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A FUZZY SET–VALUED STOCHASTIC FRAMEWORK FOR BIRTH–AND–GROWTH PROCESS. STATISTICAL ASPECTS.

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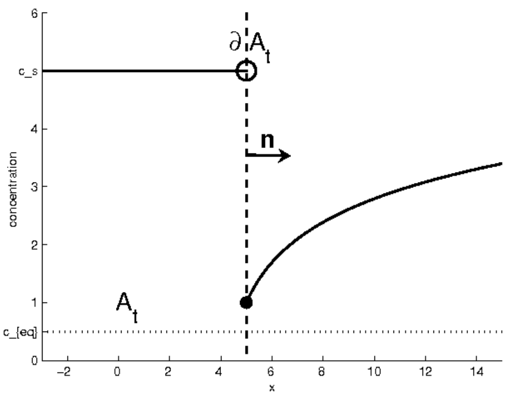
ABSTRACT

The paper considers a particular family of fuzzy monotone set–valued stochastic processes. In order to investigate suitable α–level sets of such processes, a set–valued stochastic framework is proposed for the well–posedness of birth–and–growth process. A birth–and–growth model is rigorously defined as a suitable combination, involving Minkowski sum and Aumann integral, of two very general set–valued processes representing nucleation and growth respectively. The simplicity of the proposed geometrical approach let us avoid problems arising from an analytical definition of the front growth such as boundary regularities. In this framework, growth is generally anisotropic and, according to a mesoscale point of view, is not local, i.e. for a fixed time instant, growth is the same at each point space. The proposed setting allows us to investigate nucleation and growth processes. A decomposition theorem is established to characterize nucleation and growth. As a consequence, different consistent set–valued estimators are studied for growth processes. Moreover, the nucleation process is studied via the hitting function, and a consistent estimator of the nucleation hitting function is derived.

Keywords: Random closed sets, Stochastic geometry, Birth–and–growth processes, Set–valued processes, Non additive measures, Fuzzy random sets, Fuzzy set–valued stochastic processes.

INTRODUCTION

A birth–and–growth crystal process may be studied by means of a positive time– and space–dependent stochastic function representing a concentration process as in (Aquilano *et al.*, 2009). In particular, concentration in the crystal phase takes a constant value, namely c_s (obtained from physical evidences), and outside the crystal it is represented by a sufficiently regular function c_{ex} such that $c_{ex} < c_s$; i.e., the crystal phase is more dense than the mother phase, and a jump in the concentration always occurs on the crystal boundary (Figure 1a). Figure 1b can be interpreted as a sequence of membership functions, and so, crystal growth can be seen as a fuzzy monotone set–valued stochastic process; where "monotone" means that every α–level set at each time is included in the α–level set at successive times.



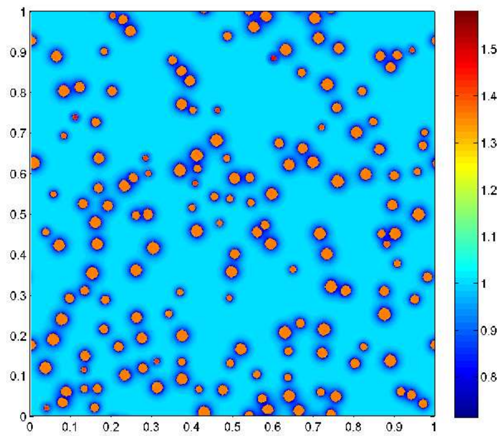


Fig. 1. *(Credits to (Aquilano et al., 2009)). A 1D sketch of the concentration for an analytical growth model, and a 2D simulation of a crystallization process on a square grid where the color scale represents the concentration. The figures may be interpreted also from a fuzzy point of view.*

In order to study some statistical aspects of the fuzzy monotone set–valued stochastic process, we notice that the α–level process is a closed set–valued stochastic process, that can be modeled as a birth–and–growth process. In this paper, we underline some geometrical properties and statistical aspects of birth–and–growth processes.

The importance of nucleation and growth processes is well known. They arise in several natural and technological applications (cf. (Capasso, 2003b;a) and references therein) such as, for example, solidification and phase–transition of materials, semiconductor crystal growth, biomineralization, and DNA replication, e.g. (Herrick *et al.*, 2002). During the years, several authors studied stochastic spatial processes (cf. (Cressie, 1991; Stoyan *et al.*, 1995; Molchanov, 1997) and references therein) nevertheless they essentially consider static approaches modeling real phenomena. For what concerns the dynamical point of view, a parametric *birth–and–growth process* was studied in (Møller, 1992; 1995). A birth–and–growth process is a random closed sets (RaCS) family given by $\Theta_t = \bigcup_{n:T_n\le t}\Theta^t_{T_n}(X_n)$, for $t \ge 0$, where $\Theta^t_{T_n}(X_n)$ is the RaCS obtained as the evolution up to time $t > T_n$ of the germ born at (random) time T_n in (random) location X_n, according to some growth model. An analytical approach is often used to model birth–and–growth process, in particular it is assumed that the growth of a spherical nucleus of infinitesimal radius is driven according to a non negative normal velocity, i.e. for every instant t, a border point of the crystal $x \in \partial\Theta_t$ "grows" along the outwards normal unit, e.g. (Frost and Thompson, 1987; Burger *et al.*, 2006; 2007; Chiu, 2004; Aquilano *et al.*, 2009). In view of the chosen framework, different parametric and non parametric estimations are proposed over the years, cf. (Møller and Sørensen, 1994; Molchanov and Chiu, 2000; Erhardsson, 2001; Capasso, 2003a; Capasso and Villa, 2005; Aletti and Saada, 2008; Chiu *et al.*, 2003) and references therein. Note that the existence of the outwards normal vector imposes a regularity condition on $\partial\Theta_t$ (and also on the nucleation process; it cannot be a point process).

In this paper, we summarize recent results obtained by Aletti *et al.* (2008a;b). In fact, in order to avoid regularity assumptions describing birth–and–growth processes, the authors offer an original approach based on a purely stochastic geometric point of view that leads to different and significant statistical results . In (Aletti *et al.*, 2008a), they derive a computationally tractable mathematical model (based on Minkowski sum and Aumann integral) rigorously defined as a suitable combination of two very general set–valued processes representing nucleation $\{B_t\}_{t\in[t_0,T]}$ and growth $\{G_t\}_{t\in[t_0,T]}$ respectively. In (Aletti *et al.*, 2008b), different set–valued parametric estimators of the rate of growth of the process are introduced. These are consistent as the observation window expands to the whole space. Moreover, keeping in mind that distributions of random closed sets are determined by hitting functions and that the nucleation process cannot be observed directly, an estimation procedure of the hitting function of the nucleation process is provided.

PRELIMINARY RESULTS

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of all non negative integer and real numbers respectively, and let $\mathfrak{X} = \mathbb{R}^d$. Let $\mathbb{F}$ be the family of all closed subsets of $\mathfrak{X}$ and $\mathbb{F}' = \mathbb{F}\setminus\{\emptyset\}$. The subscripts b, k and c denote boundedness, compactness and convexity properties respectively (e.g. $\mathbb{F}_{kc}$ denotes the family of all compact convex subsets of $\mathfrak{X}$). For all $A,B \subseteq \mathfrak{X}$ and $\alpha \ge 0$, let us consider

$$\begin{aligned} A+B &= \{a+b : a\in A,\ b\in B\} = \textstyle\bigcup_{b\in B} b+A,\\ A\ominus B &= \left(A^C+B\right)^C = \textstyle\bigcap_{b\in B} b+A,\\ \check{A} &= \{-a : a\in A\}, \end{aligned}$$

where $A^C = \mathfrak{X}\setminus A$, $x+A$ means $\{x\}+A$, and, by definition, $\emptyset + A = \emptyset = \alpha\emptyset$. In the following, we deal with closed sets; in particular, whenever sum between sets occurs, a closed bounded sets (of $\mathbb{R}^d$) is involved. The following result is applied: if $A\in\mathbb{F}$ and $B\in\mathbb{F}_k$ then $A+B\in\mathbb{F}$ (Serra, 1984).

For any $A,B\in\mathbb{F}'$ the *Hausdorff distance* is defined by

$$\delta_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathfrak{X}}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{\mathfrak{X}}\Big\}.$$

A measurable closed set–valued map X is a function defined on a finite measure space $(\Omega,\mathfrak{F},\mu)$ with values in $\mathbb{F}$ such that $\{\omega\in\Omega : X(\omega)\cap K\neq\emptyset\}$ is measurable for each compact set K in $\mathfrak{X}$. If μ is a probability measure, then X is a random closed set (RaCS).

Let X be a RaCS, then $\{T_X(K) = \mathbb{P}(X\cap K\neq\emptyset), K\in\mathbb{F}_k\}$, is its *hitting function*. The well known Choquet–Kendall–Matheron Theorem states that, the probability law $\mathbb{P}_X$ of any RaCS X is uniquely determined by its hitting function (Matheron, 1975) and hence by $Q_X(K) = 1 - T_X(K)$. A RaCS X is *stationary* if the probability laws of X and $X+v$ coincide for every $v\in\mathfrak{X}$; i.e. $T_X(K) = T_X(K+v)$ for each $K\in\mathbb{F}_k$ and $v\in\mathfrak{X}$. A stationary RaCS X is *ergodic*, if, for all $K_1,K_2\in\mathbb{F}$,

$$\frac{1}{|W_i|}\int_{W_i} Q_X((K_1+v)\cup K_2)dv \underset{i\to\infty}{\longrightarrow} Q_X(K_1)Q_X(K_2),$$

where $\{W_i\}_{i\in\mathbb{N}}$ is a *convex averaging sequence of sets* in $\mathfrak{X}$ (Daley and Vere-Jones, 2003), i.e. each $\{W_i\}$ is convex and compact, $W_i\subset W_{i+1}$ for all $i\in\mathbb{N}$ and

$$\sup\{r\ge 0 : B(x,r)\subset W_i \text{ for some } x\in W_i\}\uparrow\infty, \text{ as } i\to\infty$$

(we shall write $W_i\uparrow\mathfrak{X}$).

Let $(\Omega,\mathfrak{F},\mu)$ be a finite measure space. The *Aumann integral* of a non empty measurable closed set–valued map X is defined by

$$\int_\Omega X d\mu = \left\{\int_\Omega x d\mu : x\in S_X\right\},$$

where $S_X = \{x\in L^1[\Omega;\mathfrak{X}] : x\in X\ \mu\text{–a.e.}\}$ and $\int_\Omega x d\mu$ is the usual Bochner integral in $L^1[\Omega;\mathfrak{X}]$. Moreover, $\int_A X d\mu = \{\int_A x d\mu : x\in S_X\}$ for $A\in\mathfrak{F}$.

GEOMETRIC RANDOM PROCESS

Here, $\mathscr{F}$ denotes the family of all fuzzy sets $\nu : \mathfrak{X} \to [0,1]$. A fuzzy random set is a measurable map $X : \Omega \to \mathscr{F}$, where Ω and $\mathscr{F}$ are endowed with the relevant σ–algebra's (Li *et al.*, 2002). A family of fuzzy random sets $\{X_t\}_{t\geq 0}$ is called a fuzzy set–valued stochastic process. For $\beta \in (0,1]$, we call *β–fuzzy monotone set–valued stochastic process* a fuzzy set–valued stochastic process X such that, for every $\omega \in \Omega$ and $t_1, t_2 \geq 0$ with $t_1 \leq t_2$,

$$X_\alpha(\omega,t_1) \subseteq X_\alpha(\omega,t_2), \text{ for each } \alpha \in (0,1] \text{ with } \beta \leq \alpha$$

where $X_\alpha(\omega,t) = \{x \in \mathfrak{X} : X(\omega,t)(x) \geq \alpha\}$ is the α–level set of the fuzzy set $X(\omega,t)$. In other words, a β–fuzzy monotone set–valued stochastic process is a time dependent fuzzy random set for which every α–level processes are non decreasing RaCS processes, for any $\beta \leq \alpha$. Clearly, the associated α–level set stochastic processes X_α are useful in order to study a fuzzy monotone set–valued stochastic process X. In the following, we deal with 1–fuzzy monotone set–valued stochastic process. A set–valued stochastic process is modeled to describe $\Theta = X_1$ process and, in the next section, analyzed from a statistical point of view. In particular, we describe here the main results of (Aletti *et al.*, 2008a) in which the interested reader can find the detailed proofs.

(A-0) $[t_0,T] \subset \mathbb{R}$ is the *time interval*, and $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\in[t_0,T]},\mathbb{P})$ is a filtered probability space, where the filtration $\{\mathfrak{F}_t\}_{t\in[t_0,T]}$ is assumed to have the usual properties.

Let B and G be two processes, *Nucleation* and *Growth Process* respectively, defined on $\Omega \times [t_0,T]$ with non empty closed set values, for which the following assumptions hold.

(A-1) For every $t \in [t_0,T]$, $B_t = B(\cdot,t)$ is a RaCS defined on $(\Omega,\mathfrak{F}_t,\mathbb{P})$, i.e. B_t is an *adapted*, with respect to (w.r.t.) $\{\mathfrak{F}_t\}_{t\in[t_0,T]}$, RaCS process.

(A-2) B_t is *non decreasing*: for every $t,s \in [t_0,T]$ with $s < t$, $B_s \subseteq B_t$.

(A-3) For every $\omega \in \Omega$ and $t \in [t_0,T]$, $0 \in G(\omega,t)$.

(A-4) For every $\omega \in \Omega$ and $t \in [t_0,T]$, $G(\omega,t)$ is convex, i.e. $G : \Omega \times [t_0,T] \to \mathbb{F}'_c$.

(A-5) There exists $K \in \mathbb{F}'_b$ such that $G(\omega,t) \subseteq K$ for every $t \in [t_0,T]$ and $\omega \in \Omega$.

Let $\mathscr{P}$ denote the *previsible* (or *predictable*) σ–algebra on $\Omega \times [t_0,T]$ generated by the processes $\{X_t\}_{t\in[t_0,T]}$ adapted, w.r.t. $\{\mathfrak{F}_t\}_{t\in[t_0,T]}$, with left Hausdorff–continuous trajectories on $[t_0,T]$. Thus, let us assume the following fact,

(A-6) G is $\mathscr{P}$–measurable.

It can be proved that, for any $a,b \in [t_0,T]$, $G_{a,b} = \int_a^b G(\omega,\tau)d\tau$ is a non empty bounded (compact) convex RaCS. For every $t \in [t_0,T] \subset \mathbb{R}$, $n \in \mathbb{N}$, and $\Pi = (t_i)_{i=0}^n$ partition of $[t_0,t]$, let us define

$$s_\Pi(t) = \big(B_{t_0} + \int_{t_0}^t G(\tau)d\tau\big) \cup \bigcup_{i=1}^n \big(\Delta B_{t_i} + \int_{t_i}^t G(\tau)d\tau\big) \quad (1)$$

$$S_\Pi(t) = \big(B_{t_0} + \int_{t_0}^t G(\tau)d\tau\big) \cup \bigcup_{i=1}^n \big(\Delta B_{t_i} + \int_{t_{i-1}}^t G(\tau)d\tau\big) \quad (2)$$

where $\Delta B_{t_i} = B_{t_i} \setminus B^o_{t_{i-1}}$ ($B^o_{t_{i-1}}$ denotes the interior set of $B_{t_{i-1}}$) and where the integral is in the Aumann sense w.r.t. the Lebesgue measure $d\tau = d\mu_\lambda$. We write s_Π and S_Π instead of $s_\Pi(t)$ and $S_\Pi(t)$ when the dependence on t is clear.

Proposition 3.1 guarantees that both s_Π and S_Π are well defined RaCS, further, Proposition 3.2 shows $s_\Pi \subseteq S_\Pi$ as a consequence of different time intervals integration. Proposition 3.3 means that $\{s_\Pi\}$ ($\{S_\Pi\}$) does not decrease (does not increase) whenever a refinement of Π is considered. At the same time, Proposition 3.4 implies that s_Π and S_Π become closer each other (in the Hausdorff distance sense) when partition Π becomes finer. The "limit" is independent on the choice of the refinement as consequence of Proposition 3.5. Corollary 3.6 means that, given any $\{\Pi_j\}_{j\in\mathbb{N}}$ refinement sequence of $[t_0,t]$, the RaCS s_{Π_j} and S_{Π_j} play the same role that lower sums and upper sums have in classical analysis when we define the Riemann integral. In fact, if Θ_t denotes their limit value (cf. Definition 3.7), s_{Π_j} and S_{Π_j} are a lower and an upper approximation of Θ_t respectively. This argument prevents problems that may arise considering uncountable unions in (1), (2) instead of countable unions.

Proposition 3.1 Let Π be a partition of $[t_0,t]$. Both s_Π and S_Π, defined in (1) and (2), are RaCS.

Proposition 3.2 Let Π be a partition of $[t_0,t]$. Then $s_\Pi \subseteq S_\Pi$ almost surely.

Proposition 3.3 Let Π and Π' be two partitions of $[t_0,t]$ such that Π' is a refinement of Π. Then, almost surely, $s_\Pi \subseteq s_{\Pi'}$ and $S_{\Pi'} \subseteq S_\Pi$.

Proposition 3.4 Let $\{\Pi_j\}_{j\in\mathbb{N}}$ be a refinement sequence of $[t_0,t]$ (i.e. $|\Pi_j| \to 0$ if $j \to \infty$). Then, almost surely, $\lim_{j\to\infty} \delta_H\left(s_{\Pi_j}, S_{\Pi_j}\right) = 0$.

Proposition 3.5 Let $\{\Pi_j\}_{j\in\mathbb{N}}$ and $\{\Pi'_l\}_{l\in\mathbb{N}}$ be two distinct refinement sequences of $[t_0,t]$, then, almost surely,

$$\lim_{\substack{j\to\infty\\ l\to\infty}} \delta_H(s_{\Pi_j}, s_{\Pi'_l}) = 0 \quad\text{and}\quad \lim_{\substack{j\to\infty\\ l\to\infty}} \delta_H(S_{\Pi_j}, S_{\Pi'_l}) = 0.$$

Corollary 3.6 For every $\{\Pi_j\}_{j\in\mathbb{N}}$ refinement sequence of $[t_0,t]$, the following limits exist

$$\overline{\bigcup_{j\in\mathbb{N}} s_{\Pi_j}},\ \overline{(\lim_{j\to\infty} s_{\Pi_j})},\ \lim_{j\to\infty} S_{\Pi_j},\ \bigcap_{j\in\mathbb{N}} S_{\Pi_j},$$

and they are equals almost surely. The convergences is taken w.r.t. the Hausdorff distance.

We are now ready to define continuous time stochastic processes.

Definition 3.7 For every $t\in[t_0,T]$, let $\{\Pi_j\}_{j\in\mathbb{N}}$ be a refinement sequence of the time interval $[t_0,t]$ and let Θ_t be the RaCS defined by

$$\overline{\bigcup_{j\in\mathbb{N}} s_{\Pi_j}(t)} = \overline{(\lim_{j\to\infty} s_{\Pi_j}(t))} = \Theta_t = \lim_{j\to\infty} S_{\Pi_j}(t) = \bigcap_{j\in\mathbb{N}} S_{\Pi_j}(t),$$

then, $\Theta = \{\Theta_t : t\in[t_0,T]\}$ is called *geometric random process G-RaP* (on $[t_0,T]$).

As a consequence, Θ is an a.s. non decreasing process, i.e. $\mathbb{P}(\Theta_s \subseteq \Theta_t,\ \forall t_0\le s<t\le T) = 1$. Further, Θ is adapted w.r.t. $\{\mathfrak{F}_t\}_{t\in[t_0,T]}$.
We want to point out that, assumptions we considered on $\{B_t\}$ and $\{G_t\}$ are so general, that a wide family of classical random sets and evolution processes can be described accordingly (e.g., Boolean model can be seen as a G-RaP with “null growth”).
The presented setting allows us to justify, also with an abuse of notations, the following infinitesimal and differential model formulations. In particular, for $t\in[t_0,T]$,

$$\Theta_t = \left(B_{t_0} + \int_{t_0}^t G(\tau)d\tau\right) \cup \bigcup_{s=t_0}^t \left(dB_s + \int_s^t G(\tau)d\tau\right),$$
$$d\Theta_t = +G_t dt \cup dB_t \quad\text{or}\quad \Theta_{t+dt} = (\Theta_t + G_t dt)\cup dB_t.$$

Roughly speaking, an increment $d\Theta_t$, during an infinitesimal time interval dt, is an enlargement due to an infinitesimal addend $G_t dt$ followed by the union with the infinitesimal nucleation dB_t. Note that, as a consequence of the definition of $+$, at any instant t, each point $x\in\Theta_t$ (and then each point $x\in\partial\Theta_t$) grows up by $G_t dt$ and no regularity boundary assumptions are required. Then we deal with *non local* growth; i.e. growth is the same addend for every $x\in\Theta_t$. Nevertheless, under mesoscale hypotheses we may only consider constant growth region as described, for example, in (Burger *et al.*, 2006). On the other hand, growth is anisotropic whenever G_t is not a ball.

STATISTICAL ASPECTS

With simple observations and a suitable change of notations, it is easy to derive the following discrete time formulation of above model

$$\Theta_n = \begin{cases} (\Theta_{n-1} + G_n)\cup B_n, & n\ge 1,\\ B_0, & n=0.\end{cases}$$

In view of applications, note that a sample of a birth–and–growth process is usually a time sequence of pictures that represent process Θ at different temporal step; namely Θ_{n-1}, Θ_n that, for the sake of simplicity, we shall also denote by X and Y respectively.
In (Aletti *et al.*, 2008b), the rate growth of Θ and the hitting function of B_n are estimated. In fact, G_n is not identified univocally, while the RaCS $Y\ominus\check{X}$ (denoted, from now on and with an abuse of notation, by G) is unique, since it is the greatest RaCS, w.r.t. set inclusion, for which $(X+G)\subseteq Y$. Let us assume the following facts.

(A-7) There exists $K\in\mathbb{F}'_b$ such that $G\subseteq K$.

(A-8) For every $n\ge 1$, $\left(B_n\ominus\check{\Theta}_{n-1}\right) = \emptyset$ a.s.

Roughly speaking, Assumption (A-7) means that process Θ does not grow too “fast”, whilst Assumption (A-8) means that it cannot born something that, up to a translation, is larger than (or equal to) what there already exists.
In practical cases, data are bounded by some observation window and edge effects may cause problems estimating G. As the standard statistical scheme for spatial processes suggests (Molchanov, 1997), we wonder if there exists a consistent estimator of G as $W_i\uparrow\mathfrak{X}$. Thus, let $W\in\{W_i\}$ and let us set $Y_W = Y\cap W$. Edge effects are reduced by considering the following estimators of G

$$\widehat{G}^1_W = \left(Y_W \ominus \check{X}_{W\ominus\check{K}}\right)\cap K,$$
$$\widehat{G}^2_W = \left(\left[Y_W \cup \left(\partial^{+K}_W X_W\right)\right]\ominus\check{X}_W\right)\cap K;$$

where K is given in Assumption (A-7) and where $\left(\partial^{+K}_W X_W\right) = \overline{(X_W+K)\setminus W}$. The following results hold.

Proposition 4.1 Let Y, X be RaCS, let $0\in G = Y\ominus\check{X}\subseteq K$. Thus, for any $W_2\supseteq W_1$, $G\subseteq\widehat{G}^1_{W_2}\subseteq\widehat{G}^1_{W_1}$. In particular, $\bigcap_{i\in\mathbb{N}}\widehat{G}^1_{W_i} = G$ and $\lim_{i\to\infty}\delta_H(\widehat{G}^1_{W_i}, G) = 0$. Moreover, for every $W\in\mathbb{F}'$, $G\subseteq\widehat{G}^2_W\subseteq\widehat{G}^1_W$. Thus, $\widehat{G}^2_W$ is consistent too (i.e. if $W\uparrow\mathfrak{X}$ $\widehat{G}^2_W\downarrow G$).

In Figure 2, for two different time instants (X and Y) pictures of a simulated birth–and–growth process, we show the magnified pictures of: the true growth used for the simulation, the computed $\widehat{G}^2_W$, $\widehat{G}^1_W$ and $\widehat{G}^1_{W\ominus\check{K}}$. Propositions 4.1 is satisfied since $\widehat{G}^1_{W\ominus\check{K}} \supseteq \widehat{G}^1_W \supseteq \widehat{G}^2_W$.

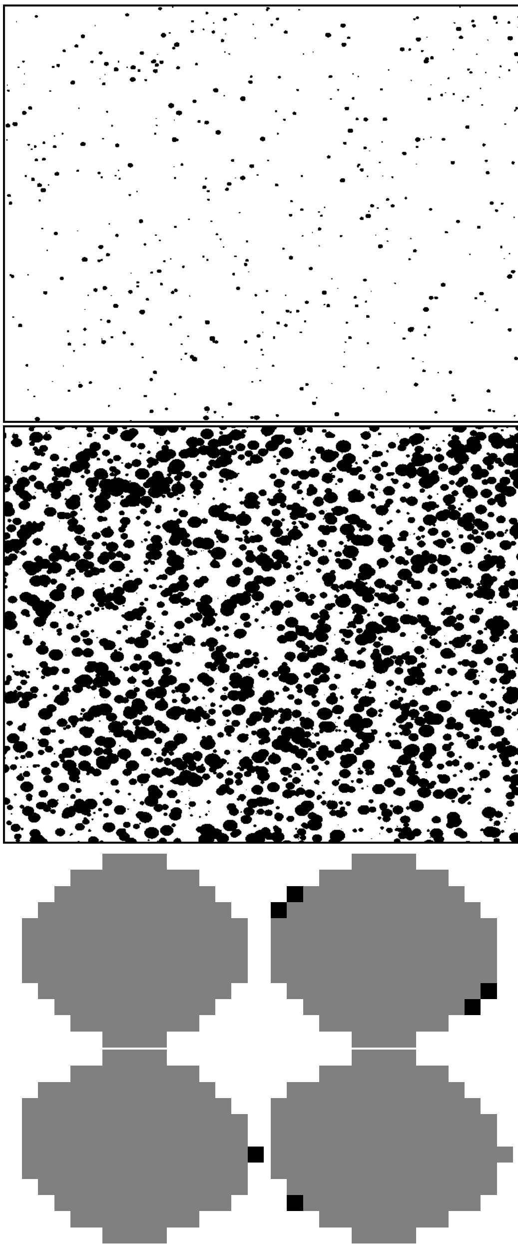


Fig. 2. *Two different time instants (X and Y) pictures of a simulated birth–and–growth process. The magnified pictures of the true growth used for the simulation, the computed* $\widehat{G}^2_W$, $\widehat{G}^1_W$ *and* $\widehat{G}^1_{W\ominus\check{K}}$.

From the birth–and–growth process point of view, it is also interesting to test whenever the nucleation process $B = \{B_n\}_{n\in\mathbb{N}}$ is a specific RaCS (for example a Boolean model or a point process). In general, we cannot directly observe the n–th nucleation B_n since it can be overlapped by other nuclei or by their evolutions. Nevertheless, we shall infer on the hitting function associated to the nucleation process $T_{B_n}(\cdot)$.

A *regular closed* set in $\mathfrak{X}$ is a closed set $G \in \mathbb{F}'$ for which $G = \overline{\text{Int}\, G}$.
For any $K \in \mathbb{F}_k$, let $\widetilde{Q}_{B,W}(K) = \widehat{Q}_{Y,W}(K)/\widehat{Q}_{X+\widehat{G}_W,W}(K)$, where $\widehat{Q}_{(\cdot)} = 1 - \widehat{T}_{(\cdot)}$ is defined in (Molchanov, 1997) and $\widehat{G}_W$ is one between $\widehat{G}^2_W$ and $\widehat{G}^1_W$.

Theorem 4.2 Let X, Y be a.s. regular closed. Let G, B be two RaCS such that $Y = (X + G) \cup B$, with B a stationary ergodic RaCS independent on G and X, and with G a.s. regular closed. Then, for any $K \in \mathbb{F}_k$,

$$\left|\widetilde{Q}_{B,W}(K) - Q_B(K)\right| \underset{W\uparrow\mathfrak{X}}{\longrightarrow} 0, \quad \text{a.s.}$$

CONCLUSIONS

Fuzzy monotone set–valued stochastic processes can be used to describe crystal growth processes. In this framework, α–level sets, modeled as birth–and–growth processes, are considered to analyze statistical aspects of crystal processes.
In this paper, a continuous time set–valued stochastic process modeling birth–and–growth process is defined. Statistical aspects of α–level sets have been considered; in particular, consistent estimators have been provided for a general birth–and–growth stochastic process. A pure geometrical approach reduces the estimation of growth process to simple operations among sets. At the same time, consistent estimators for the hitting function of nucleation process have been also provided.
Finally, we want to suggest some possible future developments. It may be interesting to define new mathematical models for fuzzy monotone set–valued stochastic process, in order to study distributions of estimators and to construct confidence intervals for the model parameters.

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