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DYNAMIC PROGRAMMING FOR OPTIMAL CONTROL PROBLEMS WITH DELAYS IN THE CONTROL VARIABLE*

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Abstract. We study a class of optimal control problems with state constraint, where the state equation is a differential equation with delays in the control variable. This class of problems arises in some economic applications, in particular in optimal advertising problems. The optimal control problem is embedded in a suitable Hilbert space, and the associated Hamilton–Jacobi–Bellman (HJB) equation is considered in this space. It is proved that the value function is continuous with respect to a weak norm and that it solves in the viscosity sense the associated HJB equation. The main results are the proof of a directional C^1-regularity for the value function and the feedback characterization of optimal controls.

Key words. Hamilton–Jacobi–Bellman equation, optimal control, delay equations, viscosity solutions, regularity, verification theorem

AMS subject classifications. 34K35, 49L25, 49K25

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1. Introduction.

This paper is devoted to studying a class of state constrained optimal control problems with distributed delay in the control variable and the associated Hamilton–Jacobi–Bellman (HJB) equation. The main result is the proof of a C^1 directional regularity for the value function associated to the control problem, which is the starting point from which to prove a smooth verification theorem.

The study of control problems with delays is motivated by economic (see, e.g., [2, 4, 5, 13, 32, 26, 36, 37, 43, 46]) and engineering applications (see, e.g., [40]). Concerning the economic applications, which are the main motivation of our study, we observe that there is a wide variety of models with memory structures considered by the economic literature. We refer, for instance, to models where the memory structure arises in the state variable as growth models with time-to-build in production (see [2, 4, 5]); to models where the memory structure arises in the control variable as vintage capital models (see [13, 26]); to advertising models (see [32, 36, 37, 43, 46]); and to growth models with time-to-build in investment (see [41, 48]).

From a mathematical point of view, our aim is to study the optimal control of the one-dimensional delay differential equation

$$y'(t) = a_0 y(t) + b_0 u(t) + \int_{-r}^{0} b_1(\xi) u(t+\xi) d\xi, \tag{1}$$

where $r > 0$, $a_0, b_0 \in \mathbb{R}$, and $b_1 : [-r, 0] \to \mathbb{R}$, subject to the state constraint $y(\cdot) > 0$ [1] and to the control constraint $u(\cdot) \in U \subset \mathbb{R}$. The objective is to maximize a functional

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[1] We remark that the case of state constraint $y(\cdot) \geq 0$ can be treated with similar arguments.

of the form

$$\int_0^{+\infty} e^{-\rho t}\big(g_0(y(t)) \,-\, h_0(u(t))\big)dt,$$

where $\rho > 0$ is a discount factor and $g_0 : \mathbb{R}^+ \to \mathbb{R}$, $h_0 : U \to \mathbb{R}$ are measurable functions.[2] Our ultimate goal is to get the synthesis of optimal controls for this problem, i.e., to produce optimal feedback controls by means of the dynamic programming approach.

The presence of the delay in the state equation (1) renders applying the dynamic programming techniques to the problem in its current form impossible. A general way to tackle control problems with delay consists in representing the controlled system in a suitable infinite dimensional space (see [12, Part II, Ch. 1] for a general theory). In this way the delay is absorbed in the infinite dimensional state, but, on the other hand, the price to pay is that the resulting system is infinite dimensional, and so the value function V is defined on an infinite dimensional space. Then the core of the problem becomes the study of the associated infinite dimensional HJB equation for V: the optimal feedback controls will be given in terms of the first space derivatives of V through the so-called verification theorem.

Sometimes explicit solutions to the (infinite dimensional) HJB equation are available (see [5, 26, 31]). In this case the optimal feedback controls are explicitly given, and the verification theorem can be proved in a quite standard way. However, in most cases the explicit solutions are not available, and then one has to try to prove a regularity result for the solutions of the HJB equation in order to be able to define formally optimal feedback controls and check its optimality through the verification theorem. This is due to the fact that, to obtain an optimal control in feedback form, one needs the existence of an appropriately defined gradient of the solution. It is possible to prove verification theorems and representation of optimal feedbacks in the framework of viscosity solutions, even if the gradient is not defined in classical sense (see, e.g., [10, 50] in finite dimension and [27, 42] in infinite dimension), but this is usually not satisfactory in applied problems since the closed loop equation becomes very hard to treat in such cases. The C^1-regularity of solutions to HJB equations is particularly important in infinite dimension, since in this case verification theorems in the framework of viscosity solutions contained in the aforementioned references are rather weak. For this reason, the main goal of the present paper is to prove a C^1-regularity result for the value function V of our problem.

To the best of our knowledge, C^1-regularity for a first order HJB equation was first proved by Barbu and Da Prato [6] using methods of convex regularization and was then developed by various authors (see, e.g., [7, 8, 9, 22, 23, 28, 34, 35]) in the case without state constraints and without applications to problems with delay. In the papers [16, 17, 29] a class of optimal control problems with state constraints is treated using again methods of convex regularization, but the C^1-regularity is not proved. To the best of our knowledge, the only paper proving a C^1-type regularity result for the solutions to HJB equations arising in optimal control problems with delay and state constraints is [30]. There a method introduced in finite dimension by Cannarsa and Soner [18] (see also [10]) and based on the concept of viscosity solution has been generalized in infinite dimension to get an ad hoc directional regularity result for viscosity solutions of the HJB equation.

[2] In economic applications typically they are, respectively, a production function and a cost function.

In our paper we want to further exploit the method of [18] to get a C^1-type regularity result for our problem. The main difference between our paper and [30, 31] is that in the latter papers the delay is in the state variable, while here the delay is in the control variable. The case of problems with delay in the control variable is harder to treat. First, if we tried a standard infinite dimensional representation as in the case of state delay problems, we would get a *boundary control problem* in infinite dimension (see [39]). However, this first difficulty can be overcome when the original state equation is linear using a suitable transformation leading to the construction of the so-called *structural state* (see [49]), and this is why, differently from [30], here we treat only the case of a linear state equation. But, once we have done that, if we try to follow the approach of [30] to prove a C^1-regularity result for the value function, it turns out that much more care is needed in the choice of the space where the infinite dimensional representation is performed. While in [30] the product space $\mathbb{R} \times L^2$ is used to represent the delay system, here we need to use the more regular product space $\mathbb{R} \times W^{1,2}$ for reasons that are explained in the crucial Remark 5.4. We observe that the theory of the infinite dimensional representation of delay systems has been developed mainly in spaces of continuous function or in product spaces of type $\mathbb{R} \times L^2$ (see the aforementioned reference [12]). Therefore, we restate the infinite dimensional representation in $\mathbb{R} \times W^{1,2}$ and carefully adapt the regularity method of [30] in such a context. So we get the desired C^1-type regularity result (Theorem 6.8), which exactly as in [30] just allows us to define an optimal feedback map in a classical sense (see Corollary 6.9 and (51)). Finally, we exploit this regularity to prove a verification theorem (Theorem 7.2) and the existence, uniqueness, and characterization of optimal feedback controls (Corollary 7.8). The main problem for this part is represented by the study of the closed loop equation (52), which, unlike [30], has to be approached in infinite dimension. Since we can only prove continuity of the feedback map, we need to work with Peano's theorem. Unfortunately Peano's theorem fails in general in infinite dimension (see [33]). There are available in the literature some results derived under stronger assumptions than just continuity and/or for weaker concepts of solutions (see [1, 3, 19, 21, 38, 47, 52]). However, our case is somehow different (also since we work with mild solutions), so we prove the result directly (Propositions 7.3 and 7.7).

The paper is structured as follows. In section 2 we give the definition of some spaces and state some notation. In section 3 we give a precise formulation of the optimal control problem. In section 4 we rephrase the problem in infinite dimension and state the equivalence with the original one (Theorem 4.4). In section 5 we prove that the value function is continuous in the interior of its domain with respect to a weak norm (Proposition 5.8). In section 6 we show that the value function solves in the viscosity sense the associated HJB equation (Theorem 6.5), and then we provide the main result; i.e., we prove that it has continuous classical derivative along a suitable direction in the space $\mathbb{R} \times W^{1,2}$ (Theorem 6.8). In section 7 we exploit such a result to prove the existence of a unique global mild solution to the closed loop equation (52) and the verification theorem, obtaining the existence and uniqueness of optimal feedback controls.

2. Spaces and notation. Let $r > 0$. Throughout paper we use the Lebesgue space $L^2_r := L^2([-r, 0]; \mathbb{R})$, endowed with inner product $\langle f, g\rangle_{L^2_r} := \int_{-r}^0 f(\xi)g(\xi)d\xi$, which renders it a Hilbert space. We shall use the Sobolev spaces (see [14])

$$W_r^{k,2} := W^{k,2}([-r, 0]; \mathbb{R}),$$

where $k \geq 1$, endowed with the inner products $\langle f, g\rangle_{W_r^{k,2}} := \sum_{i=0}^k \int_{-r}^0 f^{(i)}(\xi)g^{(i)}(\xi)d\xi$, which render them Hilbert spaces. We have the Sobolev inclusions (see [14])

$$W_r^{k,2} \hookrightarrow C^{k-1}([-r,0];\mathbb{R}),$$

with continuous embedding. Throughout the paper we shall confuse the elements of $W_r^{k,2}$, which are classes of equivalence of functions, with their (unique) representatives in $C^{k-1}([-r,0];\mathbb{R})$. Given that, we define the spaces, for $k \geq 1$,

$$W_{r,0}^{k,2} := \{f \in W_r^{k,2} \mid f^{(i)}(-r) = 0 \ \forall i = 0, 1, \ldots, k-1\} \subset W_r^{k,2}.$$

We notice that in our definition of $W_{r,0}^{k,2}$ the boundary condition is only required at $-r$. The spaces $W_{r,0}^{k,2}$ are Hilbert spaces as they are closed subsets of the Hilbert spaces $W_r^{k,2}$. On these spaces we can also consider the inner products

$$\langle f, g\rangle_{W_{r,0}^{k,2}} := \int_{-r}^0 f^{(k)}(\xi)g^{(k)}(\xi)d\xi. \tag{2}$$

Due to the boundary condition in the definition of the subspaces $W_{r,0}^{k,2}$, the inner products $\langle \cdot, \cdot\rangle_{W_{r,0}^{k,2}}$ are equivalent to the original inner products $\langle \cdot, \cdot\rangle_{W_r^{k,2}}$ on $W_{r,0}^{k,2}$ in the sense that they induce equivalent norms. For this reason, dealing with topological concepts, we shall consider the simpler inner products (2) on $W_{r,0}^{k,2}$. Finally, we consider the space H defined as

$$H := \mathbb{R} \times W_{r,0}^{1,2}.$$

This is a Hilbert space when endowed with the inner product

$$\langle \eta, \zeta\rangle := \eta_0\zeta_0 + \langle \eta_1, \zeta_1\rangle_{W_{r,0}^{1,2}},$$

which induces the norm

$$\|\eta\|^2 = |\eta_0|^2 + \int_{-r}^0 |\eta_1'(\xi)|^2 d\xi.$$

This is the Hilbert space where our infinite dimensional system will take values. We consider the following partial order relation in H: given η, $\zeta \in H$, we say that

$$\eta \geq \zeta \ \text{ if } \ \eta_0 \geq \zeta_0 \ \text{ and } \ \eta_1 \geq \zeta_1 \ \text{ a.e. in } [-r,0]; \tag{3}$$

moreover, we say that

$$\eta > \zeta \ \text{ if } \ \eta \geq \zeta \text{ and, moreover, } \eta_0 > \zeta_0 \text{ or } \mathrm{Leb}\,\{\eta_1 > \zeta_1\} > 0. \tag{4}$$

3. The optimal control problem. In this section we give the precise formulation of the optimal control problem that we study.

Given $y_0 \in (0, +\infty)$ and $\delta \in L_r^2$, we consider the optimal control of the following one-dimensional ordinary differential equation with delay in the control variable:

$$\begin{cases} y'(t) = a_0 y(t) + b_0 u(t) + \displaystyle\int_{-r}^0 b_1(\xi)u(t+\xi)d\xi, \\ y(0) = y_0, \quad u(s) = \delta(s), \ s \in [-r, 0), \end{cases} \tag{5}$$

with state constraint $y(\cdot) > 0$ and control constraint $u(\cdot) \in U \subset \mathbb{R}$. The value $y_0 \in (0, +\infty)$ in the state equation (5) represents the initial state of the system, while the function δ represents the past of the control, which is considered as a given datum.

Concerning the control set U, we assume the following, which will be a standing assumption throughout the paper.

HYPOTHESIS 3.1. *$U = [0, \bar{u}]$, where $\bar{u} \in [0, +\infty]$. When $\bar{u} = +\infty$, the set U is intended as $U = [0, +\infty)$.*

Concerning the parameters appearing in (5), we make the following assumptions, which will be standing throughout the paper as well.

HYPOTHESIS 3.2.

(i) $a_0, b_0 \in \mathbb{R}$;

(ii) $b_1 \in W^{1,2}_{r,0}$, *and* $b_1 \neq 0$.

The fact that $b_1 \neq 0$ means that the delay really appears in the state equation. We set

$$b := (b_0, b_1(\cdot)) \in H.$$

Given $u \in L^2_{loc}([0, +\infty); \mathbb{R})$, there exists a unique locally absolutely continuous solution $y : [0, +\infty) \to \mathbb{R}$ of (5), provided explicitly by the variation of constants formula

$$y(t) = y_0 e^{a_0 t} + \int_0^t e^{a_0(t-s)} f(s) ds,$$

where

$$f(s) = b_0 u(s) + \int_{-r}^0 b_1(\xi) u(s + \xi) d\xi, \qquad u(s) = \delta(s),\ s \in [-r, 0].$$

We notice that f is well defined, as b_1 is bounded and $u \in L^2_{loc}([0, +\infty), \mathbb{R})$. We denote by $y(t; y_0, \delta(\cdot), u(\cdot))$ the solution to (5) with initial datum $(y_0, \delta(\cdot))$ and under the control $u \in L^2_{loc}([0, +\infty); \mathbb{R})$. We notice that $t \mapsto y(t; y_0, \delta(\cdot), u(\cdot))$ solves the delay differential equation (5) only for almost every $t \geq 0$.

The set of admissible controls for the problem with state constraint $y(\cdot) > 0$ is

$$\mathcal{U}(y_0, \delta(\cdot)) := \{u \in L^2_{loc}([0, +\infty); U) \mid y(t; y_0, \delta(\cdot), u(\cdot)) > 0 \ \ \forall t \geq 0\}.$$

The set $\mathcal{U}(y_0, \delta(\cdot))$ is immediately seen to be convex, due to the linearity of the state equation. We define the objective functional

$$J_0(y_0, \delta(\cdot); u(\cdot)) = \int_0^{+\infty} e^{-\rho t} \big(g_0(y(t; y_0, \delta(\cdot), u(\cdot))) - h_0(u(t))\big) dt, \tag{6}$$

where $\rho > 0$ and $g_0 : [0, +\infty) \to \mathbb{R}$, $h_0 : U \to \mathbb{R}$ are functions satisfying Hypothesis 3.3 below, which will be standing assumptions throughout the paper. The optimization problem consists in the maximization of the objective functional J_0 over the set of all admissible controls $\mathcal{U}(y_0, \delta(\cdot))$:

$$\max_{u \in \mathcal{U}(y_0, \delta(\cdot))} J_0(y_0, \delta(\cdot); u(\cdot)). \tag{7}$$

HYPOTHESIS 3.3.

(i) $g_0 \in C([0,+\infty);\mathbb{R})$, *and it is concave, nondecreasing, and bounded from above. Without loss of generality we assume that* $g_0(0) = 0$ *and set*

$$\bar{g}_0 := \lim_{y\to+\infty} g_0(y).$$

(ii) $h_0 \in C(U) \cap C^1(U^\circ)$, *where* U° *denotes the interior part of* U. *Moreover, it is nondecreasing, convex, and not constant. Without loss of generality we assume* $h_0(0) = 0$.

Remark 3.4.

(i) The assumption that g_0 is bounded from above (Hypothesis 3.3(i)) is quite unpleasant if we think about the applications. This assumption is taken here just for convenience in order to avoid further technical complications diverting from our theoretical aim, which is the proof of a regularity result. However, we stress that it can be replaced (as is usual in this kind of problem) with a suitable assumption on the growth of g_0, relating it to the requirement of a large enough discount factor ρ. This is quite easy in the case $\bar{u} < +\infty$, as in this case we have a straightforward estimate on the maximal growth of y. In the case $\bar{u} = +\infty$, the estimates require much more care, as they require dealing with the trade-off between the "profit" coming from large values of y and the "cost" coming from large values of u.

(ii) We consider a delay $r \in [0,+\infty)$. However, one can obtain the same results even allowing $r = +\infty$, suitably redefining the boundary conditions as limits. In the definition of the Sobolev spaces $W^{k,2}_{-\infty,0}$, the boundary conditions required would become

$$W^{k,2}_{-\infty,0} := \left\{ f \in W^{k,2} \mid \lim_{r\to+\infty} f^{(i)}(-r) = 0 \ \forall i = 0,1,\dots,k-1 \right\} \subset W^{k,2}_{-\infty}.$$

Also, for some results, we shall make use of the following assumptions.

HYPOTHESIS 3.5.

(i) $a_0 \neq 0$;

(ii) $b > 0$ *(in the sense of* (4)*)*.

HYPOTHESIS 3.6.

(i) g_0 *is strictly increasing;*

(ii) h_0 *is strictly convex;*

(iii) $\lim_{u\downarrow 0} h_0'(u) = 0$, $\lim_{u\uparrow \bar{u}} h_0'(u) = +\infty$.

HYPOTHESIS 3.7. *Either* (i) $\bar{u} < +\infty$ *or*

$$\text{(8)} \qquad \text{(ii)} \quad \bar{u} = +\infty \quad \textit{and} \quad \exists\, \alpha > 0 \quad \textit{such that} \ \liminf_{u\to+\infty} \frac{h_0(u)}{u^{1+\alpha}} > 0.$$

Remark 3.8. Hypotheses 3.6(iii) corresponds to the Inada assumptions.

Remark 3.9. We notice that Hypothesis 3.7(ii) is just slightly stronger than the assumption $\lim_{u\to+\infty} h_0'(u) = +\infty$ in Hypothesis 3.6(iii).

4. Representation in infinite dimension. In this section we restate the delay differential equation (5) as an abstract evolution equation in a suitable infinite dimensional space. The infinite dimensional setting is represented by the Hilbert space $H = \mathbb{R} \times W^{1,2}_{r,0}$. The following argument is just a suitable rewriting in $\mathbb{R} \times W^{1,2}_{r,0}$

of the method illustrated in [12] in the framework of the product space $\mathbb{R} \times L^2$. We will use some basic concepts from the semigroups theory, for which we refer the reader to [25].

Let A be the unbounded linear operator

$$A: \mathcal{D}(A) \subset H \to H, \quad (\eta_0, \eta_1(\cdot)) \mapsto (a_0\eta_0 + \eta_1(0), -\eta_1'(\cdot)), \tag{9}$$

where a_0 is the constant appearing in (5), defined on

$$\mathcal{D}(A) = \mathbb{R} \times W^{2,2}_{r,0}.$$

It is possible to show by direct computations that A is a (densely defined) closed operator and generates a C_0-semigroup $(S_A(t))_{t\geq 0}$ in H. However, we provide the proof of this fact in Appendix A.1 by using some known facts from the semigroups theory. The explicit expression of $S_A(t)\eta$, where $\eta = (\eta_0, \eta_1(\cdot)) \in H$, is (see Appendix A.1)

$$S_A(t)\eta = \left(\eta_0 e^{a_0 t} + \int_{(-t)\vee(-r)}^0 \eta_1(\xi) e^{a_0(\xi+t)} d\xi,\ \eta_1(\cdot - t)\mathbf{1}_{[-r,0]}(\cdot - t)\right). \tag{10}$$

We define the bounded linear operator

$$B : \mathbb{R} \longrightarrow H, \qquad u \longmapsto ub = (ub_0, ub_1(\cdot)).$$

Often we will identify the operator $B \in \mathcal{L}(\mathbb{R}; H)$ with $b \in H$.

Given $u \in L^2_{loc}([0, +\infty), \mathbb{R})$ and $\eta \in H$, we consider the abstract equation in H

$$\begin{cases} Y'(t) = AY(t) + Bu(t), \\ Y(0) = \eta. \end{cases} \tag{11}$$

We will use two concepts of solution to (11), which in our case coincide. (For details we refer the reader to [42, Ch. 2, sec. 5].) In the definitions below the integral in dt is intended as a Bochner integral in the Hilbert space H.

Definition 4.1.

(i) *We call mild solution of* (11) *the function* $Y \in C([0, +\infty); H)$ *defined as*

$$Y(t) = S_A(t)\eta + \int_0^t S_A(t - \tau)Bu(\tau)d\tau, \quad t \geq 0. \tag{12}$$

(ii) *We call weak solution of* (11) *a function* $Y \in C([0, +\infty); H)$ *such that, for any* $\phi \in \mathcal{D}(A^*)$,

$$\langle Y(t), \phi\rangle = \langle \eta, \phi\rangle + \int_0^t \langle Y(\tau), A^*\phi\rangle d\tau + \int_0^t \langle Bu(\tau), \phi\rangle d\tau \quad \forall t \geq 0.$$

From now on we denote by $Y(\cdot; \eta, u(\cdot))$ the mild solution of (11). We note that $Y(t; \eta, u(\cdot))$, $t \geq 0$, lies in H, so it has two components:

$$Y(t; \eta, u(\cdot)) = (Y_0(t; \eta, u(\cdot)), Y_1(t; \eta, u(\cdot))).$$

The definition of mild solution is the infinite dimensional version of the variation of constants formula. By a well-known result (see [42, Ch. 2, Prop. 5.2]), the mild solution is also the (unique) weak solution.

4.1. Equivalence with the original problem. In order to state the equivalence between the controlled delay differential equation and the abstract evolution equation (11), we need to link the canonical $\mathbb{R}$-valued integration with the $W^{1,2}_{r,0}$-valued integration. This is provided by the following lemma whose proof is standard. We omit it for brevity.

LEMMA 4.2. *Let* $0 \leq a < b$ *and* $f \in L^2\big([a,b]; W^{1,2}_{r,0}\big)$. *Then*

$$\left(\int_a^b f(t)dt\right)(\xi) = \int_a^b f(t)(\xi)dt \qquad \forall \xi \in [-r,0],$$

where the integral in dt *in the left-hand side is intended as a Bochner integral in the space* $W^{1,2}_{r,0}$.

We need to study the adjoint operator A^* in order to use the concept of weak solution of (11).

PROPOSITION 4.3. *We have*

$$\mathcal{D}(A^*) = \{\phi = (\phi_0, \phi_1(\cdot)) \in H \mid \phi_1 \in W^{2,2}_r, \quad \phi_1(-r) = 0, \quad \phi_1'(0) = 0\}$$

and

$$A^*\phi = (a_0\phi_0,\ \xi \mapsto \phi_1'(\xi) + \phi_0(\xi + r) - \phi_1'(-r)), \quad \phi \in \mathcal{D}(A^*). \tag{13}$$

Proof. See Appendix A.2. ∎

Let $v \in L^2_r$, and consider its convolution with b_1,

$$(v * b_1)(\xi) = \int_{-r}^{\xi} b_1(\tau)v(\tau - \xi)d\tau, \quad \xi \in [-r,0].$$

Recalling that $b_1(-r) = 0$, we can extend b_1 to a function of $W^{1,2}(\mathbb{R};\mathbb{R})$ and equal to 0 in $(-\infty, -r]$. Extend v to a function of $L^2(\mathbb{R};\mathbb{R})$, simply defining it as equal to 0 out of the interval $[-r,0]$. Then the convolution above can be rewritten as

$$(v * b_1)(\xi) = \int_{\mathbb{R}} b_1(\tau)v(\tau - \xi)d\tau, \quad \xi \in [-r,0].$$

Since $v \in L^2(\mathbb{R};\mathbb{R})$ and $b_1 \in W^{1,2}(\mathbb{R};\mathbb{R})$, the result [14, Lemma 8.4] and the fact that $(v * b_1)(-r) = 0$ yield $v * b_1 \in W^{1,2}_{r,0}$ and

$$(v * b_1)'(\xi) = \int_{-r}^{\xi} b_1'(\tau)v(\tau - \xi)d\tau. \tag{14}$$

Consider still v extended to 0 out of $[-r,0]$, and set $v_\xi(\tau) := v(\tau - \xi)$, $\tau \in [-r,0]$, for $\xi \in [-r,0]$. Of course $v_\xi \in L^2_r$ and $\|v_\xi\|_{L^2_r} \leq \|v\|_{L^2_r}$ for every $\xi \in [-r,0]$. Then, due to (14) and by Holder's inequality we have

$$\begin{aligned} \|v * b_1\|^2_{W^{1,2}_{r,0}} &= \int_{-r}^0 \left|\int_{-r}^{\xi} b_1'(\tau)v(\tau-\xi)d\tau\right|^2 d\xi = \int_{-r}^0 \left|\int_{-r}^{\xi} b_1'(\tau)v_\xi(\tau)d\tau\right|^2 d\xi \\ &\leq \int_{-r}^0 \left(\int_{-r}^0 |b_1'(\tau)v_\xi(\tau)|d\tau\right)^2 d\xi \ \leq \int_{-r}^0 \left(\|b_1'\|^2_{L^2_r}\|v_\xi\|^2_{L^2_r}\right) d\xi \leq r\|b_1'\|^2_{L^2_r}\|v\|^2_{L^2_r}. \end{aligned} \tag{15}$$

Let us now introduce the linear operator

$$
\begin{aligned}
M : \mathbb{R} \times L^2([-r,0];\mathbb{R}) &\longrightarrow H \\
(z,v) &\longmapsto (z, v * b_1) = \left(z, \int_{-r}^{\cdot} b_1(\tau) v(\tau - \cdot) d\tau\right).
\end{aligned} \tag{16}
$$

Due to (15), M is bounded. Call

$$\mathcal{M} := Range(M).$$

Of course $\mathcal{M}$ is a linear subspace of H (it is possible, using [11], to show that is not closed, so $\mathcal{M} \neq H$).

THEOREM 4.4. *Let $y_0 \in \mathbb{R}$, $\delta \in L^2_r$, $u(\cdot) \in L^2_{loc}([0,+\infty),\mathbb{R})$. Set*

$$\eta := M(y_0, \delta(\cdot)) \in \mathcal{M}, \quad Y(t) := Y(t;\eta,u(\cdot)), \quad t \geq 0. \tag{17}$$

Then $Y(t) = (Y_0(t), Y_1(t)(\cdot))$ belongs to $\mathcal{M}$ for every $t \geq 0$ and

$$(Y_0(t), Y_1(t)(\cdot)) = M(Y_0(t), u(t+\cdot)) \qquad \forall t \geq 0. \tag{18}$$

Moreover, let $y(\cdot) := y(\cdot; y_0, \delta, u(\cdot))$ be the unique solution to (5)*. Then*

$$y(t) = Y_0(t) \qquad \forall t \geq 0. \tag{19}$$

Proof. Let Y be the mild solution defined by (12) with initial condition η given by (17). On the second component (12) reads

$$
\begin{aligned}
Y_1(t) &= T(t)\eta_1 + \int_0^t [T(t-s)b_1]u(s)ds \\
&= \mathbf{1}_{[-r,0]}(\cdot - t)\eta_1(\cdot - t) + \int_0^t \mathbf{1}_{[-r,0]}(\cdot - t + s) b_1(\cdot - t + s) u(s) ds,
\end{aligned} \tag{20}
$$

where $(T(t))_{t\geq 0}$ is the semigroup of truncated right shifts on $W^{1,2}_{r,0}$, that is,

$$[T(t)\phi](\xi) = \mathbf{1}_{[-r,0]}(\xi - t)\phi(\xi - t), \quad \xi \in [-r,0].$$

We recall that by assumption $\eta = M(y_0, \delta(\cdot))$, and so

$$\eta_1(\xi) = \int_{-r}^{\xi} b_1(\alpha)\delta(\alpha - \xi)d\alpha.$$

Then, by (20) and due to Lemma 4.2,

$$Y_1(t)(\xi) = \mathbf{1}_{[-r,0]}(\xi - t)\int_{-r}^{\xi - t} b_1(\alpha)u(\alpha - \xi + t)d\alpha + \int_0^t \mathbf{1}_{[-r,0]}(\xi - t + s) b_1(\xi - t + s) u(s) ds. \tag{21}$$

Taking into account that $0 \leq s \leq t$, we have $\xi - t \leq \xi - t + s \leq \xi$, so that, doing the substitution $\alpha = \xi - t + s$ in the second term of the right-hand side of (21), it becomes

$$
\begin{aligned}
Y_1(t)(\xi) &= \mathbf{1}_{[-r,0]}(\xi - t)\int_{-r}^{\xi - t} b_1(\alpha)u(\alpha - \xi + t)d\alpha \\
&\quad + \int_{\xi - t}^{\xi} \mathbf{1}_{[-r,0]}(\alpha) b_1(\alpha) u(\alpha - \xi + t) d\alpha \\
&= \int_{-r}^{(\xi - t)\vee(-r)} b_1(\alpha)u(\alpha - \xi + t)d\alpha + \int_{(\xi - t)\vee(-r)}^{\xi} b_1(\alpha)u(\alpha - \xi + t)d\alpha \\
&= \int_{-r}^{\xi} b_1(\alpha)u(\alpha + t - \xi)d\alpha.
\end{aligned} \tag{22}
$$

Therefore, due to (16), the identity (18) is proved.

Let us now show (19). Setting $\xi = 0$ in (22), we get

$$Y_1(t)(0) = \int_{-r}^{0} b_1(\alpha)u(t+\alpha)d\alpha. \tag{23}$$

Now we use the fact that Y is also a weak solution of (11). From Proposition 4.3 we know that

$$(1,0) \in \mathcal{D}(A^*), \quad A^*(1,0) = (a_0,\ \xi \mapsto \xi + r)\,. \tag{24}$$

Therefore, taking into account (24) and (23) and Definition 4.1(ii), we have for almost every $t \geq 0$

$$\begin{aligned} Y_0'(t) = \frac{d}{dt}\langle Y(t), (1,0)\rangle &= \langle Y(t), A^*(1,0)\rangle + \langle Bu(t), (1,0)\rangle \\ &= a_0 Y_0(t) + \int_{-r}^{0} Y_1(t)'(\xi)d\xi + b_0 u(t) \\ &= a_0 Y_0(t) + Y_1(t)(0) - Y_1(t)(-r) + b_0 u(t) \\ &= a_0 Y_0(t) + \int_{-r}^{0} b_1(\xi)u(t+\xi)d\xi + b_0 u(t). \end{aligned}$$

Hence, $Y_0(t)$ solves (5) with initial data $(y_0, \delta(\cdot))$, so it must coincide with $y(t)$. ☐

We can use the above result to reformulate the optimization problem (6) in the space H. Let

$$H_+ := (0, +\infty) \times W^{1,2}_{r,0}.$$

Let $\eta \in H$, and define the (possibly empty) set

$$\begin{aligned} \mathcal{U}(\eta) &:= \{u \in L^2_{loc}([0,+\infty); U) \mid Y_0(t;\eta,u(\cdot)) > 0\ \forall\, t \geq 0\} \\ &= \{u \in L^2_{loc}([0,+\infty); U) \mid Y(t;\eta,u(\cdot)) \in H_+\ \forall\, t \geq 0\}. \end{aligned}$$

Given $u \in \mathcal{U}(\eta)$, define

$$J(\eta; u(\cdot)) = \int_0^{+\infty} e^{-\rho t}\Big(g(Y(t;\eta,u(\cdot))) - h_0(u(t))\Big)dt, \tag{25}$$

where

$$g : H_+ \longrightarrow \mathbb{R}, \quad g(\eta) := g_0(\eta_0). \tag{26}$$

Due to (19), if $\eta = M(y_0, \delta(\cdot))$, then

$$\mathcal{U}(\eta) = \mathcal{U}(y_0, \delta(\cdot))$$

and

$$J(\eta; u(\cdot)) = J_0(y_0, \delta(\cdot); u(\cdot)),$$

where J_0 is the objective functional defined in (6). Therefore, we have reduced the original problem (7) to

$$\max_{u \in \mathcal{U}(y_0,\delta(\cdot))} J(\eta; u(\cdot)), \qquad \eta = M(y_0, \delta(\cdot)) \in \mathcal{M}.$$

Although we are interested in solving the problem for initial data $\eta \in \mathcal{M}$, as these are the initial data coming from the real original problem, we enlarge the problem to data $\eta \in H$ and consider the functional (25) defined also for these data. So the problem we consider in the next sections is

$$\max_{u \in \mathcal{U}(\eta)} J(\eta; u(\cdot)), \quad \eta \in H. \tag{27}$$

5. The value function in the Hilbert space *H*. In this section we study some qualitative properties of the value function V associated to the optimization problem (27) in the space H. For $\eta \in H$ the value function of our problem is the function

$$V : H \longrightarrow \mathbb{R}, \quad V(\eta) := \sup_{u(\cdot) \in \mathcal{U}(\eta)} J(\eta, u(\cdot))$$

with the convention $\sup \emptyset = -\infty$. We notice that V is bounded from above due to Hypothesis 3.3. More precisely,

$$V(\eta) \le \int_0^{+\infty} e^{-\rho t} \bar{g}_0 dt = \frac{1}{\rho} \bar{g}_0 \quad \forall \eta \in \mathcal{D}(V).$$

The domain of the value function V is defined as

$$\mathcal{D}(V) := \{\eta \in H \mid V(\eta) > -\infty\}.$$

Of course,

$$\mathcal{D}(V) \subset \{\eta \in H \mid \mathcal{U}(\eta) \neq \emptyset\}.$$

Before proceeding, we introduce a weaker norm in H, which is the natural norm to deal with the unbounded linear term in the study of the HJB equation.

5.1. The norm $\|\cdot\|_{-1}$. We will deal with a norm weaker than the natural norm $\|\cdot\|$ To define this norm, we will need Hypothesis 3.5(i), which we assume to be holding true from now on.

Remark 5.1. Hypothesis 3.5(i) is taken to define the operator A^{-1} below whose definition requires $a_0 \neq 0$. We notice that this assumption is not very restrictive for the applications, as the coefficient a_0 in the model often represents some depreciation factor (so $a_0 < 0$) or some growth rate (so $a_0 > 0$). However, the case $a_0 = 0$ can be treated by translating the problem as follows. Take $a_0 = 0$. The state equation in infinite dimension is (11) with

$$A : (\phi_0, \phi_1(\cdot)) \mapsto \Big(\phi_1(0), -\phi_1'(\cdot)\Big).$$

We can rewrite it as

$$Y'(t) = \tilde{A} Y(t) - P_0 Y(t) + B u(t),$$

where

$$P_0 : H \to H, \quad P_0 \phi = (\phi_0, 0), \quad \tilde{A} = A + P_0.$$

Then everything we will do can be suitably replaced by dealing with this translated equation and with $\tilde{A}$ in place of A.

Due to Hypothesis 3.5(i), the inverse operator of A is well defined. It is a bounded linear operator $(H, \|\cdot\|) \to (D(A), \|\cdot\|)$ whose explicit expression is

$$A^{-1}\eta = \left(\frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{a_0}, -\int_{-r}^{\xi} \eta_1(s)ds \right). \tag{28}$$

We define in H the norm $\|\cdot\|_{-1}$ as

$$\|\eta\|_{-1} := \|A^{-1}\eta\|,$$

and so

$$\|\eta\|_{-1}^2 = \left| \frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{a_0} \right|^2 + \int_{-r}^0 |\eta_1(s)|^2 ds. \tag{29}$$

We consider the space

$$X := \mathbb{R} \times L_r^2.$$

This is a Hilbert space when endowed with the inner product

$$\langle \eta, \zeta \rangle_X := \eta_0 \zeta_0 + \langle \eta_1, \zeta_1 \rangle_{L_r^2},$$

where $\eta = (\eta_0, \eta_1(\cdot))$ is the generic element of X. The norm on this space associated to $\langle \cdot, \cdot \rangle_X$ is

$$\|\eta\|_X^2 = |\eta_0|^2 + \|\eta_1\|_{L_r^2}^2.$$

Lemma 5.2. *The norms $\|\cdot\|_{-1}$ and $\|\cdot\|_X$ are equivalent in H.*

Proof. See Appendix A.2. □

From Lemma 5.2 we get the following corollary.

Corollary 5.3. *There exists a constant $C_{a_0,r} > 0$ such that*

$$|\eta_0| \le C_{a_0,r} \|\eta\|_{-1} \quad \forall \eta \in H. \tag{30}$$

Remark 5.4. Corollary 5.3 represents a crucial issue and motivates our choice of working in the product space $\mathbb{R} \times W_{r,0}^{1,2}$ in place of the more usual product space $\mathbb{R} \times L_r^2$. Indeed, embedding the problem in $\mathbb{R} \times L_r^2$ and defining everything in the same way in this bigger space, we would not be able to have an estimate of type (30) controlling $|\eta_0|$ by $\|\eta\|_{-1}$. But this estimate is necessary to prove the continuity of the value function with respect to $\|\cdot\|_{-1}$, since in this way g is continuous in $(H_+, \|\cdot\|_{-1})$. On the other hand, the continuity of V with respect to $\|\cdot\|_{-1}$ is necessary to have a suitable property for the superdifferential of V (see Proposition 5.11), allowing us to handle the unbounded linear term in the HJB equation.

We show with an example that an estimate like (30) cannot be obtained if we set the infinite dimensional problem in the space $\mathbb{R} \times L_r^2$. In this case, the unbounded operator to use for the infinite dimensional representation (see, e.g., [49]) is still

$$A : \mathcal{D}(A) \to \mathbb{R} \times L_r^2, \quad (\eta_0, \eta_1(\cdot)) \mapsto (a\eta_0 + \eta_1(0), -\eta_1'(\cdot))$$

but defined on

$$\mathcal{D}(A) = \mathbb{R} \times W_{r,0}^{1,2} \subset \mathbb{R} \times L_r^2.$$

The bounded inverse operator $A^{-1} : \mathbb{R} \times L^2_r \to D(A)$ has the same explicit expression given by (28). So, dealing within the framework of the space $\mathbb{R} \times L^2_r$, it holds that

$$\|A^{-1}\eta\|^2_{\mathbb{R}\times L^2_r} = \left| \frac{\eta_0 + \int_{-r}^0 \eta_1(s)ds}{a_0} \right|^2 + \int_{-r}^0 \left| - \int_{-r}^{\xi} \eta_1(s) \right|^2 ds. \tag{31}$$

The latter norm does not control $|\eta_0|$. Indeed, consider in $\mathbb{R} \times L^2_r$ the sequence

$$\eta^n = (\eta_0^n, \eta_1^n(\cdot)), \quad \eta_0^n := 1, \ \eta_1^n(\cdot) = -n\mathbf{1}_{[-1/n,0]}(\cdot), \quad n \geq 1.$$

Supposing without loss of generality that $1/n < r$, by (31) we have

$$\|A^{-1}\eta^n\|^2_{\mathbb{R}\times L^2_r} = 0 + \int_{-\frac{1}{n}}^0 \left| \int_{-\frac{1}{n}}^{\xi} n ds \right|^2 d\xi = \int_{-\frac{1}{n}}^0 n^2 \left(\xi + \frac{1}{n}\right)^2 d\xi = \frac{1}{3n} \longrightarrow 0.$$

Therefore, we have $|\eta_0^n| = 1$ and $\|A^{-1}\eta^n\|_{\mathbb{R}\times L^2_r} \to 0$.

5.2. Concavity and $\|\cdot\|_{-1}$-continuity of the value function. We are going to prove that V is concave and continuous with respect to $\|\cdot\|_{-1}$. First, we introduce the spaces

$$H_+ := (0, +\infty) \times W^{1,2}_{r,0},$$
$$\mathcal{G} := \{\eta \in H_+ \mid 0 \in \mathcal{U}(\eta)\},$$
$$\mathcal{F} := \left\{ \eta \in H_+ \,\middle|\, \eta_0 + \int_{-\xi}^0 \eta_1(s) e^{a_0 s} ds > 0 \quad \forall \xi \in [-r, 0] \right\},$$
$$H_{++} := (0, +\infty) \times \{\eta_1 \in W^{1,2}_{r,0} \mid \eta_1(\cdot) \geq 0 \quad \text{a.e.}\}.$$

At the end, we will fully solve the problem for initial data in H_{++}, which are the most meaningful data from the economic point of view.

PROPOSITION 5.5.

(i) $H_{++} \subset \mathcal{F} = \mathcal{G} \subset \mathcal{D}(V) \subset H_+$ *and* $V(\eta) \geq 0$ *for every* $\eta \in \mathcal{F} = \mathcal{G}$.

(ii) $\mathcal{F}$ *is open with respect to* $\|\cdot\|_{-1}$.

Proof. See Appendix A.2. ∎

Remark 5.6. $\mathcal{F}$ is open also with respect to $\|\cdot\|$.

PROPOSITION 5.7. *The set* $\mathcal{D}(V)$ *is convex and the value function* V *is concave on* $\mathcal{D}(V)$.

Proof. See Appendix A.2. ∎

COROLLARY 5.8. V *is continuous with respect to* $\|\cdot\|_{-1}$ *in* $\mathcal{F}$.

Proof. The function V is concave, finite, and bounded from below in the $\|\cdot\|_{-1}$ open set $\mathcal{F}$. Therefore, the claim follows by a result of convex analysis (see, e.g., [24, Ch. 1, Cor. 2.4]). ∎

5.3. Monotonicity of the value function. The following monotonicity result holds true.

PROPOSITION 5.9. *The value function* V *is nondecreasing with respect to the partial order relation* (3). *Moreover, for all* $\eta \in \mathcal{D}(V)$ *and* $h > 0$ *(in the sense of* (4)*), we have*

$$\lim_{s\to+\infty} V(\eta + sh) = \frac{1}{\rho}\bar{g}_0. \tag{32}$$

Proof. See Appendix A.2. ∎

PROPOSITION 5.10. *Let Hypothesis* 3.6(i) *hold. We have the following:*

(i) $V(\eta) < \frac{1}{\rho}\bar{g}_0$ *for every* $\eta \in \mathcal{D}(V)$.

(ii) *For every* $\eta \in \mathcal{D}(V)$ *and* $h \in H$ *with* $h > 0$ *in the sense of* (4), *the function*

$$[0, +\infty) \to \mathbb{R}, \ s \mapsto V(\eta + sh) \tag{33}$$

is strictly increasing.

Proof. See Appendix A.2. ∎

5.4. Superdifferential of concave $\|\cdot\|_{-1}$-continuous function. Motivated by Proposition 5.7 and Corollary 5.8, in this subsection we focus on the properties of the superdifferential of concave and $\|\cdot\|_{-1}$-continuous functions. This will be useful in proving the regularity result in the next section. We recall first some definitions and basic results from nonsmooth analysis concerning the generalized differentials. For details we refer the reader to [44].

Let v be a concave continuous function defined and finite on some open convex subset $\mathcal{A}$ of H. Given $\eta \in \mathcal{A}$, the superdifferential of v at η is the set

$$D^+v(\eta) = \{p \in H \mid v(\zeta) - v(\eta) \leq \langle \zeta - \eta, p\rangle \ \ \forall \zeta \in \mathcal{A}\}.$$

The set of the *reachable gradients* at $\eta \in \mathcal{A}$ is defined as

$$D^*v(\eta) := \{p \in H \mid \exists (\eta_n) \subset \mathcal{A}, \ \eta_n \to \eta, \text{ such that } \exists \nabla v(\eta_n) \text{ and } \nabla v(\eta_n) \rightharpoonup p\}.$$

As we know (see [44, Ch. 1, Prop. 1.11]), $D^+v(\eta)$ is a closed, convex, not empty subset of H. Moreover, the set-valued map $\mathcal{A} \to \mathcal{P}(H)$, $\eta \mapsto D^+v(\eta)$ is locally bounded (see [44, Ch. 1, Prop. 1.11]). Also we have the representation (see [15, Cor. 4.7])

$$D^+v(\eta) = \overline{co}(D^*v(\eta)), \quad \eta \in \mathcal{A}. \tag{34}$$

Given $p, h \in H$, with $\|h\| = 1$, we denote

$$p_h := \langle p, h\rangle.$$

We introduce the directional superdifferential of v at η along the direction h

$$D_h^+v(\eta) := \{\alpha \in \mathbb{R} \mid v(\eta + \gamma h) - v(\eta) \leq \gamma\alpha \ \forall \gamma \in \mathbb{R}\}.$$

This set is a nonempty, closed, and bounded interval $[a, c] \subset \mathbb{R}$. More precisely, since $v(\eta)$ is concave, we have

$$a = v_h^+(\eta), \quad c = v_h^-(\eta),$$

where $v_h^+(\eta)$ and $v_h^-(\eta)$ denote, respectively, the right and left derivatives of the real function $s \mapsto v(\eta + sh)$ at $s = 0$. By definition of $D^+v(\eta)$, the projection of $D^+v(\eta)$ onto h must be contained in $D_h^+v(\eta)$, that is,

$$D_h^+v(\eta) \supset \left\{p_h \mid p \in D^+v(\eta)\right\}. \tag{35}$$

On the other hand, Proposition 2.24 in [44, Ch. 1] states that

$$a = \inf\{\langle p, h\rangle \mid p \in D^+v(\eta)\} \quad c = \sup\{\langle q, h\rangle, \mid q \in D^+v(\eta)\},$$

and the sup and inf above are attained. This means that there exist $p, q \in D^+v(\eta)$ such that

$$a = \langle p, h\rangle, \quad c = \langle q, h\rangle.$$

Since $D^+v(\eta)$ is convex, we see that also the converse inclusion of (35) is true. Therefore,

$$D_h^+ v(\eta) = \left\{ p_h \mid p \in D^+ v(\eta) \right\}. \tag{36}$$

PROPOSITION 5.11. *Let $v : \mathcal{F} \longrightarrow \mathbb{R}$ be a concave function continuous with respect to $\|\cdot\|_{-1}$, and let $\eta \in \mathcal{F}$, $p \in D^* v(\eta)$. Then*

(i) *$p \in \mathcal{D}(A^*)$;*

(ii) *there exists a sequence $\eta_n \to \eta$ such that for each $n \in \mathbb{N}$ there exists $\nabla v(\eta_n) \in \mathcal{D}(A^*)$; and*

(iii) *$\nabla v(\eta_n) \rightharpoonup p$ and $A^* \nabla v(\eta_n) \rightharpoonup A^* p$.*

Proof. See [30, Prop. 3.12(4)] and [31, Rem. 2.11]. ∎

6. Dynamic programming and the HJB equation. We are ready to approach the problem by dynamic programming. From now on, just for convenience, we assume without loss of generality that $\|b\| = 1$.

THEOREM 6.1 (dynamic programming principle). *For any $\eta \in \mathcal{D}(V)$ and for any $\tau \geq 0$,*

$$V(\eta) = \sup_{u(\cdot) \in \mathcal{U}(\eta)} \left[\int_0^\tau e^{-\rho t} \left(g\left(Y(t;\eta,u(\cdot))\right) - h_0(u(t)) \right) dt + e^{-\rho\tau} V\left(Y(\tau;\eta,u(\cdot))\right) \right].$$

Proof. See, e.g., [42, Th. 1.1, Ch. 6]. ∎

The differential version of the dynamic programming principle is the HJB equation. We consider this equation in the set $\mathcal{F}$, where it reads as

$$\rho v(\eta) = \langle A\eta, \nabla v(\eta)\rangle + g(\eta) + \sup_{u \in U} \left\{ \langle Bu, \nabla v(\eta)\rangle - h_0(u) \right\}, \qquad \eta \in \mathcal{F}. \tag{37}$$

Define the Legendre transform of h_0 as

$$\mathcal{H}(p_0) := \sup_{u \in U} \{ u p_0 - h_0(u) \}. \tag{38}$$

Since

$$\sup_{u \in U} \left\{ \langle Bu, p\rangle - h_0(u) \right\} = \sup_{u \in U} \left\{ \langle u, B^* p\rangle - h_0(u) \right\},$$

taking into account that $B^* p = \langle b, p\rangle$, (37) can be rewritten as

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta)\rangle + g(\eta) + \mathcal{H}(\langle \nabla v(\eta), b\rangle), \qquad \eta \in \mathcal{F}. \tag{39}$$

We note that the nonlinear term in (39) can be defined without requiring the full regularity of v but only the C^1-smoothness of v with respect to the direction b. Indeed, denoting coherently with (36) by v_b the directional derivative of v with respect to b, we can intend the nonlinear term in (39) as $\mathcal{H}(v_b(\eta))$. So we can write (39) as

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta)\rangle + g(\eta) + \mathcal{H}(v_b(\eta)), \qquad \eta \in \mathcal{F}. \tag{40}$$

6.1. The HJB equation: Viscosity solutions. In this subsection we prove that the value function V is a viscosity solution of the HJB equation (40). To this end, we need to define a suitable set of regular test functions. This is the set

$$\mathcal{T} := \Big\{ \varphi \in C^1(H) \mid \nabla\varphi(\cdot) \in \mathcal{D}(A^*),\ A^*\nabla\varphi : H \to H \text{ is continuous} \Big\}.$$

Let us define, for $u \in U$, the differential operator $\mathcal{L}^u$ on $\mathcal{T}$ by

$$[\mathcal{L}^u\varphi](\eta) := -\rho\varphi(\eta) + \langle \eta, A^*\nabla\varphi(\eta)\rangle + u\langle \nabla\varphi(\eta), b\rangle. \tag{41}$$

The proof of the following chain rule can be found in [42, Ch. 2, Prop. 5.5].

LEMMA 6.2. *Let $\eta \in H$, $\varphi \in \mathcal{T}$, and $u \in L^2_{loc}([0,+\infty);\mathbb{R})$, and set for $t \geq 0$ $Y(t) := Y(t;\eta,u(\cdot))$. Then the following chain rule holds:*

$$e^{-\rho t}\varphi(Y(t)) - \varphi(\eta) = \int_0^t e^{-\rho s}[\mathcal{L}^{u(s)}\varphi](Y(s))ds \quad \forall\, t \geq 0.$$

DEFINITION 6.3.

(i) *A continuous function $v : \mathcal{F} \to \mathbb{R}$ is called a viscosity subsolution of* (40) *if, for each couple $(\eta_M, \varphi) \in \mathcal{F} \times \mathcal{T}$ such that $v - \varphi$ has a local maximum at η_M, we have*

$$\rho v(\eta_M) \leq \langle \eta_M, A^*\nabla\varphi(\eta_M)\rangle + g(\eta_M) + \mathcal{H}(\varphi_b(\eta_M)).$$

(ii) *A continuous function $v : \mathcal{F} \to \mathbb{R}$ is called a viscosity supersolution of* (40) *if, for each couple $(\eta_m, \varphi) \in \mathcal{F} \times \mathcal{T}$ such that $v - \varphi$ has a local minimum at η_m, we have*

$$\rho v(\eta_m) \geq \langle \eta_m, A^*\nabla\varphi(\eta_m)\rangle + g(\eta_m) + \mathcal{H}(\varphi_b(\eta_m)).$$

(iii) *A continuous function $v : \mathcal{F} \to \mathbb{R}$ is called a viscosity solution of* (40) *if it is both a viscosity subsolution and supersolution of* (40).

LEMMA 6.4. *Let Hypothesis* 3.7 *hold. Then, for every $\eta \in \mathcal{F}$, $\varepsilon > 0$, there exists M_ε such that*

$$\int_0^{+\infty} e^{-\rho t}|u(t)|^{1+\alpha}dt \leq M_\varepsilon \quad \forall\, u^\varepsilon(\cdot) \in \mathcal{U}(\eta) \ \varepsilon\text{-optimal for } \eta.$$

Proof. See Appendix A.2. □

THEOREM 6.5. *Let Hypothesis* 3.7 *hold. Then V is a viscosity solution of* (40).

Proof. Subsolution property. Let $(\eta_M, \varphi) \in \mathcal{F} \times \mathcal{T}$ be such that $V - \varphi$ has a local maximum at η_M. Without loss of generality we can suppose $V(\eta_M) = \varphi(\eta_M)$. Let us suppose, by contradiction, that there exists $\nu > 0$ such that

$$2\nu \leq \rho V(\eta_M) - (\langle \eta_M, A^*\nabla\varphi(\eta_M)\rangle + g(\eta_M) + \mathcal{H}(\varphi_b(\eta_M))).$$

Let us define the function

$$\tilde{\varphi}(\eta) := V(\eta_M) + \langle \nabla\varphi(\eta_M), \eta - \eta_M\rangle, \quad \eta \in H. \tag{42}$$

We have

$$\nabla\tilde{\varphi}(\eta) = \nabla\varphi(\eta_M) \quad \forall\, \eta \in H.$$

Thus $\tilde{\varphi} \in \mathcal{T}$ and we have as well

$$2\nu \leq \rho V(\eta_M) - (\langle \eta_M, A^*\nabla\tilde{\varphi}(\eta_M)\rangle + g(\eta_M) + \mathcal{H}(\tilde{\varphi}_b(\eta_M))).$$

Now, we know that V is a concave function and that $V - \varphi$ has a local maximum at η_M, so that

$$V(\eta) \leq V(\eta_M) + \langle \nabla\varphi(\eta_M), \eta - \eta_M\rangle. \tag{43}$$

Thus, by (42) and (43)

$$\tilde{\varphi}(\eta_M) = V(\eta_M), \quad \tilde{\varphi}(\eta) \geq V(\eta) \quad \forall\, \eta \in \mathcal{F}. \tag{44}$$

Let $\mathcal{B}_\varepsilon := \mathcal{B}_{(H,\|\cdot\|)}(\eta_M, \varepsilon)$ be the ball of radius $\varepsilon > 0$ centered at η_M. Due to the properties of the functions belonging to $\mathcal{T}$, we can find $\varepsilon > 0$ such that

$$\nu \leq \rho V(\eta) - \left(\langle \eta, A^* \nabla \tilde{\varphi}(\eta)\rangle + g(\eta) + \mathcal{H}\left(\tilde{\varphi}_b(\eta_M)\right)\right) \quad \forall \eta \in \mathcal{B}_\varepsilon.$$

Take a sequence $\delta_n > 0$ such that $\delta_n \to 0$. For each $n \in \mathbb{N}$, take a δ_n-optimal control $u_n \in \mathcal{U}(\eta)$, and set $Y^n(\cdot) := Y(\cdot; \eta_M, u_n(\cdot))$. Define

$$t_n := \inf\left\{t \geq 0 \mid \|Y^n(t) - \eta_M\| = \varepsilon\right\} \wedge 1,$$

with the agreement that $\inf \emptyset = +\infty$. Of course, t_n is well defined and belongs to $(0,1]$. Moreover, by continuity of $t \mapsto Y^n(t)$, we have $Y^n(t) \in \mathcal{B}_\varepsilon$ for $t \in [0, t_n)$. By definition of δ_n-optimal control, we have as a consequence of Theorem 6.1 that

$$\delta_n \geq -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - h_0(u_n(t))\right] dt - \left(e^{-\rho t_n} V(Y(t_n)) - V(\eta_M)\right). \tag{45}$$

Therefore, by (44) and (45),

$$\begin{aligned}
\delta_n &\geq -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - h_0(u_n(t))\right] dt - \left(e^{-\rho t_n}\left(\tilde{\varphi}(Y^n(t_n))\right) - \tilde{\varphi}(\eta_M)\right)\\
&= -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - h_0(u_n(t)) + [\mathcal{L}^{u_n(t)}\tilde{\varphi}](Y^n(t))\right] dt\\
&\geq -\int_0^{t_n} e^{-\rho t}\left[g(Y^n(t)) - \rho\tilde{\varphi}(Y^n(t)) + \langle A^*\nabla\tilde{\varphi}(Y^n(t)), Y^n(t)\rangle + \mathcal{H}(\tilde{\varphi}_b(Y^n(t)))\right] dt\\
&\geq t_n \nu.
\end{aligned}$$

Therefore, since $\delta_n \to 0$ we also have $t_n \to 0$. We claim that $t_n \to 0$ implies

$$\|Y^n(t_n) - \eta_M\| \longrightarrow 0. \tag{46}$$

This would be a contradiction of the definition of t_n, concluding the proof. Let us prove (46). Using the definition of mild solution (4.1) of $Y^n(t_n)$, we have

$$\begin{aligned}
\|Y^n(t_n) - \eta_M\| &= \left\| S_A(t_n)\eta_M + \int_0^{t_n} S_A(t_n - \tau) B u_n(\tau) d\tau \; - \; \eta_M \right\|\\
&\leq \left\|\left(S_A(t_n) - I\right)\eta_M\right\| + \left\|\int_0^{t_n} S_A(t_n - \tau) B u_n(\tau) d\tau\right\|\\
&\leq \left\|\left(S_A(t_n) - I\right)\eta_M\right\| + \int_0^{t_n} \|S_A(t_n - \tau)\|_{\mathcal{L}(H)} \|B\| |u_n(\tau)| d\tau.
\end{aligned}$$

Since S_A is strongly continuous and considering (75), in order to prove that the right-hand side of above inequality converges to 0, it suffices to prove that

$$\int_0^{t_n} |u_n(s)| ds \to 0. \tag{47}$$

We have to distinguish the two cases. If (i) of Hypothesis 3.7 holds true, since $t_n \to 0$ we have directly (47). If (ii) of Hypothesis 3.7 holds true, set $\beta := 1 + \alpha > 1$. By Hölder's inequality,

$$\int_0^{t_n} |u_n(s)| ds \le \left(\int_0^{t_n} |u_n(\tau)|^\beta d\tau \right)^{\frac{1}{\beta}} t_n^{\frac{\beta-1}{\beta}}.$$

Since by Lemma 6.4 we know that $(\int_0^{t_n} |u_n(\tau)|^\beta d\tau)^{\frac{1}{\beta}}$ is bounded and since $t_n \to 0$, we have again (47). So the proof of this part is complete.

Supersolution property. The proof that V is a viscosity supersolution is more standard, and we refer the reader to [42, Ch. 6, Th. 3.2] for this part. ∎

6.2. Smoothness of viscosity solutions. In this subsection we are going to show our first main result, that is, the proof of a C^1 directional regularity result for viscosity solutions to the HJB equation (37). We start by observing that, if Hypothesis 3.6(iii) holds, it is easily checked that

$$\begin{cases} \mathcal{H}(p_0) = 0 & \text{if } p_0 \le 0, \\ \mathcal{H}(p_0) > 0 & \text{if } p_0 > 0. \end{cases}$$

PROPOSITION 6.6. *Let Hypothesis* 3.6(iii) *hold true. Then the function* $\mathcal{H}$ *is finite and strictly convex in* $(0, +\infty)$.

Proof. Let $\tilde{U} := [-\bar{u}, \bar{u}]$. If $\bar{u} = +\infty$, the set $\tilde{U}$ is intended as $\mathbb{R}$. Let

$$\tilde{h}_0(u) := \begin{cases} h_0(u) & \text{if } u \in [0, \bar{u}], \\ h_0(-u) & \text{if } u \in [-\bar{u}, 0]. \end{cases}$$

The Legendre transform of $\tilde{h}_0$ is

$$\tilde{\mathcal{H}}(p_0) := \sup_{u \in \tilde{U}} \{ up - \tilde{h}_0(u) \}.$$

Due to [45, Cor. 26.4.1], the function $\tilde{\mathcal{H}}$ is finite and strictly convex in $\mathbb{R}$. In order to get the claim, we only need to prove that for $p_0 > 0$ we have $\tilde{\mathcal{H}}(p_0) = \mathcal{H}(p_0)$. Indeed, if $p_0 > 0$, then

$$\tilde{\mathcal{H}}(p_0) = \sup_{u \in \tilde{U}} \{ up_0 - \tilde{h}_0(u) \} = \sup_{u \in U} \{ up_0 - \tilde{h}_0(u) \} = \sup_{u \in U} \{ up_0 - h_0(u) \} = \mathcal{H}(p_0),$$

where the second equality follows from Hypothesis 3.6(iii). ∎

LEMMA 6.7. *Let* $v : \mathcal{F} \longrightarrow \mathbb{R}$ *be a concave* $\|\cdot\|_{-1}$*-continuous function and suppose that* $\eta \in \mathcal{F}$ *is a differentiability point for* v *and that* $\nabla v(\eta) = \xi$. *Then the following hold:*

1. *There exists a function* $\varphi \in \mathcal{T}$ *such that* $v - \varphi$ *has a local maximum at* η *and* $\nabla \varphi(\eta) = \xi$.
2. *There exists a function* $\varphi \in \mathcal{T}$ *such that* $v - \varphi$ *has a local minimum at* η *and* $\nabla \varphi(\eta) = \xi$.

Proof. See [30, Lemma 4.5]. ∎

THEOREM 6.8. *Let Hypothesis* 3.6(iii) *hold. Let* v *be a concave* $\|\cdot\|_{-1}$*-continuous viscosity solution of* (40) *on* $\mathcal{F}$ *strictly increasing along the direction* b. *Then* v *is differentiable along* b *at each* $\eta \in \mathcal{F}$ *and* $v_b(\eta) \in (0, +\infty)$. *Moreover, the function*

$$(\mathcal{F}, \|\cdot\|) \longrightarrow \mathbb{R}, \quad \eta \longmapsto v_b(\eta)$$

is continuous.

Proof. Let $\eta \in \mathcal{F}$ and $p, q \in D^*v(\eta)$. Due to Proposition 5.11, there exist sequences (η_n), $(\tilde{\eta}_n) \subset \mathcal{F}$ such that

1. $\eta_n \to \eta$, $\tilde{\eta}_n \to \eta$;
2. $\nabla v(\eta_n)$ and $\nabla v(\tilde{\eta}_n)$ exist for all $n \in \mathbb{N}$ and $\nabla v(\eta_n) \rightharpoonup p$, $\nabla v(\tilde{\eta}_n) \rightharpoonup q$; and
3. $A^*\nabla v(\eta_n) \rightharpoonup A^*p$ and $A^*\nabla v(\tilde{\eta}_n) \rightharpoonup A^*q$.

Recall that, given $\eta \in H$, we have defined

$$\eta_b := \langle \eta, b \rangle.$$

Due to Lemma 6.7 and Theorem 6.5 we can write, for each $n \in \mathbb{N}$,

$$\rho v(\eta_n) = \langle \eta_n, A^*\nabla v(\eta_n) \rangle + g(\eta_n) + \mathcal{H}(v_b(\eta_n)),$$

$$\rho v(\tilde{\eta}_n) = \langle \eta_n, A^*\nabla v(\tilde{\eta}_n) \rangle + g(\tilde{\eta}_n) + \mathcal{H}(v_b(\tilde{\eta}_n)).$$

So, letting $n \to +\infty$, we get

$$\langle \eta, A^*p \rangle + g(\eta) + \mathcal{H}(p_b) = \rho v(\eta) = \langle \eta, A^*q \rangle + g(\eta) + \mathcal{H}(q_b). \tag{48}$$

On the other hand, $\lambda p + (1-\lambda)q \in D^+v(\eta)$ for any $\lambda \in (0,1)$, so that we have the subsolution inequality

$$\rho v(\eta) \leq \langle \eta, A^*[\lambda p + (1-\lambda)q] \rangle + g(\eta) + \mathcal{H}(\lambda p_b + (1-\lambda)q_b)\ \forall\, \lambda \in (0,1). \tag{49}$$

Combining (48) and (49), we get

$$\mathcal{H}(\lambda p_b + (1-\lambda)q_b) \geq \lambda \mathcal{H}(p_b) + (1-\lambda)\mathcal{H}(q_b). \tag{50}$$

Notice that, since $p, q \in D^*v(\eta)$, we have also $p, q \in D^+v(\eta)$. On the other hand, since v is concave and strictly increasing along b, we must have $p_b, q_b \in (0, +\infty)$. Therefore, taking into account that $\mathcal{H}$ is strictly convex on $(0,+\infty)$ (Proposition 6.6), (50) yields $p_b = q_b$. So, we have shown that the projection of $D^*v(\eta)$ onto b is a singleton. Due to (34), this implies that also the projection of $D^+v(\eta)$ onto b is a singleton. Due to (36), we have that $D_b^+v(\eta)$ is a singleton, too. Since v is concave, this is enough to conclude that it is differentiable along the direction b at η and that $v_b(\eta) \in (0,+\infty)$.

Now we prove the second claim. The topological notions are intended in the norm $\|\cdot\|$. Take $\eta \in \mathcal{F}$ and a sequence $(\eta^n) \subset \mathcal{F}$ such that $\eta^n \to \eta$. Since v is concave, by (36) there exists for each $n \in \mathbb{N}$ an element $p_n \in D^+v(\eta_n)$ such that $\langle p_n, b \rangle = v_b(\eta_n)$.

The superdifferential set-valued map $\mathcal{F} \to H$, $\eta \mapsto D^+v(\eta)$ is locally bounded (see [44, Ch. 1, Prop. 1.11]). Therefore, from each subsequence (p_{n_k}) we can extract a subsubsequence $(p_{n_{k_h}})$ such that $p_{n_{k_h}} \rightharpoonup p$ for some limit point p. Due to the concavity of v, the set-valued map $\eta \mapsto D^+v(\eta)$ is norm-to-weak upper semicontinuous (see [44, Ch. 1, Prop. 2.5]), so $p \in D^+v(\eta)$. Since $v_b(\eta)$ exists, by (36) we get

$$\langle p, b \rangle = v_b(\eta).$$

With this argument we have shown that, from each subsequence $(v_b(\eta^{n_k}))$, we can extract a subsubsequence $(v_b(\eta^{n_{k_h}}))$ such that

$$v_b(\eta^{n_{k_h}}) = \langle p_{n_{k_h}}, b \rangle \to \langle p, b \rangle = v_b(\eta).$$

The claim follows by the usual argument on subsequences. ∎

Corollary 6.9. *Let Hypotheses* 3.5, 3.6(i), (iii), and 3.7 *hold. Then V is differentiable along b at any point $\eta \in \mathcal{F}$ and $V_b(\eta) \in (0, +\infty)$. Moreover, the function $(\mathcal{F}, \|\cdot\|) \to \mathbb{R}$, $\eta \mapsto V_b(\eta)$ is continuous.*

Proof. Due to Proposition 5.7, the function V is concave in $\mathcal{F}$, and due to Corollary 5.8, it is continuous therein. Moreover, since $b > 0$, due to Proposition 5.10, it is strictly increasing along b. Finally, by Theorem 6.5 it is a viscosity solution of the HJB equation (40). Therefore, Theorem 6.8 applies to V, and we have the claim. ∎

Remark 6.10. Notice that in the assumption of Theorem 6.8 we do not require that v be the value function but only that it be a concave $\|\cdot\|_{-1}$-continuous viscosity solution of (40) strictly increasing along the direction b.

We also notice that the claim of continuity of $\eta \mapsto V_b(\eta)$ can be made stronger: indeed, one has that this map is continuous with respect to $\|\cdot\|_{-1}$. However, we do not need this stronger continuity property.

7. Verification theorem and optimal feedback. In this section we assume that all the assumptions of Corollary 6.9 (Hypotheses 3.5, 3.6(i), (iii), and 3.7) hold, and we do not repeat them. Moreover, we also assume that Hypothesis 3.6(ii) holds true, and we do not repeat it. In view of Corollary 6.9, we can define a feedback map in classical form. Indeed, we can define the map

$$\mathcal{P}(\eta) := \operatorname{argmax}_{u \in U} \{uV_b(\eta) - h_0(u)\}, \quad \eta \in \mathcal{F}. \tag{51}$$

Existence and uniqueness of the argmax follow from (38) and Hypothesis 3.6(ii), (iii). Continuity in $\mathcal{F}$ of $\mathcal{P}$ follows from Corollary 6.9. With this map at hand, we can study the associated closed loop equation and prove a verification theorem stating the existence of optimal feedback controls, as is done in [31]. Unlike [31], we approach this equation directly in infinite dimension, where it reads as

$$\begin{cases} Y'(t) = AY(t) + B\mathcal{P}(Y(t)), \\ Y(0) = \eta. \end{cases} \tag{52}$$

By mild solution to (52) we intend a continuous function $Y^* : [0, +\infty) \to \mathcal{F}$ such that

$$Y^*(t) = S_A(t)\eta + \int_0^t S_A(t-\tau)B\mathcal{P}(Y^*(\tau))d\tau, \quad t \geq 0. \tag{53}$$

We notice also that the local existence of such a solution is not immediate, as the map $B\mathcal{P}$ is known to be just continuous (not Lipschitz continuous) in $\mathcal{F} \subset H$, and Peano's theorem fails in infinite dimension in general (see [33]). However, the map $B\mathcal{P}$ has finite dimensional range, and this allows us to get the existence.

7.1. Verification theorem. Assuming the existence of a mild solution to (52) (it will be proved in subsection 7.2), we prove a verification theorem yielding optimal synthesis for the control problem. We start by giving the definition of optimal control.

Definition 7.1. *Let $\eta \in \mathcal{D}(V)$. A control $u^* \in \mathcal{U}(\eta)$ is said to be optimal for the initial state η if $J(\eta; u^*(\cdot)) = V(\eta)$.*

Theorem 7.2 (verification theorem). *Let $\eta \in \mathcal{F}$, and let $Y^*(\cdot)$ be a mild solution of* (52). *Define the feedback control*

$$u^*(t) := \mathcal{P}(Y^*(t)). \tag{54}$$

Then $u^* \in \mathcal{U}(\eta)$ *and is an optimal control for the initial state* η.

Proof. Admissibility. Let us consider the mild solution $Y(t;\eta,u^*(\cdot))$ to (11) starting at η and with control $u(\cdot) = u^*(\cdot)$. By (53) also Y^* is a mild solution to the same equation, and so we get by uniqueness of mild solutions

$$Y^*(t) = Y(t;\eta,u^*(\cdot)). \tag{55}$$

Now notice that $Y^*(t) \in \mathcal{F}$ for each $t \geq 0$, since, as solution to (53), it must lie in the domain of $\mathcal{P}$, which is indeed $\mathcal{F}$. So, from the equality (55) we deduce that also $Y(t;\eta,u^*(\cdot)) \in \mathcal{F}$ for each $t \geq 0$. By definition of $\mathcal{F}$, this implies that $Y_0(t;\eta,u^*(\cdot)) > 0$ for each $t \geq 0$, so we conclude that $u^* \in \mathcal{U}(\eta)$.

Optimality. Arguing as in [31, Th. 3.2] (we omit the proof for brevity), we get

$$J(\eta;u^*(\cdot)) \geq V(\eta). \tag{56}$$

Then the optimality of $u^*(\cdot)$ follows. ∎

7.2. Closed loop equation: Local existence and uniqueness. In order to apply Theorem 7.2 and construct optimal feedback controls, we need to prove the existence of mild solutions to the closed loop equation (52). To this end, we note that there are available in the literature some results on Peano's theorems in infinite dimensional spaces (see [1, 3, 19, 21, 38, 47, 52]). We could appeal to such results and their proofs. However, our case is slightly different since we need to work with mild solutions (due to the presence of the unbounded operator A). So, for the sake of completeness, we provide the proof here.

PROPOSITION 7.3. *For each* $\eta \in \mathcal{F}$, *the closed loop equation* (52) *admits a local mild solution; i.e., there exist* $\tau > 0$ *and a continuous function* $Y^* : [0,\tau) \to \mathcal{F}$ *such that* (53) *holds for every* $t \in [0,\tau)$.

Proof. Let $\eta \in \mathcal{F}$, $\varepsilon > 0$, and $M := \sup_{\xi \in \mathcal{B}(\eta,\varepsilon)} |\mathcal{P}(\xi)|$, where $\mathcal{B}(\eta,\varepsilon)$ is the ball of radius $\varepsilon > 0$ centered at η. We take ε small enough so that $M < +\infty$ (this is possible since $\mathcal{P}$ is continuous, so locally bounded). Let

$$N_\alpha := \{Y \in C([0,\alpha],H) : \|Y(t) - \eta\| \leq \varepsilon\},$$

where α has to be determined, and G is the operator defined as

$$\begin{aligned} G: \quad & N_\alpha \longrightarrow C([0,\alpha];H), \\ & Y(\cdot) \longmapsto S_A(\cdot)\eta + \int_0^{\cdot} S_A(\cdot - s)B\mathcal{P}(Y(s))ds. \end{aligned}$$

We have for all $t \in [0,\alpha]$

$$\begin{aligned} \|(GY)(t) - \eta\| &\leq \|(GY)(t) - S_A(t)\eta\| + \|S_A(t)\eta - \eta\| \\ &\leq M\int_0^\alpha \|S_A(t-s)\|ds + \|S_A(t)\eta - \eta\| \\ &\leq C_0 M\alpha e^{\omega\alpha} + \|S_A(t)\eta - \eta\|. \end{aligned}$$

Due to the strong continuity of the semigroup S_A, we see form the estimate above that if α is small enough, then the operator G maps N_α into itself. We fix, from now on, such an α. Let $\{Y^n(\cdot)\}_{n\geq 0}$ be the sequence of functions from $[0,\alpha]$ to H defined recursively as

$$Y^0(\cdot) \equiv \eta, \quad Y^{n+1} = GY^n, \quad n \in \mathbb{N}. \tag{57}$$

Since G maps N_α into itself, we have

$$\{Y^n\}_{n\geq 0} \subset N_\alpha. \tag{58}$$

Then, due to (58) and since $\mathcal{P}$ is bounded in $\mathcal{B}(\eta,\varepsilon)$, we see that

(1) there exist a dense subset $E \subset [0,\alpha]$ and $f : E \to \mathbb{R}$, such that, extracting a subsequence if necessary, we have the convergence

$$\mathcal{P}(Y^n(s)) \overset{n\to\infty}{\longrightarrow} f(s) \quad \forall\, s \in E; \textit{ and} \tag{59}$$

(2) by definition of G, the family $\{Y^n(\cdot)\}_{n\geq 0}$ is a family of equiuniformly continuous functions.

By these two facts, arguing as in the usual proof of the Ascoli–Arzelà theorem (see, e.g., [51, Ch. III, pag. 85]), we can prove the existence of a function $\bar{f} \in C([0,\alpha];\mathbb{R})$ extending f to the whole interval $[0,\alpha]$ such that

$$\mathcal{P}(Y^n(\cdot)) \longrightarrow \bar{f}(\cdot) \quad \text{uniformly in } [0,\alpha].$$

Therefore,

$$Y^n(\cdot) \longrightarrow \bar{Y}(\cdot) := S_A(t)\eta + \int_0^t S_A(t-s)B\bar{f}(s)ds \quad \text{uniformly in } [0,\alpha].$$

By construction, $\bar{Y}(\cdot)$ is the solution we were looking for. Indeed, on one hand,

$$GY^n(t) = Y^{n+1}(t) \longrightarrow \bar{Y}(t) \quad \forall\, t \in [0,\alpha],$$

and, on the other hand,

$$GY^n(t) = S_A(t)\eta + \int_0^t S_A(t-s)B\mathcal{P}(Y^n(s))ds \longrightarrow S_A(t)\eta + \int_0^t S_A(t-s)B\mathcal{P}(\bar{Y}(s))ds.$$

Hence

$$\bar{Y}(t) = S_A(t)\eta + \int_0^t S_A(t-s)B\mathcal{P}(\bar{Y}(s))ds,$$

the claim. ∎

Proposition 7.3 provides a way to locally construct controls in the following sense. Let $\tau \geq 0$, $\eta \in \mathcal{F}$, and let us define the convex set $\mathcal{U}_\tau(\eta)$ as the set of restrictions of the functions of $\mathcal{U}(\eta)$ to the interval $[0,\tau)$, i.e.,

$$\mathcal{U}_\tau(\eta) := \{u_\tau(\cdot) = u(\cdot)|_{[0,\tau)} \mid u \in \mathcal{U}(\eta)\}, \tag{60}$$

and let us consider the following functional on $\mathcal{U}_\tau(\eta)$:

$$J_\tau(\eta; u_\tau(\cdot)) := \int_0^\tau e^{-\rho t}\big(g(Y(t;\eta,u_\tau(\cdot))) - h_0(u_\tau(t))\big)dt + e^{-\rho\tau}V(Y(\tau;\eta,u_\tau(\cdot))). \tag{61}$$

Definition 7.4. *We say that a control in $\mathcal{U}_\tau(\eta)$ is a τ-locally optimal control for η if it maximizes $J_\tau(\eta;\cdot)$ over $\mathcal{U}_\tau(\eta)$.*

Let Y^* be a mild solution in the interval $[0,\tau)$ starting at η in $\mathcal{F}$, and consider the control $u^*(\cdot) = \mathcal{P}(Y^*(\cdot))$. Then, arguing as in the proof of the verification theorem, Thorem 7.2, one sees that $u^*(\cdot)$ is τ-local optimal for η. This establishes a connection between the existence of a local mild solution to the closed loop equation (52) and the existence of a locally optimal control. On the other hand, we can address the question of the connection between the uniqueness of mild solutions to the closed loop equation (52) and the uniqueness of locally optimal controls. In this case we follow the inverse path; i.e., we first prove the uniqueness of locally optimal controls and then derive from it the uniqueness of mild solutions to the closed loop equation (52).

PROPOSITION 7.5. *Let $\eta \in \mathcal{F}$ and $\tau > 0$. The functional $J_\tau(\eta; \cdot)$ is strictly concave over $\mathcal{U}_\tau(\eta)$ and, consequently, there exists at most one τ-locally optimal control for η.*

Proof. The claim follows from the concavity of g (Hypothesis 3.3(i) and (26)), the concavity of V (Proposition 5.7), the hypothesis of strict convexity of h_0 (Hypothesis 3.6(ii)), and the linearity of the state equation. ∎

PROPOSITION 7.6. *Let $\eta \in \mathcal{F}$. If Y_1, Y_2 are two mild solutions of* (52) *on some interval* $[0, \tau)$*, then $Y_1 \equiv Y_2$ in* $[0, \tau)$*.*

Proof. Let $\tau > 0$, and let us suppose that Y_1, Y_2 are two local mild solutions of (52), defined on $[0, \tau)$. Due to the argument above, the controls

$$u_{1,\tau}(t) := \mathcal{P}(Y_1(t)), \quad u_{2,\tau}(t) := \mathcal{P}(Y_2(t)), \quad t \in [0, \tau),$$

are both τ-locally optimal for η. By Proposition 7.5, we must have

$$\mathcal{P}(Y_1(t)) = \mathcal{P}(Y_2(t)) \quad \forall\, t \in [0, \tau).$$

Then, setting $u_\tau^*(t) := \mathcal{P}(Y_1(t)) = \mathcal{P}(Y_2(t))$, we have

$$Y_1(t) = S_A(t)\eta + \int_0^t S_A(t-s) B u_\tau^*(s) ds = Y_2(t) \quad \forall t \in [0, \tau),$$

the claim. ∎

7.3. Closed loop equation: Global existence. The following result shows the existence of a unique mild global solution of the closed loop equation (52). We consider only the case when $\bar{u} < +\infty$, i.e., case (i) of Hypothesis 3.7.

PROPOSITION 7.7. *Let $\bar{u} < +\infty$. For each $\eta \in H_{++}$ there exists a unique mild solution of* (52) *in* $[0, +\infty)$.

Proof. Uniqueness. The proof follows from Proposition 7.6.

Existence. Let $Y^*(\cdot)$ be the unique mild solution of (52) starting at η, provided by Propositions 7.3 and 7.6, and let $[0, \tau_{\max})$ be its maximal interval of definition. Assume by contradiction that $\tau_{\max} < +\infty$. Using (74), we have the estimate

$$\begin{aligned} &\left\| \int_0^t S(t-r) B\mathcal{P}(Y^*(r)) dr - \int_0^s S(s-r) B\mathcal{P}(Y^*(r)) dr \right\| \\ &\le \left\| \int_s^t S(t-r) B\mathcal{P}(Y^*(r)) dr \right\| \le |\bar{u}| \|b\| \int_s^t \|S(t-r)\| dr \le C_0 |t-s| \end{aligned} \tag{62}$$

for all $0 \le s \le t \le \tau_{\max}$, where $C_0 = |\bar{u}| \|b\| M e^{\omega t}$. Therefore,

$$Y^*(t) = S_A(t)\eta + g(t), \qquad 0 \le t < \tau_{\max},$$

where $g(t) := \int_0^t S_A(t-s) B\mathcal{P}(Y^*(s)) ds$ is uniformly continuous from $[0, \tau_{\max})$ into $\mathcal{F}$. Since also the function $t \mapsto S_A(t)\eta$ is uniformly continuous from $[0, \tau_{\max})$ into $\mathcal{F}$, we have that $Y^* : [0, \tau_{\max}) \to \mathcal{F}$ is uniformly continuous. It follows that there exists

$$Y^*(\tau_{\max}) := \lim_{t \uparrow \tau_{\max}} Y^*(t) \in \bar{\mathcal{F}}. \tag{63}$$

We claim that $Y^*(\tau_{\max}) \in H_{++}$. By (10) we have

$$[S_A(t)\eta]_0 \geq \eta_0 e^{a_0 t} \qquad \forall t \in [0, \tau_{\max}), \tag{64}$$

$$[S_A(t)\eta]_1(\xi) \geq 0 \qquad \forall t \in [0, \tau_{\max}),\ \forall \xi \in [-r, 0]. \tag{65}$$

On the other hand, since $\mathcal{P} \geq 0$ and $b \geq 0$ (in the sense of (3)), and since S_A is positive preserving (see (76)), we also have $g(t) \geq 0$ for all $t \in [0, \tau_{max})$ (in the sense of (3)). Combining this fact with (64)–(65) and taking into account that $\tau_{max} < +\infty$, we see that $Y^*(\tau_{max}) \in H_{++}$, as claimed.

Now let us consider the following equation:

$$\begin{cases} Y'(t) = AY(t) + B\mathcal{P}(Y(t)), & t \geq \tau_{\max}, \\ \\ Y(\tau_{\max}) = Y^*(\tau_{\max}). \end{cases} \tag{66}$$

We know that $Y^*(\tau_{\max}) \in H_{++} \subset \mathcal{F}$. Since our system is autonomous in time, the above equation admits a (unique) mild solution in the interval $[\tau_{\max}, \tau)$ for some $\tau > \tau_{\max}$ by Proposition 7.3. Therefore, there exists a function $Y^{**} : [\tau_{\max}, \tau) \to \mathcal{F}$ such that

$$Y^{**}(t) = S_A(t - \tau_{\max})Y^*(\tau_{\max}) + \int_{\tau_{\max}}^{\tau} S_A(\tau - s)B\mathcal{P}(Y^{**}(s))ds \quad \forall\, t \in [\tau_{\max}, \tau).$$

Now we consider the function $\bar{Y} : [0, \tau) \to H$ defined as

$$\bar{Y}(t) := \begin{cases} Y^*(t), & 0 \leq t < \tau_{\max}, \\ Y^{**}(t), & \tau_{\max} \leq t < \tau. \end{cases}$$

Clearly $\bar{Y}$ solves in the mild sense (52) in the interval $[0, \tau_{\max})$. On the other hand, for $t \geq \tau_{\max}$ we have, using the semigroup property of $S_A(\cdot)$,

$$\begin{aligned} \bar{Y}(t) = Y^{**}(t) &= S_A(t - \tau_{\max})Y^*(\tau_{\max}) + \int_{\tau_{\max}}^{t} S_A(t - s)B\mathcal{P}(Y^{**}(s))ds \\ &= S_A(t - \tau_{\max}) \left(S_A(\tau_{\max})\eta + \int_0^{\tau_{\max}} S_A(\tau_{\max} - r)B\mathcal{P}(Y^*(r))dr \right) \\ &+ \int_{\tau_{\max}}^{t} S_A(t - s)B\mathcal{P}(Y^{**}(s))ds \\ &= S_A(t)\eta + \int_0^{\tau_{\max}} S_A(t - s)B\mathcal{P}(Y^*(s))ds + \int_{\tau_{\max}}^{t} S_A(t - s)B\mathcal{P}(Y^{**}(s))ds \\ &= S_A(t)\eta + \int_0^{t} S_A(t - s)B\mathcal{P}(\bar{Y}(s))ds. \end{aligned} \tag{67}$$

It follows that $\bar{Y}$ is a mild solution of (52) on the interval $[0, \tau)$, contradicting the maximality of the interval $[0, \tau_{\max})$, and the proof is complete. ∎

Combining Theorem 7.2, Proposition 7.5, and Proposition 7.7, we get the second main result of the paper.

COROLLARY 7.8. *Assume that $\bar{u} < +\infty$. Let $\eta \in H_{++}$, and let Y^* be the unique mild solution to the closed loop equation* (52) *starting at η. Then the unique optimal control starting at η is*

$$u^*(t) = \mathcal{P}(Y^*(t)), \quad t \geq 0.$$

Remark 7.9. The results proved in this section give as consequences "half" of a comparison theorem for viscosity solutions of our HJB equation (39). Indeed, suppose that in the definition of the feedback map (51) and in the proof of all the results of this section we replace V_b with v_b, where v is another viscosity solution of the HJB equation (39) such that v is concave, $\|\cdot\|_{-1}$-continuous, and strictly increasing along the direction b (consider also Remark 6.10). Working with this feedback map, we would obtain (56) with v in place of V, and the inequality $V \geq v$ follows immediately.

Appendix.

A.1. The semigroup S_A in the space H. Hereafter, given $f \in L_r^2$, when needed, we shall intend it extended to $\mathbb{R}$ setting $f \equiv 0$ outside of $[-r, 0]$. Consider the space $X = \mathbb{R} \times L_r^2$ endowed with the inner product

$$\langle \cdot, \cdot \rangle_X = \langle \cdot, \cdot \rangle_{\mathbb{R}} + \langle \cdot, \cdot \rangle_{L_r^2},$$

which makes it a Hilbert space. On this space we consider the unbounded linear operator

$$\bar{A}^* : \mathcal{D}(\bar{A}^*) \subset X \longrightarrow X, \quad (\eta_0, \eta_1(\cdot)) \longmapsto (a_0\eta_0, \eta_1'(\cdot)), \tag{68}$$

defined on the domain

$$\mathcal{D}(\bar{A}^*) = \left\{\eta = (\eta_0, \eta_1(\cdot)) \,|\, \eta_1 \in W_r^{1,2},\ \eta_1(0) = \eta_0\right\}.$$

It is well known (see [25]) that $\bar{A}^*$ is a closed operator which generates a C_0-semigroup $S_{\bar{A}^*}$ on X. More precisely, the explicit expression of $S_{\bar{A}^*}(t)$ acting on $\psi = (\psi_0, \psi_1(\cdot)) \in X$ is

$$S_{\bar{A}^*}(t)\psi = \left(e^{a_0 t}\psi_0, \mathbf{1}_{[-r,0]}(t+\xi)\psi_1(t+\xi) + \mathbf{1}_{[0,+\infty)}(t+\xi)e^{a_0(t+\xi)}\psi_0\big|_{\xi\in[-r,0]}\right). \tag{69}$$

On the other hand, it is possible to show (see, e.g., [30]) that $\bar{A}^*$ is the adjoint in X of

$$\begin{aligned} &\bar{A} : \mathcal{D}(\bar{A}) \subset X \longrightarrow X, \\ &(\eta_0, \eta_1(\cdot)) \longmapsto (a_0\eta_0 + \eta_1(0), -\eta_1'(\cdot)), \end{aligned} \tag{70}$$

where

$$\mathcal{D}(\bar{A}) = \mathbb{R} \times W_{r,0}^{1,2} = H.$$

It follows (see [25]) that $\bar{A}$ generates on X a C_0-semigroup $S_{\bar{A}}$ which is nothing else than the adjoint (taken in the space X) of $S_{\bar{A}^*}$, i.e.,

$$S_{\bar{A}}(t) = S_{\bar{A}^*}(t)^* \quad \forall t \geq 0.$$

We can compute the explicit expression of the semigroup $S_{\bar{A}}$ through the relation, which must hold for each $t \geq 0$,

$$\langle S_{\bar{A}}(t)\phi, \psi\rangle_X = \langle \phi, S_{\bar{A}^*}(t)\psi\rangle_X \quad \forall\, \phi = (\phi_0, \phi_1(\cdot)) \in X \ \ \forall\, \psi = (\psi_0, \psi_1(\cdot)) \in X.$$

By (69), we calculate

$$
\begin{aligned}
&\langle \phi, S_{\bar{A}^*}(t)\psi\rangle_X \\
&= \phi_0 e^{a_0 t}\psi_0 + \int_{-r}^{(-t)\vee(-r)} \phi_1(\xi)\psi_1(t+\xi)d\xi + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi)\psi_0 e^{a_0(t+\xi)}d\xi \\
&= \phi_0 e^{a_0 t}\psi_0 + \int_{(-r+t)\wedge 0}^{0} \phi_1(\xi - t)\psi_1(\xi)d\xi + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi)e^{a_0(\xi+t)}\psi_0 d\xi.
\end{aligned}
\tag{71}
$$

So we can write the explicit form of the operator $S_{\bar{A}}(t)$ as

$$
S_{\bar{A}}(t)\phi = \left(\phi_0 e^{a_0 t} + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi)e^{a_0(\xi+t)}d\xi,\, T(t)\phi_1\right), \qquad \phi = (\phi_0, \phi_1(\cdot)) \in X,
$$

where $(T(t))_{t\geq 0}$ is the semigroup of truncated right shifts in L^2_r defined as

$$
[T(t)f](\xi) = \begin{cases} f(\xi - t), & -r \leq \xi - t, \\ 0, & \text{otherwise}, \end{cases} \qquad f \in L^2_r. \tag{72}
$$

So, we can rewrite the above expression as

$$
S_{\bar{A}}(t)\phi = \left(\phi_0 e^{a_0 t} + \int_{(-t)\vee(-r)}^{0} \phi_1(\xi)e^{a_0(\xi+t)}d\xi,\; \phi_1(\cdot - t)\right), \quad (\phi_0, \phi_1(\cdot)) \in X. \tag{73}
$$

We have defined the semigroup $S_{\bar{A}}$ and its infinitesimal generator $(\bar{A}, \mathcal{D}(\bar{A}))$ in the space X. Therefore, by well-known results (see [25, Ch. II, p. 124]), we get that $\bar{A}|_{\mathcal{D}(\bar{A}^2)}$ is the generator of a C_0-semigroup on $(\mathcal{D}(\bar{A}), \|\cdot\|_{\mathcal{D}(\bar{A})})$, which is nothing but the restriction of $S_{\bar{A}}$ to this subspace. Now we notice that

$$
\mathcal{D}(\bar{A}) = H, \;\; \|\cdot\|_{\mathcal{D}(\bar{A})} \sim \|\cdot\|, \;\; \mathcal{D}(\bar{A}^2) = \mathbb{R} \times W^{2,2}_{r,0} = \mathcal{D}(A), \;\; \bar{A}|_{\mathbb{R}\times W^{2,2}_{r,0}} = A,
$$

where A is the operator defined in (9). Hence, we conclude that A generates a C_0-semigroup on H, whose expression is the same given in (73). We denote such a semigroup by S_A. We recall (see, e.g., [42, Ch. 2, Prop. 4.7]) that if S is a C_0-semigroup on a Banach space $\mathcal{B}$, then there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that

$$
\|S(t)\|_{\mathcal{L}(\mathcal{B})} \leq Me^{\omega t}, \qquad t \geq 0. \tag{74}
$$

In this case, using Holder's inequality and taking into account that $\phi_1(-r) = 0$, as $\phi_1 \in W^{1,2}_{r,0}$, we compute for every $t \geq 0$

$$
\begin{aligned}
&\left|\phi_0 e^{a_0 t} + \int_{(-t)\vee(-r)}^0 \phi_1(\xi) e^{a_0(\xi+t)} d\xi\right|^2 \\
&\leq 2e^{2a_0 t}|\phi_0|^2 + 2e^{2a_0 t}\left(\int_{-r}^0 |\phi_1(\xi)| d\xi\right)^2 \\
&\leq 2e^{2a_0 t}|\phi_0|^2 + 2e^{2a_0 t} r \left(\int_{-r}^0 |\phi_1(\xi)|^2 d\xi\right) \\
&\leq 2e^{2a_0 t}|\phi_0|^2 + 2e^{2a_0 t} r \left(\int_{-r}^0 \left|\int_{-r}^{\xi} \phi_1'(s) ds\right|^2 d\xi\right) \\
&\leq 2e^{2a_0 t}|\phi_0|^2 + 2e^{2a_0 t} r \left(\int_{-r}^0 (r+\xi)\left(\int_{-r}^{\xi} |\phi_1'(s)|^2 ds\right) d\xi\right) \\
&\leq 2e^{2a_0 t}|\phi_0|^2 + e^{2a_0 t} r^3 \|\phi_1\|^2_{W^{1,2}_{r,0}}.
\end{aligned}
$$

Moreover,

$$
\|T(t)\|_{\mathcal{L}(W^{1,2}_{r,0})} \leq 1 \quad \forall t \in [0,r]; \quad \|T(t)\|_{\mathcal{L}(W^{1,2}_{r,0})} = 0, \ \forall t > r.
$$

The computations above show that in our case

$$
\|S_A(t)\|_{\mathcal{L}(H)} \leq (2 + r^3)^{1/2} e^{a_0 t} \quad \forall t \geq 0. \tag{75}
$$

Finally, we notice that clearly S_A is positive preserving, i.e.,

$$
\eta \geq 0 \implies S_A(t)\eta \geq 0 \quad \forall t \geq 0. \tag{76}
$$

A.2. Proofs of technical results. Here we provide the proofs of some results we have not proved in the main text.

Proof of Proposition 4.3. Let

$$
\mathcal{D} := \left\{\phi = (\phi_0, \phi_1(\cdot)) \in H \mid \phi_1 \in W^{2,2}_r, \ \ \phi_1(-r) = 0, \ \ \phi_1'(0) = 0\right\}.
$$

First we notice that, defining $A^*\phi$ on $\mathcal{D}$ as in (13), we have $A^*\phi \in H$. Now notice that

$$
\psi_1'(-r) = 0, \quad \psi_1(0) = \int_{-r}^0 \psi_1'(\xi) d\xi \quad \forall \psi \in \mathcal{D}(A). \tag{77}
$$

Therefore, taking into account (77), we have for every $\psi \in \mathcal{D}(A)$ and every $\phi \in \mathcal{D}$

$$
\begin{aligned}
&\langle A\psi, \phi\rangle \\
&= a_0\psi_0\phi_0 + \psi_1(0)\phi_0 - \int_{-r}^0 \psi_1''(\xi)\phi_1'(\xi) d\xi \\
&= a_0\psi_0\phi_0 + \left(\int_{-r}^0 \psi_1'(\xi) d\xi\right)\phi_0 - \psi_1'(0)\phi_1'(0) + \psi_1'(-r)\phi_1'(-r) + \int_{-r}^0 \psi_1'(\xi)\phi_1''(\xi) d\xi \\
&= a_0\psi_0\phi_0 + \int_{-r}^0 \psi_1'(\xi)\left(\phi_0 + \phi_1''(\xi)\right) d\xi = \langle \psi, A^*\phi\rangle,
\end{aligned} \tag{78}
$$

where

$$\begin{cases}(A^*\phi)_0 = a_0\phi_0,\\ (A^*\phi)_1(\xi) = \int_{-r}^{\xi}(\phi_0+\phi_1''(s))ds = \phi_0\cdot(\xi+r)+\phi_1'(\xi)-\phi_1'(-r).\end{cases}$$

The equality above shows that $\mathcal{D}\subset\mathcal{D}(A^*)$ and that A^* acts as claimed in (13) on the elements of $\mathcal{D}$.

Now we have to show that $\mathcal{D}=\mathcal{D}(A^*)$. For the sake of brevity here we only sketch the proof of this fact,[3] as a complete proof would require a study of the adjoint semigroup $S_{A^*}(t)$ in the space H. We observe that $\mathcal{D}$ is dense in H. Moreover, an explicit computation of the adjoint semigroup would show that $S_{A^*}(t)\mathcal{D}\subset\mathcal{D}$ for any $t\geq 0$. Hence, by [20, Th. 1.9, p. 8], $\mathcal{D}$ is dense in $\mathcal{D}(A^*)$ endowed with the graph norm. Finally, using (13) it is easy to show that $\mathcal{D}$ is closed in the graph norm of A^* and therefore $\mathcal{D}(A^*)=\mathcal{D}$. ∎

Proof of Lemma 5.2. Let $\eta=(\eta_0,\eta_1)\in H$. Taking into account (29) and by Hölder's inequality, we have

$$\begin{aligned}\|\eta\|_X^2 &= |\eta_0|^2+\int_{-r}^0|\eta_1(\xi)|^2d\xi\\ &= \left|\eta_0+\int_{-r}^0\eta_1(\xi)d\xi-\int_{-r}^0\eta_1(\xi)d\xi\right|^2+\int_{-r}^0|\eta_1(\xi)|^2d\xi\\ &\leq 2\left|\eta_0+\int_{-r}^0\eta_1(\xi)d\xi\right|^2+2\left|\int_{-r}^0\eta_1(\xi)d\xi\right|^2+\int_{-r}^0|\eta_1(\xi)|^2d\xi\\ &\leq 2\left|\eta_0+\int_{-r}^0\eta_1(\xi)d\xi\right|^2+2\left(\int_{-r}^0|\eta_1(\xi)|d\xi\right)^2+\int_{-r}^0|\eta_1(\xi)|^2d\xi\\ &\leq 2a_0^2\left|\frac{\eta_0+\int_{-r}^0\eta_1(\xi)d\xi}{a_0}\right|^2+2r^2\int_{-r}^0|\eta_1(\xi)|^2d\xi+\int_{-r}^0|\eta_1(\xi)|^2d\xi\ \leq C\|\eta\|_{-1}^2,\end{aligned}$$

where $C=\max\{2a_0^2,2r^2+1\}$.

On the other hand, still using (29) and Hölder's inequality, we have

$$\|\eta\|_{-1}^2=\left|\frac{\eta_0+\int_{-r}^0\eta_1(s)ds}{a_0}\right|^2+\int_{-r}^0|\eta_1(s)|^2ds\leq\frac{2}{a_0^2}|\eta_0|^2+\int_{-r}^0|\eta_1(s)|^2ds\leq C'\|\eta\|_X^2,$$

where $C'=\max\left\{\frac{2}{{a_0}^2},1\right\}$. So the claim is proved. ∎

Proof of Proposition 5.5. (i) The inclusions

$$H_{++}\subset\mathcal{F}\subset H_+$$

are obvious. Let $\eta\in H$, and set $Y(\cdot):=Y(\cdot;\eta,0)$. Due to Definition 4.1(i), we have

$$\begin{aligned}Y_0(t) = [S_A(t)\eta]_0 &= \eta_0e^{a_0t}+\int_{(-t)\vee(-r)}^0e^{a_0(t+\xi)}\eta_1(\xi)d\xi\\ &= e^{a_0t}\left(\eta_0+\int_{(-t)\vee(-r)}^0e^{a_0\xi}\eta_1(\xi)d\xi\right)\ \forall t\geq 0.\end{aligned}$$

[3] To this end we observe that we use in this paper only the fact $\mathcal{D}\subset\mathcal{D}(A^*)$ and that (13) holds true on the elements of $\mathcal{D}$, which has been proven rigorously. More precisely, we use the fact that $(1,0)\in\mathcal{D}\subset\mathcal{D}(A^*)$ in the proof of Theorem 4.4.

So we see that $\mathcal{F} = \mathcal{G}$. Now let $\eta \in \mathcal{G}$. Since g_0 is nondecreasing and $g_0(0) = 0$, $h_0(0) = 0$, we have

$$V(\eta) \geq J(\eta, 0) = \int_0^{+\infty} e^{-\rho t} \left(g_0(Y_0(t; \eta, 0)) - h_0(0) \right) dt \geq 0.$$

As a byproduct this shows that $V(\eta) \geq 0$ on $\mathcal{G}$ and that $\mathcal{G} \subset \mathcal{D}(V)$, so the proof of item (i) is complete.

(ii) Let $\bar{\eta} \in \mathcal{F}$. We have to prove that

$$\exists\, \varepsilon \text{ such that } \mathcal{B}_{\|\cdot\|_{-1}}(\bar{\eta}, \varepsilon) \subset \mathcal{F}. \tag{79}$$

Due to Lemma 5.2, (79) is equivalent to

$$\exists\, \varepsilon \text{ such that } \mathcal{B}_{\|\cdot\|_{X}}(\bar{\eta}, \varepsilon) \subset \mathcal{F}. \tag{80}$$

Let $\varepsilon > 0$ and $\eta \in \mathcal{B}_{\|\cdot\|_{X}}(\bar{\eta}, \varepsilon)$. Then we have

$$|\eta_0 - \bar{\eta}_0| < \varepsilon, \qquad \|\eta_1 - \bar{\eta}_1\|_{L^2_r} < \varepsilon.$$

Therefore,

$$\begin{aligned} &\left| \left(\eta_0 + \int_{-\xi}^{0} e^{a_0 s} \eta_1(s) ds \right) - \left(\bar{\eta}_0 + \int_{-\xi}^{0} e^{a_0 s} \bar{\eta}_1(s) ds \right) \right| \\ &\leq |\eta_0 - \bar{\eta}_0| + \left| \int_{-\xi}^{0} e^{a_0 s} \left(\eta_1(s) - \bar{\eta}_1(s) \right) ds \right| \\ &\leq |\eta_0 - \bar{\eta}_0| + r^{1/2} e^{|a_0| r} \|\eta_1 - \bar{\eta}_1\|_{L^2} < \left(1 + r^{1/2} e^{|a_0| r} \right) \varepsilon, \end{aligned} \tag{81}$$

where the second inequality follows from Holder's inequality. Then (80) straightly follows from (81) taking a sufficiently small $\varepsilon > 0$, and so the proof is complete. ∎

Proof of Proposition 5.7. Let $\eta, \bar{\eta} \in \mathcal{D}(V)$, and set, for $\lambda \in [0, 1]$, $\eta_\lambda := \lambda \eta + (1 - \lambda) \bar{\eta}$. For $\varepsilon > 0$, let $u^\varepsilon(\cdot) \in \mathcal{U}(\eta)$ and $\bar{u}^\varepsilon(\cdot) \in \mathcal{U}(\bar{\eta})$ be two controls ε-optimal for the initial states $\eta, \bar{\eta}$, respectively, i.e., such that

$$J(\eta; u^\varepsilon(\cdot)) > V(\eta) - \varepsilon, \quad J(\bar{\eta}; \bar{u}^\varepsilon(\cdot)) > V(\bar{\eta}) - \varepsilon.$$

Set

$$y(\cdot) := y(\cdot; \eta, u^\varepsilon(\cdot)), \;\; \bar{y}(\cdot) := \bar{y}(\cdot; \bar{\eta}, \bar{u}^\varepsilon(\cdot)), \;\; u^\lambda(\cdot) := \lambda u^\varepsilon(\cdot) + (1 - \lambda) \bar{u}^\varepsilon(\cdot).$$

Finally, set $y_\lambda(\cdot) := \lambda y(\cdot) + (1 - \lambda) \bar{y}(\cdot)$. The function h_0 is convex, and so one has

$$h_0(u^\lambda(t)) \leq \lambda h_0(u^\varepsilon(t)) + (1 - \lambda) h_0(\bar{u}^\varepsilon(t)), \;\; t \geq 0.$$

Moreover, by linearity of the state equation, we have

$$Y(t; \eta_\lambda, u^\lambda(\cdot)) = \lambda Y(t; \eta, u^\varepsilon(\cdot)) + (1 - \lambda) Y(t; \bar{\eta}, \bar{u}^\varepsilon(\cdot)).$$

Hence, by concavity of g (due to Hypothesis 3.6(i) and (26)) we have

$$g(Y(t; \eta_\lambda, u^\lambda(\cdot))) \geq \lambda g(Y(t; \eta, u^\varepsilon(\cdot))) + (1 - \lambda) g(Y(t; \bar{\eta}, \bar{u}^\varepsilon(\cdot))), \;\; t \geq 0.$$

So, we have

$$\begin{aligned} V(\eta_\lambda) \geq\ & J(\eta_\lambda, u^\lambda(\cdot)) = \int_0^{+\infty} e^{-\rho t} \left(g(Y(t;\eta_\lambda, u^\lambda(\cdot))) - h_0(u^\lambda(t))\right) dt \\ \geq & \int_0^{+\infty} e^{-\rho t}(\lambda g(Y(t;\eta, u^\varepsilon(\cdot)))+(1-\lambda)g(Y(t;\bar\eta, \bar u^\varepsilon(\cdot)))-\lambda h_0(u^\varepsilon(t))-(1-\lambda)h_0(\bar u^\varepsilon(t)))\, dt \\ = & \lambda J(\eta, u^\varepsilon(\cdot)) + (1-\lambda)J(\bar\eta, \bar u^\varepsilon(\cdot)) > \lambda V(\eta) + (1-\lambda)V(\bar\eta) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows both claims. $\square$

Proof of Proposition 5.9. Let $\eta, \zeta \in \mathcal{D}(V)$ with $\eta \geq \zeta$. Let $u \in \mathcal{U}(\eta)$, and consider $Y(\cdot;\eta, u(\cdot))$. Since S_A is positive preserving (see (76)), we have

$$Y(t;\eta, u(\cdot)) - Y(t;\zeta, u(\cdot)) = S_A(t)\,(\eta - \zeta) \geq 0 \quad \forall t \geq 0. \tag{82}$$

Therefore,

$$Y_0(t;\eta, u(\cdot)) \geq Y_0(t;\zeta, u(\cdot)).$$

This shows that $u \in \mathcal{U}(\eta)$. Hypothesis 3.6(i) implies that g is nondecreasing with respect to the order relation defined in (3). Set

$$\beta(t) := \int_0^t S_A(t-\tau)Bu(\tau)d\tau.$$

Then, also taking into account (82),

$$\begin{aligned} & J(\eta; u(\cdot)) - J(\zeta; u(\cdot)) \\ & = \int_0^{+\infty} e^{-\rho t}\,(g(Y(t;\eta, u(\cdot))) - g(Y(t;\zeta, u(\cdot))))\, dt \\ & = \int_0^{+\infty} e^{-\rho t}\,(g(S_A(t)(\eta) + \beta(t)) - g(S_A(t)(\zeta) + \beta(t)))\, dt. \end{aligned}$$

So, by the arbitrariness of $u \in \mathcal{U}(\eta 0$, we get $V(\eta) \geq V(\zeta)$, the first part of the claim.

Let us show the second part. Since $h(0) = 0$, we have

$$\begin{aligned} V(\eta + sh) \geq J(\eta + sh; 0) &= \int_0^{+\infty} e^{-\rho t} g\left(S_A(t)(\eta + sh)\right) dt \\ &= \int_0^{+\infty} e^{-\rho t} g_0\left([S_A(t)(\eta + sh)]_0\right) dt. \end{aligned}$$

By (10) we have

$$[S_A(t)(\eta + sh)]_0 \uparrow +\infty \text{ as } s \uparrow +\infty \quad \forall t \geq 0.$$

So, since g_0 is nondecreasing, by monotone convergence, we get

$$\lim_{s\to+\infty} V(\eta + sh) = \int_0^{+\infty} e^{-\rho t}\bar g_0 dt = \frac{1}{\rho}\bar g_0,$$

the claim. $\square$

Proof of Proposition 5.10. (i) Let $u \in \mathcal{U}(\eta)$. Set $C := (2+r^3)^{\frac{1}{2}}$. By (75) we have

$$\begin{aligned} \|Y(t)\| &\le Ce^{a_0 t}\|\eta\| + \int_0^t Ce^{a_0(t-\tau)}\|b\||u(\tau)|d\tau \\ &\le Ce^{a_0 t}\left(\|\eta\| + \|b\|\int_0^t |u(\tau)|d\tau\right). \end{aligned} \tag{83}$$

With regard to the structure of U, we distinguish the two cases $\bar{u} < +\infty$, $\bar{u} = +\infty$.

Case $\bar{u} < +\infty$. In this case (83) yields

$$|Y_0(t)| \le \|Y(t)\| \le Ce^{a_0 t}\left(\|\eta\| + \frac{1}{a_0}\|b\|\bar{u}t\right) \quad \forall t \ge 0. \tag{84}$$

Let δ be such that

$$g_0\left(Ce^{a_0}\left(\|\eta\| + \frac{1}{a_0}\|b\|\bar{u}\right)\right) = \bar{g}_0 - \delta. \tag{85}$$

Since g_0 is strictly increasing, we have $\delta > 0$. Then, for every $u \in \mathcal{U}(\eta)$, we have due to (84) and (85) and since $h_0 \ge 0$

$$\begin{aligned} J(\eta; u(\cdot)) &= \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - h_0(u(t))\right)dt \le \int_0^{+\infty} e^{-\rho t} g(Y_0(t))dt \\ &\le \int_0^1 e^{-\rho t} g_0\left(Ce^{a_0}\left(\|\eta\| + \frac{1}{a}\|b\|\bar{u}\right)\right)dt + \int_1^{+\infty} e^{-\rho t}\bar{g}_0 dt \\ &\le \frac{1}{\rho}(\bar{g}_0 - \delta)\left(1 - e^{-\rho}\right) + \frac{1}{\rho}e^{-\rho}\bar{g}_0. \end{aligned}$$

Taking the supremum over $u \in \mathcal{U}(\eta)$, in the inequality above we get

$$V(\eta) = \sup_{u \in \mathcal{U}(\eta)} J(\eta; u(\cdot)) \le \frac{1}{\rho}(\bar{g}_0 - \delta)\left(1 - e^{-\rho}\right) + \frac{1}{\rho}e^{-\rho}\bar{g}_0 < \frac{1}{\rho}\bar{g}_0,$$

obtaining the claim in this case.

Case $\bar{u} = +\infty$. By Hypothesis 3.3(ii) there exist C_0, C_1 constant such that

$$h_0(u) \ge C_0 u - C_1 \quad \forall u \in U. \tag{86}$$

Given (83), we want to find an upper bound for $|Y_0(t)|$ like (84), in order to argue as before and get the claim. In this case, since $\bar{u} = +\infty$, we do not directly have this upper bound over all $u \in \mathcal{U}(\eta)$, but only on "good" controls (ε-optimal, which still suffices). Let $\varepsilon > 0$, and let $u \in \mathcal{U}(\eta)$ be an ε-optimal control for η. Then by (86),

$$\begin{aligned} V(\eta) - \varepsilon < J(\eta; u(\cdot)) &= \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - h_0(u(t))\right)dt \\ &\le \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - C_0 u(t) + C_1\right)dt. \end{aligned}$$

So we have

$$\begin{aligned} C_0 \int_0^1 e^{-\rho t}|u(t)|dt &\leq C_0 \int_0^{+\infty} e^{-\rho t}|u(t)|dt \\ &< \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) + C_1\right) dt - V(\eta) + \varepsilon \\ &\leq \int_0^{+\infty} e^{-\rho t}\left(\bar{g}_0 + C_1\right) dt - V(\eta) + \varepsilon = \frac{1}{\rho}(\bar{g}_0 + C_1) - V(\eta) + \varepsilon \\ &< M. \end{aligned}$$

This means that there exists some $M' > 0$ such that

$$\int_0^1 |u(t)|dt \leq M' \qquad \forall\, u \in \mathcal{U}(\eta)\ \varepsilon\text{-optimal.}$$

Therefore, an upper bound like (84) holds true for $t \in [0,1]$ for the controls $u \in \mathcal{U}(\eta)$ which are ε-optimal. This allows us to conclude as before.

(ii) By Propositions 5.7 and 5.9, we know that the real function defined by (33) is concave and nondecreasing. Then, assuming by contradiction that it is not strictly increasing, there must exist $\bar{s} \geq 0$ such that $V(\eta + \bar{s}h)$ is constant on the half line $[\bar{s}; +\infty)$. This fact would contradict claim (i) and (32), so we conclude. ∎

Proof of Lemma 6.4. If Hypothesis 3.7(i) holds, the proof is trivial. So, let Hypothesis 3.7(ii) hold true. By such an assumption, there exist some constants $M_0, M_1 > 0$ such that

$$h_0(u) \geq M_0 u^{1+\alpha} - M_1.$$

Let $u^\varepsilon \in \mathcal{U}(\eta)$ be an ε-optimal control for η. Then

$$\begin{aligned} V(\eta) - \varepsilon < J(\eta; u^\varepsilon(\cdot)) &= \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - h_0(u(t))\right) dt \\ &\leq \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) - M_0|u(t)|^{1+\alpha} + M_1\right) dt. \end{aligned} \tag{87}$$

From (87) we get

$$\begin{aligned} M_0 \int_0^{+\infty} e^{-\rho t}|u(t)|^{1+\alpha}dt &\leq \int_0^{+\infty} e^{-\rho t}\left(g(Y(t)) + M_1\right) dt - V(\eta) + \varepsilon \\ &\leq \int_0^{+\infty} e^{-\rho t}\left(\bar{g}_0 + M_1\right) dt - V(\eta) + \varepsilon < \frac{\bar{g}_0 + M_1}{\rho} - V(\eta) + \varepsilon =: M_\varepsilon. \end{aligned}$$

So the claim is proved. ∎

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REFERENCES

[1] O. Arino, S. Gautier, and J. P. Penot, *A fixed point theorem for sequentially continuous mapping with applicartion to ordinary differential equations*, Funkcialaj Ekvacioj, 27 (1984), pp. 273–279.

[2] P. K. Asea and P. J. Zak, *Time-to-build and cycles*, J. Econom. Dynam. Control, 23 (1999), pp. 1155–1175.

[3] K. Astala, *On Peano's theorem in locally convex spaces*, Studia Math., 73 (1982), pp. 213–223.

[4] M. Bambi, *Endogenous growth and time to build: The AK case*, J. Econom. Dynam. Control, 32 (2008), pp. 1015–1040.

[5] M. Bambi, G. Fabbri, and F. Gozzi, *Optimal policy and consumption smoothing effects in the time-to-build AK model*, Econom. Theory, 50 (2012), pp. 635–669.

[6] V. Barbu and G. Da Prato, *Hamilton-Jacobi Equations in Hilbert Spaces*, Pitman, London, 1983.

[7] V. Barbu and G. Da Prato, *Hamilton-Jacobi equations in Hilbert spaces: Variational and semigroup approach*, Ann. Mat. Pura Appl., 142 (1985), pp. 303–349.

[8] V. Barbu and G. Da Prato, *A note on a Hamilton-Jacobi equation in Hilbert space*, Nonlinear Anal., 9 (1985), pp. 1337–1345.

[9] V. Barbu, G. Da Prato, and C. Popa, *Existence and uniqueness of the dynamic programming equation in Hilbert spaces*, Nonlinear Anal., 7 (1983), pp. 283–299.

[10] M. Bardi amd I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*, Birkhäuser Boston, Boston, 1997.

[11] R. Bekes, *The range convolution operators*, Pacific J. Math., 110 (1984), pp. 257–271.

[12] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional System*, 2nd ed., Birkhäuser Boston, Boston, 2007.

[13] R. Boucekkine, O. Licandro, A. Puch Luis, and F. Fernando Del Rio, *Vintage capital and the dynamicsof the AK model*, J. Econom. Theory, 20 (2005), pp. 39–72.

[14] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[15] P. Cannarsa, *Regularity properties of solutions to Hamilton-Jacobi-Bellman equations in infinite dimensions and nonlinear optimal control*, Differential Integral Equations, 2 (1989), pp. 479–493.

[16] P. Cannarsa and G. Di Blasio, *A direct approach to infinite dimensional Hamilton–Jacobi equations and applications to convex control with state constraints*, Differential Integral Equations, 8 (1995), pp. 225–246.

[17] P. Cannarsa and G. Di Blasio, *Dynamic programming for an abstract second order evolution equation with convex state constraints*, in Proceedings of the Control of Partial Differential Equations, IFIP WG 7.2 Conference, Villa Madruzzo, Trento, Italy, 1993, pp. 51–61.

[18] P. Cannarsa and H. M. Soner, *Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications*, Nonlinear Anal., 13 (1989), pp. 305–323.

[19] S.-N. Chow and J. D. Schuur, *An existence theorem for ordinary differential equations in Banach spaces*, Bull. Amer. Math. Soc., 77 (1971), pp. 1018–1020.

[20] E. B. Davies, *One Parameter Semigroups*, Academic Press, New York, 1980.

[21] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Math. 596, Springer-Verlag, New York, 1977.

[22] G. Di Blasio, *Global solutions for a class of Hamilton-Jacobi equations in Hilbert spaces*, Numer. Funct. Anal. Optim., 8 (1985/86), pp. 261–300.

[23] G. Di Blasio, *Optimal control with infinite horizon for distributed parameter systems with constrained controls*, SIAM J. Control Optim., 29 (1991), pp. 909–925.

[24] I. Ekeland and R. Temam, *Convex Analysis and Variational Problem*, North–Holland, Amsterdam, 1976.

[25] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Spinger-Verlag, New York, 2000.

[26] G. Fabbri and F. Gozzi, *Solving optimal growth models with vintage capital: The dynamic programming approach*, J. Econom. Theory, 143 (2008), pp. 331–373.

[27] G. Fabbri, F. Gozzi, and A. Swiech, *Verification theorems and construction of ε-optimal controls*, J. Convex Anal., 17 (2010), pp. 611–642.

[28] S. Faggian, *Regular solutions of Hamilton–Jacobi equations arising in economics*, Appl. Math. Optim., 51 (2005), pp. 123–162.

[29] S. Faggian, *Hamilton–Jacobi equations arising from boundary control problems with state constraints*, SIAM J. Control Optim., 47 (2008), pp. 2157–2178.

[30] S. Federico, B. Goldys, and F. Gozzi, *HJB equations for the optimal control of differential equations with delays and state constraints,* I: *Regularity of viscosity solutions*, SIAM J. Control Optim., 48 (2010), pp. 4910–4937.
[31] S. Federico, B. Goldys, and F. Gozzi, *HJB equations for the optimal control of differential equations with delays and state constraints,* II*: Verification and optimal feedbacks*, SIAM J. Control Optim., 49 (2011), pp. 2378–2414.
[32] G. Feichtinger, R. Hartl, and S. Sethi, *Dynamic optimal control models in advertising: Recent developments*, Management Sci., 40 (1994), pp.195–226.
[33] A. N. Godunov, *On Peano's theorem in Banach spaces*, Funct. Anal. Appl., 9 (1975), pp. 53–55.
[34] F. Gozzi, *Some results for an optimal control problem with a semilinear state equation* II, SIAM J. Control Optim., 29 (1991), pp. 751–768.
[35] F. Gozzi, *Some results for an infinite horizon control problem governed by a semilinear state equation*, in Control and Estimation of Distributed Parameter Systems, Internat. Ser. Numer. Math. 91, F. Kappel, K. Kunisch, and W. Schappacher, eds., Birkäuser, Basel, 1989, pp. 145–163.
[36] F. Gozzi and C. Marinelli, *Stochastic optimal control of delay equations arising in advertising models*, in Stochastic Partial Differential Equations and Applications—VII, Lect. Notes Pure Appl. Math. 245, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp.133–148.
[37] F. Gozzi, C. Marinelli, and S. Savin, *On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects*, J. Optim. Theory Appl., 142 (2009), pp. 291–321.
[38] H. G. Hashem Hind, *Weak Solutions of Differential Equations in Banach Space*, J. Fract. Calc. Appl., 3 (2012), pp. 1–9.
[39] A. Ichikawa, *Quadratic control of evolution equations with delays in control*, SIAM J. Control Optim., 20 (1982), pp. 645–668.
[40] H. N. Koivo and E. B. Lee, *Controller synthesis for linear system with retarded state and control variables and quadratic cost*, Automatica J. IFAC, 8 (1972), pp. 203–208.
[41] F. E. Kydland and E. C. Prescott, *Time-to-build and aggregate fluctuations*, Econometrica, 50 (1982), pp. 1345–1370.
[42] X. Li and J. Yong, *Optimal Control Theory for Infinite-Dimensional Systems*, Birkhäuser Boston, Boston, 1995.
[43] W. Pauwels, *Optimal dynamic advertising policies in the presence of continuously distributed time lags*, J. Optim. Theory Appl., 22 (1977), pp. 79–89.
[44] R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Math. 1364, Springer-Verlag, New York, 1993.
[45] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[46] S. P. Sethi, *Sufficient conditions for the optimal control of a class of systems with continuous lags*, J. Optim. Theory Appl., 13 (1974), pp. 545–552.
[47] V. Teixeira Eduardo, *Strong solutions for differential equations in abstract spaces*, J. Differential Equations, 214 (2005), pp. 65–91.
[48] J. Tsoukalas, *Time to build capital: Revisiting investment-cash flow sensitivities*, J. Econom. Dynam. Control, 35 (2011), pp. 1000–1016.
[49] R. B. Vinter and R. H. Kwong, *The infinite time quadratic control problem for linear systems with state and control delays: An evolution equation approach*, SIAM J. Control Optim., 19 (1981), pp. 139–153.
[50] J.Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB equations*, Springer-Verlag, Berlin, New York, 1999.
[51] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[52] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. Vol* 1*. Fixed-Point Theorems*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986.