ON THE VANISHING PRIME GRAPH OF SOLVABLE GROUPS

SILVIO DOLFI, EMANUELE PACIFICI, LUCIA SANUS, AND PABLO SPIGA

Dedicated to the memory of Maria Silvia Lucido

Abstract. Let G be a finite group, and $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. We say that an element $g \in G$ is a *vanishing element* of G if there exists χ in $\mathrm{Irr}(G)$ such that $\chi(g) = 0$. In this paper, we consider the set of orders of the vanishing elements of a group G, and we define the *prime graph* on it, which we denote by $\Gamma(G)$. Focusing on the class of solvable groups, we prove that $\Gamma(G)$ has at most two connected components, and we characterize the case when it is disconnected. Moreover, we show that the diameter of $\Gamma(G)$ is at most 4. Examples are given to round the understanding of this matter. Among other things, we prove that the bound on the diameter is best possible, and we construct an infinite family of examples showing that there is no universal upper bound on the size of an independent set of $\Gamma(G)$.

Keywords: finite groups, irreducible characters, prime graph.

Introduction

Given a finite group G, it is possible to recognize several sets of positive integers arising from the group structure of G. Just to mention some of the most relevant and classical instances, one can consider the set $\mathrm{o}(G)$ consisting of the orders of the elements of G, or the set $\mathrm{cs}(G)$ of conjugacy class sizes of G, or the set $\mathrm{cd}(G)$ whose elements are the degrees of the irreducible complex characters of G.

In this context, it appears natural to ask to what extent the group structure of G is reflected and influenced by sets of positive integers as above, and a useful tool in this kind of investigation is the so-called *prime graph*. Given a finite set of positive integers X, the prime graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p, and two distinct vertices p, q are adjacent if and only if there exists an element of X divisible by pq.

The graph $\Pi(\mathrm{o}(G))$, which in this paper we shall denote by $\Pi(G)$, is also known as the *Gruenberg-Kegel graph* of G, and has been extensively studied. Let us focus our attention on the class of finite solvable groups, which will be our environment

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throughout the whole paper. For every group G belonging to this class, it is known that $\Pi(G)$ cannot have independent sets of size greater than 2 (that is, for every choice of three distinct vertices of $\Pi(G)$, there is an edge of $\Pi(G)$ joining two of them. See Proposition 1.5). From this fact the following properties can be derived at once: the number of connected components of $\Pi(G)$ is at most 2, if $\Pi(G)$ is disconnected then each connected component is a complete graph, and the diameter of $\Pi(G)$ (i.e. the maximum distance between two vertices lying in the same connected component, which is set to be 0 if there are no vertices) is at most 3. The latter bound is attained, as shown in [8, Example 3]. Moreover, the solvable groups G such that $\Pi(G)$ is disconnected were classified by Gruenberg and Kegel (see Theorem 1.3).

Our aim in this paper is to analyze, via the prime graph, a particular subset of $\mathrm{o}(G)$, which we denote by $\mathrm{vo}(G)$ and which encodes information coming from the set $\mathrm{Irr}(G)$ of irreducible complex characters of G. Let $\mathrm{Van}(G)$ denote the set of *vanishing elements* of G, that is, $\mathrm{Van}(G) = \{g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}$. We define $\mathrm{vo}(G)$ to be the set $\{o(g) \mid g \in \mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$. In a sense, $\mathrm{vo}(G)$ can be regarded as $\mathrm{o}(G)$ filtered by the irreducible characters of G. Given that, we define the *vanishing prime graph* $\Gamma(G)$ of G as the prime graph $\Pi(\mathrm{vo}(G))$.

As might be expected, $\Gamma(G)$ shares some properties with $\Pi(G)$. One of the main results of this paper, which should be compared with Theorem 1.3, is the following Theorem A. We say that G is a *nearly 2-Frobenius group* if there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G, furthermore G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group (see Definition 1.1 for the definition of a 2-Frobenius group).

Theorem A. *Let G be a finite solvable group. Then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius or a nearly 2-Frobenius group.*

It is worth mentioning that the bound on the number of connected components provided by Theorem A can be attained both by Frobenius groups and by nearly 2-Frobenius groups, as shown respectively in [1, Example 2] and in Example 3.6 of this paper.

In order to obtain the bound on the number of connected components in Theorem A, we cannot use the simple argument which yields the same conclusion for $\Pi(G)$. In fact, quite surprisingly, not only $\Gamma(G)$ can have independent sets of size 3, but we can construct an infinite family of examples showing that there is no upper bound on the size of such independent sets.

Theorem B. *Let k be a positive integer. There exists a finite solvable group G such that $\Gamma(G)$ is connected, and it has an independent set of size k.*

The above result appears to be remarkable, also taking into account that both the prime graphs made on $\mathrm{cs}(G)$ and on $\mathrm{cd}(G)$ cannot have independent sets of size greater than 2 if G is solvable (see [2] and [10]).

As the last main result of this paper, we also prove the following theorem concerning the diameter of $\Gamma(G)$ (note that, by a classical result of Burnside, the diameter of $\Gamma(G)$ is 0 if and only if G is abelian). This result should be compared with Corollary 1.6.

Theorem C. *Let G be a finite solvable group. Then the diameter of $\Gamma(G)$ is at most 4. Moreover, if $\Gamma(G)$ is disconnected, then the diameter of $\Gamma(G)$ is at most 1.*

The bound provided by Theorem C is sharp, as shown by Example 5.2.

To conclude, we mention that the vertex set of $\Gamma(G)$ can be strictly smaller than the one of $\Pi(G)$ (consider for instance the symmetric group S_3). Namely, [3, Theorem A] provides a necessary condition for this to happen: if a prime p is not a vertex of $\Gamma(G)$, then G has a normal Sylow p-subgroup (this holds in fact also for nonsolvable groups). Moreover, if p and q are vertices of $\Gamma(G)$ which are adjacent in $\Pi(G)$, then they are not necessarily adjacent in $\Gamma(G)$ (in other words, $\Gamma(G)$ is not an induced subgraph of $\Pi(G)$). As an example, consider the group $G = S_3 \times D_{10}$, where D_{10} is the dihedral group of order 10: it can be checked that 3 and 5 are vertices of $\Gamma(G)$ which are linked in $\Pi(G)$, but not in $\Gamma(G)$.

Throughout the whole paper, every abstract group will be assumed to be a finite group.

1. Preliminaries

We start by recalling a theorem originally due to Gruenberg and Kegel. To do this, we first recall the definition of 2*-Frobenius group.*

Definition 1.1. A group G is said to be a 2-Frobenius group if there exist two normal subgroups F and L of G with the following properties: L is a Frobenius group with kernel F, and G/F is a Frobenius group with kernel L/F.

Remark 1.2. It is worth noticing that every 2-Frobenius group is solvable. More precisely, assuming the setting of Definition 1.1, the groups L/F and G/L are cyclic. In fact, the group L/F is a Frobenius kernel and it is also isomorphic to a Frobenius complement. Then L/F is nilpotent and all its Sylow subgroups are either cyclic or generalized quaternion groups. If $|L/F|$ is even, then the unique involution of L/F is central in G/F, a contradiction. So, $|L/F|$ is odd and hence L/F is cyclic. The group G/L is isomorphic to a subgroup of $\mathrm{Aut}(L/F)$, hence

G/L is abelian. Now, G/L is isomorphic to a Frobenius complement of G/F, so it is cyclic.

Theorem 1.3. ([11, Theorem A]). *Let G be a solvable group. Then $\Pi(G)$ has at most two connected components. Moreover, if $\Pi(G)$ is disconnected, then G is either a Frobenius or a 2-Frobenius group.*

The following lemma concerning group actions of Frobenius groups will often come in useful.

Lemma 1.4. ([7, 8.3.5]). *Let G be a Frobenius group with kernel F and complement H. Assume that G acts on the abelian group A with $(|A|,|F|) = 1$ and $\mathbf{C}_A(F) = 1$. Then $\mathbf{C}_A(H) \neq 1$.*

As mentioned in the Introduction, the bound on the number of connected components provided by Theorem 1.3 can be easily deduced (together with the other properties listed in Corollary 1.6) from the next proposition, originally appeared as Proposition 1 in [8] (for the sake of completeness, we present a proof here). As customary, given an integer n, we denote by $\pi(n)$ the set of prime numbers dividing n and, given a group G, we write $\pi(G)$ for $\pi(|G|)$. Note that $\pi(G)$ is the vertex set of $\Pi(G)$.

Proposition 1.5. *Let G be a solvable group, and let p, q, r be distinct vertices of $\Pi(G)$. Then at least two among p, q and r are adjacent in $\Pi(G)$.*

Proof. Let G be a counterexample of minimal order. Since G is solvable, the existence of Hall subgroups and the minimality of G imply that $\pi(G) = \{p,q,r\}$. Further, a minimal normal subgroup A of G must be, say, a Sylow p-subgroup of G. Let K be a p-complement of G and observe that the Fitting subgroup $\mathbf{F}(K)$ of K (denote it by F) has prime power order: F is, say, a q-group. Then, if H is a Sylow r-subgroup of K, the group FH is a Frobenius group with kernel F and complement H. So Lemma 1.4 yields $\mathbf{C}_A(H) \neq 1$, whence pr divides the order of some element of G, a contradiction. ∎

Corollary 1.6. *Let G be a solvable group. Then the diameter of $\Pi(G)$ is at most 3. Moreover, if $\Pi(G)$ is disconnected, then the diameter of $\Pi(G)$ is at most 1.*

In Proposition 1.7, as a kind of converse to Theorem 1.3, we gather some information on the connected components of $\Pi(G)$ when G is a solvable Frobenius group, or a 2-Frobenius group.

Proposition 1.7. (a) *Let G be a solvable Frobenius group with kernel F and complement H. The graph $\Pi(G)$ has two connected components, whose vertex sets are $\rho_1 = \pi(F)$ and $\rho_2 = \pi(H)$, and which are both complete graphs.*

(b) *Let G be a 2-Frobenius group, and F, L be as in Definition 1.1. The graph $\Pi(G)$ has two connected components, whose vertex sets are $\rho_1 = \pi(L/F)$ and $\rho_2 = \pi(G/L) \cup \pi(F)$, and which are both complete graphs.*

Proof. By the structure of Frobenius and 2-Frobenius groups, in both cases (a) and (b) every nonidentity element of G is either a ρ_1-element or a ρ_2-element and $\rho_1 \cap \rho_2 = \emptyset$. Hence $\Pi(G)$ is disconnected. By Theorem 1.3 and Corollary 1.6 (together with Remark 1.2), the connected components of $\Pi(G)$ are complete graphs with vertex sets ρ_1, ρ_2. ∎

As shown in Section 2, the Fitting subgroup $\mathbf{F}(G)$ of G plays a prominent role when dealing with vanishing elements. The next proposition, which is a well known fact concerning this subgroup, requires some notation. Let G be a group, and let V be a chief factor of G (that is, there are two normal subgroups $A \geq B$ of G such that $V = A/B$ is a minimal normal subgroup of G/B). In what follows, we denote by $\mathbf{C}_G(V)$ the unique subgroup C of G containing B and such that $C/B = \mathbf{C}_{G/B}(V)$. Also, let M be a normal subgroup of G. If $V = A/B$ is a chief factor of G such that $A \leq M$, then we say that V is a *G-chief factor of M*. Finally, we say that $\{V_1, ..., V_n\}$ is a *set of G-chief factors of M* if $V_i = M_i/M_{i-1}$ where $1 = M_0 \leq M_1 \leq \cdots \leq M_n = M$ is part of a chief series of G passing through M.

Proposition 1.8. *Let G be a solvable group, and $\{V_1, ..., V_n\}$ a set of G-chief factors of $\mathbf{F}(G)$. Then we have $\mathbf{F}(G) = \bigcap_{i=1}^{n} \mathbf{C}_G(V_i)$.*

Proof. Set $C = \bigcap_{i=1}^{n} \mathbf{C}_G(V_i)$, $F = \mathbf{F}(G)$, and note that $F \leq C$ by [5, III, Satz 4.3]. Note also that, given any set $\{W_1, ..., W_m\}$ of G-chief factors of F, by the Jordan-Hölder Theorem we have $m = n$ and $C = \bigcap_{i=1}^{m} \mathbf{C}_G(W_i)$. Assume for a contradiction that $F < C$, and let D/F be a minimal normal subgroup of G/F with $D \leq C$. Since G is solvable, D/F is a p-group for some prime p. Write $L = \mathbf{O}_{p'}(F) \trianglelefteq G$ and let P be a Sylow p-subgroup of D. By assumption, P centralizes the elements of a set of G-chief factors of L and hence, by coprimality, P centralizes L. This yields $P \leq \mathbf{O}_p(D) \leq \mathbf{O}_p(G) \leq F$, a contradiction. ∎

2. Vanishing elements and the Fitting subgroup

We first consider a situation in which a vanishing element of a group G lies in $\mathbf{F}(G)$.

Proposition 2.1. *Let G be a group, and assume that $\mathbf{F}(G)$ contains an element of $\mathrm{Van}(G)$. Then there exists $g \in \mathbf{F}(G) \cap \mathrm{Van}(G)$ such that $\pi(o(g)) = \pi(\mathbf{F}(G))$.*

Proof. Let us set $F = \mathbf{F}(G)$. Denote by x an element in $F \cap \mathrm{Van}(G)$, and by τ the set $\pi(o(x))$. Also, let N and M be respectively the Hall τ-subgroup and the Hall τ'-subgroup of F. We claim that, for every y in M, the element $g = xy$ lies in $\mathrm{Van}(G)$. Given that, it will be enough to choose y in M with $\pi(o(y)) = \pi(M)$.

In fact, let χ be an irreducible character of G such that $\chi(x) = 0$, and let α be an irreducible constituent of χ_N. We get

$$0 = \chi(x) = e \sum_{i=1}^{s} \alpha^{g_i}(x),$$

where $e = \langle \chi_N, \alpha \rangle \neq 0$, and $\{g_1, ..., g_s\}$ is a right transversal for the inertia subgroup $I_G(\alpha)$ in G.

Consider now the irreducible character $\alpha \times 1_M$ of F. It is easy to check that $I_G(\alpha \times 1_M) = I_G(\alpha)$. Let γ be an irreducible character of $I_G(\alpha \times 1_M)$ lying over $\alpha \times 1_M$, and set $\psi = \gamma^G$. Notice that, by the Clifford Correspondence, we get $\psi \in \mathrm{Irr}(G)$. Then, setting $f = \langle \psi_F, \alpha \times 1_M \rangle$, for every y in M we have

$$\psi(xy) = f \sum_{i=1}^{s} (\alpha \times 1_M)^{g_i}(xy) = f \sum_{i=1}^{s} \alpha^{g_i}(x) = 0\,,$$

and xy is in $\mathrm{Van}(G)$, as claimed. ∎

On the other hand, we now focus on elements lying *outside* the Fitting subgroup. We start with an observation that will be repeatedly used.

Remark 2.2. We point out two statements concerning the vanishing elements of G and of the quotients of G.

First, if N is a normal subgroup of G, then any character of G/N can be viewed, by inflation, as a character of G. In particular, if $xN \in \mathrm{Van}(G/N)$, then $xN \subseteq \mathrm{Van}(G)$.

Second, if M, N are normal subgroups of G and if there exists $\psi \in \mathrm{Irr}(N)$ which vanishes on $N \setminus M$, then every element in $N \setminus M$ is a vanishing element of G. In fact, for every n in $N \setminus M$ and for every g in G, clearly we have $n^g \in N \setminus M$: choosing $\chi \in \mathrm{Irr}(G)$ which lies over ψ, we have that $\chi(n)$ is a sum of values of the kind $\psi(n^g)$, which are all 0.

As a consequence of Remark 2.2 we get the following lemma.

Lemma 2.3. *If G is a Frobenius or a 2-Frobenius group, then $G \setminus \mathbf{F}(G) \subseteq \mathrm{Van}(G)$.*

Lemma 2.3 describes one particular situation in which it is easily seen that every element outside $\mathbf{F}(G)$ is a vanishing element of G. In fact, although at the time of this writing it is not known whether this property holds for every solvable group, a great deal of information in this respect is provided by an important result by Isaacs, Navarro and Wolf, which we state next.

Theorem 2.4. ([6, Theorem D]). *Let G be a solvable group. If x is a nonvanishing element of G, then $x\mathbf{F}(G)$ is a 2-element of $G/\mathbf{F}(G)$.*

The main purpose of this section is to prove two lemmas (Lemma 2.10 and Lemma 2.11) concerning the structure of a solvable group G such that $\Gamma(G)$ is disconnected, which will be useful for the proof of Theorem A.

Lemma 2.5. *Let G be a solvable group, and V a chief factor of G. Let $N \trianglelefteq G$ be such that $N\mathbf{C}_G(V)/\mathbf{C}_G(V)$ is abelian. Then $N \setminus \mathbf{C}_G(V) \subseteq \mathrm{Van}(G)$.*

Proof. Let A and B be normal subgroups of G such that $V = A/B$. Replacing G with the quotient group G/B, it is easily seen that we can assume $B = 1$, so that V is a minimal normal subgroup of G and it can be regarded as a simple G-module over a suitable prime field $\mathrm{GF}(p)$. Set $\bar{G} = G/\mathbf{C}_G(V)$, and adopt the bar convention. By Clifford's theorem, V is a semisimple (and faithful) $\bar{N}$-module. This forces $\mathbf{O}_p(\bar{N})$ to be trivial, and therefore, as $\bar{N}$ is abelian, we get $(|\bar{N}|, |V|) = 1$. As is well known, it follows that there exists a regular orbit in the action of $\bar{N}$ on the elements of V. Applying the Glauberman Correspondence ([4, Theorem 18.9]), there exists $\phi \in \mathrm{Irr}(V)$ such that $I_{\bar{N}}(\phi) = 1$, whence $I_N(\phi) = N \cap \mathbf{C}_G(V)$.

Now, let χ be an irreducible character of G lying over ϕ. By the Clifford Correspondence we have $\chi = \psi^G$, where ψ is an irreducible character of $I = I_G(\phi)$. For every g in G, we get

$$I^g \cap N = (I \cap N)^g = I_N(\phi)^g \leq \mathbf{C}_G(V)^g = \mathbf{C}_G(V).$$

As χ vanishes in $G \setminus \bigcup_{g\in G} I^g$, the desired conclusion follows. ■

Lemma 2.6. *Let G be a solvable group, and N a normal subgroup of G. If $N/\mathbf{F}(N)$ is abelian, then $N \setminus \mathbf{F}(N) \subseteq \mathrm{Van}(G)$.*

Proof. Let $\{V_1, V_2, \ldots, V_n\}$ be a set of G-chief factors of $\mathbf{F}(N)$, and $C = \bigcap_{i=1}^n \mathbf{C}_N(V_i)$. Clearly C centralizes a set of N-chief factors of $\mathbf{F}(N)$. By Proposition 1.8, then $C \leq \mathbf{F}(N)$. Further, $\mathbf{F}(N) \leq \mathbf{F}(G)$ and hence $\mathbf{F}(N) \leq C$, so $C = \mathbf{F}(N)$.

Now observe that, for every $i \in \{1, ..., n\}$, the group $N\mathbf{C}_G(V_i)/\mathbf{C}_G(V_i)$ is isomorphic to a quotient of $N/\mathbf{F}(N)$, therefore it is abelian and we can apply Lemma 2.5. We conclude that every element in $N \setminus \bigcap_{i=1}^n \mathbf{C}_G(V_i) = N \setminus \mathbf{F}(N)$ lies in $\mathrm{Van}(G)$. ■

Lemma 2.7. *Let M, N be normal subgroups of G such that $M < N$, M is nilpotent and N/M is a p-group for some prime p. Assume that $(p, |M|) = 1$, and that $\mathbf{C}_N(M) \leq M$. Assume also that if $x \in \mathrm{Van}(G)$, then either $(o(x), |M|) = 1$ or $(o(x), p) = 1$ holds. Then N is a Frobenius group with kernel M.*

Proof. By the Schur-Zassenhaus Theorem, there exists a complement P of M in N. If A is a characteristic abelian subgroup of P, then $AM \trianglelefteq G$ and, by Lemma 2.6, $AM \setminus M \subseteq \mathrm{Van}(G)$. Thus, our assumption on the orders of the elements in $\mathrm{Van}(G)$ implies that $\mathbf{C}_{AM}(m) \leq M$ for every nontrivial $m \in M$. So AM is a Frobenius group with kernel M. It follows that A is cyclic. This shows that every characteristic abelian subgroup of P is cyclic.

By [9, Theorem 1.2], we can write $P = ET$ with $E \cap T = Z \leq \mathbf{Z}(P)$, $|Z| = p$, and $[E, T] = 1$. The group E is extraspecial or $E = Z$. There exists a cyclic subgroup $U \leq T$ with $|T : U| \leq 2$ and $U = \mathbf{C}_T(U)$. Further, U and EU are characteristic subgroups of P.

In the case when $E \neq Z$ we have $(EU)' = Z$ and $U = \mathbf{Z}(EU)$, whence, by [4, Theorem 7.5], every nonlinear character of EU vanishes on $EU \setminus U$.

Write $\overline{G} = G/M$ and adopt the bar convention. Observe that $\overline{U}, \overline{EU} \trianglelefteq \overline{G}$. By the previous paragraph and by Remark 2.2, we have $EUM \setminus UM \subseteq \mathrm{Van}(G)$. Further, applying again Lemma 2.6, we see that $UM \setminus M \subseteq \mathrm{Van}(G)$. Therefore, $EUM \setminus M \subseteq \mathrm{Van}(G)$. It follows that $\mathbf{C}_{EUM}(m) \leq M$ for every nontrivial $m \in M$, and hence EUM is a Frobenius group with kernel M.

Now, if p is odd we get $P = EU$, therefore $N = EUM$, and we are done.

If $p = 2$, then EU is either cyclic or generalized quaternion. Assume first that EU is generalized quaternion. Since EU/Z is abelian, and since the index of the derived subgroup of a generalized quaternion group is 4, it follows that $EU = E \simeq Q_8$, and $U = Z$. In particular, $T = \mathbf{C}_T(Z) = \mathbf{C}_T(U) = U$. Hence, in this case we get $P = EU$, therefore $N = EUM$, and we are done.

Finally, if EU is cyclic, then $EU = U$ and $P = T$. Now, U is an abelian normal subgroup of index at most 2 in P, so that every nonlinear character of P vanishes on $P \setminus U$. By Remark 2.2, it follows that $N \setminus UM \subseteq \mathrm{Van}(G)$. Hence, as $UM \setminus M \subseteq \mathrm{Van}(G)$, we get that $N \setminus M \subseteq \mathrm{Van}(G)$. Thus, $\mathbf{C}_N(m) \leq M$ for every nontrivial $m \in M$, that is, N is a Frobenius group with kernel M. ∎

Lemma 2.8. *Let G be a solvable nonnilpotent group with Fitting subgroup F, and assume $(|F|, |G/F|) = 1$. If $(o(g), |F|) = 1$ for every $g \in \mathrm{Van}(G) \setminus F$, then G is a Frobenius group with kernel F.*

Proof. By the Schur-Zassenhaus Theorem, there exists a complement H of F in G. Let p be a prime such that $P = \mathbf{O}_p(H)$ is nontrivial. We have that $PF \trianglelefteq G$, F is a nilpotent p'-group, and $\mathbf{C}_{PF}(F) \leq F$. Thus, by Lemma 2.7, we see that PF is a Frobenius group with kernel F. In particular, P is cyclic or generalized quaternion.

So, $L = \mathbf{F}(H) = C \times Q$ with C cyclic of odd order and Q either cyclic or generalized quaternion.

If $Q = 1$, then L clearly has a characteristic series whose factors are cyclic of prime order. Indeed, the same conclusion holds also if $Q \neq 1$ and $Q \not\simeq Q_8$, as in this case L has a unique cyclic subgroup of index 2. If $Q \simeq Q_8$, then L has a characteristic series whose factors are cyclic of prime order or isomorphic to $C_2 \times C_2$. Therefore, in any case, every H-chief factor of L is cyclic or isomorphic to $C_2 \times C_2$. Let now V be any of these factors and $D = H/\mathbf{C}_H(V)$.

If V is cyclic, then D is abelian, and an application of Lemma 2.5 yields that $H \setminus \mathbf{C}_H(V)$ lies in $\mathrm{Van}(H)$.

If $V \simeq C_2 \times C_2$, then D is isomorphic to a subgroup of S_3. If D has order less than 6, then we can argue as in the paragraph above, getting $H \setminus \mathbf{C}_H(V) \subseteq \mathrm{Van}(H)$.

In the case when D is isomorphic to S_3, the group D has an abelian normal subgroup $A/\mathbf{C}_H(V)$ of index 2. By Lemma 2.5, $A\setminus\mathbf{C}_H(V)$ lies in $\mathrm{Van}(H)$. Moreover, every element of D not lying in $A/\mathbf{C}_H(V)$ is in $\mathrm{Van}(D)$, and we conclude, applying Remark 2.2, that also in this case $H \setminus \mathbf{C}_H(V)$ lies in $\mathrm{Van}(H)$.

Since Proposition 1.8 yields $L = \bigcap_{i=1}^{n} \mathbf{C}_H(V_i)$, where $\{V_1, ..., V_n\}$ is a set of H-chief factors of L, we get $H \setminus L \subseteq \mathrm{Van}(H)$ and hence $G \setminus LF \subseteq \mathrm{Van}(G)$.

We now show that $LF \setminus F \subseteq \mathrm{Van}(G)$. Recall that $L = C \times Q$, where C is cyclic of odd order and Q is either a cyclic 2-group or a generalized quaternion group. We define a cyclic characteristic subgroup U of L as follows: we set $U = \mathbf{Z}(L)$ if Q is cyclic or isomorphic to Q_8, otherwise we set U to be the unique cyclic subgroup of index 2 in L. Given that, every element in $UF \setminus F$ is in $\mathrm{Van}(G)$ by Lemma 2.6. Moreover, in the cases when $U < L$, there exists a nonlinear irreducible character ψ of L such that ψ vanishes on $L \setminus U$. Now Remark 2.2 yields $LF \setminus UF \subseteq \mathrm{Van}(G)$.

The conclusion so far is that every element in $G \setminus F$ is a vanishing element of G. Now, if there exists a nonidentity element $h \in H$ which centralizes a nonidentity element $f \in F$, then hf is an element not lying in F (hence a vanishing element) whose order is not coprime with $|F|$. We reached a contradiction, and the proof is complete. ■

Remark 2.9. If π is a subset of the vertex set of the graph Γ, then we define the *induced subgraph* Γ_π as the subgraph of Γ with vertex set π and such that two vertices in π are adjacent in Γ_π if and only if they are adjacent in Γ. It will often be useful to note what follows. Let G be a solvable group, and let F denote its Fitting subgroup. By [3, Theorem A], every prime in $\pi = \pi(G/F)$ is a vertex of $\Gamma(G)$. Moreover, if s and t are primes in π which are adjacent in $\Pi(G/F)$, then they are adjacent in $\Gamma(G)$ as well. In fact, take x in G such that $o(xF)$ is divisible by st: as xF is clearly not a 2-element of G/F, the element x is in $\mathrm{Van}(G)$ by Theorem 2.4, and its order is divisible by st. Summing up, this says that $\Pi(G/F)$ is a subgraph of $\Gamma(G)_\pi$.

Lemma 2.10. *Let G be a solvable group, and assume that $\Gamma(G)$ is disconnected. If $\Pi(G/\mathbf{F}(G))$ is connected, then G is a Frobenius group with kernel F.*

Proof. Setting $F = \mathbf{F}(G)$ and $\pi = \pi(G/F)$ we note that, by Remark 2.9 and by our assumptions, $\Gamma(G)_\pi$ is connected.

We claim that F contains some vanishing element of G. In fact, assume $F \cap \mathrm{Van}(G) = \emptyset$, and take a vertex r of $\Gamma(G)$. There exists an element g of $\mathrm{Van}(G)$ such that r divides $o(g)$ and, since g is in $G \setminus F$, $o(g)$ must be divisible by a suitable prime in π. This argument shows that every vertex in $\Gamma(G)$ is adjacent to a prime in π. But, $\Gamma(G)_\pi$ is connected, and this leads to a contradiction.

Now, by Proposition 2.1, every prime in $\sigma = \pi(F)$ is a vertex of $\Gamma(G)$, and $\Gamma(G)_\sigma$ is a complete graph. It is then clear that G is not nilpotent, and that we must have $(|F|, |G/F|) = 1$. Also, our assumptions imply that every $g \in \mathrm{Van}(G) \setminus F$ is such that $(o(g), |F|) = 1$. We are in a position to apply Lemma 2.8, which yields the conclusion. ■

Lemma 2.11. *Let G be a solvable group, and assume that $\Gamma(G)$ is disconnected. If $\Pi(G/\mathbf{F}(G))$ is also disconnected, then $G/\mathbf{F}(G)$ is a Frobenius group and every element in $G \setminus \mathbf{F}(G)$ lies in* $\mathrm{Van}(G)$.

Proof. Let us denote by F and K respectively the first and the second term of the upper Fitting series of G. Since the prime graph of G/F is disconnected, by Theorem 1.3 we have that G/F is either a Frobenius group or a 2-Frobenius group. In order to treat the two cases simultaneously, we shall set $L = G$ if G/F is a Frobenius group, whereas L will denote the third term of the Fitting series of G if G/F is 2-Frobenius. Therefore, in any case L/F is a Frobenius group with kernel K/F. Also, given a prime p which divides $|K/F|$, we shall set $P/F = \mathbf{O}_p(K/F)$.

Consider the quotient group $\bar{G} = G/\Phi(G)$, and adopt the bar convention. The group $\bar{F}$ can be viewed as a semisimple $\bar{G}$-module (possibly in mixed characteristic), whence, by Clifford's Theorem ([4, 2.14]), it is a semisimple $\bar{P}$-module. Let q be a prime dividing the order of $[\bar{P}, \bar{F}]$, and let $\bar{Q}$ be a Sylow q-subgroup of $\bar{F}$. We get $[\bar{P}, \bar{F}] = [\bar{P}, \bar{Q}] \cdot [\bar{P}, \mathbf{O}_{q'}(\bar{F})]$, whence our assumption that q divides the order of $[\bar{P}, \bar{F}]$ forces $[\bar{P}, \bar{Q}]$ to be nontrivial. Note that, since $\mathbf{O}_p(\bar{F})$ is a semisimple module under the action of the p-group P/F, we get $[\bar{P}, \mathbf{O}_p(\bar{F})] = [P/F, \mathbf{O}_p(\bar{F})] = 1$. In particular, $\bar{Q}$ is not $\mathbf{O}_p(\bar{F})$ (i.e. $q \neq p$) and, by coprimality, $\mathbf{C}_{[\bar{P},\bar{Q}]}(\bar{P}) = 1$.

Consider now a Frobenius complement H/F of L/F, and observe that HP/F is a Frobenius group with kernel P/F. Moreover, $[\bar{P}, \bar{Q}]$ is an HP/F-module in characteristic q (which is coprime to $|P/F|$) and in which no element is fixed by P/F. We are in a position to apply Lemma 1.4 (note that $[\bar{P}, \bar{Q}]$ is certainly abelian, since it is a module), obtaining that there exists a nontrivial element in $[\bar{P}, \bar{Q}]$ which is fixed by H/F. As a consequence, there exists an element g in $L \setminus K$ whose order is divisible by q. Recalling now that every element in $L \setminus K$ lies in $\mathrm{Van}(G)$ (see Remark 2.2 and Lemma 2.3), we have that the given prime q is adjacent in $\Gamma(G)$ to a prime in $\pi(L/K)$.

Set $\sigma = \pi(K/F) \cup \pi(G/L)$, and $\tau = \pi(L/K)$. Recall that, by Remark 2.9 and Proposition 1.7, $\Gamma(G)_\sigma$ and $\Gamma(G)_\tau$ are complete graphs. We next show the following: let x be an element in $\mathrm{Van}(G)$ whose order is divisible by p, then $o(x)$ is coprime to the order of $[\bar{P}, \bar{F}]$ (recall that $p \in \sigma$). In fact, assume by contradiction that there exist a prime $q \in \pi([\bar{P}, \bar{F}])$, and $x \in \mathrm{Van}(G)$ such that $o(x)$ is divisible by pq. In particular, q is adjacent in $\Gamma(G)$ to the prime p of σ. On the other hand, by the conclusion in the paragraph above, q is adjacent in $\Gamma(G)$ to a prime in τ. Since $\Gamma(G)_\sigma, \Gamma(G)_\tau$ are connected, we have that all the primes in $\pi(G/F)$ lie in the same connected component of $\Gamma(G)$, and this forces F to contain some vanishing element. Then, by Proposition 2.1, all the primes in $\pi(F)$ (including q)

are vertices of $\Gamma(G)$ which are pairwise adjacent, and this leads to the contradiction that $\Gamma(G)$ is connected.

Since, as already observed, we have $[\bar{P}, \mathbf{O}_p(\bar{F})] = 1$, it is easily seen that $\bar{F} = [\bar{P}, \bar{F}] \times \mathbf{C}_{\bar{F}}(\bar{P})$. We are then in a position to apply Lemma 2.7 to the group $\bar{G}/\mathbf{C}_{\bar{F}}(\bar{P})$, with $\bar{F}/\mathbf{C}_{\bar{F}}(\bar{P})$ playing the role of M and $\bar{P}/\mathbf{C}_{\bar{F}}(\bar{P})$ playing the role of N, obtaining that $\bar{P}/\mathbf{C}_{\bar{F}}(\bar{P})$ is a Frobenius group with kernel $\bar{F}/\mathbf{C}_{\bar{F}}(\bar{P})$. In particular P/F, which is isomorphic to a Frobenius complement of that group, is cyclic or generalized quaternion. We have already observed that, still denoting by H/F a Frobenius complement of K/F in L/F, the group H/F acts fixed-point freely (by conjugation) on P/F. But now P/F cannot be a 2-group, otherwise its unique involution would be centralized by H/F. We conclude that K/F has odd order and, since it is nilpotent with cyclic Sylow subgroups, it is cyclic. Moreover, G/K acts faithfully on K/F, whence it is abelian, and L is forced to be G. The group G/F is then a Frobenius group. Since K/F has odd order, Lemma 2.3 together with Theorem 2.4 yield that every element in $G \setminus F$ lies in $\mathrm{Van}(G)$, thus completing the proof. ■

Remark 2.12. We note that Lemma 2.10 and Lemma 2.11 show, among other things, that if G is solvable and $\Gamma(G)$ is disconnected, then $G \setminus \mathbf{F}(G) \subseteq \mathrm{Van}(G)$. We invite the reader to compare this remark with Theorem 2.4.

3. Connected components: a proof of Theorem A

In this section we shall prove Theorem A. We start by recalling the definition of nearly 2-Frobenius group.

Definition 3.1. A group G is said to be a nearly 2-Frobenius group if there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G, furthermore G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group.

Note that, by Remark 1.2, every nearly 2-Frobenius group is solvable.

Proposition 3.2. *Let G be a solvable group with Fitting subgroup F. Let p_1, p_2, r be distinct primes such that $p_1, p_2 \in \pi(G/F)$ are not adjacent in $\Pi(G/F)$, and r lies in $\pi(F)$. Then the following conclusions hold.*

(a) *There exist g in G and $i \in \{1, 2\}$ such that $p_i r$ is a divisor of $o(g)$, and p_i is a divisor of $o(gF)$.*

(b) *If, in addition, $\Pi(G/F)$ is disconnected, then an element g as in* (a) *can be chosen to be in* $\mathrm{Van}(G)$.

Proof. (a): Somewhat more generally, we shall prove the claim with any nilpotent normal subgroup N of G in place of F, and we shall argue by induction on the order of the group. Denoting by H a Hall $\{p_1, p_2\}$-subgroup of N, we see that G/H and N/H satisfy the hypothesis. If $H \neq 1$, by induction we find an element gH in G/H of order divisible by $p_i r$, and such that the coset of gH modulo N/H has order divisible by p_i. The element g has the required properties, thus we may assume $H = 1$.

Now, we have that $\{p_1, p_2, r\}$ is a set of three distinct primes dividing the order of the solvable group G, and therefore, by Proposition 1.5, G contains an element g whose order is divisible by the product of two of these primes. If p_1p_2 divides $o(g)$ then it also divides the order of $gN \in G/N$, a contradiction. Hence the order of g is divisible by $p_1 r$ or by $p_2 r$ and, since p_1 and p_2 are not in $\pi(N)$, the order of gN in G/N must be divisible by p_1 or by p_2.

(b): In view of part (a) and Theorem 2.4, we only have to focus (up to changing p_1 with p_2) on the case when $p_1 = 2$, there are elements in $G \setminus F$ whose order is divisible by $2r$ (whereas there is no element of order divisible by $p_2 r$), and for every such element g we get that gF is a 2-element of G/F.

By Theorem 1.3, G/F is either a Frobenius group or a 2-Frobenius group. Denote by K the second term of the Fitting series of G, and set $\pi = \pi(K/F)$. If $2 \notin \pi$, then every element in G/F of even order lies outside K/F, whence it is a vanishing element of G by Remark 2.2 and Lemma 2.3, and we are done. Therefore we may assume that $2 \in \pi$, so that $\mathbf{O}_2(G/F) > 1$. Let $Z/F = \mathbf{Z}(\mathbf{O}_2(G/F))$. Since Z/F is abelian, by Lemma 2.6 every element in $Z \setminus F$ lies in $\mathrm{Van}(G)$. Hence it suffices to find an element $x \in Z \setminus F$ of order divisible by $2r$.

Let $Q \in \mathrm{Syl}_{p_2}(G)$ and $H = ZQ$. Since p_2 and 2 are not adjacent in $\Pi(G/F)$, we have that H/F is a Frobenius group with kernel Z/F, and it is easy to see that $F = \mathbf{F}(H)$. Clearly there are no elements of order $p_2 r$ in H. By part (a), we have that there is $x \in H \setminus F$ of order divisible by $2r$. Then $xF \in H/F$ is a 2-element and hence $xF \in Z/F$. Thus $x \in Z \setminus F \subseteq \mathrm{Van}(G)$, as desired. ■

Theorem A. *Let G be a solvable group. Then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius or a nearly 2-Frobenius group.*

Proof. Set $F = \mathbf{F}(G)$, and assume that $\Gamma(G)$ is disconnected. If $\Pi(G/F)$ is connected then, by Lemma 2.10, G is a Frobenius group.

What is left is to consider the case when $\Pi(G/F)$ is disconnected. Under this assumption, Lemma 2.11 yields that G/F is a Frobenius group and that every element in $G \setminus F$ is a vanishing element of G. Let L/F denote the Frobenius kernel of G/F, and define $\rho_1 = \pi(G/L)$, $\rho_2 = \pi(L/F)$. For $i \in \{1, 2\}$, set

$$\pi_i = \rho_i \cup \{t \text{ prime} \mid t \text{ divides } o(g) \text{ where } gF \text{ is a nonidentity } \rho_i\text{-element}\}.$$

We claim that the vertex set of $\Gamma(G)$ is $\pi_1 \cup \pi_2$. Since $G \setminus F \subseteq \mathrm{Van}(G)$, the set $\pi_1 \cup \pi_2$ is a subset of the vertex set of $\Gamma(G)$. Let r be in $\pi(F)$, and assume

$r \notin \rho_1 \cup \rho_2$. Choose any p_i in ρ_i, for $i = 1, 2$. By Proposition 3.2(a), we get an i in $\{1, 2\}$ and an element g in $G \setminus F$ whose order is divisible by $p_i r$ and gF is a ρ_i-element. Hence $r \in \pi_i$. Thus the vertex set of $\Gamma(G)$ is indeed $\pi_1 \cup \pi_2$. Notice that, g being a vanishing element of G, we also proved that r is adjacent in $\Gamma(G)$ to every prime in ρ_i.

Next, we claim that π_1, π_2 are the connected components of $\Gamma(G)$. Recall that, by Proposition 1.7 and Remark 2.9, $\Gamma(G)_{\rho_1}, \Gamma(G)_{\rho_2}$ are complete subgraphs of $\Gamma(G)$. As seen in the paragraph above, every prime in $\pi_i \setminus \rho_i$ is adjacent in $\Gamma(G)$ to some prime in ρ_i (in fact, to *every* prime in ρ_i), and so $\Gamma(G)_{\pi_i}$ is connected. This concludes the proof of Theorem A as regards the bound on the number of connected components.

Finally, let F_i be a Hall π_i-subgroup of F, so that $F = F_1 \times F_2$, and let H be a Hall π_1-subgroup of G. We see that H/F_1 acts fixed-point freely on L/F_1, otherwise we would get an element gF_1 in G/F_1 whose order is divisible by a prime in π_1 and a prime in π_2: of course g is not in F, so it is in $\mathrm{Van}(G)$, and this yields a contradiction. Whence $G/F_1 = HL/F_1$ is a Frobenius group with kernel L/F_1. Furthermore, the same argument shows that L/F_2 is a Frobenius group with kernel F/F_2, therefore G/F_2 is a 2-Frobenius group. We conclude that G is a nearly 2-Frobenius group, and the proof is complete. ∎

As a kind of converse to Theorem A, in the rest of this section we gather some information on the connected components of $\Gamma(G)$ when G is a solvable Frobenius group, or a nearly 2-Frobenius group.

Lemma 3.3. *Let G be a solvable Frobenius group with Frobenius kernel F and Frobenius complement H. If $F \cap \mathrm{Van}(G) \neq \emptyset$, then we have $\Gamma(G) = \Pi(G)$, otherwise $\Gamma(G)$ coincides with the connected component of $\Pi(G)$ with vertex set $\pi(H)$.*

Proof. By Remark 2.9, the connected component of $\Pi(G)$ with vertex set $\pi(H)$ is a connected component of $\Gamma(G)$. Now Proposition 2.1 yields the conclusion. ∎

It may be worth mentioning that both the situations outlined in the conclusions of Lemma 3.3 can occur. An example of the latter situation is S_3, whereas an infinite family of examples of the former is provided in [1, Example 2].

Lemma 3.4. *Let G be a nearly 2-Frobenius group. If $\Gamma(G)$ is disconnected, then each connected component is a complete graph.*

Proof. Let F_1, F_2, F, L be as in Definition 3.1. Write $\pi_1 = \pi(F_1)$, $\pi_2 = \pi(F_2)$, $\sigma = \pi(L/F)$, $\tau = \pi(G/L)$. Set $\overline{G} = G/F_2$ and adopt the bar convention. Since $\overline{G}$ is a 2-Frobenius group, as observed in Remark 1.2 we have that $\overline{L}/\overline{F}$ and $\overline{G}/\overline{L}$ are cyclic.

We claim that there exists an element $g \in G \setminus L$ such that $\pi(o(g)) \supseteq \pi_1 \cup \tau$. Let $\overline{x}\overline{L}$ be a generator of $\overline{G}/\overline{L}$. We can clearly assume that there exists $q \in \pi_1 \setminus \pi(o(\overline{x}))$. Let Z_q be the centre of the Sylow q-subgroup of $\overline{F}$. Then the Frobenius group $\overline{G}/\overline{F}$ acts on Z_q and Lemma 1.4 yields $\mathbf{C}_{Z_q}(\overline{x}) \neq 1$. Therefore, there exists an element $\overline{y} \in \mathbf{Z}(\overline{F}) \cap \mathbf{C}_{\overline{F}}(\overline{x})$ such that $\pi(o(\overline{y})) = \pi_1 \setminus \pi(o(\overline{x}))$. Considering $\overline{g} = \overline{xy}$, we see that $\pi(o(g)) \supseteq \pi_1 \cup \tau$.

By Remark 2.2 and Lemma 2.3, the element g is a vanishing element of G, and so $\Gamma(G)_{\pi_1 \cup \tau}$ is a complete subgraph of $\Gamma(G)$.

The group L/F_1 is a nilpotent normal subgroup of G/F_1. Since $L/F \neq 1$, there exists an element $h \in L \setminus F$ such that $\pi_2 \cup \sigma = \pi(L/F_1) \subseteq \pi(o(h))$. By Remark 2.2 and Lemma 2.3, the element h is a vanishing element of G and so $\Gamma(G)_{\pi_2 \cup \sigma}$ is a complete subgraph of $\Gamma(G)$.

Finally, if $\Gamma(G)$ is not connected, then clearly $\pi_1 \cup \tau, \pi_2 \cup \sigma$ are the connected components of $\Gamma(G)$. ■

Remark 3.5. Let G be a solvable group such that $\Gamma(G)$ is disconnected. Lemma 3.3 (together with Proposition 1.7) and Lemma 3.4 show that the connected components of $\Gamma(G)$ are complete graphs.

We conclude the section with an example of a nearly 2-Frobenius group G such that $\Gamma(G)$ is disconnected (whereas $\Pi(G)$ is connected, as G is not a 2-Frobenius group). This, together with the previous theorem and [1, Example 2], completes the picture.

Example 3.6. Let T be the normal 2-complement of the affine semilinear group $\mathrm{A\Gamma L}(1, 5^3)$, whence $|T| = 3 \cdot 5^3 \cdot 31$, and consider the (unique up to equivalence) nontrivial action of T on a cyclic group C of order 79. Call G the semidirect product $C \rtimes T$ formed accordingly. We have that $\mathbf{F}(G)$ has order $5^3 \cdot 79$, and it is abelian. We claim that $\mathrm{Van}(G) = G \setminus \mathbf{F}(G)$. In fact, $G \setminus \mathbf{F}(G)$ is contained in $\mathrm{Van}(G)$; moreover, no divisor of $|G/\mathbf{F}(G)|$ is a linear combination of 5 and 79 with nonnegative integer coefficients, thus we can apply [1, Theorem 2.3] to get that $\mathbf{F}(G) \cap \mathrm{Van}(G) = \emptyset$. Given that, we have that $\Gamma(G)$ is disconnected, with connected components $\{3, 5\}, \{31, 79\}$.

4. Independent sets: a proof of Theorem B

Recall that, given a graph Γ, a subset π of the vertex set of Γ is said to be *independent* if no two elements of π are adjacent in Γ. In this section we consider the question whether there exists a universal upper bound for the size of an independent set in $\Gamma(G)$. As mentioned in the Introduction, we shall see that the answer is negative even in the class of solvable groups. Note that this is remarkably in contrast to what happens for the Gruenberg-Kegel graph (see Proposition 1.5) and for many other graphs attached to a solvable group G, such as the prime graph on $\mathrm{cs}(G)$ or on $\mathrm{cd}(G)$ (see [2] and [10]).

We start with some preliminary results.

Lemma 4.1. *Given a set of prime numbers $\{p_1, ..., p_k\}$, there exists a set of prime numbers $\{q_1, ..., q_k\}$ such that p_j divides $q_i - 1$ for every pair of distinct i, j in $\{1, ..., k\}$ and such that $p_1, \ldots, p_k, q_1, \ldots, q_k$ are pairwise distinct.*

Proof. For every $i \in \{1, \ldots, k\}$, let n_i be defined as $\prod_{j \neq i} p_j$. By Dirichlet's Theorem on primes in arithmetic progression, there exists a prime q_i such that $q_i \equiv 1 \bmod n_i$. In particular, we have that p_j divides $q_i - 1$ for every $j \neq i$. Since we have infinitely many choices for each q_i, we can choose them so that $p_1, \ldots, p_k, q_1, \ldots, q_k$ are pairwise distinct. ∎

Given $p_1, \ldots, p_k, q_1, \ldots, q_k$ as in Lemma 4.1, we establish some notation. Set $n = \prod p_i$, and $m = \prod q_i$. Also, define $n_i = n/p_i$ and $m_i = m/q_i$. Now, let a_i be an invertible element of $\mathbb{Z}/q_i\mathbb{Z}$ of order n_i (such an a_i does exist, as n_i divides $q_i - 1$). By the Chinese Remainder Theorem, we can find an integer a such that $a \equiv a_i \bmod q_i$ for every $i = 1, \ldots, k$.

Next, let $C_n = \langle x \rangle$ be a cyclic group of order n, and $C_m = \langle y \rangle$ a cyclic group of order m. Note that, by our assumption on m, n and a, the map $y \mapsto y^a$ defines an automorphism of C_m of order n. Consider the semidirect product $G = C_m \rtimes C_n$ formed according to the action given by $y^x = y^a$.

Finally, let $A = \{\alpha_1, \ldots, \alpha_k\}, B = \{\beta_1, \ldots, \beta_k\}$ be two disjoint sets. We denote by Γ_k the graph with vertex set $A \cup B$ and with edge set

$$\{\{\alpha_i, \alpha_j\}, \{\beta_i, \beta_j\}, \{\alpha_i, \beta_i\} \mid i \neq j\}.$$

Note that Γ_k is a prism where each of the two bases is a complete graph.

Lemma 4.2. *The graph $\Pi(G)$ is isomorphic to Γ_k.*

Proof. Since C_n, C_m are cyclic subgroups of G, we have that the p_i's are pairwise adjacent in $\Pi(G)$, and the same holds for the q_i's.

It remains to prove that, for every i and j in $\{1, ..., k\}$, the vertices p_i and q_j are adjacent in $\Pi(G)$ if and only if $i = j$. Now, by construction, the element x acts (by conjugation) on $\langle y^{m_j} \rangle$ as an automorphism of order n_j. Therefore, $\mathbf{C}_{C_n}(y^{m_j})$ has order $n/n_j = p_j$, and the claim is proved. ∎

From now on, we shall assume that $k \geq 3$, and that $q_i > n$ for every $i \in \{1, ..., k\}$. As regards the latter assumption, Dirichlet's theorem on primes in arithmetic progression certainly enables us to make it.

Lemma 4.3. *The graphs $\Pi(G)$ and $\Gamma(G)$ have the same vertex set. Furthermore, the edge set of $\Gamma(G)$ is $\{\{p_i, p_j\}, \{p_i, q_i\} \mid i \neq j\}$.*

Proof. We prove that $\{p_i, p_j\}$ and $\{p_i, q_i\}$ are edges of $\Gamma(G)$. Let M be the normal subgroup of G of order $q_i q_j$ and N be the subgroup of G containing C_m such that $|N : C_m| = p_i p_j$. Note that since $k \geq 3$ we have $C_m > M$. Clearly, $M \leq N$ are normal subgroups of G and by construction N/M is a Frobenius group with Frobenius kernel C_m/M and with Frobenius complement of order $p_i p_j$. By Remark 2.2 and Lemma 2.3, the elements in $N \setminus C_m$ are vanishing elements for G. Therefore $\{p_i, p_j\}$ and $\{p_i, q_i\}$ are edges of $\Gamma(G)$.

To conclude it is enough to prove that $C_m \cap \mathrm{Van}(G) = \emptyset$. Let χ be in $\mathrm{Irr}(G)$, and $g \in C_m$ such that $\chi(g) = 0$. Now, $\chi_{C_m} = \lambda_1 + \cdots + \lambda_s$, where λ_i is an irreducible character of C_m and s divides n. In particular, since $\chi(g) = 0$, we get a vanishing sum of s m-th roots of unity. Therefore, by [1, Theorem 2.3], s must be a linear combination with nonnegative integer coefficients of $q_1, \ldots, q_k$. Clearly, this contradicts the fact that we chose $q_i > n$ for every $i \in \{1, ..., k\}$. ■

Theorem B, which we state again, follows at once by the discussion above.

Theorem B. *Let k be a positive integer. There exists a finite solvable group G such that $\Gamma(G)$ is connected, and it has an independent set of size k.*

Proof. Construct G as in the previous discussion. Then $\Gamma(G)$ is connected and, by Lemma 4.3, the set $\{q_1, \ldots, q_k\}$ is an independent set of $\Gamma(G)$. ■

5. Diameter of the graph: a proof of Theorem C

Let G be a solvable group, and let p, q be vertices lying in the same connected component of $\Gamma(G)$. In what follows, we shall write $d(p, q)$ to denote the distance between p and q in $\Gamma(G)$ (that is, the minimum length of a path joining p and q). We shall also use the symbol $p \sim q$ to mean that p and q are adjacent vertices of $\Gamma(G)$ or that $p = q$. Finally, we define the diameter of $\Gamma(G)$ as follows.

$$\mathrm{diam}(\Gamma(G)) = \max\{d(p, q) \mid p, q \text{ lie in the same connected component of } \Gamma(G)\}.$$

The last preliminary remark.

Remark 5.1. We note that $d(p, q) \leq 3$ whenever p and q lie in the same connected component of $\Pi(G/F)$ (here $F = \mathbf{F}(G)$). Moreover, if $\Pi(G/F)$ is disconnected, then $d(p, q) \leq 1$. This follows at once from Remark 2.9 together with Corollary 1.6.

We can now prove Theorem C, which we state again.

Theorem C. *Let G be a solvable group. Then the diameter of $\Gamma(G)$ is at most 4. Moreover, if $\Gamma(G)$ is disconnected, then the diameter of $\Gamma(G)$ is at most 1.*

Proof. If $\Gamma(G)$ is disconnected, then the result follows by Remark 3.5. Thus we may assume that $\Gamma(G)$ is connected and, given two vertices p, q of $\Gamma(G)$, we have to prove that $d(p, q) \leq 4$. Set $F = \mathbf{F}(G)$.

Assume $\Pi(G/F)$ connected and $\mathrm{Van}(G) \cap F \neq \emptyset$. If $p, q \in \pi(G/F)$, then by Remark 5.1 $d(p,q) \leq 3$. If $p, q \in \pi(F)$, then by Proposition 2.1 $d(p,q) \leq 1$. Up to relabelling, it remains to consider the case $p \in \pi(G/F)$ and $q \in \pi(F)$. Since $\Gamma(G)$ is connected, there is an element g in $\mathrm{Van}(G)$ of order divisible by sr for some $s \in \pi(G/F)$ and $r \in \pi(F)$. Since $p, s \in \pi(G/F)$ and $r, q \in \pi(F)$, by Remark 5.1 and Proposition 2.1 there exists a path $p \sim s_1 \sim s_2 \sim s \sim r \sim q$ with $s_1, s_2 \in \pi(G/F)$. Observe that, if the vertices in this path are not pairwise distinct, then we are done. Now, either p, s_2 are odd or s_1, s are odd. In the former case, by Proposition 3.2(a) and Theorem 2.4, either $p \sim q$ (so $d(p,q) \leq 1$) or $s_2 \sim q$ (so $d(p,q) \leq 3$). The latter case is similar.

Assume $\Pi(G/F)$ connected and $\mathrm{Van}(G) \cap F = \emptyset$. If $p, q \in \pi(G/F)$ then, as above, $d(p,q) \leq 3$. If $p, q \in \pi(F)$, then, as $\mathrm{Van}(G) \cap F = \emptyset$, there exist $s_p, s_q \in \pi(G/F)$ such that $p \sim s_p$ and $q \sim s_q$. Therefore, by Remark 5.1, there exists a path $p \sim s_p \sim s_1 \sim s_2 \sim s_q \sim q$ with $s_1, s_2 \in \pi(G/F)$. Now, either s_p, s_2 are odd or s_1, s_q are odd. In the former case, by Proposition 3.2(a) and Theorem 2.4, either $s_p \sim q$ (so $d(p,q) \leq 2$) or $s_2 \sim q$ (so $d(p,q) \leq 4$). The latter case is similar. Up to relabelling, it remains to consider the case $p \in \pi(G/F)$ and $q \in \pi(F)$. As $\mathrm{Van}(G) \cap F = \emptyset$, there exists $s_q \in \pi(G/F)$ such that $q \sim s_q$. So $d(p, s_q) \leq 3$ and $d(p,q) \leq 4$.

For the rest of the proof we shall assume that $\Pi(G/F)$ is disconnected.

Assume $\mathrm{Van}(G) \cap F \neq \emptyset$. If p, q are in $\pi(G/F)$ and lie in the same connected component of $\Pi(G/F)$, then $d(p,q) \leq 1$. So we may assume that p, q lie in distinct connected components of $\Pi(G/F)$. Since $\Gamma(G)$ is connected, there exist primes s_p and s_q of $\pi(G/F)$, in the connected component of $\Pi(G/F)$ containing respectively p and q, and primes r_p, r_q in $\pi(F)$ such that $s_p \sim r_p$, $s_q \sim r_q$. Consider the path $p \sim s_p \sim r_p \sim r_q \sim s_q \sim q$. By Proposition 3.2(b) either $p \sim r_p$ (so $d(p,q) \leq 4$) or $q \sim r_p$ (so $d(p,q) \leq 3$). If $p, q \in \pi(F)$, then by Proposition 2.1 $d(p,q) \leq 1$. Up to relabelling, it remains to consider the case $p \in \pi(G/F)$ and $q \in \pi(F)$. Since $\Gamma(G)$ is connected, there exists a prime $s \in \pi(G/F)$ in the same connected component of $\Pi(G/F)$ containing p, and a prime r in $\pi(F)$ such that $s \sim r$. Now, $p \sim s \sim r \sim q$ and $d(p,q) \leq 3$.

Assume $\mathrm{Van}(G) \cap F = \emptyset$. We claim that there exist $r \in \pi(F)$ and s_1, s_2 in distinct connected components of $\Pi(G/F)$, such that $s_1 \sim r \sim s_2$. In fact, let s_1, s_2 be vertices lying in distinct connected components of $\Pi(G/F)$ such that $d(s_1, s_2)$ is as small as possible. Let $s_1 \sim r_1 \sim r_2 \sim \cdots \sim r_k \sim s_2$ be a path of minimal length connecting them. By Remark 5.1, it is clear that $r_i \notin \pi(G/F)$ for every $i = 1, \ldots, k$. On the other hand, since $\mathrm{Van}(G) \cap F = \emptyset$, for every $g \in \mathrm{Van}(G)$ the order of g is divisible by some prime in $\pi(G/F)$. It follows that $k = 1$ and the claim is proved. If p, q are in $\pi(G/F)$ and lie in the same connected component of $\Pi(G/F)$, then $d(p,q) \leq 1$. So we may assume that p lies in the connected component of $\Pi(G/F)$ containing s_1 and q lies in the connected component of $\Pi(G/F)$ containing s_2. Hence we have the path $p \sim s_1 \sim r \sim s_2 \sim q$ and $d(p,q) \leq 4$. If $p, q \in \pi(F)$, then, as $\mathrm{Van}(G) \cap F = \emptyset$, there exist $s_p, s_q \in \pi(G/F)$ such that $p \sim s_p$ and $q \sim s_q$. If s_p and s_q lie in the same connected component of $\Pi(G/F)$, then $p \sim s_p \sim s_q \sim q$ and $d(p,q) \leq 3$. So we may assume that s_p lies in the connected component of $\Pi(G/F)$ containing s_1 and s_q lies in the connected component of $\Pi(G/F)$ containing s_2. Now, by Proposition 3.2(b) either $s_p \sim q$ (so $d(p,q) \leq 2$) or $s_2 \sim q$. In the latter case $p \sim s_p \sim s_1 \sim r \sim s_2 \sim q$ and $d(p,q) \leq 5$. Applying again Proposition 3.2(b) we have that either $s_1 \sim p$ (so $d(p,q) \leq 4$) or $s_2 \sim p$ (so $d(p,q) \leq 2$). Up to relabelling, it remains to consider the case $p \in \pi(G/F)$ and $q \in \pi(F)$. As $\mathrm{Van}(G) \cap F = \emptyset$, there exists $s_q \in \pi(G/F)$ such that $q \sim s_q$. If s_q is in the same connected component of $\Pi(G/F)$ containing p, then $d(p,q) \leq 2$. So we may assume that p lies in the connected component of $\Pi(G/F)$ containing s_1 and that s_q lies in the connected component of $\Pi(G/F)$ containing s_2. Hence we have the path $p \sim s_1 \sim r \sim s_2 \sim s_q \sim q$. By Proposition 3.2(b) either $p \sim q$ (so $d(p,q) \leq 1$) or $s_2 \sim q$ (so $d(p,q) \leq 4$). ∎

We conclude providing an infinite family of solvable groups G with $\Gamma(G)$ of diameter 4. Indeed, $\Gamma(G)$ will be a path of length 4.

Example 5.2. Let q_1 be an odd prime which is not a Mersenne prime. Let p_1 be a prime different from q_1, and let p_2 be a Zsigmondy prime divisor of $q_1^{p_1} - 1$ (which certainly exists by our assumptions on q_1). It is an elementary exercise to show that p_1 divides $p_2 - 1$. Let p_3 be a prime dividing $q_1 - 1$ and q_2 a prime such that $q_2 > p_1 p_2 p_3$ and $p_1 p_3 \mid q_2 - 1$. Note that the existence of the prime q_2 is guaranteed by Dirichlet's theorem on primes in arithmetic progression.

The semilinear group $\Gamma\mathrm{L}(1, q_1^{p_1})$ contains a subgroup K isomorphic to the non-abelian group $(C_{p_2} \rtimes C_{p_1}) \times C_{p_3}$. Pick V an elementary abelian q_1-group of rank p_1 and C_{q_2} a cyclic group of order q_2. Consider the natural action of K on V as a subgroup of $\mathrm{GL}(p_1, q_1)$, and the action of K on C_{q_2} so that C_{p_2} centralizes C_{q_2} and K/C_{p_2} acts fixed-point-freely on C_{q_2}. Set $G = (V \times C_{q_2}) \rtimes K$.

Clearly, $\mathbf{F}(G) = V \times C_{q_2}$ and $G \setminus \mathbf{F}(G) \subseteq \mathrm{Van}(G)$. By construction, no divisor of $|G/\mathbf{F}(G)|$ is a linear combination of q_1, q_2 with nonnegative integer coefficients, thus we can apply [1, Theorem 2.3] to get $\mathbf{F}(G) \cap \mathrm{Van}(G) = \emptyset$. Hence, $\mathrm{Van}(G) = G \setminus \mathbf{F}(G)$. Now, it is easy to check that $\Gamma(G)$ is the path $q_1 \sim p_1 \sim p_3 \sim p_2 \sim q_2$.

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Silvio Dolfi, Dipartimento di Matematica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: `dolfi@math.unifi.it`

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
E-mail address: `emanuele.pacifici@unimi.it`

Lucia Sanus, Departament d'Àlgebra, Facultat de Matemàtiques,
Universitat de València, 46100 Burjassot, València, Spain.
E-mail address: `lucia.sanus@uv.es`

Pablo Spiga, Dipartimento di Matematica Pura ed Applicata,
Università degli Studi di Padova, via Trieste, 63 35121 Padova, Italy.
E-mail address: `spiga@math.unipd.it`