the desired solution operator is most easily presented in characteristic coordinates. We consider the homeomorphism $\Phi : \mathbb{R} \times [0, +\infty) \to \mathcal{H} = \Phi(\mathbb{R} \times [0, +\infty))$ defined by

$$\Phi(x, y) = (\xi(x, y), \eta(x, y)) = \left((x + x_0) - \frac{2}{3}(-y)^{3/2}, (x + x_0) + \frac{2}{3}(-y)^{3/2}\right) \tag{3.5}$$

which also translates $A(-x_0, 0)$ to the origin $(\xi, \eta) = (0, 0)$ and $B(x_0, 0)$ to $(\xi, \eta) = (l, l)$ with $l = 2x_0$. Φ is a C^∞ diffeomorphism on the interior $\mathbb{R} \times (0, +\infty)$ of its domain. The image $\mathcal{H}$ is the half-space $\{(\xi, \eta) : \eta \geq \xi\}$ and the inverse map $\Psi : \mathcal{H} \to \mathbb{R} \times [0, +\infty)$ is given by

$$\Psi(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta)) = \left(\frac{1}{2}(\xi + \eta) - x_0, -\left(\frac{3}{4}(\eta - \xi)\right)^{2/3}\right). \tag{3.6}$$

The image of characteristic triangle $\Omega = ABC$ under Φ is the triangle

$$\Delta = \{(\xi, \eta) : \ 0 < \xi < l \text{ and } \xi < \eta < l\}. \tag{3.7}$$

With this change of variables, the partial differential equation $Tu - \lambda u = f$ then transforms into

$$w_{\xi\eta} + \frac{1}{6(\eta - \xi)}(w_\xi - w_\eta) - \lambda C_0 \frac{1}{(\eta - \xi)^{2/3}} w = C_0 \frac{1}{(\eta - \xi)^{2/3}} \tilde{f} \tag{3.8}$$

where

$$w(\xi, \eta) = (u \circ \Psi)(\xi, \eta), \qquad \tilde{f}(\xi, \eta) = (f \circ \Psi)(\xi, \eta), \qquad C_0 = \frac{1}{4}\left(\frac{4}{3}\right)^{2/3}. \tag{3.9}$$

The Riemann–Hadamard function associated to (3.8) when $\lambda = 0$ has the following expression which is well defined for $(\xi', \eta'; \xi, \eta) \in (\Delta \times \Delta) \setminus \{\eta' = \xi\}$

$$R(\xi', \eta'; \xi, \eta) = \begin{cases} R^+(\xi', \eta'; \xi, \eta) = \left(\dfrac{\eta' - \xi'}{\eta - \xi}\right)^{1/6} F\left(\dfrac{1}{6}, \dfrac{5}{6}, 1; s\right) & \eta' > \xi \\ R^-(\xi', \eta'; \xi, \eta) = k \dfrac{(\eta' - \xi')(\eta - \xi)^{2/3}}{(\eta - \eta')^{5/6}(\xi - \xi')^{5/6}} F\left(\dfrac{5}{6}, \dfrac{5}{6}, \dfrac{5}{3}; \dfrac{1}{s}\right) & \eta' < \xi \end{cases} \tag{3.10}$$

where

$$s = \frac{(\xi - \xi')(\eta - \eta')}{(\eta' - \xi')(\eta - \xi)}, \qquad k = \frac{\Gamma(5/6)}{\Gamma(1/6)\Gamma(5/3)}. \tag{3.11}$$

$F(a, b, c; \zeta)$ is the standard hypergeometric function of Gauss and Γ is the gamma function of Euler. See section II.2 of Smirnov [23] for a discussion of R and its basic properties.

It is known that sufficiently regular solutions v to the Goursat problem (2.18) with $\gamma = 0$

$$\begin{cases} Tv - \lambda v = f & \text{in } \Omega \\ v = 0 & \text{on } \Gamma, \end{cases} \tag{3.12}$$

have an explicit integral representation formula in terms of the Riemann–Hadamard function, as is shown in Theorem 1 of Moiseev [24]. However, it is not shown there (or anywhere else to our knowledge) that the converse is true. That is, given f sufficiently regular does the representation formula yield a solution to the problem in some reasonable sense? We will show that the formula of Moiseev gives a weak solution which is continuous up to the boundary for any given $f \in C^0(\overline{\Omega})$. For $\lambda = 0$, sufficiently regular solutions v of (3.12) are given by $v = w \circ \Phi$ where $w = w(z)$ is defined by

$$w(z) = \int_{\Delta} C(z')R(z';z)\tilde{f}(z')\,dz', \quad z = (\xi, \eta) \in \Delta, \tag{3.13}$$

Δ is defined in (3.7), $\tilde{f}$ is defined in (3.9) and

$$C(z') = C(\xi', \eta') = -C_0(\eta' - \xi')^{-2/3}, \tag{3.14}$$

with C_0 the constant defined in (3.9). We remark that for each fixed z in the open triangle Δ, the integral kernel is defined almost everywhere; that is, for $z' \in \Delta_z^+ \cup \Delta_z^-$ where

$$\Delta_z^+ = \{z' = (\xi', \eta') : \eta' > \xi\} \quad \text{and} \quad \Delta_z^- = \{z' = (\xi', \eta') : \eta' < \xi\}. \tag{3.15}$$

Hence the representation formula (3.13) can be used to define w on Δ for a given g. In order to extend the representation formula to $z \in \partial\Omega$, we define the integral kernel $K : \overline{\Delta} \times \overline{\Delta} \to \mathbb{R}$ by

$$K(z'; z) = \chi_{\Delta_z^+ \cup \Delta_z^-}(z')C(z')R(z'; z) \tag{3.16}$$

where χ_E is the characteristic function associated to $E \subset \mathbb{R}^2$ and R is a suitable extension of (3.10) to $(\Delta_z^+ \cup \Delta_z^-) \times \overline{\Delta}$. We define R on $(\Delta_z^+ \cup \Delta_z^-) \times \partial\Delta$ in the following way. For $z = (0, \eta)$ with $\eta \in (0, l]$ and for $z = (\xi, l)$ with $\xi \in (0, l)$, the functions on the right hand side of (3.10) continue to be well defined. Notice that since $\Delta^+_{(0,\eta)} \cup \Delta^-_{(0,\eta)} = \emptyset$, $K(z'; 0, \eta) = 0$ for each $\xi \in (0, l)$ as desired so that $w(0, \eta) \equiv 0$. In order to ensure that w also vanishes for $z = (\xi, \xi)$ with $\xi \in [0, l]$ we merely define

$$R(z'; \xi, \xi) = 0 \quad \text{for every } z' \in \Delta_z^+ \cup \Delta_z^-, \xi \in [0, l]. \tag{3.17}$$

Finally, we define the operator $S_0 : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ by

$$S_0(f) = w \circ \Phi$$

where Φ is the homeomorphism (3.5) and

$$w(z) = \widetilde{S}_0(\tilde{f}) = \int_{\Delta} \chi_{\Delta_z^+ \cup \Delta_z^-}(z')C(z')R(z'; z)\tilde{f}(z')\,dz', \quad z = (\xi, \eta) \in \overline{\Delta}. \tag{3.18}$$

Recall that $\tilde{f} = f \circ \Psi$ is defined by (3.6) and (3.9), R is defined by (3.10) and (3.17) and C defined by (3.14). Since Φ and Ψ are continuous, S_0 will be well defined and compact provided that $\tilde{S}_0$ is a compact operator on $C^0(\overline{\Delta})$.

Step 2. (*Compactness of* S_0). The key technical step is the following lemma which completes the claim (3.2). The proof will be given in Appendix A.

Lemma 3.3. *The operator $\widetilde{S}_0 : C^0(\overline{\Delta}) \to C^0(\overline{\Delta})$ is well defined, linear, continuous and compact.*

Step 3. (*The case* $\lambda, \gamma = 0$). We begin by noting that if $f \in C^2(\overline{\Omega})$, then by the result of Nakhushev (see Theorem 2 of [11]) there exists a classical solution $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the Goursat problem (3.12) and that $v \in C^1(\overline{\Omega})$ with $\|v\|_{C^1(\overline{\Omega})} \leq C\|f\|_{C^2(\overline{\Omega})}$ for some constant C independent of v (see also Theorem 4.2 of [25]). Hence $v \in H^1_\Gamma(\Omega)$ and $v = u$ a.e. in Ω where u is the weak solution of (3.12).

As mentioned in Step 1 above, Theorem 1 of Moiseev [24] shows that classical solutions v of (3.12) satisfy the representation formula $v = S_0(f)$ in Ω and our extension of Step 1 ensures that this is also true at the boundary. Hence we have (3.3) for $f \in C^2(\overline{\Omega})$. The validity of (3.3) for $f \in C^0(\overline{\Omega})$ and the existence of a continuous representative u_* when $\lambda = 0$ then follows. Indeed, approximate $f \in C^0(\overline{\Omega}) \subset L^2(\Omega)$ with a sequence $\{f_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that

$$\|f_k - f\|_{L^2(\Omega)} \leq |\Omega|^{1/2}\|f_k - f\|_{C^0(\overline{\Omega})} \to 0 \quad \text{as } k \to +\infty.$$

By the continuity of S_0 and $S_\Gamma^{0,0}$ one has

$$\|S_0(f_k) - S_0(f)\|_{L^2(\Omega)} \leq |\Omega|^{1/2}\|S_0(f_k - f)\|_{C^0(\overline{\Omega})} \to 0$$

and

$$\|S_\Gamma^{0,0}(f_k) - S_\Gamma^{0,0}(f)\|_{L^2(\Omega)} \leq C_P\|S_\Gamma^{0,0}(f_k - f)\|_{H^1_\Gamma(\Omega)} \to 0.$$

But $S_0(f_k) = S_\Gamma^{0,0}(f_k)$ in $L^2(\Omega)$ and hence $S_0(f) = S_\Gamma^{0,0}(f)$ in $L^2(\Omega)$ by the uniqueness of the limit in $L^2(\Omega)$. Hence $S_0(f) = S_\Gamma^{0,0}(f)$ a.e. in Ω and the unique weak solution $u \in H^1_\Gamma(\Omega)$ admits the continuous representative $u_* = S_0(f)$ if $f \in C^0(\overline{\Omega})$, where $u^* = v$ is the solution of (3.1). Notice that also (3.4) holds.

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Step 4. (*The case* $\lambda < 0$ *and* $\gamma \in \mathbb{R}$). Recalling that the weak solution operator $S_\Gamma^{0,0} : L^2(\Omega) \to H^1_\Gamma(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ is compact as an operator on $L^2(\Omega)$, by the Fredholm alternative one has

$$\left(I - \lambda S_\Gamma^{0,0}\right) : L^2(\Omega) \to L^2(\Omega) \text{ is invertible for each } \lambda \leq 0. \tag{3.19}$$

Indeed, if (3.19) were false, then there would be a non zero solution $w \in H^1_\Gamma(\Omega)$ to the problem

$$\begin{cases} Tw - \lambda w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma, \end{cases}$$

but by Theorem 2.2, $w = 0$ is the only weak solution in $H^1_\Gamma(\Omega)$.

By (3.3), it follows that $\{0\} = \ker(I - \lambda S_0) \subset C^0(\overline{\Omega})$ for each $\lambda < 0$. Applying the Fredholm alternative to S_0 on $C^0(\overline{\Omega})$, one has that for each $\lambda < 0$, $\gamma \in \mathbb{R}$ and $f \in C^0(\overline{\Omega})$ there exists a unique $v \in C^0(\overline{\Omega})$ satisfying

$$(I - \lambda S_0)v = S_0(f + \lambda\gamma) \quad \text{in } C^0(\overline{\Omega}) \subset L^2(\Omega).$$

Hence

$$v = S_0(f + \lambda\gamma + \lambda v) \quad \text{in } C^0(\overline{\Omega}) \subset L^2(\Omega); \tag{3.20}$$

that is, there exists a unique $v \in C^0(\overline{\Omega})$ solution to (3.1) as claimed. Again using (3.3), one has that $v \in H^1_\Gamma(\Omega)$ is a weak solution of

$$\begin{cases} Tv - \lambda v = f + \lambda\gamma & \text{in } \Omega \\ v = 0 & \text{on } \Gamma, \end{cases}$$

and hence $u_* = v + \gamma \in H^1_\Gamma(\Omega) \cap C^0(\overline{\Omega})$ is a weak solution of

$$\begin{cases} Tu_* - \lambda u_* = f & \text{in } \Omega \\ u_* = \gamma & \text{on } \Gamma. \end{cases}$$

By the uniqueness of the weak solution, $u = u_*$ in $H^1_\Gamma(\Omega)$ and so $u = u_*$ a.e. in Ω. Notice also that since $u_* = v = \gamma$, (3.4) follows from (3.20). $\square$

3.2. Maximum principles

As noted above, combining the regularity results of the previous subsection with a variant of the classical maximum principle yields a comparison principle for weak solutions compatible with the solvability theory. The main result is the following comparison principle for weak solutions. We recall that $y_C = -(3x_0/2)^{2/3}$ is the y-coordinate of C where $\Omega = ABC$.

Theorem 3.4. *Let $\lambda \in [-5/(16y_C^2), 0]$, $\gamma \in \mathbb{R}$ and $f \in L^2(\Omega)$ be given. Let $u \in H^1(\Omega)$ be the unique weak solution to the problem* (2.18)*; that is,*

$$\begin{cases} Tu - \lambda u = f & \text{in } \Omega \\ u = \gamma & \text{on } \Gamma. \end{cases}$$

(a) *If $f \geq 0$ a.e. in Ω and $\gamma \geq 0$ then $u \geq 0$ a.e. in Ω;*
(b) *If $f \leq 0$ a.e. in Ω and $\gamma \leq 0$ then $u \leq 0$ a.e. in Ω.*

A similar statement holds for $u \in H^1(\Omega)$ the unique weak solution to the adjoint problem with $\Gamma^* = BC \cup AB$ in place of $\Gamma = AC \cup AB$.

Proof. The proof follows closely that of Theorem 3.1 in [10]. For completeness, we will give the outline of the main ideas in the case of the problem (2.18) with $\gamma \geq 0$ and $f \in L^2(\Omega)^+ := \{f \in L^2(\Omega) : f \geq 0\}$, where partial ordering $f \geq 0$ is the standard one; that is $f(x) \geq 0$ for almost every $x \in \Omega$.

Step 1: (*Maximum principle for regular solutions*). Define the first order differential operators

$$D_\pm = D_y \pm \sqrt{-y}D_x, \tag{3.21}$$

which are essentially the directional derivatives along characteristic directions.

Lemma 3.5. *Let $\lambda \in [-5/(16y_C^2), 0]$ and $\gamma \in \mathbb{R}$ be as in* Theorem 3.4 *and $f \in C^0_0(\Omega)$. Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical solution of* (2.18) *which satisfies*

$$\lim_{R \to P} D_- u(R) = 0 \quad \textit{for each } P \in AC \setminus \{A, C\}. \tag{3.22}$$

(a) *Let* $f \geq 0$. *Then the minimum* $m = \min_{\overline{\Omega}} u$ *is realized on* Γ *if* $m \leq 0$ *and one has the lower bound*

$$u \geq \min\{\gamma, 0\} \quad \text{on } \overline{\Omega}. \tag{3.23}$$

In particular, if $\gamma \geq 0$, *then the comparison principle* $u \geq 0$ *on* $\overline{\Omega}$ *holds.*

(b) *Let* $f \leq 0$. *Then the maximum* $M = \max_{\overline{\Omega}} u$ *is realized on* Γ *if* $M \geq 0$ *and one has the upper bound*

$$u \leq \min\{\gamma, 0\} \quad \text{on } \overline{\Omega}. \tag{3.24}$$

In particular, if $\gamma \leq 0$, *then the comparison principle* $u \leq 0$ *on* $\overline{\Omega}$ *holds.*

(c) *In the case* $\lambda = 0$ *the hypothesis that* m/M *is non positive/non negative is not needed and* (3.23) *and* (3.24) *become*

$$u \geq \gamma \quad \text{on } \overline{\Omega}$$

and

$$u \leq \gamma \quad \text{on } \overline{\Omega}.$$

Parts (a) and (b) of Lemma 3.5 are variants of the classical result of Agmon, Nirenberg and Protter [20]. The weaker regularity condition (3.22) combined with f having compact support replaces the additional regularity assumption $u \in C^1(\overline{\Omega} \setminus \{A, B\})$ used by them. The requirement that u is constant on Γ implies that u is monotone on the characteristic AC which is required by them. The condition (3.22) was introduced in [10] to prove the analogous lemma for the mixed type Tricomi problem in the case $\gamma = 0$. For completeness, a sketch of the proof will be given in Appendix B.

Step 2. (*Approximation and solvability*). Using non negative cutoff functions and standard mollifiers, one can approximate $f \in L^2(\Omega)^+$ by $f_n \in C_0^\infty(\Omega)$ such that $f_n \geq 0$ in Ω and

$$\mathrm{supp}(f_n) \subset \Omega_n := \{(x, y) \in \Omega : \mathrm{dist}((x, y), \partial\Omega) > 1/n\}.$$

Since $\lambda \leq 0$, by Theorem 2.5, there exists a unique generalized solution $u_n = S_\Gamma^{\lambda,\gamma}(f_n) \in H^1(\Omega)$ to the problem (2.18) with $f = f_n$.

Step 3: (*Regularity of the approximate solution*). We can apply the comparison principle of Lemma 3.5(a) to the approximate solutions u_n provided that $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and u_n satisfies the condition (3.22). Since $f_n \in C_0^\infty(\Omega)$, by Theorems 3.1 and 3.2 we have $u_n \in H^1(\Omega) \cap C^\infty(\Omega) \cap C^0(\overline{\Omega})$. Using $u_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f_n \in C_0^0(\Omega)$, it is not difficult to show that (3.22) holds. For the case $\lambda = 0 = \gamma$ this was proven in Lemma 3.2 of [10] and a simple examination of the proof shows that the argument carries over to the other cases $\gamma \neq 0$ or $\lambda < 0$. Hence we may apply part (a) of Lemma 3.5 to conclude that $u_n \geq \min\{\gamma, 0\} \geq 0$ in Ω since $u_n = \gamma \geq 0$ on Γ.

Step 4. (*Continuity of the solution operator* $S_\Gamma^{\lambda,\gamma}$). Since $u_n \in H^1(\Omega) \cap C^0(\overline{\Omega}) \subset L^2(\Omega)$ satisfies $u_n \geq 0$ in Ω, one has

$$u = \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} S_\Gamma^{\lambda,\gamma}(f_n) = S_\Gamma^{\lambda,\gamma}(f) \quad \text{in } H^1(\Omega)$$

from which it follows that $u \geq 0$ a.e. in Ω. □

4. Nonlinear theory: existence of solutions

The nonlinear results we will obtain rely on Theorem 2.2 which says that the linear problem (2.1) admits a continuous solution operator $S_\Gamma : L^2(\Omega) \to H_\Gamma^1(\Omega)$ and hence

$$S_\Gamma : L^2(\Omega) \to H_\Gamma^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \quad \forall\, p \in [1, +\infty),$$

as noted in Remark 2.3. Moreover, since Ω is bounded, we have $L^p(\Omega) \subset L^2(\Omega)$ for each $p \geq 2$. Hence we may reformulate the question of finding a weak solution $u \in H_\Gamma^1(\Omega)$ to the semilinear Goursat problem (1.1)

$$\begin{cases} Tu = f(x, y, u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases} \tag{4.1}$$

as a fixed point problem. Namely look for $u \in L^p(\Omega)$ such that

$$u = G(u) := S_\Gamma \circ f_\#(u) \tag{4.2}$$

where $f_\# : u \mapsto f(\cdot, u(\cdot))$ is the *Nemytskii operator* associated to f. One knows that

$$f_\# : L^p(\Omega) \to L^2(\Omega)$$

is continuous and maps bounded sets to bounded sets provided that $f = f(x, y, s) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the *Carathéodory conditions*

$$f \text{ is measurable in } (x, y) \text{ for each } s \in \mathbb{R} \text{ and continuous in } s \quad \text{for a.e. } (x, y) \in \Omega \tag{4.3}$$

and the *natural growth bound*

$$|f(x,y,s)| \leq a(x,y) + b|s|^{p/2}, \quad a \geq 0 \text{ in } L^2(\Omega),\ b \geq 0 \text{ in } \mathbb{R}, \tag{4.4}$$

which needs to hold for almost every $(x,y) \in \Omega$ and each $s \in \mathbb{R}$. These claims on $f_\#$ are standard (see e.g. Vainberg [26]).

Since the image of G lies in the subspace $H^1_\Gamma(\Omega)$ such fixed points will be weak solutions to (4.1) in the sense that $u \in H^1_\Gamma(\Omega)$ and the natural analog of (2.10) holds; that is

$$\mathcal{B}(u,v) := \int_\Omega \left(yu_xv_x + u_yv_y\right)\,dxdy = \int_\Omega f_\#(u)v\,dxdy, \quad \forall\, v \in C^\infty_{\Gamma^*}(\overline{\Omega}).$$

Remark 4.1. For each $\lambda \leq 0$, one can obviously use $T - \lambda I$ in place of T in the problem (1.1) since the corresponding linear solution operator

$$S^{\lambda,0}_\Gamma : L^2(\Omega) \to H^1_\Gamma(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \quad \forall\, p \in [1,+\infty),$$

gives a compact linear map. We will exploit this fact later in the use of monotone methods by adding a term ωu to both sides of (1.1) with a suitable $\omega > 0$ so that the Nemytskii operator associated to $f(x,y,s) + \omega s$ will be a monotone operator.

We will divide the results into two cases on the basis of whether $q := p/2 \leq 1$ or $q > 1$ in (4.4); that is, into the cases of *sublinear* or *superlinear growth* at infinity.

4.1. Sublinear growth

Our first result concerns the case of strictly sublinear growth; that is $q = p/2 < 1$ in (4.4) and is a simple application of the Leray–Schauder principle.

Theorem 4.2. *If f satisfies the Carathéodory conditions (4.3) and the growth bound (4.4) with $q = p/2 \in [0,1)$, then there exists at least one weak solution $u \in H^1_\Gamma(\Omega)$ to the problem (4.1).*

Proof. First notice that f also satisfies (4.4) with $q = 1$ since $|f(x,y,s)| \leq \tilde{a}(x,y) + b|s|$ with $\tilde{a} = a + b \in L^2(\Omega)$. Hence $G : L^2(\Omega) \to L^2(\Omega)$ is compact. There will be a fixed point of G provided one has the a priori bound: there exists a constant $C > 0$ such that

$$u = tG(u),\ t \in (0,1) \Rightarrow \|u\|_{L^2(\Omega)} \leq C. \tag{4.5}$$

The case $p = 0$ is obvious. Using $t \in (0,1)$ and the boundedness of S_Γ on $L^2(\Omega)$ one has

$$\|u\|_{L^2(\Omega)} \leq \|S_\Gamma\|_{\text{op}}\, \|f_\#(u)\|_{L^2(\Omega)}.$$

A standard calculation using (4.4), $a \in L^2(\Omega)$ and Hölder's inequality yields constants C_1, C_2 and C_3 such that

$$\|u\|_{L^2(\Omega)} \leq \|S_\Gamma\|_{\text{op}} \left[C_1 + C_2\|u\|^q_{L^2(\Omega)} + C_3\|u\|^{2q}_{L^2(\Omega)}\right]^{1/2}. \tag{4.6}$$

If (4.5) were to fail, then there would be a sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying $u_n = t_nG(u_n)$ with $t_n \in (0,1)$ and $\|u_n\|_{L^2(\Omega)} \to +\infty$. This contradicts (4.6) for $q \in (0,1)$. □

On the other hand, for f with at most linear growth but satisfying a suitable Lipschitz condition, the contraction mapping principle gives the existence of a unique solution.

Theorem 4.3. *If f satisfies the Carathéodory conditions (4.3), the growth bound (4.4) with $q = p/2 \in [0,1]$ and the estimate*

$$|f(x,y,s) - f(x,y,t)| \leq C_L|s-t|, \quad \text{for a.e. } (x,y) \in \Omega \text{ and each } s,t \in \mathbb{R} \tag{4.7}$$

with the Lipschitz constant satisfying

$$0 < C_L < M_0^{-1}, \tag{4.8}$$

where $M_0 = \|S_\Gamma\|_{\text{op}}$ as discussed in Remark 2.6, then there exists a unique solution $u \in H^1_\Gamma(\Omega)$ of the problem (4.1).

Proof. As in the proof of Theorem 4.2, G is well defined and continuous on $L^2(\Omega)$. Using (4.7), one has the estimate

$$\|G(u) - G(v)\|_{L^2(\Omega)} \leq \|S_\Gamma\|_{\text{op}} C_L \|u - v\|_{L^2(\Omega)},$$

and G will be a contraction on $L^2(\Omega)$ if C_L satisfies (4.8). Hence G admits a unique fixed point u, which lies in $H^1_\Gamma(\Omega) \subset L^2(\Omega)$. □

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We conclude the discussion of this case with a pair of remarks.

Remark 4.4. If $f(x,y,0)=0$ for almost every $(x,y)\in\Omega$, then clearly $u=0$ is a trivial solution of (4.1). On the other hand, if $f(x,y,0)\neq 0$ on a set of positive measure, then the solutions in Theorems 4.2 and 4.3 are nontrivial. This observation also applies to Theorem 4.7 in the superlinear case below.

Remark 4.5. Analogs of Theorems 4.2 and 4.3 have been shown for mixed type operators including T: for the Tricomi problem in [1] and the Dirichlet problem in [3]. In the case of the Dirichlet problem, the linear part is self-adjoint and many additional existence results have been obtained for asymptotically linear nonlinearities f in [4].

4.2. Superlinear cases

In this section, we will treat the case of superlinear growth $q=p/2>1$ in (4.4). Exploiting the comparison principle of Theorem 3.4, we will make use of *monotone iteration* to show the existence of a fixed point for the equation (4.2). As noted in Remark 4.1, it will be useful to rewrite the problem (4.1) as

$$\begin{cases} Tu+\omega u = f_\omega(x,y,u) & \text{in } \Omega \\ u=0 & \text{on } \Gamma, \end{cases} \tag{4.9}$$

where

$$f_\omega(x,y,s)=f(x,y,s)+\omega s \quad \text{and} \quad \omega\in[0,5/(16y_C^2)].$$

Note that the maximum principle holds for $(T+\omega I)=(T-\lambda I)$ if $\lambda=-\omega\in[-5/(16y_C^2),0]$. Using

$$K_\omega := S_\Gamma^{-\omega,0} : L^2(\Omega)\to H^1_\Gamma(\Omega)\hookrightarrow\hookrightarrow L^p(\Omega),\quad \forall\, p\in[1,+\infty)$$

the fixed point problem associated to (4.9) becomes: look for a solution $u\in L^p(\Omega)$ of

$$u=G_\omega(u):=K_\omega\circ(f_\omega)_\#(u),$$

where

$$(f_\omega)_\# : L^p(\Omega)\to L^2(\Omega)$$

is well defined, continuous and maps bounded sets to bounded sets if f satisfies (4.3) and (4.4) since f_ω will as well. In particular, with $p>2$ one has

$$|f_\omega(x,y,s)|\le a(x,y)+b|s|^{p/2}+\omega|s|\le (a(x,y)+\omega)+(b+\omega)|s|^{p/2}. \tag{4.10}$$

The basic tool is the following (see Corollary 6.2 of [27]).

Lemma 4.6. *Let E be an ordered Banach space with positive cone P. If $[\underline{u},\overline{u}]$ is a non empty order interval such that $G:[\underline{u},\overline{u}]\to E$ is increasing and compact and*

$$\underline{u}\le G(\underline{u}) \quad \text{and} \quad G(\overline{u})\le\overline{u}, \tag{4.11}$$

then G has both a minimal and maximal fixed point u_*, u^* given by monotone iteration

$$G^k(\underline{u})\nearrow u_* \quad \text{and} \quad G^k(\overline{u})\searrow u^*.$$

Using Lemma 4.6, it suffices to place suitable hypotheses on f and ω so that G_ω admits an ordered pair $\underline{u},\overline{u}$ satisfying (4.11) and that G_ω is compact and increasing on $[\underline{u},\overline{u}]$. To this end we will assume that

$$f \text{ satisfies (4.3) and (4.4) with } q=p/2>1, \tag{4.12}$$

that there exist constants $c_1,c_2\in\mathbb{R}$ with

$$c_1<0<c_2 \quad \text{and} \quad f(x,y,c_2)\le 0\le f(x,y,c_1) \quad \text{for a.e. } (x,y)\in\Omega \tag{4.13}$$

and that there exists $\omega\in(0,5/(16y_C^2)]$ such that

$$f(x,y,s)-f(x,y,t)\ge-\omega(s-t), \tag{4.14}$$

for each $s,t\in[c_1,c_2]$ with $s\ge t$ and for a.e. $(x,y)\in\Omega$.

Theorem 4.7. *If f satisfies (4.12)–(4.14), then there exists at least one weak solution $u\in H^1_\Gamma(\Omega)$ to the problem (4.1).*

Proof. We will work in the ordered Banach space $L^p(\Omega)$ with $p>2$ so that $L^p(\Omega)\subset L^2(\Omega)$ and the positive cone $L^p(\Omega)^+ = L^p(\Omega)\cap L^2(\Omega)^+$ is normal. Hence each order interval $[\underline{u},\overline{u}]$ is bounded (see Theorem 1.5 of [27]). As already noted, the hypothesis (4.12) ensures that f_ω satisfies (4.10) and that $(f_\omega)_\# : L^p(\Omega) \to L^2(\Omega)$ is continuous. The hypotheses (4.13) and (4.14) show that

$$(f_\omega)_\# : [c_1, c_2] \to L^p(\Omega) \text{ is increasing}$$

since for each $s, t \in [c_1, c_2]$ with $s \geq t$ and for a.e. $(x, y) \in \Omega$ one has

$$f(x, y, s) + \omega s - [f(x, y, t) + \omega t] \geq -\omega(s - t) + \omega(s - t) = 0.$$

For each $p \geq 2$, one has $G_\omega : L^p(\Omega) \to L^p(\Omega)$ is compact. Indeed, for $\omega \geq 0$ one has

$$G_\omega : L^p(\Omega) \subset L^2(\Omega) \to H^1_\gamma(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega),$$

where

$$G_\omega = (T + \omega I)^{-1} = (T - \lambda I)^{-1} \quad \text{with } \lambda = -\omega \leq 0.$$

It follows that G_ω is increasing on $[c_1, c_2]$. Indeed, for each pair $\varphi, \psi \in L^p(\Omega)$ such that

$$c_1 \leq \varphi \leq \psi \leq c_2 \quad \text{in } L^p(\Omega)$$

one has

$$v := G_\omega(\varphi) \leq G_\omega(\psi) := w$$

since $z = w - v$ is the unique weak solution in $H^1_\Gamma(\Omega)$ of

$$\begin{cases} (T + \omega I)z = \psi - \varphi \geq 0 & \text{in } \Omega \\ z = 0 & \text{on } \Gamma, \end{cases}$$

with $\omega \in (0, 5/(16y_C^2)]$. The comparison principle (Theorem 3.4) yields that $z = w - v \geq 0$ a.e. in Ω and hence $z \geq 0$ in $L^p(\Omega)$.

It remains only to show that $\underline{u} = c_1$ and $\overline{u} = c_2$ satisfy (4.11) with $G = G_\omega$; that is,

$$c_1 \leq G_\omega(c_1) \quad \text{and} \quad G_\omega(c_2) \leq c_2 \quad \text{in } L^p(\Omega). \tag{4.15}$$

With $\underline{u} = c_1 < 0$ one has $f(x, y, c_1) \geq 0$ for a.e. $(x, y) \in \Omega$ by (4.13) and $u := G_\omega(c_1)$ is the unique weak solution in $H^1_\Gamma(\Omega)$ of

$$\begin{cases} (T + \omega I)u = f(x, y, c_1) + \omega c_1 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases}$$

The function $v := u - c_1 \in H^1(\Omega)$ and is the unique weak solution of

$$\begin{cases} (T + \omega I)v = f(x, y, c_1) + \omega c_1 - \omega c_1 \geq 0 & \text{in } \Omega \\ v = -c_1 > 0 & \text{on } \Gamma = AC \cup AB, \end{cases}$$

Again by Theorem 3.4 one has $v \geq -c_1$ a.e. in Ω so that $u \geq 0$ a.e. in Ω. Hence

$$c_1 < 0 \leq u = G_\omega(c_1) \quad \text{a.e. in } \Omega,$$

which is the first inequality in (4.15). An analogous argument with $\overline{u} := c_2 > 0$ and using $f(\cdot, \cdot, c_2) \leq 0$ a.e. yields the second inequality in (4.15). □

5. Nonlinear theory: uniqueness of the trivial solution

In this section, we examine the question of uniqueness of the trivial solution $u = 0$ to the semilinear Goursat problem (1.1) when the nonlinearity is homogeneous;[3] that is, $f = f(u)$ and vanishes to high enough order in $u = 0$. In particular, we will consider weak solutions $u \in H^1_\Gamma(\Omega)$ to the problem

$$\begin{cases} Lu + F'(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases} \tag{5.1}$$

where $L = -T = yD_x^2 + D_y^2$ and $A = (-2x_0, 0)$, $B = (0, 0)$ and $f = F' \in C^0(\mathbb{R})$ with primitive $F(s) := \int_0^s f(t)\, dt$ satisfying

$$F(0) = 0. \tag{5.2}$$

[3] In this section, we will denote by $f(u)$ the values of the Nemytskii operator $f_\#$ acting on u.

Notice that we have merely rewritten the PDE and exploited translation invariance in x in order to represent the problem as in [5–7], where the same question for regular solutions was considered. For example, if f takes pure power form

$$f(s) = s|s|^{p-2} \tag{5.3}$$

with supercritical growth

$$p \geq 2^*(1, 1) = 10, \tag{5.4}$$

then the only solution $u \in C^2(\overline{\Omega})$ to (5.1) is the trivial solution $u = 0$. The case $p > 10$ was treated in [5] and the extension to $p \geq 10$ for Dirichlet boundary conditions was treated in [6] and for various "open" boundary conditions in [8]. Here we wish to relax the regularity assumption to $u \in H^1_\Gamma(\Omega)$, the space in which we can find solutions as in Section 4. The main result is the following theorem in the pure power case (5.3)–(5.4). Various generalizations will be discussed at the end of this section (see Remark 5.4).

Theorem 5.1. *If $f = F'$ satisfies (5.3)–(5.4), then the only solution $u \in H^1_\Gamma(\Omega)$ of (5.1) is the trivial solution $u = 0$.*

The proof, which will be given in the following two subsections, splits into two cases; namely the *supercritical case* with $p > 10$ and the *critical case* with $p = 10$. We begin with two preliminary facts. Both cases rely on the following Pohožaev type identity which will be applied along a suitably regularized approximating sequence which satisfies the boundary conditions.

Lemma 5.2. *Let $f \in C^0(\mathbb{R})$ with primitive F satisfying (5.2). For any $u \in C^\infty(\overline{\Omega})$ such that $u_{|\Gamma} = 0$ one has*

$$\int_\Omega \left(Mu + \frac{1}{2}u\right)\left(Lu + F'(u)\right) dxdy = \int_\Omega \left[\frac{1}{2}uF'(u) - 5F(u)\right] dxdy + \int_{BC} \left[Mu + \frac{1}{2}u\right](yu_x, u_y) \cdot \nu \, ds \tag{5.5}$$

where ν is the exterior unit normal, ds the arc length element and

$$M = -3xD_x - 2yD_y. \tag{5.6}$$

This is Theorem 3.3 of [5], where $Mu + u/2$ is the infinitesimal generator of an anisotropic dilation invariance for L. One merely multiplies $Lu + F'(u)$ by $Mu + u/2$, applies the divergence theorem and uses $u = F(u) = 0$ on $AB \cup AC$ and the geometry of Ω.

In the pure power case (5.3), the integral over Ω on the right hand side of (5.5) has the sign of $p - 10$, while the boundary integral is non-negative due to a sharp Hardy–Sobolev inequality (see Lemma 4.3 of [5]). In order to also treat the critical case $p = 10$, we will make use of a related inequality with remainder term. First we fix a few notations which will be used in the rest of this section. The characteristics AC and BC are given by (compare with (2.2))

$$\begin{cases} AC : x + 2x_0 - g(y) = 0 \quad \text{and} \quad BC : \ x + g(y) = 0, \quad y \in [y_C, 0] \\ \quad \text{where } g(y) = \dfrac{2}{3}(-y)^{3/2} \quad \text{and} \quad y_C = -\left(\dfrac{3x_0}{2}\right)^{2/3}. \end{cases} \tag{5.7}$$

Parameterizing BC by $\beta(t) = (-g(t), t)$ with $t \in [y_C, 0]$ and setting $w(t) = u(\beta(t))$, one finds that the boundary integral in (5.5) is

$$\int_{BC} \left[Mu + \frac{1}{2}u\right](yu_x, u_y) \cdot \nu \, ds = \int_{y_C}^0 \left[4(-t)^{3/2}w'(t)^2 - \frac{1}{4}(-t)^{-1/2}w(t)^2\right] dt. \tag{5.8}$$

The Hardy–Sobolev inequality with remainder that we will ~~be~~ recorded in the following lemma.

Lemma 5.3. *Let $w \in C^1([a, 0])$ satisfy $w(a) = 0$. Then*

$$\int_a^0 (-t)^{3/2} w'(t)^2 \, dt \geq \frac{1}{16}\int_a^0 (-t)^{-1/2} w(t)^2 \, dt + \frac{4}{a^2}\int_a^0 (-t)^{3/2} w(t)^2 \, dt. \tag{5.9}$$

Proof. We follow the approach of Chen and Shen [28]. Starting from the easily established identity

$$\int_a^0 (-t)^{3/2}(w')^2 \, dt - \frac{1}{16}\int_a^0 (-t)^{-1/2} w^2 \, dt = \int_a^0 (-t)\left[((-t)^{1/4}w)'\right]^2 dt,$$

and setting $v(t) = (-t)^{1/4}w(t)$, it is enough to establish the Hardy–Sobolev inequality

$$\int_a^0 (-t)(v')^2 \, dt \geq \frac{4}{a^2}\int_a^0 (-t)v^2 \, dt. \tag{5.10}$$

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To establish (5.10), one makes use of $v(a) = 0$, the fundamental theorem of calculus and Hölder's inequality to find

$$\begin{aligned}
\int_a^0 (-t)v^2\,dt &= \int_a^0 -t\left(\int_a^t v'(s)\,ds\right)^2 dt \\
&\le \int_a^0 -t\left(\int_a^t -s(v')^2\,ds\right)\left(\int_a^t (-s)^{-1}\,ds\right) dt \\
&\le \left[\int_a^0 -s(v')^2\,ds\right]\left[\int_a^0 (t\log(-t) - t\log(-a))\;dt\right] \\
&= \frac{a^2}{4}\left[\int_a^0 -s(v')^2\,ds\right]. \quad \square
\end{aligned}$$

We conclude these preliminary observations by noting that if $u \in C^2(\overline{\Omega})$ is a solution of (5.1) in the supercritical case ($p > 10$), and assuming that u is nontrivial, then combining (5.5) with (5.8) and (5.9) yields

$$0 > \frac{10-p}{2p}\int_\Omega |u|^p\,dxdy = \int_{BC}\left[Mu + \frac{1}{2}u\right](yu_x, u_y)\cdot\nu\,ds \ge 0,$$

which contradicts u being nontrivial.

Remark 5.4. At least for $C^2(\overline{\Omega})$ solutions, the same argument gives the uniqueness of the trivial solution for nonlinearities $f \in C^0(\mathbb{R})$ whose primitive F with $F(0) = 0$ satisfies

$$10F(s) - sf(s) < 0 \quad \text{for } s \neq 0, \tag{5.11}$$

since this ensures that the integral over Ω on the right hand side of (5.5) is negative. For example, the condition (5.11) is satisfied by

$$f(s) = Cs|s|^{p-2} + \lambda s \quad \text{with } C > 0 \tag{5.12}$$

provided $p > 10$ and $\lambda \le 0$ or $p = 10$ and $\lambda < 0$. In addition, the result applies to

$$f(s) = C|s|^{p-1} + \lambda s \quad \text{with } C > 0 \tag{5.13}$$

provided that $0 \ge \lambda \ge -5/(16y_C^2)$. Indeed, the maximum principle of Theorem 3.4 then yields $u \ge 0$ a.e. and one may replace f given by (5.13) with that of (5.12).

5.1. Proof of Theorem 5.1 in the supercritical case

For $u \in H^1_\Gamma(\Omega)$ we will exploit the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, \infty)$ and mollifying procedures which are well calibrated to the geometry of Ω and the boundary conditions. To this end, fix a canonical mollifier $j \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp}(j) \subset (-1, 1), \quad j \text{ even}, j \ge 0, \quad \int_{\mathbb{R}} j(t)\,dt = 1. \tag{5.14}$$

For each $\varepsilon = (\varepsilon_1, \varepsilon_2)$ satisfying

$$0 < \varepsilon_1 \le \varepsilon_2,$$

define the mollified function $u_\varepsilon = J_\varepsilon u$ on $\overline{\Omega}$ by

$$J_\varepsilon u(x, y) = \int_\Omega \Phi_\varepsilon(x, y; \bar{x}, \bar{y})u(\bar{x}, \bar{y})\,d\bar{x}d\bar{y}, \tag{5.15}$$

where the mollifier kernel is defined by

$$\Phi_\varepsilon(x, y; \bar{x}, \bar{y}) = \frac{1}{\varepsilon_1\varepsilon_2} j\left(\frac{\bar{x}-x}{\varepsilon_2} + \xi\right)\left[j\left(\frac{y-\bar{y}}{\varepsilon_1}\right) - j\left(\frac{y+\bar{y}}{\varepsilon_1}\right)\right] \tag{5.16}$$

with

$$\xi \ge 1 + \xi_0, \qquad \xi_0 = \|g\|_{\text{Lip}([y_C, 0])} = \left(\frac{3x_0}{2}\right)^{1/3} \tag{5.17}$$

and g which defines the characteristics AC and BC by (5.7). Notice that for each $(x, y) \in \mathbb{R}^2$ fixed, $\Phi_\varepsilon(x, y; \cdot, \cdot) \in C_0^\infty(\mathbb{R}^2)$. In addition, since Ω is a Lipschitz domain, for any $u \in H^1_\Gamma(\Omega) \subset H^1(\Omega)$, we can extend u to an element of $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$

for each $q \in [1, \infty)$. Using this extension, we can also use (5.15)–(5.16) to define $J_\varepsilon u$ in a neighborhood of Ω and make the change variables $s = x - \bar{x}$, $t = y - \bar{y}$ when desired, without having to work on a subdomain Ω_ε of Ω. In a similar way, one defines the adjoint integral operator J_ε^* with kernel

$$\Phi_\varepsilon^*(x, y; \bar{x}, \bar{y}) = \Phi_\varepsilon(\bar{x}, \bar{y}; x, y) = \frac{1}{\varepsilon_1\varepsilon_2} j\left(\frac{x - \bar{x}}{\varepsilon_2} + \xi\right)\left[j\left(\frac{y - \bar{y}}{\varepsilon_1}\right) - j\left(\frac{y + \bar{y}}{\varepsilon_1}\right)\right], \tag{5.18}$$

where we recall that j is even. We record the following elementary properties of these mollification operators.

Lemma 5.5. *Let $u \in H^1(\Omega)$ and $0 < \varepsilon_1 \leq \varepsilon_2$. Then*

(a) $\|J_\varepsilon u\|_{L^p(\Omega)} \leq 4\|j\|^2_{L^\infty(\mathbb{R})} \|u\|_{L^p(\Omega)}$, *for each $p \in [1, \infty)$;*
(b) $\|J_\varepsilon u - u\|_{H^1(\Omega)} \to 0$ *as $\varepsilon \to 0$ along $0 < \varepsilon_1 \leq \varepsilon_2$, and hence there is also convergence in $L^p(\Omega)$ for each $p \in [1, \infty)$;*
(c) $J_\varepsilon u, J_\varepsilon^* u \in C^\infty(\overline{\Omega})$ *and*

$$J_\varepsilon u_{|\Gamma} = 0, \quad \Gamma = AC \cup AB. \tag{5.19}$$

Moreover, the same properties are satisfied by the adjoint integral operator J_ε^* where in place of (5.19), one has

$$J_\varepsilon^* u_{|\Gamma^*} = 0, \quad \Gamma^* = BC \cup AB.$$

Proof. To prove the L^p bound in part (a), notice that for each choice of ξ and for each $(x, y) \in \Omega$ fixed, the effective domain of integration in the formula (5.15) is

$$\Omega_{x,y} = \{(\bar{x}, \bar{y}) \in \Omega : \bar{y} \in [y - \varepsilon_1, y + \varepsilon_1],\ \bar{x} \in [x + \varepsilon_2\xi - \varepsilon_2, x + \varepsilon_2\xi + \varepsilon_2]\}, \tag{5.20}$$

which has measure $\left|\Omega_{x,y}\right| \leq 4\varepsilon_1\varepsilon_2$, while

$$|\Phi_\varepsilon(x, y; \bar{x}, \bar{y})| \leq \frac{1}{\varepsilon_1\varepsilon_2}\|j\|^2_{L^\infty(\mathbb{R})}, \quad (x, y), (\bar{x}, \bar{y}) \in \Omega.$$

Hence

$$\sup_{(x,y),(\bar{x},\bar{y})\in\Omega}\left\{\int_\Omega |\Phi_\varepsilon(x, y; \bar{x}, \bar{y})|\, d\bar{x}d\bar{y}, \int_\Omega |\Phi_\varepsilon(x, y; \bar{x}, \bar{y})|\, dxdy\right\} \leq 4\|j\|^2_{L^\infty(\mathbb{R})}$$

and hence the L^p-bound of part (a). That $J_\varepsilon u, J_\varepsilon^* u \in C^\infty(\overline{\Omega})$ and the convergence in $H^1(\Omega)$-norm are shown in the standard way by making use of Lebesgue's dominated convergence theorem. It remains only to show the vanishing claim (5.19).

Since j is even, one has

$$\Phi_\varepsilon(x, 0; \bar{x}, \bar{y}) = 0 = \Phi_\varepsilon^*(x, 0; \bar{x}, \bar{y}) \quad \text{for each } (\bar{x}, \bar{y}) \in \Omega$$

and hence $J_\varepsilon u, J_\varepsilon^* u$ vanish on AB where $y = 0$. To see that $J_\varepsilon u = 0$ on AC, for $(x, y) \in AC$ one has the relation (5.7) and

$$-2x_0 < \bar{x} \pm g(\bar{y}) < 0 \quad \text{for each } (\bar{x}, \bar{y}) \in \Omega.$$

Hence

$$\frac{\bar{x} - x}{\varepsilon_2} > \frac{-2x_0 + g(\bar{y}) + 2x_0 - g(y)}{\varepsilon_2} \geq -\frac{\xi_0|y - \bar{y}|}{\varepsilon_2}, \quad \forall\, (x, y) \in AC,\ (\bar{x}, \bar{y}) \in \Omega. \tag{5.21}$$

As noted in (5.20), one has $|y - \bar{y}| \leq \varepsilon_1$ on the effective domain of integration $\Omega_{x,y}$. Combining this fact with (5.17) and (5.21) yields

$$\frac{\bar{x} - x}{\varepsilon_2} + \xi \geq -\frac{\xi_0\varepsilon_1}{\varepsilon_2} + \xi \geq -\xi_0 + \xi \geq 1, \quad \forall\, (x, y) \in AC,\ (\bar{x}, \bar{y}) \in \Omega_{x,y}, \tag{5.22}$$

and hence $\Phi_\varepsilon(x, y; \cdot, \cdot) \equiv 0$ on Ω if $(x, y) \in AC$. A similar argument shows that $\Phi_\varepsilon^*(x, y; \cdot, \cdot) \equiv 0$ on Ω if $(x, y) \in BC$. □

Using this mollifying procedure, we can apply the dilation identity (5.5) with $F(u) = |u|^p/p$ to $u_\varepsilon = J_\varepsilon u$ since $u_\varepsilon \in C^\infty(\overline{\Omega})$ and vanishes on Γ to find

$$\int_\Omega \left(Mu_\varepsilon + \frac{1}{2}u_\varepsilon\right)\left(Lu_\varepsilon + F'(u_\varepsilon)\right) dxdy \geq \frac{p - 10}{2p}\int_\Omega F(u_\varepsilon)\, dxdy, \tag{5.23}$$

where the boundary integral on BC is non-negative by Lemma 5.3. We will exploit the following representation formula.

Lemma 5.6. *Let $u \in H^1_\Gamma(\Omega)$ be a weak solution of (5.1), M defined by (5.6) and $F(u) = |u|^p/p$. Then*

$$Lu_\varepsilon + F'(u_\varepsilon) = A_\varepsilon + B_\varepsilon \quad \text{a.e. in } \Omega, \tag{5.24}$$

where

$$A_\varepsilon(x, y) = \int_\Omega (y - \bar{y}) D^2_{\bar{x}} \Phi_\varepsilon(x, y; \bar{x}, \bar{y}) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y}, \tag{5.25}$$

$$B_\varepsilon = (J_\varepsilon u)|J_\varepsilon u|^{p-2} - J_\varepsilon(u|u|^{p-2}), \tag{5.26}$$

Φ_ε is given by (5.16) and $u_\varepsilon = J_\varepsilon u$ defined in (5.15).

Proof. Since $u \in H^1_\Gamma(\Omega)$ is a weak solution, one has

$$\int_\Omega \left(y u_x \varphi_x + u_y \varphi_y - F'(u)\varphi\right) dxdy = 0, \quad \forall\, \varphi \in H^1_{\Gamma^*}(\Omega).$$

Insert $\varphi = J^*_\varepsilon v$ with $v \in C^\infty_0(\Omega)$ an arbitrary test function, integrate by parts and apply Fubini's theorem to find

$$\begin{aligned} 0 &= \int_\Omega \left(u \left[y (J^*_\varepsilon v)_{xx} + (J^*_\varepsilon v)_{yy}\right] + F'(u) J^*_\varepsilon v\right) dxdy \\ &= \int_\Omega u(x, y) \left(\int_\Omega \left\{L\left[\Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})\right] + \Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})|u|^{p-2}(x, y)\right\} v(\bar{x}, \bar{y})\, d\bar{x} d\bar{y}\right) dxdy \\ &= \int_\Omega v(\bar{x}, \bar{y}) \left(\int_\Omega \left\{L\left[\Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})\right] + \Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})|u|^{p-2}(x, y)\right\} u(x, y)\, dxdy\right) d\bar{x} d\bar{y}, \end{aligned}$$

where $L = yD^2_x + D^2_y$. Since $v \in C^\infty_0(\Omega)$ is arbitrary, for a.e. $(\bar{x}, \bar{y}) \in \Omega$ one has

$$\int_\Omega \left\{\Psi_\varepsilon(x, y; \bar{x}, \bar{y}) + \Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})|u|^{p-2}(x, y)\right\} u(x, y)\, dxdy = 0. \tag{5.27}$$

where $\Psi_\varepsilon(x, y, \bar{x}, \bar{y}) := L\left[\Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})\right] = \left(yD^2_x + D^2_y\right)\left[\Phi^*_\varepsilon(x, y, \bar{x}, \bar{y})\right]$. Interchanging the roles of (x, y) and $(\bar{x}, \bar{y})$ and recalling (5.18), the relation (5.27) yields

$$\int_\Omega \left\{\Psi_\varepsilon(\bar{x}, \bar{y}; x, y) + \Phi_\varepsilon(x, y, \bar{x}, \bar{y})|u|^{p-2}(\bar{x}, \bar{y})\right\} u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y} = 0, \quad \text{a.e. } (x, y) \in \Omega. \tag{5.28}$$

Simple calculations show that

$$\Psi_\varepsilon(\bar{x}, \bar{y}; x, y) = \left(\bar{y} D^2_{\bar{x}} + D^2_{\bar{y}}\right)\left[\Phi_\varepsilon(x, y, \bar{x}, \bar{y})\right] \tag{5.29}$$

and inserting (5.29) into (5.28) shows that for a.e. $(x, y) \in \Omega$ one has

$$\begin{aligned} &-J_\varepsilon\left(u|u|^{p-2}\right)(x, y) - \int_\Omega \bar{y} D^2_{\bar{x}} \Phi_\varepsilon(x, y, \bar{x}, \bar{y}) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y} \\ &\quad = \int_\Omega D^2_{\bar{y}} \Phi_\varepsilon(x, y, \bar{x}, \bar{y}) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y} = \int_\Omega D^2_y \Phi_\varepsilon(x, y, \bar{x}, \bar{y}) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y}. \end{aligned} \tag{5.30}$$

Finally, since

$$L(J_\varepsilon u) + (J_\varepsilon u)\, |J_\varepsilon u|^{p-2} = \int_\Omega \left(yD^2_x + D^2_y\right)\left[\Phi_\varepsilon(x, y, \bar{x}, \bar{y})\right] u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y} + (J_\varepsilon u)\, |J_\varepsilon u|^{p-2}, \tag{5.31}$$

combining (5.31) with (5.30) yields (5.24) with A_ε and B_ε defined by (5.25)–(5.26). □

Using Lemma 5.6, we may rewrite (5.23) as

$$\left(Mu_\varepsilon + \frac{1}{2} u_\varepsilon, A_\varepsilon + B_\varepsilon\right) \geq \frac{p - 10}{2p} \int_\Omega |u_\varepsilon|^p\, dxdy, \tag{5.32}$$

where $(\cdot, \cdot)$ is the L^2-scalar product. By Lemma 5.5(b), for $\varepsilon \to 0$ along $0 < \varepsilon_1 \leq \varepsilon_2$ we have

$$\int_\Omega |u_\varepsilon|^p\, dxdy \to \int_\Omega |u|^p\, dxdy, \tag{5.33}$$

$$Mu_\varepsilon + \frac{1}{2} u_\varepsilon \to Mu + \frac{1}{2} u \quad \text{in } L^2(\Omega), \tag{5.34}$$

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and

$$B_\varepsilon = (J_\varepsilon u)|J_\varepsilon u|^{p-2} - J_\varepsilon(u|u|^{p-2}) \to 0 \quad \text{in } L^2(\Omega), \tag{5.35}$$

where we note that $B_\varepsilon = f_\#(J_\varepsilon u) - J_\varepsilon(f_\#(u))$ where $f_\# : L^{2p-2}(\Omega) \to L^2(\Omega)$ is a continuous Nemytskii operator, since $f(s) = s|s|^{p-1}$ satisfies (4.4) with $2p-2$ in place of p. Hence

$$J_\varepsilon(f_\#(u)) \to f_\#(u) \quad \text{and} \quad f_\#(J_\varepsilon u) \to f_\#(u) \quad \text{in } L^2(\Omega)$$

by Lemma 5.5(b) and the continuity of $f_\#$. We claim that

$$A_\varepsilon \to 0 \quad \text{in } L^2(\Omega). \tag{5.36}$$

Given the claim, Theorem 5.1 in the supercritical case follows by taking the limit in (5.32) and using (5.33)–(5.36) to find

$$0 \ge \frac{p-10}{2p} \int_\Omega |u|^p \, dxdy,$$

and hence $u = 0$ in $H^1(\Omega)$ if $p > 10$.

Proof of claim (5.36). For $u \in H^1_\Gamma(\Omega)$ we can integrate by parts once in the definition (5.25) of A_ε to find

$$A_\varepsilon(x,y) = \int_\Omega (y-\bar{y}) D_{\bar{x}} \Phi_\varepsilon(x,y;\bar{x},\bar{y}) D_{\bar{x}} u(\bar{x},\bar{y}) \, d\bar{x}d\bar{y} + \int_{BC} (y-\bar{y}) D_{\bar{x}} \Phi_\varepsilon u \nu_1 \, ds(\bar{x},\bar{y}),$$

but for $(\bar{x},\bar{y}) \in BC$ we have

$$D_{\bar{x}}\Phi_\varepsilon(x,y;\bar{x},\bar{y}) = \frac{1}{\varepsilon_1\varepsilon_2^2} j'\left(\frac{\bar{x}-x}{\varepsilon_2} + \xi\right)\left[j\left(\frac{y-\bar{y}}{\varepsilon_1}\right) - j\left(\frac{y+\bar{y}}{\varepsilon_1}\right)\right] = 0, \quad \forall\,(x,y) \in \Omega,$$

by using an argument similar to that which leads to (5.22) in the proof of Lemma 5.5. Since $u_x \in L^2(\Omega)$, we need only show that

$$\tilde{J}_\varepsilon v \to 0 \quad \text{in } L^2(\Omega) \text{ for each } v \in L^2(\Omega) \tag{5.37}$$

where $\tilde{J}_\varepsilon$ is the mollifying operator with kernel

$$\tilde{\Phi}_\varepsilon(x,y,\bar{x},\bar{y}) = (\bar{y}-y) D_{\bar{x}}\Phi_\varepsilon(x,y;\bar{x},\bar{y}).$$

The family $\tilde{J}_\varepsilon$ is uniformly bounded in ε on $L^2(\Omega)$ since the kernel is pointwise bounded by $|\bar{y}-y|\varepsilon_1^{-1}\varepsilon_2^{-2}\|j\|_{L^\infty(\mathbb{R})}\|j'\|_{L^\infty(\mathbb{R})}$ and supported on a rectangle of measure $4\varepsilon_1\varepsilon_2$ on which $|\bar{y}-y| \le \varepsilon_1 \le \varepsilon_2$. Using this uniform boundedness and the density of $C_0^\infty(\Omega)$, it is enough to verify the limit claim (5.37) for $v \in C_0^\infty(\Omega)$. For $v \in C_0^\infty(\Omega)$, one integrates by parts to find

$$\tilde{J}_\varepsilon v(x,y) = \int_\Omega (y-\bar{y}) \Phi_\varepsilon(x,y,;\bar{x},\bar{y}) v_x(\bar{x},\bar{y}) \, d\bar{x}d\bar{y}.$$

Estimating as before, one finds pointwise convergence

$$\left|\tilde{J}_\varepsilon v(x,y)\right| \le 4\varepsilon_1 \|v_x\|_{L^\infty(\Omega)} \|j\|^2_{L^\infty(\mathbb{R})} \to 0$$

and a uniform L^p bound for $\varepsilon_1 \le 1$

$$\left|\tilde{J}_\varepsilon v(x,y)\right|^p \le 4^p \|v_x\|^p_{L^\infty(\Omega)} \|j\|^{2p}_{L^\infty(\mathbb{R})}.$$

Hence the claim follows by Lebesgue's dominated convergence theorem.

5.2. Proof of Theorem 5.1 in the critical case

In the critical case $p = 10$, we will show that weak solutions $u \in H^1_\Gamma(\Omega)$ must have zero trace also on BC and hence $u \in H^1_0(\Omega)$ and so u solves the characteristic Cauchy problem with $u_{|AC\cup BC} = 0$. An additional multiplier identity using the y-translation multiplier $D_y u$ in place of the dilation multiplier Mu along a suitably regularized sequence will yield the conclusion $u = 0$ in $H^1(\Omega)$. This was what was done for $C^2(\overline{\Omega})$ solutions in [6], but additional work is needed to extend the result to weak solutions.

We begin by showing that u has zero trace on BC. We again apply the dilation identity (5.5) to $u_\varepsilon = J_\varepsilon u$, but we keep the non-negative boundary integral on BC and combine this with the representation formula (5.24) to find the following variant of (5.32)

$$\begin{aligned}\left(Mu_\varepsilon + \frac{1}{2}u_\varepsilon, A_\varepsilon + B_\varepsilon\right) &= \int_{BC}\left(Mu_\varepsilon + \frac{1}{2}u_\varepsilon\right)\left(yD_xu_\varepsilon, D_yu_\varepsilon\right)\cdot \nu\, ds\\ &= \int_{y_C}^0\left[4(-t)^{3/2}\psi_\varepsilon'(t)^2 - \frac{1}{4}(-t)^{-1/2}\psi_\varepsilon(t)^2\right] dt\\ &\geq \frac{16}{y_C^2}\int_{y_C}^0 (-t)^{3/2}\psi_\varepsilon(t)^2\, dt \geq 0,\end{aligned} \tag{5.38}$$

where we have also applied the Hardy–Sobolev inequality with remainder (5.9) to $\psi_\varepsilon(t) = u_{\varepsilon|BC}(t) = u_\varepsilon(\beta(t))$ with $\beta(t) = (-g(t), t)$ as in Lemma 5.2. Taking the limit in (5.38) as $\varepsilon = (\varepsilon_1, \varepsilon_2) \to 0$ along $0 < \varepsilon_1 \leq \varepsilon_2$ yields

$$\frac{16}{y_C^2}\int_{y_C}^0 (-t)^{3/2}\psi(t)^2\, dt = 0,$$

since $u_\varepsilon \to u$ in $H^1(\Omega)$ implies that $Mu_\varepsilon \to Mu$ and $A_\varepsilon, B_\varepsilon \to 0$ in $L^2(\Omega)$ as well as

$$\psi_\varepsilon = \mathrm{tr}_{|BC}u_\varepsilon \to \mathrm{tr}_{|BC}u = \psi \quad \text{in } L^2(BC).$$

In order to set up the mollifying scheme needed for the D_yu multiplier identity and in order to exploit fully the vanishing of u on BC, the following reformulation of weak solutions to (5.1) will be used.

Lemma 5.7. *Let $u \in H^1_\Gamma(\Omega)$ be a weak solution to (5.1) with $F'(u) = u|u|^{p-2}$. Then*

$$\int_\Omega \left(yu_xv_x + u_yv_y - u|u|^{p-2}v\right) dxdy = 0 \quad \text{for each } v \in C^\infty_{AB}(\overline{\Omega}), \tag{5.39}$$

where $C^\infty_{AB}(\overline{\Omega}) = \{v \in C^\infty(\overline{\Omega}) :\ v = 0 \text{ in a neighborhood of } AB\}$.

Proof. Given $v \in C^\infty_{AB}(\overline{\Omega})$, then $v(x,y) = 0$ for each $(x,y) \in \Omega$ with $y \geq -\delta$ for some $\delta = \delta(v) > 0$. Select a cutoff profile $\phi \in C^\infty(\mathbb{R})$ such that

$$\phi(s) = 0 \quad \text{for } s \leq 1/3 \quad \text{and} \quad \phi(s) = 1 \quad \text{for } s \geq 2/3.$$

For each $\sigma > 0$, define the function

$$v_\sigma(x,y) = v(x,y)\phi(h_\sigma(x,y)), \qquad h_\sigma(x,y) = \left(\frac{-x-g(y)}{\sigma}\right), \quad g(y) = \frac{2}{3}(-y)^{3/2}.$$

Since $v_\sigma \in H^1_\Gamma(\Omega)$, by the definition of weak solutions to (5.1), the identity (5.39) holds with v_σ in place of v and hence

$$\begin{aligned}0 &= \int_\Omega \left[yu_xv_x + u_yv_y + u|u|^{p-2}v\right]\phi(h_\sigma)\, dxdy + \int_\Omega \left[-yu_xv + (-y)^{1/2}u_yv\right]\frac{1}{\sigma}\phi'(h_\sigma)\, dxdy\\ &:= A_\sigma + B_\sigma.\end{aligned}$$

Applying the dominated convergence theorem to A_σ, one will have the identity (5.39) provided that $B_\sigma \to 0$ for $\sigma \to 0^+$. Integrating by parts and taking into account where u, v and ϕ vanish, one finds

$$\begin{aligned}B_\sigma &= \int_\Omega \left(yu\left[\frac{1}{\sigma}v\phi'(h_\sigma)\right]_x - u\left[\frac{1}{\sigma}(-y)^{1/2}v\phi'(h_\sigma)\right]_y\right) dxdy\\ &= \frac{1}{2}\int_\Omega \frac{u\phi'(h_\sigma)}{\sigma}\left[(-y)^{-1/2}v + 2yv_x - 2(-y)^{1/2}v_y\right] dxdy,\end{aligned}$$

where the support of the integrand is contained in

$$\Omega_{\sigma,\delta} = \{(x,y) \in \Omega :\ 0 \leq -x-g(y) \leq \sigma,\ y \leq -\delta < 0\}.$$

Hence there exists a constant $C_1 = C_1\left(\delta, \|\phi'\|_{C^1(\mathbb{R})}, \|v\|_{C^1(\overline{\Omega})}\right)$ such that

$$|B_\sigma| \leq C_1\int_{\Omega_{\sigma,\delta}} \frac{|u|}{\sigma}\, dxdy \leq C_1\int_{y_C}^{-\delta}\left(\int_{-\sigma-g(y)}^{-g(y)} \frac{|u(x,y)|}{-x-g(y)}\, dx\right) dy. \tag{5.40}$$

Making the change of variables $t = -x-g(y)$, the inner integral in (5.40) becomes

$$\int_{-\sigma-g(y)}^{-g(y)} \frac{|u(x,y)|}{-x-g(y)}\, dx = \int_0^\sigma \frac{|w(t)|}{t}\, dt$$

where $w(t) = u(-t - g(y), y)$. This can be estimated by the Hardy–Sobolev inequality

$$\int_0^\sigma \frac{|w|}{t}\, dt \le 2\sigma^{1/2} \|w'\|_{L^2([0,\sigma])}, \quad w \in C^1([0,\sigma]),\, w(0) = 0$$

to yield

$$|B_\sigma| \le 2C_1 \sigma^{1/2} \left(\int_{\Omega_{\sigma,\delta}} |u_x|^2\, dxdy \right)^{1/2} \le 2C_1 \sigma^{1/2} \|u_x\|_{L^2(\Omega)} \to 0 \quad \text{for } \sigma \to 0^+. \quad \square$$

Next we introduce a family of mollifiers which is well calibrated to the characteristic Cauchy problem, at least on the part of the domain where L is strictly hyperbolic. A natural family of mollifiers will require a Lipschitz bounds like (5.17) on the inverse function to g, which fails to be Lipschitz along AB where L degenerates. Hence, for each $\tau < 0$ fixed and small in absolute value, we will work on the domain

$$\Omega^\tau = \{(x,y) \in \Omega : y < \tau\}$$

and we will show that u must vanish in $H^1(\Omega_\tau)$ for each τ. Notice that $\Omega^\tau = A^\tau B^\tau C$ is also a characteristic triangle where

$$A^\tau = (-2x_0 + g(\tau), \tau) \quad \text{and} \quad B^\tau = (-g(\tau), \tau) \quad \text{with } g(\tau) = \frac{2}{3}(-\tau)^{3/2}$$

and $C = (-x_0, y_C)$ as before. Consider the inverse function h to $-g$; that is,

$$h(x) = -\left(\frac{3x}{2}\right)^{2/3}.$$

One has

$$A^\tau C : \ y = h(x + 2x_0), \quad x \in [-2x_0 + g(\tau), -x_0] \quad \text{and} \quad B^\tau C : \ y = h(x), \quad x \in [-x_0, -g(\tau)]$$

and the following Lipschitz bounds on h

$$\eta_\tau = \|h\|_{\mathrm{Lip}([-2x_0+g(\tau),-x_0])} = \|h\|_{\mathrm{Lip}([-x_0,-g(\tau)])} = (-\tau)^{-1/2}.$$

Now, for each $\tau < 0$ and each $\varepsilon > 0$, define the mollified function $u_\varepsilon^\tau = J_\varepsilon^\tau u$ on $\overline{\Omega}$ by

$$J_\varepsilon^\tau u(x,y) = \frac{1}{\varepsilon^2} \int_{\Omega^\tau} j\left(\frac{x - \bar{x}}{\varepsilon}\right) j\left(\frac{y - \bar{y}}{\varepsilon} - \eta\right) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y}, \tag{5.41}$$

where $j \in C_0^\infty(\mathbb{R})$ satisfies (5.14) and

$$\eta \ge 1 + \eta_\tau, \qquad \eta_\tau = (-\tau)^{-1/2}.$$

We record the following properties of this family of mollifiers which will be used in the limiting argument.

Lemma 5.8. *Let $u \in H^1(\Omega)$, $\varepsilon > 0$ and $\tau < 0$. Then*

(a) $\|J_\varepsilon u\|_{L^p(\Omega^\tau)} \le 4\|j\|^2_{L^\infty(\mathbb{R})} \|u\|_{L^p(\Omega^\tau)}$, *for each* $p \in [1, \infty)$;
(b) $\|J_\varepsilon u - u\|_{H^1(\Omega^\tau)} \to 0$ *as* $\varepsilon \to 0^+$, *and hence there is also convergence in* $L^p(\Omega^\tau)$ *for each* $p \in [1, \infty)$;
(c) $J_\varepsilon \in C^\infty(\overline{\Omega})$ *and*

$$J_\varepsilon^\tau u_{|A^\tau C \cup B^\tau C} = 0. \tag{5.42}$$

Proof. Parts (a) and (b) and the smoothness claim of part (c) proceed in the same manner as the corresponding statements in Lemma 5.5. The claim (5.42) is also similar. For example, if $(x,y) \in B^\tau C$ and $(\bar{x}, \bar{y}) \in \Omega^\tau$ then $|x - \bar{x}| < \varepsilon$ on the support of the integrand in (5.41) and hence

$$\frac{y - \bar{y}}{\varepsilon} - \eta = \frac{h(x) - \bar{y}}{\varepsilon} - \eta < \frac{h(x) - h(\bar{x})}{\varepsilon} - \eta \le \eta_\tau \frac{|x - \bar{x}|}{\varepsilon} - \eta \le -1,$$

and hence the claim $J_\varepsilon^\tau u = 0$ on $B^\tau C$. The proof that $J_\varepsilon^\tau u = 0$ on $A^\tau C$ is analogous. $\square$

We are now ready to complete the proof. Given $u \in H_0^1(\Omega)$ a weak solution to (5.1) with $F'(u) = u|u|^8$, then $u_\varepsilon^\tau \in C^\infty(\overline{\Omega})$ satisfies the following y-translation identity (see formula (3.7) of [29])

$$\begin{aligned} \int_{\Omega^\tau} D_y u_\varepsilon^\tau \left(L u_\varepsilon^\tau + F'(u_\varepsilon^\tau) \right) dxdy &= \frac{1}{2} \int_{\Omega^\tau} |D_x u_\varepsilon^\tau|^2\, dxdy \\ &\quad + \int_{\partial\Omega^\tau} \left(y D_x u_\varepsilon^\tau D_y u_\varepsilon^\tau, \frac{1}{2}\left[(D_y u_\varepsilon^\tau)^2 - y (D_x u_\varepsilon^\tau)^2 \right] + F(u_\varepsilon^\tau) \right) \cdot \nu\, ds. \end{aligned} \tag{5.43}$$

Since $u_\varepsilon^\tau = F(u_\varepsilon^\tau) = 0$ on $A^\tau C \cup B^\tau C$ one easily shows that the integral over $\partial\Omega^\tau$ reduces to that over $A^\tau B^\tau$ where $y = \tau < 0$, $\nu = (0, 1)$ and

$$\frac{1}{2}\left[(D_y u_\varepsilon^\tau)^2 - \tau(D_x u_\varepsilon^\tau)^2\right] + \frac{1}{10}|u_\varepsilon^\tau|^{10} \geq 0. \tag{5.44}$$

The following representation formula analogous to that of Lemma 5.6 holds:

$$Lu_\varepsilon^\tau + F'(u_\varepsilon^\tau) = A_\varepsilon^\tau + B_\varepsilon^\tau \quad \text{a.e. in } \Omega^\tau, \tag{5.45}$$

where

$$A_\varepsilon^\tau(x, y) = \int_{\Omega^\tau} (y - \bar{y}) D_{\bar{x}}^2 \Phi_\varepsilon^\tau(x, y; \bar{x}, \bar{y}) u(\bar{x}, \bar{y})\, d\bar{x} d\bar{y},$$
$$B_\varepsilon^\tau = (J_\varepsilon^\tau u)|J_\varepsilon^\tau u|^{p-2} - J_\varepsilon^\tau(u|u|^{p-2})$$

and $\Phi_\varepsilon^\tau = \varepsilon^{-2} j((x - \bar{x})/\varepsilon) j((y - \bar{y})/\varepsilon - \eta)$ is the kernel of J_ε^τ appearing in (5.41). The proof is completely analogous to that of Lemma 5.6 after one notes that for any $v \in C_0^\infty(\Omega^\tau)$ the function

$$J_\varepsilon^{\tau^*} v(x, y) = \frac{1}{\varepsilon^2} \int_{\Omega^\tau} j\left(\frac{\bar{x} - x}{\varepsilon}\right) j\left(\frac{\bar{y} - y}{\varepsilon} - \eta\right) v(\bar{x}, \bar{y})\, d\bar{x} d\bar{y}$$

satisfies

$$J_\varepsilon^{\tau^*} v \in C^\infty(\overline{\Omega}) \quad \text{and} \quad J_\varepsilon^{\tau^*} v = 0 \quad \text{on } \{(x, y) \in \Omega^\tau : y \geq \tau - \varepsilon\} \text{ for each } \tau \in (-1, 0).$$

That is, $J_\varepsilon^{\tau^*} v \in C_{AB}^\infty(\overline{\Omega})$ and hence one may apply Lemma 5.7 with $J_\varepsilon^{\tau^*} v$ in place of v. Combining (5.43) with (5.45) and (5.44) one has

$$\left(D_y u_\varepsilon^\tau, A_\varepsilon^\tau + B_\varepsilon^\tau\right) \geq \frac{1}{2} \int_{\Omega^\tau} \left|D_x u_\varepsilon^\tau\right|^2 dxdy. \tag{5.46}$$

Using Lemma 5.8, one has that

$$\begin{cases} A_\varepsilon^\tau \to 0 \\ B_\varepsilon^\tau \to 0 \end{cases} \quad \text{and} \quad \begin{cases} D_x u_\varepsilon^\tau \to D_x u \\ D_y u_\varepsilon^\tau \to D_y u \end{cases} \quad \text{in } L^2(\Omega^\tau) \text{ for } \varepsilon \to 0^+$$

and then taking the limit in (5.46) yields

$$0 \geq \frac{1}{2} \int_{\Omega^\tau} |D_x u|^2\, dxdy.$$

Hence $D_x u = 0$ a.e. in Ω^τ. However, since Ω^τ is convex in the x-direction and u has zero trace along $A^\tau C$, one has the Poincarè inequality

$$\|u\|_{L^2(\Omega^\tau)} \leq \sqrt{2} x_0 \|D_x u\|_{L^2(\Omega^\tau)}$$

by integrating along segments with y constant and using Hölder's inequality. Hence $u = 0$ in $H^1(\Omega^\tau)$ for each $\tau < 0$, which completes the proof.

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Appendix A. Proof of Lemma 3.3

We consider the integral operator $\widetilde{S}_0$ defined in (3.18) with kernel $K(z'; z) : \overline{\Delta} \times \overline{\Delta} \to \mathbb{R}$ defined in (3.16) as the product of $\chi_{\Delta_z^+ \cup \Delta_z^-}(z')$ (with $\Delta_z^\pm$ defined in (3.15)), $C(z')$ defined in (3.14) and $R(z'; z) : \Delta_z^+ \cup \Delta_z^- \times \overline{\Omega} \to \mathbb{R}$ the Riemann–Hadamard function defined in (3.10) and (3.17). We want to show that $\widetilde{S}_0 : C^0(\overline{\Delta}) \to C^0(\overline{\Delta})$ is well-defined, linear, continuous and compact. To do this, it suffices to show that the map

$$z \mapsto \|K(\cdot; z)\|_{L^1(\overline{\Delta})} \text{ is well-defined and continuous for } z \in \overline{\Delta}, \tag{A.1}$$

as is well known (see Theorem 3′ of [30] for example).

Proof. We will use standard properties of the hypergeometric functions involved in the kernel, elementary but careful estimates and standard convergence theorems for the Lebesgue integral. To aid the reader, we briefly recall those known properties of $F(a, b, c, ; \zeta)$ that we will use, all of which can be found in [31], for example. If $c \neq 0, -1, -2, \dots$ then

$$F(a, b, c; \zeta) = \sum_{n=0}^{+\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{\zeta^n}{n!},$$

where $(a, 0) = 1$ and $(a, n) = \Gamma(a + n)/\Gamma(a) = a(a + 1) \cdots (a + n - 1)$ for $n \in \mathbb{N}$ and the series converges absolutely for $\zeta \in \mathbb{C}$ with $|\zeta| < 1$ and also for $|\zeta| = 1$ if $\mathrm{Re}(c - a - b) > 0$. If $-1 < \mathrm{Re}(c - a - b) \leq 0$ then the series converges conditionally for $|\zeta| = 1$ with $\zeta \neq 1$ and the asymptotic behavior in $\zeta = 1$ given by

$$F(a, b, c; \zeta) \sim \begin{cases} \dfrac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}(1 - \zeta)^{c-a-b} & \text{if } \mathrm{Re}(c - a - b) < 0 \\ \dfrac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \log(1/(1 - s)) & \text{if } c = a + b \end{cases} \quad \text{for } \zeta \to 1^-. \tag{A.2}$$

We will consider only real values of ζ of the form s or $1/s$ where $s(z'; z)$ is defined in (3.11) so that

$$0 < s(z'; z) = \frac{(\xi - \xi')(\eta - \eta')}{(\eta' - \xi')(\eta - \xi)} < 1 \quad \text{for } z' = (\xi', \eta') \in \Delta_z^+ \tag{A.3}$$

and

$$0 < 1/s(z'; z) = \frac{(\eta' - \xi')(\eta - \xi)}{(\xi - \xi')(\eta - \eta')} < 1 \quad \text{for } z' = (\xi', \eta') \in \Delta_z^-. \tag{A.4}$$

Hence the hypergeometric functions used in (3.10) and (3.17) to define $R^{\pm}$ are given by convergent power series with $a + b = c$ in both cases. Consequently, the second asymptotic formula of (A.2) is relevant. Notice that $s(z'; z) = 1$ along the interface between Δ_z^+ and Δ_z^-, where $\eta' = \xi$. In addition, if $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ we have the Euler representation

$$F(a, b, c; \zeta) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - \zeta t)^{-a}\, dt, \tag{A.5}$$

which shows that if $c > b > 0$ then

$$\phi(r) := F(a, b, c; r) \text{ is an increasing function for } r \in (0, 1). \tag{A.6}$$

Then, using (A.5), one has

$$\phi(0) := F(a, b, c; 0) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}\, dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} B[b, c - b] = 1, \tag{A.7}$$

where

$$B[p, q] = \int_0^1 \frac{dt}{t^{1-p}(1 - t)^{1-q}} \tag{A.8}$$

is the beta function.

As a final preliminary regarding the map (A.1), for each $z = (\xi, \eta) \in \overline{\Delta}$ we denote by

$$I(z) = \|K(\cdot; z)\|_{L^1(\overline{\Delta})} = \|C(\cdot)R(\cdot; z)\|_{L^1(\Delta_z^+ \cup \Delta_z^-)} = I^+(z) + I^-(z), \tag{A.9}$$

where

$$I^+(z) = \int_0^{\xi} \left(\int_{\xi}^{\eta} C_0(\eta' - \xi')^{-\frac{2}{3}} R^+(z'; z)\, d\eta' \right) d\xi' \tag{A.10}$$

and

$$I^-(z) = \int_0^{\xi} \left(\int_{\xi'}^{\xi} C_0(\eta' - \xi')^{-\frac{2}{3}} R^-(z'; z)\, d\eta' \right) d\xi', \tag{A.11}$$

with $C_0 = 4^{-1}(4/3)^{2/3}$ as defined in (3.9) and $R^{\pm}$ as defined for $z \in \Delta$ in (3.10) and for $z \in \partial\Delta$ in (3.17) and the discussion leading up to that formula.

Step 1. (*Estimates for R*) Using the properties mentioned above, one has the following basic estimate. For each $\sigma > 0$ there exists $C_\sigma > 0$ such that for each $z = (\xi, \eta) \in \Delta$ one has

$$|R(z'; z)| \le C_\sigma \begin{cases} \dfrac{(\eta' - \xi')^{\frac{1}{6}}}{(\eta - \xi)^{\frac{1}{6}}} \left(1 + \dfrac{(\xi - \xi')^\sigma}{(\eta' - \xi)^\sigma}\right) & z' \in \Delta_z^+ \\ \dfrac{(\eta' - \xi')(\eta - \xi)^{\frac{2}{3}}}{(\eta - \eta')^{\frac{5}{6}} (\xi - \xi')^{\frac{5}{6}}} \left(1 + \dfrac{(\eta' - \xi')^\sigma}{(\xi - \eta')^\sigma}\right) & z' \in \Delta_z^-. \end{cases} \tag{A.12}$$

In fact, for the first estimate, with $z' \in \Delta_z^+$ we have $s \in (0, 1)$ by (A.3) and $R^+(z'; z) = ((\eta' - \xi')/(\eta - \xi))^{\frac{1}{6}} \phi(s)$ with $\phi(s) = F(1/6, 5/6, 1; s)$. Combining (A.2) with (A.6) and (A.7) shows that there exist C_1 and C_2 such that

$$|R^+(z'; z)| \le \left(\frac{\eta' - \xi'}{\eta - \xi}\right)^{1/6} \left(C_1 + C_2 \log\left(\frac{1}{1 - s}\right)\right) \quad \text{with } \frac{1}{1 - s} = \frac{(\eta' - \xi')(\eta - \xi)}{(\eta - \xi')(\eta' - \xi)} \tag{A.13}$$

but for $z' \in \Delta_z^+$ one has $0 < (\eta - \xi)/(\eta - \xi') \le 1$ and hence

$$\log\left(\frac{1}{1 - s}\right) \le \log\left(\frac{\eta' - \xi'}{\eta' - \xi}\right) = \log\left(1 + \frac{\xi - \xi'}{\eta' - \xi}\right) = \log(1 + t) \tag{A.14}$$

with $t \in (0, +\infty)$. Given $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$\log(1 + t) \le C_\sigma (1 + t^\sigma). \tag{A.15}$$

Combining (A.13)–(A.15) yields the first estimate in (A.12). Notice that in (A.14), $\xi - \xi' = 0$ at the boundary point $(\xi', \eta') = (\xi, \xi) \in \Delta_z^+$ while $\eta' - \xi = 0$ along the boundary segment $(\xi', \eta') = (\xi', \xi) \in \Delta_z^+$ where $s = 1$. A similar argument starting from (A.4) yields the second estimate in (A.12).

Step 2. (*Boundedness of I*) One has $\sup_{z \in \Delta} I(z) < +\infty$. In particular, there exists $\widetilde{C} > 0$ such that

$$I(z) \le \widetilde{C} (\eta - \xi)^{\frac{2}{3}} (\xi^{\frac{1}{2}} + \xi^{\frac{2}{3}}). \tag{A.16}$$

Indeed, splitting I as in (A.9)–(A.11) and using the estimates (A.12), for each $\sigma > 0$ one has

$$I^+(z) \le C_0 C_\sigma \int_0^\xi \frac{1}{(\xi - \xi')^{\frac{1}{6}}} \int_\xi^\eta \frac{1}{(\eta' - \xi')^{\frac{1}{3}} (\eta - \eta')^{\frac{1}{6}}} \left(1 + \frac{(\xi - \xi')^\sigma}{(\eta' - \xi)^\sigma}\right) d\eta' \, d\xi'.$$

Using that $(1/(\eta' - \xi')) \le 1/(\xi - \xi')$ since $0 < \xi' < \xi$ and $\xi < \eta' < \eta$ for $z' \in \Delta_z^+$, one has Q3

$$I^+(z) \le C_0 C_\sigma \int_0^\xi \frac{1}{(\xi - \xi')^{\frac{1}{2}}} \left(\frac{6}{5} (\eta - \xi)^{\frac{5}{6}} + (\xi - \xi')^\sigma \int_\xi^\eta \frac{d\eta'}{(\eta - \eta')^{\frac{1}{6}} (\eta' - \xi)^\sigma}\right) d\xi'. \tag{A.17}$$

The change of variables $t = (\eta' - \xi)/(\eta - \xi)$ yields

$$\int_\xi^\eta \frac{d\eta'}{(\eta - \eta')^{\frac{1}{6}} (\eta' - \xi)^\sigma} = (\eta - \xi)^{\frac{5}{6} - \sigma} \int_0^1 \frac{dt}{t^\sigma (1 - t)^{\frac{1}{6}}} = (\eta - \xi)^{\frac{5}{6} - \sigma} B[1 - \sigma, 5/6], \tag{A.18}$$

with B defined by (A.8). Combining (A.17) and (A.18) yields the existence of $\widetilde{C}_\sigma^+$ such that

$$I^+(z) \le \widetilde{C}_\sigma^+ (\eta - \xi)^{\frac{5}{6} - \sigma} \left(\xi^{\frac{1}{2}} + \xi^{\sigma + \frac{1}{2}}\right). \tag{A.19}$$

A similar argument for $I^-(z)$ yields the existence of $\widetilde{C}_\sigma^- > 0$ such that

$$I^-(z) \le \widetilde{C}_\sigma^- (\eta - \xi)^{\frac{2}{3}} \left(\xi^{\frac{2}{3}} + \eta^{\frac{1}{6}} \xi^{\frac{1}{2}}\right), \tag{A.20}$$

where $\eta^{1/6} \le l^{1/6}$. Choosing $\sigma = 1/6$ in (A.19) and combining with (A.20) yields (A.16).

Step 3. (*Continuity of I for $z = (\xi, \eta) \in \partial\Delta$ with $\xi = 0$ or $\xi = \eta$*) In fact, from (A.16) one has

$$\begin{cases} \lim\limits_{(\xi, \eta) \to (0, \eta_0)} I(\xi, \eta) = 0 & \forall\, \eta_0 \in [0, l] \\ \lim\limits_{(\xi, \eta) \to (\xi_0, \xi_0)} I(\xi, \eta) = 0 & \forall\, \xi_0 \in [0, l], \end{cases} \tag{A.21}$$

while

$$K(\cdot,z)=0 \quad \text{if } z=(0,\eta_0) \text{ or } z=(\xi_0,\xi_0) \tag{A.22}$$

by the definition of K and R (see (3.16)–(3.17) and the discussion there). Combining (A.21) and (A.22) yields the claim.

Step 4. (*Interior continuity of* I) For $z_1, z_2 \in \Delta$, one wants to show that $|I(z_2)-I(z_1)|$ is small if $|z_2-z_1|$ is small. Recall that

$$I(z)=\int_{\Delta_z^+\cup\Delta_z^-} C(z')R(z';z)\,dz'.$$

Since the form of $R(z';z)$ depends on whether z' belongs to Δ_z^+ or Δ_z^-, the increment

$$J(z_1,z_2):=I(z_2)-I(z_1)$$

will take on various forms depending on how $z_k=(\xi_k,\eta_k)$ are situated relative to one another. We will decompose the analysis into various pieces by splitting J into a sum of terms where the increment is taken in only one variable (ξ or η) and the corresponding domains of integration are nested. Notice that if $\Delta_{z_1}\subset\Delta_{z_2}$ then

$$J(z_1,z_2)=\int_{\Delta_{z_2}\setminus\overline{\Delta}_{z_1}} C(z')R(z';z_2)\,dz'+\int_{\Delta_{z_1}} C(z')[R(z';z_2)-R(z';z_1)]\,dz'.$$

We now discuss the reduction. By exchanging the roles of z_1 and z_2 we may assume that $\xi_1\le\xi_2$ and since $z_k\in\Delta$ we have $\xi_k<\eta_k$ for $k=1,2$. There are four non equivalent possibilities:

(1) $\xi_1<\xi_2$ and $\eta_1<\eta_2$;
(2) $\xi_1<\xi_2$ and $\eta_2<\eta_1$;
(3) $\xi_2=\xi_1<\eta_1<\eta_2$;
(4) $\xi_1<\xi_2<\eta_2=\eta_1$.

In cases (3) and (4), the increment appears in only one variable and $\Delta_{z_1}\subset\Delta_{z_2}$, while in the first cases, the insertion of a $z_3=(\xi_1,\eta_2)$ allows one to write

$$J(z_1,z_2)=I(z_2)-I(z_3)+I(z_3)-I(z_1)=J(z_2,z_3)+J(z_3,z_1) \tag{A.23}$$

where these two increments are with respect to a single variable and the domains are nested with $\Delta_{z_3}\subset\Delta_{z_2}$ and $\Delta_{z_1}\subset\Delta_{z_3}$. In the second case, picking $z_3=(\xi_1,\eta_2)$ also yields (A.23) where $\Delta_{z_2}\subset\Delta_{z_3}$ and $\Delta_{z_3}\subset\Delta_{z_1}$.

Hence it is enough to show that for each fixed $z=(\xi,\eta)\in\Delta$ one has

$$F(\delta):=\int_{\Delta_{z_\delta}\setminus\overline{\Delta}_z} C(z')R(z';z_\delta)\,dz'\to 0 \quad \text{as } \delta\to 0^+, \tag{A.24}$$

$$G(\delta):=\int_{\Delta_z} C(z')[R(z';z_\delta)-R(z';z)]\,dz'\to 0 \quad \text{as } \delta\to 0^+, \tag{A.25}$$

where

$$z_\delta=(\xi_\delta,\eta):=(\xi+\delta,\eta) \quad \text{or} \quad z_\delta=(\xi,\eta_\delta):=(\xi,\eta+\delta), \tag{A.26}$$

with $\delta>0$ and small enough so that $z_\delta\in\Delta$. The limit claims (A.24)–(A.25) follow from standard analysis and the estimates (A.12) on R where the details differ only slightly for the two cases of z_δ given in (A.26).

The limit of F *with* $z_\delta=(\xi_\delta,\eta)$: We split $F(\delta)=F^+(\delta)+F^-(\delta)$ by integrating over $\Delta_{z_\delta}^+$ and $\Delta_{z_\delta}^-$ respectively; that is,

$$F^+(\delta)=\int_\xi^{\xi_\delta}\left(\int_{\xi_\delta}^{\eta} C(z')R^+(z';z_\delta)\,d\eta'\right)d\xi' \tag{A.27}$$

and

$$F^-(\delta)=\int_\xi^{\xi_\delta}\left(\int_{\xi'}^{\xi_\delta} C(z')R^-(z';z_\delta)\,d\eta'\right)d\xi'. \tag{A.28}$$

To estimate (A.27), we use both the first estimate of (A.12) and the relation (A.3) with $z=z_\delta=(\xi_\delta,\eta)$ to find

$$F^+(\delta)\le C_0C_\sigma\int_\xi^{\xi_\delta}\frac{1}{(\xi_\delta-\xi')^{\frac16}}\left(\int_{\xi_\delta}^{\eta}\frac{1}{(\eta'-\xi')^{\frac13}(\eta-\eta')^{\frac16}}\left(1+\frac{(\xi_\delta-\xi')^\sigma}{(\eta'-\xi_\delta)^\sigma}\right)d\eta'\right)d\xi'. \tag{A.29}$$

Estimating as was done in (A.17)–(A.19), one finds

$$F^+(\delta)\le C_\sigma^{(1)}(\eta-\xi-\delta)^{\frac56-\delta}\left[\delta^{\frac12}+\delta^{\frac12+\sigma}\right]\to 0 \quad \text{as } \delta\to 0^+.$$

To estimate (A.28), we use the second estimate of (A.12) with $z = z_\delta = (\xi_\delta, \eta)$ and the fact that $\eta' \le \xi_\delta$ to find

$$F^-(\delta) \le C_0 C_\sigma (\eta - \xi_\delta)^{\frac{2}{3}} \int_\xi^{\xi_\delta} \frac{1}{(\xi_\delta - \xi')^{\frac{1}{2}}} \left(\int_{\xi'}^{\xi_\delta} \left(\frac{1}{(\eta - \eta')^{\frac{5}{6}}} + \frac{(\xi_\delta - \xi')^\sigma}{(\eta - \eta')^{\frac{5}{6}} (\xi_\delta - \eta')^\sigma} \right) d\eta' \right) d\xi'.$$

Calculating the integral in η' (which involves the beta function $B[1/6, 1-\sigma]$), using the inequality $(\eta - \xi')^{1/6} - (\eta - \xi_\delta)^{1/6} \le \eta^{1/6}$ and then computing the integral in ξ' yields

$$F^-(\delta) \le C_\sigma^{(2)} \eta^{\frac{1}{6}} (\eta - \xi - \delta)^{\frac{2}{3}} \left[\delta^{\frac{1}{2}} + \delta^{\frac{2}{3}} \right] \to 0 \quad \text{as } \delta \to 0^+.$$

The limit of F with $z_\delta = (\xi, \eta_\delta)$: In this case, there is no need to split F since $\eta' > \xi$ everywhere on $\Delta_{z_\delta} \setminus \overline{\Delta}_z$. Proceeding as was done to arrive at (A.29), one finds

$$F(\delta) \le C_0 C_\sigma \int_0^\xi \frac{1}{(\xi_\delta - \xi')^{\frac{1}{6}}} \left(\int_\eta^{\eta_\delta} \frac{1}{(\eta' - \xi')^{\frac{1}{3}} (\eta - \eta')^{\frac{1}{6}}} \left(1 + \frac{(\xi - \xi')^\sigma}{(\eta' - \xi)^\sigma} \right) d\eta' \right) d\xi'.$$

Using $\eta' - \xi' \ge \eta - \xi' \ge \xi - \xi'$ and $\eta' - \xi \ge \eta' - \eta$ one easily finds

$$F(\delta) \le C_\sigma^{(3)} \delta^{\frac{5}{6} - \sigma} \left[\xi^{\frac{1}{2}} \delta^\sigma + \xi^{\frac{1}{2} + \sigma} \right] \to 0 \quad \text{as } \delta \to 0^+,$$

by choosing $\sigma < 5/6$.

The limit of G with $z_\delta = (\xi_\delta, \eta)$: With $z = (\xi, \eta) \in \Delta$ fixed but arbitrary, we will verify that $G(1/k) \to 0$ for $k \to +\infty$ by showing that the sequence of functions

$$f_k(z') := C(z') R(z'; \xi + 1/k, \eta)$$

satisfies

$$\int_{\Delta_z^\pm} f_k(z')\, dz' \to \int_{\Delta_z^\pm} f(z')\, dz' \quad \text{where } f(z') := C(z') R(z'; \xi, \eta). \tag{A.30}$$

Indeed, f_k are defined a.e. on Δ_z (except for the segments $\eta' = \xi + 1/k$) and satisfy

$$f_k(z') \to f(z') \quad \text{for a.e. } z' \in \Delta_z.$$

By the dominated convergence theorem, one has (A.30) if there exist g_k and $g \in L^1(\Delta_z^\pm)$ such that

$$|f_k(z')| \le g_k(z') \quad \text{a.e. on } \Delta_z^\pm; \tag{A.31}$$

$$\int_{\Delta_z^\pm} \left| g_k(z') - g(z') \right| dz' \to 0 \quad \text{as } k \to +\infty. \tag{A.32}$$

We define for each $k > d := (\eta - \xi)/2$

$$g_k(z') = \frac{C_0 C_\sigma}{(\eta' - \xi')^{\frac{1}{2}}} \begin{cases} \dfrac{1}{(\eta - \xi - d)^{\frac{1}{6}}} \left(1 + \dfrac{(\eta' - \xi')^\sigma}{|\eta' - \xi - 1/k|^\sigma} \right), & z' \in \Delta_z^+, \\ \dfrac{1}{(\eta - \xi - 1/k)^{\frac{1}{6}}} \left(1 + \dfrac{(\eta' - \xi')^\sigma}{(\xi + 1/k - \eta')^\sigma} \right), & z' \in \Delta_z^- \end{cases} \tag{A.33}$$

$$g(z') = \frac{C_0 C_\sigma}{(\eta' - \xi')^{\frac{1}{2}}} \begin{cases} \dfrac{1}{(\eta - \xi - d)^{\frac{1}{6}}} \left(1 + \dfrac{(\eta' - \xi')^\sigma}{(\eta' - \xi)^\sigma} \right), & z' \in \Delta_z^+, \\ \dfrac{1}{(\eta - \xi)^{\frac{1}{6}}} \left(1 + \dfrac{(\eta' - \xi')^\sigma}{(\xi - \eta')^\sigma} \right), & z' \in \Delta_z^- \end{cases} \tag{A.34}$$

where in Δ_z^+, each g_k is defined a.e. (for $z' \ne \xi + 1/k$), C_0 and C_σ are as above and $\sigma > 0$ will be chosen suitably to ensure (A.32). Indeed, the estimates (A.12) imply that (A.31) holds and the claim that $g_k \to g$ a.e. in $\Delta_z^\pm$ is obvious from the definitions (A.33) and (A.34). On Δ_z^-, the needed limit (A.32) follows by the dominated convergence theorem since in Δ_z^- one has

$$|g_k(z')| \le \frac{C_0 C_\sigma}{(\eta - \xi - d)^{\frac{1}{6}}} \left[\frac{1}{(\eta' - \xi')^{\frac{1}{2}}} + \frac{1}{(\eta' - \xi')^{\frac{1}{2} - \sigma}} \frac{1}{(\xi - \eta')^\sigma} \right] \in L^1(\Delta_z^-),$$

provided that one chooses $\sigma < 1/4$. On Δ_z^+, one needs only verify that

$$\int_{\Delta_z^+} \frac{1}{(\eta' - \xi')^{\frac{1}{2} - \sigma}} \left| \frac{1}{|\eta' - \xi - 1/k|^\sigma} - \frac{1}{|\eta' - \xi|^\sigma} \right| dz' \to 0.$$

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We are free to choose $\sigma = 1/2$ and we have

$$h_k(z') := \frac{1}{|\eta' - \xi - 1/k|^\sigma} \to h(z') := \frac{1}{|\eta' - \xi|^\sigma} \quad \text{a.e. } in \Delta_z^+. \tag{A.35}$$

Using the "missing term in the Fatou lemma" (see p. 21 of [32]) it is enough to show that

$$\int_{\Delta_z^+} h_k(z')\,dz' \to \int_{\Delta_z^+} h(z')\,dz', \tag{A.36}$$

where we note that $h_k, h \geq 0$ by (A.35). One merely calculates the integrals to verify that (A.36) holds.

The limit of G with $z_\delta = (\xi, \eta_\delta)$: The analogous argument in this case is somewhat simpler since

$$f_k(z') := C(z')R(z'; \xi, \eta + 1/k)$$

has its only singularity along the interface $\eta' = \xi$ between Δ_z^+ and Δ_z^-. The limit f is as in the previous case and one easily checks that by defining

$$g_k(z') = \frac{C_0 C_\sigma}{(\eta' - \xi')^{\frac{1}{2}}(\eta + 1/k - \xi)^{\frac{1}{6}}} \times \begin{cases} 1 + \dfrac{(\xi - \xi')^\sigma}{(\eta' - \xi)^\sigma}, & z' \in \Delta_z^+ \\ 1 + \dfrac{(\eta' - \xi')^\sigma}{(\xi - \eta')^\sigma}, & z' \in \Delta_z^- \end{cases}$$

$$g(z') = \frac{C_0 C_\sigma}{(\eta' - \xi')^{\frac{1}{2}}(\eta - \xi)^{\frac{1}{6}}} \times \begin{cases} 1 + \dfrac{(\xi - \xi')^\sigma}{(\eta' - \xi)^\sigma}, & z' \in \Delta_z^+ \\ 1 + \dfrac{(\eta' - \xi')^\sigma}{(\xi - \eta')^\sigma}, & z' \in \Delta_z^- \end{cases}$$

one has the bounds (A.31) and the limit (A.32) in this case by choosing $\sigma > 0$ suitably small.

Step 5. (*Continuity of I for $z = (\xi_0, \eta) \in \partial\Delta$ with $\xi_0 \in (0, l)$*)

This is now easy since we can embed Δ in a larger triangle (by taking $\tilde{l} > l$ in place of l) and use the interior continuity of I on the larger triangle. $\square$

Appendix B. Proof of Lemma 3.5

We will first justify parts (a) and (b) of the lemma; that is, if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies (3.22) and

$$\begin{cases} Tu - \lambda u = f & \text{in } \Omega \\ u = \gamma & \text{on } \Gamma, \end{cases}$$

with $0 \leq f \in C_0^0(\Omega)$ and $\lambda \in [-5/(16y_C^2), 0]$, then the minimum m of u on $\overline{\Omega}$ be realized on $\Gamma = AC \cup AB$ if $m \leq 0$ and hence (3.23) holds. An analogous argument shows that when $f \leq 0$ then the maximum M of u must be realized on Γ if $M \geq 0$ and hence (3.24) holds.

In what follows, if P and Q are points in the half-space $y \leq 0$ that are connected by a characteristic of positive/negative slope and whose y-coordinates satisfy $y_P < y_Q \leq 0$, we will denote by ~~and~~ $[P, Q]_\pm$ the characteristic segment joining them (including the endpoints) and $(P, Q)_\pm$ the segment with endpoints removed. We will also denote by $\mathcal{F}_k^\pm(P)$ the flow from $P \in \Omega \cup AC \cup BC$ along the characteristic of positive/negative slope ~~or~~ arc length $k > 0$. The key point is to integrate the following differential identity over characteristic segments:

$$D_+(gD_-u) = g(-Tu) + D_+gD_+u \quad \text{where } g(y) = (-y)^{-1/4}, \tag{B.1}$$

and the differential operators $D_\pm = D_y \pm \sqrt{-y}D_x$ have been introduced in (3.21). In this context, one has the following version of the fundamental theorem of calculus

$$\int_R^S D_+w\,dy = w\,|_R^S = w(S) - w(R), \quad w \in C^1([R, S]_+), \tag{B.2}$$

where by $\int_R^S \alpha$ we intend the line integral of the differential 1-form α along the oriented characteristic interval $[R, S]_+$. Formula (B.2) gives rise to the integration by parts formula

$$\int_R^S vD_+w\,dy = vw\,|_R^S - \int_R^S wD_+v\,dy, \quad v, w \in C^1([R, S]_+). \tag{B.3}$$

Proof of part (a). Assume the contrary; that is, there exists $Q \in \Omega \cup (C, B)_+$ such that $u(Q) = \min_{\overline{\Omega}} u \le 0$ and $u(Q) < \gamma$.

Case 1. ($Q \in \Omega$)

If u assumes its negative minimum at Q in the hyperbolic interior Ω, one joins Q to some $P \in AC \setminus \{A, C\}$ with a characteristic segment $[P, Q]_+$. We will show that $D_-u(Q) < 0$, and hence Q cannot be a location of a minimum. One multiplies the identity (B.1) by the 1-form dy and integrates along the oriented characteristic segment $[P, Q]_+$ by splitting the line integral into two pieces at an intermediate point S close enough to P so that S is outside the support of f. The proof of Lemma 3.4 of [10] shows that the integral along $[P, S]_+$ is a convergent (perhaps improper) integral while the other is a proper integral. This yields

$$\begin{aligned}\int_P^Q D_+(gD_-u)\,dy &= \int_P^Q g(-Tu)\,dy + \int_P^Q D_+gD_+u\,dy \\ &\le \int_P^Q g(-\lambda u)\,dy + \int_P^Q D_+gD_+u\,dy,\end{aligned}$$

since $g > 0$ in Ω and $-Tu = -\lambda u - f \le -\lambda u$ in Ω. Integration by parts (B.3) on the last integral above yields

$$\int_P^Q D_+(gD_-u)\,dy \le \int_P^Q g(-\lambda u)\,dy + uD_+g\,|_P^Q - \int_P^Q uD_+^2g\,dy,$$

which by the fundamental theorem of calculus (B.2) gives

$$g(Q)D_-u(Q) \le \int_P^Q u(-\lambda g - D_+^2 g)\,dy + u(Q)D_+g(Q) - u(P)D_+g(P). \tag{B.4}$$

Noticing that

$$u(Q)D_+g(Q) = u(Q)\int_P^Q (\lambda g + D_+^2 g)\,dy - \lambda u(Q)\int_P^Q g\,dy + u(Q)D_+g(P) \tag{B.5}$$

and inserting (B.5) into (B.4) one obtains

$$g(Q)D_-u(Q) \le \int_P^Q (u(Q) - u)(\lambda g + D_+^2 g)\,dy - \lambda u(Q)\int_P^Q g\,dy + (u(Q) - u(P))D_+g(P). \tag{B.6}$$

If u has a minimum in Q, then $D_-u(Q) = 0$ and hence the right hand side of (B.6) must be non negative. However, the third term on the left is negative since $u(Q) - u(P) = u(Q) - \gamma < 0$. The second term is non positive since $g > 0$, $-\lambda \ge 0$ and $u(Q) \le 0$. The first term is also non negative since $u(Q) - u \le 0$ provided that

$$0 \le \lambda g + D^2 g = \lambda(-y)^{-1/4} + \frac{5}{16}(-y)^{-9/4},$$

which happens precisely when $\lambda \ge -5/(16y_C^2)$.

Case 2. ($Q \in (C, B)_+$)

Assuming that u assumes a non positive minimum on at Q on the characteristic arc $(C, B)_+$, one can integrate dy times the identity (B.1) along a characteristic segment $[P, Q']_+$ where $Q' = \mathcal{F}_k^-(Q)$ and $P = \mathcal{F}_k^-(C)$ with k chosen small enough so that $[P, Q']_+ \cap \operatorname{supp}(f) = \emptyset$. One obtains

$$D_-u(Q') = \frac{1}{g(Q')}\left[\int_P^{Q'} \left(u(Q') - u\right)(\lambda g + D_+^2 g)\,dy - \lambda u(Q')\int_P^{Q'} g\,dy + (u(Q') - u(P))D_+g(P)\right], \tag{B.7}$$

by repeating the argument leading to (B.6). One can view (B.7) as a family of formulas in terms of the characteristic distance k from BC; that is

$$\begin{aligned}D_-u(\mathcal{F}_k^-(Q)) = \frac{1}{g(\mathcal{F}_k^-(Q))}\Bigg[&\int_{\mathcal{F}_k^-(C)}^{\mathcal{F}_k^-(Q)} \left(u(\mathcal{F}_k^-(Q)) - u\right)(\lambda g + D_+^2 g)\,dy \\ &- \lambda u(\mathcal{F}_k^-(Q))\int_{\mathcal{F}_k^-(C)}^{\mathcal{F}_k^-(Q)} g\,dy + (u(\mathcal{F}_k^-(Q)) - u(\mathcal{F}_k^-(C)))D_+g(\mathcal{F}_k^-(C))\Bigg],\end{aligned} \tag{B.8}$$

for $k \in (0, \bar{k}]$ and $\bar{k}$ small enough as to ensure that $[\mathcal{F}_k^-(C), \mathcal{F}_k^-(Q)]_+$ lies outside of the support of f. Since all of the relevant objects on the right hand side of (B.8) are continuous, one finds that

$$\lim_{k\to 0^+} D_-(\mathcal{F}_k^-(Q)) = \frac{1}{g(Q)}\left[\int_C^Q (u(Q) - u)(\lambda g + D_+^2 g)\,dy - \lambda u(Q)\int_C^Q g\,dy + (u(Q) - u(C))D_+g(C)\right], \tag{B.9}$$

where again the right hand side of (B.9) must be strictly negative. Indeed, $u(Q) - u(C) = u(Q) - \gamma < 0$ and $D_+ g(C) > 0$ makes the third term negative, while the first two terms are non positive as in case 1. Hence u is strictly increasing as R tends to Q along $\Gamma_-(Q)$, which contradicts u having a minimum in Q. □

Finally, we justify part (c) of the lemma, where again it suffices to consider the case when $f \geq 0$.

Proof of part (c). If $\lambda = 0$ we merely repeat the argument above. In Case 1, the formula (B.6) becomes

$$g(Q)D_- u(Q) \leq \int_P^Q (u(Q) - u) D_+^2 g \, dy + (u(Q) - u(P)) D_+ g(P). \tag{B.10}$$

If u has a minimum in $Q \in \Omega$, the left hand side of (B.10) vanishes while the right hand side is negative since $u(Q) - u(P) = u(Q) - \gamma < 0$ and $u(Q) - u \leq 0$ along $[P, Q]_+$. In case 2, formula (B.9) becomes

$$\lim_{k \to 0^+} D_-(\mathcal{F}_k^-(Q)) = \frac{1}{g(Q)} \left[\int_C^Q (u(Q) - u) \, D_+^2 g \, dy + (u(Q) - u(C)) D_+ g(C) \right],$$

which is negative. Hence Q cannot be a location of a minimum. □

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