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APPLYING THE HADAMARD PRODUCT TO DECOMPOSE GINI, CONCENTRATION, REDISTRIBUTION AND RE-RANKING INDEXES

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ABSTRACT

Gini and concentration indexes are well known useful tools in analysing redistribution and re-ranking effects of taxes with respect to a population of income earners. There are several attempts in the literature to decompose Gini and re-ranking indices to analyse potential redistribution effects and the unfairness of a tax systems, including ones that consider contiguous income groups being created by dividing the pre-tax income parade according to the same bandwidth. However, earners may be very often split into groups characterized by social and demographic aspects or by other characteristics: in these circumstances groups can easily overlap. In this paper we consider a more general situation that takes into account overlapping among groups; we obtain matrix compact forms for Gini and concentration indexes, and consequently, for redistribution and re-ranking indexes. In deriving formulae the so called matrix Hadamard product is extensively used. Matrix algebra allows to write indexes aligning incomes in a non decreasing order either with respect to post-tax income or to pre-tax incomes. Moreover, matrix compact formulae allow an original discussion for the signs of the *within group*, *across group*, *between* and *transvariation* components into which the Atkinson-Plotnick-Kakwany re-ranking index can split.

Key words: Gini and concentration indexes decompositions, Tax redistributive effects, Tax re-ranking effects, Hadamard product.

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Introduction

It is known that, dealing with a transferable phenomenon where units are classifiable into groups, Gini index fails to decompose additively into a between and a within component if the group ranges overlap. Following Bahattacharya and Mahalanobis (1967), a number of Gini decompositions was proposed (Rao (1969), Pyatt (1976), Mookherjee and Shorrocks (1982), Silber (1989), Yitzhaki and Lerman (1991), Lambert and Aronson (1993), Yitzhaki (1994), Dagum (1997)) and after Lambert and Aronson (1993), the third component of the conventional Gini index decomposition is denoted by overlapping term.

Monti (2007) shows that the conventional and the Dagum (1997) decomposition are identical, so that an alternative way to calculate the overlapping term can be derived from the decomposition suggested by this author.

Aronson, Johnson and Lambert (1994), Urban and Lambert (2008), use Gini and concentration index decomposition to identify and evaluate potential distributive effects and unfairness in a tax system. These authors consider contiguous income groups created by dividing the pre-tax income parade according to an identical bandwidth, so that the pre-tax income parade excludes overlapping by construction.

In the present paper we consider incomes gathered into groups characterized by social, demographic or income sources characteristics, so that overlapping among groups need not to be excluded. Our results are obtained using the Gini index decomposition derived from Dagum decomposition (Monti and Santoro 2007, Monti 2008).

Making use of the Hadamard product, in the first section we present Gini and concentration indexes in compact matrix forms. In the second section we introduce groups, present Gini and concentration indexes and show how within groups, across, between groups and transvariation components can be written in matrix compact forms. Links from matrix compact forms and scalar forms are reported: some scalar expressions are well known in literature, while others appears as modifications of already well known forms.

Section 3 presents matrix forms for redistribution and re-ranking indexes, together with their within, across, between groups and transvariation components.

In the fourth section we show how the signs of Atkinson-Plotnick-Kakwani (Plotnick 1981) re-ranking index components can be analysed, thanks to the algebraic tools presented in the paper.

1. Matrix forms for concentration and Gini indexes

Let X and Y be two real non negative statistical variables that describe a transferable phenomenon for a population of K units, $K \in \mathrm{N}$. In this paper we suppose that X represents income before taxation and Y after-tax income; not infrequently the pair (x_i , y_i) has associated a weight p_i (i=1,……,K), $\sum_{i=1}^{K} p_i = N$. Furthermore, in measuring concentration we generally need to rank either x_i or y_i in a non-decreasing order: when the X elements are ranked in a non-decreasing order, the sequence of (x_i, y_i, p_i) triplets will be indicated as $\{(x_i, y_i, p_i)\}_X$; analogously, $\{(x_i, y_i, p_i)\}_Y$ will denote the sequence of (x_i, y_i, p_i), when the Y elements are ranked in a non-decreasing order.

The concentration index [1] for Y, in the ordering $\{(x_i, y_i, p_i)\}_X$, is defined as [2]

$$C_{Y|X} = 1 - \sum_{i=1}^{K}\left(\sum_{j=1}^{i}\frac{y_j p_j}{\mu_Y N} + \sum_{j=1}^{i-1}\frac{y_j p_j}{\mu_Y N}\right)\frac{p_i}{N} = \frac{1}{\mu_Y N^2}\sum_{i=1}^{K}\sum_{j=1}^{i-1}\left(y_i - y_j\right)p_i p_j =$$

$$= \frac{1}{2\mu_Y N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(y_i - y_j\right)p_i p_j I_{i-j} \tag{1}$$

$$I_{i-j} = \begin{cases} 1: & i-j>0 \\ 0: & i-j=0 \\ -1: & i-j<0 \end{cases}$$

where μ_Y is the weighed mean of the observations on Y. Obviously, in the ordering $\{(x_i, y_i, p_i)\}_Y$, the concentration index $C_{Y|Y}$ coincides with the Gini index G_Y and, analogously in the ordering $\{(x_i, y_i, p_i)\}_X$, $C_{X|X} \equiv G_X$ [3]. Generally, when tax effects are analyzed, one considers the Gini index for the pre-tax distribution G_X , the Gini index for the post-tax distribution G_Y, and the concentration index for the post tax distributions, $C_{Y|X}$, with incomes ranked according to the $\{(x_i, y_i, p_i)\}_X$ ordering.

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[1] The author is in debt with Maria Monti for the suggestion to express the concentration index by differences between incomes: this suggestion is at the basis of this paper. It can be shown that in expressions (1) the first formula is equal to the second one: the proof can be easily obtained following the demonstration that Landenna (1994, Ch. 4, § 4.4.) gives for the Gini index. In the right hand side of (1), the first component calculates the normalized concentration. In the case where the y's are in a non decreasing order, the second one is the normalized mean absolute difference, that is $G_Y = \left(1/2\mu_Y N^2\right)\sum_{i=1}^{K}\sum_{j=1}^{K}\left|y_i - y_j\right| p_i p_j = \Delta/2\mu_Y$.

[2] The indicator function I_{i-j} is a particular case of generalized functions considered in Faliva (2000): this article can be consulted for I_{i-j} properties.

[3] For definitions concerning concentration indexes and their relations with Gini indexes, see e.g. Kakwani (1980), in particular Ch. 5 and 8.

In order to pass to a matrix representation, we stack the *K* observations on *X*, *Y* and the weights *P* into *K*×1 vectors: when referring to the ordering $\{(x_i, y_i, p_i)\}_X$, the vectors will be indicated as $\mathbf{x}$, $\mathbf{y}_X$ and $\mathbf{p}_X$, while, referring to the ordering $\{(x_i, y_i, p_i)\}_Y$, the vectors will be labelled as $\mathbf{x}_Y$, $\mathbf{y}$ and $\mathbf{p}_Y$, that is, when elements in a vector are ranked in a non-decreasing order no label will be added, conversely, when they are ordered according to a non-decreasing order for another variable, this variable will be explicitly indicated.

We also introduce the following definitions:

$\mathbf{S} = \left[s_{i,j} \right]$ will denote a *K*×*K* semi-symmetric matrix with diagonal elements equal to zero, super-diagonal elements equal to 1 and sub-diagonal elements equal to −1;

$\mathbf{j}$ for a *K*×1 vector that has entries equal to 1;

$\mathbf{D}_X$ and $\mathbf{D}_Y$ will denote the *K*×*K* matrices $\mathbf{D}_X = (\mathbf{jx}' - \mathbf{xj}')$, $\mathbf{D}_Y = (\mathbf{jy}' - \mathbf{yj}')$.

Then, by making use of the Hadamard product $\odot$, we can express the indexes G_Y and G_X as follows [1]:

$$G_Y = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y' (\mathbf{S} \odot \mathbf{D}_Y) \mathbf{p}_Y \qquad G_X = \frac{1}{2\mu_X N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_X) \mathbf{p}_X \tag{2}$$

where μ_Y and μ_X are the weighed mean of the observations on *Y* and on *X*, respectively.

In addition, by introducing the *K*×*K* matrix $\mathbf{D}_{Y|X} = (\mathbf{jy}_X' - \mathbf{y}_X \mathbf{j}')$, we can write the concentration index in compact form as

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' (\mathbf{S} \odot \mathbf{D}_{Y|X}) \mathbf{p}_X \tag{3}$$

The transformation from vectors $\mathbf{y}$ and $\mathbf{p}_Y$ to vectors $\mathbf{y}_X$ and $\mathbf{p}_X$ can be performed by a proper *K*×*K* permutation matrix $\mathbf{E}$. The reverse transformation from $\mathbf{y}_X$ and $\mathbf{p}_X$ to $\mathbf{y}$ and $\mathbf{p}_Y$ can be obtained through the matrix $\mathbf{E}^{-1}$ which is equal to $\mathbf{E}'$. Formally

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[1] The Hadamard product for two matrices **A** and **B** is defined if both of them have the same number of rows and the same number of columns: $\left[a_{i,j} \right] \odot \left[b_{i,j} \right] = \left[a_{i,j} \cdot b_{i,j} \right]$. For the definition and properties of the Hadamard product see, e.g., Faliva (1983, Appendix) and (1987, Ch. 3), Schott (2005, Ch. 5).

$$\begin{cases} \mathbf{y}_X = \mathbf{E}\mathbf{y}, & \mathbf{y} = \mathbf{E}'\mathbf{y}_X \\ \mathbf{x} = \mathbf{E}\mathbf{x}_Y, & \mathbf{x}_Y = \mathbf{E}'\mathbf{x} \\ \mathbf{p}_X = \mathbf{E}\mathbf{p}_Y, & \mathbf{p}_Y = \mathbf{E}'\mathbf{p}_X \end{cases} \tag{4}$$

We shall show that, with some suitable algebraic permutations of the elements of **S**, it is possible to reformulate both the matrices **D** and the vectors **p** in (2) and (3) according either to the $\{(x_i, y_i, p_i)\}_X$ or to the $\{(x_i, y_i, p_i)\}_Y$ ordering, maintaining both Gini and concentration indexes unchanged. This leads to rewrite the expressions of formula (2) as

$$G_Y = \frac{1}{2\mu_Y N^2}\mathbf{p}_X'\left(\mathbf{ESE}' \odot \mathbf{D}_{Y|X}\right)\mathbf{p}_X \quad \text{and} \quad G_X = \frac{1}{2\mu_X N^2}\mathbf{p}_X'\left(\mathbf{S} \odot \mathbf{D}_X\right)\mathbf{p}_X \tag{5}$$

or as

$$G_Y = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left(\mathbf{S} \odot \mathbf{D}_Y\right)\mathbf{p}_Y \quad \text{and} \quad G_X = \frac{1}{2\mu_X N^2}\mathbf{p}_Y'\left(\mathbf{E}'\mathbf{SE} \odot \mathbf{D}_{X|Y}\right)\mathbf{p}_Y \tag{6}$$

where $\mathbf{D}_{Y|X} = \left(\mathbf{j}\mathbf{y}_X' - \mathbf{y}_X\mathbf{j}'\right)$ and $\mathbf{D}_{X|Y} = \left(\mathbf{j}\mathbf{x}_Y' - \mathbf{x}_Y\mathbf{j}'\right)$, respectively.

Moreover, $C_{Y|X}$ can be given in the following alternative form:

$$C_{Y|X} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left(\mathbf{E}'\mathbf{SE} \odot \mathbf{D}_Y\right)\mathbf{p}_Y \tag{7}$$

Proof

Consider G_X as specified in (2) and (6). As $\mathbf{EE}' = \mathbf{E}'\mathbf{E} = \mathbf{I}$, the following holds:

$$\mathbf{p}_X'\left(\mathbf{S} \odot \mathbf{D}_X\right)\mathbf{p}_X = \mathbf{p}_X'\mathbf{EE}'\left(\mathbf{S} \odot \mathbf{D}_X\right)\mathbf{EE}'\mathbf{p}_X = \mathbf{p}_Y'\left(\mathbf{E}'\mathbf{SE} - \mathbf{E}'\mathbf{D}_X\mathbf{E}\right)\mathbf{p}_Y$$

by keeping in mind the noteworthy property of the Hadamard product, $\mathbf{E}'\left(\mathbf{S} \odot \mathbf{D}_X\right)\mathbf{E} = \left(\mathbf{E}'\mathbf{SE}\right) \odot \left(\mathbf{E}'\mathbf{D}_X\mathbf{E}\right)$ (Faliva 1996, property *vii,* page. 157).

Noticing that $\mathbf{E}'\mathbf{D}_X\mathbf{E} = \mathbf{E}'\left(\mathbf{jx}' - \mathbf{xj}'\right)\mathbf{E} = \left(\mathbf{jx}'\mathbf{E} - \mathbf{E}'\mathbf{xj}'\right) = \left(\mathbf{j}\mathbf{x}_Y' - \mathbf{x}_Y\mathbf{j}'\right) = \mathbf{D}_{X|Y}$, as $\mathbf{E}'\mathbf{j} = \mathbf{j}$ and $\mathbf{j}'\mathbf{E} = \mathbf{j}'$, the equivalence of expression (2) and expression (6) for G_X is proved.

The equivalence of expressions (2) and (5) for G_Y can be likewise proved. Indeed, the following holds:

$$\mathbf{p}_Y'\left(\mathbf{S} \odot \mathbf{D}_Y\right)\mathbf{p}_Y = \mathbf{p}_Y'\mathbf{E}'\mathbf{E}\left(\mathbf{S} \odot \mathbf{D}_Y\right)\mathbf{E}'\mathbf{E}\mathbf{p}_Y = \mathbf{p}_X'\left(\mathbf{ESE}' - \mathbf{D}_{Y|X}\right)\mathbf{p}_X$$

upon noticing that $\mathbf{E}\mathbf{D}_Y\mathbf{E}' = \mathbf{jy}'\mathbf{E}' - \mathbf{Eyj}' = \mathbf{j}\mathbf{y}_X' - \mathbf{y}_X\mathbf{j}' = \mathbf{D}_{Y|X}$.

As far as $C_{Y|X}$ is concerned, expression (3) turns out to be equivalent to expression (7), upon noticing that

$$\mathbf{E'D}_{Y|X}\mathbf{E} = \mathbf{jy}_X\mathbf{'E} - \mathbf{E'y}_X\mathbf{j'} = \mathbf{jy'} - \mathbf{yj'} = \mathbf{D}_Y\,.$$

2. Introducing groups

A population of income earners can be partitioned into *H* groups, $H \in Nn$, which can be characterized by income sources or by social and demographic aspects: typical group characterizations are family composition, dependent/non-dependent worker, men/women, geographic area, etc.

Dagum (1997) decomposes the Gini coefficient into *within groups* (henceforth *W*) and an *across groups* (henceforth *AG*) component. Dagum calls this latter component *gross between*).

Hence $G_Y = G_Y^W + G_Y^{AG}$. In addition, Dagum splits the *AG* component into a *between* and a *transvariation* component: $G_Y^{AG} = G_Y^B + G_Y^T$. The between component G_Y^B is the Gini (weighed) index which results when all values within the same group are replaced by their (weighed) average; the transvariation component G_Y^T measures the overlapping among groups: it is zero when no overlapping exists and it is equal to G_Y^{AG} when all group averages are equal [1]. Extending Dagum's decompositions to concentration indexes, we can split $C_{Y|X}$ into the two components *W* and *AG*, and write $C_{Y|X} = C_{Y|X}^W + C_{Y|X}^{AG}$, accordingly with

$$C_{Y|X}^W = \frac{1}{2\mu_Y N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(y_i - y_j\right)p_i p_j \cdot I_{i,j\in h} \cdot I_{i-j} \tag{8}$$

$$C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\left(y_i - y_j\right)p_i p_j \cdot \left(1 - I_{i,j\in h}\right) \cdot I_{i-j} \tag{9}$$

In (8) and (9) I_{i-j} is as defined in (1) above, and $I_{i,j\in h}$ is an indicator function: $I_{i,j\in h} = 1$ if both y_i and y_j belong to the same group *h* (*h*=1,2,...,*H*), $I_{i,j\in h} = 0$ if y_i and y_j do not.

[1] For more details on the expression of the Gini components in the Dagum decomposition, see e.g. Monti (2008).

Similar expressions hold for $C_{Y|Y}^{W} = G_{Y}^{W}$, $C_{Y|Y}^{AG} = G_{Y}^{AG}$ and $C_{X|X}^{W} = G_{X}^{W}$, $C_{X|X}^{AG} = G_{X}^{AG}$. In particular, for what concerns G^{W} and G^{AG}, the product $\left(y_i - y_j\right) \cdot I_{i-j}$ can be replaced by the absolute difference $\left|y_i - y_j\right|$.

In order to formalize compact matrix forms for $C_{Y|X}^{W}$ and $C_{Y|X}^{AG}$, it is worth to introduce a proper notation. More precisely, $\mathbf{J}$ will denote a $K \times K$ matrix with all elements equal to one, $\mathbf{W}_X = \sum_{h=1}^{H} \mathbf{w}_{X,h} \mathbf{w}_{X,h}'$ a $K \times K$ matrix in the $\left\{(x_i, y_i, p_i)\right\}_X$ ordering, where $\mathbf{w}_{X,h}$ stands for a K×1 vector with the i-th entry equal to one if the income in the i-th position belongs to group h (h=1,2,..,H), whereas it is zero otherwise. The matrix $\mathbf{W}_X$, when applied to $\mathbf{S} \circ \mathbf{D}_{Y|X}$ in expression (3) allows to detect the $\sum_{h=1}^{H} K_h^2$ differences belonging to the same group from the whole K^2 $\left[s_{i,j} \cdot \left(y_i - y_j\right)\right]$ income differences. Conversely, the matrix $\left(\mathbf{J} - \mathbf{W}_X\right)$, when applied to $\mathbf{S} \circ \mathbf{D}_{Y|X}$, allows to detect the $\left(K^2 - \sum_{h=1}^{H} K_h^2\right)$ differences between incomes belonging to different groups. Consider now the following expressions for the W and AG components of $C_{Y|X}$:

$$C_{Y|X}^{W} = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' \left(\mathbf{W}_X \circ \mathbf{S} \circ \mathbf{D}_{Y|X}\right) \mathbf{p}_X \tag{10}$$

$$C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2} \mathbf{p}_X' \left[\left(\mathbf{J} - \mathbf{W}_X\right) \circ \mathbf{S} \circ \mathbf{D}_{Y|X}\right] \mathbf{p}_X \tag{11}$$

It is immediate to verify that $C_{Y|X} = C_{Y|X}^{W} + C_{Y|X}^{AG}$. Similar expressions for $G_Y^{W} = C_{Y|Y}^{W}$ and for $G_Y^{AG} = C_{Y|Y}^{AG}$ can be obtained by substituting $\mathbf{p}_X$ with $\mathbf{p}_Y$, $\mathbf{W}_X$ with $\mathbf{W}_Y$ and $\mathbf{D}_{Y|X}$ with $\mathbf{D}_Y$. Likewise, the corresponding expressions for $G_X^{W} = C_{X|X}^{W}$ and $G_X^{AG} = C_{X|X}^{AG}$ are obtained by replacing μ_Y with μ_X, and $\mathbf{D}_X$ with $\mathbf{D}_{Y|X}$. Observe also [1] that $\mathbf{W}_Y = \mathbf{E}'\mathbf{W}_X\mathbf{E}$ and $\mathbf{W}_X = \mathbf{E}\mathbf{W}_Y\mathbf{E}'$.

Moreover, Dagum (1997) splits G_Y^{AG} into the components G_Y^{B} and G_Y^{T}, bringing subdivision to the fore. Let uss now label each subject triplet of observations on X, Y and P by a pair of indexes (h,i), instead of one as before: h refers to the group (h=1,2,….,H), whereas i (i=1,2,….,K_h) refers to the position

[1] $\mathbf{w}_{X,h} = \mathbf{E}\mathbf{w}_{Y,h}$ and $\mathbf{w}_{Y,h} = \mathbf{E}'\mathbf{w}_{X,h}$.

that the subject occupies within the h-th group; note that $\sum_{i=1}^{K_h} p_{h,i} = N_h$ and $\sum_{h=1}^{H}\sum_{i=1}^{K_h} p_{h,i} = \sum_{h=1}^{H} N_h = N$.

Dagum's representations are:

$$G_Y = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g=1}^{H}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left|y_{h,i}-y_{g,j}\right| p_{h,i}p_{g,j}\right) \tag{12}$$

$$G_Y^W = \frac{1}{2\mu N^2}\sum_{h=1}^{H}\sum_{i=1}^{K_h}\sum_{j=1}^{K_h}\left|y_{h,i}-y_{h,j}\right| p_{h,i}p_{h,j} \tag{13}$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g\neq h}\sum_{g=1}^{H}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left|y_{h,i}-y_{g,j}\right| p_{h,i}p_{g,j}\right) \tag{14}$$

$$G_Y^B = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g=1}^{H}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left|\mu_{Yh}-\mu_{Yg}\right| p_{h,i}p_{g,j}\right) = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g=1}^{H}\left|\mu_{Yh}-\mu_{Yg}\right| \overline{p}_h\overline{p}_g \tag{15a}$$

where μ_{Yh} represents the income average of the h-th group (h=1, 2, ...,H).

$$G_Y^B = \frac{1}{\mu_Y N^2}\sum_{h=2}^{H}\sum_{g=1}^{h-1}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left(y_{h,i}-y_{g,j}\right) p_{h,i}p_{g,j}\right) \tag{15b}$$

$$G_Y^T = \frac{2}{\mu_Y N^2}\sum_{h=2}^{H}\sum_{g=1}^{h-1}\left(\sum_{i}^{K_g}\sum_{j}^{K_h}\sum_{\{y_{h,i}<y_{g,j}\}}\left|y_{h,i}-y_{g,j}\right| p_{h,i}p_{g,j}\right) \tag{16}$$

where $\overline{p}_h = \sum_{i=1}^{K_h} p_{h,i}$ and $\overline{p}_g = \sum_{j=1}^{K_g} p_{g,j}$.

We refer to Monti and Santoro (2007), formula (6) in particular, for the derivation of expression (15b). Expressions (12) (13), (14) and the first term on the right hand side in (15a) do not need ranking Y values; whereas (15b) and (16) need groups to be ranked according to their averages.

Let us now order the Y values (and the related P and, possibly, X values) so that

(i) within each group they are ranked in a non-decreasing order;

(ii) groups are aligned in a non-decreasing order with respect to the their averages.

Then the Y values parade becomes

$$\mathbf{y}_A' = \left[\left(y_{1,1}, y_{1,2}, \ldots y_{1,K_1}\right), \ldots, \left(y_{h,1}, y_{h,2}, \ldots y_{h,K_h}\right), \ldots, \left(y_{H,1}, y_{H,2}, \ldots y_{H,K_H}\right)\right] \tag{17}$$

$y_{h,i} \le y_{h,i+1}$ (i=1,2,…,K_h) and $\mu_{Yh} \le \mu_{Yh+1}$ (h=1,2,….H) [1].

We shall denote the ordering given by (17) as the $\{(x_i, y_i, p_i)\}_{AY}$ ordering.

The $\{(x_i, y_i, p_i)\}_{AX}$ ordering can be introduced likewise: according to this ordering the X values, together with the related Y and P values, are distributed into the H groups such that

(i) within each group the x's are ranked in a non-decreasing order;

(ii) groups are in a non-decreasing order with respect to their X averages.

Thus, for what concerns the X values, the $\{(x_i, y_i, p_i)\}_{AX}$ ordering will appear as

$$\mathbf{x}_A{}' = \left[\left(x_{1,1}, x_{1,2}, \ldots x_{1,K_1} \right), \ldots, \left(x_{h,1}, x_{h,2}, \ldots x_{h,K_h} \right), \ldots, \left(x_{H,1}, x_{H,2}, \ldots x_{H,K_H} \right) \right] \qquad (18)$$

$x_{h,i} \le x_{h,i+1}$ (i=1,2,…,K_h) and $\mu_{Xh} \le \mu_{Xh+1}$ (h=1,2,….H) [2].

The vectors $\mathbf{y}_A$ in (17) and $\mathbf{x}_A$ in (18) can be expressed as functions of $\mathbf{y}$ and $\mathbf{x}$ respectively, by introducing proper $K \times K$ permutation matrices $\mathbf{A}_Y$ and $\mathbf{A}_X$, such that $\mathbf{y}_A = \mathbf{A}_Y \mathbf{y}$ and $\mathbf{x}_A = \mathbf{A}_X \mathbf{x}$. Since $\mathbf{A}_Y$ and $\mathbf{A}_X$ are permutation matrices, the following holds: $\mathbf{A}_Y{}^{-1} = \mathbf{A}_Y{}'$ and $\mathbf{A}_X{}^{-1} = \mathbf{A}_X{}'$.

The Y vector corresponding to the $\{(x_i, y_i, p_i)\}_{AX}$ ordering can be obtained as $\mathbf{y}_{AX} = \mathbf{A}_X \mathbf{y}_X$, and likewise $\mathbf{p}_{AX} = \mathbf{A}_X \mathbf{p}_X$.

Also $\mathbf{x}_{AY} = \mathbf{A}_Y \mathbf{x}_X$ and $\mathbf{p}_{AY} = \mathbf{A}_Y \mathbf{p}_X$ contain the Y and the P elements, respectively, aligned according to the $\{(x_i, y_i, p_i)\}_{AY}$ ordering.

If we work out (3), (10) and (11), by making use of the property $\mathbf{A}_X{}'\mathbf{A}_X = \mathbf{I}$, we get

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}{}' \left(\mathbf{A}_X \mathbf{S} \mathbf{A}_X{}' \odot \mathbf{D}_{Y|AX} \right) \mathbf{p}_{AX} \qquad (19)$$

$$C^W_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}{}' \left(\mathbf{W}_{AX} \odot \mathbf{S} \odot \mathbf{D}_{Y|AX} \right) \mathbf{p}_{AX} \qquad (20)$$

$$C^{AG}_{Y|X} = \frac{1}{2\mu_Y N^2} \mathbf{p}_{AX}{}' \left[\left(\mathbf{J} - \mathbf{W}_{AX} \right) \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X{}' \odot \mathbf{D}_{Y|AX} \right] \mathbf{p}_{AX} \qquad (21)$$

where $\mathbf{W}_{AX} = \mathbf{A}_X \mathbf{W}_X \mathbf{A}_X{}'$ and $\mathbf{D}_{Y|AX} = \left(\mathbf{j} \mathbf{y}_X{}' \mathbf{A}_X{}' - \mathbf{A}_X \mathbf{y}_X \mathbf{j}' \right) = \left(\mathbf{j} \mathbf{y}_{AX}{}' - \mathbf{y}_{AX} \mathbf{j}' \right)$.

For what concerns $C^W_{Y|X}$ in (21), it is shown in Appendix A2 that

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[1] It is not excluded that $y_{h,i} > y_{g,j}$, $g > h$.

[2] Here also it is not excluded that $x_{h,i} > x_{g,j}$, $g > h$.

$$\mathbf{W}_{AX} \circ \mathbf{A}_X\mathbf{S}\mathbf{A}_X' = \mathbf{W}_{AX} \circ \mathbf{S}\,.$$

Focusing on $C_{Y|X}^{AG}$ decomposition, notice that:

$$C_{Y|X}^{B} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AX}'\left[(\mathbf{J}-\mathbf{W}_{AX})\circ \mathbf{S}\circ \mathbf{D}_{Y|AX}\right]\mathbf{p}_{AX} \tag{22}$$

$$C_{Y|X}^{T} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AX}'\left[(\mathbf{J}-\mathbf{W}_{AX})\circ (\mathbf{A}_X\mathbf{S}\mathbf{A}_X'-\mathbf{S})\circ \mathbf{D}_{Y|AX}\right]\mathbf{p}_{AX} \tag{23}$$

Summing (22) and (23) yields (21).

Should $C_{Y|Y} \equiv G_Y$, $C_{Y|Y}^{W} \equiv G_Y^{W}$, $C_{Y|Y}^{AG} \equiv G_Y^{AG}$, $C_{Y|Y}^{B} \equiv G_Y^{B}$ and $C_{Y|Y}^{T} \equiv G_Y^{T}$, then (19), (20), (21), (22) and (23) would take the following forms:

$$G_Y = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AY}'\left(\mathbf{A}_Y\mathbf{S}\mathbf{A}_Y'\circ \mathbf{D}_{AY}\right)\mathbf{p}_{AY} \tag{24}$$

$$G_Y^{W} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AY}'\left(\mathbf{W}_{AY}\circ \mathbf{S}\circ \mathbf{D}_{AY}\right)\mathbf{p}_{AY} \tag{25}$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AY}'\left[(\mathbf{J}-\mathbf{W}_{AY})\circ \mathbf{A}_Y\mathbf{S}\mathbf{A}_Y'\circ \mathbf{D}_{AY}\right]\mathbf{p}_{AY} \tag{26}$$

$$G_Y^{B} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AY}'\left[(\mathbf{J}-\mathbf{W}_{AY})\circ \mathbf{S}\circ \mathbf{D}_{AY}\right]\mathbf{p}_{AY} \tag{27}$$

$$G_Y^{T} = \frac{1}{2\mu_Y N^2}\mathbf{p}_{AY}'\left[(\mathbf{J}-\mathbf{W}_{AY})\circ (\mathbf{A}_Y\mathbf{S}\mathbf{A}_Y'-\mathbf{S})\circ \mathbf{D}_{AY}\right]\mathbf{p}_{AY} \tag{28}$$

where $\mathbf{D}_{AY} = (\mathbf{j}\mathbf{y}'\mathbf{A}_Y' - \mathbf{A}_Y\mathbf{y}\mathbf{j}') = (\mathbf{j}\mathbf{y}_A' - \mathbf{y}_A\mathbf{j}')$ and $\mathbf{W}_{AY} = \mathbf{A}_Y\mathbf{W}_Y\mathbf{A}_Y'$.

The matrix compact forms (24), (25), (26) (27) and (28) correspond to the scalar expressions (19), (20), (21), (22) and (23), respectively.

We conclude this section by providing closed-form expressions for $C_{Y|X}^{B}$ and $C_{Y|X}^{T}$, by bearing in mind $C_{Y|X}^{AG}$, as specified in (11), under the $\{(x_i, y_i, p_i)\}_X$ ordering:

$$C_{Y|X}^{B} = \frac{1}{2\mu_Y N^2}\mathbf{p}_X'\left[(\mathbf{J}-\mathbf{W}_X)\circ \mathbf{A}_X'\mathbf{S}\mathbf{A}_X\circ \mathbf{D}_{Y|X}\right]\mathbf{p}_X \tag{29}$$

$$C_{Y|X}^{T} = \frac{1}{2\mu_Y N^2}\mathbf{p}_X'\left[(\mathbf{J}-\mathbf{W}_X)\circ (\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\circ \mathbf{D}_{Y|X}\right]\mathbf{p}_X \tag{30}$$

3. Redistribution and re-ranking indexes

The redistributive effect of a tax system can be measured by the difference between the Gini index for the pre-tax income distribution *X* and the Gini index for the post- tax income distribution *Y* [1]: following e.g. Urban and Lambert (2008), we shall denote difference by the acronym *RE*.

The Atkinson-Plotnick-Kakwani index is generally applied to measure the re-ranking effect generated by a tax system; it is defined as the difference between the Gini index for the post-tax income distribution and the concentration index for net incomes *Y* in the $\left\{\left(x_i, y_i, p_i\right)\right\}_X$ ordering [2]. The Atkinson, Plotnick; Kakwani index is usually denoted by the acronym *R*.

In considering the effects of a tax, it may be interesting to evaluate how *RE* and *R* act within and across groups and, eventually, also how they modify both group average positions and group intersections. This can be attained by splitting either *RE* or *R* into the within groups, across groups, between groups and transvariation components, introduced in the previous section.

One of the advantages of the compact expressions introduced in the previous sections is that all indexes can be calculated either aligning incomes according to the pre-tax or according to the post-tax ranking. We will present the *RE* and the *R* indexes by writing **D** matrices and **p** vectors either according to the $\left\{\left(x_i, y_i, p_i\right)\right\}_X$ or the $\left\{\left(x_i, y_i, p_i\right)\right\}_Y$ orderings, when individual income units are considered. Here, for the sake of shortness, the decompositions of *RE* will be reported only according to the $\left\{\left(x_i, y_i, p_i\right)\right\}_X$ ordering, and, conversely, *R* decompositions will be written according to the $\left\{\left(x_i, y_i, p_i\right)\right\}_Y$ ordering. All indexes could be also represented either according to the $\left\{\left(x_i, y_i, p_i\right)\right\}_{AX}$ or to the $\left\{\left(x_i, y_i, p_i\right)\right\}_{AY}$ orderings [3].

3.1.The RE index

From the definition of *RE* we can write

$$RE = G_X - G_Y = \left(G_X^W + G_X^{AG}\right) - \left(G_Y^W + G_Y^{AG}\right) = \left(G_X^W + G_X^B + G_X^T\right) - \left(G_Y^W + G_Y^B + G_Y^T\right)$$

Rearranging terms we get

$$RE = \left(G_X^W - G_Y^W\right) + \left(G_X^{AG} - G_Y^{AG}\right) = RE^W + RE^{AG} \tag{31}$$

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[1] See e.g. Lambert (2001, Ch. 2, Section 2.5).
[2] Plotnick (1981), Lambert (2001, Ch. 2, Section 2.5).
[3] The formulae that are not reported in this article, will be provided to anyone on request.

Here, in what concerns RE^{AG}, bearing in mind that $G^{AG} = G^B + G^T$, we get

$$RE^{AG} = \left(G_X^B - G_Y^B\right) + \left(G_X^T - G_Y^T\right) = RE^B + RE^T \tag{32}$$

From (5) ad (6) it follows that $RE = G_X - G_Y =$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_X' \left[\mu_Y \left(\mathbf{S} \circ \mathbf{D}_X\right) - \mu_X \left(\mathbf{ESE}' \circ \mathbf{D}_{Y|X}\right)\right] \mathbf{p}_X \tag{33a}$$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_Y' \left[\mu_Y \left(\mathbf{E'SE} \circ \mathbf{D}_{X|Y}\right) - \mu_X \left(\mathbf{S} \circ \mathbf{D}_Y\right)\right] \mathbf{p}_Y \tag{33b}$$

The RE^W components can be written, according to (32) and bearing in mind (10) as $RE^W = G_X^W - G_Y^W =$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_X' \left\{\mathbf{W}_X \circ \left[\mu_Y \left(\mathbf{S} \circ \mathbf{D}_X\right) - \mu_X \left(\mathbf{ESE}' \circ \mathbf{D}_{Y|X}\right)\right]\right\} \mathbf{p}_X \tag{34}$$

Likewise the RE^{AG} components can be written as $RE^{AG} = G_X^{AG} - G_Y^{AG} =$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_X' \left\{\left(\mathbf{J} - \mathbf{W}_X\right) \circ \left[\mu_Y \left(\mathbf{S} \circ \mathbf{D}_X\right) - \mu_X \left(\mathbf{ESE}' \circ \mathbf{D}_{Y|X}\right)\right]\right\} \mathbf{p}_X \tag{35}$$

Resorting to (29) and (30), RE^B and RE^T can be rewritten as

$$RE^B = G_X^B - G_Y^B =$$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_X' \left\{\left(\mathbf{J} - \mathbf{W}_X\right) \circ \left[\mu_Y \left(\mathbf{A}_X'\mathbf{S}\mathbf{A}_X \circ \mathbf{D}_X\right)\right.\right.$$

$$\left.\left. - \mu_X \left(\mathbf{E}\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y\mathbf{E}' \circ \mathbf{D}_{Y|X}\right)\right]\right\} \mathbf{p}_X \tag{36}$$

$$RE^T = G_X^T - G_Y^T =$$

$$= \frac{1}{2\mu_X \mu_Y N^2} \mathbf{p}_X' \left\{\left(\mathbf{J} - \mathbf{W}_X\right) \circ \left[\mu_Y \left(\mathbf{S} - \mathbf{A}_X'\mathbf{S}\mathbf{A}_X\right) \circ\right.\right.$$

$$\left.\left. \mathbf{D}_X - \mu_X \mathbf{E}\left(\mathbf{S} - \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y\right)\mathbf{E}' \circ \mathbf{D}_{Y|X}\right]\right\} \mathbf{p}_X \tag{37}$$

3.2. The R (Atkinson-Plotnick-Kakwani) index

From the definition of R we can write

$$R = G_Y - C_{Y|X} = \left(G_Y^W + G_Y^{AG}\right) - \left(C_{Y|X}^W + C_{Y|X}^{AG}\right) = \left(G_Y^W + G_Y^B + G_Y^T\right) - \left(C_{Y|X}^W + C_{Y|X}^B + C_{Y|X}^T\right)$$

Rearranging the terms we get

$$R = \left(G_Y^W - C_{Y|X}^W\right) + \left(G_Y^{AG} - C_{Y|X}^{AG}\right) = R^W + R^{AG} \tag{38}$$

and in particular, for what concerns R^{AG}, we have

$$R^{AG} = \left(G_Y^B - C_{Y|X}^B\right) + \left(G_Y^T - C_{Y|X}^T\right) = R^B + R^T \tag{39}$$

When considering income units individually, from (2), (3), (5) and (7) the index R, and its components, can be written as follows

$$R = G_Y - C_{Y|X} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{S} - \mathbf{E'SE}\right) \odot \mathbf{D}_Y\right]\mathbf{p}_Y \tag{40a}$$

$$= \frac{1}{2\mu_Y N^2}\mathbf{p}_X'\left[\left(\mathbf{ESE'} - \mathbf{S}\right) \odot \mathbf{D}_{Y|X}\right]\mathbf{p}_X \tag{40b}$$

From (10) and (40a) it follows that

$$R^W = G_Y^W - C_{Y|X}^W = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\mathbf{W}_Y \odot \left(\mathbf{S} - \mathbf{E'SE}\right) \odot \mathbf{D}_Y\right]\mathbf{p}_Y \tag{41}$$

From (11) and (40a) it follows that

$$R^{AG} = G_Y^{AG} - C_{Y|X}^{AG} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J} - \mathbf{W}_Y\right) \odot \left(\mathbf{S} - \mathbf{E'SE}\right) \odot \mathbf{D}_Y\right]\mathbf{p}_Y \tag{42}$$

From (29) the component R^B of R can be expressed as

$$R^B = G_Y^B - C_{Y|X}^B =$$

$$\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J} - \mathbf{W}_Y\right) \odot \left(\mathbf{A}_Y'\mathbf{SA}_Y - \mathbf{E'A}_X'\mathbf{SA}_X\mathbf{E}\right) \odot \mathbf{D}_Y\right]\mathbf{p}_Y \tag{43}$$

From (30) the component R^T can be expressed as

$$R^T = G_Y^T - C_{Y|X}^T =$$

$$\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left\{\left(\mathbf{J} - \mathbf{W}_Y\right) \odot \left[\left(\mathbf{S} - \mathbf{A}_Y'\mathbf{SA}_Y\right) - \mathbf{E}'\left(\mathbf{S} - \mathbf{A}_X'\mathbf{SA}_X\right)\mathbf{E}\right] \odot \mathbf{D}_Y\right\}\mathbf{p}_Y \tag{44}$$

Either from the definitions of R^{AG} and R^{B} or by rearranging the terms in (44), R^{T} can be given the following representations:

$$R^{T} = \left(R^{AG} - R^{B}\right) = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\Big\{\left(\mathbf{J}-\mathbf{W}_Y\right)\circ$$

$$\Big[\left(\mathbf{S}-\mathbf{E'SE}\right)-\left(\mathbf{A}_Y'\mathbf{SA}_Y-\mathbf{E'A}_X'\mathbf{SA}_X\mathbf{E}\right)\Big]\circ\ \mathbf{D}_Y\Big\}\mathbf{p}_Y \tag{45}$$

4. The issue of the signs of *R* and its components

We will now analyse the signs of *R* and of its decompositions by making use of the matrix tools introduced in the previous sections. Although most of the results presented in this section are available in the specialized literature[1], we think that our reappraisal of the issue through a tailor-made matrix toolkit provides some additional insights on the matter. Demonstrations will be carried out by inspecting the quadratic form which the *R* index and its decompositions are proportional to.

R

It is well known that for the concentration *C* index the property $-G \le C \le +G$ holds[2], from which it follows that $R = G_Y - C_{Y|X} \ge 0$. This result will be proved considering expression (40a).

Statement 1

The quadratic form $\mathbf{p}_Y'\left[\left(\mathbf{S}-\mathbf{E'SE}\right)\circ\ \mathbf{D}_Y\right]\mathbf{p}$ is non-negative definite.

Proof

Recall that (*i*) matrix $\mathbf{S}=\left[s_{i,j}\right]$ has all super-diagonal elements equal to +1 and sub-diagonal ones equal to −1; (*ii*) the elements of $\mathbf{E'SE}=\left[s^{e}_{i,j}\right]$ may not necessarily respect the same repartition as in $\mathbf{S}$, due to permutations performed by $\mathbf{E}$. Thus, for all entries of $\mathbf{S}$ and $\mathbf{E'SE}$ which present the same values, $s_{i,j}-s^{e}_{i,j}=0$, otherwise for *i*<*j* we would have $s_{i,j}-s^{e}_{i,j}=2$ and, for *i*>*j*, $s_{i,j}-s^{e}_{i,j}=-2$. Bearing in mind that for *i*<*j*, the matrix $\mathbf{D}_Y=\left[d^{Y}_{i,j}\right]$ has super-diagonal elements non-negative and sub-diagonal ones non-positive, the product

[1] Mussini (2008, Ch. 6, § 6.1, page 92) discusses the signs of *R* and its components R^W, R^B and R^T. The author observes also that R^T can be positive, null or negative in the framework of non contiguous pre-tax income groups: the proofs reported here complete the author's statements, especially in what concerns R^T. See also Vernizzi (2007) for considerations on *G* and *C* components especially for pre-tax non overlapping groups.

[2] Kakwani (1980, Corollary 8.7, page 175).

$\left(s_{i,j} - s_{i,j}^{e}\right) \cdot d_{i,j}^{Y}$ will in any case result to be non-negative, which proves the Statement.

R^W and R^{AG}

We will prove that $R^W = G_Y^W - C_{Y|X}^W \geq 0$ and $R^{AG} = G_Y^{AG} - C_{Y|X}^{AG} \geq 0$, by considering expressions (41) and (42) respectively.

<u>Statement 2</u>

The quadratic forms

$\mathbf{p}_Y'\left[\mathbf{W}_Y \circ (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \circ \mathbf{D}_Y\right]\mathbf{p}_Y$ and $\mathbf{p}_Y'\left[(\mathbf{J} - \mathbf{W}_Y) \circ (\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}) \circ \mathbf{D}_Y\right]\mathbf{p}_Y$

are non-negative definite. Statement 2 is just a corollary of Statement 1.

R^B

We now prove that $R^B = G_Y^B - C_{Y|X}^B \geq 0$. In order to carry out the proof as for the previous Statements, it is convenient to consider a matrix compact form that corresponds in a straightforward manner to the second term in the right hand side of (15a). Let us define the $H\times1$ vector $\boldsymbol{\mu}_Y = \left[\mu_{Y1}, \mu_{Y2}, \ldots \mu_{YH}\right]'$ of group averages, $\mu_{Yh} \leq \mu_{Yh+1}$ (h=1,2,….H), the $H\times1$ vector $\overline{\mathbf{p}}_Y = \left[\overline{p}_1, \overline{p}_2, \ldots \overline{p}_H\right]'$ of group weights $\overline{p}_h = \sum_{i=1}^{K_h} p_{h,i}$ and the $H\times H$ matrix $\overline{\mathbf{D}}_Y = \left(\mathbf{1}\boldsymbol{\mu}_Y' - \boldsymbol{\mu}_Y\mathbf{1}'\right)$ of group average differences. Then

$$G_Y^B = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g=1}^{H}\left|\mu_{Y.h} - \mu_{Y.g}\right|\overline{p}_h\overline{p}_g = \frac{1}{2\mu_Y N^2}\overline{\mathbf{p}}_Y'\left[\mathbf{S} \circ \overline{\mathbf{D}}_Y\right]\overline{\mathbf{p}}_Y \qquad (46)$$

where $\mathbf{S}$ is now an $H\times H$ matrix.

After having defined $\boldsymbol{\mu}_{Y|X}$ and $\overline{\mathbf{p}}_X$, respectively, as the $H\times1$ vector of μ_{Yh} and the $H\times1$ vector of $\overline{p}_h$, aligned according to the $\left\{(x_i, y_i, p_i)\right\}_{AX}$ order, and the $H\times H$ matrix $\overline{\mathbf{D}}_{Y|X} = \left(\mathbf{1}\boldsymbol{\mu}_{Y|X}' - \boldsymbol{\mu}_{Y|X}\mathbf{1}'\right)$, (22) can be rewritten in this way:

$$C_Y^B = \frac{1}{2\mu_Y N^2}\overline{\mathbf{p}}_X'\left[\mathbf{S} \circ \overline{\mathbf{D}}_{Y|X}\right]\overline{\mathbf{p}}_X \qquad (47)$$

Finally, by denoting by $\overline{\mathbf{E}}$ the $H\times H$ full rank permutation matrix such that $\boldsymbol{\mu}_{Y|X} = \overline{\mathbf{E}}\boldsymbol{\mu}_Y$, $\boldsymbol{\mu}_Y = \overline{\mathbf{E}}'\boldsymbol{\mu}_{Y|X}$, $\overline{\mathbf{p}}_X = \overline{\mathbf{E}}\overline{\mathbf{p}}_Y$ and $\overline{\mathbf{p}}_Y = \overline{\mathbf{E}}'\overline{\mathbf{p}}_X$, R^B can be rewritten as

$$R^B = G_Y^B - C_Y^B = \frac{1}{2\mu_Y N^2}\overline{\mathbf{p}}_Y'\left[\left(\mathbf{S} - \overline{\mathbf{E}}'\mathbf{S}\overline{\mathbf{E}}\right) \circ \overline{\mathbf{D}}_Y\right]\overline{\mathbf{p}}_Y \qquad (48)$$

Statement 4
The quadratic form
$\overline{\mathbf{p}}_Y'\left[\left(\mathbf{S}-\overline{\mathbf{E}}'\mathbf{S}\overline{\mathbf{E}}\right)\circ\ \overline{\mathbf{D}}_Y\right]\overline{\mathbf{p}}_Y$ is n. n. definite.
Proof
Considerations analogous to those reported above hold for $\left(\mathbf{S}-\overline{\mathbf{E}}'\mathbf{S}\overline{\mathbf{E}}\right)\circ\ \overline{\mathbf{D}}_Y$. In $\overline{\mathbf{D}}_Y$ the super-diagonal entries are non-negative, the sub-diagonal entries are non-positive: while the former are multiplied either by 0 or by +2 entries which are in the super-diagonal part of $\left(\mathbf{S}-\overline{\mathbf{E}}'\mathbf{S}\overline{\mathbf{E}}\right)$, the latter by 0 or by –2 entries which are in the sub-diagonal part of $\left(\mathbf{S}-\overline{\mathbf{E}}'\mathbf{S}\overline{\mathbf{E}}\right)$, and hence it is proved that $R^B\geq 0$.

R^T
Differently from R, R^W, G^T and R^B, that are all non-negative, R^T can be either positive or negative, and, obviously, equal to zero.
Statement 5
In expression (44) the quadratic form

$$\mathbf{p}_Y'\left\{(\mathbf{J}-\mathbf{W}_Y)\circ\ \left[(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y)-\mathbf{E}'(\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}\right]\circ\ \mathbf{D}_Y\right\}\mathbf{p}_Y$$

can be zero, positive or negative.
Proof
Both in matrix $(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y)=\left[{}_Y\omega_{i,j}\right]$ and in matrix $(\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)=\left[{}_X\omega_{i,j}\right]$ non zero super-diagonal entries are +2, non zero sub-diagonal are –2. Due to permutation performed by $\mathbf{E}'$ and $\mathbf{E}$, $\mathbf{E}'(\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}=\left[{}_X\omega_{i,j}^e\right]$ can present some –2 as super-diagonal entries and, symmetrically, some +2 as sub-diagonal entries: hence, not considering the cases when both ${}_Y\omega_{i,j}$ and ${}_X\omega_{i,j}^e$ are zero, the super-diagonal differences in $\left[(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y)-\mathbf{E}'(\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}\right]=\left\{\left[{}_Y\omega_{i,j}\right]-\left[{}_X\omega_{i,j}^e\right]\right\}$ may assume values $[2]-[2]=0$, $[2]-[0]=2$, $[2]-[-2]=4$, $[0]-[-2]=2$, $[0]-[2]=-2$. It follows that non-negative super-diagonal entries of $\mathbf{D}_Y$ can be multiplied by a negative value. Symmetrically, sub-diagonal entries of $\left[(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y)-\mathbf{E}'(\mathbf{S}-\mathbf{A}_X'\mathbf{S}\mathbf{A}_X)\mathbf{E}\right]$ can now be equal not only to $[-2]-[0]=-2$, $[-2]-[2]=-4$, and to $[0]-[2]=-2$, but also to $[0]-[-2]=2$, so that non-positive sub-diagonal entries of $\mathbf{D}_Y$ can be multiplied by a positive value, which proves the Statement.

Conclusions

By use of the Hadamard product, an elegant compact representation in matrix notation has been obtained not only for Gini, concentration indexes and for their decompositions, but for redistribution and re-ranking indexes and their decompositions as well. The matrix toolkit introduced in this paper paves the way to obtain informative expressions for both the said indexes and their components, with incomes aligned either according to the pre-tax non-decreasing order or to the post-tax non-decreasing order.

Moreover, the compact representation introduced in this paper leads to establish in a straightforward manner the signs of the Atkinson-Plotnick-Kakwani index and of its components. We prove that R, R^W, R^{AG} and R^B are non-negative quantities, both when pre-tax income groups do overlap and when do not. In the latter case $R^T \equiv G^T$ ($R^T \equiv R^{AJL}$, following Urban and Lambert, 2008, notation) is non-negative, whereas in the former case we show R^T can be either positive or negative. Even if it is well known that R and $G^T \equiv R^{AJL}$ are non-negative, the proofs presented in this paper are new.

Appendix

On simplifying C_W

We will prove the simplification used in formula (20), that is

$$\mathbf{W}_{AX} \odot \mathbf{A}_X \mathbf{S} \mathbf{A}_X' = \mathbf{W}_{AX} \odot \mathbf{S} \tag{A1}$$

<u>*Proof*</u>

The elements $w_{i,j}$ of matrix $\mathbf{W}_X$ and the elements $w^a_{l,m} = \mathbf{a}_l' \mathbf{W}_X \mathbf{a}_m$ of matrix $\mathbf{W}_{AX}$ are equal to 1 if the associated pair of incomes, x_i and x_j, belong to the same group, they are zero otherwise. As all super-diagonal elements in matrix $\mathbf{S}$ are plus 1 and sub-diagonal elements are -1, we have to prove that all super-diagonal elements of matrix $\mathbf{A}_X \mathbf{S} \mathbf{A}_X'$, that are selected by $\mathbf{W}_{AX}$, are 1, and all sub-diagonal elements of $\mathbf{A}_X \mathbf{S} \mathbf{A}_X'$ selected by $\mathbf{W}_{AX}$ are -1.

Observe that incomes belonging to the same group remain ranked in a non decreasing order within each group, also according to the $\{(x_i, y_i, p_i)\}_{AX}$ ordering: therefore

(i) if in the $\{(x_i, y_i, p_i)\}_X$ ordering x_i occupies the *i*-th position and x_j the *j-th* one, with *i*<*j*, in the $\{(x_i, y_i, p_i)\}_{AX}$ ordering, x_i will occupy the *l*-th position and x_j the *m*-th one with *l*<*m*;

(ii) symmetrically, in the $\{(x_i, y_i, p_i)\}_X$ ordering, all pairs of incomes $x_i > x_j$, belonging to the same group, will respectively be in positions *i* and *j*, *i*>*j*, and in the $\{(x_i, y_i, p_i)\}_{AX}$ ordering, in positions *l* and *m*, *l*>*m*, respectively.

This implies that the entry $s_{i,j}$ of $\mathbf{S}$ will be shifted to the entry $s^a_{l,m}$ of $\mathbf{A}_X\mathbf{S}\mathbf{A}_X'$, with *l*<*m* if *i*<*j*, and *l*>*m* if *i*>*j*, so that in the super-diagonal part of $\mathbf{W}_{AX} \odot \mathbf{A}_X\mathbf{S}\mathbf{A}_X'$ all elements will be equal to 1, and in the sub-diagonal part, all elements will be equal to −1, which proves (A3).

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