

On an eigenvalue problem involving the Hardy potential

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Abstract

In this note we consider the eigenvalue problem for the Laplacian with the Neumann and Robin boundary conditions involving the Hardy potential. We prove the existence of eigenfunctions of the second eigenvalue for the Neumann problem and of the principal eigenvalue for the Robin problem in "high" dimensions.

1 Introduction

In this note we assume that $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded, connected open set with a smooth boundary $\partial\Omega$. We consider the following eigenvalue problem

$$(1.1) \quad \begin{cases} -\Delta u &= \mu \frac{1}{|x|^2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega. \end{cases}$$

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Solutions to this problem are sought in the Sobolev space $W^{1,2}(\Omega)$. We recall that $W^{1,2}(\Omega)$ is the Sobolev space equipped with norm

$$\|u\|_{W^{1,2}}^2 = \int_{\Omega} (|\nabla u|^2 + u^2) dx.$$

$W_0^{1,2}(\Omega)$ is a subspace of $W^{1,2}(\Omega)$ consisting of functions vanishing in the sense of trace on the boundary $\partial\Omega$ of Ω . The norm $\|u\|_{W^{1,2}}$ on this subspace is equivalent to

$$\|u\|_{W_0^{1,2}}^2 = \int_{\Omega} |\nabla u|^2 dx.$$

Problem (1.1) is closely related to the Hardy inequality. The Hardy inequality in $W_0^{1,2}(\Omega)$ has the following form: let $0 \in \Omega$, then

$$(1.2) \quad \int_{\Omega} \frac{u^2}{|x|^2} dx \leq \frac{1}{\Lambda_N} \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W_0^{1,2}(\Omega)$, where $\Lambda_N = \frac{(N-2)^2}{4}$. The constant Λ_N is an optimal constant and there is no nonzero function in $W_0^{1,2}(\Omega)$ which changes this inequality into equality (see [10], Theorem 4.1). In recent years the Hardy inequality in $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$ and the related problems for semilinear elliptic equation have attracted considerable interest. We refer to papers [6] and [7], where further bibliographical references can be found. For further generalizations and applications of the Hardy - Sobolev type inequalities, we refer to the papers [1], [2], [3], [4], [16]. The value of the minimum and its attainability of the Rayleigh with respect to the boundary conditions has been investigated in [5]. According to the results of this paper the optimal constant in $W^{1,2}(\Omega)$ is zero and is attained by constant functions. Thus to obtain an extension of (1.2) to $W^{1,2}(\Omega)$ involving the whole norm, it is necessary to add a new positive term to the right hand side of (1.2). This question is discussed in the papers [6] and [7]. In this note we need the following extension of (1.2) to the space $W^{1,2}(\Omega)$.

Proposition 1.1 *Let $0 \in \bar{\Omega}$. Then for every $\delta > 0$ there exists a constant $A = A(\delta, \Omega)$ such that*

$$(1.3) \quad \int_{\Omega} \frac{u^2}{|x|^2} dx \leq \left(\frac{1}{\Lambda_N} + \delta\right) \int_{\Omega} |\nabla u|^2 dx + A \int_{\Omega} u^2 dx$$

for every $u \in W^{1,2}(\Omega)$.

The proof of (1.3) can be found in papers [13] and [14]. The case $0 \in \Omega$ is considered in [14] and the case $0 \in \partial\Omega$ in [13] (see also [19]).

The paper is organized as follows. In Section 2 we discuss the inequalities of Hardy - Poincaré type. These inequalities lead to the eigenvalue problem (1.1). A sufficient condition for the existence of the second eigenvalue and the corresponding eigenfunction is presented in Section 3, namely that the corresponding Rayleigh quotient lies strictly below the Hardy constant Λ_N (see (3.1)). In Section 4 we give examples of sets which satisfy this condition

provided the dimension is sufficiently large. The eigenfunctions we find may be singular at the origin. Section 5 is devoted to the study of the asymptotic behavior of the second eigenfunction of (1.1) around 0. Since Λ_N is not attained on the space $W_0^{1,2}(\Omega)$ there is no analogue to problem (1.1) with Dirichlet boundary conditions. In Section 6 we briefly discuss the eigenvalue problem involving the Hardy potential with Robin boundary conditions. We show that the smallest eigenvalue for this problem is attained if it lies below the Hardy constant Λ_N ; again, this is the case if the space-dimension is sufficiently large. In the final Section 7 we formulate some open problems related to this paper.

Problems (1.1) and (6.1) investigated in this paper are eigenvalue problems with weight functions. The weight function $\frac{1}{|x|^2}$ considered here is not in $L^{\frac{N}{2}}(\Omega) \cap L^\infty(\Omega)$ and consequently the functional $u \rightarrow \int_\Omega \frac{u^2}{|x|^2} dx$ is not completely continuous in $W^{1,2}(\Omega)$ unlike in papers [8], [11] and [25]. For a good survey of eigenvalue problems for the p -Laplacian, however without the Hardy potential, we refer to paper [20].

Throughout this paper in a given Banach space X we denote strong convergence by " \rightarrow " and weak convergence by " \rightharpoonup ". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$, are denoted by $\|\cdot\|_p$.

2 The inequality of the Hardy - Poincaré type

We denote by $L^2(\Omega, \frac{1}{|x|^2} dx)$ the weighted Lebesgue space equipped with norm

$$\|u\|_*^2 = \int_\Omega \frac{u^2}{|x|^2} dx.$$

The corresponding scalar product on $L^2(\Omega, \frac{1}{|x|^2})$ is denoted by $(\cdot, \cdot)_*$.

Theorem 2.1 *Suppose that $0 \in \bar{\Omega}$ and that $g \in L^2(\Omega, \frac{1}{|x|^2})$ with $\int_\Omega \frac{g}{|x|^2} dx = 1$. We assume that Ω is an open, connected and bounded set with a smooth boundary. Then there exists a constant $H = H(g, \Omega)$ such that*

$$(2.1) \quad \int_\Omega \left(u - \int_\Omega \frac{ug}{|x|^2} dx \right)^2 \frac{dx}{|x|^2} \leq H \int_\Omega |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$.

The proof is similar to that of Theorem 8.11 in [21] and is omitted. We list below some consequences of Theorem 2.1.

Corollary 2.2 *We make now specific choices of g .*

(a) If $g = \frac{1}{\int_{\Omega} \frac{dx}{|x|^2}}$, then (2.1) takes the form: there exists a constant $H > 0$ such that

$$\int_{\Omega} \left(u - \frac{1}{\int_{\Omega} \frac{dx}{|x|^2}} \int_{\Omega} \frac{u}{|x|^2} dx \right)^2 \frac{dx}{|x|^2} \leq H \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$.

(b) If $g = \frac{|x|^2}{|\Omega|}$, then (2.1) takes the form: there exists a constant $H > 0$ such that

$$\int_{\Omega} \left(u - \frac{1}{|\Omega|} \int_{\Omega} u dx \right)^2 \frac{dx}{|x|^2} \leq H \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$.

(c) If $g = |x|^2 h$ with $h \in L^2(\Omega)$ and $\int_{\Omega} h dx = 1$, then (2.1) takes the form: there exists a constant $H > 0$ such that

$$\int_{\Omega} \left(u - \int_{\Omega} u h dx \right)^2 \frac{dx}{|x|^2} \leq H \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$.

From the case (c) we derive the following inequality

$$\frac{1}{d^2} \int_{\Omega} \left(u - \int_{\Omega} u h dx \right)^2 dx \leq \int_{\Omega} \left(u - \int_{\Omega} u h dx \right)^2 \frac{dx}{|x|^2} \leq H \int_{\Omega} |\nabla u|^2 dx,$$

where $d = \text{diam } \Omega$.

If we restrict (2.1) to the subspace of $W^{1,2}(\Omega)$ consisting of functions orthogonal to g in $L^2(\Omega, \frac{1}{|x|^2} dx)$, then (2.1) is reduced to the following inequality: given $g \in L^2(\Omega, \frac{1}{|x|^2})$ with $\int_{\Omega} \frac{g}{|x|^2} dx = 1$, then there exists a constant $H > 0$ such that

$$\int_{\Omega} \frac{u^2}{|x|^2} dx \leq H \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$ with $(u, g)_* = 0$. This observation can be applied to the cases (a), (b) and (c) of Corollary 2.2. In particular, from the case (a) we obtain the following inequality: there exists a constant $H > 0$ such that

$$(2.2) \quad \int_{\Omega} \frac{u^2}{|x|^2} dx \leq H \int_{\Omega} |\nabla u|^2 dx$$

for every $u \in W^{1,2}(\Omega)$ with $\int_{\Omega} \frac{u}{|x|^2} dx = 0$. The reciprocal of the smallest constant H satisfying inequality (2.2) is the second eigenvalue of the eigenvalue problem (1.1). The first eigenvalue $\lambda_1 = 0$ and the corresponding eigenfunctions are constant.

Using the the triangle inequality we deduce the following estimate from (2.1): given $g \in L^2(\Omega, \frac{1}{|x|^2})$ with $\int_{\Omega} \frac{g}{|x|^2} dx = 1$, there exists a constant $H > 0$ such that

$$\left(\int_{\Omega} \frac{u^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq H^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} \frac{dx}{|x|^2} \right)^{\frac{1}{2}} \left| \int_{\Omega} \frac{ug}{|x|^2} dx \right|$$

for every $u \in W^{1,2}(\Omega)$. In the case (a) of Corollary 2.2 this inequality takes the form: there exists a constant $H > 0$ such that

$$(2.3) \quad \left(\int_{\Omega} \frac{u^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq H^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} \frac{dx}{|x|^2} \right)^{-\frac{1}{2}} \left| \int_{\Omega} \frac{u}{|x|^2} dx \right|$$

for every $u \in W^{1,2}(\Omega)$. Obviously, similar inequalities can be established in the cases (b) and (c).

3 The second eigenvalue

The second eigenvalue λ_2^H of the eigenvalue problem (1.1) is defined by

$$\lambda_2^H = \inf_{u \in W^{1,2}(\Omega) - \{0\}, \int_{\Omega} \frac{u}{|x|^2} dx = 0} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}.$$

Theorem 3.1 *Suppose that*

$$(3.1) \quad \lambda_2^H < \Lambda_N.$$

Then there exists an eigenfunction φ_2 corresponding to the eigenvalue λ_2^H .

Proof Let $\{u_n\}$ be a minimizing sequence for λ_2^H : $\int_{\Omega} |\nabla u_n|^2 dx \rightarrow \lambda_2^H$ with $\int_{\Omega} \frac{u_n}{|x|^2} dx = 0$ and $\int_{\Omega} \frac{u_n^2}{|x|^2} dx = 1$ for each n . We may assume that $u_n \rightharpoonup u$ in $W^{1,2}(\Omega)$ and $L^2(\Omega, \frac{dx}{|x|^2})$ and $u_n \rightarrow u$ a.e. on Ω . By the P.L. Lions' concentration - compactness principle [23] there exist constants $\mu_{\circ}, \nu_{\circ} \geq 0$ such that

$$(3.2) \quad |\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \mu_{\circ} \delta_{\circ},$$

$$(3.3) \quad \frac{u_n^2}{|x|^2} \rightharpoonup \nu = \frac{u^2}{|x|^2} + \nu_{\circ} \delta_{\circ}$$

in the sense of measures and

$$(3.4) \quad \nu_{\circ} \Lambda_N \leq \mu_{\circ},$$

where δ_\circ is the Dirac measure assigned to 0. First, we show that $u \not\equiv 0$. Arguing by contradiction assume that $u \equiv 0$ on Ω . Then $\nu_\circ = 1$ and $\Lambda_N \leq \mu_\circ$. From (3.2) we derive that

$$\Lambda_N \leq \mu_\circ \leq \lambda_2^H < \Lambda_N$$

which is impossible. If $\nu_\circ = 0$ we are done. So it remains to consider the case $0 < \nu_\circ$. In this case we have

$$\begin{aligned} \lambda_2^H &\geq \int_\Omega |\nabla u|^2 dx + \mu_\circ \geq \int_\Omega |\nabla u|^2 dx + \nu_\circ \Lambda_N \\ &\geq \int_\Omega |\nabla u|^2 dx + \Lambda_N \left(1 - \int_\Omega \frac{u^2}{|x|^2} dx\right). \end{aligned}$$

This yields

$$(3.5) \quad \lambda_2^H - \Lambda_N \geq \int_\Omega |\nabla u|^2 dx - \Lambda_N \int_\Omega \frac{u^2}{|x|^2} dx.$$

Since $\int_\Omega \frac{u}{|x|^2} dx = 0$, u cannot be a constant function. Hence $\int_\Omega |\nabla u|^2 dx > 0$. From (3.5) we deduce that

$$\Lambda_N + \frac{\lambda_2^H - \Lambda_N}{\int_\Omega \frac{u^2}{|x|^2} dx} \geq \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx} \geq \lambda_2^H.$$

Hence

$$\Lambda_N - \lambda_2^H \geq \frac{\Lambda_N - \lambda_2^H}{\int_\Omega \frac{u^2}{|x|^2} dx}.$$

This implies that $\int_\Omega \frac{u^2}{|x|^2} dx \geq 1$ which is impossible because $\int_\Omega \frac{u^2}{|x|^2} dx < 1$. To complete the proof we show that u is a weak solution to problem (1.2). Let $\phi \in W^{1,2}(\Omega)$ with $\int_\Omega \frac{\phi}{|x|^2} dx = 0$. We put

$$h(t) = \frac{\int_\Omega |\nabla(u + t\phi)|^2 dx}{\int_\Omega \frac{(u+t\phi)^2}{|x|^2} dx}$$

for $t \in \mathbb{R}$. Since the function h attains a minimum at $t = 0$ we get

$$\int_\Omega \nabla u \nabla \phi dx - \lambda_2^H \int_\Omega \frac{u\phi}{|x|^2} dx = 0.$$

To extend this identity for arbitrary $\phi \in W^{1,2}(\Omega)$ we set $\psi = \phi - \frac{\int_\Omega \frac{\phi}{|x|^2} dx}{\int_\Omega \frac{1}{|x|^2} dx}$. Then $\int_\Omega \frac{\psi}{|x|^2} dx = 0$ and

$$\int_\Omega \nabla u \nabla \psi dx = \int_\Omega \nabla u \nabla \phi dx = \lambda_2^H \int_\Omega \frac{u\psi}{|x|^2} dx = \lambda_2^H \int_\Omega \frac{u\phi}{|x|^2} dx$$

since u is orthogonal to 1 in $L^2(\Omega, \frac{dx}{|x|^2})$. □

Remark 3.2 If $0 < \mu < \lambda_2^H$, then problem (1.1) has only the trivial solution in $W^{1,2}(\Omega)$. Indeed, suppose that $u \not\equiv 0$ belonging to $W^{1,2}(\Omega)$ satisfies (1.1). Then $\int_{\Omega} \frac{u}{|x|^2} dx = 0$ and

$$\frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} \frac{v^2}{|x|^2}} = \mu < \lambda_2^H$$

which is impossible.

4 Examples of sets satisfying (3.1)

In this section we give examples of domains satisfying (3.1). These domains have some sort of symmetry or satisfy a pinching condition, and they are rather high dimensional.

1. Let $\Omega = B(0, R)$. First of all we observe that there are two radial solutions of equation (1.1) for $0 < \lambda < \Lambda_N$:

$$u_1 = r^{-\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}} \quad \text{and} \quad u_2 = r^{-\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}}.$$

Since $u_2 \notin W^{1,2}(B(0, R))$ we cannot use these radial functions to construct a solution of (1.1) for $\lambda = \lambda_2^H$. To verify (3.1) one can use the coordinate function $u = x_j$. It is easy to show that $\lambda_2^H \leq N$. Hence (3.1) is satisfied for $N \geq 8$. This can be slightly improved by taking as a test function $v(x) = \frac{x_N}{|x|}$. We then have

$$\begin{aligned} \int_{B(0,R)} |\nabla v(x)|^2 dx &= \int_{B(0,R)} \left(\frac{1}{|x|^2} - \frac{x_N^2}{|x|^4} \right) dx = \left(1 - \frac{1}{N}\right) \int_{B(0,R)} \frac{1}{|x|^2} dx \\ &= \omega_{N-1} R^{N-2} \frac{N-1}{N(N-2)}. \end{aligned}$$

Similarly, we have

$$\int_{B(0,R)} \frac{v(x)^2}{|x|^2} dx = \int_{B(0,R)} \frac{x_N^2}{|x|^4} dx = \frac{1}{N} \int_{B(0,R)} \frac{1}{|x|^2} dx = \omega_{N-1} \frac{R^{N-2}}{N(N-2)}.$$

Hence $\lambda_2^H \leq N - 1$ and (3.1) is satisfied for $N \geq 7$.

2. This observation can be extended to sets which are close to balls, i.e. satisfying a pinching condition. Let Ω be a bounded domain in \mathbb{R}^N . Suppose that there exists a ball $B(0, R) \subset \Omega$ satisfying the following condition

$$(*) \quad |\Omega| < \frac{1}{N} \left(\frac{N-2}{2} \right)^2 |B(0, R)|.$$

Then (3.1) holds. This condition is meaningful if $\frac{1}{N} \left(\frac{N-2}{2} \right)^2 > 1$, that is $N \geq 8$. Obviously we can assume that $B(0, R)$ is largest ball contained in Ω and satisfying (*). We distinguish

two cases: (i) $\int_{\Omega} \frac{x_j}{|x|^2} dx = 0$ for some $1 \leq j \leq N$ and (ii) $\int_{\Omega} \frac{x_j}{|x|^2} dx \neq 0$ for every $j = 1, \dots, N$. In the first case take $u = x_j$; then $\int_{\Omega} |\nabla(x_j)|^2 dx = |\Omega|$ and

$$\int_{\Omega} \frac{x_j^2}{|x|^2} dx \geq \int_{B(0,R)} \frac{x_j^2}{|x|^2} dx = \frac{1}{N} |B(0, R)|.$$

Hence $\lambda_2^H \leq \frac{N|\Omega|}{|B(0,R)|}$. If (ii) holds we put $u(x) = \alpha_k x_j - \alpha_j x_k$, where $\alpha_j = \int_{\Omega} \frac{x_j}{|x|^2} dx$ and $\alpha_k = \int_{\Omega} \frac{x_k}{|x|^2} dx$. It is clear that $\int_{\Omega} \frac{u}{|x|^2} dx = 0$. We also have $\int_{\Omega} |\nabla u|^2 dx = (\alpha_k^2 + \alpha_j^2) |\Omega|$ and

$$\begin{aligned} \int_{\Omega} \frac{u^2}{|x|^2} dx &\geq \int_{B(0,R)} \frac{\alpha_k^2 x_j^2 + \alpha_j^2 x_k^2 - 2\alpha_j \alpha_k x_j x_k}{|x|^2} dx \\ &= \frac{1}{N} (\alpha_k^2 + \alpha_j^2) |B(0, R)|. \end{aligned}$$

Again we obtain the estimate $\lambda_2^H \leq \frac{N|\Omega|}{|B(0,R)|}$. Since (*) holds we get in both cases condition (3.1).

3. We now consider sets having a symmetry with respect to the coordinate x_N . Set $x = (x', x_N)$ and let $\Omega = \{x' \in D; -f(x') < x_N < f(x')\}$, where $f: \bar{D} \rightarrow (0, \infty)$ is a smooth function satisfying

$$(**) \quad \int_D f(x')^3 dx' > \frac{(12)^{\frac{3}{2}} d^3 |D|}{(N-2)^3}.$$

We show that (3.1) is satisfied. Take $u = x_N$; it is clear that $\int_{\Omega} \frac{x_N}{|x|^2} dx = 0$. We have

$$\int_{\Omega} |\nabla(x_N)|^2 dx = \int_D dx' \int_{-f(x')}^{f(x')} dx_N = 2 \int_D f(x') dx'$$

and

$$\int_{\Omega} \frac{x_N^2}{|x|^2} dx \geq \frac{1}{d^2} \int_{\Omega} x_N^2 dx = \frac{2}{3d^2} \int_D f(x')^3 dx'.$$

Hence with the aid of the Hölder inequality and (**) we obtain

$$\lambda_2^H \leq \frac{\int_{\Omega} |\nabla(x_N)|^2 dx'}{\int_{\Omega} \frac{x_N^2}{|x|^2} dx'} \leq \frac{3d^2 \int_D f(x') dx'}{\int_D f(x')^3 dx'} \leq 3d^2 |D|^{\frac{2}{3}} \left(\int_D f(x')^3 dx' \right)^{-\frac{2}{3}} < \Lambda_N.$$

Let $\text{diam } D \leq d$, ($d = \text{diam } \Omega$). Since $\max_{x' \in D} f(x') \leq \frac{d}{2}$ we have

$$\int_D f(x')^3 dx' \leq \left(\frac{d}{2}\right)^3 |D|.$$

Thus condition (**) is meaningful if

$$\left(\frac{d}{2}\right)^3 |D| > \frac{(12)^{\frac{3}{2}} d^3 |D|}{(N-2)^3},$$

that is $N > 2 + 2\sqrt{12}$, ($N \geq 9$).

4. Suppose that $\Omega' \subset \Omega$, with Ω' as in example 3, that is, $\Omega' = \{x' \in D; -f(x') < x_N < f(x')\}$, where $f : \bar{D} \rightarrow (0, \infty)$ is a smooth function. Assume that

$$(***) \quad |\Omega| < \frac{2}{3d_1^2} \left(\frac{N-2}{2}\right)^2 \int_D f(x')^3 dx',$$

where $d_1 = \text{diam } \Omega'$. Then (3.1) holds. To show this we distinguish two cases:

(i) $\int_\Omega \frac{x_N}{|x|^2} dx = 0$ and (ii) $\int_\Omega \frac{x_N}{|x|^2} dx \neq 0$. Assuming (i), we choose $u = x_N$; then $\int_\Omega |\nabla(x_N)|^2 dx = |\Omega|$ and

$$\int_\Omega \frac{x_N^2}{|x|^2} dx \geq \frac{1}{d_1^2} \int_{\Omega'} x_N^2 dx = \frac{2}{3d_1^2} \int_D f(x')^3 dx'.$$

Condition (3.1) follows from (***). In the case (ii) we test λ_2^H with

$$u(x) = x_N - \frac{\int_\Omega \frac{x_N}{|x|^2} dx}{\int_\Omega \frac{dx}{|x|^2}}.$$

It is clear that $\int_\Omega \frac{u}{|x|^2} dx = 0$ and $\int_\Omega |\nabla u|^2 dx = |\Omega|$. Letting $a = \frac{\int_\Omega \frac{x_N}{|x|^2} dx}{\int_\Omega \frac{dx}{|x|^2}}$ we get

$$\begin{aligned} \int_\Omega \frac{u^2}{|x|^2} dx &\geq \int_{\Omega'} \frac{u^2}{|x|^2} dx = \int_{\Omega'} \frac{x_N^2}{|x|^2} dx + a^2 \int_{\Omega'} \frac{dx}{|x|^2} - 2a \int_{\Omega'} \frac{x_N}{|x|^2} dx \\ &= \int_{\Omega'} \frac{x_N^2}{|x|^2} dx + a^2 \int_{\Omega'} \frac{dx}{|x|^2} \\ &\geq \frac{2}{3d_1^2} \int_D f(x')^3 dx'. \end{aligned}$$

The estimate (3.1) follows from (***) .

5. Let $\Omega = \{x; (x_1 - R)^2 + x_2^2 + \dots + x_N^2 \leq R^2\}$. In this example $0 \in \partial\Omega$. We test λ_2^H with $u(x) = x_j$ for some $2 \leq j \leq N$. We have $\int_\Omega \frac{x_j}{|x|^2} dx = 0$, $\int_\Omega |\nabla(x_j)|^2 dx = |\Omega|$ and

$$\int_\Omega \frac{x_j^2}{|x|^2} dx \geq \frac{1}{4R^2} \int_\Omega x_j^2 dx = \frac{1}{4R^2} \int_{B(0,R)} x_j^2 dx = \frac{1}{4NR^2} \int_{B(0,R)} |x|^2 dx = \frac{\omega_{N-1}R^N}{4N(N+2)}.$$

Thus

$$\lambda_2^H \leq \frac{4N(N+2)|\Omega|}{\omega_{N-1}R^N} = 4(N+2)$$

and (3.1) holds provided $4(N+2) < \left(\frac{N-2}{2}\right)$, that is, $N \geq 22$.

6. We now give an example of a domain satisfying (3.1) which does not have a symmetry with respect to 0. Let Ω be a domain from example 3. We put

$$\Omega_\delta = \{x; x' \in D, -f(x') < x_N < (1 + \delta)f(x')\},$$

where $\delta > 0$. Let $0 < m = \min_{x' \in \bar{D}} f(x')$ and $M = \max_{x' \in \bar{D}} f(x')$. Suppose that $\text{diam } D \leq 2M$ and that

$$(\ast \ast \ast) \quad \frac{m^3}{6M^3} \left(\frac{N-2}{2} \right)^2 > 2 + \delta.$$

(This condition makes sense for $N \geq 9$). Then condition (3.1) holds. First we observe that $\int_{\Omega_\delta} \frac{x_N}{|x|^2} dx = \int_{\Omega_\delta - \Omega} \frac{x_N}{|x|^2} dx > 0$. So we introduce a function

$$v(x) = x_N - \alpha_\delta, \quad \alpha_\delta = \frac{\int_{\Omega_\delta} \frac{x_N}{|x|^2} dx}{\int_{\Omega_\delta} \frac{dx}{|x|^2}}.$$

It is clear that $\int_{\Omega_\delta} \frac{v(x)}{|x|^2} dx = 0$ and

$$\int_{\Omega_\delta} |\nabla v|^2 dx = |\Omega_\delta| = (2 + \delta) \int_D f(x') dx'.$$

Letting $d = \text{diam } \Omega$ we get

$$\begin{aligned} \int_{\Omega_\delta} \frac{v^2}{|x|^2} dx &\geq \int_{\Omega} \frac{v^2}{|x|^2} dx = \int_{\Omega} \frac{x_N^2 + \alpha_\delta^2 - 2\alpha_\delta x_N}{|x|^2} dx = \int_{\Omega} \frac{x_N^2 + \alpha_\delta^2}{|x|^2} dx \\ &\geq \frac{2}{3d^2} \int_D f(x')^3 dx'. \end{aligned}$$

Since $d = 2M$ we obtain the following estimate for λ_2^H

$$\begin{aligned} \lambda_2^H &\leq \frac{(2 + \delta) \int_D f(x') dx'}{\frac{2}{3d^2} \int_D f(x')^3 dx'} \\ &\leq \frac{3(2 + \delta)}{2} (2M)^2 \frac{M}{m^3} = 6(2 + \delta) \frac{M^3}{m^3}. \end{aligned}$$

Condition (3.1) follows from (****).

According to example 1 we have $\lambda_2^H = O(N - 1)$, as $N \rightarrow \infty$, if $\Omega = B(0, R) \subset \mathbb{R}^N$. Obviously it is not clear whether this asymptotic behaviour is optimal. Below we give an example of a domain for which λ_2^H remains bounded as $N \rightarrow \infty$.

7. Let $\Omega = \{x' \in B(0, 1); -f(x') < x_N < f(x')\}$, where $f : B(0, 1) \rightarrow [1, \infty)$ is a smooth function. We test λ_2^H with $v(x) = x_N|x|$. It is clear that $\int_{\Omega} \frac{v}{|x|^2} dx = 0$. Moreover, we have

$$\int_{\Omega} |\nabla(x_N|x|)|^2 dx = \int_{\Omega} (4x_N^2 + |x'|^2) dx = \frac{8}{3} \int_{B(0,1)} f(x')^3 dx' + 2 \int_{B(0,1)} |x'|^2 f(x') dx'$$

and

$$\int_{\Omega} \frac{v^2}{|x|^2} dx = \int_{\Omega} x_N^2 dx = \frac{2}{3} \int_{B(0,1)} f(x')^3 dx'.$$

Hence

$$\lambda_2^H \leq \frac{2 \int_{B(0,1)} |x'|^2 f(x') dx' + \frac{8}{3} \int_{B(0,1)} f(x')^3 dx'}{\frac{2}{3} \int_{B(0,1)} f(x')^3 dx'}$$

Since $f(x') \geq 1$ on $B(0, 1)$, we derive from this estimate that $\lambda_2^H \leq 7$ for every N . Condition (3.1) is satisfied for $N \geq 10$. In the Appendix we give a slightly better estimate of λ_2^H .

5 Asymptotic behaviour of the second eigenfunction around 0

To examine the behaviour around 0 of the second eigenfunction of (1.1) we use some ideas from [15], [18]. We denote by $S = S(\Omega)$ the best Sobolev constant for the embedding of $W^{1,2}(\Omega)$ into $L^{2^*}(\Omega)$, that is,

$$S = \inf_{u \in W^{1,2}(\Omega) - \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + u^2) dx}{\left(\int_{\Omega} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}.$$

We only consider the case $0 \in \Omega$.

Proposition 5.1 *Suppose that $\lambda_2^H < \Lambda_N$. Then there exists $\delta_0 \in (0, 1)$ such that the eigenfunction φ_2 (constructed in Theorem 3.1) belongs to $L^{(1+\delta_0)2^*}(\Omega)$.*

Proof To simplify notation we set $\lambda = \lambda_2^H$ and $u = \varphi_2$. Let $\phi = u \min(|u|, L)^{p-2} = uu_L^{p-2}$, where $p > 2$, $L > 0$. Testing equation (1.1) we obtain

$$(5.1) \quad \int_{\Omega} |\nabla u|^2 u_L^{p-2} dx + (p-2) \int_{\Omega} \nabla u \nabla u_L u_L^{p-2} dx = \lambda \int_{\Omega} \frac{u^2 u_L^{p-2}}{|x|^2} dx.$$

On the other hand we have

$$\begin{aligned} \int_{\Omega} |\nabla (uu_L^{\frac{p}{2}-1})|^2 dx &= \int_{\Omega} |\nabla u|^2 u_L^{p-2} dx + \frac{(p-2)^2}{4} \int_{\Omega} |\nabla u_L|^2 u_L^{p-2} dx \\ &+ (p-2) \int_{\Omega} \nabla u \nabla u_L u_L^{p-2} dx \\ &= \int_{\Omega} |\nabla u|^2 u_L^{p-2} dx + \frac{p^2-4}{4} \int_{\Omega} |\nabla u_L|^2 u_L^{p-2} dx. \end{aligned}$$

It then follows from (5.1) that

$$\int_{\Omega} |\nabla u|^2 u_L^{p-2} dx + \frac{p^2-4}{4} \int_{\Omega} \nabla u \nabla u_L u_L^{p-2} dx \leq \frac{\lambda(p+2)}{4} \int_{\Omega} \frac{u^2 u_L^{p-2}}{|x|^2} dx.$$

Hence

$$(5.2) \quad \int_{\Omega} |\nabla (uu_L^{\frac{p}{2}-1})|^2 dx \leq \frac{\lambda(p+2)}{4} \int_{\Omega} \frac{u^2 u_L^{p-2}}{|x|^2} dx.$$

We now apply inequality (1.3) to $uu_L^{\frac{p}{2}-1}$. For a given $\epsilon > 0$ we can find a constant $A = A(\epsilon, \Omega) > 0$ such that

$$(5.3) \quad \int_{\Omega} |\nabla (uu_L^{\frac{p}{2}-1})|^2 dx \leq \frac{\lambda(p+2)}{4} \left(\frac{1}{\Lambda_N} + \epsilon \right) \int_{\Omega} |\nabla (uu_L^{\frac{p}{2}-1})|^2 dx + \frac{\lambda A(p+2)}{4} \int_{\Omega} u^2 u_L^{p-2} dx.$$

We now take $p = 2 + \delta$ with $0 < \delta < 1$ chosen so that

$$\frac{\lambda}{\Lambda_N} \left(1 + \frac{\delta}{4}\right) < 1.$$

Next we choose ϵ small enough so that

$$\frac{\lambda(p+2)}{4} \left(\frac{1}{\Lambda_N} + \epsilon\right) = \left(1 + \frac{\delta}{4}\right) \left(\frac{\lambda}{\Lambda_N} + \lambda\epsilon\right) < 1.$$

With these choices of δ and ϵ we derive from (5.3) that

$$S \left(1 - \frac{\lambda(p+2)}{4} \left(\frac{1}{\Lambda_N} + \epsilon\right)\right) \left(\int_{\Omega} |(uu_L^{\frac{p}{2}-1})|^{2^*} dx\right)^{\frac{2}{2^*}} \leq \left(\frac{A\lambda(p+2)}{4} + 1\right) \int_{\Omega} u^2 u_L^{p-2} dx.$$

Letting $L \rightarrow \infty$ the result follows with $\delta_o = \frac{\delta}{2}$. \square

Theorem 5.2 *Suppose that $0 \in \Omega$ and $\lambda_2^H < \lambda_N$. Then there exists a ball $B(0, \rho_o) \subset \Omega$ such that*

$$|\varphi_2(x)| \leq C|x|^{-(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} \text{ for } x \in B(0, \rho_o) - \{0\},$$

where $C > 0$ is a constant and φ_2 is an eigenfunction from Theorem 3.1.

Proof We use again the notation $\lambda = \lambda_2^H$ and we set $u = \varphi_2$. We put

$$v(x) = |x|^{(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} u(x).$$

It is clear that $v \in W^{1,2}(\Omega, |x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} dx)$. By straightforward calculations we check that

$$(5.4) \quad \operatorname{div} (|x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} \nabla v) = 0 \text{ in } \Omega.$$

Let $0 < r < \rho$ and $B(0, \rho) \subset \Omega$. We put $\phi = \eta^2 v v_l^{2(s-1)}$, where $l, s > 1$, $v_l = \min(|v|, l)$ and η is a C^1 -function such that $\eta = 1$ on $B(0, r)$, $\eta = 0$ on $\Omega - B(0, \rho)$ and $|\nabla \eta| \leq \frac{4}{\rho-r}$ on Ω . Testing (5.4) with ϕ we obtain

$$(5.5) \quad \int_{\Omega} |x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} (2\eta v v_l^{2(s-1)} \nabla \eta \nabla v + \eta^2 v_l^{2(s-1)} |\nabla v|^2 + 2(s-1)\eta^2 v_l^{2(s-1)} |\nabla v_l|^2) dx = 0.$$

For every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\begin{aligned} 2 \int_{\Omega} |x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} \eta v v_l^{2(s-1)} \nabla \eta \nabla v dx &\leq \epsilon \int_{\Omega} |x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} \eta^2 v_l^{2(s-1)} |\nabla v|^2 dx \\ &+ C(\epsilon) \int_{\Omega} |x|^{-2(\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx. \end{aligned}$$

Taking $\epsilon = \frac{1}{2}$ we derive from (5.5) that

$$(5.6) \quad \int_{\Omega} |x|^{-2} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) (\eta^2 v_l^{2(s-1)} |\nabla v|^2 + 2(s-1) \eta^2 v_l^{2(s-1)} |\nabla v_l|^2) dx \\ \leq C \int_{\Omega} |x|^{-2} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx.$$

In the next step we use the Caffarelli - Kohn - Nirenberg inequality [12]:

$$(5.7) \quad \left(\int_{\Omega} |x|^{-bp} |w|^p dx \right)^{\frac{2}{p}} \leq C_{a,b} \int_{\Omega} |x|^{-2a} |\nabla w|^2 dx$$

for every $w \in W_{\circ}^{1,2}(\Omega, |x|^{-2a} dx)$, where $-\infty < a < \frac{N-2}{2}$, $a \leq b \leq a+1$, $p = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b} > 0$ is a constant depending on a and b . We choose $a = b = \sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H} < \frac{N-2}{2}$. In this case we have $p = 2^*$. We then deduce from (5.6) and (5.7) with $w = \eta v v_l^{s-1}$ that

$$(5.8) \quad \left(\int_{\Omega} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\eta v v_l^{s-1}|^{2^*} dx \right)^{\frac{2}{2^*}} \\ \leq C_{a,a} \int_{\Omega} |x|^{-2} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\nabla(\eta v v_l^{s-1})|^2 dx \\ \leq 2C_{a,a} \int_{\Omega} |x|^{-2} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) (|\nabla \eta|^2 v^2 v_l^{2(s-1)} + \eta^2 v_l^{2(s-1)} |\nabla v|^2 \\ + (s-1)^2 \eta^2 v_l^{2(s-1)} |\nabla v_l|^2) dx \\ \leq Cs \int_{\Omega} |x|^{-2} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx.$$

Since

$$\int_{\Omega} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\eta|^{2^*} v^2 v_l^{2^* s-2} dx \leq \int_{\Omega} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\eta v v_l^{s-1}|^{2^*} dx$$

we can rewrite (5.8) as

$$\left(\int_{\Omega} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\eta|^{2^*} v^2 v_l^{2^* s-2} dx \right)^{\frac{2}{2^*}} \leq Cs \int_{\Omega} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) |\nabla \eta|^2 v^2 v_l^{2(s-1)} dx.$$

Due to the properties of the function η the above inequality becomes

$$(5.9) \quad \left(\int_{B(0,r)} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) v^2 v_l^{2^* s-2} dx \right)^{\frac{2}{2^*}} \leq \frac{Cs}{(\rho-r)^2} \int_{B(0,\rho)} |x|^{-2^*} (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H}) v^2 v_l^{2s-2} dx.$$

We now choose $\frac{N}{N-2} < s^* < (1 + \delta_{\circ}) \frac{N}{N-2}$ and define the sequence

$$s_j = s^* \left(\frac{2^*}{2} \right)^j, \quad j = 0, 1, 2, \dots$$

Letting $s = s_j$ in (5.9) we obtain

$$\begin{aligned} & \left(\int_{B(0,r)} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2^* s_{j+1} - 2} dx \right)^{\frac{1}{2s_{j+1}}} \\ & \leq \left(\frac{C s_j}{(\rho - r)^2} \right)^{\frac{1}{2s_j}} \left(\int_{B(0,\rho)} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2s_j - 2} dx \right)^{\frac{1}{2s_j}}. \end{aligned}$$

We put $r_j = \rho_o(1 + \rho_o^j)$, $j = 0, 1, 2, \dots$ with $\rho_o > 0$ small so that $\overline{B(0, 2\rho_o)} \subset \Omega$. Substituting in the above inequality $\rho = r_j$ and $r = r_{j+1}$ we obtain

$$(5.10) \quad \begin{aligned} & \left(\int_{B(0,r_{j+1})} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2^* s_{j+1} - 2} dx \right)^{\frac{1}{2s_{j+1}}} \\ & \leq \left(\frac{C s_j}{(\rho_o - \rho_o^2)^2 \rho_o^{2j}} \right)^{\frac{1}{2s_j}} \left(\int_{B(0,r_j)} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2s_j - 2} dx \right)^{\frac{1}{2s_j}}. \end{aligned}$$

Iterating we obtain

$$(5.11) \quad \begin{aligned} & \left(\int_{B(0,r_{j+1})} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2^* s_{j+1} - 2} dx \right)^{\frac{1}{2s_{j+1}}} \\ & \leq \left(\frac{C}{\rho_o - \rho_o^2} \right)^{\sum_{j=0}^{\infty} \frac{1}{s_j} - \sum_{j=0}^{\infty} \frac{1}{s_j}} \prod_{j=0}^{\infty} s_j^{\frac{1}{2s_j}} \left(\int_{B(0,r_0)} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2s^* - 2} dx \right)^{\frac{1}{2s^*}}. \end{aligned}$$

We now notice that the infinite sums and the infinite product in the above inequality are finite. Since $2^* < 2s^* < (1 + \delta_o)2^*$ we get that

$$(5.12) \quad \begin{aligned} & \int_{B(0,r_o)} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v^{2s^* - 2} dx \\ & \leq \int_{B(0,r_o)} |x|^{(2s^* - 2^*) (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} |u|^{2s^*} dx \\ & \leq d^{(2s^* - 2^*) (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} \int_{B(0,r_o)} |u|^{2s^*} dx < \infty. \end{aligned}$$

We now deduce from (5.11) and (5.12) that

$$\begin{aligned} \|v_l\|_{L^{2s_{j+1}}(B(0,r_o))} & \leq \|v_l\|_{L^{2s_{j+1}}(B(0,r_{j+1}))} \\ & \leq d^{\frac{2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})}{2s_{j+1}}} \left(\int_{B(0,r_{j+1})} |x|^{-2^* (\sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_2^H})} v^2 v_l^{2s_{j+1} - 2} dx \right)^{\frac{1}{2s_{j+1}}}. \end{aligned}$$

Letting $l \rightarrow \infty$ and $s_j \rightarrow \infty$ the result easily follows. \square

6 The Robin boundary conditions

The approach from Section 3 to the eigenvalue problem (1.1) can be extended to the eigenvalue problem with Robin boundary conditions

$$(6.1) \quad \begin{cases} -\Delta u & = \mu \frac{1}{|x|^2} u \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u & = 0 \text{ on } \partial\Omega, \end{cases}$$

where β is a continuous nonnegative function on $\partial\Omega$ with $\beta \not\equiv 0$ on $\partial\Omega$. We assume that $0 \in \Omega$. However, we consider here the smallest eigenfunction

$$(6.2) \quad \lambda_1^R = \inf_{u \in W^{1,2}(\Omega) - \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} \beta u^2 ds}{\int_{\Omega} \frac{u^2}{|x|^2} dx}.$$

First we show that $\lambda_1^R > 0$. Arguing by contradiction assume $\lambda_1^R = 0$. Then there exists a sequence $\{u_n\} \subset W^{1,2}(\Omega)$ such that $\int_{\Omega} |\nabla u_n|^2 dx + \int_{\partial\Omega} \beta u_n^2 ds \rightarrow 0$ as $n \rightarrow \infty$ and $\int_{\Omega} \frac{u_n^2}{|x|^2} dx = 1$ for every n . Since $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$ we may assume that $u_n \rightharpoonup u$ in $W^{1,2}(\Omega)$ and $u_n \rightarrow u$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$. We now observe that $\int_{\Omega} |\nabla u_n|^2 dx \rightarrow 0$, so $u = l$ (constant). Since $\beta \not\equiv 0$, $l = 0$. Thus $u_n \rightarrow 0$ in $W^{1,2}(\Omega)$. This contradicts the fact that $\int_{\Omega} \frac{u_n^2}{|x|^2} dx = 1$ for every n . Hence $\lambda_1^R > 0$. Obviously, we have

$$\lambda_1^R \leq \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx} = \Lambda_N.$$

We now recall the following inequality: there exists a constant $C = C(\Omega) > 0$ such that

$$(6.3) \quad \Lambda_N \int_{\Omega} \frac{u^2}{|x|^2} dx \leq \int_{\Omega} |\nabla u|^2 dx + C \int_{\partial\Omega} u^2 ds$$

for every $u \in W^{1,2}(\Omega)$ (see Theorem 1.1 (inequality (1.4)) in [7]). Actually, this is a simplified version of inequality (1.4) in [7], which contains an extra positive term on the left hand side. Let C be a constant from inequality (6.3). If $C \leq \beta(s)$ on $\partial\Omega$, then $\lambda_1^R \geq \Lambda_N$. Thus if $C \leq \beta(s)$ on $\partial\Omega$ then $\lambda_1^R = \Lambda_N$.

Theorem 6.1 *Suppose that*

$$(6.4) \quad \lambda_1^R < \Lambda_N.$$

Then problem (6.2) admits a minimizer.

The proof is a minor modification of the proof of Theorem 3.1 and is omitted.

Using as a test function $u(x) \equiv 1$ on Ω we derive the following condition guaranteeing the validity of (6.4): if

$$(6.5) \quad \int_{\partial\Omega} \beta(s) ds < \Lambda_N \int_{\Omega} \frac{dx}{|x|^2}$$

then (6.4) holds. If $\Omega = B(0, R)$, and $B = \max_{s \in \partial\Omega} \beta(s)$, then

$$\frac{\int_{\partial\Omega} \beta(s) ds}{\int_{\Omega} \frac{dx}{|x|^2}} \leq BR(N-2)$$

and (6.5) becomes

$$R < \frac{N-2}{4B}.$$

Proposition 6.2 *Suppose that (6.4) holds. Then the first eigenvalue λ_1^R is simple and the corresponding eigenfunction does not change sign on $\Omega - \{0\}$.*

Proof If a function $u \in W^{1,2}(\Omega)$ is a minimizer for λ_1^R , then $|u|$ is also a minimizer for λ_1^R . Therefore we can assume that u is nonnegative on $\Omega - \{0\}$. The fact that $u > 0$ on $\Omega - \{0\}$ follows from the Harnack inequality (see Theorem 8.20 in [17]). To prove that λ_1^R is simple we follow some ideas from [9] and [24]. We use the Picone identity. Let

$$L(u, v) = |\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2\frac{u}{v} \nabla u \nabla v$$

and

$$R(u, v) = |\nabla u|^2 - \nabla v \nabla \left(\frac{u^2}{v} \right).$$

If $u, v \in C^1(\Omega - \{0\}) \cap W^{1,2}(\Omega)$, with $u \geq 0$, $v > 0$, then $R(u, v) = L(u, v) \geq 0$. Moreover, $R(u, v) = L(u, v) = 0$ if and only if $u = kv$ for some $k \in \mathbb{R}$. Suppose that ϕ_1 and v_1 are positive eigenfunctions corresponding to λ_1^R . We choose a sequence $\psi_n \in C(\Omega) \cap W^{1,2}(\Omega)$ such that $\psi_n \rightarrow \phi_1$ in $W^{1,2}(\Omega)$. We then have

$$\begin{aligned} (6.6) \quad 0 &= \int_{\Omega} |\nabla \phi_1|^2 dx + \int_{\partial\Omega} \beta \phi_1^2 ds - \lambda_1^R \int_{\Omega} \frac{\phi_1^2}{|x|^2} dx \\ &= \lim_{n \rightarrow \infty} \left[\int_{\Omega} |\nabla \psi_n|^2 dx + \int_{\partial\Omega} \beta \psi_n^2 ds - \lambda_1^R \int_{\Omega} \frac{\psi_n^2}{|x|^2} dx \right]. \end{aligned}$$

Testing the equation for v_1 with $v = \frac{\psi_n^2}{v_1 + \frac{1}{n}}$ we get

$$- \int_{\partial\Omega} \beta \psi_n^2 \frac{v_1}{(v_1 + \frac{1}{n})} ds + \lambda_1^R \int_{\Omega} \frac{v_1}{(v_1 + \frac{1}{n})} \frac{\psi_n^2}{|x|^2} dx = \int_{\Omega} \nabla v_1 \nabla \left(\frac{\psi_n^2}{v_1 + \frac{1}{n}} \right) dx.$$

Combining this with (6.6) we derive

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} \left(|\nabla \psi_n|^2 - \nabla v_1 \nabla \left(\frac{\psi_n^2}{v_1 + \frac{1}{n}} \right) \right) dx = \lim_{n \rightarrow \infty} \int_{\Omega} L(\psi_n, v_1) dx = \int_{\Omega} L(\phi_1, v_1) dx.$$

Thus $\phi_1 = cv_1$ for some $c \in \mathbb{R}$. □

The asymptotic estimates from Theorem 5.2 continue to hold for the first eigenfunction of problem (6.1).

Proposition 6.3 *Suppose that (6.4) holds and let v be a minimizer of problem (6.2). Then there exist positive constants C_1 , C_2 and ρ_o such that $B(0, \rho_o) \subset \Omega$ and*

$$C_1|x|^{-\left(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^R}\right)} \leq v(x) \leq C_2|x|^{-\left(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^R}\right)}$$

for every $x \in B(0, \rho_o) - \{0\}$.

Proof First we observe that Proposition 5.1 remains true for v . Hence we can repeat the proof of Theorem 5.2 to obtain the upper bound. The lower bound follows from Proposition 2.2 in [15]. \square .

We now consider the Robin problem (6.1) with a non-positive coefficient β with $\beta \not\equiv 0$. We rewrite this problem as

$$(6.7) \quad \begin{cases} -\Delta u & = \mu \frac{1}{|x|^2} u \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} - \beta u & = 0 \text{ on } \partial\Omega, \end{cases}$$

where β is a continuous non-negative function on $\partial\Omega$ with $\beta \not\equiv 0$. We put

$$(6.8) \quad \lambda_1^{-R} = \inf_{u \in W^{1,2}(\Omega) - \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} \beta u^2 ds}{\int_{\Omega} \frac{u^2}{|x|^2} dx}.$$

Theorem 6.4 *Suppose that β is a continuous non-negative function on $\partial\Omega$ with $\beta \not\equiv 0$. Then problem (6.8) admits a minimizer.*

Proof We commence by showing that $-\infty < \lambda_1^{-R} < 0$. Testing λ_1^{-R} with $u \equiv 1$ we get $\lambda_1^{-R} < 0$. To show that λ_1^{-R} is finite we recall the following version of the trace inequality in $W^{1,2}(\Omega)$: for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$(6.9) \quad \int_{\partial\Omega} u^2 ds \leq \epsilon \int_{\Omega} |\nabla u|^2 dx + C(\epsilon) \int_{\Omega} u^2 dx$$

for every $u \in W^{1,2}(\Omega)$. We choose $\epsilon > 0$ so that $\epsilon \|\beta\|_{\infty} < 1$. With this choice of ϵ we obtain the following estimate

$$\int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} \beta u^2 ds \geq (1 - \epsilon \|\beta\|_{\infty}) \int_{\Omega} |\nabla u|^2 dx - \|\beta\|_{\infty} C(\epsilon) \int_{\Omega} u^2 dx.$$

We may assume that $\int_{\Omega} \frac{u^2}{|x|^2} dx = 1$. Hence $1 = \int_{\Omega} \frac{u^2}{|x|^2} dx \geq \frac{1}{d^2} \int_{\Omega} u^2 dx$. We then derive the following estimate

$$\int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} \beta u^2 ds \geq -d^2 \|\beta\|_{\infty} C(\epsilon).$$

This argument also shows that every minimizing sequence for λ_1^{-R} must be bounded in $W^{1,2}(\Omega)$. Let $\{u_n\} \subset W^{1,2}(\Omega)$ be a minimizing sequence for λ_1^{-R} such that $\int_{\Omega} \frac{u_n^2}{|x|^2} dx = 1$ for every n . We may assume that $u_n \rightharpoonup u$ in $W^{1,2}(\Omega)$ and $L^2(\Omega, \frac{1}{|x|^2})$ and $u_n \rightarrow u$ in $L^2(\Omega)$

and $L^2(\partial\Omega)$. We claim that $u \not\equiv 0$. In the contrary case by the P.L. Lions concentration - compactness principle we get (see the proof of Theorem 3.1) $0 > \lambda_1^{-R} \geq \mu_o$ which is impossible. To complete the proof we show that $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u|^2 dx = 0$. Arguing by contradiction assume $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u|^2 dx = c > 0$. Since $\{u_n\}$ is a minimizing sequence for λ_1^{-R} we obtain

$$\begin{aligned} \lambda_1^{-R} &= \int_{\Omega} |\nabla u_n|^2 dx - \int_{\partial\Omega} \beta u_n^2 ds + o(1) = c + \int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} \beta u^2 ds + o(1) \\ &\geq c + \lambda_1^{-R} \int_{\Omega} \frac{u^2}{|x|^2} dx + o(1). \end{aligned}$$

Hence

$$c \leq \lambda_1^{-R} \left(1 - \int_{\Omega} \frac{u^2}{|x|^2} dx\right) \leq 0$$

and we have arrived at a contradiction. \square

We close this section with remarks on the Robin problem with the coefficient β changing sign on $\partial\Omega$, that is, we assume $\beta^+ \not\equiv 0$ and $\beta^- \not\equiv 0$ on $\partial\Omega$. We distinguish two cases: (I) $\int_{\partial\Omega} \beta dx \leq 0$ and (II) $\int_{\partial\Omega} \beta ds > 0$. Using the trace inequality (6.9) we can show that λ_1^R is finite.

Case (I) In this case $\lambda_1^R < 0$. This is obvious if $\int_{\Omega} \beta dx < 0$. If $\int_{\partial\Omega} \beta ds = 0$, we test λ_1^R with a function $u(x) = v(x) + t$, where $t > 0$ is a constant and $v \in C^1(\bar{\Omega})$, $v \geq 0$ on Ω , $\int_{\partial\Omega} \beta^- v ds > 0$ and $\text{supp } v \cap \text{supp } \beta^+ = \emptyset$. Thus

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} \beta u^2 ds &= \int_{\Omega} |\nabla v|^2 dx + \int_{\partial\Omega} \beta v^2 ds + 2t \int_{\partial\Omega} \beta v ds + t^2 \int_{\partial\Omega} \beta ds \\ &= \int_{\Omega} |\nabla v|^2 dx - \int_{\partial\Omega} \beta^- v^2 ds - 2t \int_{\partial\Omega} \beta^- v ds < 0 \end{aligned}$$

provided t is sufficiently large. Repeating the proof of Theorem 6.4 we obtain the existence of a minimizer for λ_1^R .

Case (II) In this case

$$\lambda_1^R \leq \frac{\int_{\partial\Omega} \beta ds}{\int_{\Omega} \frac{dx}{|x|^2}}.$$

To obtain the existence of a minimizer for λ_1^R we impose the following condition

$$\int_{\partial\Omega} \beta ds < \Lambda_N \int_{\Omega} \frac{dx}{|x|^2}.$$

Obviously this condition is not needed if it occurs that $\lambda_1^R \leq 0$.

7 Open problems

Listed below are some open problems.

- 1) Do there exist second eigenfunctions in balls with dimensions less than or equal to 6?
- 2) Do there exist other eigenvalues below the Hardy constant Λ_N and possibly above Λ_N ?
- 3) Find conditions for the mixed Dirichlet - Neumann problem guaranteeing that the corresponding minimum of the Rayleigh quotient is strictly less than Λ_N and is attained.

8 Appendix

Let Ω be a domain from example 7 in Section 4. We test λ_2^H with $v(x) = x_N^{\frac{2k+1}{2k+3}}|x|$, where $k \geq 1$ is an integer. We have

$$\nabla v = \left(x_N^{\frac{2k+1}{2k+3}} \frac{x_1}{|x|}, \dots, x_N^{\frac{2k+1}{2k+3}} \frac{x_{N-1}}{|x|}, \frac{2k+1}{2k+3} x_N^{\frac{-2}{2k+3}} |x| + x_N^{\frac{2k+1}{2k+3}} \frac{x_N}{|x|} \right).$$

Thus

$$\begin{aligned} \int_{\Omega} |\nabla v|^2 dx &= \int_{\Omega} \left(x_N^{\frac{4k+2}{2k+3}} + \left(\frac{2k+1}{2k+3} \right)^2 x_N^{\frac{-4}{2k+3}} |x|^2 + \frac{4k+2}{2k+3} x_N^{\frac{4k+2}{2k+3}} \right) dx \\ &= \int_{\Omega} \left(x_N^{\frac{4k+2}{2k+3}} + \left(\frac{2k+1}{2k+3} \right)^2 x_N^{\frac{4k+2}{2k+3}} + \left(\frac{2k+1}{2k+3} \right)^2 x_N^{\frac{-4}{2k+3}} |x'|^2 + \frac{4k+2}{2k+3} x_N^{\frac{4k+2}{2k+3}} \right) dx \\ &= 2 \left(\frac{2k+3}{6k+5} \int_{|x'| \leq 1} f(x')^{\frac{6k+5}{2k+3}} dx' + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{6k+5} \int_{|x'| \leq 1} f(x')^{\frac{6k+5}{2k+3}} dx' \right. \\ &\quad \left. + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{2k-1} \int_{|x'| \leq 1} |x'|^2 f(x')^{\frac{2k-1}{2k+1}} dx' + \frac{4k+2}{6k+5} \int_{|x'| \leq 1} f(x')^{\frac{6k+5}{2k+3}} dx' \right) \\ &\leq 2 \left(\frac{2k+3}{6k+5} + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{6k+5} + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{2k-1} + \frac{4k+2}{6k+5} \right) \\ &\quad \times \int_{|x'| \leq 1} f(x')^{\frac{6k+5}{2k+3}} dx'. \end{aligned}$$

We also have

$$\int_{\Omega} \frac{v^2}{|x|^2} dx \leq \int_{\Omega} x_N^{\frac{4k+2}{2k+3}} dx = 2 \frac{2k+3}{6k+5} \int_{\Omega} f(x')^{\frac{6k+5}{2k+3}} dx'.$$

Since $\int_{\Omega} \frac{v}{|x'|^2} dx = 0$, v is a legitimate function to test λ_2^H and we get

$$\lambda_2^H \leq \left(\frac{2k+3}{6k+5} + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{6k+5} + \left(\frac{2k+1}{2k+3} \right)^2 \frac{2k+3}{2k-1} + \frac{4k+2}{6k+5} \right) \frac{6k+5}{2k+3}.$$

The sequence on the right - hand side of this inequality tends to 7 as $k \rightarrow \infty$. However, the terms of this sequence are strictly less than 6 for $k = 2, 3, 4, 5$. For example for $k = 2$ we have

$$\lambda_2^H \leq 1 + \frac{25}{49} + \frac{25}{49} \cdot \frac{17}{3} + \frac{10}{7} < 5 \frac{122}{147} < 6.$$

In particular, condition (3.1) is satisfied for $N \geq 7$.

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