Corrigenda to “Reducible Veronese surfaces”

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Abstract. We correct the definition and the list of all reducible Veronese surfaces in our previous paper “Reducible Veronese surfaces”, Adv. Geom. 10 (2010), 719–735.

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1 Introduction

In [1] we claimed to give the complete list of reducible Veronese surfaces according to the following definition.

Definition 1. For any positive integer \( n \geq 1 \), we will call reducible Veronese surface any algebraic surface \( X \subset \mathbb{P}^{n+4}(\mathbb{C}) \) such that:

i) \( X \) is a non-degenerate, reduced, reducible surface of pure dimension 2;

ii) \( \deg(X) = n + 3 \) and \( \text{cod}(X) = n + 2 \), so that \( X \) is a minimal degree surface;

iii) \( \dim[\text{Sec}(X)] \leq 4 \), so that it is possible to choose a generic linear space \( \mathcal{L} \) of dimension \( n - 1 \) in \( \mathbb{P}^{n+4} \) such that \( \pi_{\mathcal{L}}(X) \) is isomorphic to \( X \), where \( \pi_{\mathcal{L}} \) is the the rational projection \( \pi_{\mathcal{L}} : \mathbb{P}^{n+4} \to \Lambda \) from \( \mathcal{L} \) to a generic target \( \Lambda \simeq \mathbb{P}^4 \);

iv) \( X \) is connected in codimension 1, i.e. if we drop any finite number (possibly 0) of points \( P_1, \ldots, P_r \) from \( X \) then \( X \setminus \{P_1, \ldots, P_r\} \) is connected;

v) \( X \) is a locally Cohen–Macaulay surface.

Condition iii) deserves particular attention. When \( \dim[\text{Sec}(X)] \leq 4 \), for a generic linear \( (n - 1) \)-dimensional linear space \( \mathcal{L} \) we have that \( \pi_{\mathcal{L}|X} \) is injective. However this condition, obviously necessary, is not sufficient to get that \( \pi_{\mathcal{L}|X} \) is an isomorphism. The condition \( \dim[\text{Sec}(X)] \leq 4 \) is in fact equivalent to have that \( \pi_{\mathcal{L}|X} \) is only a J-embedding.

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according to the definition of Johnson (see [5], 1.2, and Proposition 1.5 of [6], chapter II, p. 37). To have that $X$ is a reducible Veronese surface, i.e. to have that $\pi_{\mathcal{L}|X}$ is an isomorphism, instead of iii) we need to use a different condition:

$$\text{iii') dim}[\text{Sec}(X)] \leq 4 \text{ and dim}\left[\bigcup_{x \in X} \langle T_x(X) \rangle \right] \leq 4,$$

where $T_x(X)$ is the Zariski tangent space to $X$ at $x$ and $\langle V \rangle$ is the linear span of a variety $V$ in a projective space. See [2] for the proof of the equivalence. From now on a reducible Veronese surface will be a surface satisfying conditions i), ii), iii'), iv) and v).

Throughout [1], to get condition iii) for the members of our list, we used the condition on $\text{dim}[\text{Sec}(X)]$ and, independently, the fact that $\pi_{\mathcal{L}|X}$ has to be an isomorphism, see for instance the proof of Lemma 4. As the condition on $\text{dim}[\text{Sec}(X)]$ is necessary for iii'), it follows that to classify reducible Veronese surfaces, according to the above new definition, we have to check the list of [1] and we have to exclude surfaces for which $\text{dim}\left[\bigcup_{x \in X} \langle T_x(X) \rangle \right] \leq 4$ does not hold.

In this note we perform this check and we also fix some mistakes in the proof of Proposition 2 of [1].

### 2 Refining and completing the list

The list in [1] contained three types of surfaces $X$:

- $a_n$) for any integer $n \geq 1$, a suitable union of $n + 3$ planes which sits as a linearly normal scheme in $\mathbb{P}^{n+4}$ (see Definition 2 of [1] for a precise description); these surfaces were introduced in [4].
- b) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface $Q$ in $\mathbb{P}^3$ and two planes $X_1$ and $X_2$ sitting as a linearly normal scheme in $\mathbb{P}^5$; $X_1$ and $X_2$ cut $Q$, respectively, along two lines $L_1$, $L_2$, intersecting at a point $P := X_1 \cap X_2$, and $L_1 = \langle Q \rangle \cap X_1$, $L_2 = \langle Q \cup X_1 \rangle \cap X_2$.
- c) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface $Q$ in $\mathbb{P}^3$ and two planes $X_1$ and $X_2$, sitting as a linearly normal scheme in $\mathbb{P}^5$; $X_1$, $X_2$ and $Q$ intersect pairwise transversally along a unique line $L := Q \cap X_1 \cap X_2$ and $L = \langle Q \rangle \cap X_1 \cap X_2$.

It is easy to see that $\text{dim}\left[\bigcup_{x \in X} \langle T_x(X) \rangle \right] \leq 4$ in both cases $a_n$) and b). In contrast, if we consider points $x \in L$ in case c), the tangent space at $x$ to $X$ is $\langle T_x(Q) \cup X_1 \cup X_2 \rangle \simeq \mathbb{P}^4$ and $\bigcup_{x \in L} \langle T_x(Q) \cup X_1 \cup X_2 \rangle = \mathbb{P}^5$, so that there is no point $\mathcal{L} \in \mathbb{P}^5$ such that $\pi_{\mathcal{L}|X}$ is an isomorphism.

Unfortunately, there exist two other surfaces to check, i.e. two surfaces satisfying conditions i), ii), iii), iv), v) but not considered in [1]. These surfaces sit as linearly normal schemes, respectively, in $\mathbb{P}^5$ and $\mathbb{P}^6$:

- d) $X = S \cup X_1$ where $S$ is a smooth rational cubic scroll in $\mathbb{P}^4$ having a line $L$ as fundamental section and $X_1$ is a plane such that $S \cap X_1 = \langle S \rangle \cap X_1 = L$.
- e) $X = S \cup X_1 \cup X_2$ where $S \cup X_1$ is a surface as in d) and $X_2$ is a plane such that $S \cap X_1 \cap X_2 = \langle S \cup X_1 \rangle \cap X_2 = L$.

Obviously conditions i), ii) and iv) are satisfied. Condition v) is satisfied by arguing as in Lemma 1 of [1]. For a surface $X$ as in d) we have $\text{dim}[\text{Sec}(X)] \leq 4$ by direct cal-
ulation with a computer algebra system or by considering that every line joining generic points of $S$ and $X_1$ is contained in the 4-dimensional quadric cone having $X_1$ as vertex and the smooth conic $\Gamma$ as base, where $\Gamma$ is the smooth conic generating $S$ with $L$. For a surface $X$ as in e) we have $\dim[\text{Sec}(X)] \leq 4$ by looking at every pair of irreducible components of $X$.

A surface $X$ as in d) can also be isomorphically projected in $\mathbb{P}^4$ because one has $\dim[\bigcup_{x \in X} \langle T_x(X) \rangle] \leq 4$. In contrast, if we consider points $x \in L$ in case e), the tangent space at $x$ to $X$ is $\langle T_x(S) \cup X_1 \cup X_2 \rangle \simeq \mathbb{P}^4$ and $\bigcup_{x \in L} \langle T_x(S) \cup X_1 \cup X_2 \rangle$ is a quadric cone in $\mathbb{P}^6$, so that its dimension is 5, hence, for any line $L \in \mathbb{P}^6$, $\pi_L|_X$ cannot be an isomorphism.

Now we prove that there are no other reducible Veronese surfaces up to those above. In Proposition 2 of [1] we claimed that every irreducible component of a reducible Veronese surface $X$ can be only a plane, a smooth quadric in $\mathbb{P}^3$ or a quadric in $\mathbb{P}^3$ having rank 3. With this assumption we get only the surfaces $a_i$, b), c) as it is proved in [1]. However there are other possibilities for the irreducible components of $X$: by Theorem 1 of [1], they are reduced surfaces of minimal degree in their spans, and the classification of such surfaces is quoted in Theorem 0.1 of [3] where “rational normal scroll” for 2-dimensional varieties means: a smooth rational normal scroll or a cone over a smooth rational normal curve. Not all these surfaces were well considered in Proposition 2 of [1], so we have to fill this gap.

Let us consider cones $Y$ over smooth rational normal curves and let $E$ be the vertex of a cone $Y$. The tangent space at $E$ to $Y$, which is $\langle Y \rangle$, cannot have dimension bigger than 4 otherwise condition iii) would be not satisfied, so that $\deg(Y) \leq 3$. If $\deg(Y) = 2$ the other irreducible components of $X$ must be planes (see the final part of the proof of Proposition 2 in [1]) and the union of a rank 3 quadric cone in $\mathbb{P}^3$ and planes can be excluded by arguing as in Case 1) of the proof of Theorem 3 in [1]. It follows that here we have to consider only the case $\deg(Y) = 3$. By contradiction, let us assume that an irreducible component of a reducible Veronese surface $X$ is a degree 3 cone $Y$ as above, having vertex $E$. Let $X_i$ be another component of $X$. To satisfy condition iii) we must have $E \notin X_i$ so that $Y \cap X_i = \langle Y \rangle \cap \langle X_i \rangle$ is a single point $P \in Y$, $P \neq E$, by Corollary 2 of [1]. If $X_i$ is not a plane, the join of $Y$ and $X_i$ has dimension 5, hence $\dim[\text{Sec}(X)] \geq 5$, which is a contradiction. If $X_i$ is a plane, any projection $\pi_L$ of $Y \cup X_i$ in $\mathbb{P}^4$ cannot be an isomorphism because $\pi_L(Y) \cap \pi_L(X_i)$ cannot be a single point.

Now let us consider smooth rational normal scrolls of dimension 2. As no smooth surface can be isomorphically projected in $\mathbb{P}^4$ with the exception of the Veronese surface, we have to consider only smooth rational cubic scrolls $S$ in $\mathbb{P}^4$ (other than smooth quadrics in $\mathbb{P}^3$ examined in [1]). In spite of what we said in the proof of Proposition 2 of [1], p. 126, lines 13–18, also a smooth rational cubic scroll $S$ in $\mathbb{P}^4$ can be an irreducible component of a reducible Veronese surface $X$. The correct part of the proof of Proposition 2 in [1] shows that this is possible only when all other components of $X$ are planes cutting $\langle S \rangle$ and $S$ only along a line $L$ which is its fundamental section. This line escaped the analysis made in [1], where only the fibres of the scroll were considered. All other possibilities, involving planes and quadrics, are considered and correctly excluded in Proposition 2 of [1].
As we have seen, the union of a smooth cubic scroll $S$ in $\mathbb{P}^4$ and one or two planes, cutting $\langle S \rangle$ and $S$ along its fundamental section $L$, gives rise to two surfaces to be checked. No other plane can be admitted by Lemma 3 of [1] and condition iii)'.

In conclusion: the surfaces a), b) and d) can be isomorphically projected in $\mathbb{P}^4$, but not c) and e). This is the complete list of reducible Veronese surfaces with the correct condition iii)' instead of iii).

Remark 1. This note is also a correction of the list of reducible Veronese surfaces quoted in Theorem 1 of [2] and never used in that paper.

References


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